1.1: The Pythagorean Theorem

Learning Objectives

- Use the Pythagorean Theorem to determine if a triangle is a right triangle.
- Use the Pythagorean Theorem to determine the length of one side of a right triangle.
- Use the distance formula to determine the distance between two points on the coordinate plane.

Recall the following definitions from elementary geometry:

a. An angle is **acute** if it is between 0° and 90°.
b. An angle is a **right angle** if it equals 90°.
c. An angle is **obtuse** if it is between 90° and 180°.
d. An angle is a **straight angle** if it equals 180°.

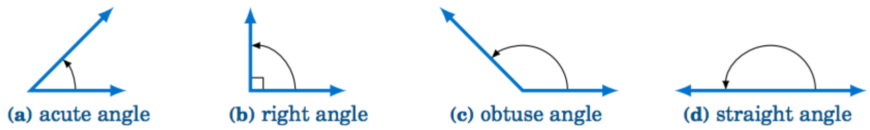

Figure 1.1.1 *Types of angles*

In elementary geometry, angles are always considered to be positive and not larger than 360°. For now we will only consider such angles. The following definitions will be used throughout the text:

a. Two acute angles are **complementary** if their sum equals 90°. In other words, if $0^\circ \leq \angle A, \angle B \leq 90^\circ$ then $\angle A$ and $\angle B$ are complementary if $\angle A + \angle B = 90^\circ$.
b. Two angles between 0° and 180° are **supplementary** if their sum equals 180°. In other words, if $0^\circ \leq \angle A, \angle B \leq 180^\circ$ then $\angle A$ and $\angle B$ are supplementary if $\angle A + \angle B = 180^\circ$.
c. Two angles between 0° and 360° are **conjugate** (or **explementary**) if their sum equals 360°. In other words, if $0^\circ \leq \angle A, \angle B \leq 360^\circ$ then $\angle A$ and $\angle B$ are conjugate if $\angle A + \angle B = 360^\circ$.

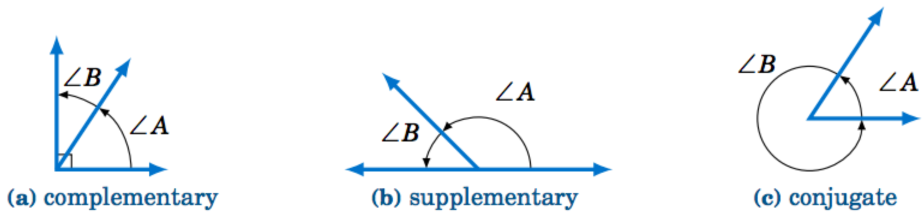

Figure 1.1.2 Types of pairs of angles

Instead of using the angle notation $\angle A$ to denote an angle, we will sometimes use just a capital letter by itself (e.g. A, B, C) or a lowercase variable name (e.g. x, y, t). It is also common to use letters (either uppercase or lowercase) from the Greek alphabet, shown in the table below, to represent angles:

Table 1.1 The Greek alphabet

Letters		Name	Letters		Name	Letters		Name
A	α	alpha	I	ι	iota	P	ρ	rho
B	β	beta	K	κ	kappa	Σ	σ	sigma
Γ	γ	gamma	Λ	λ	lambda	T	τ	tau
Δ	δ	delta	M	μ	mu	Υ	υ	upsilon
E	ϵ	epsilon	N	ν	nu	Φ	ϕ	phi
Z	ζ	zeta	Ξ	ξ	xi	X	χ	chi
H	η	eta	O	o	omicron	Ψ	ψ	psi
Θ	θ	theta	Π	π	pi	Ω	ω	omega

In elementary geometry you learned that the sum of the angles in a triangle equals 180°, and that an **isosceles triangle** is a triangle with two sides of equal length. Recall that in a **right triangle** one of the angles is a right angle. Thus, in a right triangle one of the angles is 90° and the other two angles are acute angles whose sum is 90° (i.e. the other two angles are complementary angles).

$$\alpha + 3\alpha + \alpha = 180^\circ \Rightarrow 5\alpha = 180^\circ \Rightarrow \alpha = 36^\circ \Rightarrow \boxed{X = 36^\circ,\ Y = 3 \times 36^\circ = 108^\circ,\ Z = 36^\circ}$$

QED

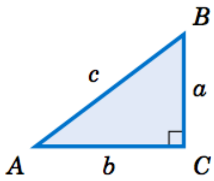

Figure 1.1.3

By knowing the lengths of two sides of a right triangle, the length of the third side can be determined by using the **Pythagorean Theorem**:

$$a^2 + b^2 = c^2$$

The square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of its legs.

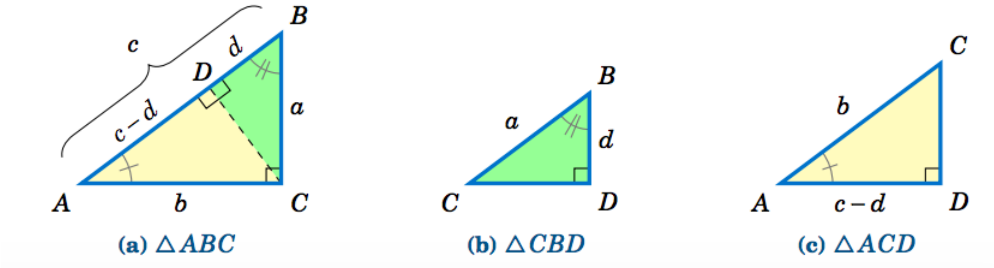

Figure 1.1.4 Similar triangles $\triangle ABC, \triangle CBD, \triangle ACD$

Recall that triangles are **similar** if their corresponding angles are equal, and that similarity implies that corresponding sides are proportional. Thus, since $\triangle ABC$ is similar to $\triangle CBD$, by proportionality of corresponding sides we see that

$$\overline{AB} \text{ is to } \overline{CB} \text{ (hypotenuses) as } \overline{BC} \text{ is to } \overline{BD} \text{ (vertical legs)} \quad \Rightarrow \quad \frac{c}{a} = \frac{a}{d} \quad \Rightarrow \quad cd = a^2 .$$

Since $\triangle ABC$ is similar to $\triangle ACD$, comparing horizontal legs and hypotenuses gives

$$\frac{b}{c-d} = \frac{c}{b} \quad \Rightarrow \quad b^2 = c^2 - cd = c^2 - a^2 \quad \Rightarrow \quad a^2 + b^2 = c^2 . \textbf{ QED}$$

Note: The symbols $\perp$ and $\sim$ denote perpendicularity and similarity, respectively. For example, in the above proof we had $\overline{CD} \perp \overline{AB}$ and $\triangle ABC \sim \triangle CBD \sim \triangle ACD$.

For triangle $\triangle ABC$, the Pythagorean Theorem says that

$$a^2 + 4^2 = 5^2 \quad \Rightarrow \quad a^2 = 25 - 16 = 9 \quad \Rightarrow \quad \boxed{a = 3}.$$

For triangle $\triangle DEF$, the Pythagorean Theorem says that

$$e^2 + 1^2 = 2^2 \quad \Rightarrow \quad e^2 = 4 - 1 = 3 \quad \Rightarrow \quad \boxed{e = \sqrt{3}}.$$

For triangle $\triangle XYZ$, the Pythagorean Theorem says that

$$1^2 + 1^2 = z^2 \quad \Rightarrow \quad z^2 = 2 \quad \Rightarrow \quad \boxed{z = \sqrt{2}}.$$

Let h be the height at which the ladder touches the wall. We can assume that the ground makes a right angle with the wall, as in the picture on the right. Then we see that the ladder, ground, and wall form a right triangle with a hypotenuse of length 17 ft (the length of the ladder) and legs with lengths 8 ft and h ft. So by the Pythagorean Theorem, we have

$$h^2 + 8^2 = 17^2 \quad \Rightarrow \quad h^2 = 289 - 64 = 225 \quad \Rightarrow \quad \boxed{h = 15 \text{ ft}}.$$

Determining the Distance Using the Pythagorean Theorem

You can use the Pythagorean Theorem is to find the distance between two points.

Consider the points (-1, 6) and (5, -3). If we plot these points on a grid and connect them, they make a diagonal line. Draw a vertical line down from (-1, 6) and a horizontal line to the left of (5, -3) to make a right triangle.

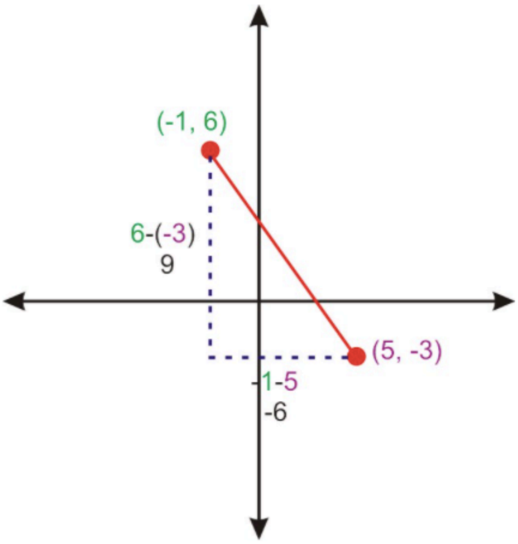

Figure 1.1.5

Now we can find the distance between these two points by using the vertical and horizontal distances that we determined from the graph.

$$\begin{aligned} 9^2 + (-6)^2 &= d^2 \\ 81 + 36 &= d^2 \\ 117 &= d^2 \\ \sqrt{117} &= d \\ 3\sqrt{13} &= d \end{aligned}$$

Notice, that the x−values were subtracted from each other to find the horizontal distance and the y−values were subtracted from each other to find the vertical distance. If this process is generalized for two points $(x_1, y_1)$and (x_2, y_2), the **Distance Formula** is derived.

$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = d^2$$

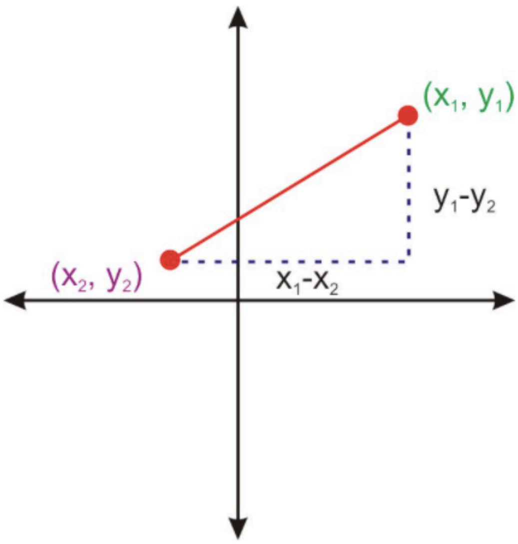

Figure 1.1.6

This is the Pythagorean Theorem with the vertical and horizontal differences between (x_1, y_1) and (x_2, y_2). Taking the square root of both sides will solve the right hand side for d, the distance.

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = d$$

This is the Distance Formula.

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