Graph and discrete structure

Louis Esperet (GSCOP) - 4 lectures.
Lecture 1

Reminder

Greedy algorithm

The greedy algorithm colors every graph of max degree Δ with at most $\Delta + 1$ colors.

Brooks

If G has max degree Δ, and is neither an odd cycle nor a conplete graph, then $\chi(G) \leq \Delta$.

Definition

d-degenerate

A graph G is d-degenerate if there is an order $v_1, \cdots, v_n$ on it's vertices s.t. for any i the number of vertices $v_j (j < i)$ adjeacend to v_i is at most d.

Corollary of the greedy algorithm

If G is d-degenerate, then $\chi(G) \leq d + 1$.

Theorem

Let C be a class of graphs such that for any $G \in C$, and for any vertex $v \in G, G - v \in C$.

C is said to be hereditary.

Exemple

True

k-colorable, biparptite, interval graph, triangle-free graph.

False

cycles, connected, complete graph.

Theorem next

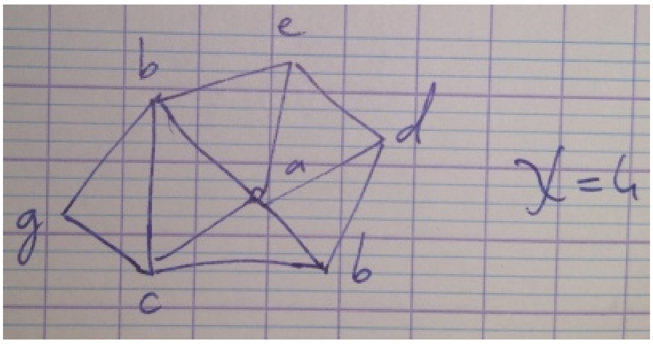
Assume that any graph of C has a vertex of degree $\leq d$, then any graph of C is d-degenerate.

Proof by induction

Proof by induction on the number of vertices of G.

G has a vertex v of degree $\leq d$.

- $G - v_n \in C$, by induction $G - v_n$ is d-degenerate leq d.
- $G - v_n$ has an ordre $v_1, v_2, \cdots, v_n$ just add v_n at the end.



Note : you recalculate the degree at each step.

Planar graphs

Definition planar graph

A graph is planar if it has a planar drawing.

Definition planar drawing

A drawing in the plane (or the sphere) such that edges don't cross.

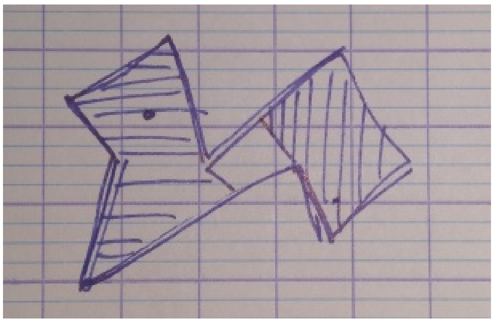
Theorem (fary 48) on straiht line

For any planar drawing of a planar graph G there is a an equivalent planar drawing in which all the edges are straiht line segments.

The art gallery problem

Question

How many gards can you put in the gallery s.t. they see everything ?



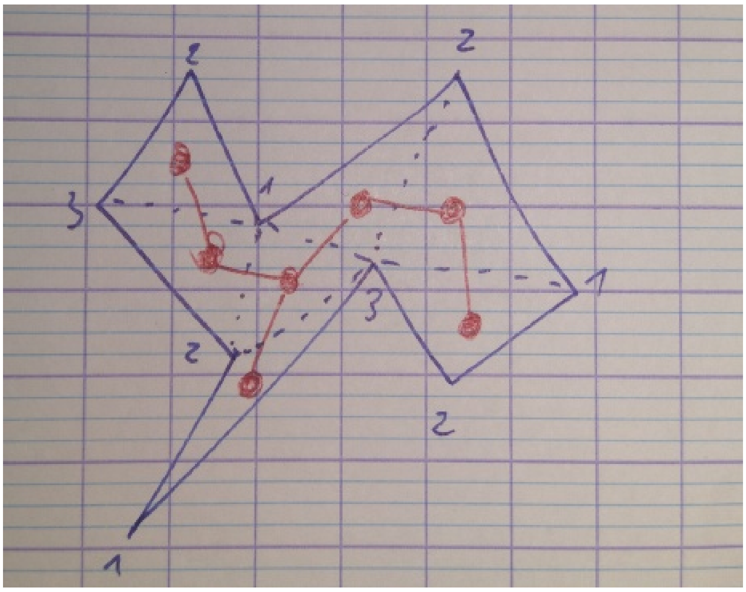
Ground $\lfloor \frac{n}{3} \rfloor$ guards are enough.

Proof

- Any triangulated polygon is 3-colorable.
- Triangulation :
  - triangulate the polygon.

- 3-color it.
- One color has size $\leq \lfloor \frac{n}{3} \rfloor$.
- Place the guards at the locations of those vertices.

Proof with chordal graph.



The red graph is a tree.

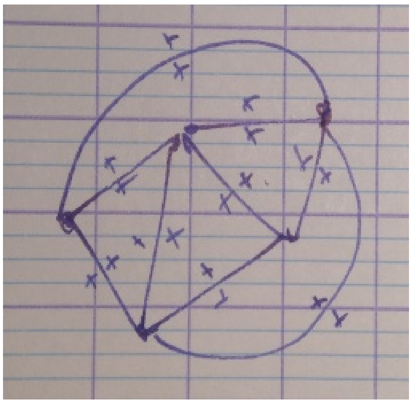
Any face of G corresponding to a face of the red tree contains a vertex of degree 2, whose removal yields a triangulated polygon.

Color this by induction with 3 colors, and color v with color $\{1, 2, 3\}$ distinct from it's 2 neightbours.

Proof of Fary's theorem

- Triangulate all faces.
- Euler's formula :
  - G is connected with a planar drawing then $\#vertices - \#edges + \#faces = 2$
  - $n - m + f = 2$.

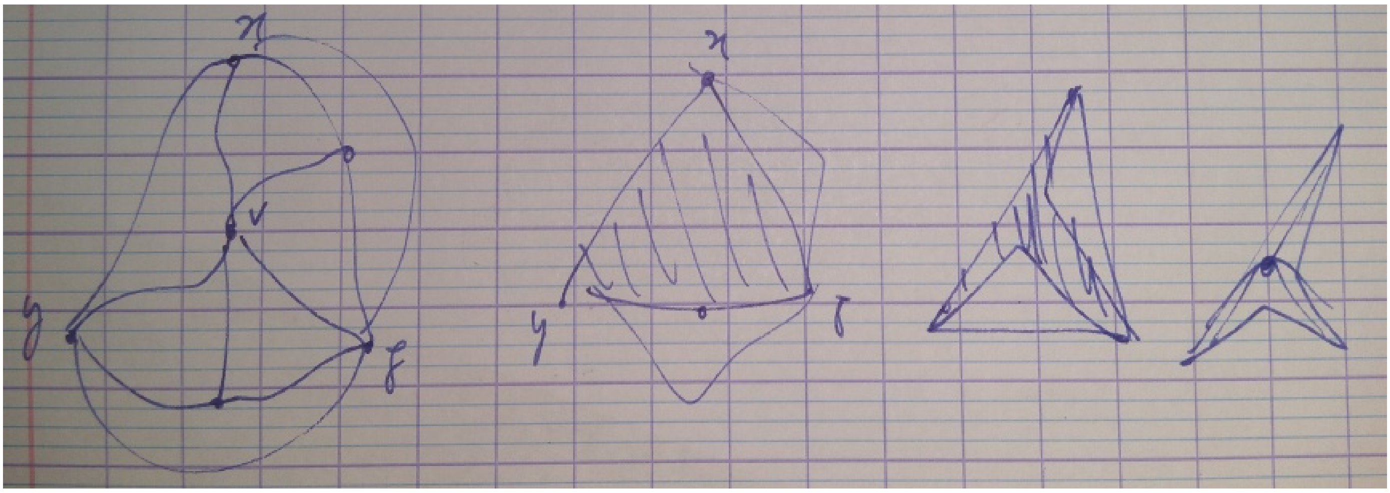
If all faces are triangle : $2m = 3f \Rightarrow m = 3n - 6$.



$\sum_v d(v) = 2m = 6m - 12$
$\sum(d(v) - 6) = -12$

$\Rightarrow$ For any x, y, z there is a vertex v distinct from x, y, z such that $d(v) \leq 5$.

Final step of the proof


x
v
y
z
x
y
z