# MA 8251 - Engineering Mathematics - II

## Unit - I Matrices

### Part A

1. Find the sum and product of the eigen values of the matrix A = $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$. (April/May 2015)

   **Solution :** Sum of the eigen values = Sum of the diagonal elements = $8 + 7 + 3 = 18$.

   $$\text{Product of the eigen values} = |A| = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$$
   $$= 8(21 - 16) + 6(-18 + 8) + 2(24 - 14)$$
   $$= 40 - 60 + 20 = 0.$$

2. Find the sum and product of the eigen values of a $3 \times 3$ matrix $A$ whose characterisitic equation is $\lambda^3 - 7\lambda^2 + 36 = 0$. (Nov./Dec. 2017)

   **Solution :** The characterisitc equation of a 3×3 matrix $A$ is given by $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$, where $S_1$ = Sum of the leading diagonal elements, $S_2$ = Sum of the minors of the leading diagonal elements and $S_3 = |A|$. Here $S_1 = 7, S_2 = 0, S_3 = -36$.
   Hence Sum of the eigen values = Sum of the diagonal elements = 7
   Product of the eigen values = $|A| = -36$.

3. If the eigen values of the matrix $A$ of order $3 \times 3$ are $2, 3$ and $1$, then find the determinant of $A$. (Apr./May 2019)

   **Solution :** Given the eigen values of $A$ are $2, 3, 1$.
   By property, Product of the eigen values of $A = |A|$.
   $\Rightarrow |A| = (2)(3)(1) = 6$.

4. The product of two eigen values of the matrix $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ is 16. Find the third eigen value. (Jan. 2012)

   **Solution:** Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of A.
   Given product of two eigen values = 16 (i.e.)$\lambda_1\lambda_2 = 16$

   $$\text{We know that product of the eigen values} = |A| = \begin{vmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{vmatrix}$$
   $$= 6(9 - 1) + 2(-6 + 2) + 2(2 - 6)$$
   $$= 48 - 8 - 8 = 32.$$
   $$\text{(i.e.)}\lambda_1\lambda_2\lambda_3 = 32$$
   $$\Rightarrow 16\lambda_3 = 32 \quad \Rightarrow \lambda_3 = 2.$$

   $\therefore$ The third eigen value is $\lambda_3 = 2$.

5. If the sum of two eigen values and trace of a 3 x 3 matrix $A$ are equal, find $|A|$. (May 2008, Nov./Dec. 2016)

**Solution:** Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of $A$.

By property, sum of the eigen values of the matrix $A$ = trace of the matrix $A$.

$$\text{(i.e.)} \lambda_1 + \lambda_2 + \lambda_3 = \text{trace of } A.$$

$$\text{Given that the sum of two eigen values} = \lambda_1 + \lambda_2 = \text{trace of } A.$$

$$\therefore \lambda_1 + \lambda_2 + \lambda_3 = \lambda_1 + \lambda_2 \quad \Rightarrow \lambda_3 = 0$$

$$\therefore |A| = \text{Product of the eigen values} = \lambda_1\lambda_2\lambda_3 = 0.$$

6. Two eigen values of the matrix A = $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$ are 3 and 0. What is the third eigen value? What is the product of the eigen values of A?. (April/May 2018)

**Solution :** Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of $A$ and let $\lambda_1 = 3$ and $\lambda_2 = 0$.

By property, sum of the eigen values of the matrix $A$ = trace of the matrix $A$.

$$\text{(i.e.)} \lambda_1 + \lambda_2 + \lambda_3 = \text{trace of } A.$$

$$\therefore 3 + 0 + \lambda_3 = 8 + 7 + 3$$

$$\Rightarrow \lambda_3 = 15$$

$$\therefore |A| = \text{Product of the eigen values} = \lambda_1\lambda_2\lambda_3 = 0.$$

7. Show that the eigen values of a null matrix are zero. (April/May 2018)

**Solution :** Let A= $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Then $|A - \lambda I| = \begin{vmatrix} 0-\lambda & 0 & 0 \\ 0 & 0-\lambda & 0 \\ 0 & 0 & 0-\lambda \end{vmatrix} = 0$

$\Rightarrow (0-\lambda)(0-\lambda)(0-\lambda) = 0$

$\Rightarrow \lambda^3 = 0$

Therefore eigen values of $A$ are $0, 0, 0$.

8. If 2, 3 are the eigen values of $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ b & 0 & 2 \end{pmatrix}$, then find the value of '$b$' . (Nov/Dec. 2014)

**Solution:** Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of $A$ and let $\lambda_1 = 2$ and $\lambda_2 = 3$.

We know that, sum of the eigen values of the matrix $A$ = trace of the matrix $A$.

$$\text{(i.e.)} \lambda_1 + \lambda_2 + \lambda_3 = 2 + 2 + 2$$

$$\Rightarrow 2 + 3 + \lambda_3 = 6 \Rightarrow \lambda_3 = 1$$

$$\text{Also, Product of the eigen values} = |A| = \begin{vmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ b & 0 & 2 \end{vmatrix} = 2(4-0) + 1(0-2b) = 8 - 2b$$

$$\text{(i.e.)} \lambda_1\lambda_2\lambda_3 = 8 - 2b$$

$$\therefore (3)(2)(1) = 8 - 2b \Rightarrow 2b = 8 - 6 \Rightarrow b = 1.$$

9. For a given matrix A of order 3, $|A|$= 32 and two of its eigen values are 8 and 2. Find the sum of the eigen values. (Jan. 2009)

**Solution:** Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen values of $A$.

Given that two of its eigen values are 8 and 2. Let $\lambda_1 = 8$, $\lambda_2 = 2$.

$$\begin{aligned}\text{Given } |A| &= \text{Product of the eigen values} = 32\\ \text{(i.e.)}\lambda_1\lambda_2\lambda_3 &= 32\\ \Rightarrow (8)(2)\lambda_3 &= 32\\ \Rightarrow 16\lambda_3 &= 32\\ \Rightarrow \lambda_3 &= 2\end{aligned}$$

$\therefore$ The eigen values of A are 8, 2, 2.

Hence the sum of the eigen values = 8 + 2 + 2 = 12.

10. Find the values of '$a$' and '$b$' such that the matrix $\begin{pmatrix} a & 4 \\ 1 & b \end{pmatrix}$ has 3 and −2 as its eigen values. (April/May 2011, April/May 2017)

**Solution:** Let A = $\begin{bmatrix} a & 4 \\ 1 & b \end{bmatrix}$.

Since sum of the eigen values = sum of the diagonal elements,

$$a + b = 3 - 2 = 1 \qquad (1)$$

Since product of eigen values is determinant value of matrix,

$$ab - 4 = 3(-2) = -6$$
$$\therefore ab = -2 \qquad (2)$$

Solving (1) and (2), we get the values of $a$ and $b$. (i.e) $a = -1, b = 2$ or $a = 2, b = -1$.

11. If $\lambda$ is an eigen value of the matrix $A$, then prove that $\lambda^2$ is an eigen value of $A^2$. (Jan. 2014, May/June 2016, Apr./May 2019)

**Solution:** Let $\lambda$ be an eigen value of $A$. Then we can find a non-zero column vector $X$, such that $AX = \lambda X$.

$$\text{Now } A(AX) = A(\lambda X) = \lambda(AX) = \lambda(\lambda X)$$
$$\Rightarrow A^2X = \lambda^2 X$$

$\therefore$ $\lambda^2$ is an eigen value of $A^2$.

12. Prove that any square matrix $A$ and its transpose have the same eigen values. (Nov./Dec. 2019)

**Solution:** The eigenvalues of a matrix are roots of its characteristic polynomial. Hence if the matrices $A$ and $A^T$ have the same characteristic polynomial, then they have the same eigenvalues.

$$\begin{aligned}\text{The characteristic polynomial of } A^T &= \left|A^T - \lambda I\right|\\ &= \left|A^T - \lambda I^T\right|\\ &= \left|(A - \lambda I)^T\right|\\ &= |A - \lambda I|, \text{since } \left|B^T\right| = |B|\\ &= \text{The characteristic polynomial of A}\end{aligned}$$

Therefore the eigenvalues of $A$ and $A^T$ are the same.

13. Given A = $\begin{pmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 4 & 2 \end{pmatrix}$. Find the Eigen values of $A^2$. (Jan. 2010)

**Solution**: Given A is a lower triangular matrix. We know, the eigen values of a lower triangular matrix are the leading diagonal elements of the matrix.

$\therefore$ the eigen values are $-1, -3, 2$. (i.e.) The eigen values of the given matrix A are $-1, -3, 2$.

By the property, the eigen values of $A^2$ are $(-1)^2, (-3)^2, (2)^2$. (i.e.) $1, 9, 4$.

14. Two eigen values of the matrix $P = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are equal to 1 each. Find the eigen values of $P^3$. (Nov./Dec. 2007)

**Solution**: Given two eigen values of $P$ are 1, 1. Let $\lambda$ be the third eigen value.

$\therefore\ 1 + 1 + \lambda = 2 + 3 + 2 \Rightarrow \lambda = 7 - 2 = 5$

The eigen values of $P$ are $1, 1, 5$. $\Rightarrow$ The eigen values of $P^3$ are $1^3, 1^3, 5^3$. (i.e.) $1, 1, 125$.

15. Given that A=$\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$, find the eigen values of $A^3$. (May/June 2007)

**Solution**: The characteristic equation of $A$ is given by $|A - \lambda I| = 0$

$$\text{(i.e.)} \begin{vmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (4-\lambda)(2-\lambda) - 3 = 0$$

$$8 - 6\lambda + \lambda^2 - 3 = 0$$

$$\lambda^2 - 6\lambda + 5 = 0$$

$$(\lambda - 1)(\lambda - 5) = 0$$

$$\lambda = 1, 5$$

The eigen values of $A$ are $1, 5$. Therefore, the eigen values of $A^3$ are $1^3, 5^3$. (i.e.) $1, 125$.

16. If $\lambda (\neq 0)$ is an eigen value of a square matrix $A$, then show that $\lambda^{-1}$ is also an eigen value of $A^{-1}$. (Nov./Dec. 2017)

**Solution:** Let $\lambda$ be an eigen value of $A$. Then we can find a non-zero column vector $X$, such that $AX = \lambda X$.

$$\text{Now } A^{-1}AX = A^{-1}\lambda X = \lambda(A^{-1}X)$$

$$\Rightarrow IX = \lambda(A^{-1}X)$$

$$\Rightarrow \frac{1}{\lambda}X = A^{-1}X. (i.e.) A^{-1}X = \frac{1}{\lambda}X$$

$\therefore \frac{1}{\lambda}$ is an eigen value of $A^{-1}$.

17. If 3 and 6 are two eigen values of $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$, write down all the eigen values of $A^{-1}$. (June 2012)

**Solution :** Let $\lambda$ be the third eigen value of $A$. We know that,

$$\text{Sum of the eigen values} = \text{Trace of} A$$

$$3 + 6 + \lambda = 1 + 5 + 1$$

$$\Rightarrow \lambda = 7 - 9 = -2.$$

Therefore the eigen values of $A$ are $3, 6, -2$. Hence, the eigen values of $A^{-1}$ are $\frac{1}{3}, \frac{1}{6}, \frac{1}{-2}$.

18. Find the eigen values of the inverse of the matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{pmatrix}$. (May/June 2014)

**Solution :** Given matrix $A$ is an upper-triangular matrix. We know the eigen values of an upper triangular matrix are the leading diagonal elements of the matrix.

$\therefore$ the eigen values of $A$ are $2, 3, 4$. $\Rightarrow$ the eigen values of $A^{-1}$ are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$.

19. If $A = \begin{pmatrix} 5 & 0 & 0 \\ 2 & -1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$, then find the eigen values of adj $A$.

**Solution :** Given matrix $A$ is a lower-triangular matrix. We know the eigen values of a lower triangular matrix are the leading diagonal elements of the matrix.

$\therefore$ the eigen values of $A$ are $5, -1, 1$. $\Rightarrow$ the eigen values of $A^{-1}$ are $\frac{1}{5}, -1, 1$.

We have $A^{-1} = \frac{1}{|A|}\text{adj(A)} \Rightarrow \text{adj (A)} = |A|A^{-1}$

The eigen values of adj (A) are $\frac{1}{5} \times |A|, -1 \times |A|, 1 \times |A|$

Now, $|A| = \begin{vmatrix} 5 & 0 & 0 \\ 2 & -1 & 0 \\ 3 & 2 & 1 \end{vmatrix} = -5$

$\therefore$ The eigen values of adj (A) are $-1, 5, -5$.

20. If the eigen values of the matrix $A$ of order $3 \times 3$ are $2, 3$, and $1$, then find the eigen values of adjoint of $A$. (Jan. 2014, May/June 2016)

**Solution :** Given the eigen values of $A$ are $2, 3, 1$. $\Rightarrow$ the eigen values of $A^{-1}$ are $\frac{1}{2}, \frac{1}{3}, 1$.

We have $A^{-1} = \frac{1}{|A|}\text{adj(A)} \Rightarrow \text{adj (A)} = |A|A^{-1}$

The eigen values of adj (A) are $\frac{1}{2} \times |A|, \frac{1}{3} \times |A|, 1 \times |A|$

Now, $|A|$ = product of the eigen values = $(2)(3)(1) = 6$.

$\therefore$ The eigen values of adj (A) are $\frac{1}{2} \times 6, \frac{1}{3} \times 6, 1 \times 6 = 3, 2, 6$.

21. Prove that, if $X$ is an eigen vector of $A$ corresponding to an eigen value $\lambda$, then any non-zero scalar multiple of $X$ is also an eigen vector of $A$. (April/May 2005)

**Solution:** Given $X$ is an eigen vector of $A$ corresponding to the eigenvalue $\lambda$. $\Rightarrow AX = \lambda X$. Multiply both sides by non-zero scalar $k$.

$$kAX = k\lambda X.$$

$$A(kX) = \lambda(kX)$$

$\Rightarrow kX$ is also an eigen vector of $A$ corresponding to $\lambda$.

$\Rightarrow$ Eigen vector $X$ corresponding to an eigen value is not unique.

22. Prove that an eigen vector cannot correspond to two different eigen values.

**Solution :** Let $\lambda_1$ and $\lambda_2$ ($\lambda_1 \neq \lambda_2$) be two different eigen values of a matrix $A$ and $X$ be the eigen vector corresponding to both $\lambda_1$ and $\lambda_2$.

$$\therefore AX = \lambda_1 X \text{ \& } AX = \lambda_2 X.$$

$$\Rightarrow \lambda_1 X = \lambda_2 X.$$

$$\Rightarrow (\lambda_1 - \lambda_2) X = 0.$$

Since $\lambda_1 \neq \lambda_2$, we have $X = 0$, which is a contradiction (since an eigen vector is a non-zero vector). Therefore an eigen vector cannot correspond to two different eigen values.

23. Find the eigen values of $3A + 2I$, where $A = \begin{pmatrix} 5 & 4 \\ 0 & 2 \end{pmatrix}$. (Dec./Jan. 2016)

**Solution :** The eigen values of $A$ are 5, 2.

$\therefore$ The eigen values of $3A + 2I$ are $3(5) + 2$, $3(2) + 2$. i.e The eigen values of $3A + 2I$ are 17, 8.

24. If $2, -1, -3$ are the eigen values of the matrix $A$, then find the eigen values of the matrix $A^2 - 2I$. (May/June 2014)

**Solution :** The eigen values of $A$ are $2, -1, -3$.

$\therefore$ The eigen values of $A^2 - 2I$ are $2^2 - 2, (-1)^2 - 2, (-3)^2 - 2$. i.e The eigen values of $A^2 - 2I$ are $2, -1, 7$.

25. If X = (-1, 0, 1)$^T$ is an eigen vector of the matrix A = $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$, find the corresponding eigen value. (Dec. 2006)

**Solution :** Let A = $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$ and $X = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. Let $\lambda$ be the corresponding eigen value.

The eigen vector of the matrix $A$ is given by $(A - \lambda I)\,X = 0$.

$$\text{Therefore} \begin{pmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$(1-\lambda)(-1) + 1(0) + 3(1) = 0 \Rightarrow -1 + \lambda + 0 + 3 = 0 \Rightarrow \lambda + 2 = 0$$

$$\therefore \lambda = -2 \text{ is the corresponding eigen value.}$$

26. For what values of '$c$' the eigen values of the matrix $\begin{pmatrix} 1 & 2 \\ c & 4 \end{pmatrix}$ are real and unequal, real and equal, complex conjugates? (Jan. 2011)

**Solution:** Given matrix is $\begin{pmatrix} 1 & 2 \\ c & 4 \end{pmatrix}$. The characteristic equation is given by $\lambda^2 - S_1\lambda + S_2 = 0$.

$$\lambda^2 - 5\lambda + (4 - 2c) = 0. \quad \Rightarrow A = 1, B = -5, C = 4 - 2c.$$

Since the characteristic equation is quadratic, the roots can be classified based on the value of

$$B^2 - 4AC = 25 - 4(1)(4 - 2c)$$
$$= 25 - 16 + 8c = 9 + 8c.$$

The roots are real and unequal when $B^2 - 4AC > 0 \Rightarrow c > \frac{-9}{8}$.

The roots are real and equal when $B^2 - 4AC = 0 \Rightarrow c = \frac{-9}{8}$.

The roots are complex conjugates when $B^2 - 4AC < 0 \Rightarrow c < \frac{-9}{8}$.

27. If $A$ is an $n \times n$ real symmetric matrix, $D$ is an $n \times n$ diagonal matrix whose diagonal elements are the eigen values of the matrix $A$ and $P$ is an $n \times n$ orthogonal diagonalizing matrix whose columns are the normalized eigen vectors of the matrix $A$, satisfying the similarity transformation $D = P^{-1}AP$, then find the matrix $A^k, k$ is a positive integer. (Jan. 2011)

**Solution:** Given $D = P^{-1}AP$. Pre-multiply by $P$ and post-multiply by $P^{-1}$, we get

$$PDP^{-1} = PP^{-1}APP^{-1}$$
$$\text{(i.e.)} A = PDP^{-1}$$
$$A^2 = \left(PDP^{-1}\right)\left(PDP^{-1}\right) = PD\left(P^{-1}P\right)DP^{-1} = PD^2P^{-1}.$$

Similarly, $A^3 = PD^3P^{-1}. \Rightarrow A^k = PD^kP^{-1}$.

28. Can $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ be diagonalized? Why? (Jan. 2012)

**Solution:** Yes, given $A$ is diagonalizable because it is non-singular matrix. It is already in the diagonalized form.

29. If $A$ is an orthogonal matrix prove that $|A| = \pm 1$. ( Jan. 2006)

**Solution :** $A$ is orthogonal $\Rightarrow AA^T = I = A^TA$.

$$\text{Now} \qquad \left|AA^T\right| = |I| \ \Rightarrow |A|\left|A^T\right| = 1 \ \Rightarrow |A|\,|A| = 1 \ \Rightarrow |A|^2 = 1 \Rightarrow |A| = \pm 1.$$

30. Check whether the matrix $B = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is orthogonal? Justify. (Jan. 2009)

**Solution:** Condition for orthogonality is $BB^T = I = B^TB$.

$$BB^T = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I$$

Similarly $B^TB = I$. Therefore, $B$ is orthogonal.

31. Show that similar matrices have same eigen values.

**Solution :** Let $A$ and $B$ be two similar matrices. Since $A$ and $B$ are similar there exists a non singular matrix $P$ such that $B = P^{-1}AP$.

$$\Rightarrow \quad B - \lambda I = P^{-1}AP - \lambda I = P^{-1}AP - P^{-1}\lambda IP = P^{-1}(A - \lambda I)P.$$

$$\Rightarrow \quad |B - \lambda I| = \left|P^{-1}\right||A - \lambda I|\,|P| = |A - \lambda I|\left|P^{-1}P\right| = |A - \lambda I|\,|I| = |A - \lambda I|.$$

Therefore $A, B$ have the same characteristic polynomial and hence characteristic roots. (i.e.) they have same eigen values.

32. Find the symmetric matrix $A$ whose eigen values are 1 & 3 with corresponding eigen vectors $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. (Jan. 2013)

**Solution:** Here $D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$, $N = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ and $N^T = \dfrac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$$A = NDN^T = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

33. State Cayley-Hamilton theorem. (May/June 2009 & 2010, Nov./Dec. 2014)

**Solution:** Every square matrix satisfies its own characteristic equation.

34. Give two uses of Cayley-Hamilton theorem. (May 2008)

**Solution:** (i) To calculate the positive integral powers of $A$.

(ii) To find the inverse of the square matrix $A$.

35. Verify Cayley-Hamilton for the matrix $A = \begin{pmatrix} 2 & -3 \\ 4 & 1 \end{pmatrix}$. (Nov./Dec. 2007)

**Solution:** The characteristic equation of $A$ is given by $\lambda^2 - S_1\lambda + S_2 = 0$.

$$S_1 = 2 + 1 = 3$$
$$S_2 = 2 + 12 = 14$$
$$\text{(i.e.) } \lambda^2 - 3\lambda + 14 = 0.$$

By Cayley-Hamilton theorem, $A^2 - 3A + 14I = 0$.

$$A^2 = \begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} -8 & -9 \\ 12 & -11 \end{bmatrix}$$

$$A^2 - 3A + 14I = \begin{bmatrix} -8 & -9 \\ 12 & -11 \end{bmatrix} - 3\begin{bmatrix} 2 & -3 \\ 4 & 1 \end{bmatrix} + 14\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Hence Cayley-Hamilton theorem is verified.

36. If $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$, then find $2A^2 - 8A - 10I$, where $I$ is the unit matrix. (Nov./Dec. 2019)

**Solution:** The characteristic equation of $A$ is given by $\lambda^2 - S_1\lambda + S_2 = 0$.

$$S_1 = 1 + 3 = 4$$
$$S_2 = 3 - 8 = -5$$

(i.e.) $\lambda^2 - 4\lambda - 5 = 0$.

By Cayley-Hamilton theorem, $A^2 - 4A - 5I = 0$.
Therefore $2A^2 - 8A - 10I = 2(A^2 - 4A - 5I) = 2(0) = 0$.

37. Use Cayley-Hamilton theorem to find $A^4 - 4A^3 - 5A^2 + A + 2I$ when $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$. (Jan. 2005)

**Solution:** The characteristic equation of $A$ is given by $\lambda^2 - S_1\lambda + S_2 = 0$.

$$S_1 = 1 + 3 = 4$$
$$S_2 = 3 - 8 = -5$$
$$\text{(i.e.) } \lambda^2 - 4\lambda - 5 = 0.$$

By Cayley-Hamilton theorem, $A^2 - 4A - 5I = 0$.

$$\text{Now, } A^4 - 4A^3 - 5A^2 + A + 2I = A^2\left(A^2 - 4A - 5I\right) + A + 2I = 0 + \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 3 & 2 \\ 4 & 5 \end{bmatrix}$$

38. If the matrix $A$ is given by $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2+i & -1 & 0 & 0 \\ -3 & 2i & i & 0 \\ 4 & -i & 1 & -i \end{pmatrix}$ where $i = \sqrt{-1}$, then by using Cayley-Hamilton theorem prove that $A^4 = I$. (June 2011)

**Solution:** Given matrix $A$ is a lower triangular matrix. We know the eigen values of a lower triangular matrix are the leading diagonal elements of the matrix.
$\therefore$ the eigen values of $A$ are $1, -1, i, -i$.

Hence the characteristic equation is $(\lambda - 1)(\lambda + 1)(\lambda - i)(\lambda + i) = 0$

$$\Rightarrow (\lambda^2 - 1)(\lambda^2 - i^2) = 0$$
$$\Rightarrow (\lambda^2 - 1)(\lambda^2 + 1) = 0$$

$$\Rightarrow \lambda^4 - 1 = 0$$

By Cayley-Hamilton theorem, $A^4 - I = 0. \Rightarrow A^4 = I.$

39. Write down the matrix corresponding to the quadratic form $2x^2 + 2y^2 + 3z^2 + 2xy - 4xz - 4yz$. (May 2009)

**Solution:** The matrix of the quadratic form is $A = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{pmatrix}$

40. Write down the quadratic form corresponding to the matrix $\begin{pmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{pmatrix}$. (June 2012)

**Solution :** The quadratic form is $y^2 + 2z^2 + 10xy - 2xz + 12yz$.

41. Write down the quadratic form corresponding to the matrix $\begin{pmatrix} 2 & 0 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & -2 \end{pmatrix}$. (Jan. 2013)

**Solution :** $2x_1^2 + 2x_2^2 - 2x_3^2 - 4x_1x_3 + 2x_2x_3$.

42. What is the nature of the quadratic form $x^2 + y^2 + z^2$ in four variables. (Dec./Jan. 2016)
**Solution :** Since the quadratic form contains only square terms, it is in canonical form (in four variables). The eigen values are the coefficients of the square terms. $\therefore \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1, \lambda_4 = 0$.
Thus the quadratic form is positive semi-definite.

43. Give the nature of the quadratic form whose matrix is $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$. (Apr./May. 2015)

**Solution :** Given matrix is a diagonal matrix. The eigen values are the leading diagonal elements. $\therefore$ the eigen values are $-1, -1, -2$(all are negative). Thus the quadratic form is negative definite.

44. State the nature of the quadratic form $2xy + 2yz + 2zx$. (Dec. 2008)

**Solution :** The matrix of the quadratic form is $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

The principal minors are $D_1 = 0$,

$$D_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1 < 0, D_3 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = -1(-1) + 1(1) = 1 + 1 = 2 > 0.$$

Since $D_1 = 0, D_2 < 0, D_3 > 0$, the quadratic form is indefinite.

45. Find the index and signature of the quadratic form $x_1^2 + 2x_2^2 - 3x_3^2$. (Apr./May 2011)

**Solution :** Matrix of the quadratic form is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$.

The eigen values are $1, 2, -3$. Rank of quadratic form $r = 3$, Index of quadratic form $p = 2$, Signature $= 2p - r = 4 - 3 = 1$.

46. If the sum of the eigen values of the matrix of the quadratic form is equal to zero, then what will be the nature of the quadratic form?
**Solution :** Since sum of the eigen values is zero, all the eigen values of the matrix are not positive. It can be observed that few eigen values are positive and few are negative. Thus the quadratic form is indefinite in nature.

**Part B(Assignment Questions)**

1. Find the eigen values and corresponding eigen vectors of the matrix $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$. (Apr./May 2019)

   **Ans:** Eigen values are $0, 3, 15$. Eigen vectors are $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$.

2. Find the eigen values and corresponding eigen vectors of the matrix $\begin{pmatrix} 11 & -4 & -7 \\ 7 & -2 & -5 \\ 10 & -4 & -6 \end{pmatrix}$. (Apr./May 2018)

   **Ans:** Eigen values are $0, 1, 2$. Eigen vectors are $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$.

3. Find the eigen values and eigen vectors of $\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$. (May 2009, Jan. 2010, Jan. 2011, May/June 2014)

   **Ans:** Eigen values are -3,-3,5. Eigen vectors are $\begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$.

4. Find the eigen values and corresponding eigen vectors of the matrix $\begin{pmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 2 \end{pmatrix}$.(Apr. 2015, Dec. 2016)

   **Ans:** Eigen values are 1, 2, 3. Eigen vectors are $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

5. Find the eigen values and corresponding eigen vectors of the matrix $\begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$. (Nov./Dec. 2019)

   **Ans:** Eigen values are 2, 3, 6. Eigen vectors are $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$.

6. Find the Eigen values and the eigen vectors of the matrix $\begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$. (June 2012, May/June 2015)

   **Ans:** Eigen values are 8,2,2. Eigen vectors are $\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

7. Verify Cayley-Hamilton theorem for the matrix $\begin{pmatrix} 2 & -1 & 2 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}$ and hence find $A^{-1}$ and $A^4$.

(Nov./Dec. 2014)

**Ans:** $A^{-1} = \frac{1}{3}\begin{pmatrix} 3 & 0 & -3 \\ 1 & 2 & 0 \\ -1 & 1 & 3 \end{pmatrix}$, $A^4 = \begin{pmatrix} 124 & -123 & 162 \\ -95 & 96 & -123 \\ 95 & -95 & 124 \end{pmatrix}$

8. Using Cayley-Hamilton theorem find the inverse of the given matrix $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix}$.

(Apr./May 2018)

**Ans:** $A^{-1} = \frac{1}{5}\begin{pmatrix} -5 & 5 & 0 \\ 5 & -2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$.

9. Using Cayley-Hamilton theorem find $A^{-1}$ if $A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{pmatrix}$. (Apr./May 2019)

**Ans:** $A^{-1} = \begin{pmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{pmatrix}$.

10. Find the matrix $A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I$ if $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$, using Cayley-Hamilton theorem. (June 2009)

**Ans:** $A^8 - 5A^7 + 7A^6 - 3A^5 + 8A^4 - 5A^3 + 8A^2 - 2A + I = \begin{pmatrix} 295 & 285 & 285 \\ 0 & 10 & 0 \\ 285 & 285 & 295 \end{pmatrix}$

11. Find $A^n$ using Cayley-Hamilton theorem taking $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$. Hence find $A^3$. (Jan. 2012)

**Ans:** $A^n = \left(\frac{5^n - (-1)^n}{6}\right)\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} + \left(\frac{5^n + 5(-1)^n}{6}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A^3 = \begin{pmatrix} 41 & 84 \\ 42 & 83 \end{pmatrix}$

12. If $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Then show that $A^n = A^{n-2} + A^2 - I$ for $n \geq 3$ using Cayley-Hamilton theorem.

13. Diagonalise the matrix $A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$ by similarity transformation.

**Ans:** $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{pmatrix}$.

14. Reduce the quadratic form $2xy-2yz+2xz$ into a canonical form by an orthogonal reduction. (Apr./May 2019)
**Ans:** Canonical form : $-2y_1^2 + y_2^2 + y_3^2$.

15. Reduce the given quadratic form $x^2 + 5y^2 + z^2 + 2xy + 2yz + 6xz$ to its canonical form using orthogonal transformation. (Apr./May 2014)(May/June 2014)
**Ans:** Canonical form : $-2y_1^2 + 3y_2^2 + 6y_3^2$.

16. Reduce the given quadratic form $2x^2 + 5y^2 + 3z^2 + 4xy$ to its canonical form using orthogonal transformation and state its nature. (Apr./May 2018)
**Ans:** Canonical form : $y_1^2 + 3y_2^2 + 6y_3^2$, Nature is Positive definite.

17. Reduce the given quadratic form $6x^2 + 3y^2 + 3z^2 - 4xy - 2yz + 4xz$ into a canonical by an orthogonal reduction. Find its nature. (Nov./Dec. 2019)
**Ans:** Canonical form : $8y_1^2 + 2y_2^2 + 2y_3^2$, Nature is Positive definite.

18. Reduce the given quadratic form $2x_1^2 + 6x_2^2 + 2x_3^2 + 8x_1x_3$ to its canonical form using orthogonal transformation and state its nature. (Apr/May 2003,2011,2015)(May/June 2014)
**Ans:** Canonical form : $-2y_1^2 + 6y_2^2 + 6y_3^2$, Nature is Indefinite.

19. The eigen vectors of a 3 x 3 real symmetric matrix $A$ corresponding to the eigen value 1, 2, 3 are $(1, -1, 0)^T, (0, 0, 1)^T, (1, 1, 0)^T$ respectively. Find the matrix $A$ by an orthogonal transformation.
**Ans:** $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

# MA 8251 - Engineering Mathematics - II

## Unit -II Vector Calculus

### Part A

1. **Find** $\nabla r$ **where** $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ **and** $r = |\vec{r}|$ **.**

   **Solution:** $\nabla r = \vec{i}\dfrac{\partial r}{\partial x} + \vec{j}\dfrac{\partial r}{\partial y} + \vec{k}\dfrac{\partial r}{\partial z} = \dfrac{x}{r}\vec{i} + \dfrac{y}{r}\vec{j} + \dfrac{z}{r}\vec{k} = \dfrac{1}{r}(x\vec{i} + y\vec{j} + z\vec{k}) = \dfrac{\vec{r}}{r}$ .

2. **Prove that** $\nabla(r^n) = nr^{n-2}\vec{r}$ **.**

   **Solution:** $\nabla r^n = nr^{n-1}\nabla r = nr^{n-1}\dfrac{\vec{r}}{r} = nr^{n-2}\vec{r}$ .

3. **Find the unit normal vector to the surface xy+ yz +zx =3 at (1,1,1).**

   **Solution:** Let $\phi = xy + yz + zx - 3$.

   $\nabla\phi = \vec{i}\dfrac{\partial\phi}{\partial x} + \vec{j}\dfrac{\partial\phi}{\partial y} + \vec{k}\dfrac{\partial\phi}{\partial z} = \vec{i}(y+z) + \vec{j}(z+x) + \vec{k}(y+x) \implies \nabla\phi_{(1,1,1)} = 2\vec{i} + 2\vec{j} + 2\vec{k}$

   $\therefore$ Unit normal $= \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{2\vec{i} + 2\vec{j} + 2\vec{k}}{\sqrt{4+4+4}} = \dfrac{\vec{i} + \vec{j} + \vec{k}}{\sqrt{3}}$.

4. **Find a unit normal to the surface** $x^3 + y^2 = z$ **at** $(1,1,2)$ **.**

   **Solution:** Let $\phi = x^3 + y^2 - z$ .

   $\nabla\phi = \vec{i}\dfrac{\partial\phi}{\partial x} + \vec{j}\dfrac{\partial\phi}{\partial y} + \vec{k}\dfrac{\partial\phi}{\partial z} = 3x^2\vec{i} + 2y\vec{j} - 1\vec{k} \implies \nabla\phi_{(1,1,2)} = 3\vec{i} + 2\vec{j} - 1\vec{k}$

   $\therefore$ Unit normal $= \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{3\vec{i} + 2\vec{j} - 1\vec{k}}{\sqrt{9+4+1}} = \dfrac{3\vec{i} + 2\vec{j} - 1\vec{k}}{\sqrt{14}}$

5. **Find** $|\nabla\phi|$**, if** $\phi = 2xz^4 - x^2y$ **at** $(2,-2,-1)$ .

   **Solution:** $\phi = 2xz^4 - x^2y$ .

   $\nabla\phi = \vec{i}\dfrac{\partial\phi}{\partial x} + \vec{j}\dfrac{\partial\phi}{\partial y} + \vec{k}\dfrac{\partial\phi}{\partial z} = \vec{i}(2z^4 - 2xy) + \vec{j}(-x^2) + \vec{k}(8xz^3)$

   $\implies \nabla\phi_{(2,-2,-1)} = 10\vec{i} - 4\vec{j} - 16\vec{k} \implies |\nabla\phi| = \sqrt{100+16+256} = \sqrt{372}$ units.

6. **What is the greatest rate of increase of** $\phi = xyz^2$ **at** $(1,0,3)$ **.**

   **Solution:** $\phi = xyz^2$

   $\nabla\phi = \vec{i}\dfrac{\partial\phi}{\partial x} + \vec{j}\dfrac{\partial\phi}{\partial y} + \vec{k}\dfrac{\partial\phi}{\partial z} = \vec{i}\left(yz^2\right) + \vec{j}(xz^2) + \vec{k}(2xyz) \implies \nabla\phi_{(1,0,3)} = 9\vec{j}$ . $\therefore |\nabla\phi| = 9$ units.

7. **If the directional derivative of the function** $x^2 + y^2 + z^2$ **at** $(1,2,3)$ **in the direction of** $a\vec{i} + \vec{j} + \vec{k}$ **is** $2$**, then find** $a$ **.**

   **Solution:** $\phi = x^2 + y^2 + z^2$.

   $\implies \nabla\phi = \vec{i}\dfrac{\partial\phi}{\partial x} + \vec{j}\dfrac{\partial\phi}{\partial y} + \vec{k}\dfrac{\partial\phi}{\partial z} = \vec{i}(2x) + \vec{j}(2y) + \vec{k}(2z) \implies \nabla\phi_{(1,2,3)} = 2\vec{i} + 4\vec{j} + 6\vec{k}$.

   Given Directional derivative of $\phi$ in the given direction $= \nabla\phi \cdot \dfrac{a\vec{i} + \vec{j} + \vec{k}}{\left|a\vec{i} + \vec{j} + \vec{k}\right|} = 2$ units.

$$\therefore \quad \frac{\left(2\vec{i}+4\vec{j}+6\vec{k}\right)\cdot(a\vec{i}+\vec{j}+\vec{k})}{\sqrt{a^2+1+1}} = 2 \text{ units} \implies \frac{2a+4+6}{\sqrt{a^2+2}} = 2 \implies 10+2a = 2\sqrt{a^2+2}.$$

Squaring on both sides and solving we get $a = -\dfrac{23}{10} = -2.3$

8. **Find the maximum directional derivative of** $\phi = zyx^2 + 4xyz^2$ **at** $(1,-2,1)$ **.**

**Solution:** $\phi = zyx^2 + 4xyz^2$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z} = \vec{i}(2xyz+4yz^2)+\vec{j}(zx^2+4xz^2)+\vec{k}(yx^2+8xyz) \implies \nabla\phi_{(1,-2,1)} = -12\vec{i}+5\vec{j}-18\vec{k}$$

The maximum of directional derivative of $\phi$ is $|\nabla\phi| = \sqrt{144+25+324} = \sqrt{493}$ units.

9. **In what direction from** $(2,1,-1)$ **is the directional derivative of** $\phi = x^2y^2z^4$ **maximum ?**

**Solution:** $\phi = x^2y^2z^4$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z} = \vec{i}(2xy^2z^4)+\vec{j}(x^22yz^4)+\vec{k}(x^2y^24z^3) \implies \nabla\phi_{(2,1,-1)} = 4\vec{i}+8\vec{j}-16\vec{k}$$

10. **If** $\phi = 3xy - yz$ **, find grad at** $(1,1,1)$ **.**

**Solution:** $\phi = 3xy - yz$

$$grad\ \phi = \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z} = \vec{i}(3y)+\vec{j}(3x-z)+\vec{k}(-y) \implies \nabla\phi_{(1,1,1)} = 3\vec{i}+2\vec{j}-\vec{k}.$$

11. **Find the directional derivative of** $\phi = x^2yz + 4xz^2$ **at** $(1,1,1)$ **in the direction of** $\vec{i}+\vec{j}+\vec{k}$ **.**

**Solution:** $\phi = x^2yz + 4xz^2$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x}+\vec{j}\frac{\partial\phi}{\partial y}+\vec{k}\frac{\partial\phi}{\partial z} = \vec{i}(2xyz+4z^2)+\vec{j}(x^2z)+\vec{k}(x^2y+8zx) \implies \nabla\phi_{(1,1,1)} = 6\vec{i}+\vec{j}+9\vec{k}$$

Directional derivative of $\phi$ in the given direction $= \nabla\phi\cdot\dfrac{\vec{i}+\vec{j}+\vec{k}}{\left|\vec{i}+\vec{j}+\vec{k}\right|}$

$$= \frac{\left(6\vec{i}+\vec{j}+9\vec{k}\right)\cdot(\vec{i}+\vec{j}+\vec{k})}{\sqrt{1+1+1}} = \frac{16}{\sqrt{3}} \text{ units.}$$

12. **Define divergence and curl of a vector.**

**Solution:** The divergence of $\vec{V} = V_1\vec{i}+V_2\vec{j}+V_3\vec{k}$ is defined as $\nabla\cdot\vec{V} = \dfrac{\partial V_1}{\partial x}+\dfrac{\partial V_2}{\partial y}+\dfrac{\partial V_3}{\partial z}$

The curl of $\vec{V} = V_1\vec{i}+V_2\vec{j}+V_3\vec{k}$ is defined as

$$\nabla\times\vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix} = \vec{i}\left(\frac{\partial V_3}{\partial y}-\frac{\partial V_2}{\partial z}\right)-\vec{j}\left(\frac{\partial V_3}{\partial x}-\frac{\partial V_1}{\partial z}\right)+\vec{k}\left(\frac{\partial V_2}{\partial x}-\frac{\partial V_1}{\partial y}\right)$$

13. **If** $\vec{F} = 3\vec{i}+z\vec{j}+y\vec{k}$ **, show that** $curl\vec{F} = \vec{0}$ **.**

**Solution:** To prove: $\nabla\times\vec{F}=\vec{0}$

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 3 & z & y\end{vmatrix}=\vec{i}\left(\frac{\partial}{\partial y}(y)-\frac{\partial}{\partial z}(z)\right)-\vec{j}\left(\frac{\partial}{\partial x}(y)-\frac{\partial}{\partial z}(3)\right)+\vec{k}\left(\frac{\partial}{\partial x}(z)-\frac{\partial}{\partial y}(3)\right)=\vec{0}.$$

14. **If** $\vec{F}=x^3\vec{i}+y^3\vec{j}+z^3\vec{k}$ **, then find** *div curl* $\vec{F}$ **.**

**Solution:**

$$curl\,\vec{F}=\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x^3 & y^3 & z^3\end{vmatrix}=\vec{i}\left(\frac{\partial}{\partial y}\left(z^3\right)-\frac{\partial}{\partial z}\left(y^3\right)\right)-\vec{j}\left(\frac{\partial}{\partial x}\left(z^3\right)-\frac{\partial}{\partial z}\left(x^3\right)\right)+\vec{k}\left(\frac{\partial}{\partial x}\left(y^3\right)-\frac{\partial}{\partial y}\left(x^3\right)\right)=\vec{0}.$$

Therefore, *div curl* $\vec{F}=\nabla\cdot\left(\nabla\times\vec{F}\right)=\nabla\cdot\left(\vec{0}\right)=0.$

15. **Prove that** *div curl* $\vec{F}=\vec{0}$ **.**

**Solution:** To prove: $\nabla\cdot\left(\nabla\times\vec{F}\right)=0$ Let $\vec{F}=f_1\vec{i}+f_2\vec{j}+f_3\vec{k}.$

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ f_1 & f_2 & f_3\end{vmatrix}=\vec{i}\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)-\vec{j}\left(\frac{\partial f_3}{\partial x}-\frac{\partial f_1}{\partial z}\right)+\vec{k}\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)$$

$$\begin{aligned}\nabla.\left(\nabla\times\vec{F}\right)&=\frac{\partial}{\partial x}\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)-\frac{\partial}{\partial y}\left(\frac{\partial f_3}{\partial x}-\frac{\partial f_1}{\partial z}\right)+\frac{\partial}{\partial z}\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)\\&=\frac{\partial^2 f_3}{\partial x\partial y}-\frac{\partial^2 f_2}{\partial x\partial z}-\frac{\partial^2 f_3}{\partial y\partial x}+\frac{\partial^2 f_1}{\partial y\partial z}+\frac{\partial^2 f_2}{\partial z\partial x}-\frac{\partial^2 f_1}{\partial z\partial y}\\&=0 \qquad\qquad \left(\text{since}\quad \frac{\partial^2 f}{\partial y\partial x}=\left(\frac{\partial^2 f}{\partial x\partial y}\right)\right)\end{aligned}$$

16. **Prove that the vector** $\vec{F}=(3x+2y+4z)\vec{i}+(2x+5y+4z)\vec{j}+(4x+4y-8z)\vec{k}$ **is both solenodial and irrotaional.**

**Solution:** To prove: $\nabla\cdot\vec{F}=0$ (solenoidal), $\nabla\times\vec{F}=\vec{0}$ (irrotational)

$$\nabla\cdot\vec{F}=\frac{\partial}{\partial x}(3x+2y+4z)+\frac{\partial}{\partial y}(2x+5y+4z)+\frac{\partial}{\partial z}(4x+4y-8z)=3+5-8=0$$

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 3x+2y+4z & 2x+5y+4z & 4x+4y-8z\end{vmatrix}$$

$$\begin{aligned}&=\vec{i}\left(\frac{\partial}{\partial y}(4x+4y-8z)-\frac{\partial}{\partial z}(2x+5y+4z)\right)-\vec{j}\left(\frac{\partial}{\partial x}(4x+4y-8z)-\frac{\partial}{\partial z}(3x+2y+4z)\right)\\&\qquad+\vec{k}\left(\frac{\partial}{\partial x}(2x+5y+4z)-\frac{\partial}{\partial y}(3x+2y+4z)\right)\\&=(4-4)\vec{i}-(4-4)\vec{j}+(2-2)\vec{k}=\vec{0}.\end{aligned}$$

17. **If** $\vec{V} = (x + 3y)\vec{i} + (y - 2z)\vec{j} + (x + 2\alpha z)\vec{k}$ **has divergence zero, find the value of** $\alpha$ **.**

**Solution:** Given $\nabla \cdot \vec{V} = 0$

$$\Longrightarrow \frac{\partial}{\partial x}(x + 3y) + \frac{\partial}{\partial y}(y - 2z) + \frac{\partial}{\partial z}(x + 2\alpha z) = 1 + 1 + 2\alpha = 2 + 2\alpha = 0 \quad \therefore \alpha = -1.$$

18. **Find** $\alpha$ **such that** $\vec{V} = (3x - 2y + z)\vec{i} + (4x + \alpha y - z)\vec{j} + (x - y + 2z)\vec{k}$ **is solenoidal.**

**Solution:** Given $\nabla \cdot \vec{V} = 0$

$$\Longrightarrow \frac{\partial}{\partial x}(3x - 2y + z) + \frac{\partial}{\partial y}(4x + \alpha y - z) + \frac{\partial}{\partial z}(x - y + 2z) = 3 + \alpha + 2 = 5 + \alpha = 0. \quad \therefore \alpha = -5.$$

19. **Find the value of** $a, b, c$ **so that the vector**

$\vec{F} = (x + y + az)\vec{i} + (bx + 2y - z)\vec{j} + (-x + cy + 2z)\vec{k}$ **may be irrotational.**

**Solution:** Given $\nabla\times\vec{F} = \vec{0}$

$$\Longrightarrow \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x + y + az & bx + 2y - z & -x + cy + 2z \end{vmatrix} = \vec{0}$$

$$\Longrightarrow \vec{i}\left(\frac{\partial}{\partial y}(-x + cy + 2z) - \frac{\partial}{\partial z}(bx + 2y - z)\right) - \vec{j}\left(\frac{\partial}{\partial x}(-x + cy + 2z) - \frac{\partial}{\partial z}(x + y + az)\right)$$
$$+\vec{k}\left(\frac{\partial}{\partial x}(bx + 2y - z) - \frac{\partial}{\partial y}(x + y + az)\right) = \vec{0}$$

$$\Longrightarrow (c - 1)\vec{i} - (1 + a)\vec{j} + (b - 1)\vec{k} = \vec{0} = 0\vec{i} + 0\vec{j} + 0\vec{k}$$

Equating the coefficients of $\vec{i}, \vec{j}, \vec{k}$ , we get $c = 1, a = -1, b = 1$.

20. **Find** $a$ **so that the vector** $\vec{A} = \left(ax^2 - y^2 + x\right)\vec{i} - (2xy)\vec{j}$ **is irrotational.**

**Solution:** Given $\nabla\times\vec{A} = \vec{0}$

$$\Longrightarrow \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ ax^2 - y^2 + x & -2xy & 0 \end{vmatrix} = \vec{0}$$

$$\Longrightarrow \vec{i}\left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(2xy)\right) - \vec{j}\left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(ax^2 - y^2 + x\right) + \vec{k}\left(\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}\left(ax^2 - y^2 + x\right)\right) = \vec{0}$$

$$\Longrightarrow (0)\vec{i} - (0)\vec{j} + (-2y + 2y)\vec{k} = \vec{0}.$$

$$\Longrightarrow \textit{For all values of } a, \ \vec{A} \textit{ is irrotational.}$$

21. **If** $\nabla\phi = yz\vec{i} + xz\vec{j} + xy\vec{k}$ **find** $\phi$.

**Solution:** $\nabla\phi = yz\vec{i} + xz\vec{j} + xy\vec{k}$

$$\therefore \frac{\partial\phi}{\partial x} = yz \quad \Rightarrow \phi = xyz + f(y, z)$$
$$\frac{\partial\phi}{\partial y} = xz \quad \Rightarrow \phi = xyz + f(x, z)$$
$$\frac{\partial\phi}{\partial x} = xy \quad \Rightarrow \phi = xyz + f(x, y)$$
$$\therefore \phi = xyz + c.$$

22. **Find** $\nabla \cdot \left(\frac{\vec{r}}{r}\right)$

**Solution:** $\nabla \cdot \left(\frac{\vec{r}}{r}\right) = \frac{1}{r}\nabla \cdot \vec{r} + \nabla\left(\frac{1}{r}\right) \cdot \vec{r} = \frac{1}{r}(3) - \frac{1}{r^2}\nabla r \cdot \vec{r} = \frac{3}{r} - \frac{1}{r^2}\frac{\vec{r}}{r} \cdot \vec{r} = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}.$

23. **If $\nabla\phi^2 = 0$ , show that $\nabla\phi$ is solenoidal**.

**Solution:** Given $\nabla\phi^2$ =0. But $\nabla\phi^2 = \nabla.\nabla\phi = 0$, this implies $\nabla\phi$ is solenoidal.

24. **If $\vec{A}$ and $\vec{B}$ are irrotational , prove that $\vec{A}\times\vec{B}$ is solenoidal.**

**Solution:** Given $\nabla\times\vec{A} = \vec{0}, \nabla\times\vec{B} = \vec{0}$

Now, $\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B}) = \vec{B}\cdot\vec{0} - \vec{A}\cdot\vec{0} = 0$. $\therefore\ \vec{A}\times\vec{B}$ is solenoidal.

25. **If $\vec{F} = 5xy\vec{i}+2y\vec{j}$, find $\int_c \vec{F}.d\vec{r}$ where c is the part of the curve $y = x^3$ between x=1and x=2.**

**Solution:** $\vec{F}.d\vec{r} = 5xydx + 2ydy$. Along the curve $c$, $y = x^3$, $dy = 3x^2dx$

$$\therefore \int_c \vec{F}.d\vec{r} = \int_1^2 5x^4dx + 6x^5dx = 94\ units.$$

26. **If $\vec{F} = x^2\vec{i} + xy^2\vec{j}$, evaluate the line integral $\int_c \vec{F}\cdot d\vec{r}$ from $(0,0)$ to $(1,1)$ along the path $y = x$ .**

**Solution:** $\vec{F}\cdot d\vec{r} = x^2dx + xy^2dy$. Along $y = x, dy = dx,\ \ x \to 0\ to\ 1$

$$\therefore \int_C \vec{F}\cdot d\vec{r} = \int_0^1 x^2dx + xx^2dx = \int_0^1 \left(x^2 + x^3\right)dx = \frac{7}{12}units.$$

27. **If $\vec{F} = 3x^2+6y\vec{i}+14yz\vec{j}+20xz^2\vec{k}$, evaluate $\int_c \vec{F}\cdot d\vec{r}$ from $(0,0,0)$ to $(1,1,1)$ over the curve $x = t, y = t^2, z = t^3$.**

**Solution:** $\vec{F}\cdot d\vec{r} = (3x^2 + 6y)dx + 14yzdy + 20xz^2dz$.

Along $x = t, y = t^2, z = t^3 dx = dt, dy= 2tdt, dz= 3t^2dt$, t varies from $0\ to\ 1$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (3t^2+6t^2)dt+14t^2t^3 2tdt+20tt^6 3t^2dt = \int_0^1 (9t^2+28t^6+60t^9)dt = 13\ units.$$

28. **Determine whether $\vec{F}=\left(2xy+z^3\right)\vec{i}+\left(x^2\right)\vec{j}+\left(3xz^2\right)\vec{k}$ is a conservative force field.**

**Solution:** To prove: $\nabla\times\vec{F} = \vec{0}$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+z^3 & x^2 & 3xz^2 \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}\left(3xz^2\right) - \frac{\partial}{\partial z}\left(x^2\right)\right) - \vec{j}\left(\frac{\partial}{\partial x}\left(3xz^2\right) - \frac{\partial}{\partial z}\left(2xy+z^3\right)\right) + \vec{k}\left(\frac{\partial}{\partial x}\left(x^2\right) - \frac{\partial}{\partial y}\left(2xy+z^3\right)\right)$$

$$= (0-0)\,\vec{i} - \left(3z^2-3z^2\right)\vec{j} + (2x-2x)\,\vec{k} = \vec{0}.$$

$\therefore\ \vec{F} = \nabla\phi$ , $\vec{F}$ is a conservative vector field.

29. **State Green's theorem in a plane.**

**Solution:** If C is a simple closed curve in the XY-plane and R be the region bounded by

C, then $\int_c udx + vdy = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy$, where $u, v, u_y, v_x$ are continuous functions inside and on C.

30. **Using Green's theorem, prove that the area enclosed by a simple closed curve c is** $\frac{1}{2}\int xdy - ydx.$

**Solution:** By Green's theorem, $u = -\frac{y}{2}, v = -\frac{x}{2}, \frac{\partial u}{\partial y} = -\frac{1}{2}, \frac{\partial v}{\partial x} = \frac{1}{2}.$

$$\frac{1}{2}\int_c xdy - ydx = \iint_R \left(\frac{1}{2} + \frac{1}{2}\right) dxdy = \iint_R dxdy = Area\ enclosed\ by\ a\ given\ region.$$

31. **Using Green's theorem in a plane find the area circle** $x^2 + y^2 = a^2$

**Solution:** By Green's theorem, $area = \frac{1}{2}\int_c xdy - ydx.$

$x^2 + y^2 = a^2 \Rightarrow x = a\cos\theta,\ y = a\sin\theta$ and $dx = -a\sin\theta d\theta,\ dy = a\cos\theta d\theta$

$$Area = \frac{1}{2}\int_0^{2\pi} (a^2\cos^2\theta + a^2\sin^2\theta) d\theta = \pi a^2.$$

32. **State Gauss- divergence theorem.**

**Solution:** If $\vec{F}$ is a vector point function having continuous first order partial derivatives in the volume $V$ bounded by a closed surface $S$, then $\iiint_V \nabla \cdot \vec{F}\, dv = \iint_S \vec{F} \cdot \hat{n}\, ds$, where $\hat{n}$ is the outward drawn normal vector to the surface $S$.

33. **Find** $\iint_S \vec{r} \cdot d\vec{s}$ **,where s is the surface of the unit cube in the first octant whose vertices are** $(0, 0, 0), (1, 0, 0), (0, 1, 0)$ **and** $(0, 0, 1)$**.**

**Solution:** By Gauss divergence theorem,

$$\iint_S \vec{r} \cdot d\vec{s} = \iint_S \vec{r} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{r}\, dv = \int_0^1 \int_0^1 \int_0^1 3\ dxdydz = 3\ units.$$

34. **If** $S$ **is any closed surface enclosing a volume** $V$ **and** $\vec{F} = x\vec{i} + 2y\vec{j} + 3z\vec{k}$ **, prove that** $\iint_S \vec{F} \cdot \hat{n}\, ds = 6V.$

**Solution:** By Gauss divergence theorem,

$$\iint_S \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{F}\, dv = \iiint_V 6\, dv = 6V$$

since $\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(3z)$ = 1+2+3=6.

35. **If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and S is the surface of the sphere of unit radius, find $\iint_S \vec{r} \cdot \hat{n}\, ds$.**

**Solution:** By Gauss divergence theorem,

$$\iint_S \vec{r} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \vec{r}\, dv = \iiint_V 3\, dv = 3V$$

since $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, $\nabla \cdot \vec{r} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3$.

36. **For any closed surface S, prove that $\iint_S curl\vec{F} \cdot \hat{n}\, ds = 0$.**

**Solution:** By Gauss divergence theorem,

$$\iint_S \nabla \times \vec{F} \cdot \hat{n}\, ds = \iiint_V \nabla \cdot \nabla \times \vec{F}\, dv = 0. \quad (since\ div(curl\vec{F}) = 0)$$

37. **State Stoke's theorem.**

**Solution:** If S is an open surface bounded by a simple closed curve $C$ and if a vector function $\vec{F}$ is continuous and has continuous partial derivatives in $S$ and on $C$, then $\int_c \vec{F} \cdot d\vec{r} = \iint_S (\nabla\times\vec{F}) \cdot \hat{n}\, ds$ where $\hat{n}$ is the unit vector normal to the surface.

38. **If $\vec{F}$ is irrotational and c is a closed curve, then find the value of $\int_c \vec{F} \cdot d\vec{r}$.**

**Solution:** Given $\nabla\times\vec{F} = \vec{0}$. By Stoke's theorem,

$$\int_c \vec{F} \cdot d\vec{r} = \iint_S \nabla\times\vec{F} \cdot \hat{n}\, ds = 0.$$

## Part B(Assignment Questions)

1. Find the constants $a$ and $b$ so that the surface is $5x^2 - 2yz - 9x = 0$ and $ax^2y + bz^3 = 4$ may cut orthogonally at the point $(1, -1, 2)$. **[ Ans:** $a = 4, b = 1$**]**

2. Find the value of a, b, c so that the vector $\vec{F} = (x + 2y + az)\vec{i} + (bx - 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}$ may be irrotational. For those values find its scalar potential.

   **[ Ans:** $a = 4,\ b = 2,\ c = -1$ ; $\phi = \frac{x^2}{2} - \frac{3y^2}{2} + z^2 + 2xy - yz + 4xz + c.$**]**

3. Show that $\vec{F} = \left(6xy + z^3\right)\vec{i} + \left(3x^2 - z\right)\vec{j} + \left(3xz^2 - y\right)\vec{k}$ is irrotational. And hence find its scalar potential. **[ Ans:** $\phi = 3x^2y + xz^3 - yz + c.$**]**

4. Find the angle between the normals to the surface $xy = z^2$ at the points $(1, 4, 2)$ and $(-3, -3, 3)$. **[ Ans:** $\theta = \cos^{-1}\left(\frac{1}{\sqrt{22}}\right)$.**]**

5. Find the angle between the surface $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$. **[ Ans:** $\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$.**]**

6. Find $\nabla^2(r^n)$ and hence deduce $\nabla^2\left(\frac{1}{r}\right)$ where $r = |\vec{r}|$ and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$.
   [ **Ans:** $n(n+1)r^{n-2}, 0$]

7. Prove that $\nabla \cdot r\nabla\left(\frac{1}{r^3}\right) = \frac{3}{r^4}$.

8. If $\varphi$ and $\psi$ satisfy Laplace equation, prove that the vector $\varphi\nabla\psi - \psi\nabla\varphi$ is solenoidal.

9. If $\vec{A} = \left(5xy - 6x^2\right)\vec{i} + (2y - 4xy)\vec{j}$, evaluate $\int_c \vec{A} \cdot d\vec{r}$ where $c$ is the curve $y = x^2$ in the xy-plane from the point $(1,1)$ to $(2,4)$. **[Ans:** $\frac{-597}{20}$**]**

10. Evaluate $\iint_s \vec{F} \cdot \hat{n}\, ds$ where $\vec{F} = \left(x^2 - yz\right)\vec{i} + \left(y^2 - xz\right)\vec{j} + \left(z^2 - xy\right)\vec{k}$ and $S$ is the surface bounded by $0 \le x \le a, 0 \le y \le b, 0 \le z \le c$. [ **Ans:** $abc(a+b+c)$]

11. Evaluate $\iint_s \vec{F} \cdot \hat{n}\, ds$ where $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ and $S$ is the portion of the plane $x + y + z = 1$ included in the first octant. [ **Ans:** $\frac{1}{4}$]

12. A vector field is given by $\vec{F} = \left(x^2 - y^2 + x\right)\vec{i} - (2xy + y)\vec{j}$. Show that $\vec{F}$ is irrotational and find its scalar potential. Hence evaluate the line integral from the origin to the point $(1,1)$. **[Ans:** $\varphi = \frac{x^3}{3} + \frac{x^2}{2} - xy^2 + \frac{y^2}{2} + c, \int \vec{F} \cdot d\vec{r} = \frac{1}{3}$.]

13. Using Green's theorem, evaluate $\int_c \left(3x^2 - 8y^2\right) dx + (4y - 6xy)\, dy$ where $C$ is the boundary of the region enclosed by $y = x^2$ and $x = y^2$. [ **Ans:** $\frac{3}{2}$]

14. Apply Green's theorem to evaluate $\int_c \left(2x^2 - y^2\right) dx + \left(x^2 + y^2\right) dy$ where $C$ is the boundary of the area by the x-axis and upper half of the circle $x^2 + y^2 = a^2$. [ **Ans:** $\frac{4a^3}{3}$]

15. Verify Green's theorem in the plane for $\int_C \left(xy + y^2\right) dx + x^2 dy$ where $C$ is the closed curve of the region bounded by $y = x^2$ and $x = y$. [ **Ans:** $\frac{-1}{20}$]

16. Verify Gauss Divergence theorem for $\vec{F} = x^2\vec{i} + z\vec{j} + yz\vec{k}$ over the cube formed by the planes $x = \pm 1,\ \ y = \pm 1,\ \ z = \pm 1$. [ **Ans:** 0]

17. Verify Gauss Divergence theorem for $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ over the cube bounded by $x = 0, x = a, y = 0, y = a, z = 0$ and $z = a$. [ **Ans:** $\frac{3a^4}{2}$]

18. Verify Gauss Divergence theorem for $\vec{F} = (x^2 - yz)\vec{i} + (y^2 - xz)\vec{j} + (z^2 - xy)\vec{k}$ taken over a rectangle parallelepiped $0 \le x \le 1,\ 0 \le y \le 2,\ 0 \le z \le 3$. [ **Ans:** 36]

19. Verify Gauss Divergence theorem for $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ over the cube bounded by $x = 0, x = 1, y = 0, y = 1, z = 0$ and $z = 1$. [ **Ans:** 3]

20. Verify Stoke's theorem for $\vec{F} = (y - z + 2)\vec{i} + (yz + 4)\vec{j} - xz\vec{k}$ over the surface of the cube $x = 0, x = 2, y = 0, y = 2, z = 0$ and $z = 2$ above the $XY$-plane. [ **Ans:** $-4$]

21. Verify Stoke's theorem for $\vec{F} = y^2 z\vec{i} + z^2 x\vec{j} + x^2 y\vec{k}$ over the open surface of the cube formed by the planes $x = -a, x = a, y = -a, y = a, z = -a$ and $z = a$. In which $z = -a$ is cut open.
**[ Ans:** $4a^4$**]**

22. Verify Stoke's theorem for $\vec{F} = \left(x^2 + y^2\right)\vec{i} - 2xy\vec{j}$ taken around the rectangle bounded by the lines $x = \pm a,\ y = 0, y = b.$ **[ Ans:** $-4ab^2$**]**

23. Verify Stoke's theorem for $\vec{F} = \left(x^2 - y^2\right)\vec{i} + 2xy\vec{j}$ taken around the rectangle bounded by the lines $x = 0,\ x = a,\ y = 0, y = b.$ **[ Ans:** $-2ab^2$**]**

24. Evaluate $\int_C \left(xydx + xy^2dy\right)$ where $C$ is the square in the $XY$-plane with vertices $(1, 0)$, $(-1, 0)$, $(0, 1)$ and $(0, -1)$, using Stoke's theorem. **[ Ans:** $\frac{4}{3}$**]**

25. Using Green's theorem, evaluate $\int_c (y - \sin x)\, dx + \cos x\, dy$ where $C$ is the triangle formed by $y = 0,\ x = \frac{\pi}{2},\ y = \frac{2x}{\pi}$. **[ Ans:** $-\left(\frac{\pi}{4} + \frac{2}{\pi}\right)$**]**

# MA 8251 - Engineering Mathematics - II

## Unit - III Analytic Functions

### Part A

1. **Verify f(z) = $z^3$ is analytic or not.**
   **Solution:** Given $f(z) = z^3$. Put $z = x + iy$.

$$\therefore u + iv = (x+iy)^3 = x^3 + i3x^2y - 3xy^2 - iy^3 = (x^3 - 3xy^2) + i(3x^2y - y^3)$$

$\Rightarrow u(x,y) = x^3 - 3xy^2, \;\; v(x,y) = 3x^2y - y^3$ and

$$\begin{array}{ll} u_x = 3x^2 - 3y^2 & v_x = 6xy \\ u_y = -6xy & v_y = 3x^2 - 3y^2 \end{array}$$

We see that (i) $u_x, u_y, v_x, v_y$exists and are continuous everywhere.

(ii) $u_x = v_y, \;\; u_y = -v_x$. i.e., C-R equations are satisfied. Hence f(z) = $z^3$ is analytic.

2. **Find the analytic region of f(z) = $(x - y)^2$ + 2i(x + y) .**
   **Solution**: Given w = f(z) = $(x - y)^2$ + 2i(x+y)
   i.e., u + iv = $(x - y)^2$ + 2i(x+y)$\Rightarrow u = (x-y)^2, v = 2(x+y)$

$$u_x = 2(x-y), v_x = 2, \qquad u_y = -2(x-y), v_y = 2$$

$u_x = v_y$ only if 2(x – y) = 2.i.e., if x – y = 1.

$u_y = -v_x$ only if -2(x – y) = - 2.i.e.,if x – y = 1

Hence the function is analytic on x – y =1.

3. **Test whether f(z) = $\bar{z}$ is analytic or not.**
   **Solution:** Let w = f(z) = $\bar{z}$. i.e., $u + iv = x - iy \Rightarrow u = x, \; v = -y$

$$u_x = 1, v_x = 0 \;\; \& \;\; u_y = 0, v_y = -1$$

Here $u_x \neq v_y$ and hence C-R equations are not satisfied. The function $\bar{z}$ is no where analytic.

4. **State sufficient condition for analytic function.**
   Let w = f(z) = u(x, y) + iv(x, y) . If
   (i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist and are continuous
   (ii) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at every point of R in z-plane, then the function w = f(z) is analytic in R.

5. **Test whether the function $\frac{1}{z}$ is analytic or not.**

**Solution:** Given f(z) = $\frac{1}{z}$. Let z = $re^{i\theta}$.

$$\therefore f(z) = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta} = \frac{1}{r}(\cos\theta - i\sin\theta) = \frac{1}{r}\cos\theta - \frac{i}{r}\sin\theta$$

$$\Rightarrow u(r,\theta) = \frac{1}{r}\cos\theta, \quad v(r,\theta) = -\frac{1}{r}\sin\theta;$$
$$u_r = \frac{-\cos\theta}{r^2}, \quad v_r = \frac{\sin\theta}{r^2}; \; u_\theta = \frac{-\sin\theta}{r} \quad v_\theta = \frac{-\cos\theta}{r}$$

We see (i) $u_r, u_\theta, v_r, v_\theta$ exits and are continuous everywhere except at r = 0(i.e., z = 0)

(ii) C-R equations $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}$ are satisfied.

Hence f(z) = $\frac{1}{z}$is analytic everywhere except at z = 0.

6. **Give an example of a function where u and v are harmonic but u + iv is not analytic.**

   **Solution :** Let w = $\bar{z}$. i.e., u + iv = x - iy

$$\Rightarrow u = x, v = -y \quad \begin{matrix} u_x = 1, v_x = 0 \\ u_y = 0, v_y = -1 \end{matrix}$$

   Here $u_x \neq v_y$ and hence C-R equations are not satisfied.

7. **If $f(z) = r^2(\cos 2\theta + i\sin p\theta)$ is analytic, then find the value of 'p'.**
   **solution:** Given that $f(z) = r^2(\cos 2\theta + i\sin p\theta)$
   here $u = r^2\cos 2\theta$ and $v = r^2\sin p\theta$
   by C-R equation, $u_r = \frac{1}{r}v_\theta$ and $u_\theta = -rv_r$ —- (i)
   now, $u_\theta = -2r^2 sin2\theta$ and $v_r = 2rsinp\theta$
   from equation (i),
   $-2r^2 sin2\theta = -r2rsinp\theta$
   $\implies \sin 2\theta = \sin p\theta$
   $\implies 2\theta = p\theta$
   $\therefore p = 2$

8. **Examine whether the function $u = xy^2$ can be a real part of an analytic function.**
   **solution:** If a function is analytic its real part must be a harmonic function.

   Given that $u = xy^2$
   $u_x = y^2 \quad u_y = 2xy$
   $u_{xx} = 0 \quad u_{yy} = 2x$
   $u_{xx} + u_{yy} = 2x \neq 0$ (except at $x = 0$)
   As $u$ is not harmonic, it cannot be the real part of an analytic function.

9. **State any two properties of analytic function**

   (i). The real and imaginary part of an analytic function w = u(x, y) + iv(x, y) satisfies Laplace equation.

   (ii). If w = u(x, y) + i v(x, y) is an analytic function then the curves of the families u(x, y) = a , v(x, y) = b cut each other orthogonally where a and b are varying constants.

10. **Show that $\frac{x}{x^2 + y^2}$ is harmonic.**

**Solution:** Let $u(x,y) = \frac{x}{x^2+y^2}$, $u_x = \frac{(x^2+y^2)-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$;

$$u_{xx} = \frac{(x^2+y^2)^2(-2x)-(y^2-x^2)2(x^2+y^2)2x}{(x^2+y^2)^4} = \frac{2x^3-6xy^2}{(x^2+y^2)^3}$$

$$u_y = \frac{-2xy}{(x^2+y^2)^2}; u_{yy} = \frac{(x^2+y^2)^2(-2x)+2xy.2(x^2+y^2)2y}{(x^2+y^2)^4} = \frac{6xy^2-2x^3}{(x^2+y^2)^3}$$

$\therefore u_{xx}+u_{yy} = \frac{2x^3-6xy^2}{(x^2+y^2)^3} + \frac{6xy^2-2x^3}{(x^2+y^2)^3} = 0$ u(x,y) satisies Laplace equation. Hence $u(x,y) = \frac{x}{x^2+y^2}$ is harmonic.

11. **State the points where the function f(z) =$\frac{1}{z^2+1}$is not analytic.**

**Solution:** Let f(z) = $\frac{1}{z^2+1}$ $f'(z) = \frac{-2z}{(z^2+1)^2}$

f(z) is not analytic when $f'(z)$ does not exist. i.e., when $f'(z) \to \infty$ .

$f'(z) \to \infty$ if $(z^2+1)^2 = 0$. $i.e, \ \ if \ \ z = \pm i$

12. **Verify if the function** $u(x,y) = \log\sqrt{x^2+y^2}$ **is harmonic or not.**

**Solution:** $u(x,y) = \log\sqrt{x^2+y^2} = \log(x^2+y^2)^{\frac{1}{2}} = \frac{1}{2}\log(x^2+y^2)$

$$u_x = \frac{x}{x^2+y^2}; u_{xx} = \frac{y^2-x^2}{(x^2+y^2)^2} u_y = \frac{y}{x^2+y^2}; u_{yy} = \frac{x^2-y^2}{(x^2+y^2)^2}$$

$\therefore u_{xx}+u_{yy} = \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = 0$. Hence u is harmonic.

13. **Find the value of m so that the functin** $u = 2x + x^2 - my^2$**is harmonic.**

**Solution:** $u_x = 2+2x; \ \ u_{xx} = 2$
$u_y = -2my; \ \ u_{yy} = -2m$

Since u is harmonic, it satisfies Laplace equation.

$$\therefore u_{xx}+u_{yy} = 0 \Rightarrow 2-2m = 0 \Rightarrow m = 1$$

14. **Prove that an analytic function with constant real part is constant.**

**Solution:** Let f(z) = u + iv be an analytic funtion. Given u is a constant.i.e., u = c (say).
Hence$u_x = 0$; $u_y = 0$.By C – R equations $v_y = u_x = 0$ and $v_x = -u_y = 0$.
Since the partial derivatives of v with respect to both x and y are zero.
v is a constant = $c'$(say)
Hence f(z) = c + i$c'$= constant.
Therefore an analytic function with constant real part is constant.

15. **Construct an analytic function f(z) = u + iv given that** $v = y + e^x\cos y$**.**

**Solution:** Given $v = y + e^x\cos y$; $v_x = e^x\cos y$ ; $v_y = 1 - e^x\sin y$
Let f(z)= u + iv be the required analytic function.
Then$f'(z) = u_x + iv_x = v_y + iv_x$ (by C-R equations)

$$= (1-e^x\sin y) + ie^x\cos y$$

$=(1 - e^z \sin 0) + ie^z \cos 0$ (By Milne-Thomson rule)

$$= 1 + ie^z$$

On integrating, $f(z) = z + ie^z + C$, where C is a constant.

16. **Define conformal mapping.**

A mapping or transformation w = f(z) is said to be conformal at a point, if it preserves angles between every pair of curves through that point, both in magnitude and sense.

17. **Find the points in the z-plane at which the mapping w = z + $z^{-1}$ fails to be conformal.**

**Solution:** The point at which the mapping is not conformal is the critical point.

The critical points are given by $\frac{dw}{dz} = 0$ *and* $\frac{dz}{dw} = 0$. $\frac{dw}{dz} = 1 - \frac{1}{z^2}$;

Hence $\frac{dw}{dz} = 0 \Rightarrow 1 - \frac{1}{z^2} = 0$ *.i.e.,* $z^2 = 1,\ z = \pm 1$

$\frac{dz}{dw} = 0 \Rightarrow \frac{z^2}{z^2 - 1} = 0$ *.i.e.,* $z^2 = 0,\ z = 0$. Hence the critical points are $z = 0, z = \pm 1$.

18. **Find the image of the circle** $|z| = 2$ **under the transformation w = 3z.**

**Solution:** Given curve in z-plane: $|z| = 2 \Rightarrow |x + iy| = 2 \Rightarrow x^2 + y^2 = 4$, a circle with centre origin and radius 2.

Given transformation is w = 3z $\Rightarrow u + iv = 3x + i3y \Rightarrow u = 3x;\ v = 3y \Rightarrow x = \frac{u}{3}, y = \frac{v}{3}$

Image in w-plane: $\left(\frac{u}{3}\right)^2 + \left(\frac{v}{3}\right)^2 = 4 \Rightarrow u^2 + v^2 = 36$, a circle with centre origin and radius 6.

19. **Find the image of the region x > c, where c > 0 under the transformation** $w = \frac{1}{z}$**.**

**Solution:** The transformation equations of $w = \frac{1}{z}$ are x = $\frac{u}{u^2 + v^2}$; y = $\frac{-v}{u^2 + v^2}$.

x > c implies $\frac{u}{u^2 + v^2} > c$ ie., $\frac{u}{c} > u^2 + v^2 \Rightarrow u^2 + v^2 - \frac{u}{c} < 0$. This represents the interior of the circle $u^2 + v^2 - \frac{u}{c} = 0$ with centre $\left(\frac{1}{2c}, 0\right)$ and radius $\frac{1}{2c}$.

20. **Find the critical points of the transformation** $w^2 = (z - \alpha)(z - \beta)$

**Solution:** The critical points of the transformation can be obtained from diff w. r. t. 'z' we get

2w $\frac{dw}{dz} = 2z - (\alpha + \beta)$, $\frac{dw}{dz} = \frac{2z - (\alpha + \beta)}{2w}$

$$\frac{dw}{dz} = 0 \Rightarrow z = \frac{\alpha + \beta}{2}. Also \frac{dz}{dw} = 0 \Rightarrow w = 0 \Rightarrow w^2 = 0 \Rightarrow (z - \alpha)(z - \beta) = 0 \Rightarrow z = \alpha, \beta$$

$\therefore$ The critical points are $\alpha, \beta$ and $\frac{\alpha + \beta}{2}$.

21. **Find the invariant points of the mapping w =** $\frac{1 - z}{1 + z}$**.**

**Solution:** The invariant points are given by z = $\frac{1 - z}{1 + z}$.

$$\Rightarrow z + z^2 = 1 - z \quad \Rightarrow z^2 + 2z - 1 = 0 \Rightarrow z = -1 \pm \sqrt{2}$$

22. **Find the fixed points of the transformation w =** $\frac{3z-4}{z-1}$

**Solution:** The fixed points are given by z =$\frac{3z-4}{z-1}$

$$\Rightarrow z^2 - z = 3z - 4 \Rightarrow z^2 - 4z + 4 = 0 \Rightarrow z = 2, 2$$

23. **Is the function** $f(z) = e^z$ **analytic**
**Solution:** Yes. $f(z) = e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y) \Longrightarrow f(z) = e^x \cos y + ie^x \sin y = u + iv$
$u = e^x \cos y, v = e^x \sin y \implies u_x = e^x \sin y, u_y = -e^x \sin y, v_x = e^x \sin y, v_y = e^x \cos y$
$\therefore u_x = v_y$ and $u_y = -v_x$ which satisfies C-R equation and also $u_x, u_y, v_x, v_y$ are continuous.
Hence $f(z) = e^z$ is analytic.

24. **Find the fixed point of the bilinear transformation** $w = \frac{1}{z}$
**Solution:** Given: $w = \frac{1}{z} \implies z = \frac{1}{z} \implies z^2 - 1 = 0 \implies (z+1)(z-1) = 0 \implies z = -1, 1$
$\therefore$ The fixed points of the transformation are 1 and -1.

25. **Show that the function** $f(z) = z\bar{z}$ is nowhere analytic.
**Solution:** $f(z) = z\bar{z} = (x+iy)(x-iy) = x^2+y^2 \implies f(z) = x^2+y^2+i0 = u+iv \implies u = x^2+y^2, v = 0$
$u_x = 2x, u_y = 2y, v_x = 0, v_y = 0 \therefore u_x \neq v_y$ and $u_y \neq -v_x$ which not satisfies C-R equation
Hence $f(z) = z\bar{z}$ is nowhere analytic.

**Part B (Assignment Questions)**

1. If $f(z) = u(x,y) + iv(x,y)$ is an analytic function, show that the curves $u(x,y) = c_1$ and $v(x,y) = c_2$ cut orthogonaly. **[Ans:** $m_1 m_2 = -1$]

2. Prove that $u = x^2 - y^2$ and $v = \frac{-y}{x^2+y^2}$ are harmonic functions but *f(z)=u+iv is* not an analytic function. **[Ans:** $u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$]

3. If *f (z)* is an analytic function of *z*, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) |f(z)|^2 = 4 \left|f'(z)\right|^2$.

4. If $f(z) = u + iv$ is an analytic function then prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) (u^p) = p(p-1)(u^{p-2}) \left|f'(z)\right|^2$.

5. Find the analytic function $f(z) = u + iv$ if $u - v = e^x[\cos y - \sin y]$ **[Ans:** $f(z) = e^z + c$]

6. Given that $u = \frac{\sin 2x}{\cos h2y - \cos 2x}$, find the analytic function whose real part is *u*. **[Ans:** $f(z) = \cot z + c$]

7. Find the analytic function *f(z)* whose real part is $u = e^x(x \cos y - y \sin y)$. Find also the harmonic conjugate *u*. **[Ans:** $f(z) = ze^z + c$ and $v = e^x(x \sin y + y \cos y)$]

8. Find the analytic function *f(z)* = *u*+*iv*, given that $2u + v = e^x(\cos y - \sin y)$. **[Ans:** $\frac{1+3i}{5} e^z + c$]

9. Show that the transformation $w = \frac{1}{z}$ transforms in general, circles and straight lines into circles or straight lines.

10. Find the image of the circle $|z - 2i| = 2$ under the transformation $w = \frac{1}{z}$. **[Ans:** $v = -\frac{1}{4}$]

11. Under the transformation $w = \frac{1}{z}$,find the image in w-plane of the infinite strip $\frac{1}{4} < y < \frac{1}{2}$. [**Ans:** $u^2 + (v+1)^2 = 1$ and $u^2 + (v+2)^2 = 4$]

12. Find the image of the hyperbola $x^2 - y^2 = 1$ under the transformation $w = \frac{1}{z}$. [**Ans:** $\cos 2\phi = R^2$]

13. Prove that the transformation $w = \frac{z}{z-1}$ maps the upper half of z-plane to the upper half of w-plane. What is the image of $|z| = 1$ under this transformation. [**Ans:** $v > 0,\ u > -\frac{1}{2}$]

14. Find the bilinear transformation which maps the points $z = 0, 1, -1$ onto the points $w = -1, 0, \infty$. Find also the invariant points of the transformation. [**Ans:** $w = \frac{z-1}{z+1}$ and $z = \pm i$]

15. Find the bilinear mapping which maps (-1, 0, 1) of the $z$ plane onto (-1, -$i$, 1) of the $w$ plane. Show that under this mapping, the upper half of the $z$ – plane maps onto the interior of the unit circle $|w| = 1$. [**Ans:** $= \frac{(z+1)+i(z-1)}{(z+1)-i(z-1)}$]

16. Find the bilinear transformation that maps the points 1+$i$, -$i$, 2-$i$ of the z-plane into the points 0,1,$i$ of the w-plane. [**Ans:** $w = \frac{2i(z-1-i)}{-z+5-3i-zi}$]

17. Find the bilinear transformation which maps the points $z = 1, i, -1$ onto the points $w = i, 0, -i$. Hence find the image of $|z| < 1$. [**Ans:** $w = \frac{(1-z)+i(z+1)}{z(1-i)+(1+i)}$]

18. Show that the real and imaginary parts of an analytic function are harmonic.

19. Find the bilinear map which maps the points $z = 1, -1, \infty$ in the z-plane onto the points $w = -1, -i, i$ in the w-plane. [**Ans:** $w = \frac{iz+(2+i)}{z-(3+2i)}$

20. If $f(z) = u+iv$ is an analytic function, then prove that both $u$ and $v$ are harmonic functions. Justify your answer about the converse of the statement.

# MA 8251 - Engineering Mathematics - II

## Unit - IV Complex integration

### Part A

1. State Cauchy's integral theorem.
**Solution :** If a function $f(z)$ is analytic and its derivative $f'(z)$ is continuous at all points inside and on a simple closed curve c, then $\int_c f(z)dz = 0$.

2. What is the value of $\int_c e^z dz$, where C is $|z| = 1$? (June 2003)
**Solution :** $\int_c f(z)dz = \int_c e^z dz$ where$f(z) = e^z$ and C is $|z| = 1$, which is circle with centre$(0, 0)$ and radius 1 and hence it is a closed curve. $f(z) = e^z$is analytic within and on the closed curve C. Hence by Cauchy's integral theorem, $\int_c e^z dz = 0$

3. Evaluate $\int_c sinz dz$, where C is the entire complex plane.
**Solution :** $f(z) = sinz$ is analytic everywhere in the complex plane. Hence by Cauchy's integral theorem, $\int_c sinz dz = 0$.

4. Evaluate $\int_c \frac{z}{z-2} dz$ where C is **(a)** $|z| = 1$ **(b)** $|z| = 3$. (M/J 2013)
**Solution :** $\int_c f(z)dz = \int_c \frac{z}{z-2} dz$ where $f(z) = \frac{z}{z-2}$
**(a)** C is $|z| = 1$ which is circle with centre $(0, 0)$ and radius 1 and hence it is a closed curve. The singular point is $z = 2$ , which lies outside C. Hence $\frac{z}{z-2}$ is analytic outside C. Hence by Cauchy's integral theorem, $\int_c \frac{z}{z-2} dz = 0$.
**(b)** C is $|z| = 3$ which is circle with centre $(0, 0)$ and radius 3 and hence it is a closed curve. The singular point is $z = 2$ , which lies inside C. Hence $\frac{z}{z-2}$ is analytic inside C. Hence by Cauchy's integral formula, $\int_c \frac{f(z)}{z-a} dz = 2\pi i f(a) \Rightarrow \int_c \frac{z}{z-2} dz = 2\pi i f(2)$, where $f(z) = z \Rightarrow f(2) = 2 \Rightarrow 2\pi i \times f(2) = 4\pi i$.

5. Evaluate $\int_c \frac{e^z}{z-1} dz$ where C is $|z| = 2$. (M/J 2013)
**Solution :** C is $|z| = 2$ which is circle with centre $(0, 0)$ and radius 2. The singular point is $z = 1$ which lies inside C. Hence by Cauchy's integral formula, $\int_c \frac{f(z)}{z-a} dz = 2\pi i f(a) \Rightarrow \int_c \frac{e^z}{z-1} dz = 2\pi i f(1)$, where $f(z) = e^z \Rightarrow f(1) = e \Rightarrow 2\pi i \times f(1) = 2\pi i e$.

6. If C is the circle $|z| = 3$ and if $g(z_0) = \int_c \frac{2z^2 - z - 2}{z - z_0} dz$ then find $g(2)$. (A/M 2018)
**Solution :** Given $g(z_0) = \int_c \frac{2z^2 - z - 2}{z - z_0} dz$. Now, $g(2) = \int_c \frac{2z^2 - z - 2}{z - 2} dz$. Here $a = 2$ lies

inside $|z| = 3$. Hence by Cauchy's integral formula, $\int_c \frac{2z^2 - z - 2}{z - z_0} dz = 2\pi i g(2)$, where $g(z) = 2z^2 - z - 2 \Rightarrow g(2) = 4 \Rightarrow 2\pi i \times g(2) = 8\pi i$.

7. Find the value of $\int_c \frac{3z^2 + 7z + 1}{z + 1} dz$ if C is $|z| = \frac{1}{2}$. (A/M 2018)

**Solution :** Given C is $|z| = \frac{1}{2}$ which is a circle with centre $(0, 0)$ and radius $\frac{1}{2}$. Hence $a = -1$ lies outside C. $\therefore$ by Cauchy's integral theorem, $\int_c \frac{3z^2 + 7z + 1}{z + 1} dz = 0$.

8. Evaluate $\int_c \frac{1}{(z + 1)^2 (z - 2)} dz$ where C is the circle $|z| = \frac{3}{2}$.

**Solution :** $a = -1$ lies inside $|z| = \frac{3}{2}$, $a = 2$ lies outside $|z| = \frac{3}{2}$.

$\int_c \frac{1}{(z + 1)^2 (z - 2)} dz = \int_c \frac{\frac{1}{z-2}}{(z + 1)^2} dz$. Here $f(z) = \frac{1}{z - 2}$ is analytic inside C. $f'(z) = \frac{-1}{(z - 2)^2}$.

By Cauchy's integral formula $\int_c \frac{f(z)}{(z - a)^{(n+1)}} dz = \frac{2\pi i}{n!} f^n(a)$

$\int_c \frac{1}{(z + 1)^2 (z - 2)} dz = \int_c \frac{\frac{1}{z-2}}{(z - (-1))^2} dz = \frac{2\pi i}{1!} f'(-1) = \frac{-2\pi i}{9}$.

9. Evaluate $\int_c \frac{z^2 + 1}{z^2 - 1} dz$ where C is the circle $|z - 1| = 1$, using Cauchy's integral formula.

**Solution :** Given $|z - 1| = 1$ is a circle with centre$(1, 0)$ and radius 1. $a = 1$ lies inside C and $a = -1$ lies outside C. $\int_c \frac{z^2 + 1}{z^2 - 1} dz = \int_c \frac{z^2 + 1}{(z + 1)(z - 1)} = \int_c \frac{\frac{z^2+1}{z+1}}{(z - 1)} dz$. Here $f(z) = \frac{z^2 + 1}{z + 1}$ is analytic inside C. Hence by Cauchy's integral formula $\int_c \frac{f(z)}{z - a} dz = 2\pi i f(a) \Rightarrow \int_c \frac{\frac{z^2+1}{z+1}}{(z - 1)} dz = 2\pi i f(1) = 2\pi i$.

10. Evaluate $\int_c \frac{e^z}{z + 1} dz$ where C is the circle $\left|z + \frac{1}{2}\right| = 1$.

**Solution :** $z = -1$ lies inside $\left|z + \frac{1}{2}\right| = 1$. Here $f(z) = e^z$. $\therefore \int_c \frac{e^z}{z + 1} dz = 2\pi i f(-1) = 2\pi i (e^{-1})$.

11. Evaluate 0.

**Solution :** $f(z) = \frac{1}{z + 4}$. Here $z = -4$ lies outside C. Hence by Cauchy's integral theorem, $\int_c \frac{dz}{z + 4} = 0$.

12. Evaluate $\int_c \frac{\cos \pi z^2}{(z - 1)(z - 2)} dz$ where C is $|z| = \frac{3}{2}$.

**Solution :** $a = 1$ lies inside C and $a = 2$ lies outside C. $\int_c \frac{\cos \pi z^2}{(z - 1)(z - 2)} dz = \int_c \frac{\frac{\cos \pi z^2}{(z-2)}}{(z - 1)} dz$.

Here $f(z) = \frac{\cos \pi z^2}{(z - 2)}$. Hence by Cauchy's integral formula $\int_c \frac{\frac{\cos \pi z^2}{(z-2)}}{(z - 1)} dz = 2\pi i f(1) = 2\pi i$.

13. Evaluate $\int_c \tan z dz$ where C is the circle $|z| = 2$.

**Solution :** $f(z) = \tan z$, $f'(z) = \sec^2(z)$. $|z| = 2$ is a circle with centre origin and radius 2. At

$z = 0, f'(z) = \sec^2(0) = 1$. $\therefore$ $f(z) = \tan z$ is differentiable at all points inside and on the circle $|z| = 2$. Hence by Cauchy's integral theorem, $\int_c f(z)dz = 0 \Rightarrow \int_c \tan z dz = 0$.

14. Expand $\frac{z-1}{z+1}$ as Taylor series about the point $z = 1$.
**Solution :** $f(z) = \frac{z-1}{z+1}$. $f(1) = 0, f'(z) = \frac{2}{(z+1)^2}, f'(1) = \frac{1}{2}, f''(z) = \frac{-4}{(z+1)^3}, f'(1) = \frac{-1}{2}$.
So the Taylor's series for $\frac{z-1}{z+1}$ at $z = 1$ is
$\frac{z-1}{z+1} = f(1) + \frac{f'(1)}{1!}(z-1) + \frac{f''(1)}{2!}(z-1)^2 + \ldots = 0 + \frac{1/2}{1!}(z-1) + \frac{-1/2}{2!}(z-1)^2 + \ldots$

15. State Laurent's series
**Solution :** Let $C_1$ and Let $C_2$ be two concentric circles $|z-a| = R_1$ and $|z-a| = R_2$ where $R_2 < R_1$. Let $f(z)$ be analytic on $C_1$ and $C_2$ and in the annular region R between them. Then, for any point z in R, $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$ where $a_n = \frac{1}{2\pi i}\int_{c_1} \frac{f(z)}{(z-a)^{(n+1)}}dz,\ b_n = \frac{1}{2\pi i}\int_{c_2} \frac{f(z)}{(z-a)^{(1-n)}}dz$, integrals being taken anticlockwise.

16. Find the Laurent's series expansion of $f(z) = \frac{e^{2z}}{(z-1)^3}$ about $z = 1$.
**Solution :** Let $f(z) = \frac{e^{2z}}{(z-1)^3}$. At $z = 1$ the function fails to be analytic. $\therefore$ $z = 1$ is a singular point and pole of order 3. Put $z - 1 = u \Rightarrow z = u + 1$.
$f(z) = \frac{e^{2u+2}}{u^3} = \frac{e^2}{u^3}\left[1 + \frac{2u}{1!} + \frac{(2u)^2}{2!} + \ldots\right] = e^2\left[\frac{1}{(z-1)^3} + \frac{2}{(z-1)^2} + \frac{2}{(z-1)} + \frac{8}{6} + \ldots\right]$ where $z \neq 1$.

17. Expand $\frac{1}{z(z-1)}$ as Laurent's series about $z = 0$ in the annulus $0 < |z| < 1$.
**Solution :** $\frac{1}{z(z-1)} = \frac{-1}{z} + \frac{1}{z-1} = -\frac{1}{1-z} - \frac{1}{z} = -(1-z)^{-1} - \frac{1}{z} = -\sum_{n=0}^{\infty} z^n - \frac{1}{z}$

18. Define singularity of a function $f(z)$
**Solution :** A point $z = a$ at which a function $f(z)$ fails to be analytic is called a singular point of the function.

19. Define isolated singularity and give an example.
**Solution :** A point $z = z_0$ is said to be isolated singularity of $f(z)$ if (i). $z = z_0$ should be a singular point.(i.e. $f(z)$ is not analytic at $z = z_0$). (ii). The neighbourhood of $z = z_0$ should not contain any other singular point.
Example: $f(z) = \frac{1}{z}$. This function is analytic everywhere expect at $z = z_0$. $\therefore$ $z = 0$ is an isolated singularity.

20. Define removable singularity and give an example.
**Solution :** A point $z = z_0$ is said to be isolated singularity of $f(z)$ if $\lim_{z\to z_0} f(z)$ exists finitely or if the principal part of $f(z)$ in Laurent's series contains no term (i.e. $b_n = 0$ for all n, in Laurent's series).
Example: $f(z) = \frac{\sin z}{z}$.

21. Define essential singularity and give an example.
**Solution :** If the principal part of Laurent's series contains an infinite number of terms, then $z = z_0$ is known as an essential singularity. Example: $f(z) = e^{\frac{1}{z}}$.

22. Define poles and simple poles and give an example.
**Solution :** If we can find a positive integer $n$ such that $\lim\limits_{z \to z_0} (z - z_0)^n f(z) \neq 0$, then $z = z_0$ is called a pole of order $n$ of $f(z)$. Example (i): $f(z) = \dfrac{1}{(z-4)^2(z-3)^4}$. Here $z = 4$ is a pole of order 2 and $z = 3$ is a pole of order 4. Example (ii): $f(z) = \dfrac{1}{(z-1)(z-3)^4}$. Here $z = 1$ is a pole of order one or a simple pole.

23. What is the nature of the singularity $z = 0$ of the function $f(z) = \dfrac{\sin z - z}{z^3}$ ?
**Solution :** $f(z) = \dfrac{\sin z - z}{z^3}$ is not defined at $z = 0$. By L' Hospital's rule, $\lim\limits_{z \to 0} \dfrac{\sin z - z}{z^3} = \lim\limits_{z \to 0} \dfrac{\cos z - 1}{3z^2} = \lim\limits_{z \to 0} \dfrac{-\sin z}{6z} = \lim\limits_{z \to 0} \dfrac{-\cos z}{6} = \dfrac{-1}{6}$. Since the limit exists and is finite, the singularity at $z = 0$ is a removable singularity.

24. Classify the singularity of $f(z) = \dfrac{e^{\frac{1}{z}}}{(z-a)^2}$.
**Solution :** The singular point of $f(z)$ is $z = a, z = 0$.
$$\frac{e^{\frac{1}{z}}}{(z-a)^2} = \frac{1}{(z-a)^2}\left[1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \dots\right]$$
$\Rightarrow z = 0$ is an essential singularity and $z = a$ is a pole of order two.

25. Find the singularities of $f(z) = \dfrac{\cot \pi z}{(z-\alpha)^3}$.
**Solution :** $f(z) = \dfrac{\cot \pi z}{(z-\alpha)^3} = \dfrac{\cos \pi z}{\sin \pi z (z-\alpha)^3}$. The points where the function fails to be analytic are $\sin \pi z = 0$ and $(z-\alpha)^3 = 0$. $(z-\alpha)^3 = 0 \Rightarrow z = \alpha$ is a pole of order 3. $\sin \pi z = 0 \Rightarrow \pi z = n\pi$ where $n = 0, \pm 1, \pm 2 \dots \Rightarrow z = n$ are all simple poles.

26. Find the singular points of $f(z) = \dfrac{1}{\sin\left(\dfrac{1}{z-a}\right)}$ and state their nature.
**Solution :** $f(z)$ has infinite number of poles which are given by $\dfrac{1}{z-a} = n\pi, n = \pm 1, \pm 2 \dots \Rightarrow z - a = \dfrac{1}{n\pi} \Rightarrow z = a + \dfrac{1}{n\pi}$. But $z = a$ is also a singular point. It is an essential singularity. It is a limit point of the poles. So it is a non-isolated singularity.

27. Find the singular points of $f(z) = \dfrac{1}{z \sin z}$
**Solution :** The singular points of $\dfrac{1}{z \sin z}$ are given by $z \sin z = 0 \Rightarrow z = 0, \sin z = 0 \Rightarrow z = 0, z = n\pi$ where $n$ is zero or any integer. $\therefore z = n\pi$ where $n$ is zero or any integer are poles of $f(z)$ and $z = 0$ is a double pole and $z = n\pi$ where $n \neq 0$ and $n$ is any integer are simple poles of $f(z)$.

28. Find the singular points of $f(z) = \dfrac{\sin z}{z}$.

**Solution :** The singular points of $f(z) = \dfrac{\sin z}{z}$ are given by $z = 0$. $\lim\limits_{z\to 0} \dfrac{\sin z}{z} = \lim\limits_{z\to 0} \dfrac{\cos z}{1} = 1$. Hence $z = 0$ is a removable singularity.

29. Calculate the residue of $f(z) = \dfrac{e^{2z}}{(z+1)^2}$ at its pole.

**Solution :** Given $f(z) = \dfrac{e^{2z}}{(z+1)^2}$. Here $z = -1$ is a pole of order 2.

We know that $[Resf(z)]_{z=z_0} = \dfrac{1}{(m-1)!} \lim\limits_{z\to z_0} \dfrac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)]$.

Here $m = 2$. $\therefore$ $[Resf(z)]_{z=-1} = \dfrac{1}{1!} \lim\limits_{z\to -1} \dfrac{d}{dz}\left[(z+1)^2 \dfrac{e^{2z}}{(z+1)^2}\right] = \lim\limits_{z\to -1} \dfrac{d}{dz} e^{2z} = \lim\limits_{z\to -1} 2e^{2z} = 2e^{-2}$.

30. Consider the function $f(z) = \dfrac{\sin z}{z^4}$. Find the pole and its order.

**Solution :** $f(z) = \dfrac{\sin z}{z^4} = \dfrac{z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots}{z^4} = \dfrac{1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots}{z^3}$

$\therefore$ $z = 0$ is a pole of order 3.

31. Find the principal part and residue at the pole of the function $f(z) = \dfrac{2z+3}{(z+2)^2}$.

**Solution :** The principal part $= (z+2)^{-2}$. $z = -2$ is a pole of order 2.

$[Resf(z)]_{z=-2} = \dfrac{1}{1!} \lim\limits_{z\to -2} \dfrac{d}{dz}\left[(z+2)^2 \dfrac{2z+3}{(z+2)^2}\right] = \lim\limits_{z\to -2} \dfrac{d}{dz}(2z+3) = \lim\limits_{z\to -2}(2) = 2$.

32. If $f(z) = -\dfrac{1}{z-1} - 2\left[1 + (z-1) + (z-1)^2 + \dots\right]$ find the residue of $f(z)$ at $z = 1$

**Solution :** In the Laurent's series expansion, residue at 1 = coefficient of $\dfrac{1}{z-1} = -1$.

33. State Cauchy's Residue Theorem.

**Solution :** If $f(z)$ be analytic at all points inside and on a simple closed curve c, except for a finite number of singularities $z_1, z_2, \cdots, z_n$ inside C, then $\int\limits_c f(z)dz = 2\pi i$[sum of residues of $f(z)$ at $z_1, z_2, \dots, z_n$] $= 2\pi i \sum\limits_{i=1}^{n} R_i$, where $R_i$ is the residue of $f(z)$ at $z = z_i$.

34. Find the residue of $f(z) = \dfrac{1-e^{2z}}{z^4}$ at $z = 0$.

**Solution :** $\dfrac{1-e^{2z}}{z^4} = \left(\dfrac{1 - 1 - 2z - \frac{4z^2}{2!} - \frac{8z^3}{3!} - \dots}{z^4}\right) = -\dfrac{2}{z^3} - \dfrac{2}{z^2} - \dfrac{4}{3z} - \dots$

Residue at $z = 0$ = coefficient of $\dfrac{1}{z} = -\dfrac{4}{3}$.

35. Find the residues of $f(z) = \dfrac{z^2}{(z-1)(z+2)^2}$ at its simple pole.

**Solution :** $f(z) = \dfrac{z^2}{(z-1)(z+2)^2}$. Here $z = 1$ is a simple pole and $z = -2$ is a pole of order 2. $[Resf(z)]_{z=1} = \lim\limits_{z\to 1}(z-1)f(z) = \lim\limits_{z\to 1}(z-1)\dfrac{z^2}{(z-1)(z+2)^2} = \lim\limits_{z\to 1} \dfrac{z^2}{(z+2)^2} = \dfrac{1}{9}$.

**Part B**

1. Evaluate $\int_c \frac{z+4}{z^2+2z+5}dz$ where C is **(i)** $|z+1+i|=2$ **(ii)** $|z+1-i|=2$ **(iii)** $|z+1|=1$ (Jan 2008/N/D 2011)
   **Ans:** $\left[\textbf{(i)}\ \frac{\pi}{2}(2i-3)\ \ \textbf{(ii)}\ \frac{\pi}{2}(2i+3)\ \ \textbf{(iii)}\ 0\right]$.

2. Evaluate $\int_c \frac{\cos\pi z^2+\sin\pi z^2}{(z-1)(z-2)}dz$ where C is $|z|=3$. (N/D 2011,M/J 2013)
   **Ans:** $[4\pi i]$

3. Evaluate $\int_c \frac{z^2}{(z-1)^2(z-2)}dz$ where C is $|z|=3$. (N/D 2016)
   **Ans:** $[2\pi i]$

4. Evaluate $\int_c \frac{z+1}{(z-1)(z-3)}dz$ where C is $|z|=2$ using Cauchy's integral formula. (M/J 2016)
   **Ans:** $[-2\pi i]$

5. Evaluate $\int_c \frac{z^2}{(z^2+1)^2}dz$ where C is $|z-i|=1$ using Cauchy's integral formula. (A/M 2018)
   **Ans:** $\left[\frac{\pi}{2}\right]$

6. Evaluate $\int_c \frac{z}{(z-1)^2(z+2)}dz$ where C is $|z-1|=1$ using Cauchy's integral formula. (N/D 2016)
   **Ans:** $\left[\frac{4\pi i}{9}\right]$

7. Evaluate $\int_c \frac{z}{(z-2)^2(z-1)}dz$ where C is $|z-2|=\frac{1}{2}$ using Cauchy's integral formula. (M/J 2009,N/D 2009,M/J2012,A/M 2017)
   **Ans:** $[-2\pi i]$

8. If $F(a)=\int_c \frac{3z^2+7z+1}{z-a}dz$, where C is $|z|=2$, then find $F(1-i)$ and $F'(1-i)$.(Apr/May 2019)
   **Ans:** $F(1-i)=2\pi(13+8i), F'(1-i)=-2\pi i(13-6i)$

9. Using Cauchy's Residue theorem, evaluate $\int_c \frac{z-1}{(z+1)^2(z-2)}dz$ where C is $|z-i|=2$ (M/J 2012,M/J 2014)
   **Ans:** $\left[\frac{-2\pi i}{9}\right]$

10. Using Cauchy's residue theorem evaluate $\int_c \frac{z}{(z^2+1)^2}dz$ where C is $|z-i|=1$ (N/D 2016)
    **Ans:** $[0]$

11. Using Cauchy's residue theorem evaluate $\int_c \frac{z^3}{(z-1)^4(z-2)(z-3)}dz$ where C is $|z|=2.5$ (Jan 2016)

12. Expand $f(z) = \dfrac{z^2 - 1}{(z+2)(z+3)}$ as a Taylor series valid in the region $|z| < 2$ (Jun 2010)

**Ans:** $\left[f(z) = \dfrac{-1}{6} + \dfrac{5z}{36} + \dfrac{17z^2}{216} - \dfrac{115z^3}{1296} + \dots\right]$

13. Expand $f(z) = \dfrac{1}{(z+1)(z+3)}$ as Laurent's series valid in the regions **(i)** $1 < |z| < 3$ **(ii)** $|z| > 3$ (N/D 2009, M/J 2012, A/M 2017)

**Ans:**
$\left(\textbf{(i)}\dfrac{1}{2z}\left[1 - \dfrac{1}{z} + \dfrac{1}{z^2} - \dots\right] - \dfrac{1}{6}\left[1 - \dfrac{z}{3} + \dfrac{z^2}{9} - \dots\right]\ \textbf{(ii)}\dfrac{1}{2z}\left[1 - \dfrac{1}{z} + \dfrac{1}{z^2} - \dots\right] - \dfrac{1}{2z}\left[1 - \dfrac{3}{z} + \dfrac{3}{z^2} - \dots\right]\right).$

14. Find the Laurent's expansion of $f(z) = \dfrac{7z - 2}{z(z+1)(z-2)}$ valid in the region $1 < |z+1| < 3$ (May 2010)

**Ans:** $\left[f(z) = -2(z+1)^{-1} + (z+1)^{-2} + \dots - \dfrac{2}{3}\left(1 + \dfrac{z+1}{3} + \dfrac{(z+1)^2}{9} + \dots\right)\right]$

15. Expand $f(z) = \dfrac{6z + 5}{z(z+1)(z-2)}$ as Laurent's series valid in the region $1 < |z+1| < 3$. (A/M 2018)

**Ans:** $\left(f(z) = \dfrac{-5}{2(z+1)}\left[1 + \dfrac{1}{z+1} + \dfrac{1}{(z+1)^2} + \dots\right] - \dfrac{1}{3(z+1)} - \dfrac{17}{18}\left[1 + \dfrac{z+1}{3} + \dfrac{(z+1)^2}{9} + \dots\right]\right).$

16. Find the Laurent's expansion of $f(z) = \dfrac{z}{z^2 - 3z + 2}$ valid in the regions **(i)** $1 < |z| < 2$ **(ii)** $|z| < 1$ **(iii)** $|z| > 1$ (M/J 2016)

**Ans: (i)** $\dfrac{-1}{z}\left[1 + \dfrac{1}{z} + \dfrac{1}{z^2} + \dots\right] - \left[1 + \dfrac{z}{2} + \dfrac{z^2}{4} + \dots\right]$ **(ii)** $\left[1 + z + z^2 + \dots\right] - \left[1 + \dfrac{z}{2} + \dfrac{z^2}{4} + \dots\right]$
**(iii)** $\dfrac{-1}{z}\left[1 + \dfrac{1}{z} + \dfrac{1}{z^2} + \dots\right] + \dfrac{2}{z}\left[1 + \dfrac{2}{z} + \dfrac{4}{z^2} + \dots\right].$

17. Find the Laurent's expansion of $f(z) = \dfrac{1}{z^2 + 5z + 6}$ valid in the region $1 < |z+1| < 2$ (N/D 2016)

**Ans:** $\left(\dfrac{1}{z+1}\left[1 - \dfrac{1}{z+1} + \dfrac{1}{(z+1)^2} - \dots\right] - \dfrac{1}{2}\left[1 - \dfrac{z+1}{2} + \dfrac{(z+1)^2}{4}\right]\right)$

18. Obtain the Laurent's series expansion of $f(z) = \dfrac{z^2 - 1}{(z+2)(z+3)}$ valid in the region $2 < |z| < 3$ (M/J 2014, Apr/May 2019)

**Ans:** $\left[f(z) = 1 + 2\left(z^{-1} - 2z^{-2} + 4z^{-3} - 8z^{-4} + \dots\right) - \dfrac{8}{3}\left(1 - \dfrac{z}{3} + \dfrac{z^2}{9} - \dfrac{z^3}{29} + \dots\right)\right]$

19. Find the residues of $f(z) = \dfrac{z^2}{(z+2)(z-1)^2}$ at its isolated singularities using Laurent's series expansion. (N/D 2013)

**Ans:** $\left[Residue\ at\ z = -2 = \dfrac{4}{9}, Residue\ at\ z = 1 = \dfrac{5}{9}\right]$

20. Evaluate $\displaystyle\int_0^{2\pi} \dfrac{d\theta}{13 + 5\sin\theta}$ using contour integration. (M/J 2014, Apr/May 2019)

**Ans:** $\left[\dfrac{\pi}{6}\right]$

21. Evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta} d\theta$ using contour integration. (N/D 2013, A/M 2018)

**Ans:** $\left[\frac{\pi}{6}\right]$

22. Evaluate $\int_0^{2\pi} \frac{\cos 3\theta}{5 - 4\cos\theta} d\theta$ using contour integration. (M/J 2013)

**Ans:** $\left[\frac{\pi}{12}\right]$

23. Prove that $\int_{-\infty}^{\infty} \frac{x^2}{(x^2 + a^2)(x^2 + b^2)} dx = \frac{\pi}{a + b}$; $(a > b > 0)$ using contour integration. (M/J 2013)

24. Show that $\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{(x^4 + 10x^2 + 9} dx = \frac{5\pi}{12}$ (M/J 2010, A/M 2011, N/D 2013)

25. Using contour integration, evaluate $\int_0^{\infty} \frac{dx}{(1 + x^2)^2}$ (M/J 2014,April/May 2019)

**Ans:** $\left[\frac{\pi}{4}\right]$

26. Show that $\int_0^{\infty} \frac{\cos mx}{(x^2 + a^2)} dx = \left[\frac{\pi e^{-ma}}{2a}\right]$ (M/J 2012)

27. Using contour integration, evaluate $\int_0^{\infty} \frac{dx}{(x^2 + a^2)^2} dx$, $a > 0$ (M/J 2015)

**Ans:** $\left[\frac{1}{4a^3 i}\right]$

# MA 8251 - Engineering Mathematics - II

## Unit - V LAPLACE TRANSFORMS

### Part A

1. **State the conditions for the existence of Laplace transform of a function.**
   **Solution:** The sufficient conditions for the existence of the Laplace transforms are
   (i) The function $f(t)$ should be a continuous or piecewise continuous function and
   (ii) $\lim_{t\to\infty} e^{-st} f(t)$ = finite i.e., $f(t)$ should be of exponential order.

2. **Does the Laplace transform of $\mathbf{\frac{\cos 2t}{t}}$ exists?**

   **Solution:** No. Since $\lim_{t\to 0} e^{-st}\frac{\cos 2t}{t} = \lim_{t\to 0}\frac{\cos 2t}{te^{st}} = \infty$

3. **Find the Laplace transform of** $\sin \mathbf{2t}.\sin \mathbf{3t}$
   **Solution:** $L\left[\sin 2t \sin 3t\right] = \frac{1}{2}L\left[\cos t\right] - L\left[\cos 5t\right] = \frac{1}{2}\left[\frac{s}{s^2+1} - \frac{s}{s^2+25}\right] = \frac{12s}{(s^2+1)(s^2+25)}$

4. **Show that (i)** $\mathbf{(i)L(t\sin at) = \frac{2as}{(s^2+a^2)^2}}$ **and** $\mathbf{(ii)L(t\cos at) = \frac{s^2-a^2}{(s^2+a^2)^2}}$

   **Solution:** $L[t] = \frac{1}{s^2}$
   Therefore $L[te^{iat}] = \frac{1}{(s-ia)^2} = \frac{(s+ia)^2}{((s-ia)(s+ia))^2}$
   $L\left[t(\cos at + i\sin at\right] = \frac{(s^2-a^2)+i2as}{(s^2+a^2)^2}$
   Equating the real and imaginary parts from both sides, we get the desired results.

5. **Find the Laplace transform of** $\mathbf{t}\cos \mathbf{at}$.
   **Solution:** $L[t\cos at] = -\frac{d}{ds}L[\cos at] = -\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right) = -\left[\frac{(s^2+a^2)-s.2s}{(s^2+a^2)^2}\right]$
   $= -\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right] = \frac{s^2-a^2}{(s^2+a^2)^2}$

6. **Find the Laplace transform of** $\mathbf{t}\sin \mathbf{2t}$.

   **Solution:** $L\left[\sin 2t\right] = \frac{2}{s^2+4}$
   Therefore $L\left[t\sin 2t\right] = -\frac{d}{ds}\left(\frac{2}{s^2+4}\right) = -\left[\frac{0-(2)(2s)}{(s^2+4)^2}\right] = \frac{4s}{(s^2+4)^2}$

7. **Find** $\mathbf{L}[\sin \mathbf{5t}\cos \mathbf{2t}]$.
   **Solution:**
   $$L[\sin 5t\cos 2t] = L\left[\frac{\sin 7t + \sin 3t}{2}\right] = \frac{1}{2}\left[\frac{7}{s^2+49} + \frac{3}{s^2+9}\right]$$

8. **Find** $\mathbf{L\left[t^2 2^t\right]}$
   **Solution:** $L\left[t^2 2^t\right] = L[t^2 e^{t\log 2}] = L\left[t^2\right]_{s\to s-\log 2} = \left[\frac{2}{s^3}\right]_{s\to s-\log 2} = \frac{2}{(s-\log 2)^3}$

9. **Find L[sin² t]**

**Solution:** $L[\sin^2 t] = L\left[\frac{1-\cos 2t}{2}\right] = L\left[\frac{1}{2}\right] - L\left[\frac{\cos 2t}{2}\right] = \frac{1}{2}L[1] - \frac{1}{2}L[\cos 2t]$

$= \frac{1}{2s} - \frac{1}{2}\frac{s}{s^2+2^2} = \frac{1}{2s} - \frac{s}{2(s^2+4)}$

10. **Find the Laplace transform of $f(t)$ defined as** $f(t) = \begin{cases} 1, & 0 < t \le 1 \\ t, & 1 < t \le 2 \\ 0, & t > 2 \end{cases}$

**Solution:** $L[f(t)] = \int_0^1 e^{-st}.1dt + \int_1^2 e^{-st}.tdt + \int_2^\infty e^{-st}.0dt = \left(\frac{e^{-st}}{-s}\right)_0^1 + \left(t.\frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2}\right)_1^2$

$= \frac{1-e^{-s}}{s} + \left\{\left(-\frac{2e^{-2s}}{s} - \frac{e^{-2s}}{s^2}\right) - \left(\frac{e^{-s}}{-s} - \frac{e^{-s}}{s^2}\right)\right\} = \frac{1}{s} - \frac{2e^{-2s}}{s} + \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2}$

11. **Find the Laplace transform of the function $f(t) = |t-1| + |t+1|, t \ge 0$.**

**Solution:** Given function is equivalent to $f(x) = \begin{cases} 2 & \text{for } 0 \le t < 1 \\ 2t & \text{for } t \ge 1 \end{cases}$

(Since $|t-1| = -(t-1)$ if $t-1<0$ (or) $t<1$ and $(t-1)$ if $t-1>0$ $(or) t>1$
but $|t+1| = t+1$ only for $t > -1$)

$L[f(t)] = \int_0^1 e^{-st} 2dt + \int_1^\infty e^{-st} 2tdt$

$= 2\left[\left(\frac{e^{-st}}{-s}\right)_0^1\right] + 2\left[\left(\frac{te^{-st}}{-s}\right)_1^\infty + \left(\frac{e^{-st}}{s^2}\right)_1^\infty\right] = 2\left(\frac{e^{-s}}{-s} + \frac{1}{s}\right) + 2\left(\frac{0-e^{-s}}{-s} - \frac{0-e^{-s}}{s^2}\right) = \frac{2}{s}\left(1 + \frac{e^{-s}}{s}\right).$

12. **Find the Laplace transform of the function** $f(t) = [t]$, where [] stands for the greatest integer function.

**Solution:** Given function is equivalent to $[t] = 0$ in $(0,1) + 1$ in $(1,2) + 2$ in $(2,3) + 3$ in $(3,4) + \ldots$ $L[f(t)] = \int_0^\infty e^{-st} f(t)dt = \int_0^\infty e^{-st}[t]\,dt$

$= \int_0^1 e^{-st}(0)dt + \int_1^2 e^{-st}(1)dt + \int_2^3 e^{-st}(2)dt + \int_3^4 e^{-st}(3)dt + \ldots$

$= 0 + \left(\frac{e^{-st}}{-s}\right)_1^2 + 2\left(\frac{e^{-st}}{-s}\right)_2^3 + 3\left(\frac{e^{-st}}{-s}\right)_3^4 + \ldots\infty$

$= -\frac{1}{s}\left[(e^{-2s} - e^{-s}) + 2(e^{-3s} - e^{-2s}) + 3(e^{-4s} - e^{-3s}) + \ldots\infty\right]$

$= \frac{1}{s}\left[e^{-s} + e^{-2s} + e^{-3s} + \ldots\infty\right] = \frac{1}{s}\left[\frac{e^{-s}}{1-e^{-s}}\right] = \frac{1}{s(e^s-1)}.$

13. **Find L[e$^{-at}$ sin bt]**

**Solution:** $L[e^{-at}\sin bt] = L[\sin bt]_{s\to s+a} = \left[\frac{b}{s^2+b^2}\right]_{s\to s+a} = \frac{b}{(s+a)^2+b^2}$

14. **Find L[te$^{-2t}$ sin 2t]**

**Solution:** $L[te^{-2t}\sin 2t] = -\frac{d}{ds}L[e^{-2t}\sin 2t] = -\frac{d}{ds}L[\sin 2t]_{s\to s+2} = -\frac{d}{ds}\left(\frac{2}{s^2+4}\right)_{s\to s+2}$

$= -\frac{d}{ds}\left(\frac{2}{(s+2)^2+4}\right) = \frac{4(s+2)}{(s+2)^2+4^2}$

15. **State and prove change of scale property.**

**Statement.** If $L[f(t)] = F(s)$ then $L[f(at)] = \frac{1}{a}F\left(\frac{s}{a}\right), a > 0.$

**Proof.** $L[f(t)] = \int_0^\infty e^{-st} f(t)dt = F(s)$

$$L[f(at)] = \int_0^\infty e^{-st} f(at)dt$$

Let $at = x \quad \Rightarrow t = \frac{x}{a} \Rightarrow dt = \frac{1}{a}dx$

When $t = 0, x = 0,$ when $t = \infty, x = \infty$

$$\therefore L[f(at)] = \int_0^\infty e^{-s\frac{x}{a}} f(x)\frac{dx}{a}$$
$$= \frac{1}{a}\int_0^\infty e^{-(\frac{s}{a})x} f(x)dx$$
$$= \frac{1}{a}F\Big(\frac{s}{a}\Big)$$
$$= \frac{1}{a}[F(s)]_{s\to\frac{s}{a}} = \frac{1}{a}[L[f(t)]]_{s\to\frac{s}{a}}.$$

16. **Find the laplace transform of unit step function.**

**Solution:** Unit step function$u_a(t) = \begin{cases} 0 & when \;\; t < a \\ 1 & when \;\; t > a \end{cases}$

$$\therefore L[u_a(t)] = \int_0^\infty e^{-st}u_a(t)dt = \int_0^a e^{-st}0\;dt + \int_a^\infty e^{-st}1\;dt$$
$$= \int_a^\infty e^{-st}dt = \left(\tfrac{e^{-st}}{-s}\right)_a^\infty = 0 - \left(\tfrac{e^{-as}}{-s}\right) = \tfrac{e^{-as}}{s}$$
$$\therefore L[u_a(t)] = \tfrac{e^{-as}}{s}$$

17. **Find** $L^{-1}\left\{\cot^{-1}(s)\right\}$**.**

**Solution:** Let$L^{-1}\left\{\cot^{-1}(s)\right\} = f(t)$

$L[f(t)] = \cot^{-1}(s) \Rightarrow L[tf(t)] = -\frac{d}{ds}\left\{\cot^{-1}(s)\right\} L[tf(t)] = \frac{1}{s^2+1} \Rightarrow tf(t) = L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$

Therefore$L^{-1}\left\{\cot^{-1}(s)\right\} = f(t) = \frac{\sin t}{t}$.

18. **Find** $L^{-1}\left[\frac{1}{s^2+4s+4}\right]$

**Solution:** $L^{-1}\left[\frac{1}{s^2+4s+4}\right] = L^{-1}\left[\frac{1}{(s+2)^2}\right] = e^{-2t}L^{-1}\left[\frac{1}{s^2}\right] = e^{-2t}t$

19. **Find** $L^{-1}\left[\frac{1}{s}\left(\frac{1}{s^2+\omega^2}\right)\right]$

**Solution.** $L^{-1}\Big[\frac{1}{s(s^2+\omega^2)}\Big] = L^{-1}\Big[\frac{F(s)}{s}\Big]$ where $F(s) = \frac{1}{s^2+\omega^2}$

$$= \int_0^t L^{-1}[F(s)]dt = \int_0^t L^{-1}\Big[\frac{1}{s^2+\omega^2}\Big]dt = \int_0^t \frac{\sin\omega t}{\omega}dt$$
$$= \frac{1}{\omega}\left(-\frac{\cos\omega t}{\omega}\right)_0^t = -\frac{1}{\omega^2}(\cos\omega t - 1) = \frac{1-\cos\omega t}{\omega^2}.$$

20. **Find** $L^{-1}\left[\frac{s+2}{s^2+4s+8}\right]$

**Solution:** $L^{-1}\left(\frac{s+2}{s^2+4s+8}\right) = L^{-1}\left[\frac{s+2}{(s+2)^2+4}\right] = e^{-2t}L^{-1}\left[\frac{s}{s^2+4}\right] = e^{-2t}\cos 2t$

21. **Find** $L^{-1}\left[\dfrac{s}{(s+2)^2+1}\right]$**.**

**Solution: Solution.** $L^{-1}\Big[\dfrac{s}{(s+2)^2+1}\Big] = L^{-1}\Big[s\dfrac{1}{(s+2)^2+1}\Big] = \dfrac{d}{dt}\left(L^{-1}\Big[\dfrac{1}{(s+2)^2+1}\Big]\right)$

$$= \frac{d}{dt}\left(e^{-2t}L^{-1}\Big[\frac{1}{s^2+1}\Big]\right)$$
$$= \frac{d}{dt}\left(\frac{e^{-2t}}{2}L^{-1}\Big[\frac{2}{s^2+1}\Big]\right)$$
$$= \frac{d}{dt}\left(\frac{e^{-2t}}{2}\sin t\right)$$
$$= \frac{1}{2}\left(e^{-2t}2\cos t - 2e^{-2t}\sin t\right)$$
$$= e^{-2t}(\cos t - \sin t).$$

22. **Find** $L^{-1}\left[\dfrac{1}{(s+2)^3}\right]$

**Solution:** $L^{-1}\left[\dfrac{1}{(s+2)^3}\right] = e^{-2t}L^{-1}\left[\dfrac{1}{s^3}\right] = e^{-2t}\dfrac{t^2}{2!}$

23. **Find** $L^{-1}\left[\dfrac{1}{\sqrt{s+2}}\right]$

**Solution:** $L^{-1}\left[\dfrac{1}{\sqrt{s+2}}\right] = e^{-2t}L^{-1}\left[\dfrac{1}{\sqrt{s}}\right] = e^{-2t}\dfrac{t^{-\frac{1}{2}}}{\sqrt{\pi}}$ $\left[Since\ \ L(t^{-\frac{1}{2}}) = \dfrac{\sqrt{\pi}}{\sqrt{s}}\right]$

24. **If** $L[f(t)] = \dfrac{1}{s(s+a)}$, **find** $f(0)$.

**Solution:**

$f(t) = L^{-1}\left[\dfrac{1}{s(s+a)}\right] = \int_0^t L^{-1}\left[\dfrac{1}{s+a}\right]dt = \int_0^t e^{-at}dt = \left[\dfrac{e^{-at}}{-a}\right]_0^t = -\dfrac{1}{a}[e^{-at}-1]$

$f(0) = -\frac{1}{a}[e^0 - 1] = 0.$

25. **If** $L[f(t)] = \dfrac{1}{s(s+1)}$**, find** $\lim_{t\to 0} f(t)$ **and** $\lim_{t\to\infty} f(t)$**using initial and final value theorems.**

**Solution:** $\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\dfrac{1}{s+1} = 0$

$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = \lim_{s\to 0}\dfrac{1}{s+1} = 1$

26. **Find the Laplace transform of** $\dfrac{\mathbf{1-\cos t}}{\mathbf{t}}$.

**Solution:** $L\left[\dfrac{1-\cos t}{t}\right] = \int_s^\infty L[1-\cos t]ds = \int_s^\infty L[1] - L[\cos t]ds = \int_s^\infty \left[\dfrac{1}{s} - \dfrac{s}{s^2+1}\right]ds$

$= \left[\log s - \dfrac{1}{2}\log(s^2+1)\right]_s^\infty = \left[\log s - \log(s^2+1)^{\frac{1}{2}}\right]_s^\infty$

$= \left[\log\dfrac{s}{\sqrt{s^2+1}}\right]_s^\infty = \left[\log\dfrac{1}{\sqrt{1+\dfrac{1}{s^2}}}\right]_s^\infty = \log 1 - \log\dfrac{1}{\sqrt{1+\dfrac{1}{s^2}}} = 0 - \log\dfrac{1}{\sqrt{1+\dfrac{1}{s^2}}}$

$= \log\left[\dfrac{s}{\sqrt{s^2+1}}\right]^{-1} = \log\left[\dfrac{\sqrt{s^2+1}}{s}\right]$

27. **Using Laplace transform, evaluate** $\int_0^\infty te^{-2t}\cos t dt$.

**Solution:** $\int_0^\infty te^{-2t}\cos t dt = L[t\cos t] = \left[\frac{s^2-1}{(s^2+1)^2}\right]_{s=2} = \frac{3}{25}$

28. **Prove that** $\int_0^\infty e^{-t}\frac{\sin t}{t}dt = \frac{\pi}{4}$.

**Solution:** Using Laplace transform $\int_0^\infty e^{-t}\frac{\sin t}{t}dt = L\left[\frac{\sin t}{t}\right] = \tan^{-1}\left(\frac{a}{s}\right)$

Put $s = 1, a = 1$ then $=\tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$

29. **Using Laplace transform, evaluate** $\int_0^\infty te^{-t}\sin t dt$.

**Solution:** $\int_0^\infty te^{-t}\sin t dt = L[t\sin t] = \left[-\frac{d}{ds}L[sint]\right]_{s=1} = \left[-\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right]_{s=1}$
$= \left[\frac{2s}{(s^2+1)^2}\right]_{s=1} = \frac{1}{2}$

30. **State Convolution theorem of Laplace Transformation.**

**Statement:** If *f(t)* and *g(t)* are functions defined for $t \geq 0$, then
$L[f(t) * g(t)] = L[f(t)]L[g(t)] = F(s)G(s)$

31. **Find the convolution of $t$ with $e^t$. Solution:** By the definition of convolution, we have
$t * e^t = \int_0^t te^{t-u}du = [t(-e^{t-u}) - (e^{t-u})]_0^t = (t+1)(e^t - 1)$.

## Part B(Assignment Questions)

1. Find the Laplace transform of $te^{-3t}\sin 2t$. $Ans : \left[\frac{4(s+3)}{(s^2+6s+13)^2}\right]$

2. Find the Laplace transform of $t^2e^{-3t}\sin 2t$. $Ans : \left[\frac{2\left(s^3+6s^2+9s+2\right)}{(s^2+4s+5)^3}\right]$

3. Evaluate $\int_0^\infty e^{-t}\left[\frac{\cos at - \cos bt}{t}\right]dt$ by using Laplace transformations. $Ans : \left[\frac{1}{2}\log\frac{1+b^2}{1+a^2}\right]$

4. Evaluate $\int_0^\infty\left[\frac{1-\cos 2t}{t^2}\right]dt$ by using Laplace transformations. $Ans : \pi$

5. Find the Laplace transform of the following triangular wave function given by
$f(t) = \begin{cases} t & 0 \leq t \leq a \\ 2a - t & a \leq t \leq 2a \end{cases}$ and $f(t+2a) = f(t)$ $Ans : \left[\frac{1}{s^2}\tan h\left(\frac{as}{2}\right)\right]$

6. Obtain the Laplace transform of the rectangular wave given by
$f(t) = \begin{cases} 1 & 0 < t < b \\ -1 & b < t < 2b \end{cases}$ with $f(t+2b) = f(t)$ $Ans : \left[\frac{1}{s}\tanh\left(\frac{sb}{2}\right)\right]$

7. Obtain the Laplace transform of the square wave function is given by
$f(t) = \begin{cases} k & 0 < t < a \\ -k & a < t < 2a \end{cases}$ with $f(t+2a) = f(t)$ $Ans : \left[\frac{k}{s}\tanh\left(\frac{as}{2}\right)\right]$

8. Find the Laplace transform of
$f(t) = \begin{cases} \sin\omega t & for \quad o < t < \frac{\pi}{\omega} \\ 0 & for \quad \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$ with $f\left(t + \frac{2\pi}{\omega}\right) = f(t)$. Ans: $\dfrac{\omega}{(s^2+\omega^2)\left(1 - e^{-\frac{s\pi}{\omega}}\right)}$

9. Find the Laplace transform of the full-sine wave rectifier function
$f(t) = |\sin\omega t|, t \geq 0$ with $f\left(t + \frac{t}{\omega}\right) = f(t)$. Ans: $\dfrac{\omega}{(s^2+\omega^2)} \cot h\left[\dfrac{\pi s}{2\omega}\right]$

10. Verify the initial and final value theorem for the function
$f(t) = (t+2)^2 e^{-t}$ Ans: $[IVT : 4, FVT : 0]$

11. Verify the initial and final value theorem for the function
$f(t) = L^{-1}\left[\dfrac{1}{s(s+2)^2}\right]$ Ans: $\left[IVT : 0, FVT : \dfrac{1}{4}\right]$

12. Verify the initial and final value theorem for the function
$L\left[e^{-t}\cos^2 t\right] = F(s)$ Ans: $[IVT : 1, FVT : 0]$

13. Verify the initial and final value theorem for the function
$f(t) = 1 + e^{-t}(\sin t + \cos t)$ Ans: $[IVT : 2, FVT : 1]$

**Find the inverse Laplace transformations of the following functions**

14. $\tan^{-1}\left(\frac{2}{s^2}\right)$ Ans: $\left[-\frac{2}{t}\sin t \sinh t\right]$

15. $log\left(\dfrac{s+a}{s+b}\right)$. Ans: $\left[\frac{1}{t}\left(e^{-bt} - e^{-at}\right)\right]$

16. $log\left(\dfrac{s+1}{s-1}\right)$. Ans: $\left[\frac{2\sinh t}{t}\right]$

17. $log\left(\dfrac{s+1}{(s+2)(s+3)}\right)$ Ans: $\left[e^{-t} - e^{-2t} - e^{-3t}\right]$

18. $log\left(\dfrac{s^2+1}{s(s+1)}\right)$. Ans: $\left[\frac{1}{t}\left(1 + e^{-t} - 2\cos t\right)\right]$

19. $\dfrac{1}{2} log\left(\dfrac{s^2+b^2}{s^2+a^2}\right)$. Ans: $\left[\frac{1}{t}\left(\cos at - \cos bt\right)\right]$

20. Find the inverse Laplace transform of $\dfrac{s-3}{s^2+4s+13}$ Ans: $\left[e^{-2t}\cos 3t - 5e^{-2t}\frac{\sin 3t}{3}\right]$

**Find the inverse Laplace transform of the following functions by partial fractions**

21. $\dfrac{5s+3}{(s-1)(s^2+2s+5}$ Ans: $\left[e^t - e^{-t}\cos 2t + \frac{3}{2}e^{-t}\sin 2t\right]$

22. $\dfrac{s}{s^4+4a^4}$ Ans: $\left[\frac{1}{2a^2}\sin at \sinh at\right]$

23. $\dfrac{2s^2-6s+5}{s^3-6s^2+11s-6}$ Ans: $\left[\frac{1}{2}e^t - e^{2t} + \frac{5}{2}e^{3t}\right]$

24. $\dfrac{4s+5}{(s-1)^2(s+2)}$ Ans: $\left[\frac{1}{3}e^t + 3te^t - \frac{1}{3}e^{-2t}\right]$

25. $\dfrac{s}{(2s-1)(3s-1)}$ Ans: $\left[3e^{\frac{t}{2}} + 2e^{\frac{t}{3}}\right]$

26. $\dfrac{3s+2}{s^2-s-2}$ Ans: $\left[2e^{3t} - \frac{3}{5}e^{2t} - \frac{2}{5}e^{7t}\right]$

**Evaluate the following problems by using convolution theorem.**

27. $L^{-1}\left[\dfrac{1}{(s^2+1)(s^2+9)}\right]$ Ans: $\left[\frac{1}{8}\left(\sin t - \frac{1}{3}\sin 3t\right)\right]$

28. $L^{-1}\left[\dfrac{s^2}{(s^2+4^2)^2}\right]$. Ans: $\left[\frac{1}{4}(\sin 2t + 2t\cos 2t)\right]$

29. $L^{-1}\left[\dfrac{1}{(s^2+4)^2}\right]$ Ans: $\left[\frac{1}{8}\left(\frac{\sin 2t}{2} - t\cos 2t\right)\right]$

30. $L^{-1}\left[\dfrac{s}{(s^2+1)(s^2+9)(s^2+4)}\right]$ Ans: $\left[\frac{1}{12}\cos t - \frac{1}{10}\cos 2t + \frac{1}{60}\cos 3t\right]$

31. $L^{-1}\left[\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}\right]$ Ans: $\left[\dfrac{a\sin at - b\sin bt}{a^2-b^2}\right]$

32. $L^{-1}\left[\dfrac{s}{(s^2+1)(s^2+4)}\right]$. Ans: $\left[\frac{1}{3}(\cos t - \cos 2t)\right]$

33. $L^{-1}\left[\dfrac{1}{s^3(s^2+1)}\right]$ Ans: $\left[\frac{t^2}{2} + \cos t - 1\right]$

34. $L^{-1}\left[\dfrac{s}{(s+2)(s^2+9)}\right]$ Ans: $\frac{1}{13}\left[3\sin 3t + 2\cos 2t - 2e^{-2t}\right]$

**Solve the following differential equations with initial values by Laplace Transformation.**

35. $y''' + 2y'' - y' - 2y = 0; y(0) = y'(0) = 0, y''(0) = 6$ Ans: $\left[y(t) = e^t - 3e^{-t} + 2e^{-2t}\right]$

36. $(D^3 - 3D^2 + 3D - 1)y = t^2e^t; y(0) = 1, y'(0) = 0, y''(0) = -2$ Ans: $\left[y(t) = e^t\left(1 - t - \frac{1}{2}t^2 + \frac{1}{60}t^5\right)\right]$

37. $y'' + 9y = \cos 2t; y(0) = 1, y\left(\frac{\pi}{2}\right) = -1$ Ans: $\left[y(t) = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t\right]$

38. $y'' + 25y = 10\cos 5t;\ y(0) = 2, y'(0) = 0$ Ans: $[y(t) = 2\cos 5t + t\sin 5t]$

39. $y'' - 3y' + 2y = 4\ ;\ y(0) = 2,\ y'(0) = -3.$ Ans: $\left[y = 2 + 3e^t - 3e^{2t}\right]$

40. $y'' - 3y' + 2y = e^{3t}\ ;\ y(0) = 1,\ y'(0) = 0.$ Ans: $\left[y = \dfrac{1}{2}e^{3t} - 2e^{2t} + \dfrac{5}{2}e^t\right]$

41. $y'' + y = 2e^t;\ y(0) = 1;\ y'(0) = 2.$ Ans: $[y = e^t + \sin t]$

42. $y'' - 3y' + 2y = 4t + e^{3t}; y(0) = 1, y'(0) = -1.$ Ans: $\left[y = 3 + 2t - e^t - 2e^{2t} + \frac{1}{2}e^{3t}\right]$

43. $y'' + 4y' + 3y = e^{-t}; y(0) = 1, y'(0) = 1.$ Ans: $\left[y(t) = \frac{7}{4}e^{-t} - \frac{3}{4}e^{-3t} - \frac{1}{2}te^{-t}\right]$

44. $y'' - 3y' + 2y = 4t, y(0) = 1, y'(0) = -1.$ Ans: $\left[y(t) = 3 + 2t - e^{2t} - e^t\right]$

# Multiple Choice Questions

## Unit - I Matrices

### Part A

1. The characteristic equation of the matrix $\begin{pmatrix} 0 & -1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ is

   (a) $\lambda^3-5\lambda^2+4\lambda-2=0$ (b) $\lambda^3-4\lambda^2+4\lambda-2=0$ (c) $\lambda^3-5\lambda^2+5\lambda-2=0$ (d) $\lambda^3-4\lambda^2+5\lambda-2=0$

2. The sum and product of the eigen values of the matrix A = $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$ is

   (a) $18, 0$ (b) $0, 18$ (c)$17, 18$ (d) $18, 17$

3. The sum and product of the eigen values of a $3\times 3$ matrix $A$ whose characterisitic equation is $\lambda^3 - 7\lambda^2 + 36 = 0$ is
   (a)$36, -7$ (b)$-36, 7$ (c) $-7, 36$ (d) $7, -36$

4. For the matrix P = $\begin{pmatrix} 3 & -2 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, one of the eigen values is equal to $-2$. Which of the following is an eigen vector?
   (a) $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$ (b) $\begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}$ (c) $\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ (d) $\begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix}$

5. If the product of two eigen values of the matrix $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$ is 16 then the third eigen value is
   (a)2 (b) 3 (c) 0 (d)$-1$

6. The eigen values of the matrix $P = \begin{pmatrix} 4 & 5 \\ 2 & -5 \end{pmatrix}$ are
   (a) $-7, 8$ (b) $-6, 5$ (c) $3, 4$ (d) $1, 2$

7. Eigen values of a matrix S = $\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$ are 5 and 1. What are the eigen values of the matrix $S^2$.

   (a) 1 and 25 (b) 6 and 4 (c) 5 and 1 (d) 2 and 10.

8. The sum of the eigen values of the matrix $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$ is

   (a) 5 (b) 7 (c) 9 (d) 18

9. The eigen values of the matrix $\begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 1 & 4 \end{pmatrix}$ are
(a)$2, -2, 1, -1$ (b) $2, 3, -2, 4$ (c)$2, 3, 1, 4$ (d) $1, 2, 1, 1$

10. The eigen values of $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ are
(a)$0, 0, 0$ (b) $0, 0, 1$ (c)$1, 1, 1$ (d)$0, 0, 3$

11. If $A$ be a $3x3$ matrix with eigen values $1, -1, 0$, then the determinant of $I + A^{100}$ is
(a)6 (b) 4 (c) 9 (d)100

12. The sum of the eigen values of the matrix $\begin{pmatrix} 3 & 4 \\ x & 1 \end{pmatrix}$ for real and negative values of $x$ is
(a) greater than zero (b) less than zero (c)zero (d) dependent

13. $A$ is a singular matrix of order 3 with eigen values 2 and 3. The third eigen value is
(a) 1 (b) 0 (c) 4 (d) $-1$

14. The eigen values and the corresponding eigen vectors of a $2x2$ matrix are given by $\lambda_1 = 8$, $X_1 = [11]^T$ and $\lambda_1 = 4$, $X_2 = [1 - 1]^T$. The matrix is
(a) $\begin{pmatrix} 3 & 2 \\ 2 & 6 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 6 \\ 6 & 4 \end{pmatrix}$ (c) $\begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}$ (d) $\begin{pmatrix} 4 & 8 \\ 8 & 4 \end{pmatrix}$

15. The minimum and maximum eigen values of the matrix $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$ are $-2$ and 6 respectively. What is the other eigen value?
(a) 5 (b)3 (c)1 (d)$-1$

16. If the sum of two eigen values and trace of a $3x3$ matrix $A$ are equal, then $|A|$ is
(a) 0 (b)1 (c) $-1$ (d) 3

17. The trace and determinant of a $2x2$ matrix are known to be -2 and -35 respectively. Its eigen values are
(a) $-30, -5$ (b) $-37, -1$ (c) $-7, 5$ (d) $18, -2$

18. If two eigen values of the matrix A = $\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$ are 3 and 0 then the product of the eigen values of A is
(a)0 (b) $-1$ (c)3 (d)1

19. If 2, 3 are the eigen values of $\begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ b & 0 & 2 \end{pmatrix}$, then the value of '$b$' is
(a) 1 (b)0 (c)3 (d)6

20. For a given matrix A of order 3, $|A|$= 32 and two of its eigen values are 8 and 2 then the sum of the eigen values is
(a) 12 (b)10 (c) 16 (d) 32

21. Given A = $\begin{pmatrix} -1 & 0 & 0 \\ 2 & -3 & 0 \\ 1 & 4 & 2 \end{pmatrix}$. Then the eigen values of A$^2$ are
(a) $1, 9, 4$ (b)$-1, -3, 2$ (c) $1, \frac{1}{9}, \frac{1}{4}$ (d) $1, -3, 5$

22. The eigenvalues of $\begin{pmatrix} 5 & 6 & 17 \\ 0 & -19 & 23 \\ 0 & 0 & 37 \end{pmatrix}$ are
(a) $-19, 5, 37$ (b) $19, -5, -37$ (c) $2, -3, 7$ (d) $3, -5, 37$

23. The eigen values of the matrix $\begin{pmatrix} 3 & 2 & 9 \\ 7 & 5 & 13 \\ 6 & 17 & 19 \end{pmatrix}$ are given by solving the cubic equation
(a) $\lambda^3 - 27\lambda^2 + 167\lambda - 285 = 0$ (b) $\lambda^3 - 27\lambda^2 - 122\lambda - 313 = 0$ (c) $\lambda^3 + 27\lambda^2 + 167\lambda + 285 = 0$
(d) $\lambda^3 - 27\lambda^2 - 167\lambda - 285 = 0$

24. The eigenvalues of a $4x4$ matrix $A$ are given as $2, -3, 13$ and $7$. Then $det(A)$ is
(a) 546 (b) 19 (c) 25 (d) cannot be determined

25. Two eigen values of the matrix $P = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ are equal to 1 each. Then the eigen values of $P^3$ are
(a) $1, 1, 125$ (b) $1, 1, 5$ (c) $1, 1, 1$ (d) $1, 0, 1$

26. Given that A=$\begin{pmatrix} 4 & 1 \\ 3 & 2 \end{pmatrix}$, then the eigen values of $A^3$ are
(a) $1, 125$ (b) $1, 5$ (c) $1, 0$ (d) $5, 1$

27. Let A=$\begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}$ and $v_1 = [1 \ -2]^T, v_2 = [1 \ 1]^T$. Then $v_1$ and $v_2$ are eigen vectors of $A$ corresponding to the eigen values—- and —- respectively.
(a) $1, 4$ (b) $4, 1$ (c) $3, 1$ (d) $1, 3$

28. The eigen values of the matrix $\begin{pmatrix} 1 & -8 \\ -2 & b \end{pmatrix}$ are $-3$ and $5$. Then the value of $b$ is
(a) 3 (b) 2 (c) 1 (d) 0

29. If 3 and 6 are two eigen values of $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$, then the eigen values of $A^{-1}$ are
(a) $\frac{1}{3}, \frac{1}{6}, \frac{1}{-2}$ (b) $\frac{-1}{3}, \frac{-1}{6}, \frac{1}{-2}$ (c) 3, 6 (d) 6, 3

30. The eigen values of the inverse of the matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 4 \end{pmatrix}$ are
(a) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ (b)$\frac{-1}{2}, \frac{-1}{3}, \frac{-1}{4}$ (c) 2,3,4 (d) -2,-3,-4

31. If $A = \begin{pmatrix} 5 & 0 & 0 \\ 2 & -1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$, then the eigen values of adj $A$ are
(a)$-1, 5, -5$ (b) $1, 5, -5$ (c) $1, -5, 5$ (d) $1, 5, 5$

32. Which one of the following statements is true for all real symmetric matrices?
(a) All the eigen values are real (b) All the eigen vlues are positive (c) All the eigen values are distinct (c) Sum of all the eigen values is zero

33. Let $A$ and $P$ be both $2x2$ matrices and $P$ be non-singular. If $\lambda_1, \lambda_2$ are the eigen values of $A$ then the eigen values of $P^{-1}AP$ are
(a) $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}$ (b) $\frac{\lambda_1}{\lambda_2}, \frac{\lambda_2}{\lambda_1}$ (c) $\lambda_1, \lambda_2$ (d) $\lambda_1 + \lambda_2, \lambda_1 - \lambda_2$

34. An eigen vector of $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ is

(a) $[-1\ 1\ 1]^T$ (b) $[1\ 2\ 1]^T$ (c) $[1\ -1\ 2]^T$ (d) $[2\ 1\ -1]^T$

35. Consider the matrix $\begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$. Which of the following is an eigen vector of $A$ corresponding to the eigen value $\lambda =$?

(a) $[\frac{1}{2}\ 1\ 1]^T$ (b) $[\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}]^T$ (c) $[\frac{1}{2}\ \frac{1}{2}\ 1]^T$ (d) $[1\ -1\ 1]^T$

36. Consider the matrix $\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$. Which of the following is an eigen vector of $A$?

(a) $[1\ 1\ 1]^T$ (b) $[1\ 0\ -1]^T$ (c) $[1\ -2\ 1]^T$ (d) $[1\ -1\ 0]^T$

37. If $[-4.5\ -4\ 1]^T$ is an eigen vector of $\begin{pmatrix} 8 & -4 & 2 \\ 4 & 0 & 2 \\ 0 & -2 & -4 \end{pmatrix}$, the eigen value corresponding to the eigen vector is

(a) 1 (b)4 (c) $-4.5$ (d) 6

38. If the eigen values of the matrix $A$ of order $3 \times 3$ are $2, 3,$ and $1$, then the eigen values of adjoint of $A$ are

(a) $3, 2, 6$ (b) $-3, -2, -6$ (c) $\frac{1}{3}, \frac{1}{2}, \frac{1}{6}$ (d)$\frac{-1}{3}, \frac{-1}{2}, \frac{-1}{6}$

39. If $A = \begin{pmatrix} 5 & 4 \\ 0 & 2 \end{pmatrix}$ then the eigen values of $3A + 2I$ are

(a)$17, 8$ (b) $-17, -8$ (c) $7, 18$ (d) $-7, -18$

40. If $2, -1, -3$ are the eigen values of the matrix $A$, then the eigen values of the matrix $A^2-2I$ are

(a)$2, -1, 7$ (b)$-2, 1, -7$ (c)$2, -1, -3$ (d) $-2, 1, 3$

41. If X = (-1, 0, 1)$^T$ is an eigen vector of the matrix A = $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$, then the corresponding eigen value is

(a)$-2$ (b) 2 (c) 3 (d) $-3$

42. The characterisitic equation of a $3x3$ matrix $P$ is given by $\lambda^3 + \lambda^2 + 2\lambda + 1 = 0$. If $I$ denotes the identity matrix, then the inverse of the matrix $P$ is given by

(a) $P^2 + P + 2I$ (b)$P^2 + P + I$ (c) $-(P^2 + P + I)$ (d) $-(P^2 + P + 2I)$

43. The quadratic form corresponding to the matrix $\begin{pmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{pmatrix}$ is

(a)$y^2 + 2z^2 + 10xy - 2xz + 12yz$ (b)$y^2 + z^2 + 10xy - 2xz + 12yz$ (c) $y^2 + 2z^2 + xy - 2xz + 12yz$ (d)$y^2 + 2z^2 + 10xy + 2xz + 12yz$

44. The quadratic form corresponding to the matrix $\begin{pmatrix} 2 & 0 & -2 \\ 0 & 2 & 1 \\ -2 & 1 & -2 \end{pmatrix}$ is

(a)$2x_1^2+2x_2^2-2x_3^2-4x_1x_3+2x_2x_3$ (b)$2x_1^2+x_2^2-2x_3^2-4x_1x_3+2x_2x_3$ (c) $2x_1^2+2x_2^2+2x_3^2-4x_1x_3+2x_2x_3$ (d)$2x_1^2 + 2x_2^2 - 2x_3^2 + 4x_1x_3 + 2x_2x_3$

45. The nature of the quadratic form of matrix $\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ is

(a)negative definite (b) positive definite (c) indefinite (d) positive semidefinite

46. The matrix of the quadratic form $6x_1^2 + 3x_2^2 + 14x_3^2 + 4x_1x_2 + 4x_2x_3 + 18x_3x_1$ is

(a) $\begin{pmatrix} 6 & 2 & 9 \\ 2 & 3 & 2 \\ 9 & 2 & 14 \end{pmatrix}$ (b) $\begin{pmatrix} -6 & -2 & -9 \\ -2 & -3 & -2 \\ 9 & 2 & -14 \end{pmatrix}$ (c) $\begin{pmatrix} 6 & 4 & 18 \\ 4 & 3 & 4 \\ 18 & 4 & 14 \end{pmatrix}$ (d) $\begin{pmatrix} -6 & 4 & 18 \\ 4 & -3 & 4 \\ 18 & 4 & -14 \end{pmatrix}$

47. The quadratic form of the symmetric matrix $diag[\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n]$ is

(a) $\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_n$ (b) $\lambda_1^2 x_1 + \lambda_2^2 x_2 + \lambda_3^2 x_3 + \cdots + \lambda_n^2 x_n$ (c)$\lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 + \cdots + \lambda_n x_n^2$
(d) $\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \cdots + \lambda_n x_n$

48. The nature of the quadratic form $2xy + 2yz + 2zx$ is

(a) indefinite (b)positive definite (c)positive semidefinite (d) negative semidefinite

49. The index and signature of the quadratic form $x_1^2 + 2x_2^2 - 3x_3^2$ is

(a) $2, 1$ (b) $-2, -1$ (c) $1, 2$ (d) $-1, -2$

50. If the sum of the eigen values of the matrix of the quadratic form is equal to zero, then what will be the nature of the quadratic form?

(a)indefinite (b)positive definite (c)positive semidefinite (d) negative semidefinite

## Unit -II Vector Calculus

### Part A

1. Find a vector normal to the curve $x^2y + logy - 2x = 0$ at the point $(2, 1)$.
   (a) $2\vec{i} + 5\vec{j}$
   (b) $5\vec{i} + 2\vec{j}$
   (c) $-25\vec{i} + \vec{j}$
   (d) $2\vec{i} + \vec{j}$

2. Let the temperature at the point $(x, y)$ in a flat plate be given by the function $T(x, y) = 3x^2 + 2xy$. A tub of margarine is placed at $(3, -6)$ in what direction should it be moved to cool most quickly?
   (a) $6\vec{i} + 6\vec{j}$
   (b) $\vec{i} + \vec{j}$
   (c) $-\vec{i} - \vec{j}$
   (d) $6\vec{i} - 12\vec{j}$
   (e) $(3, -6)$ is already the coolest point on the plate.

3. Find the maximum directional derivative of the function $f(x, y) = x \ln y + x^2y^2$ at the point $(-1, 1)$.
   (a) $-2\vec{i} + \vec{j}$
   (b) $\sqrt{5}(-2\vec{i} + \vec{j})$
   (c) $1$
   (d) $\sqrt{5}$
   (e) $\frac{1}{\sqrt{5}}$

4. Find the direction where the directional derivative is greatest for the function $f(x, y) = 3x^2y^2 - x^4 - y^4$ at the point $(1, 2)$.
   (a) $\frac{1}{\sqrt{2}}(\vec{i} - \vec{j})$
   (b) $\frac{1}{\sqrt{2}}(\vec{i} + \vec{j})$
   (c) $\frac{1}{\sqrt{5}}(2\vec{i} + \vec{j})$
   (d) $-\frac{1}{\sqrt{5}}(\vec{i} - \vec{j})$

5. The directional derivative of $f(x, y) = x^2y^3 + 2x^4y$ at the point $(1, -2)$ in the direction $-3\vec{i} - 4\vec{j}$ is
   (a) $14\vec{i} + 12\vec{j}$
   (b) $-96\vec{i} - 56\vec{j}$

(c) $-152$

(d) $-30.4$

(e) $-32\vec{i} + 14\vec{j}$

6. Let $f(x, y) = x + y^2$. Find the gradient vector $\nabla f(1, 1)$ at the point $(1, 1)$.

   (a) $-14\vec{i} - 12\vec{j}$

   (b) $-\vec{i} - 12\vec{j}$

   (c) $14\vec{i} + 12\vec{j}$

   (d) $\vec{i} + 12\vec{j}$

7. Let $f(x, y) = e^{x^2} cosy$. What is $\nabla f(x, y)$ ?

   (a) $e^{x^2}\vec{i} + cosy\vec{j}$

   (b) $e^{x^2} cosy\vec{i} - e^{x^2} siny\vec{j}$

   (c) $2xe^{x^2} cosye^{x^2} siny$

   (d) $e^{x^2} + cosy$

   (e) $2xe^{x^2} cosy\vec{i}e^{x^2} siny\vec{j}$

8. Find the curl of $F = x^3y^2\vec{i} - 3x^2y\vec{j} + xyz\vec{k}$.

   (a) $xz\vec{i} + yz\vec{j} + (-6xy + 2x^3y\vec{k}$

   (b) $3x^2y^2\vec{i} - 3x^2\vec{j} + xy\vec{k}$

   (c) $xz\vec{i} + yz\vec{j} - (6xy + 2x^3y\vec{k}$

   (d) $xz\vec{i} - yz\vec{j} - (6xy + 2x^3y\vec{k}$

9. Given that $F$ and $G$ are vector fields and $\phi$ is a scalar field, which of the following statements is false?

   (a) $\nabla \times (\nabla\phi) = 0$

   (b) $\nabla \cdot (\phi F) = \nabla\phi \cdot F + \phi(\nabla \cdot F)$

   (c) $\nabla \cdot (\nabla \times F) = 0$

   (d) $\nabla \cdot (F \times G) = (\nabla \times F) \cdot G + F \cdot (\nabla \times G)$

10. Given that $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$, which of the following statements is false

    (a) $\nabla(\sin(r)) = \cos(r)\frac{\vec{r}}{r}$

    (b) $\nabla(\ln(r)) = \frac{\vec{r}}{r}$

    (c) $\nabla\left(\frac{1}{r}\right) = -\frac{\vec{r}}{r^3}$

    (d) $\nabla(r^n) = nr^{n-1}\vec{r}$

11. Use Green's Theorem to find the work done by the force field $F(x, y) = y^3\vec{i} + (x^3 + 3xy^2)\vec{j}$ in moving a particle once in a counter clockwise direction around the circle of radius "a" centred at the origin.

(a) $\frac{3a^4\pi}{2}$

(b) $\frac{3a^4\pi}{4}$

(c) $\frac{a^2\pi}{2}$

(d) $a^\pi$

12. Use Green's Theorem to evaluate $\int\limits_C y^2dx+xydy$ for $C$: boundary of the region lying between the graphs of $y = 0, y = \sqrt{x}$ and $x = 9$.

(a) $-\frac{81}{2}$

(b) $\frac{81}{4}$

(c) $\frac{243}{4}$

(d) $-\frac{81}{4}$

13. Use the Divergence Theorem to calculate the flux $\int\limits_C y^2dx + xydy$ of the vector field $F = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$ out of the sphere $S$ with equation $x^2 + y^2 + z^2 = a^2$

(a) $\frac{12\pi a^5}{5}$

(b) $4\pi a^3$

(c) $2\pi^2 a^3$

(d) $\frac{6\pi a^5}{5}$

14. Let $\phi : \mathbb{R}^3 \to \mathbb{R}$ be a scalar field. Then what is the gradient of $\phi$ ?

(a) $\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$

(b) $\nabla \cdot \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$

(c) $\nabla \times \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$

(d) $|\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)|$

15. Let $\varphi$ be a scalar field on $\mathbb{R}^3$ and let $\mathbf{F}$ be a vector field on $\mathbb{R}^3$. Then what is the divergence of $\varphi\mathbf{F}$?

(a) $\nabla\varphi \cdot \mathbf{F} + \varphi\,\mathrm{div}\mathbf{F}$

(b) $\nabla\varphi \cdot \mathbf{F} - \varphi\,\mathrm{div}\mathbf{F}$

(c) $\nabla\varphi \times \mathbf{F} + \varphi\,\mathrm{curl}\mathbf{F}$

(d) $\nabla\varphi \times \mathbf{F} - \varphi\,\mathrm{curl}\mathbf{F}$

16. What is a conservative force?

(a) A force such that the work done in taking a particle between two points is independent of the path taken

(b) A force that propagates faster than the speed of light

(c) A force with zero divergence

(d) A force with zero gradient

17. Let $S$ be an open subset of $\mathbb{R}^3$ . When is a vector field $\mathbf{F} : S \to \mathbb{R}^3$ said to be conservative?

    (a) When there exists a scalar field $\varphi : S \to \mathbb{R}$ such that $\mathbf{F} = \nabla\varphi$

    (b) When there exists a vector field $\varphi : S \to \mathbb{R}^3$ such that $\mathbf{F} = \nabla \cdot \varphi$

    (c) When there exists a vector field $\varphi : S \to \mathbb{R}^3$ such that $\mathbf{F} = \nabla \times \varphi$

    (d) When there exists a scalar field $\varphi : S \to \mathbb{R}$ such that $|\mathbf{F}| = |\nabla\varphi|$

18. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field. Then what is the curl of $\mathbf{F}$ ?

    (a) $\left(\frac{\partial \mathbf{F}_2}{\partial z} - \frac{\partial \mathbf{F}_3}{\partial y}, \frac{\partial \mathbf{F}_3}{\partial x} - \frac{\partial \mathbf{F}_1}{\partial z}, \frac{\partial \mathbf{F}_1}{\partial y} - \frac{\partial \mathbf{F}_2}{\partial x}\right)$

    (b) $\left(\frac{\partial \mathbf{F}_3}{\partial y} - \frac{\partial \mathbf{F}_2}{\partial z}, \frac{\partial \mathbf{F}_1}{\partial z} - \frac{\partial \mathbf{F}_3}{\partial x}, \frac{\partial \mathbf{F}_2}{\partial x} - \frac{\partial \mathbf{F}_1}{\partial y}\right)$

    (c) $\left(\frac{\partial \mathbf{F}}{\partial x}, \frac{\partial \mathbf{F}}{\partial y}, \frac{\partial \mathbf{F}}{\partial z}\right)$

    (d) $\frac{\partial \mathbf{F}_1}{\partial x} + \frac{\partial \mathbf{F}_2}{\partial y} + \frac{\partial \mathbf{F}_3}{\partial z}$

19. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field. Then what is the divergence of $\mathbf{F}$ ?

    (a) $\left(\frac{\partial \mathbf{F}_2}{\partial z} - \frac{\partial \mathbf{F}_3}{\partial y}, \frac{\partial \mathbf{F}_3}{\partial x} - \frac{\partial \mathbf{F}_1}{\partial z}, \frac{\partial \mathbf{F}_1}{\partial y} - \frac{\partial \mathbf{F}_2}{\partial x}\right)$

    (b) $\left(\frac{\partial \mathbf{F}_3}{\partial y} - \frac{\partial \mathbf{F}_2}{\partial z}, \frac{\partial \mathbf{F}_1}{\partial z} - \frac{\partial \mathbf{F}_3}{\partial x}, \frac{\partial \mathbf{F}_2}{\partial x} - \frac{\partial \mathbf{F}_1}{\partial y}\right)$

    (c) $\left(\frac{\partial \mathbf{F}}{\partial x}, \frac{\partial \mathbf{F}}{\partial y}, \frac{\partial \mathbf{F}}{\partial z}\right)$

    (d) $\frac{\partial \mathbf{F}_1}{\partial x} + \frac{\partial \mathbf{F}_2}{\partial y} + \frac{\partial \mathbf{F}_3}{\partial z}$

20. The del operator is called as

    a) Gradient

    b) Curl

    c) Divergence

    d) Vector differential operator

21. The Laplacian operator is actually

    a) Grad(Div V)

    b) Div(Grad V)

    c) Curl(Div V)

    d) Div(Curl V)

22. The divergence of curl of a vector is zero. State True or False.

    a) Always true

    b) Always False

    c) some time it is True

    d) some time it is False

23. The curl of gradient of a vector is non-zero.

    a) Always true

b) Always False

c) some time it is True

d) some time it is False

24. Identify the correct vector identity.

a) $i.i = j.j = k.k = 0$

b) $i \times j = j \times k = k \times i = 1$

c) $Div(u \times v) = v.Curl(u)u.Curl(v)$

d) $i.j = j.k = k.i = 1$

25. A vector is said to be solenoidal when its

a) Divergence is zero

b) Divergence is unity

c) Curl is zero

d) Curl is unity

26. A field has zero divergence and it has curls. The field is said to be

a) Divergent, rotational

b) Solenoidal, rotational

c) Solenoidal, irrotational

d) Divergent, irrotational

27. When a vector is irrotational, which condition holds good?

a) Stoke's theorem gives non-zero value

b) Stoke's theorem gives zero value

c) Divergence theorem is invalid

d) Divergence theorem is valid

28. Gradient of a function is a scalar constant.

a) Always true

b) Always False

c) some time it is True

d) some time it is False

29. The mathematical perception of the gradient is said to be

a) Tangent

b) Chord

c) Slope

d) Arc

30. Divergence of gradient of a vector function is equivalent to

a) Laplacian operation

b) Curl operation

c) Double gradient operation

d) Null vector

31. The gradient of $xi + yj + zk$ is

a) 0

b) 1

c) 2

d) 3

32. Find the gradient of $t = x^2y + e^z$ at the point $p(1, 5, -2)$

a) i + 10j + 0.135k

b) 10i + j + 0.135k

c) i + 0.135j + 10k

d) 10i + 0.135j + k

33. Curl of gradient of a vector is

a) Unity

b) Zero

c) Null vector

d) Depends on the constants of the vector

34. Find the gradient of the function given by, $x^2 + y^2 + z^2$ at (1,1,1)

a) i + j + k

b) 2i + 2j + 2k

c) 2xi + 2yj + 2zk

d) 4xi + 2yj + 4zk

35. When gradient of a function is zero, the function lies parallel to the x-axis.

a) Always true

b) Always False

c) some time it is True

d) some time it is False

36. Find the gradient of the function sin x + cos y.

a) cos x i - sin y j

b) cos x i + sin y j

c) sin x i - cos y j

d) sin x i + cos y j

37. The divergence of a vector is a scalar.

a) Always true

b) Always False

c) some time it is True

d) some time it is False

38. Compute the divergence of the vector $xi + yj + zk$.

a) 0

b) 1

c) 2

d) 3

39. Find the divergence of the vector yi + zj + xk.

a) -1

b) 0

c) 1

d) 3

40. Given $D = e^{-x} sinyi - e^{-x} cosyj$ Find divergence of D.

a) 3

b) 2

c) 1

d) 0

41. Find the divergence of the vector $F = xe^{-x}i + yj - xzk$

a) $(1 - x)(1 + e^{-x})$

b) $(x - 1)(1 + e^{-x})$

c) (1 - x)(1 - e)

d) (x -1)(1 - e)

42. Determine the divergence of $F = 30i + 2xyj + 5xz^2k$ at (1,1,-0.2) and state the nature of the field.

a) 1, solenoidal

b) 0, solenoidal

c) 1, divergent

d) 0, divergent

43. Find whether the vector is solenoidal, E = yz i + xz j + xy k

a) Yes, solenoidal

b) No, non-solenoidal

c) Solenoidal with negative divergence

d) Variable divergence

44. Find the divergence of the field, $P = x^2yzi + xzk$

a) xyz + 2x

b) 2xyz + x

c) xyz + 2z

d) 2xyz + z

45. Identify the nature of the field, if the divergence is zero and curl is also zero.

a) Solenoidal, irrotational

b) Divergent, rotational

c) Solenoidal, irrotational

d) Divergent, rotational

46. The curl of curl of a vector is given by,

a) $Div(GradV) - (Del)^2V$

b) $Grad(DivV) - (Del)^2V$

c) $(Del)^2V - Div(GradV)$

d) $(Del)^2V - Grad(DivV)$

47. Which of the following theorem use the curl operation?

a) Green's theorem

b) Gauss Divergence theorem

c) Stoke's theorem

d) Maxwell equation

48. The curl of a curl of a vector gives a

a) Scalar

b) Vector

c) Zero value

d) Non zero value

49. Find the curl of the vector and state its nature at (1,1,-0.2)

F = 30 i + 2xy j + 5xz2 k

a) $\sqrt{4.01}$

b) $\sqrt{4.02}$

c) $\sqrt{4.03}$

d) $\sqrt{4.04}$

50. Find the curl of $A = (ycosax)i + (y + e^x)k$

a) $2i - e^x j - cosaxk$

b) $i - e^x j - cosaxk$

c) $2i - e^x j + cosaxk$

d) $i - e^x j + cosaxk$

51. Find the curl of the vector A = yz i + 4xy j + y k

a) xi + j + (4y - z)k

b) xi + yj + (z - 4y)k

c) i + j + (4y - z)k

d) i + yj + (4y -z)k

52. Find the value of divergence theorem for $A = xy^2 i + y^3 j + y^2 zk$ for a cuboid given by $0 < x < 1, 0 < y < 1$ and $0 < z < 1$.

a) 1

b) 4/3

c) 5/3

d) 2

53. Find the value of divergence theorem for the field $D = 2xyi + x^2 j$ for the rectangular parallelepiped given by x = 0 and 1, y = 0 and 2, z = 0 and 3.

a) 10

b) 12

c) 14

d) 16

54. The divergence theorem converts

a) Line to surface integral

b) Surface to volume integral

c) Volume to line integral

d) Surface to line integral

55. Using volume integral, which quantity can be calculated?

a) area of cube

b) area of cuboid

c) volume of cube

d) distance of vector

56. Find the Laplace equation value of the following potential field $V = x^2 - y^2 + z^2$

a) 0

b) 2

c) 4

d) 6

57. Find the Laplace equation value of the following potential field $V = \rho cos\varphi + z$

a) 0

b) 1

c) 2

d) 3

58. Find the value of Stoke's theorem for y i + z j + x k.

a) i + j

b) j + k

c) i + j + k

d) -i - j - k

59. The Stoke's theorem uses which of the following operation?

a) Divergence

b) Gradient

c) Curl

d) Laplacian

60. Which of the following theorem convert line integral to surface integral?

a) Gauss divergence and Stoke's theorem

b) Stoke's theorem only

c) Green's theorem only

d) Stoke's and Green's theorem

61. Find the value of Stoke's theorem for A = x i + y j + z k. The state of the function will be

a) Solenoidal

b) Divergent

c) Rotational

d) Curl free

62. The Stoke's theorem can be used to find which of the following?

a) Area enclosed by a function in the given region

b) Volume enclosed by a function in the given region

c) Linear distance

d) Curl of the function

63. Green's Theorem Mathematically, the functions in GreenâĂŹs theorem will be

a) Continuous derivatives

b) Discrete derivatives

c) Continuous partial derivatives

d) Discrete partial derivatives

64. Find the value of Green's theorem for F = x2 and G = y2 is

a) 0

b) 1

c) 2

d) 3

65. The path traversal in calculating the Green's theorem is

a) Clockwise

b) Anticlockwise

c) Inwards

d) Outwards

66. Calculate the Green's value for the functions $F = y^2$ and $G = x^2$ for the region x = 1 and y = 2 from origin.

a) 0

b) 2

c) -2

d) 1

67. The Green's theorem can be related to which of the following theorems mathematically?

a) Gauss divergence theorem

b) Stoke's theorem

c) Euler's theorem

d) Leibnitz's theorem

68. Gauss Divergence Theorem Gauss theorem uses which of the following operations?

a) Gradient

b) Curl

c) Divergence

d) Laplacian

69. Evaluate the surface integral $\int\int(3xi+2yj)\cdot dS$, where $S$ is the sphere given by $x^2+y^2+z^2=9$.

a) $120\pi$

b) $180\pi$

c) $240\pi$

d) $300\pi$

70. The Gauss divergence theorem converts

a) line to surface integral

b) line to volume integral

c) surface to line integral

d) surface to volume integral

71. The divergence theorem value for the function $x^2 + y^2 + z^2$ at a distance of one unit from the origin is

a) 0

b) 1

c) 2

d) 3

72. If a function is described by $F = (3x + z, y^2 - sinx^2z, xz + ye^x5)$, then the divergence theorem value in the region $0 < x < 1, 0 < y < 3$ and $0 < z < 2$ will be

a) 13

b) 26

c) 39

d) 51

73. Find the divergence theorem value for the function given by $(e^z, sinx, y^2)$

a) 1

b) 0

c) -1

d) 2

74. Divergence theorem computes to zero for a solenoidal function.

a) Always True

b) Always False

c) some time it is True

d) some time it is False

## Unit - III Analytic Functions

### Part A

1. The definite integral $\int_a^b f(x)dx$
   (a) $u_x = v_y$ and $u_y = -v_x$
   (b) $u_x = v_y$ and $u_y = v_x$
   (c) $u_x = v_x$ and $u_y = v_y$
   (d) $u_x = -v_y$ and $u_y = v_x$

2. If $f(z) = u + iv$ in polar form is analytic then $\frac{\partial u}{\partial r}$ is
   (a) $\frac{\partial v}{\partial \theta}$
   (b) $r\frac{\partial v}{\partial \theta}$
   (c) $\frac{1}{r}\frac{\partial v}{\partial \theta}$
   (d) $-\frac{\partial v}{\partial \theta}$

3. If $f(z) = u + iv$ in polar form is analytic then $\frac{\partial u}{\partial \theta}$ is
   (a) $\frac{\partial v}{\partial r}$
   (b) $-r\frac{\partial v}{\partial r}$
   (c) $-\frac{1}{r}\frac{\partial v}{\partial r}$
   (d) $-\frac{\partial v}{\partial r}$

4. A function $u$ is said to be harmonic if and only if
   (a) $u_{xx} + u_{yy} = 0$
   (b) $u_{xy} + u_{yx} = 0$
   (c) $u_x + u_y = 0$
   (d) $u_x{}^2 + u_y{}^2 = 0$

5. A function $f(z)$ is analytic function if
   (a) Real part of $f(z)$ is analytic
   (b) Imaginary part of $f(z)$ is analytic
   (c) Both real and imaginary part of $f(z)$ is analytic
   (d) None of the above

6. An analytic function with constant imaginary part is
   (a) constant
   (b) non constant

(c) real

(d) imaginary

7. The analytic region of $f(z) = (x-y)^2 + 2i(x+y)$

(a) $x + y = 1$

(b) $x - y = 1$

(c) $x - y = 0$

(d) $x + y = 0$

8. If $u$ and $v$ are harmonic functions then $f(z) = u + iv$ is

(a) Analytic function

(b) Need not be analytic function

(c) Analytic function only at $z = 0$

(d) None of the above

9. If $f(z) = x + ay + i(bx + cy)$ is analytic then $a, b, c$ equals to

(a) $c = 1$ and $a = -b$

(b) $a = 1$ and $c = -b$

(c) $b = 1$ and $a = -c$

(d) $a = b = c = 1$

10. A point at which a function ceases to be analytic is called a

(a) Singular point

(b) Non-Singular point

(c) Regular point

(d) Non-Regular point

11. The function $f(z) = |z|$ is a non-constant

(a) Analytic function

(b) Nowhere analytic function

(c) Non-Analytic function

(d) Entire function

12. A function $v$ is called a conjugate harmonic function for a harmonic function $u$ in $\omega$ whenever

(a) $f = u + iv$ is analytic

(b) $u$ is analytic

(c) $v$ is analyitc

(d) $f = u - iv$ is analytic

13. The function $f(x + iy) = x^3 + ax^2y + bxy^2 + cy^3$ is analytic only if

(a) $a = 3i, b = -3$ and $c = -i$

(b) $a = 3i, b = 3$ and $c = -i$

(c) $a = 3i, b = -3$ and $c = i$

(d) $a = -3i, b = -3$ and $c = -i$

14. There exist no analytic function $f$ such that

(a) $Ref(z) = y - 2x$

(b) $Ref(z) = y^2 - 2x$

(c) $Ref(z) = y^2 - x^2$

(d) $Ref(z) = y - x$

15. If $e^{ax} \cos y$ is harmonic, then $a$ is

(a) $i$

(b) 0

(c) $-1$

(d) 2

16. The harmonic conjugate of $2x - x^3 + 3xy^2$ is

(a) $x - 3x^2y + y^3$

(b) $2y - 3x^2y + y^3$

(c) $y + 3x^2y + y^3$

(d) $2y + 3x^2y - y^3$

17. The harmonic conjugate of $u(x, y) = 2x(1 - y)$ is

(a) $x^2 + y^2 - 2x + C$

(b) $x^2 + y^2 + 2y + C$

(c) $x^2 - y^2 - 2y + C$

(d) $x^2 - y^2 + 2y + C$

18. Harmonic conjugate of $u(x, y) = e^y \cos x$ is

(a) $e^x \cos y + C$

(b) $e^x \sin y + C$

(c) $e^y \sin x + C$

(d) $-e^y \sin x + C$

19. If the real part of an analytic function $f(z)$ is $x^2 - y^2 - y$, then the imaginary part is

(a) $2xy$

(b) $x^2 + 2xy$

(c) $2xy - y$

(d) $2xy + x$

20. If the imaginary part of an analytic function $f(z)$ os $2xy + y$, then the real part is

(a) $x^2 + y^2 - y$

(b) $x^2 - y^2 - x$

(c) $x^2 - y^2 + x$

(d) $x^2 - y^2 + y$

21. $f(z) = \bar{z}$ is differentiable

(a) nowhere

(b) only at $z = 0$

(c) everywhere

(d) only at $z = 1$

22. $f(z) = |\bar{z}|^2$ is differentiable

(a) nowhere

b) only at $z = 0$

(c) everywhere

(d) only at $z = 1$

23. $f(z) = |\bar{z}|^2$ is

(a) differentiable and analytic everywhere

(b) not differentiable at $z = 0$ but analytic at $z = 0$

(c) differentiable at $z = 1$ and not analytic at $z = 1$ only (d) differentiable at $z = 0$ but not analytic at $z = 0$

24. If $f(z) = \begin{cases} \dfrac{xy}{(x^2 + y^2)}, & if\ z \neq 0 \\ 0, & if\ z = 0 \end{cases}$ then $f(z)$ is

(a) continuous but not differentiable at $z = 0$

(b) differentiable at $z = 0$

(c) analytic everywhere except at $z = 0$

(d) not differentiable at $z = 0$

25. $f(z) = e^z$ is analytic

(a) only at $z = 0$

(b) only at $z = 1$

(c) nowhere

(d) everywhere

26. $e^x(\cos y - i \sin y)$ is

(a) analytic

(b) not analytic

(c) analytic when $z = 0$

(d) analytic when $z = i$

27. If $f(z)$ is analytic, then $f(\bar{z})$ is

(a) analytic

(b) not analytic

(c) analytic when $z = 0$

(d) analytic when $z = 1$

28. The points at which $f(z) = \dfrac{(z^2 - z)}{(z^2 - 3z + 2)}$ is not analytic are

(a) 0 and 1

(b) 1 and -1

(c) $i$ and 2

(d) 1 and 2

29. The points at which $f(z) = \dfrac{1}{z^2 + 1}$ is not analytic are

(a) 1 and -1

(b) $i$ and $-i$

(c) 1 and $i$

(d) -1 and $-i$

30. The harmonic conjugate of $u = log\sqrt{x^2 + y^2}$ is

(a) $\dfrac{x}{x^2 + y^2}$

(b) $\dfrac{y}{x^2 + y^2}$

(c) $\arctan(\dfrac{x}{y})$

(d) $\arctan(\dfrac{y}{x})$

31. $f(z) = z(2 - z)$, then $f(1 + i) =$

(a) 0

(b) $i$

(c) $-i$

(d) 2

32. $f(z) = |\bar{z}|$ then $f(3 - 4i) =$

(a) 0

(b) 5

(c) -5

(d) 12

33. Critical points of the bilinear transformation $w = \dfrac{a + bz}{c + dz}$ are

(a) $a, c$

(b) $\dfrac{c}{d}, \infty$

(c) $-\dfrac{c}{d}, \infty$

(d) None of these

34. The points coincide with their transformations are known as

(a) fixed points

(b) critical points

(c) singular points

(d) None of these

35. $w = \dfrac{a + bz}{c + dz}$ is a bilinear transformation when

(a) $ad - bc = 0$

(b) $ad - bc \neq 0$

(c) $ab - cd \neq 0$

(d) None of these

36. $w = \dfrac{1}{z}$ is known as

(a) inversion

(b) translation

(c) rotation

(d) None of these

37. $w = z + a$ is known as

(a) inversion

(b) translation

(c) rotation

(d) None of these

38. A translation of the type $w = \alpha z + \beta$ where $\alpha$ and $\beta$ are complex constants, is known as a

(a) translation

(b) magnification

(c) linear transformation

(d) bilinear transformation

39. A mapping that preserves angles between oriented curves both in magnitude and in sense is called a/an ..... mapping.

(a) informal

(b) isogonal

(c) conformal

(d) formal

40. The mapping defined by an analytic function $f(z)$ is conformal at all points $z$ except at points where

(a) $f'(z) = 0$

(b) $f'(z) \neq 0$

(c) $f'(z) > 0$

(d) $f'(z) < 0$

41. The fixed points of the transformation $w = z^2$ are

(a) $0, 1$

(b) $0, -1$

(c) $-1, 1$

(d) $-i, i$

42. The invariant points of the mapping $w = \dfrac{z}{2 - z}$ are

(a) $1, -1$

(b) $0, -1$

(c) $0, 1$

(d) $-1, -1$

43. The fixed points of $w = \dfrac{z - 1}{z + 1}$ are

(a) $\pm 1$

(b) $\pm i$

(c) $0, -1$

(d) $0, 1$

44. The mapping $w = z + \dfrac{1}{z}$ transforms circles of constant radius into

(a) confocal ellipses

(b) hyperbolas

(c) circles

(d) parabolas

45. Under the transformations $w = \dfrac{1}{z}$, the image of the line $y = \dfrac{1}{4}$ in $z$-plane is

(a) circle $u^2 + v^2 + 4v = 0$

(b) circle $u^2 + v^2 = 4$

(c) circle $u^2 + v^2 = 2$

(d) none of these

46. The bilinear transformation that maps the points $0, i, \infty$ respectively into $0, 1, \infty$ is $w =$

(a) $\frac{1}{z}$

(b) $-z$

(c) $-iz$

(d) $iz$

47. The bilinear transformation which maps the points $1, 0, 1$ of $z$- plane into $i, 0, -1$ of $w$-plane respectively is

(a) $w = iz$

(b) $w = z$

(c) $w = i(z + 1)$

(d) none of these

48. The bilinear transformation that maps the points $z_1, z_2, z_3$ of the $z$-plane onto the points $w_1, w_2, w_3$ of the $w$-plane is

(a) $\frac{(w - w_1)(w_3 - w_2)}{(w - w_1)(w_2 - w_3)} = \frac{(z - z_1)(z_3 - z_2)}{(z - z_1)(z_2 - z_3)}$

(b) $\frac{(w - w_1)(w_2 - w_3)}{(w - w_2)(w_1 - w_3)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_2)(z_1 - z_3)}$

(c) $\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}$

(d) $\frac{(w - w_3)(w_2 - w_1)}{(w - w_1)(w_2 - w_3)} = \frac{(z - z_3)(z_2 - z_1)}{(z - z_1)(z_2 - z_3)}$

49. Bilinear transformation preserves cross ratio of

(a) one point

(b) two points

(c) three points

(d) four points

50. If a bilinear transformation has three or more fixed points, then it must be

(a) one-one mapping

(b) onto mapping

(c) identity mapping

(d) one-one and onto mapping

## Unit - IV Complex integration

### Part A

1. A continuous curve which does not have a point of self intersection is called

   (a) Simple curve

   (b) Multiple curve

   (c) Integral curve

   (d) None Ans (a)

2. Simple curves are also called

   (a) Multiple curve

   (b) Jordan curve

   (c) Integral curve

   (d) None Ans (b)

3. An integral along a simple closed curve is called

   (a) Contour integral

   (b) Jordan integral

   (c) Multiple integral

   (d) None Ans (a)

4. A region which is not simply connected is called . . . region

   (a) Multiple curve

   (b) Jordan connected

   (c) Connected curve

   (d) Multi-connected Ans (d)

5. If $f(z)$ is analytic and $f'(z)$ is continuous at all points inside and on a simple closed curve C, then

   (a) $\int_c f(z)dz = 0$

   (b) $\int_c f(z)dz \neq 0$

   (c) $\int_c f(z)dz = 1$

   (d) $\int_c f(z)dz \neq 1$ Ans (a)

6. If $f(z)$ is analytic and $f'(z)$ is continuous at all points in the region bounded by the simple closed curve $C_1$ and $c_2$, then

(a) $\int_{c_1} f(z)dz = \int_{c_2} f(z)dz$

(b) $\int_{c_1} f(z)dz \neq \int_{c_2} f(z)dz$

(c) $\int_{c_1} f'(z)dz = \int_{c_2} f'(z)dz$

(d) $\int_{c_1} f'(z)dz \neq \int_{c_2} f'(z)dz$ Ans (a)

7. A point $z_0$ at which a function is not analytic is known as ... of $f(z)$.

(a) Residue

(b) Singularity

(c) Integral

(d) None Ans (b)

8. The value of $\int_c \frac{e^z}{z-3}dz$ where C is $|z| = 2$ is

(a) $2\pi i$

(b)$4\pi i$

(c) 0

(d) 2 Ans (c)

9. The value of $\int_c \frac{dz}{z+1}$ where C is the circle $|z| = 2$

(a) $4\pi i$

(b) $\pi i$

(c) 0

(d) $2\pi i$ Ans (d)

10. The value of $\int_c \frac{1}{z^2-1}dz$ where C is the circle $|z-1| = 1$

(a) $\pi i$

(b)$2\pi i$

(c) 0

(d) $4\pi i$ Ans (a)

11. The value of $\int_c e^z dz$ where C is the circle $|z| = 1$

(a) $\pi i$

(b)$2\pi i$

(c) 0

(d) $4\pi i$ Ans (c)

12. The Taylor series expansion of $f(z) = \log z$ about the point $z = 1$ upto second degree terms is

(a) 1

(b)$\frac{1}{2}$

(c) $2\pi$

(d) 0 Ans (b)

13. The Taylor series expansion of $f(z) = \cos z$ about the point $z = \frac{\pi}{2}$ upto second degree terms is

(a) 1

(b)$\frac{1}{2}$

(c) $-1$

(d) 0 Ans (c)

14. The region of validity of $\frac{1}{1+z}$ for its Taylor series expansion about $z = 0$ is

(a) $|z| < 1$

(b)$|z| > 1$

(c) $|z| = 1$

(d) None of these Ans (a)

15. If the principal part of the Laurent's series contains an infinite number of terms then it is known as

(a) Pole

(b)Essential singularity

(c) Removable singularity

(d) Isolated singularity Ans (b)

16. The singularity of $f(z) = \frac{z+3}{(z-1)(z-2)}$ are

(a) $z = 1, 3$

(b) $z = 1, 0$

(c) $z = 1, 2$

(d) $z = 2, 3$ Ans (c)

17. A zero of an analytic function $f(z)$ is a value of z for which

(a) $f(z) = 0$

(b) $f(z) = 1$

(c) $f(z) \neq 0$

(d) $f(z) \neq 1$ Ans (a)

18. The poles of $f(z) = \frac{z-2}{z^2} \sin\left(\frac{1}{z-1}\right)$ is

(a) 0

(b) 1

(c) None

(d) 2 Ans (d)

19. The poles of $f(z) = \frac{1+z^2}{1-z^2}$ is

(a) 1

(b) $-1$

(c) $\pm 1$

(d) 0 Ans (c)

20. The poles of $f(z) = \frac{1}{(z-2)^3(z-3)^2}$ is $z = 2$ and $z = 3$ of order $\cdots$ and $\cdots$ respectively

(a) $z = 2, 3$

(b) $z = 3, 2$

(c) $z = 2, 2$

(d) $z = 3, 3$ Ans (b)

21. $z = 1$ is a $\cdots$ of $f(z) = \frac{1}{z(z-1)^2}$

(a) Zero

(b) Double pole

(c) Simple pole

(d) None of these Ans (b)

22. The pole for the function $f(z) = \dfrac{\tan \frac{z}{2}}{z - (1+i)^2}$ is $(1+i)$ of order

(a) 1

(b) 2

(c) 0

(d) None of these Ans (c)

23. The residue of $f(z) = \cot z$ at each poles is

(a) 1

(b) 2

(c) 0

(d) None of these Ans (a)

24. The residue of $f(z) = \dfrac{1 + e^z}{\sin z + z\cos z}$ at the pole $z = 0$ is

(a) 0

(b) $-1$

(c) $\pm 1$

(d) 1 Ans (d)

25. The residue of $f(z) = \dfrac{1 + z}{z^2 - 2z^4}$ at a pole of order 2 is

(a) 0

(b) $-1$

(c) $\pm 1$

(d) 1 Ans (b)

26. The sum of residues of $f(z) = \dfrac{1}{z(z-1)^2}$ is

(a) 0

(b) $-1$

(c) $\pm 1$

(d) 1 Ans (a)

27. The simple poles of $f(z) = \dfrac{z^2 - 4}{z^2 + 5z + 4}$ is

(a) $1, 4$

(b) $-1, 4$

(c) $-1, -4$

(d) $1, -4$ Ans (c)

28. If $f(z) = -\dfrac{1}{z-1} - 2\left[1 + (z-1) + (z-1)^2 + \ldots\right]$ the residue of $f(z)$ at $z = 1$ is

(a) 0

(b) $-1$

(c) 6

(d) 2 Ans (b)

29. The residue of $f(z) = \dfrac{z^2}{(z-1)(z+2)^2}$ at its simple pole is

(a) $\dfrac{1}{2}$

(b) $-1$

(c) 6

(d) $\dfrac{1}{9}$ Ans (d)

30. The sum of residues of $f(z) = \dfrac{1}{(z^2+1)^3}$ is

(a) 0

(b) $-1$

(c) 2

(d) 1 Ans (a)

# Unit - V LAPLACE TRANSFORMS

## Part A

1. A Laplace Transform exists when

   A. The function is piece-wise continuous

   B. The function is of exponential order

   C. The function is piecewise discrete

   D. The function is of differential order

   a) A and B (Ans)

   b) C and D

   c) A and D

   d) B and C

2. If $f(t) = e^{t^2}$ then the value of $L[f(t)]$ is .

   a) 0

   b) Does not exist since $f(t)$ is not continuous at$t = 0$

   c) Does not exist since $f(t)$ is not of an exponential order (Ans)

   d) 1

3. Laplace transform of $f(t) = t \sin t$ is

   a) $\frac{2s}{(s^2+1)^2}$(Ans)

   b) $\frac{2}{(s^2+1)^2}$

   c) $\frac{s}{(s^2+1)^2}$

   d) $\frac{1}{(s^2+1)^2}$

4. Laplace transform of $f(t) = \delta(t)$ is

   a) 0

   b) $e^{-as}$

   c) $\infty$

   d) 1 (Ans)

5. Laplace transform of $f(t) = \frac{\cos at}{t}$ is

a) 1

b) Does not exist since $f(t)$ is not continuous at$t = 0$

c) Does not exist since $f(t)$ is not of an exponential order

d) 0

6. Laplace transform of $f(t) = \dfrac{1}{\sqrt{t}}$ is

a) Does not exist since $f(t)$ is not continuous at$t = 0$

b) Exist. Though $f(t)$ is not continuous at$t = 0$

c) Does not exist since $f(t)$ is not of an expential order

d) above all true

7. Laplace transform of $f(t) = \dfrac{1 - \cos t}{t}$ is

a) $log\left[\dfrac{s}{\sqrt{s^2 + 1}}\right]$

b) Does not exist

c) $log\left[\dfrac{\sqrt{s^2 + 1}}{s}\right]$

d) $log\left[\dfrac{s^2 + 1}{s}\right]$

8. If $f(t) = \sin 5t \cos 2t$ then the value of $L[f(t)]$ is

a) $\dfrac{1}{2}\left[\dfrac{10s^2 + 210}{s^4 + 58s^2 + 441}\right]$

b) $\dfrac{1}{2}\left[\dfrac{7}{s^2 + 49} + \dfrac{3}{s^2 + 9}\right]$

c) Does not exist

d) (a) and (b)(Ans)

9. If $f(t) = \sin^2 t$ then the value of $L[f(t)]$ is

a) $\dfrac{8}{2s^3 + 8s}$

b) Does not exist

c) $\dfrac{1}{2s} - \dfrac{s}{2(s^2 + 4)}$

d) (a) and (c)(Ans)

10. If $f(t) = t^2 2^t$ then the value of $L[f(t)]$ is

a) $\dfrac{2}{(s - \log 2)^3}$(Ans)

b) Does not exist

c) $\dfrac{2}{(s + \log 2)^3}$

d) $\dfrac{2}{(s+\log 3)^2}$

11. If $f(t) = \sin 2t . \sin 3t$ then the value of $L[f(t)]$ is

a) $\dfrac{12}{(s^2+1)(s^2+25)}$

b) $\dfrac{s}{(s^2+1)(s^2+25)}$

c) Does not exist

d) $\dfrac{12s}{(s^2+1)(s^2+25)}$(Ans)

12. If $f(t) = t\cos at$ then the value of $L[f(t)]$ is

a) $\dfrac{s^2-a^2}{(s^2+a^2)^2}$ (Ans)

b) Does not exist

c) $\dfrac{s^2+a^2}{(s^2-a^2)^2}$

d) $\dfrac{s^2}{(s^2-a^2)^2}$

13. If $f(t) = t\sin at$ then the value of $L[f(t)]$ is

a) $\dfrac{4}{(s^2+4)^2}$

b) $\dfrac{4s}{(s^2+4)^2}$ (Ans)

c) $\dfrac{s}{(s^2+4)^2}$

d) Does not exist

14. If $f(t) = e^{-2t}$ then the value of $L[f(t)]$ is

a) $\dfrac{\mathbf{1}}{\mathbf{s+2}}$(Ans)

b) $\dfrac{1}{s-2}$

c) $\dfrac{2}{s+2}$

d) $\dfrac{2}{s-2}$

15. If $f(t) = e^{3t}$ then the value of $L[f(t)]$ is

a) $\dfrac{1}{s+3}$

**b)** $\dfrac{\mathbf{1}}{\mathbf{s-3}}$

c) $\dfrac{3}{s+3}$

16. If $L[f(t)] = F(s)$ then $L[e^{at}f(t)]$ is equal to

a) $F(s+a)$

b) $F\left[\frac{s}{a}\right]$

**c) F(s – a)**

d) $\frac{3}{s+3}$

e) $\frac{3}{s-3}$

17. Laplace transform of $\sin(at)u(t)$ is

a) $\frac{s}{s^2+a^2}$

b) $\frac{a}{s^2+a^2}$

c) $\frac{s^2}{s^2+a^2}$

d) $\frac{a^2}{s^2+a^2}$

18. Laplace transform of $\cos(at)u(t)$ is

a) $\frac{s}{s^2+a^2}$

b) $\frac{a}{s^2+a^2}$

c) $\frac{s^2}{s^2+a^2}$

d) $\frac{a^2}{s^2+a^2}$

19. If $f(t) = t^n e^{at}$ then the value of $L[f(t)]$ is

a) $\frac{n!}{(s-a)^{n-1}}$

b) $\frac{n!}{(s+a)^{n-1}}$

**c)** $\boldsymbol{\frac{n!}{(s-a)^{n+1}}}$

d) $\frac{n!}{(s+a)^{n+1}}$

20. If $f(t) = t^2 e^{-3t}$ then the value of $L[f(t)]$ is

a) $\frac{2}{s^2+6s+9}$

b) $\frac{6}{s^2-6s+9}$

c) $\frac{6}{(s-3)^3}$

**d)** $\boldsymbol{\frac{2}{(s+3)^3}}$

21. If $f(t) = t^3 e^{2t}$ then find $L[f(t)]$ .

a) $\boldsymbol{\frac{6}{(s-2)^4}}$

b) $\frac{6}{(s+2)^4}$

c) $\frac{2}{s^2-4s+4}$

d) $\frac{2}{s^2+4s+4}$

22. If $f(t) = e^{at}\sin bt$ then find $L[f(t)]$ .

a) $\frac{b}{(s+a)^2+b^2}$

**b)** $\mathbf{\frac{b}{(s-a)^2+b^2}}$

c) $\frac{a}{(s+b)^2+a^2}$

d) $\frac{a}{(s-b)^2+a^2}$

23. Find $L[f(t)]$ if $f(t) = e^{-at}\sin bt$.

a) $\frac{b}{(s-a)^2+b^2}$

**b)** $\mathbf{\frac{b}{(s+a)^2+b^2}}$

c) $\frac{a}{(s-b)^2+a^2}$

d) $\frac{a}{(s+b)^2+a^2}$

24. The value of $L[f(t)]$ if $f(t) = e^{3t}\sin 5t$ is

a) $\frac{5}{s^2+6s+34}$

b) $\frac{3}{s^2+10s+34}$

**c)** $\mathbf{\frac{5}{s^2-6s+34}}$

d) $\frac{3}{s^2-10s+34}$

25. The value of $L[f(t)]$ if $f(t) = e^{-7t}\sin 9t$ is

a) $\frac{7}{s^2-18s+130}$

b) $\frac{7}{s^2+18s+130}$

c) $\frac{9}{s^2-14s+130}$

**d)** $\mathbf{\frac{9}{s^2+14s+130}}$

26. if $f(t) = e^{at}\cos bt$ then the value of $L[f(t)]$ is

a) $\frac{s-a}{(s+a)^2+b^2}$

**b)** $\mathbf{\frac{s-a}{(s-a)^2+b^2}}$

c) $\dfrac{s+a}{(s+b)^2+a^2}$

d) $\dfrac{s-a}{(s-b)^2+a^2}$

27. if $f(t) = e^{5t}\cos 6t$ then the value of $L[f(t)]$ is

a) $\dfrac{s-a}{(s+a)^2+b^2}$

**b)** $\mathbf{\dfrac{s-5}{(s-5)^2+36}}$

c) $\dfrac{s+a}{(s+b)^2+a^2}$

d) $\dfrac{s-a}{(s-b)^2+a^2}$

28. If $F(s) = \dfrac{1}{s(s+1)}$ then the value of $\lim_{t\to\infty} f(t)$ is

a) 0

b) 1 (Ans)

c) $\infty$

d) Does not exist

29. The value of $\int_0^\infty te^{-2t}\cos t dt$ by using Laplace transform is

a) $\dfrac{1}{25}$

b) $\dfrac{3}{5}$

c) Does not exist

d) $\dfrac{3}{25}$ (Ans)

30. The convolution of $t$ with $e^t$ is

a) $(t-1)(e^t-1)$

b) $(t+1)(e^{-t}-1)$

c) $(t+1)(e^t-1)$ (Ans)

d) $(t+1)(e^t+1)$

31. If $F(s) = \dfrac{1}{s^2+s}$ then the value of $\lim_{t\to 0} f(t)$ is

a) 0 (Ans)

b) 1

c) $\infty$

d) Does not exist

32. The value of $L^{-1}\left(\sqrt{t}\right)$ is

a) Does not exist

b) $\Gamma(2/3)/s^{2/3}$

c) $\Gamma(1/2)/s^{1/2}$

d) $\Gamma(3/2)/s^{3/2}$(Ans)

33. The value of $L^{-1}\left[\dfrac{1}{s^3+6s^2+12s+8}\right]$ is

a) $e^{2t}\dfrac{t^2}{2!}$

b) $e^{-2t}\dfrac{t}{2!}$

c) $e^{-2t}\dfrac{t^2}{2!}$ (Ans)

d) Does not exist

34. The value of $L^{-1}\left[\dfrac{s+2}{s^2+4s+8}\right]$ is

a) $e^{-2t}\cos 2t$ (Ans)

b) $e^{2t}\cos 2t$

c) $e^{-2t}\sin 2t$

d) $e^{2t}\sin 2t$