# Pattern Recognition Project 2

least squares regression and nearest neighbor classification

# Task 2.1:
# Least Squares Method for Polynomial Regression

# Problem Statement:

- Use method of least squares to fit polynomial model to the data
- Plot the results for $d \in \{1, 5, 10\}$
- Use the resulting model to predict missing weight values

# Method:

Estimation of weights using: $\boldsymbol{w} = \left(XX^\top\right)^{-1} X\boldsymbol{y}$

**Problem faced:**

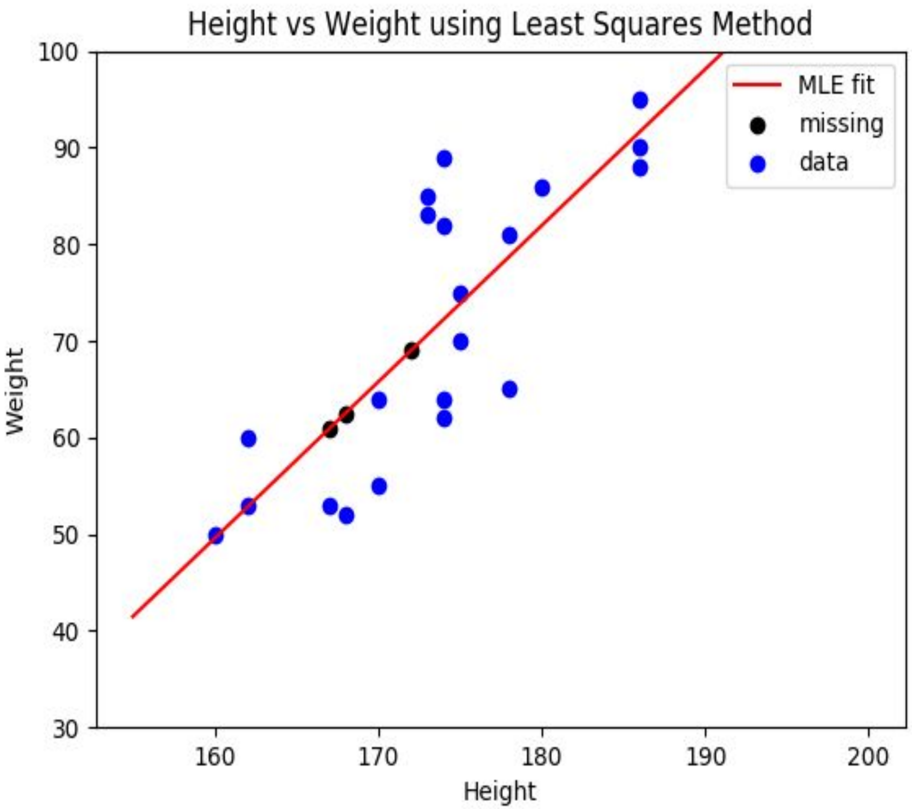
Difficulty in finding inverse of matrices with large values for d=5 and 10.

**Solution:**

To resolve we used standardization of matrix values for numerical stability

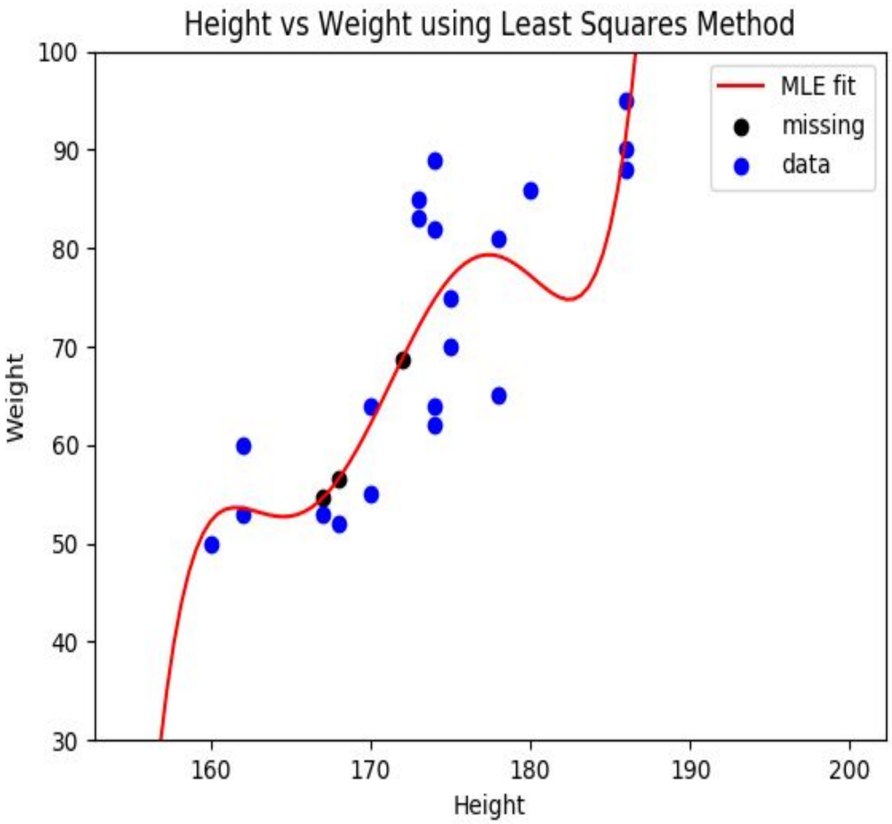
# Results for order 1:

| | | | |
|---|---|---|---|
| Height | 168.0 | 172.0 | 167.0 |
| Weight | 62.51 | 68.98 | 60.89 |



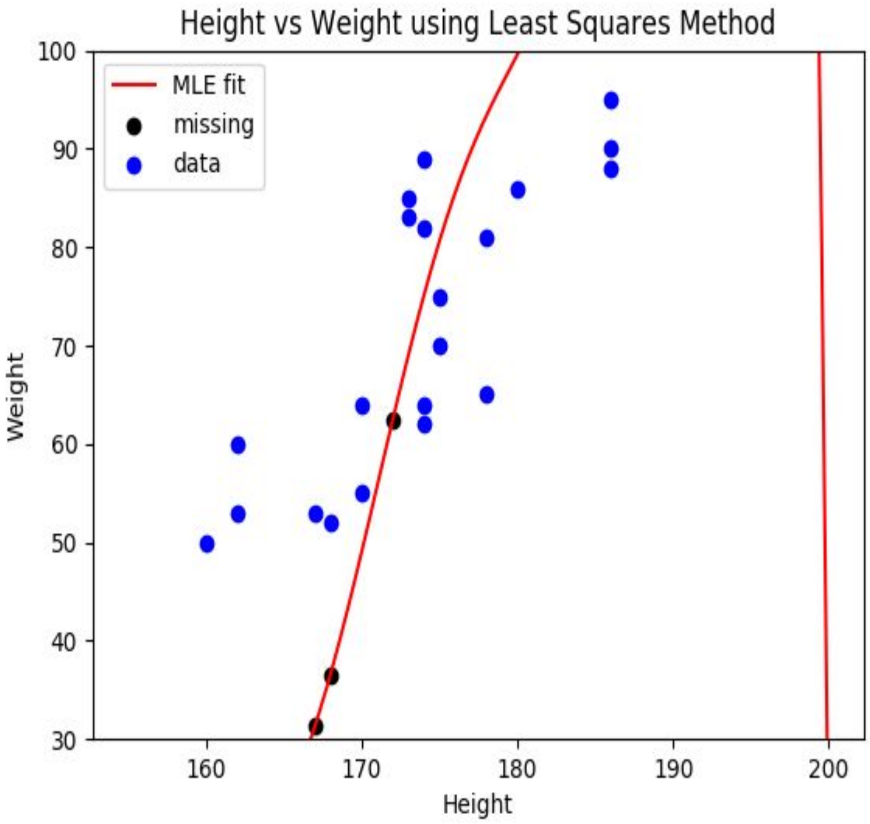
# Results for order 5:

| | | | |
|---|---|---|---|
| Height | 168.0 | 172.0 | 167.0 |
| Weight | 56.53 | 68.76 | 54.58 |



# Results for order 10:

| | | | |
|---|---|---|---|
| Height | 168.0 | 172.0 | 167.0 |
| Weight | 36.57 | 62.46 | 31.30 |



# Task 2.2:
# Conditional Expectation for Missing Value Prediction

# Problem Statement:

- To fit a Bivariate Gaussian to model the joint density of x(height) and y(weight).
- Use the resulting model to predict missing weight values

# Approach

PDF: $P(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left[-\frac{z}{2(1-\rho^2)}\right],$

Where :

$$z \equiv \frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2},$$

$$\rho \equiv \operatorname{cor}(x_1, x_2) = \frac{V_{12}}{\sigma_1\sigma_2}$$

The conditional expectation of X given Y satisfies : $\mathbf{E}[X \mid Y] = \mathbf{E}[X] + \rho\frac{\sigma_X}{\sigma_Y}(Y - \mathbf{E}[Y]).$

The conditional distribution of X given Y is normal with mean E[X | Y ] and variance σ2x^

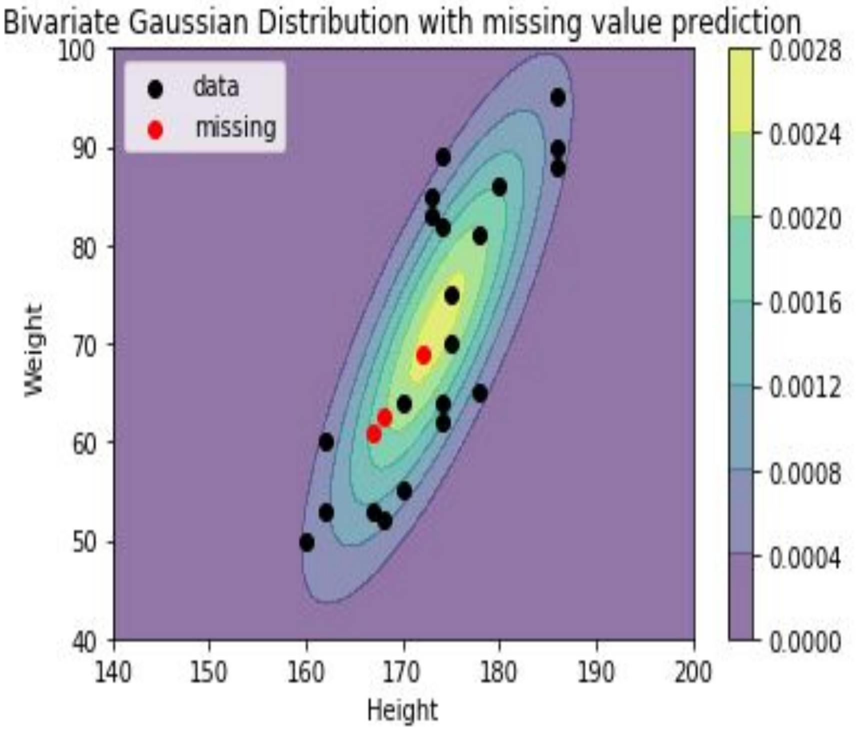
Thus the Conditional PDF:

$$f_{X,Y}(x,y) = ce^{-q(x,y)},$$

Where: $c = \frac{1}{2\pi\sqrt{1-\rho^2}\,\sigma_X\sigma_Y}.$ $q(x,y) = \frac{\frac{x^2}{\sigma_X^2} - 2\rho\frac{xy}{\sigma_X\sigma_Y} + \frac{y^2}{\sigma_Y^2}}{2(1-\rho^2)}.$

# Missing Values

| Weights | Heights |
|---|---|
| 62.50 | 168.0 |
| 68.98 | 172.0 |
| 60.89 | 167.0 |



# Exercise 3:
# Bayesian Regression

# Preprocessing Data:

- **Problem:** Fit a 5th order Polynomial using Bayesian Regression.
- Given Data:

$$\boxed{H = [h_1, h_2, ......, h_n]}$$

- Preprocessing for a 5th order Polynomial:

$$\boxed{X = [h'_1, h'_2, ......, h'_n], \text{ where } h'_i = [1, h_i, h_i^2, ..., h_i^5]^T}$$

- Standardization for numerical stability (used only for MLE):

$$\boxed{x = \frac{x - \mu_X}{\sigma_X}}$$

# Fitting Weights:

- Preprocessed Input:

$$\boxed{X = [h'_1, h'_2, ......, h'_n], \text{ where } h'_i = [1, h_i, h_i^2, ..., h_i^5]^T}$$

- Given Weights: $\boxed{W = [w_0, w_1, ......, w_5]}$

- **<u>MAP Update</u>:**

$$\boxed{W_{MAP} = (XX^T + \frac{\sigma^2}{\sigma_0^2} I)^{-1} Xy}$$

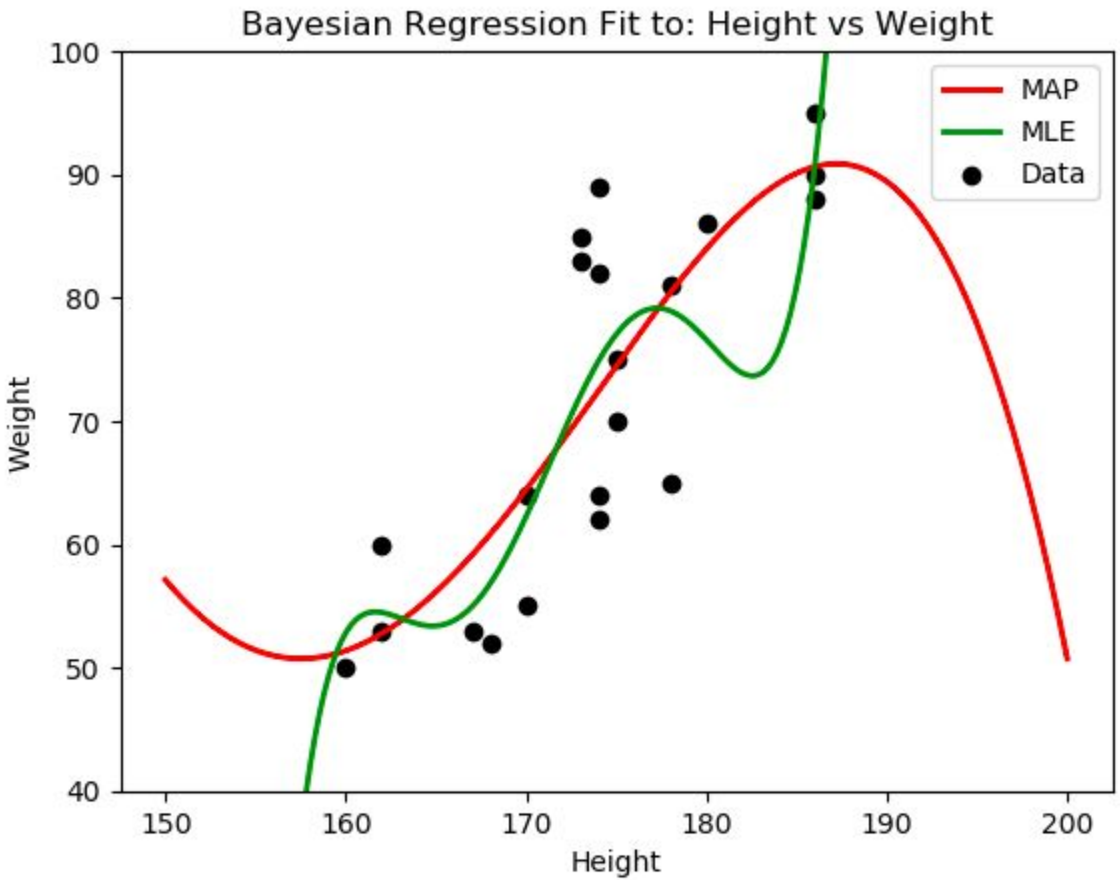
**<u>MLE Update</u>:**

$$\boxed{W_{MLE} = (XX^T)^{-1} Xy}$$

# MAP vs MLE

**Mean Squared Error Loss:**

- MLE : 62.4830
- MAP: 69.8029



# Exercise 4:
# Boolean functions and the Boolean Fourier Transform

# Problem Statement:

- Use least squares to approximate rules of a cellular automata by the multiplication of an vector with an matrix
- Two variants of matrices:
  - Usage of the matrix X specifying the rule (with entries in {-1, 1})
  - Usage of a feature matrix based on X

# Simple Model fitting

Problem: find w* that minimizes :

$$\boxed{E[w] = ||Xw - y||^2}$$

Solution from lecture:

$$\boxed{w^* = (X^T X)^{-1} X^T y}$$

Results:

$$\boxed{\begin{aligned} w^*_{110} &= (0.25, -0.25, -0.25)^T \\ \hat{y}_{110} &= (0.25, -0.25, -0.25, -0.75, 0.75, 0.25, 0.25, -0.25)^T \\ w^*_{126} &= (0.00, 0.00, 0.00)^T \\ \hat{y}_{126} &= (0.00, 0.00, 0.00, 0.00, 0.00, 0.00, 0.00, 0.00)^T \end{aligned}}$$

# Model fitting using the feature Matrix

To increase the quality of our results we replace the matrix by the feature matrix, as seen on the left

where $\varphi$ is defined as:

$$\boxed{\varphi_S(x_1, x_2, x_3) = \prod_{i \in S} x_i}$$

$$\boxed{\Phi = \begin{bmatrix} — & \varphi(X_1) & — \\ — & \varphi(X_2) & — \\ — & \varphi(X_3) & — \\ — & \varphi(X_4) & — \\ — & \varphi(X_5) & — \\ — & \varphi(X_6) & — \\ — & \varphi(X_7) & — \\ — & \varphi(X_8) & — \end{bmatrix}}$$

# Final Results

$$\boxed{\begin{aligned} w^*_{110} &= (0.25, 0.25, -0.25, -0.25, -0.25, -0.25, -0.75, 0.25)^T \\ \hat{y}_{110} &= (-1, +1, +1, -1, +1, +1, +1, -1)^T \\ w^*_{126} &= (0.50, 0.00, 0.00, -0.50, 0.00, -0.50, -0.50, 0.00)^T \\ \hat{y}_{126} &= (-1, +1, +1, +1, +1, +1, +1, -1)^T \end{aligned}}$$

# Exercise 5:

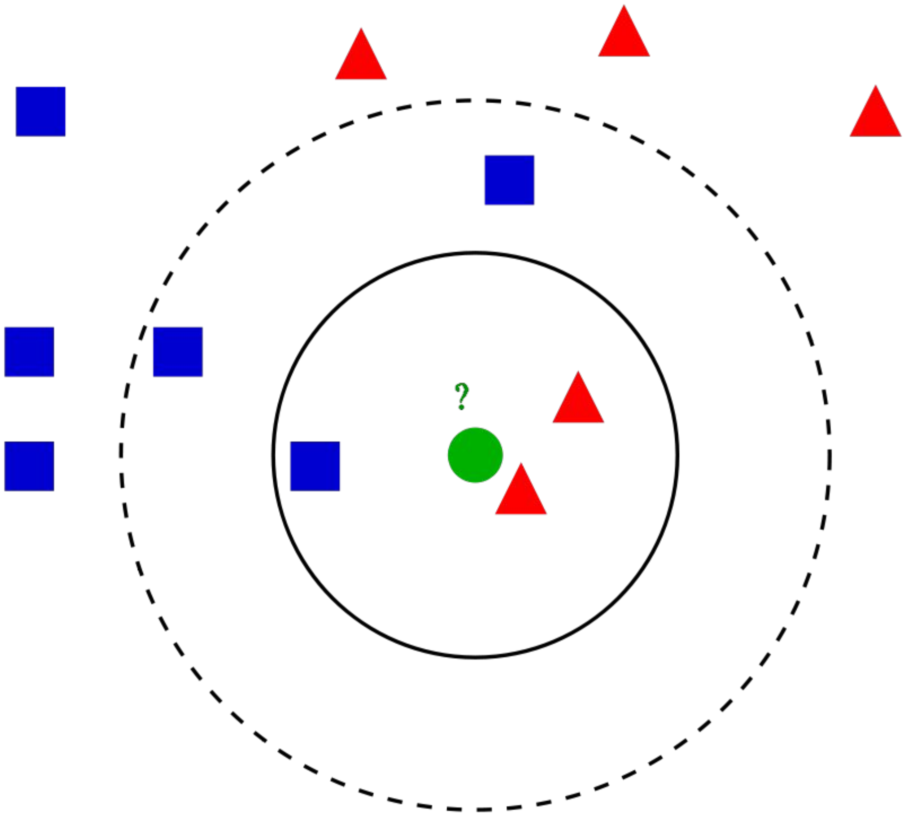
# Nearest Neighbor Classifier

# Problem Statement:

- Implement a function that realizes an n-nearest neighbor for $n \in \{1, 3, 5\}$.
- Show the percentage accuracy.
- Show the overall runtime for $n = 1$ for test data.

# What is a nearest neighbor classifier?

It's a method in which n closest points w.r.t a given test sample are searched in the training data and based on the majority, a class is assigned to it.



# Method:

- Calculate Euclidean of the test data with all of the training data
- Find the n training points with smallest distances
- Take the majority of the labels from the neighbors

# Broadcasting

- Calculation of distance involves subtracting the points and then taking the p-norm of the difference.
- How to calculate the difference between single test point and all of the training data ?
  1. Iterate over training data using for loop and subtract the test point from each training point
  2. Vectorise approach using Broadcasting

# Broadcasting

It is process of making two arrays of dissimilar dimensions compatible with each other for performing arithmetic operations.

Example

|1 2 3| + |7 8 9|
|4 5 6|

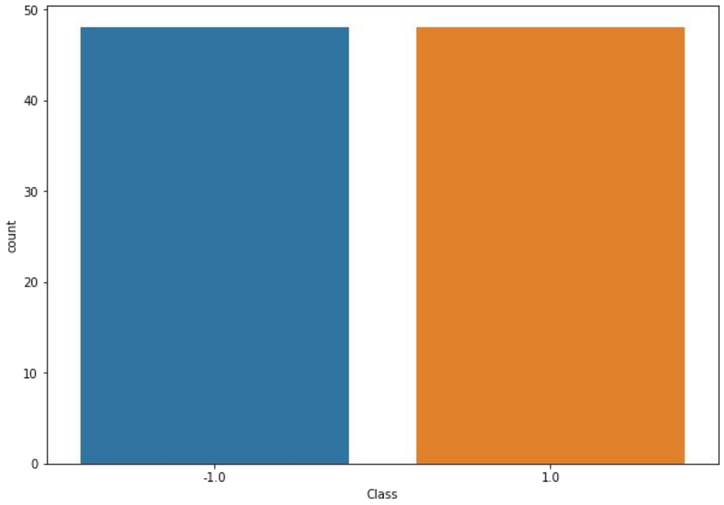
≡

|1 2 3| + |7 8 9| = |8 10 12|
|4 5 6| |7 8 9| |11 13 15|

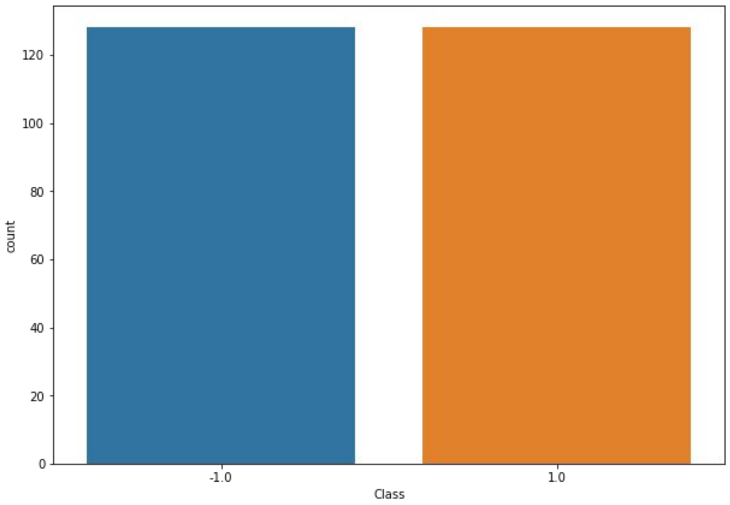
Now for subtraction of test data from training samples, a single line of code can be used with the in-built broadcasting of numpy

```
diff = X_train - data_point
```

# Visualisations

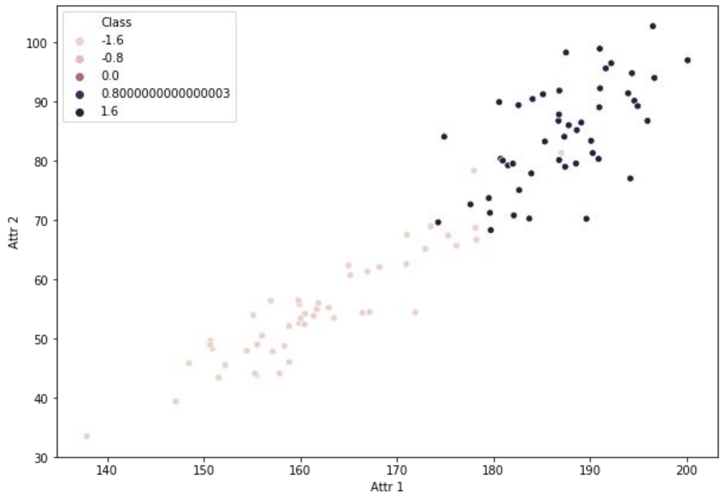



Test Data

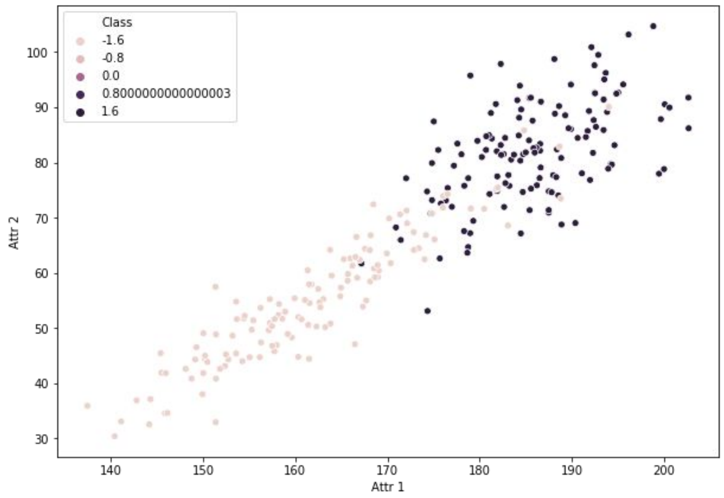



Training Data

# Visualisations



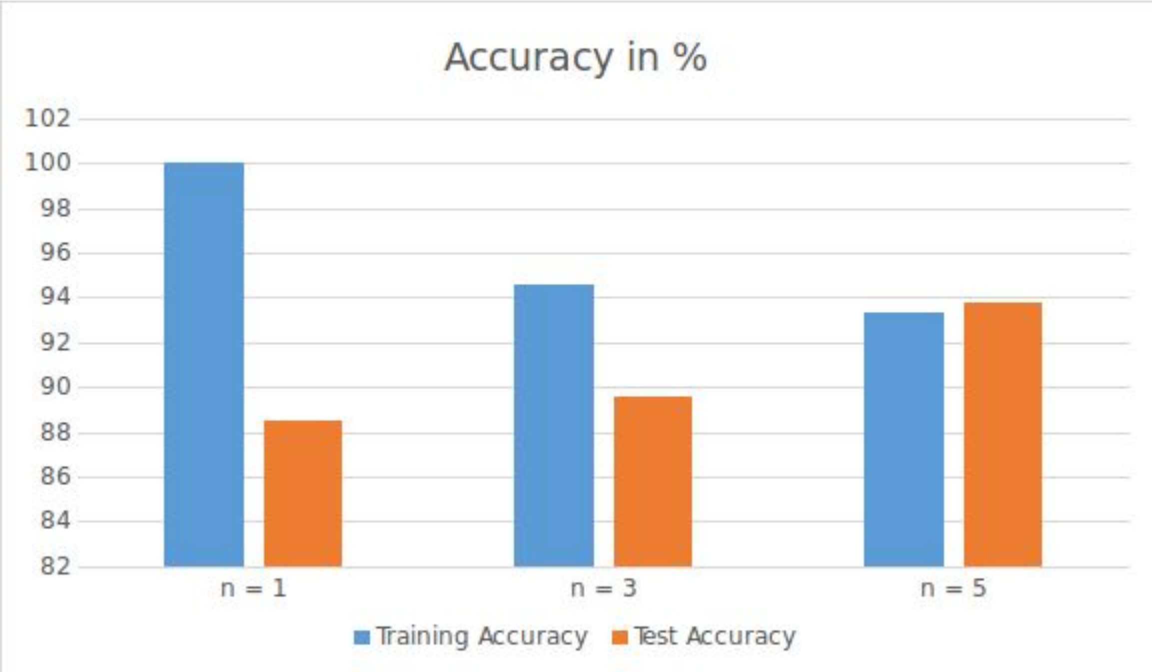

Test Data




Training Data

# Results

## Accuracy

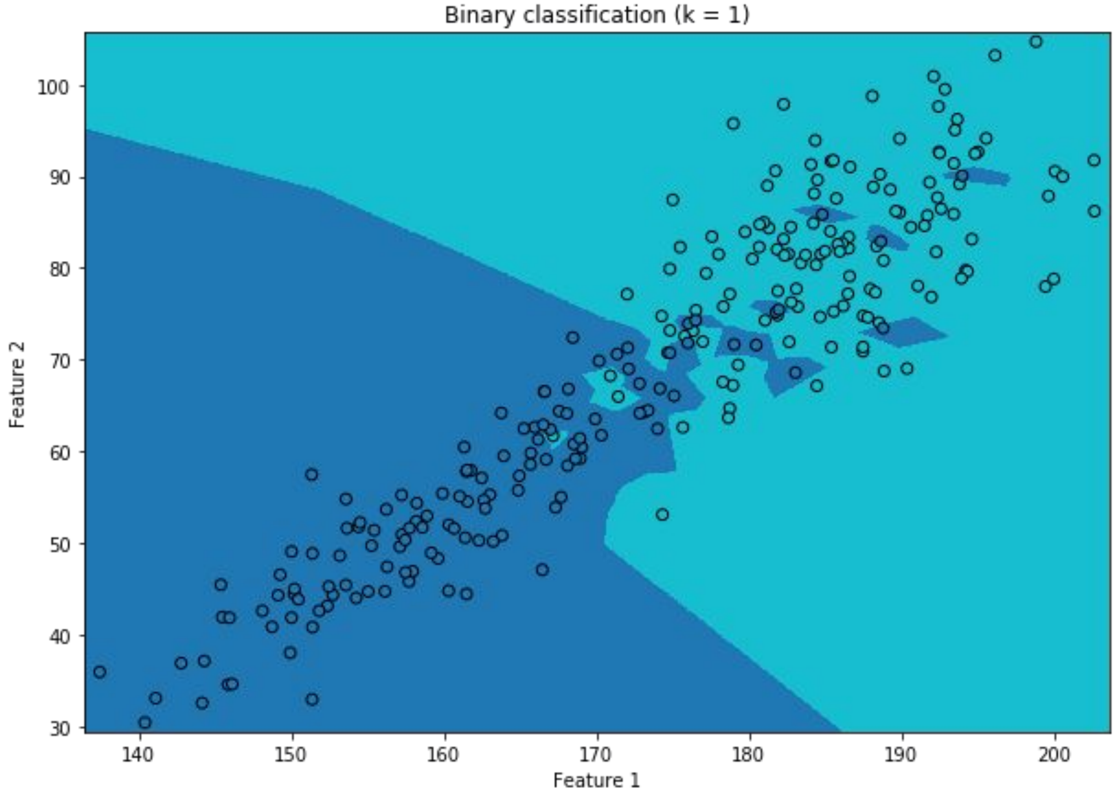


# Results

## Execution time

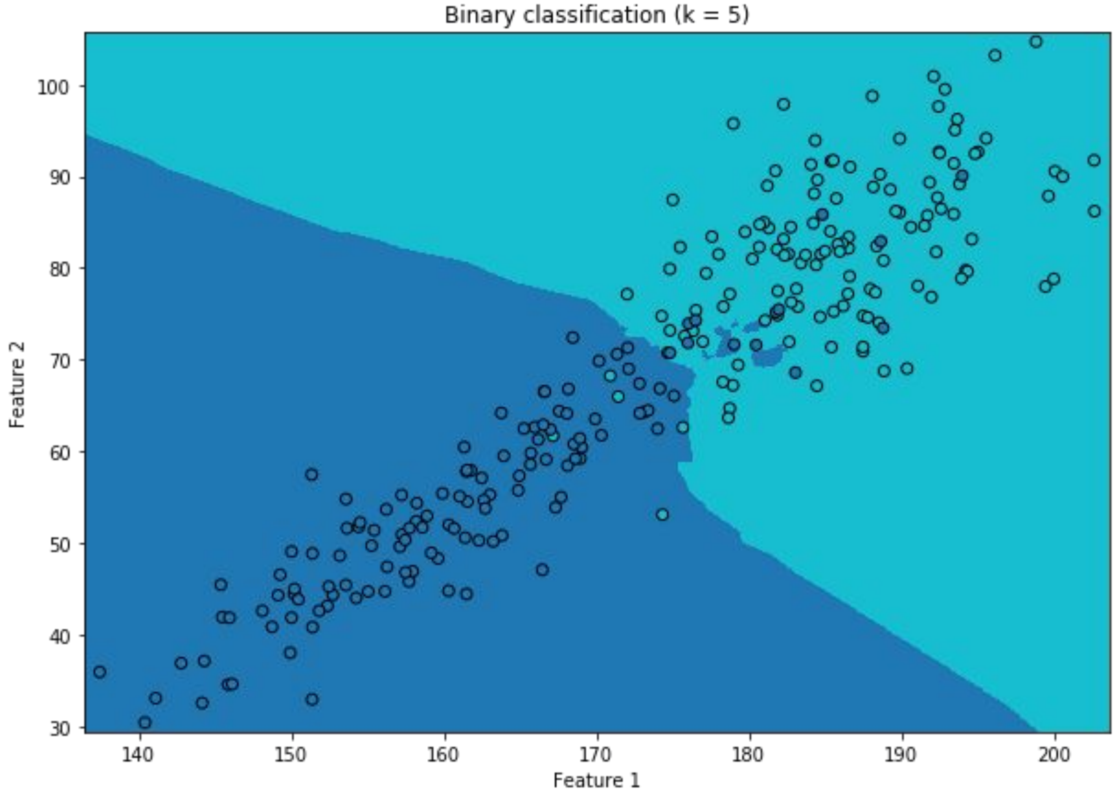
Time taken for computing the 1-nearest neighbor for the test data was 0.1654 seconds

# Decision Boundary



# Decision Boundary

# Decision Boundary



# Exercise 6:
# k-Dimensional Tree
# (k-D Tree)

# k-D Trees

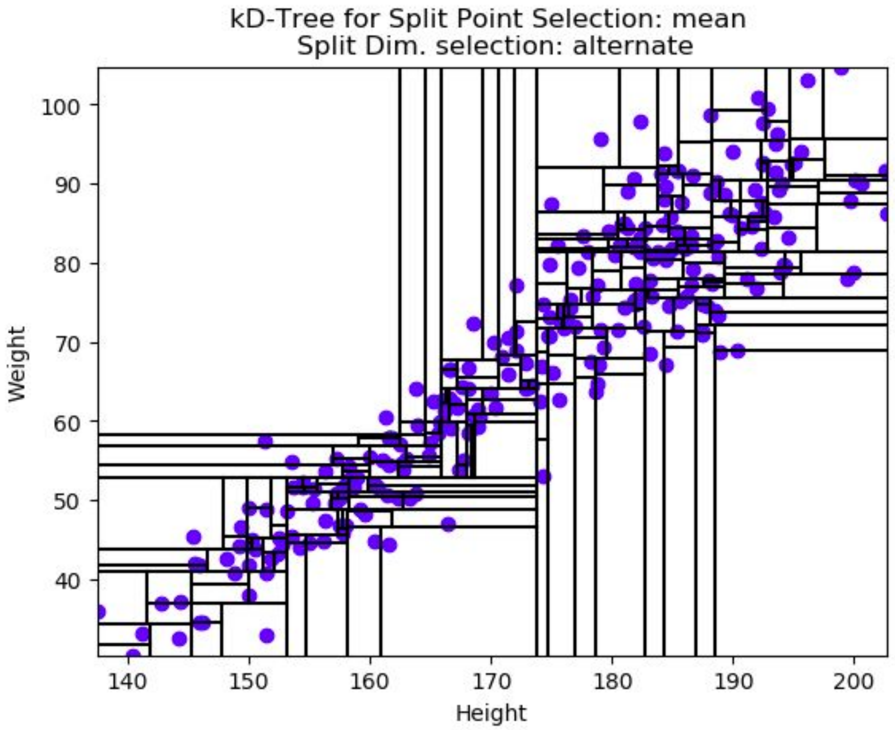
- **Def:** Space partitioning data-structure for organizing points in a k-dimensional space.
  - Data is split along a dimension at a predefined point (axis parallel lines)
- **Problem Statement:** 1-NN search on a 2-dimensional space.
  - Training : data2-train.dat
  - Testing : data2-test.dat

# k-D Trees

- <u>**Splitting Dimension**</u>**:**
  - Alternate through dimensions
  - Choose the dimension of highest variance
- <u>**Point of Split (Position of splitting plane)**</u>**:**
  - The mean of the dimension values
  - The median of the dimension values

Combination of the above results in 4 different trees.

# k-D Tree I (Mean: Alternate)

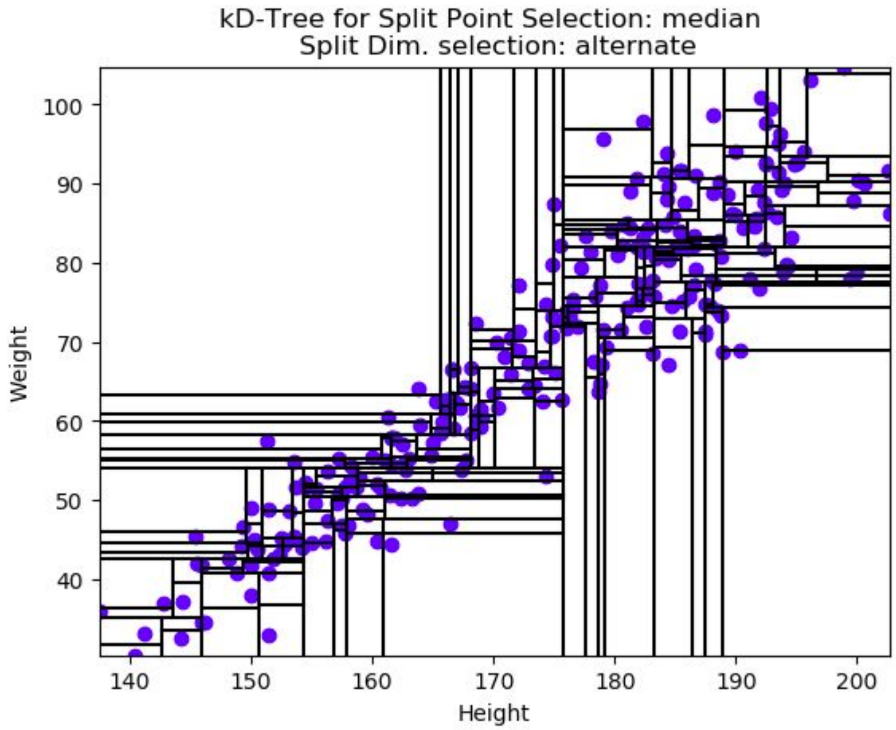



**Statistics**:

- Tree Depth = 10
- Accuracy on Test Set: 84.38%

- Creation Time : 0.005 secs
- Testing Time : 0.007 secs

# k-D Tree II (Median: Alternate)

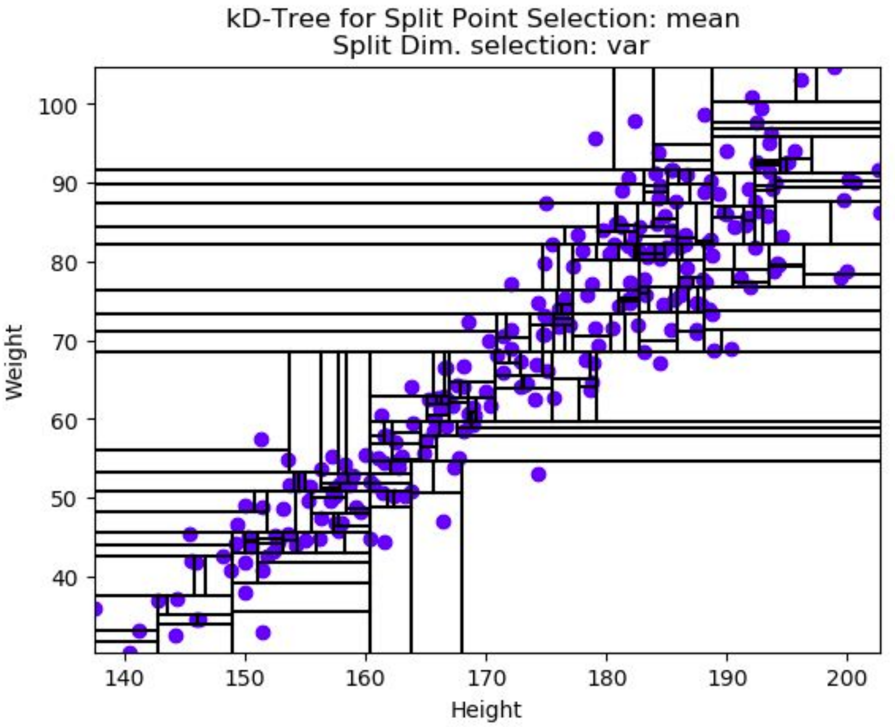



**Statistics**:

- Tree Depth = 8
- Accuracy on Test Set: 87.5%

- Creation Time : 0.014 secs
- Testing Time : 0.005 secs

# k-D Tree III (Mean: Highest Variance)

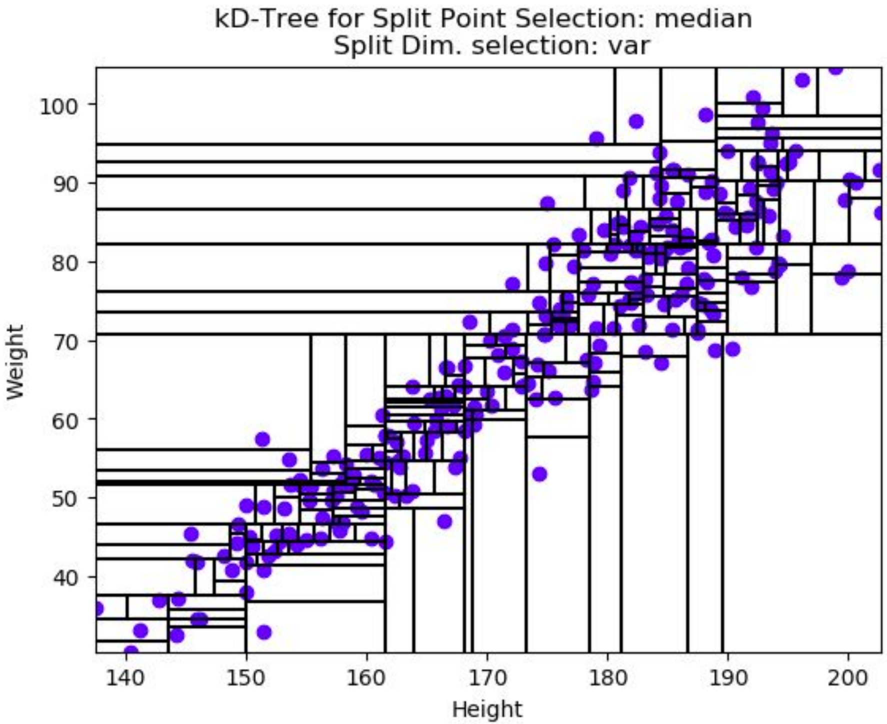



**<u>Statistics</u>**:

- Tree Depth = 10
- Accuracy on Test Set: 87.5%

- Creation Time : 0.012 secs
- Testing Time : 0.006 secs

# k-D Tree IV (Median: Highest Variance)




**Statistics**:

- Tree Depth = 8
- Accuracy on Test Set: 88.54%

- Creation Time : 0.023 secs
- Testing Time : 0.005 secs

# Performance of different k-D Trees

| Sl. | k-D Tree Type | Test Acc. | Train Time | Search Time |
|---|---|---|---|---|
| 1 | Split = mean, Dim = alternate | 84.37 | 0.005 | 0.007 |
| 2 | Split = median, Dim = alternate | 87.50 | 0.014 | 0.005 |
| 3 | Split = mean, Dim = variance | 87.50 | 0.014 | 0.007 |
| 4 | **Split = median, Dim = variance** | **88.54** | 0.025 | 0.005 |
| 5 | **Sklearn kD Tree, leaf_size = 1** | **88.54** | **0.0002** | **0.0002** |

- Highest Accuracy on Test Set: 88.54%
  - median based splitting with dimension of highest variance
  - Significantly faster than naive implementation (0.16s) search time
- Scikit-learn benchmarks:
  - Sklearn-significantly faster. But with equal classification accuracy

Thank You.