On Two Minimax Theorems in Graph

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First theorem

branching (r-arborescence) : In a directed graph $D = (V, A)$ a branching is a set of arcs not containing circuits s.t. each node of D is entered at most by one arc in A'. (So the arc in A' make up a forest).

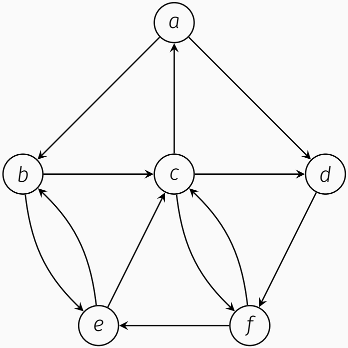

Figure 1: A digraph $D = (A, V)$

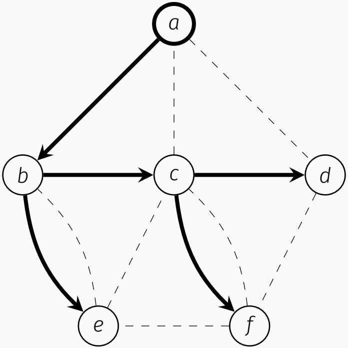

Figure 2: A branching of D routed at a.

Edge disjooint branching

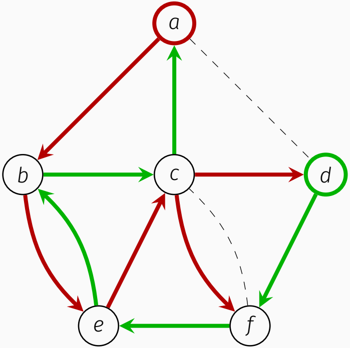

Figure 3: Two edge-disjoint branching routed respectively at *a* and *d*.

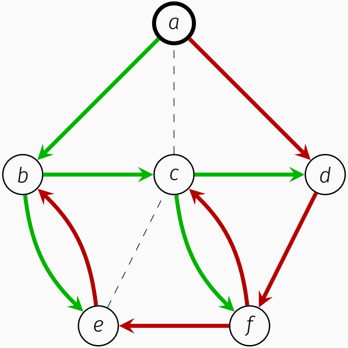

Figure 4: Two edge-disjoint *a*-routed branching.

a-cut : A $a-$cut of G determined by a set $S \subset V(G)$ is the set of edges going from S to $V(G) - S$. It will be denoted by $\Delta_G(S)$. We also set that $\delta_G(S) = |\Delta_G(S)|$.

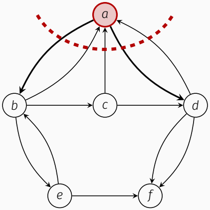

Figure 5: A a-cut of D, $S = \{a\}$, $\Delta_D(S) = \{ab, ad\}$, $\delta_D(S) = 2$.

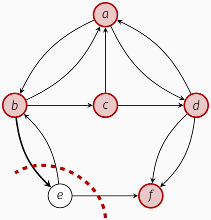

Figure 6: A minimal a-cut of D, $S = \{a, b, c, d, f\}$, $\Delta_D(S) = \{be\}$, $\delta_D(S) = 1$.

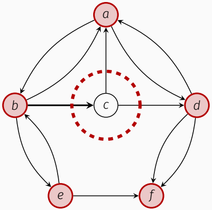

Figure 7: A minimal a-cut of D, $S = \{a, b, d, e, f\}$, $\Delta_D(S) = \{bc\}$, $\delta_D(S) = 1$.

THEOREM (Edmonds)
The maximum number of edge-disjoint branchings (rooted at a) equals the minimum number of edges in a—cuts.

Proof

Let k represent the number of edges. If $\Delta_G(S) \geq k \ \forall \ S \subset V(G), a \in S$ then there are k edge-disjoint branchings.

Let F be a set of edges such that F is an arborescence rooted at a. Then we want to show that $\Delta_{G-F}(S) \geq k - 1 \ \forall \ S \subset V(G)$, $a \in S$.

If $F = E(G)$ then it is a branching and we are done, else $G - F$ contains $k - 1$ edge-disjoint branchings and F is in the kth edge-disjoint branching. Following from this, we can see that F only covers a set of vertices $T \subset V(G)$. Therefore, there exist a vertex v that is not connected to F and we can add an edge $e \in \Delta_G(T)$. This will yield an arborescence $F + e$ such that F covers all points of G, satisfying $\Delta_{G-F}(S) \geq k - 1 \ \forall \ S \subset V(G)$, $a \in S$.

Proof

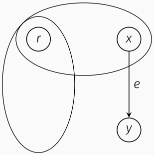

Figure 8

If we have a maximal set $A \subset V(G)$ such that A contains the root a, $A \cup T \neq V(G)$ and $\delta_{G-F}(S) = k - 1$. If no such A exist then we have set A which is not maximal. Then we have $\delta_{G-F}(A \cup T) > \delta_{G-F}(A)$ that we can add a new edge $e = (x, y)$ which belongs to $\boldsymbol{\Delta}_{G-F}(A \cup T) - \boldsymbol{\Delta}_{G-F}(A)$ such that $x \in T - A$ and $y \in V(G) - T - A$.

Now we want to show that that we can add e to F and $F + e$ will be an arborescence and satisfy $\mathbf{\Delta}_{G-F}(S) \geq k - 1$. For sure, we can see that F is an arborescence. If $e \notin \mathbf{\Delta}_G(S)$ then

$$\delta_{G-F-e}(S) = \delta_{G-F}(S) \geq k - 1$$

Now, if we have that $e \in \mathbf{\Delta}_G(S)$ then $x \in S$ and $y \in V(G) - S$. That is, x and y are in different set of vertices of G. Next, we will use the inequality;

$$\delta_{G-F}(S \cup A) + \delta_{G-F}(S \cap A) \leq \delta_{G-F}(S) + \delta_{G-F}(A) \tag{1}$$

We already know from the maximality of A that $\delta_{G-F}(A) = k - 1$, $\delta_{G-F}(S \cap A) \geq k - 1$ and $\delta_{G-F}(S \cup A) \geq k$. Then from (1) we have $\delta_{G-F}(S) \geq k$. Therefore $\delta_{G-F-e}(S) \geq k - 1$.

Second theorem

Directed cut

directed cut (dicuts) : A directed cut of a weakly connect graph G is the set $D = \Delta_G(S),\ (S \subset V(G),\ S \neq \varnothing)$ provided $\Delta_G(S),\ (V(G) - S) = \varnothing$.

dijoin : A dijoin is a subset $A' \subset A$ which covers all dicuts.

Let $\Omega(D)$ denote the maximum number of arc-disjoint dicuts in D and let $\epsilon(D)$ be the minimum cardinality of a dijoin.

THEOREM (Lucchesi and Younger)
The maximum number of disjoint directed cuts equals the minimum number of edges which cover all directed cuts.