Graph and discrete structure

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Lecture 3

Reminder

- greedy seq. algorithm
- simplicial ordering of V $v_1, v_2 ... v_n. \; v_i \; is \; simplicial \; in \; G[v_1 ... v_n]$

$\Rightarrow$ optimal coloring

Theorem : $\exists$ simplicial ordering <=> $\forall V', G[V']$ contains a simplicial vertex

Corollary : simplicial ordering $V(G) \Rightarrow$ has no induced chordless $C_k, k \geq 4$

Course

Theorem : G is chordal $\Rightarrow$ G has a simplicial ordering.
Proof : G is connected.

Lemma : $G \neq$ clique, G chordal, G connected
$\Rightarrow$ $\exists$ clique W in $G \ni GW$ is not connected.

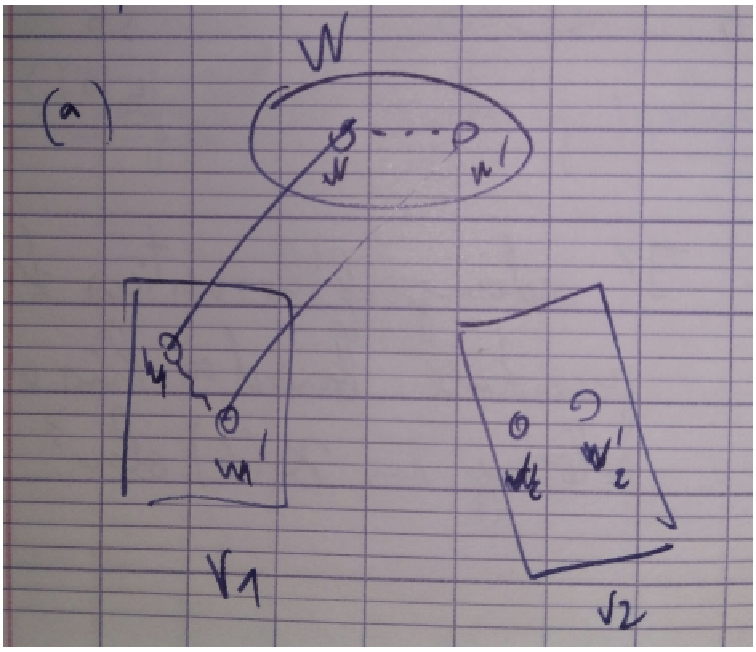
$GV \setminus \{x, y\}$
$\Rightarrow \exists W \ni G - W$ is not connected.
$\forall w \in W, G - (W \setminus \{w\})$ is connected.

$\forall w \in W$ has a neighbourg in each cc of $G \setminus W$.

Proof

Proof of the lemma

Assume W is not a clique :
$\exists w, w' \in W \ni ww' \notin E$



$\exists$ Shortest path P_1 between w and w' having all interval vertices in V_1.
$\exists$ Shortest path P_2 between w and w' having all interval vertices in V_2.

$P_1 + P_2 =$ chordless cycle χ of lenght ≥ 4.
Since V_1 and V_2 are connected by an edge.

$\Rightarrow \exists$ clique K in $G \ni GK$ not connected.

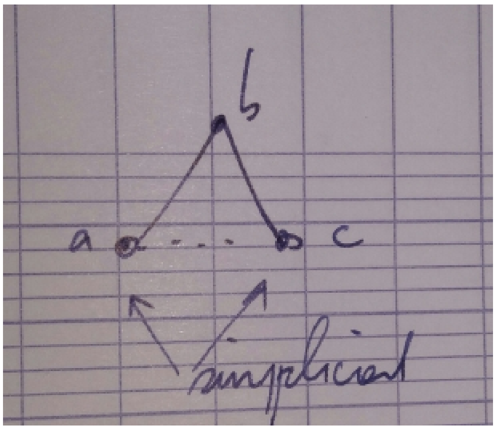
Proof of the theorem

Theorem : G chordal, connected $\Rightarrow$ simple ordering.

Any induced subgraph of a chordal graph is chrodal. So by the characterisation Th it is enough to show that a chordal connected graph contains simplicial vertex.

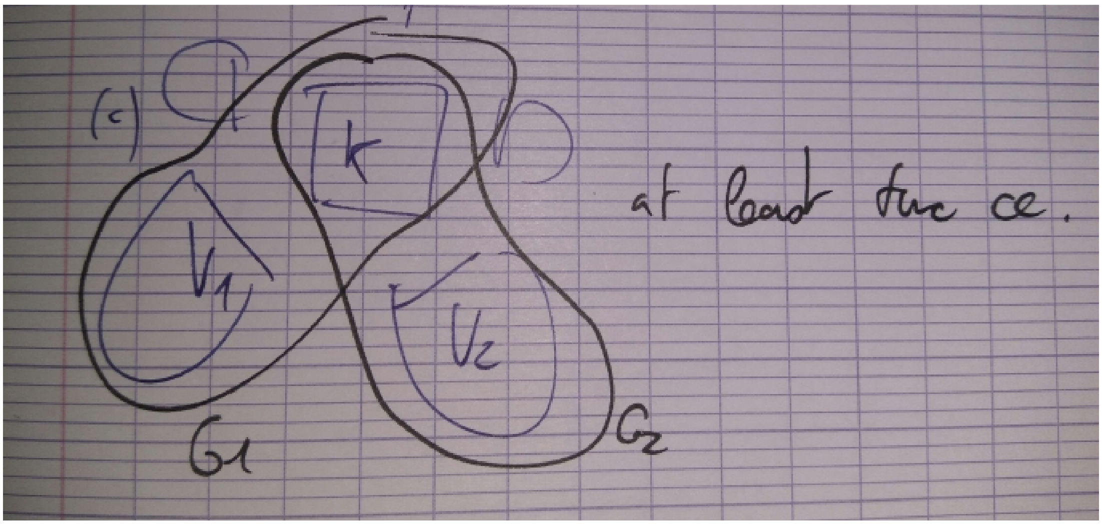
We will show that G either is a clique or it contains two non adjacent simplicial vertices.

By induction on $|V|$.



- smallest case of connected graph that is not a clique : 3 vertices, 2 edges.
- assume the propert is true for $|V| \leq n$. Let G be $\ni |V| = n + 1$.

By the lemma there exists a clique K s.t. $G - K$ is not connected.



$G_1 = G[K \cup V_1],\ G_2 := G[K \cup V_2]$

G_i chordal :

- clique $\forall v \in V[G_i]$, v is simplicial
- $\exists$ two non adjacent simple vertices.

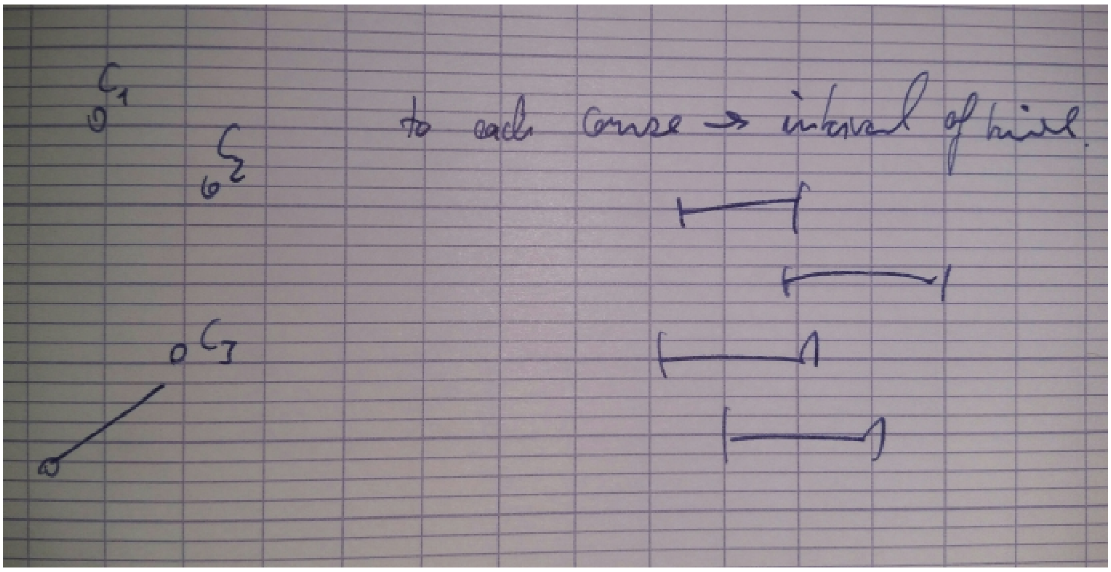
Any vertex in V_i has the same neightborhoud in G_i and in G, so our goal is to show that G_i has a simplicial vertex in V_i.

if G_i clique $\to$ all vertices in V_i are simplicial.
$G_i \neq$ clique, by the induction hypothesis.
G_i contains two adjacent simplicial vertives : Since K is a clique at least one of them is in $V_i \to V_1 \in V_1, v_2 \in V_2$: non adjacent vertices.

Corollaries

- Find an optimal colloring of a cordal graph is easy.
- Find a biggest clique is also easy.
- At most n maximum clique in a cordal graph (it can be exponnential in $|V|$ in arbitrary graphs).
  - (proof : $v_{max} = v_k, v_{max} \cup Neighborhood[v_{max}]\ is\ max.$)

Back to classroom affectation problem



Course $c_1, c_2, ..., c_n$
To each course : interval o time.
Intersection graph.

G interval Graph $\Rightarrow$ chordal.

G interval $\neq$ clique :
$\Rightarrow \exists$ 2 non adjacent simplicial vertices
$[x_1, y_1], [x_2, y_2], ..., [x_n, y_n]$

Exercice : proove it.

Definition : intersection graph

$G = (V, E)$ is an intersection graph if one can associate to every vertex v a set S_v in such a way that
$vm \in E \Leftrightarrow S_v \cap S_w \neq \emptyset$

Exercice : Every graph is an intersection graph.

Theorem : chordal graphs are intersection graphs of subtrees of a tree.

Note : Interval graph are special chrodal graphs.