Explanation of "On Two Minimax Theorems in Graph"

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October 24, 2018

Contents

1 Theorems

THEOREM 1 (Edmonds)**.** *The maximum number of edge-disjoint branchings (rooted at a) equals the minimum number of edges in $a-$cuts.*

2 Definitions

Edge-disjoint branchings

Two branching are edge-disjoint if they do not have any internal edge in common.

Branching :

Branching One In a directed graph $D = (V, A)$ a branching is a set of arcs not containing circuits s.t. each node of D is entered at most by one arc in A'. (So the arc in A' make up a forest).

Branching Two A branching B is an arc set in a digraph D that is a forest such that every node of D is the head of at most one arc of B. A branching that is a tree is called an arborescence. A branching that is a spanning tree of D is called a spanning arborescence of D. Clearly, in a spanning arborescence B of D every node of D is the head of one arc of B except for one node. This node is called the root of B. If r is the root of arborescence B we also say that B is rooted at r or $r-$rooted.

R-branching For a subset $R \subset V$, an $R-$branching in G is a spanning forest $B \subset G$ in which all vertices of $V - R$ have outdegree precisely 1.

r-branching When R just consists of a single vertex r, we refer to B as an $r-$branching.

k-connected

A graph G (digraph D) is called $k-$connected ($k-$diconnected) if every pair s, t of nodes is connected by at least k $[s,t]-$paths ($(s,t)-$dipaths) whose sets of internal nodes are mutually disjoint.

a-cut

A $a-$cut of G determined by a set $S \subset V(G)$ is the set of edges going from S to $V(G) - S$. It will be denoted by $\Delta_G(S)$. We also set that $\delta_G(S) = |\Delta_G(S)|$.

r-Cuts and r-Arborescences

Consider a connected digraph (N, A) with $r \in N$ and nonnegative integer arc lengths l_a for $a \in A$. An $r-$arborescence is a minimal arc set that contains an $rv-$dipath for every $v \in N$. It follows that an $r-$arborescence contains $|N| - 1$ arcs forming a spanning tree and each node of $N - r$ is entered by exactly one arc. The minimal transversals of the clutter of r-arborescences are called $r-$cuts.

We define a clutter C to be a family $E(C)$ of subsets of a finite ground set $V(C)$ with the property that $S_1 \nsubseteq S_2$ for all distinct $S_1, S_2 \in E(C)$. $V(C)$ is called the set of vertices and $E(C)$ the set of edges of C.

r-cut 2

An r-cut in a digraph is an edge set $\delta^+(X)$ for some $X \subset V(G)$ with $r \in X$.

r-arborescence 2

an Arborescence in a digraph $D = (V, A)$ is a set A' of arcs making up a spanning tree such that each node of D is entered by at most one arc in A'. It follows that there is exactly one node r that is not entered by any arc of A'. This node is called the root of A', and A' is called rooted in r, or an $r-$arborescence.

A note on orientation

A rooted tree is a tree in which one vertex has been designated the root. The edges of a rooted tree can be assigned a natural orientation, either away from or towards the root, in which case the structure becomes a directed rooted tree. When a directed rooted tree has an orientation away from the root, it is called an arborescence, branching, or out-tree; when it has an orientation towards the root, it is called an anti-arborescence or in-tree.

3 Examples

Simple branching

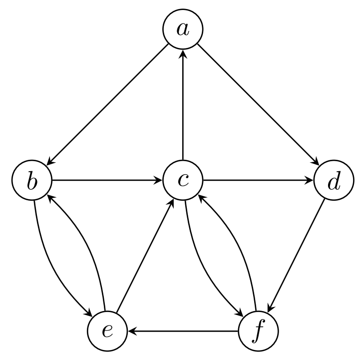

Figure 1: A digraph $G = (A, V)$

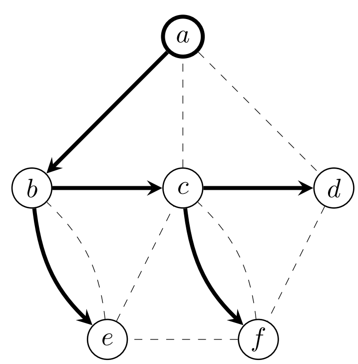

Figure 2: A branching of G routed at a.

Edge disjoint branching

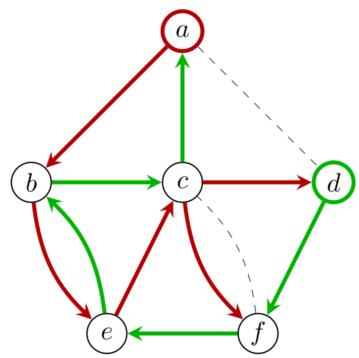

Figure 3: Two edge-disjoint branching routed respectively at a and d.

Edge disjoint a-routed branching

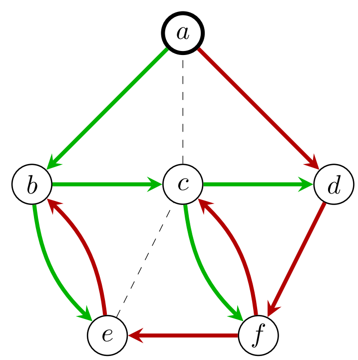

Figure 4: Two edge-disjoint a-routed branching.

a-cut