Graph and discrete structure

Louis Esperet (GSCOP) - 4 lectures.
Lecture 4

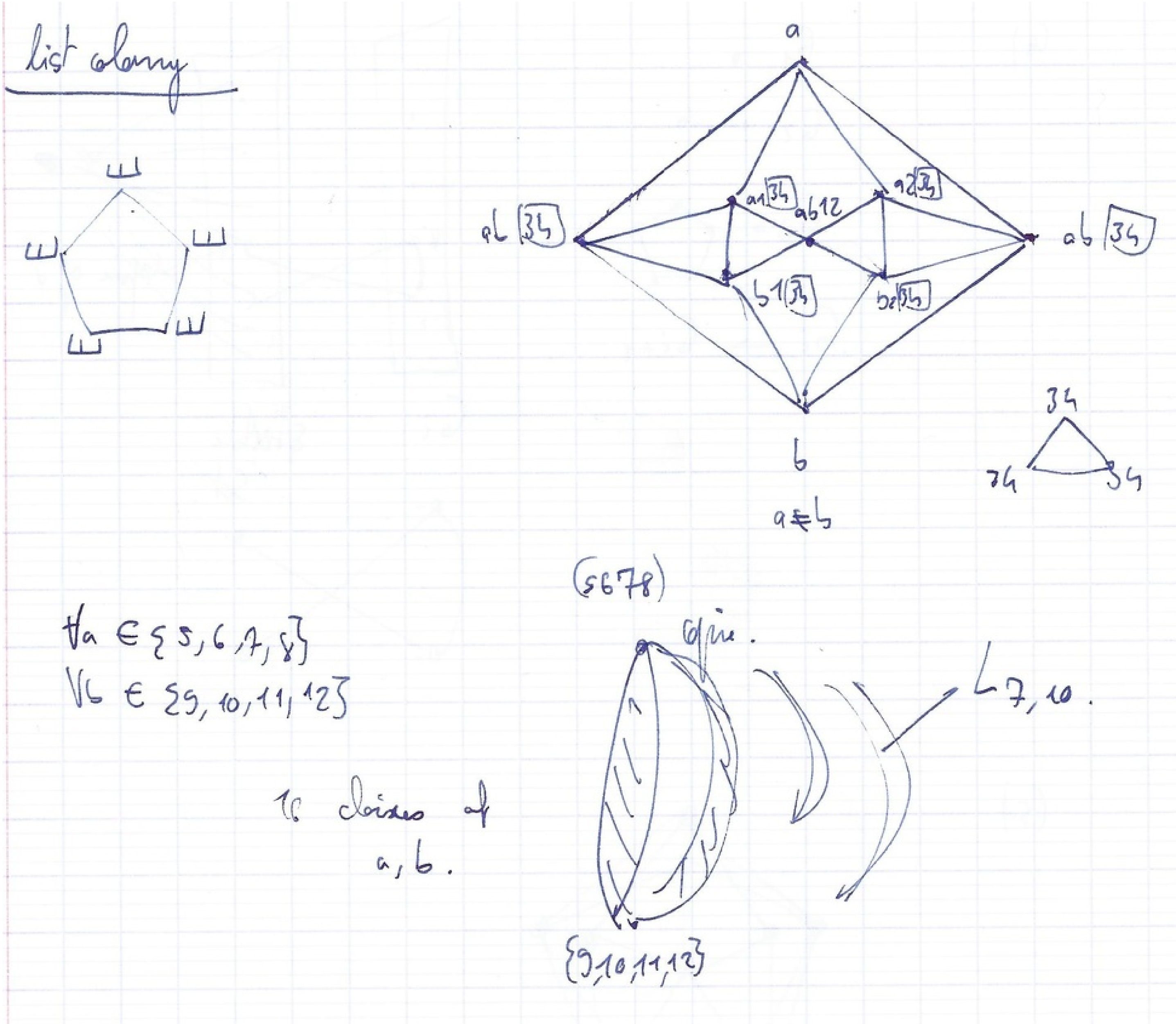
Reminder

Th: planar graphs are 5-list-colorable

Th(Voigt): There are planar graphs that are not 4-list-colorable.

Course

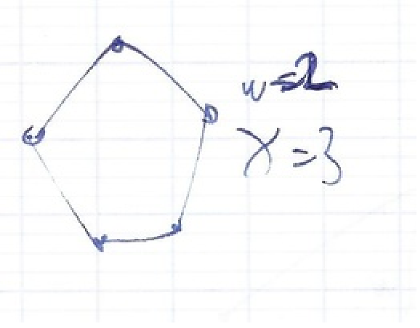
Proof with probability



Relation between ω and α

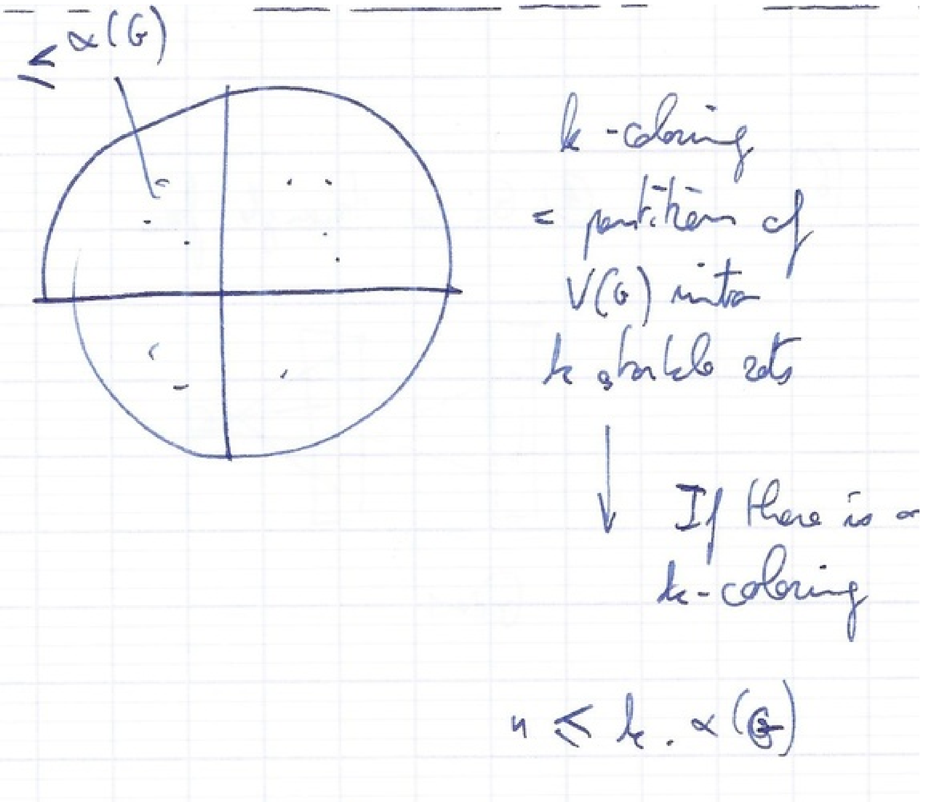
$\omega(G) =$ size of the largest clique.
$\alpha(G) =$ size of the largest stable set = independence number.

$\omega(G) \leq \chi(G)$



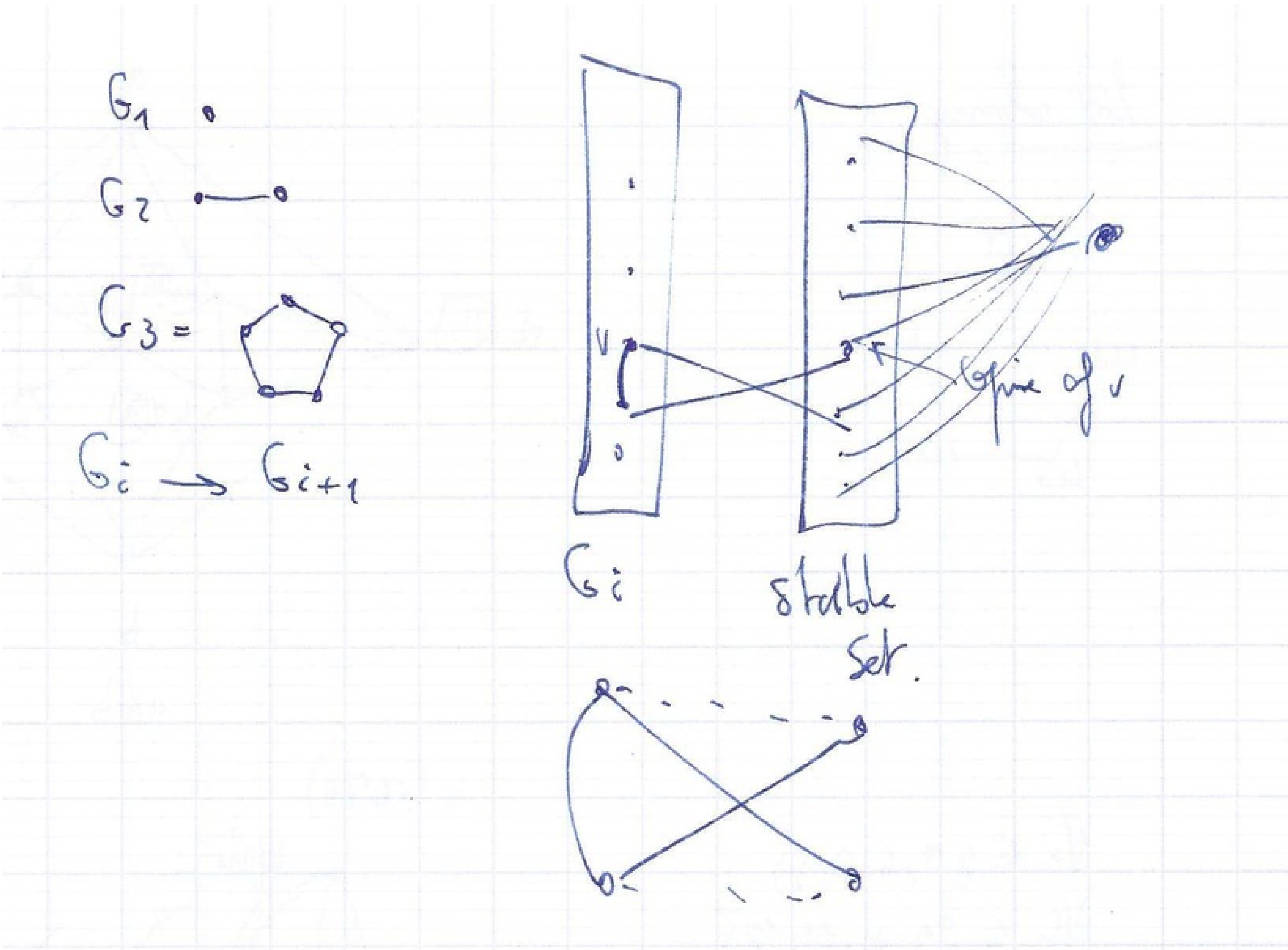
Obs : If G has n vertices, $\chi(G) \geq n/\alpha(G)$

List coloring

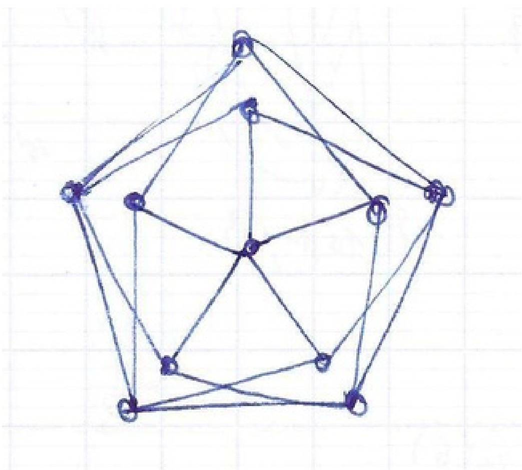


Mycielski 1955

Construction of triangle-free graphs of arbitrarily large chromatic numbers.

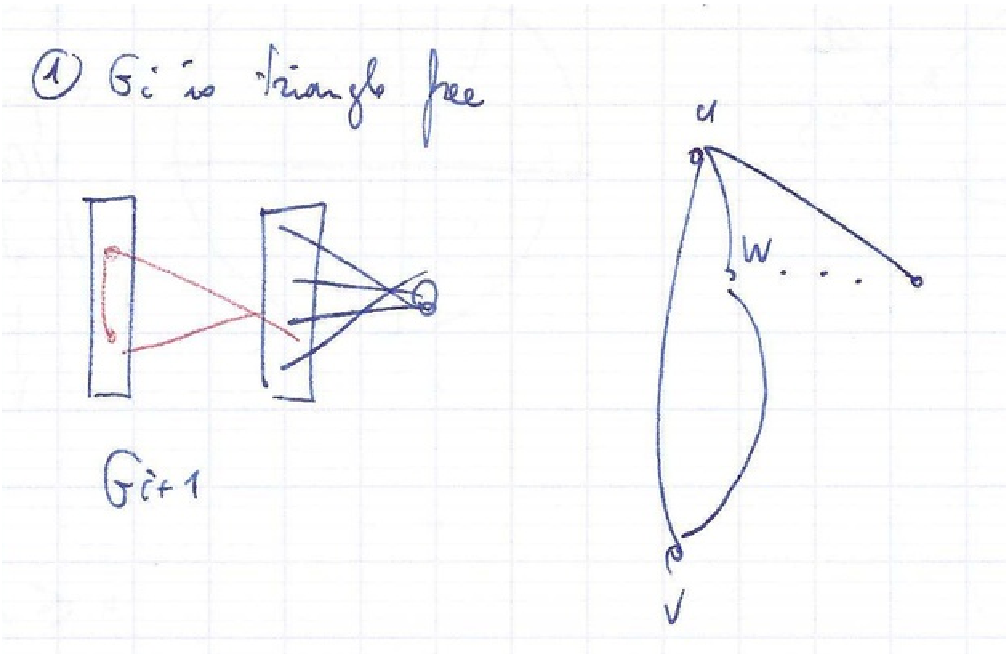


Exemple G_4



Proof of Th.

G_i is triangle free and $\chi(G_i) = i$



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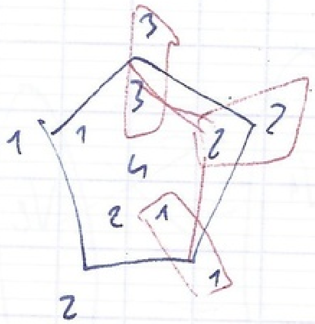
By contradiction.

$\exists$ color class C, $\forall$ vertex $v \in C$. v does not see some color class.

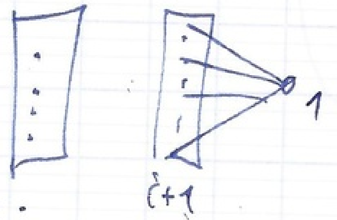
② $\chi(G_i) = i$



You have to use a new color.

②a $\chi(G_i) \le i$

by induction

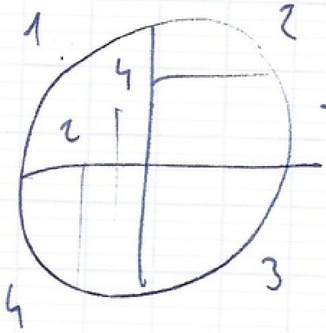


G_i

i-coloring of G_i

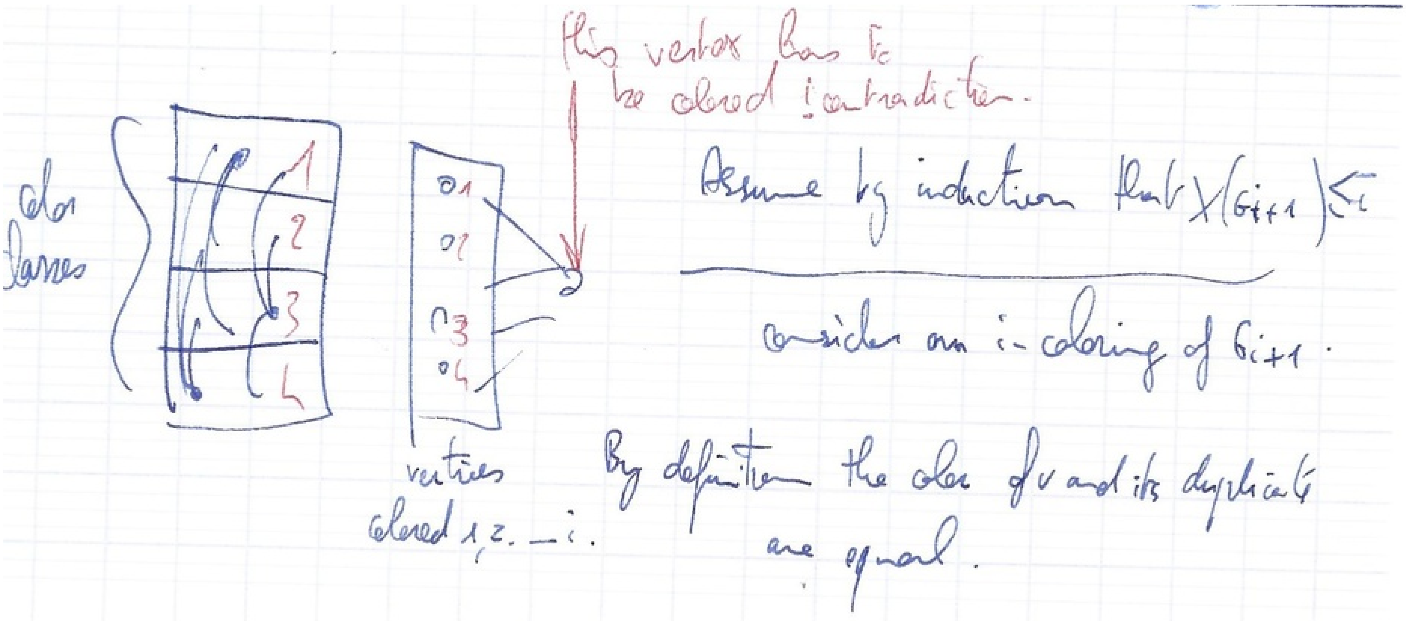
②b $\chi(G_i) \ge i$

__Lemma:__ If $\chi(G) = k$, then in any k-coloring of G, each color class contains a vertex that is adjacent to each other color class.



→ recolor v with a color it does not see

→ $\chi(G) \le k-1$ contradiction.



Erdos 49

The girth of a graph $=$ the length of the shortest cycle.

For any g and any $k, \exists$ a graph of girth $\geq g$ and chormatic number ≥ 4.

triangle-free $\Leftrightarrow$ girth $\geq$ 4.

Birth of the *probabilistic method*.

Exemple

Every graph of m edges has a partition V_1, V_2 of its vertex set with $\geq m/2$ edges between V_1 and V_2.

Probabilistic version

For each vertex put it in V_1 with prob $1/2$.

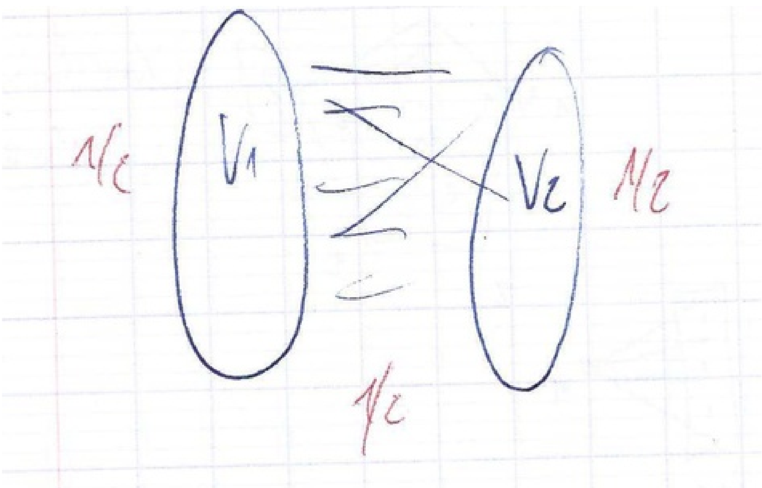
What is *expectation* of th enumber of edges between V_1 and V_2 ?

Expectation $= \#edges \times$probability that a given edge e is between V_1 and V_2.

$1/2 * 1/2 + 1/2 * 1/2 = 1/2$

We proved that the expectation is $m/2$.

There is a partition V_1, V_2 with $m/2$ edges between V_1 and V_2.



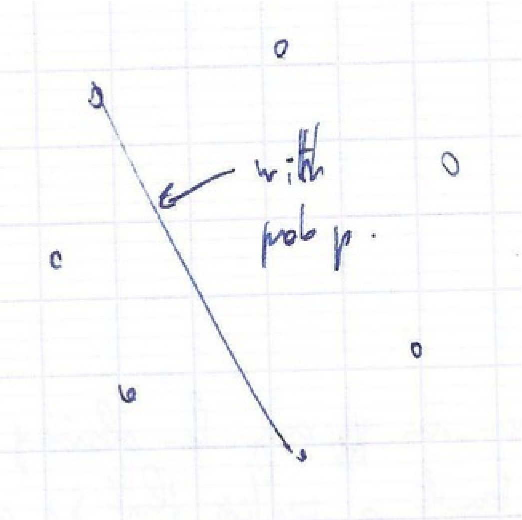
Theorem next time.

Proof for g = 4

Assignement : do it for all g.

$G(n,p)$ binomial random graph.

n vertices. For each u, v add an edge between u and v independently with prob p.



1. prove that $G(n,p$ is triangle free.
2. prove that $\chi(G(n,p)) \leq k$.

We are going to show taht $\alpha(G(n,k)) \leq n/2k$

Proof 1

$\mathbb{E}(\#\text{ triangles }) = \binom{n}{3} \times p \leq \frac{n^3}{6} \times p^3 \leq \frac{n}{6}$

$p = n^{\frac{-2}{3}}$

Markov inequality

$X \geq 0$ random variable, $\mathbb{P}(X \geq k) \leq \frac{\mathbb{E}(X)}{k}$

Proof (next)

![...]

Corollary

If $X \geq 0$ is an integer random variable.

$\mathbb{P}(X \neq 0) \leq \mathbb{E}(X)$

$X \neq 0 \Leftrightarrow X \geq 1$

$\mathbb{E}(\#\text{ triangle }) \leq \frac{n}{6}$

Markov :

$\mathbb{P}(\#$ triangle $\leq \frac{n}{2}) \leq \frac{n}{6}/\frac{n}{2} \leq \frac{1}{3}$

Proof of 2

$\mathbb{P}(G$ has a stable set of size $n/2k) \leq \mathbb{E}(X)$

$X : \#$ stable set of size $n/2k$.

$\mathbb{E}(X) = \binom{n}{\frac{n}{2k}} \times \mathbb{P}($ a given set of size $n/2k$ is a stable set $)$

$\leq n^{n/2k} \cdot (1-p)^{n/2k}$

$1 - p \leq \exp(-p)$

$\mathbb{E}(X) \leq n^{n/2k} \cdot \exp(-p \cdot n/2k(n/2k - 1) \cdot 1/2)$

...

Exercice

Give another planar graph that is not 4-list-colorable.