Graph and discrete structure

Myriam Preissman (GSCOP) - 4 lectures.
Lecture 2

Reminder

- $\chi(G)$.
- $\omega(G)$.
- Lower bound : $\chi(G) \geq \omega(G)$.
- A proper coloring of V(G).
- Decide if $\chi(G) = 2$ is easy.
- $\chi(G) = k, k \leq 3$ is hard.

A gready exponential algorithm.

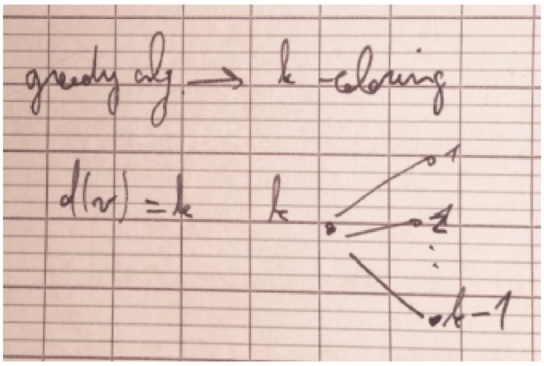
Course

From the greedy algorithm we can deduce an upperbound :

$\chi(G) \leq \Delta(G) + 1$

With : $d(v) = \#$ edges incident to v.
$\Delta(G) = \max d(v), \forall v \in V$

gready alg. $\rightarrow$ k-coloring.



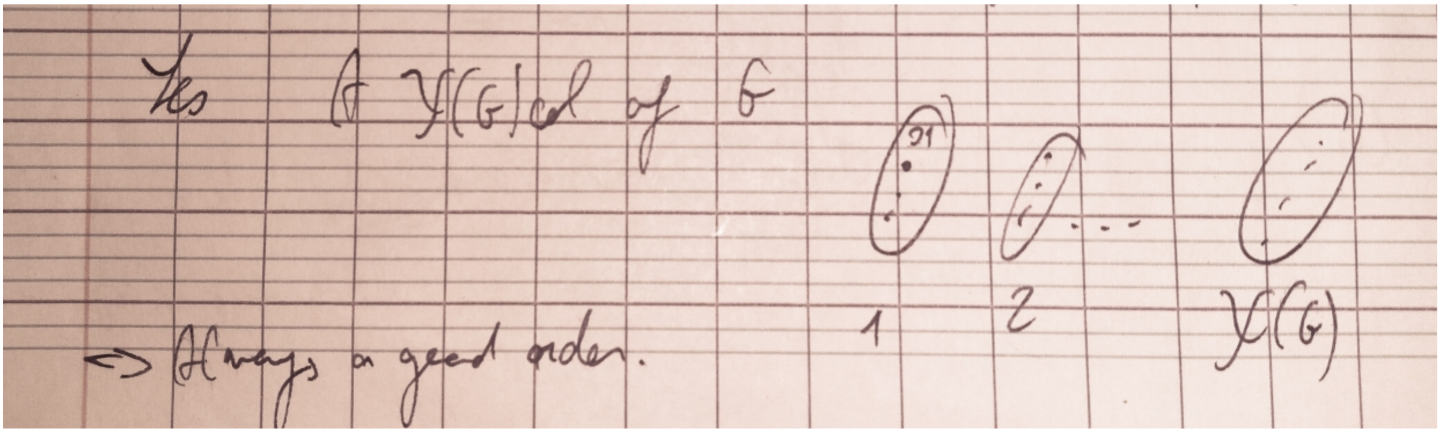
$\Delta(G) \geq d(G) \geq k - 1$
$\Rightarrow k \leq \Delta(G) + 1$

Property

Graph $\chi(G) = \omega(G) = \Delta(G) + 1$ if G is a complete graph.

Exercice

$G, \exists$ ordering of V s.t. greedy seq. alg provide a $\chi(G)$-coloring ?



⇒ always a good order.

Definition and Theorem

Definition : Simplicial

$v \in V$ is said simplicial if $N(G)$ is a clique with $N(v) = \{w \in V, vw \in E\}$.

In other words a graph G, a vertex x is simplicial if its neighbourhood $N(x)$ induces a complete ($K_{n,n}$) subgraph of G.

$v_1, v_2, \cdots v_n$ is a simplicial ordering if v_i is simplicial in $G[i]$, with $v_i = \{v_1, v_2, \cdots, v_n\}$.

Definition : Neighborhood

$N(G) = \Gamma_G(v) = \{u \in V : uv \in E\}$

The *neighbourhood* of a vertex v in a graph G is the induced subgraph of G consisting of all vertices adjacent to v.

Definition : Induced subgraph

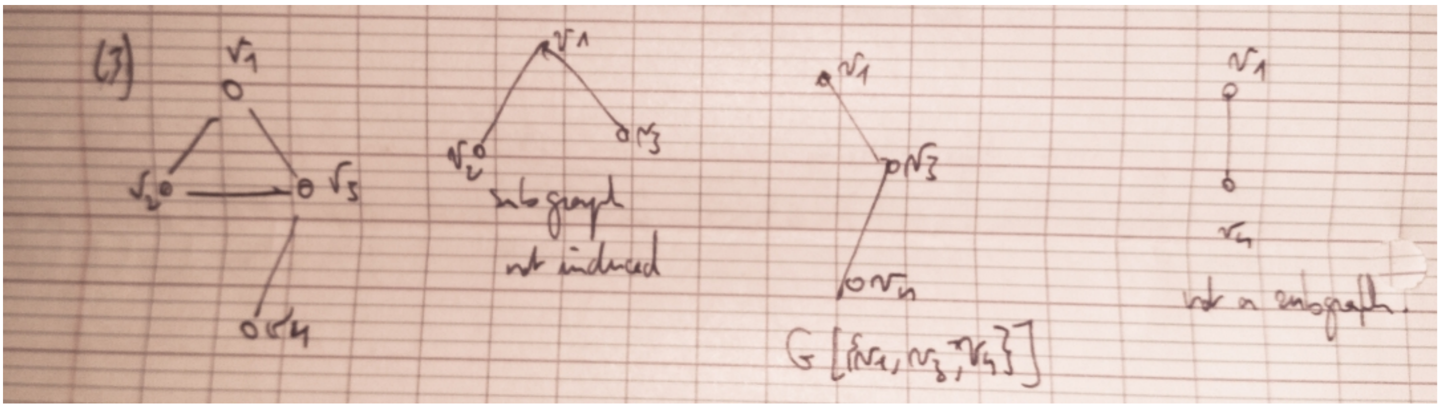
$G = (V, E)$ and $S \subset V$. Then the induced subgraph $G[S]$ is the graph whose vertex set is S and whose edge set consists of all of the edges in E that have both endpoints in S.

Definition : Subgraph

$G'(V', E')$ is a subgraph of G if $V' \subseteq V$ and $E' \subseteq E$.

$G[V]$ if $E' = \{e \in E,\ both\ extremities\ of\ e\ are\ in\ V'\}$.

G' is an inducted subgraph of G if $G[V']$.

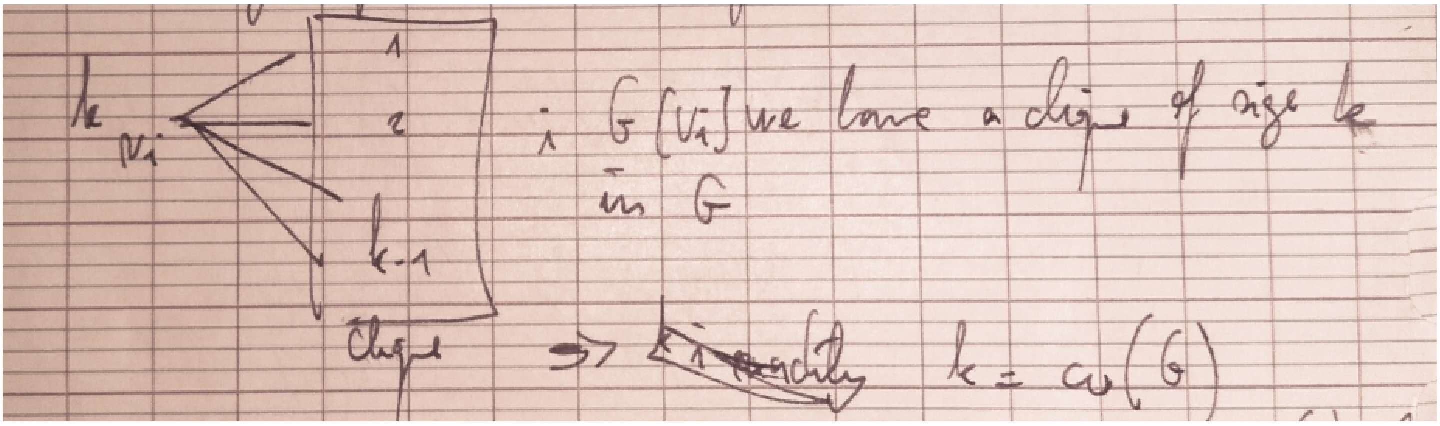


Theorem (a)

if $v_1, v_2, \cdots, v_n$ is a simplicial ordering of V then the greedy seq-alg provides a $\chi(G)$-coloring.

Proof (a)

Assume you get a k-coloring.



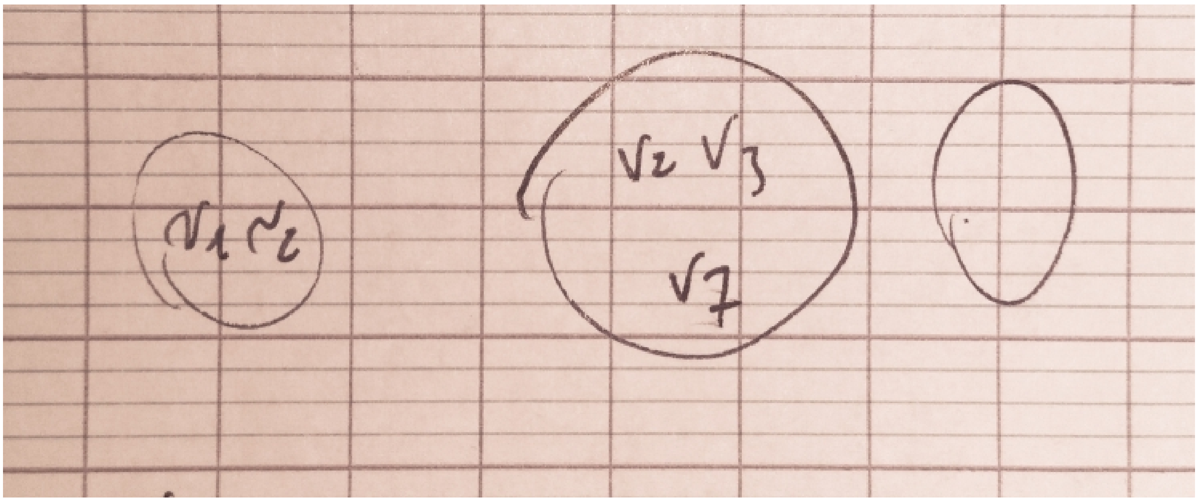
Characterization (b)

$\exists$ simplicial ordering $\Leftrightarrow \forall\, V' \subseteq V,\ G[V']$ contains at least one simplicial vertex.

Proof (b)

$\Rightarrow$

$v_1, v_2, \cdots, v_n$ simplicial ordering of V. $V' \subseteq V$.



$V' \subseteq V_j$ with $j = \max\{i \ni v_i \in V'\}$.

if v simplicial in $G[V_j], v_j\ simplicial\, G[V']$.

$\Leftarrow$

v_i is simplicial in $G[v_i], v_i = \{v_1, \cdots, v_i\}$.

v_n	G
v_{n-1}	G_{n-1}

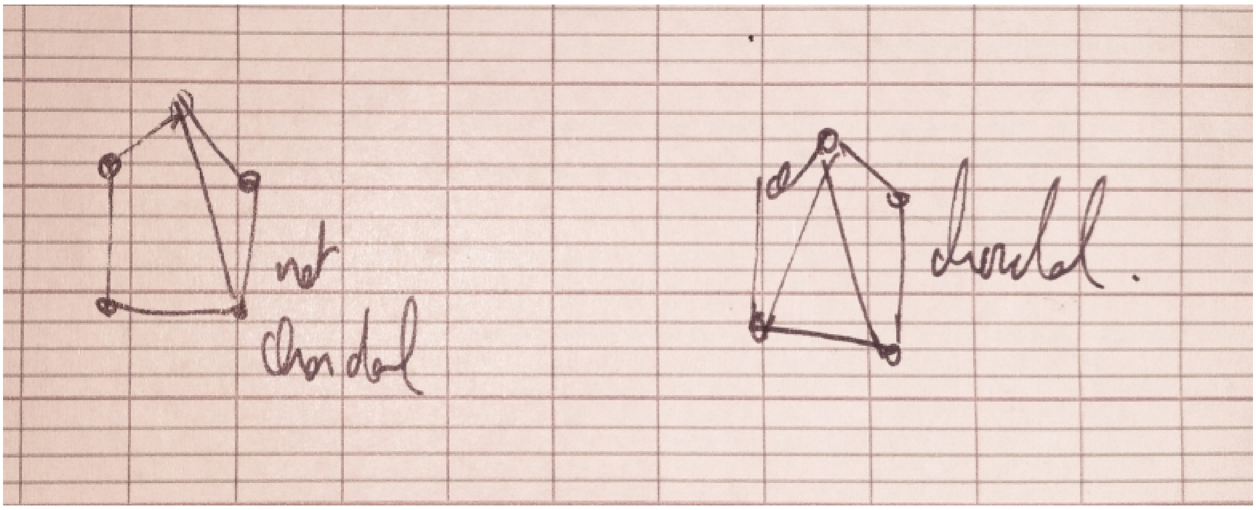
Start with V.

- Choose a simplicial vertex v_n in $G[V]$.
- Choose a simplicial vertex v_{n-1} in $G[V - v_{n-1}]$.
- Choose a simplicial vertex v_i in ...
- [...]
- Choose a simplicial vertex v_1 in ...

Characterization (c)

G has a simplicial ordering $\Rightarrow$ G has no chordless cycle.

$C_k,\ k \geq 4 =$ chordal graphs.



Definition : Chordless Cycle

A chordless cycle of a graph G is a graph cycle of length at least four in G that has no cycle chord (i.e., the graph cycle is an induced subgraph).

Definition : Chordal Graph

A chordal graph is a simple graph possessing no chordless cycles. A chordless cycle is sometimes also called a graph hole (Chvátal).

Definition : Cycle Chord

A chord of a graph cycle C is an edge not in the edge set of C whose endpoints lie in the vertex set C.

Lemma (d)

We may assume that G is connected.

$G \neq \ clique\ \Rightarrow \exists\ clique\ K \subseteq V \ni G - K$ is not connected. (clique subset).

$G \neq$ *clique*, $\exists x, y \in V \ni xy \notin E$.
$G - (V \setminus \{x, y\})$ is not connected.
$G - (V \setminus \{x, y\})$ is a separation of G.

$\exists$ a minimal separation of G.