

Paul's Online Notes

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Section 1.3 : Radicals

We'll open this section with the definition of the radical. If n is a positive integer that is greater than 1 and a is a real number then,

$$\sqrt[n]{a} = a^{\frac{1}{n}}$$

where n is called the **index**, a is called the **radicand**, and the symbol $\sqrt{}$ is called the **radical**. The left side of this equation is often called the radical form and the right side is often called the exponent form.

From this definition we can see that a radical is simply another notation for the first rational exponent that we looked at in the [rational exponents section](#).

Note as well that the index is required in these to make sure that we correctly evaluate the radical. There is one exception to this rule and that is square root. For square roots we have,

$$\sqrt[2]{a} = \sqrt{a}$$

In other words, for square roots we typically drop the index.

Let's do a couple of examples to familiarize us with this new notation.

Example 1 Write each of the following radicals in exponent form.

(a) $\sqrt[4]{16}$

(b) $\sqrt[10]{8x}$

(c) $\sqrt{x^2 + y^2}$

[Show Solution](#) ▶

As seen in the last two parts of this example we need to be careful with parenthesis. When we convert to exponent form and the radicand consists of more than one term then we need to enclose the whole radicand in parenthesis as we did with these two parts. To see why this is consider the following,

$$8x^{\frac{1}{10}}$$



From our discussion of exponents in the previous sections we know that only the term immediately to the left of the exponent actually gets the exponent. Therefore, the radical form of this is,

$$8x^{\frac{1}{10}} = 8 \sqrt[10]{x} \neq \sqrt[10]{8x}$$

So, we once again see that parenthesis are very important in this class. Be careful with them.

Since we know how to evaluate rational exponents we also know how to evaluate radicals as the following set of examples shows.

Example 2 Evaluate each of the following.

(a) $\sqrt{16}$ and $\sqrt[4]{16}$

(b) $\sqrt[5]{243}$

(c) $\sqrt[4]{1296}$

(d) $\sqrt[3]{-125}$

(e) $\sqrt[4]{-16}$

Show Discussion ▶

(a) $\sqrt{16}$ and $\sqrt[4]{16}$ **Show Solution** ▶

(b) $\sqrt[5]{243}$ **Show Solution** ▶

(c) $\sqrt[4]{1296}$ **Show Solution** ▶

(d) $\sqrt[3]{-125}$ **Show Solution** ▶

(e) $\sqrt[4]{-16}$ **Show Solution** ▶

Let's briefly discuss the answer to the first part in the above example. In this part we made the claim that $\sqrt{16} = 4$ because $4^2 = 16$. However, 4 isn't the only number that we can square to get 16. We also have $(-4)^2 = 16$. So, why didn't we use -4 instead? There is a general rule about evaluating square roots (or more generally radicals with even indexes). When evaluating square roots we ALWAYS take the positive answer. If we want the negative answer we will do the following.

$$-\sqrt{16} = -4$$

This may not seem to be all that important, but in later topics this can be very important.

Following this convention means that we will always get predictable values when evaluating

roots.

Note that we don't have a similar rule for radicals with odd indexes such as the cube root in part (d) above. This is because there will never be more than one possible answer for a radical with an odd index.

We can also write the general rational exponent in terms of radicals as follows.

$$a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(\sqrt[n]{a}\right)^m \quad \text{OR} \quad a^{\frac{m}{n}} = \left(a^m\right)^{\frac{1}{n}} = \sqrt[n]{a^m}$$

We now need to talk about some properties of radicals.

Properties

If n is a positive integer greater than 1 and both a and b are positive real numbers then,

$$1. \sqrt[n]{a^n} = a$$

$$2. \sqrt[n]{ab} = \sqrt[n]{a} \sqrt[n]{b}$$

$$3. \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}$$

Note that on occasion we can allow a or b to be negative and still have these properties work. When we run across those situations we will acknowledge them. However, for the remainder of this section we will assume that a and b must be positive.

Also note that while we can “break up” products and quotients under a radical we can't do the same thing for sums or differences. In other words,

$$\sqrt[n]{a+b} \neq \sqrt[n]{a} + \sqrt[n]{b} \quad \text{AND} \quad \sqrt[n]{a-b} \neq \sqrt[n]{a} - \sqrt[n]{b}$$

If you aren't sure that you believe this consider the following quick number example.

$$5 = \sqrt{25} = \sqrt{9+16} \neq \sqrt{9} + \sqrt{16} = 3 + 4 = 7$$

If we “break up” the root into the sum of the two pieces we clearly get different answers! So, be careful to not make this very common mistake!

We are going to be simplifying radicals shortly so we should next define **simplified radical form**. A radical is said to be in simplified radical form (or just simplified form) if each of the following are true.

Simplified Radical Form

1. All exponents in the radicand must be less than the index.
2. Any exponents in the radicand can have no factors in common with the index.
3. No fractions appear under a radical.
4. No radicals appear in the denominator of a fraction.

In our first set of simplification examples we will only look at the first two. We will need to do a little more work before we can deal with the last two.

Example 3 Simplify each of the following. Assume that x , y , and z are positive.

(a) $\sqrt{y^7}$

(b) $\sqrt[9]{x^6}$

(c) $\sqrt{18x^6y^{11}}$

(d) $\sqrt[4]{32x^9y^5z^{12}}$

(e) $\sqrt[5]{x^{12}y^4z^{24}}$

(f) $\sqrt[3]{9x^2} \sqrt[3]{6x^2}$

(a) $\sqrt{y^7}$ [Show Solution](#) ▶

(b) $\sqrt[9]{x^6}$ [Show Solution](#) ▶

(c) $\sqrt{18x^6y^{11}}$ [Show Solution](#) ▶

(d) $\sqrt[4]{32x^9y^5z^{12}}$ [Show Solution](#) ▶

(e) $\sqrt[5]{x^{12}y^4z^{24}}$ [Show Solution](#) ▶

(f) $\sqrt[3]{9x^2} \sqrt[3]{6x^2}$ [Show Solution](#) ▶

Before moving into a set of examples illustrating the last two simplification rules we need to talk briefly about adding/subtracting/multiplying radicals. Performing these operations with radicals is much the same as performing these operations with polynomials. If you don't remember how to add/subtract/multiply polynomials we will give a quick reminder here and then give a more in depth set of examples the next section.

Recall that to add/subtract terms with x in them all we need to do is add/subtract the coefficients of the x . For example,

$$4x + 9x = (4 + 9)x = 13x \qquad 3x - 11x = (3 - 11)x = -8x$$

Adding/subtracting radicals works in exactly the same manner. For instance,

$$4\sqrt{x} + 9\sqrt{x} = (4 + 9)\sqrt{x} = 13\sqrt{x} \qquad 3\sqrt[10]{5} - 11\sqrt[10]{5} = (3 - 11)\sqrt[10]{5} = -8\sqrt[10]{5}$$

We've already seen some multiplication of radicals in the last part of the previous example. If we are looking at the product of two radicals with the same index then all we need to do is use the second property of radicals to combine them then simplify. What we need to look at now are problems like the following set of examples.

Example 4 Multiply each of the following. Assume that x is positive.

(a) $(\sqrt{x} + 2)(\sqrt{x} - 5)$

(b) $(3\sqrt{x} - \sqrt{y})(2\sqrt{x} - 5\sqrt{y})$

(c) $(5\sqrt{x} + 2)(5\sqrt{x} - 2)$

Show Discussion ▶

(a) $(\sqrt{x} + 2)(\sqrt{x} - 5)$ **Show Solution** ▶

(b) $(3\sqrt{x} - \sqrt{y})(2\sqrt{x} - 5\sqrt{y})$ **Show Solution** ▶

(c) $(5\sqrt{x} + 2)(5\sqrt{x} - 2)$ **Show Solution** ▶

The last part of the previous example really used the fact that

$$(a + b)(a - b) = a^2 - b^2$$

If you don't recall this formula we will look at it in a little more detail in the next section.

Okay, we are now ready to take a look at some simplification examples illustrating the final two rules. Note as well that the fourth rule says that we shouldn't have any radicals in the denominator. To get rid of them we will use some of the multiplication ideas that we looked at above and the process of getting rid of the radicals in the denominator is called **rationalizing the denominator**. In fact, that is really what this next set of examples is about. They are really more examples of rationalizing the denominator rather than simplification examples.

Example 5 Rationalize the denominator for each of the following. Assume that x is positive.

(a) $\frac{4}{\sqrt{x}}$

(b) $\sqrt[5]{\frac{2}{x^3}}$

(c) $\frac{1}{3 - \sqrt{x}}$

(d) $\frac{5}{4\sqrt{x} + \sqrt{3}}$

Show Discussion ▶

(a) $\frac{4}{\sqrt{x}}$ **Show Solution** ▶

(b) $\sqrt[5]{\frac{2}{x^3}}$ **Show Solution** ▶

(c) $\frac{1}{3 - \sqrt{x}}$ **Show Solution** ▶

(d) $\frac{5}{4\sqrt{x} + \sqrt{3}}$ **Show Solution** ▶

Rationalizing the denominator may seem to have no real uses and to be honest we won't see many uses in an Algebra class. However, if you are on a track that will take you into a Calculus class you will find that rationalizing is useful on occasion at that level.

We will close out this section with a more general version of the first property of radicals. Recall that when we first wrote down the properties of radicals we required that a be a positive number. This was done to make the work in this section a little easier. However, with the first property that doesn't necessarily need to be the case.

Here is the property for a general a (i.e. positive or negative)

$$\sqrt[n]{a^n} = \begin{cases} |a| & \text{if } n \text{ is even} \\ a & \text{if } n \text{ is odd} \end{cases}$$

where $|a|$ is the absolute value of a . If you don't recall absolute value we will cover that in detail in a **section** in the next chapter. All that you need to do is know at this point is that absolute value always makes a a positive number.

So, as a quick example this means that,

$$\sqrt[8]{x^8} = |x| \quad \text{AND} \quad \sqrt[11]{x^{11}} = x$$

For square roots this is,

$$\sqrt{x^2} = |x|$$

This will not be something we need to worry all that much about here, but again there are topics in courses after an Algebra course for which this is an important idea so we needed to at least acknowledge it.