Graph and discrete structure

Myriam Preissman (GSCOP) - 4 lectures.
Lecture 4

Reminder

Response to question

Intersection of tree.

$G = (V, E)$

$v \in V \rightarrow S_v = \{e \in E, v \; is \; incident \; to \; e\}$

$vw \in E \Leftrightarrow S_v \cap S_w \neq \emptyset$

Special chordal graph

Interval graph is special chordal graph.

Last time

$G(V, E)$ chordal graph $\Rightarrow$ simplicial ordering of $V \Rightarrow \forall W \subseteq V$

Greedy sequencial coloring gives an optimal coloring in chordal graph (if V is simplicial in G, it is simplicial in any subgraph of G).

Course

Definition

Definition : perfect order

An order θ on $V(G)$ is a *perfect order* if $\forall W \subseteq V$ the greedy seq alg based on θ/W gives an optimal coloring.

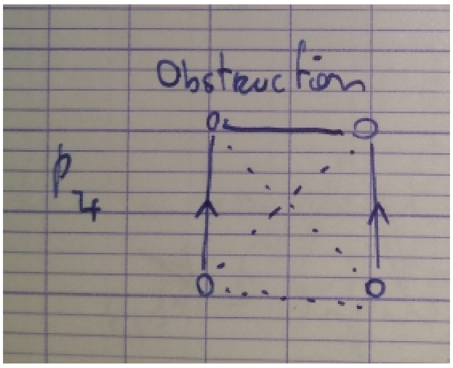
Definition : perfectly orderable

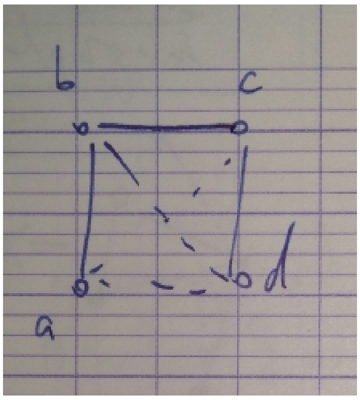
G is said *perfectly orderable* if there exists a perfect order of $V(G)$.

Remark

The smallest graph for wich there exists a non perfect order of the vertices

$\chi(P_4) = 2$

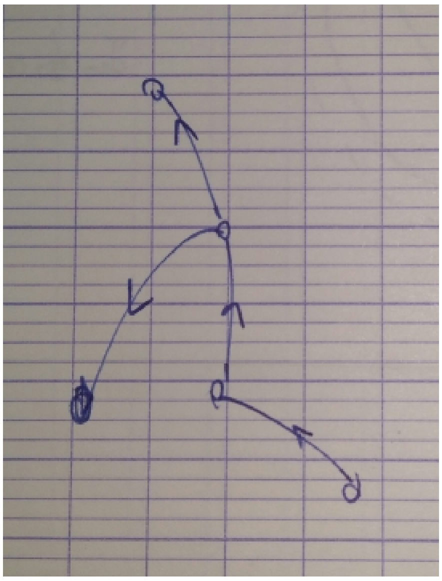




if $a < b$ and $d < c$ gives a 3-coloration.

Theorem Chatwal

G is perfectly orderable $\Leftrightarrow$ $\exists$ perfect orientation of G.



- No circuit.
- No obstruction.

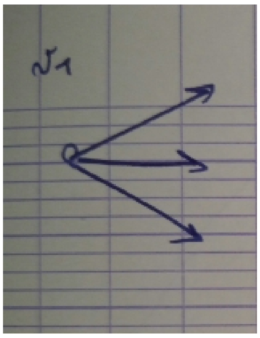
Proof

Idea of the proof

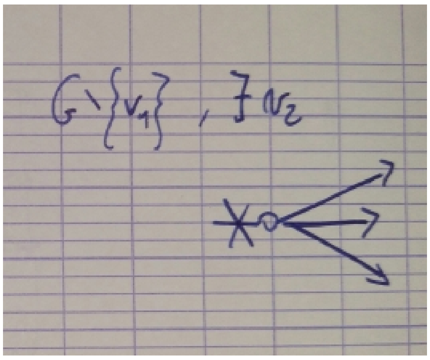
In a DAG without circuit there $\exists v$ such that no edge is entering in v.

(assume not ? c ?)

$v_1, v_2, \cdots, v_n$ on the vertices give an order witch is perfect.



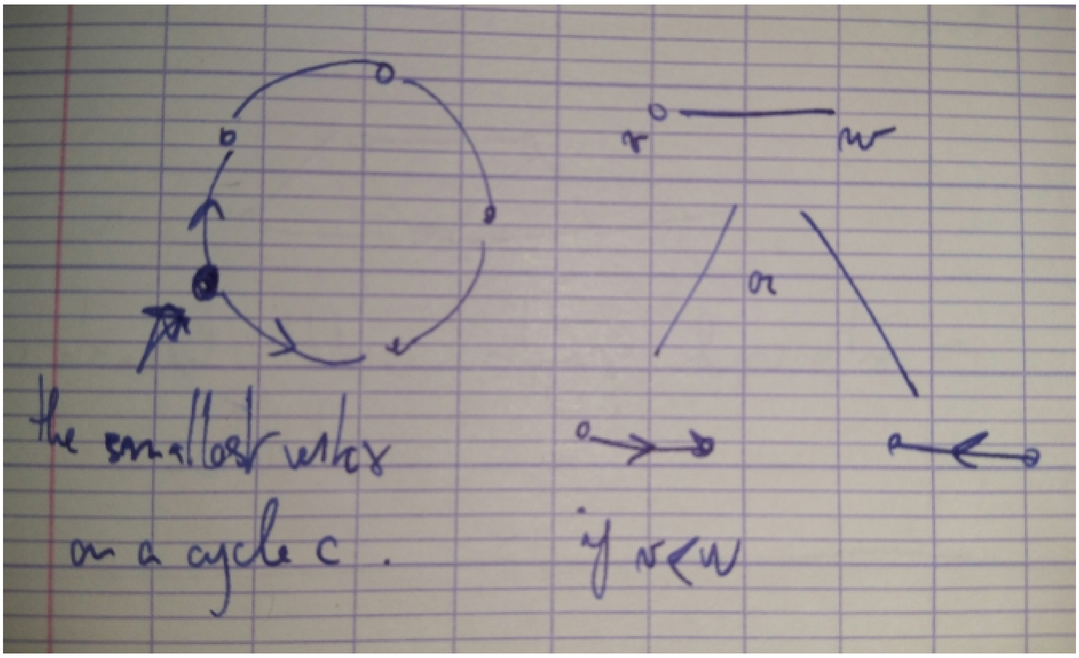
In $G \setminus \{v_1\}$



Proof $\Rightarrow$

No circuit

$\exists$ a perfect order $\Rightarrow$ orientation without circuit.



No obstruction

The order is perfect : gives an optimal coloring for all subgraph $\Rightarrow$ no obstruction.

Counter exemple : P_4 wich is not perfectly color.

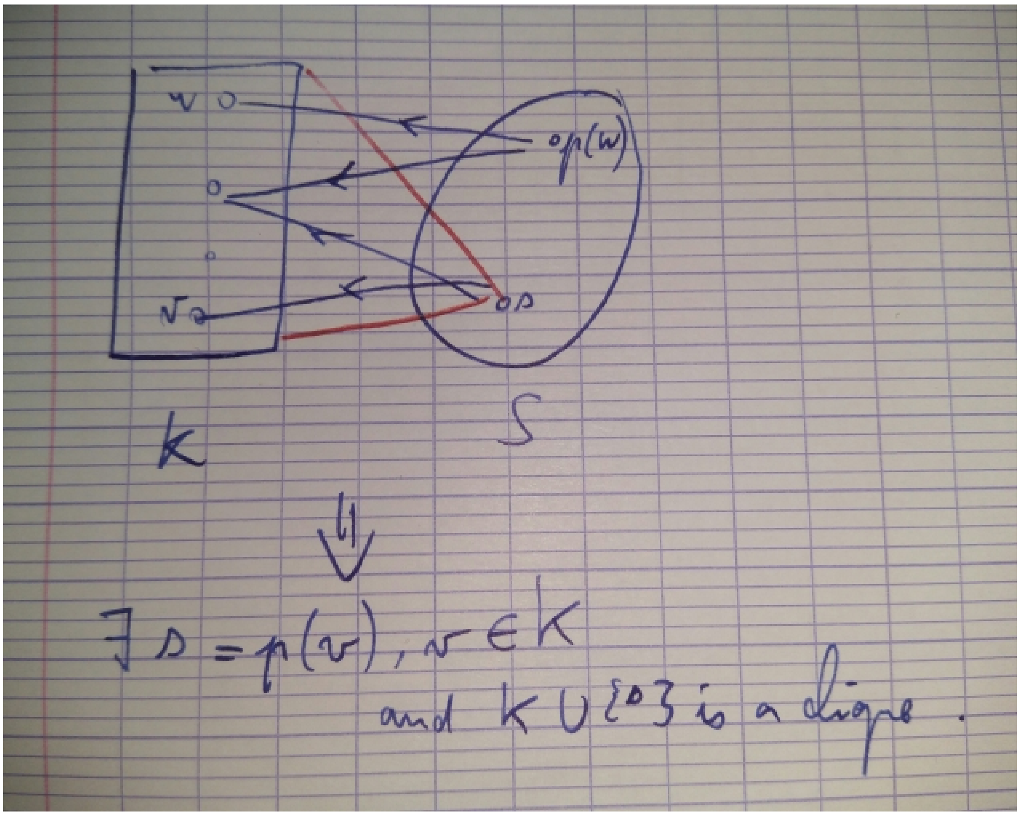
Proof $\Leftarrow$

$\overrightarrow{G}$ perfect orientation $\Rightarrow$ ordering on V.

Lemma

$\exists s \in S$ if K is a clique of G, S stable set, $p : K \mapsto S$

$p(w)w$ is a clique of s.



It is enough to prove that the greedy sequencial algorithm $v_1, v_2, \cdots, v_n$ gives an optimal coloring of G.

Assume this algorithm gives a k-coloring.

If G contains a clique of size k then the k-coloring is optimal.

$i_0 = min\{i,$ such that there exists in G a clique K made of vertices made of colors $k, k-1, k-2\}$

Assume $i_0 \geq 1$.

By the lemma there exists a vertex $s \in S \ni K \cup \{s\}$ is a clique $K \cup \{s\}$ a contradiction of i_0.

Complexity

The problem of deciding in a graph G is perfectly orderable is *NP-complete*.

Definition

G is perfect.

$\chi(G') = \omega(G'), \forall G'$ induced subgraph of G.

Two conjecture about perfect grpahs by Claude berge (1960, prooved in 2002).

- G perfect $\Leftrightarrow$ $\overline{G}$ is perfect. ($\overline{G} = G(V, \overline{E})$).
- G perfect $\Leftrightarrow$ G contains no C_{2k+1}, no $\overline{C_{2k+1}}, k \geq 2$

Theorem

There exists a polynomial algorithm to optimally color the vertices of a perfect graph to obtain a maximum clique stable set.

Polynomial algorithm : based on ellipsoid method no usable in practice.

Exercice

Witch classes does belong those graph :