Paul's Online Notes

Section 1.1 : Integer Exponents

We will start off this chapter by looking at integer exponents. In fact, we will initially assume that the exponents are positive as well. We will look at zero and negative exponents in a bit.

Let's first recall the definition of exponentiation with positive integer exponents. If a is any number and n is a positive integer then,

$$a^n = \underbrace{a \cdot a \cdot a \cdots \cdots a}_{n \text{ times}}$$

So, for example,

$$3^5 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 243$$

We should also use this opportunity to remind ourselves about parenthesis and conventions that we have in regard to exponentiation and parenthesis. This will be particularly important when dealing with negative numbers. Consider the following two cases.

$$(-2)^4 \qquad \text{and} \qquad -2^4$$

These will have different values once we evaluate them. When performing exponentiation remember that it is only the quantity that is immediately to the left of the exponent that gets the power.

In the first case there is a parenthesis immediately to the left so that means that everything in the parenthesis gets the power. So, in this case we get,

$$(-2)^4 = (-2)(-2)(-2)(-2) = 16$$

In the second case however, the 2 is immediately to the left of the exponent and so it is only the 2 that gets the power. The minus sign will stay out in front and will NOT get the power. In this case we have the following,

$$-2^4 = -\left(2^4\right) = -(2 \cdot 2 \cdot 2 \cdot 2) = -(16) = -16$$

We put in some extra parenthesis to help illustrate this case. In general, they aren't included and we would write instead,

$$-2^4 = -2 \cdot 2 \cdot 2 \cdot 2 = -16$$

The point of this discussion is to make sure that you pay attention to parenthesis. They are important and ignoring parenthesis or putting in a set of parenthesis where they don't belong can completely change the answer to a problem. Be careful. Also, this warning about parenthesis is not just intended for exponents. We will need to be careful with parenthesis throughout this course.

Now, let's take care of zero exponents and negative integer exponents. In the case of zero exponents we have,

$$a^0 = 1 \qquad \text{provided } a \neq 0$$

Notice that it is required that a not be zero. This is important since 0^0 is not defined. Here is a quick example of this property.

$$(-1268)^0 = 1$$

We have the following definition for negative exponents. If a is any non-zero number and n is a positive integer (yes, positive) then,

$$a^{-n} = \frac{1}{a^n}$$

Can you see why we required that a not be zero? Remember that division by zero is not defined and if we had allowed a to be zero we would have gotten division by zero. Here are a couple of quick examples for this definition,

$$5^{-2} = \frac{1}{5^2} = \frac{1}{25} \qquad\qquad (-4)^{-3} = \frac{1}{(-4)^3} = \frac{1}{-64} = -\frac{1}{64}$$

Here are some of the main properties of integer exponents. Accompanying each property will be a quick example to illustrate its use. We will be looking at more complicated examples after the properties.

Properties

1. $a^n a^m = a^{n+m}$ Example : $a^{-9} a^4 = a^{-9+4} = a^{-5}$

2. $\left(a^n\right)^m = a^{nm}$ Example : $\left(a^7\right)^3 = a^{(7)(3)} = a^{21}$

3. $\dfrac{a^n}{a^m} = \begin{cases} a^{n-m} \\ \dfrac{1}{a^{m-n}} \end{cases}$, $\quad a \neq 0$ Example : $\begin{aligned} \dfrac{a^4}{a^{11}} &= a^{4-11} = a^{-7} \\ \dfrac{a^4}{a^{11}} &= \dfrac{1}{a^{11-4}} = \dfrac{1}{a^7} = a^{-7} \end{aligned}$

4. $(ab)^n = a^n b^n$ Example : $(ab)^{-4} = a^{-4} b^{-4}$

5. $\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}, \quad b \neq 0$ Example : $\left(\dfrac{a}{b}\right)^8 = \dfrac{a^8}{b^8}$

6. $\left(\dfrac{a}{b}\right)^{-n} = \left(\dfrac{b}{a}\right)^n = \dfrac{b^n}{a^n}$ Example : $\left(\dfrac{a}{b}\right)^{-10} = \left(\dfrac{b}{a}\right)^{10} = \dfrac{b^{10}}{a^{10}}$

7. $(ab)^{-n} = \dfrac{1}{(ab)^n}$ Example : $(ab)^{-20} = \dfrac{1}{(ab)^{20}}$

8. $\dfrac{1}{a^{-n}} = a^n$ Example : $\dfrac{1}{a^{-2}} = a^2$

9. $\dfrac{a^{-n}}{b^{-m}} = \dfrac{b^m}{a^n}$ Example : $\dfrac{a^{-6}}{b^{-17}} = \dfrac{b^{17}}{a^6}$

10. $(a^n b^m)^k = a^{nk} b^{mk}$ Example : $\left(a^4 b^{-9}\right)^3 = a^{(4)(3)} b^{(-9)(3)} = a^{12} b^{-27}$

11. $\left(\dfrac{a^n}{b^m}\right)^k = \dfrac{a^{nk}}{b^{mk}}$ Example : $\left(\dfrac{a^6}{b^5}\right)^2 = \dfrac{a^{(6)(2)}}{b^{(5)(2)}} = \dfrac{a^{12}}{b^{10}}$

Notice that there are two possible forms for the third property. Which form you use is usually dependent upon the form you want the answer to be in.

Note as well that many of these properties were given with only two terms/factors but they can be extended out to as many terms/factors as we need. For example, property 4 can be extended as follows.

$$(abcd)^n = a^n b^n c^n d^n$$

We only used four factors here, but hopefully you get the point. Property 4 (and most of the other properties) can be extended out to meet the number of factors that we have in a given problem.

There are several common mistakes that students make with these properties the first time they see them. Let's take a look at a couple of them.

Consider the following case.

Correct : $ab^{-2} = a\dfrac{1}{b^2} = \dfrac{a}{b^2}$

Incorrect : $ab^{-2} \neq \dfrac{1}{ab^2}$

In this case only the b gets the exponent since it is immediately off to the left of the exponent and so only this term moves to the denominator. Do NOT carry the a down to the denominator with the b. Contrast this with the following case.

$$(ab)^{-2} = \frac{1}{(ab)^2}$$

In this case the exponent is on the set of parenthesis and so we can just use property 7 on it and so both the a and the b move down to the denominator. Again, note the importance of parenthesis and how they can change an answer!

Here is another common mistake.

$$\text{Correct} \ : \quad\quad \frac{1}{3a^{-5}} = \frac{1}{3}\frac{1}{a^{-5}} = \frac{1}{3}a^5$$

$$\text{Incorrect} \ : \quad\quad \frac{1}{3a^{-5}} \neq 3a^5$$

In this case the exponent is only on the a and so to use property 8 on this we would have to break up the fraction as shown and then use property 8 only on the second term. To bring the 3 up with the a we would have needed the following.

$$\frac{1}{(3a)^{-5}} = (3a)^5$$

Once again, notice this common mistake comes down to being careful with parenthesis. This will be a constant refrain throughout these notes. We must always be careful with parenthesis. Misusing them can lead to incorrect answers.

Let's take a look at some more complicated examples now.

Example 1 Simplify each of the following and write the answers with only positive exponents.

(a) $\left(4x^{-4}y^5\right)^3$

(b) $\left(-10z^2y^{-4}\right)^2\left(z^3y\right)^{-5}$

(c) $\dfrac{n^{-2}m}{7m^{-4}n^{-3}}$

(d) $\dfrac{5x^{-1}y^{-4}}{(3y^5)^{-2}x^9}$

(e) $\left(\dfrac{z^{-5}}{z^{-2}x^{-1}}\right)^6$

(f) $\left(\dfrac{24a^3b^{-8}}{6a^{-5}b}\right)^{-2}$

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(a) $\left(4x^{-4}y^5\right)^3$ *Show Solution* ▸

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(d) $\dfrac{5x^{-1}y^{-4}}{\left(3y^5\right)^{-2}x^9}$ *Show Solution* ▸

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(f) $\left(\dfrac{24a^3b^{-8}}{6a^{-5}b}\right)^{-2}$ *Show Solution* ▸

Before leaving this section we need to talk briefly about the requirement of positive only exponents in the above set of examples. This was done only so there would be a consistent final answer. In many cases negative exponents are okay and in some cases they are required. In fact, if you are on a track that will take you into calculus there are a fair number of problems in a calculus class in which negative exponents are the preferred, if not required, form.

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