Graph and discrete structure

Louis Esperet (GSCOP) - 4 lectures.
Lecture 2

Reminder

Induction

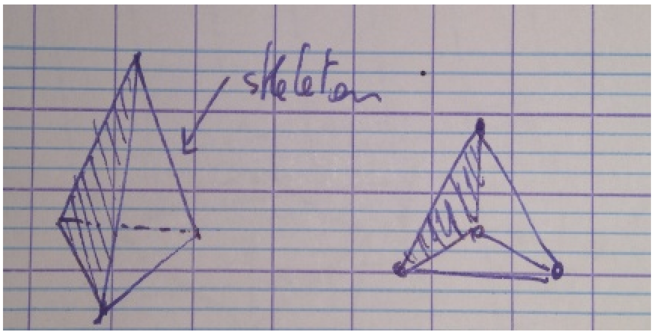
Reminder on induction.

Course

Planar graph

Steinitz (1920)

G is planar and 3-connected
$\Leftrightarrow$ G is the skeleton of a 3d polyhedron.
$\Leftrightarrow$ G has a planar drawing with strait line edges and in which edges face is convex.



k-connected

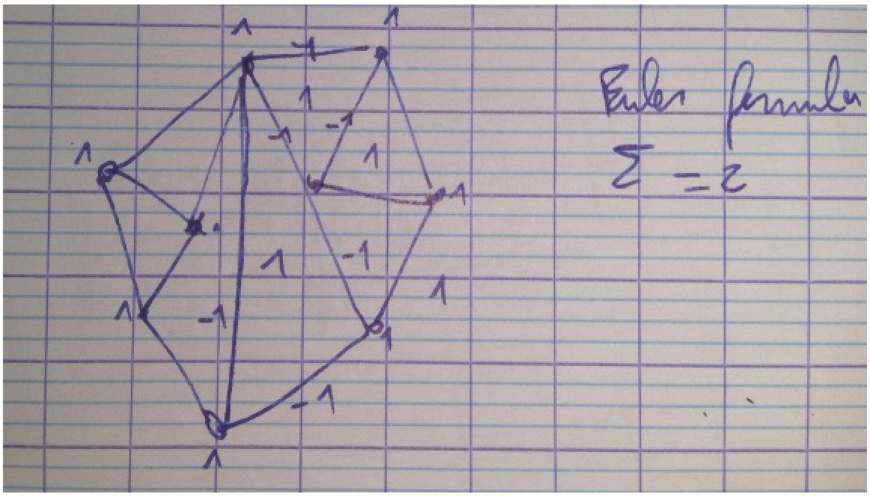
G is k-connected

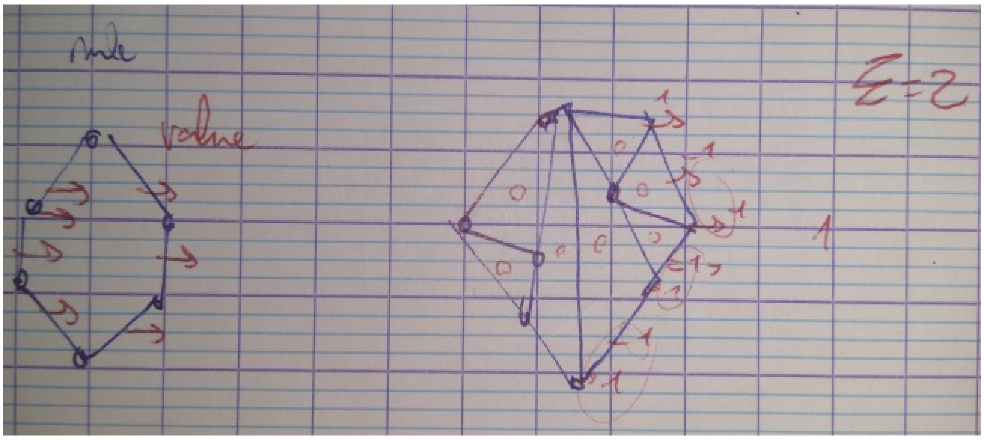
- $|V(G)| \geq k-1$ vertices (???)
- whatever we remove $\leq k-1$ vertices, the graph remains connected.

Euler formula

$n - m + f = 2$ for plannar graph.

Proof of Euler formula for 3-connected planar graph





Coloring planar graphs

Let G be a planar graph.

Assume that G is triangulated. (all the faces are triangle).

For each vertex v let $\omega(v) = d(v) - 6$

Question

What is $\sum_{v \in V(G)} \omega(v)$? -12.

$\sum(d(v)-6) + \sum(2 \cdot (e) - 6) = \sum(d(v)-6) + 0 = 2m - 6n + 4m - 6f = 6m - 6n - 6f = -6(n - m + 6) = -12$

In general, i.e. when G is not triangulated $\sum_{v \in V(G)} \omega(v) \leq -12$

(w, omega ?)

$\exists v, d(v) - 6 \leq 1$
$d(v) \leq S$

Corollary

In any planar graph, there is a vertex of degree ≤ 5.

(Another from last time)
Any planar graph is 5-degenerate and thus, 6-colorable.

Theorem : planar graph are 5 colorable

proof by induction on the number of vertices (n)

- if $n \leq 5$, it is fine.
- Assume we have proved that every planar graph for $n-1$ vertices is 5-colorable. We want to prove it also for n vertices.

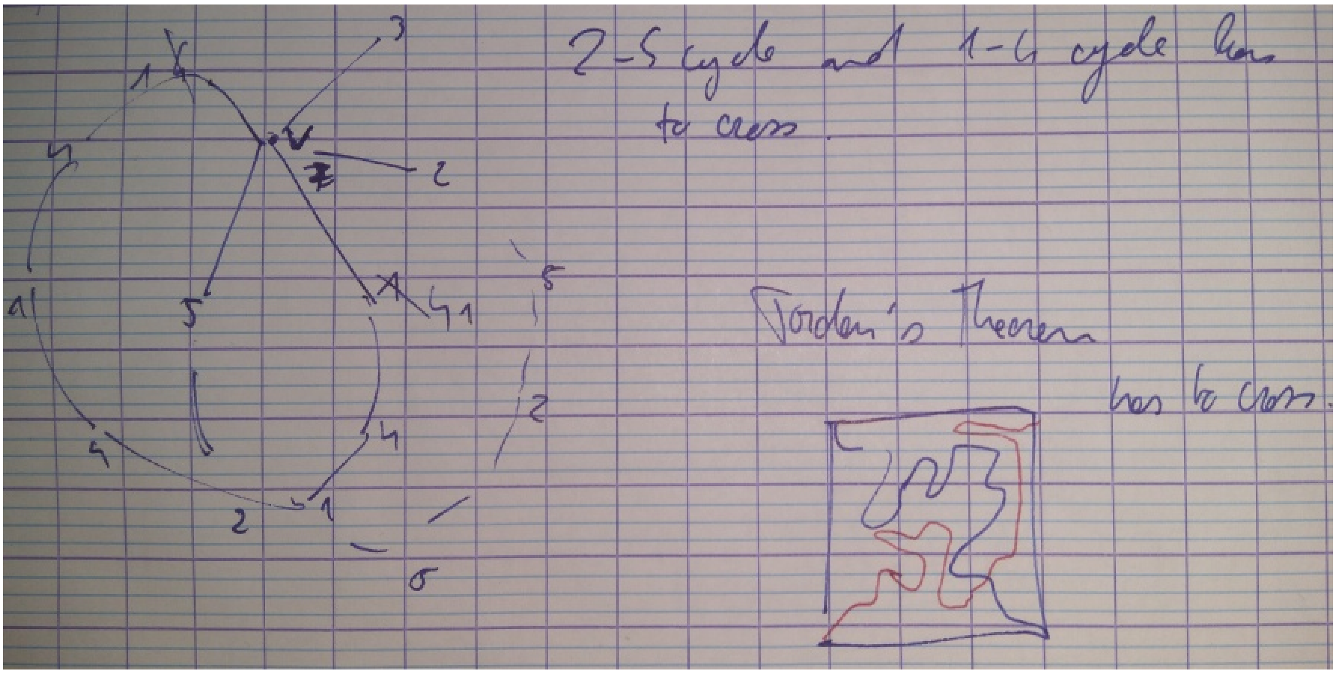
Case 1:

G has a vertex of degree \leq 4.

By induction $G - v$ is 5-colorable. Color v with a color that does not appear in its neighborhood.

Case 2:

Every vertex has degree ≥ 5.
There is a vertex of degree $= 5$. By induction $G - v$ is 5-colorable.



A 1-4 cycle cannot cross a 2-5 cycle. So, at least one of the recoloring will work and v can be recolored.

4-coloring

Proposition

Any planar graph has

- a vertex of degree ≤ 4
- on a vertex of degree 5, adjacent to vertex of degree ≤ 6.

Proof

- We can assume that G is a triangulation.
- $\forall v,\ \omega(v) = d(v) - 6,\ \sum \omega(v) = -12.$

Assume that we don't have vertices of degree ≤ 4. Every vertex ha degree at least 5.

By contradiction, assume that every vertex of degree 5 has all its neighboors that have degree ≥ 7.

$d(v)$	$\omega(v)$
5	-1
6	0
7	1

8	2
...	...

Idea : move the weight s.t. everyone ha weight ≥ 0

Rule : every vertex of degree 5 receive weight from all of it's neightboors.

After application of the rule :

Vertices of the rule, vertices of degree $d = 5$ are ok $-1 + 5 \times \frac{1}{6} = 0$