Graph and discrete structure

Myriam Preissman (GSCOP) - 4 lectures.
Lecture 1

Room affectation problem

Set of course $c_1, c_2 \cdots c_n$ on the same day.

$C_i = (B_i, K_i)$
k class rooms.

Examples

$$
\begin{aligned}
c_1 &= 08{:}00 - 11{:}00 \mapsto R_1 \\
c_2 &= 09{:}00 - 12{:}30 \mapsto R_2 \\
c_3 &= 12{:}00 - 14{:}00 \mapsto R_1 \\
c_4 &= 13{:}00 - 16{:}00 \mapsto R_2 \\
c_5 &= 10{:}00 - 11{:}00 \mapsto R_3
\end{aligned}
$$

Could it be assigned to 2 rooms instead ? No.

Graph transposition

Graph :

- $V = \{c_1, c_2 \cdots c_n\}$ set of courses.
- $c_i c_j \in E, c_i$ and c_j cannot be affected by the same room.

Definition

K-coloring

k-coloring o a graph : $cV \mapsto \{1, 2, \cdots k\} \ni c(v) \neq c(w), \forall vw \in E$

The problem is equivalent to decide if G has a k-coloring.

Decide if G has a k-coloring

Decide if G has a k-coloring is NP-complete for arbitrary G and k.

Remarks

- $k \geq |V| \Rightarrow \exists$ *k-coloring*.
- any k*-coloring* is also a k*'-coloring* for $k' \geq k$.

Chromatic number

- $\chi(G)$: chromatic number of a graph $= min\{k, G\ is\ a\ k\text{-}coloriable\}$

Clique

$\omega(G)$ the clique number of G $= max\{|W|, W \leq V \ni ww' \in E, \forall ww' \in W\}$

So :

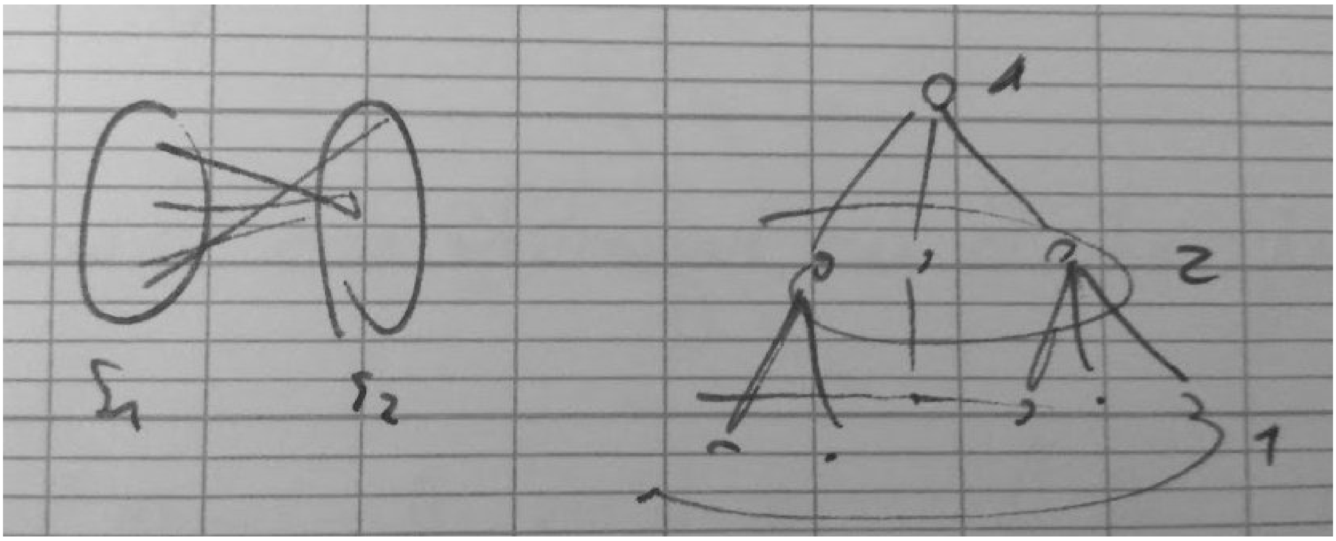
$\omega(G) \leq \chi(G) \leq |V|$

Question

Exist a 3 coloring graph with $\omega(G) = 2$? Yes, if an odd cycle graph.

Case $k = 2$

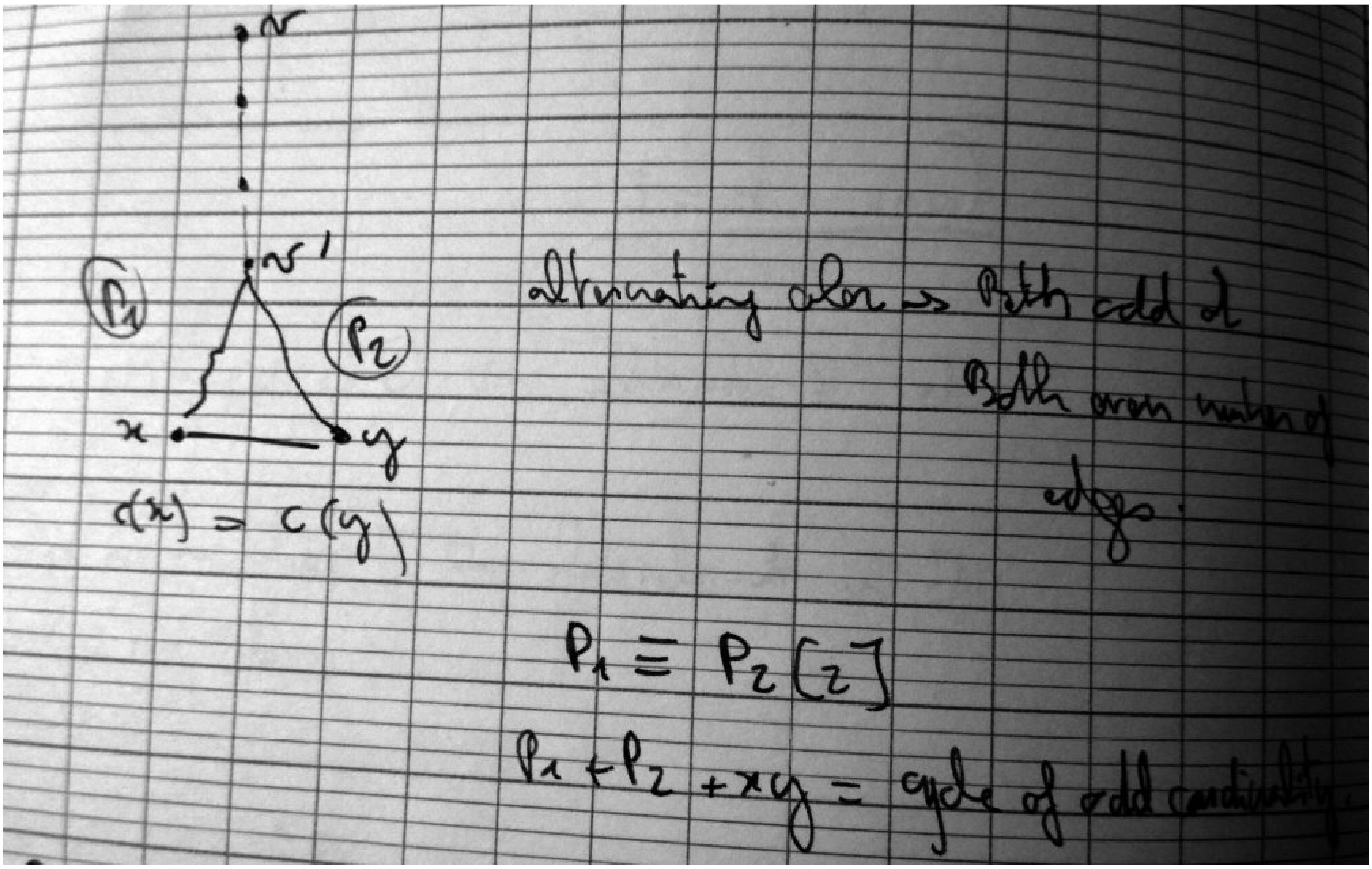
G 2-colorable $\Leftrightarrow$ G is bipartite.
G k-colorable $\Leftrightarrow$ $\exists$ partition of $V(G)$ into k-stable sets ($w \subseteq V$ containing no edge).



Algorithm = Give color 1 to an arbitrary vertex as long as there is an edge xy where x has a colors and y no. BFS exploration.

$c(x) = 1 \Rightarrow c(y) := 2$
$c(x) = 2 \Rightarrow c(y) := 1$

If at the end of the algorithm there is an edge between two vertices of the same color we want to show that there is an odd cycle in G.



Proof that G is bipartite

G is bypartite ⇔ G contains no odd cycle.

Greedy algorithm to obtain a proper coloring of the vertices

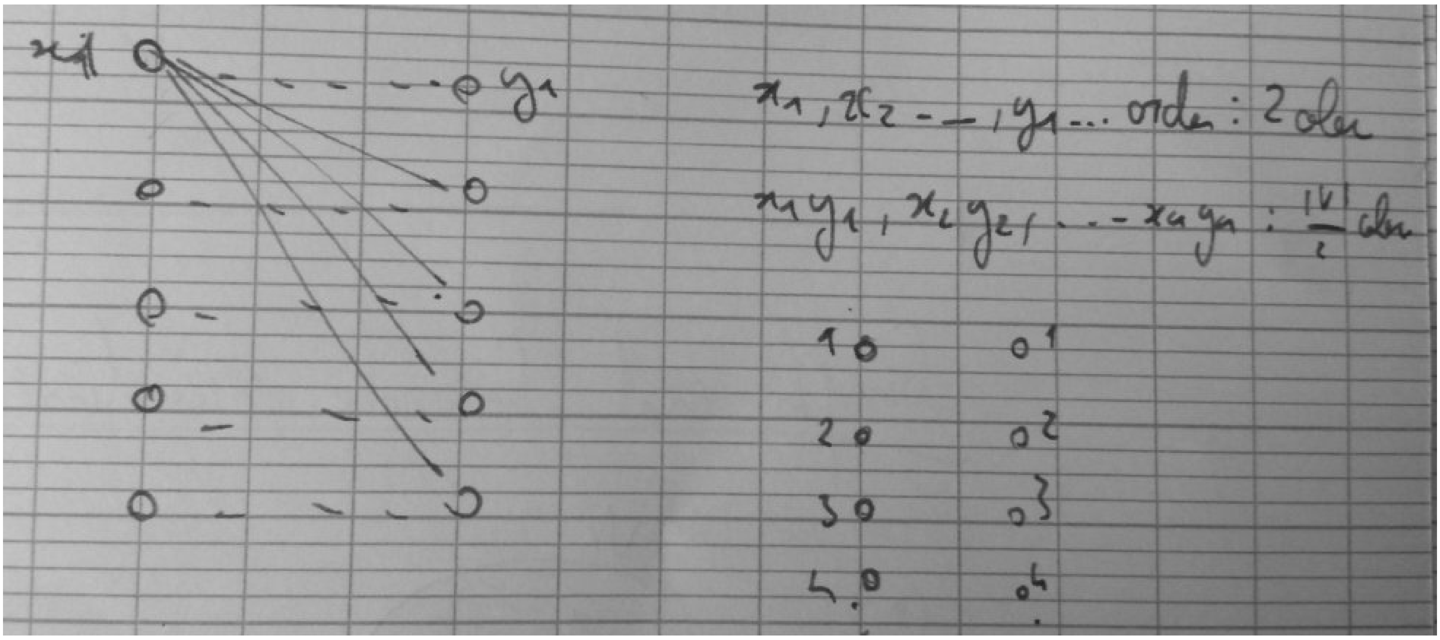
Greedy procedure :

Start with an ordering of the vertices $v_1, v_2 \cdots v_n$.

$$c(v_1) = 1$$
$$c(v_i) = \min\{ji \ni \textit{there is no neighbour of } v_i \textit{ that is colored } j\}$$

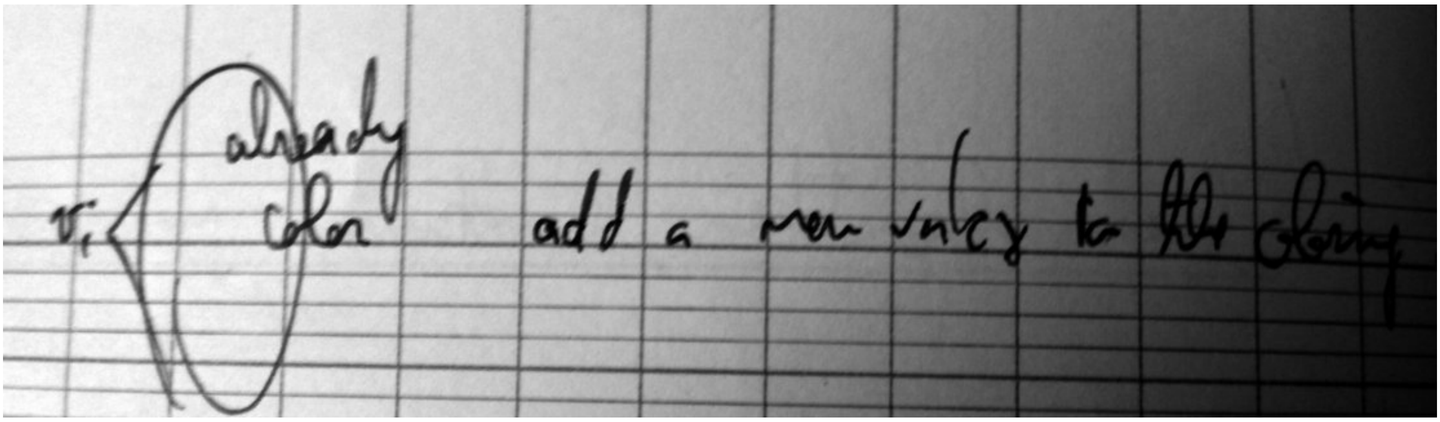
Bipartite Graph

Bad coloring exemple



Exercice

Exists and ordering of the vertices s.t. the greedy procedure based on this order give coloring in $\chi(G)$ color ?



Until v_i use only k-color, if there is a clique of size k, giving a new color is OK $\rightarrow$ The greedy procedure will give a coloring in minimum number of color.

Simplicial Vertex

A Simplicial Vertex of a graph G is a vertex such that $N(v)$ is clique of G. (with $N(G) =$ neighborhood)