Explanation of paper: "On Two Minimax Theorems in Graph"

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Contents

1 Definitions

Branching and a-cut (from the paper)

branching In a directed graph $D = (V, A)$ a branching is a set of arcs not containing circuits s.t. each node of D is entered at most by one arc in A'. (So the arc in A' make up a forest).

a-cut A $a-$cut of G determined by a set $S \subset V(G)$ is the set of edges going from S to $V(G) - S$. It will be denoted by $\Delta_G(S)$. We also set that $\delta_G(S) = |\Delta_G(S)|$.

Branching and a-cut (from books)

branching A branching B is an arc set in a digraph D that is a forest such that every node of D is the head of at most one arc of B. A branching that is a tree is called an arborescence. A branching that is a spanning tree of D is called a spanning arborescence of D. Clearly, in a spanning arborescence B of

D every node of D is the head of one arc of B except for one node. This node is called the root of B. If r is the root of arborescence B we also say that B is rooted at r or $r-$rooted.

r-arborescence An arborescence in a digraph $D = (V, A)$ is a set A' of arcs making up a spanning tree such that each node of D is entered by at most one arc in A'. It follows that there is exactly one node r that is not entered by any arc of A'. This node is called the root of A', and A' is called rooted in r, or an $r-$arborescence.

r-cut An r-cut in a digraph is an edge set $\delta^+(X)$ for some $X \subset V(G)$ with $r \in X$.

edge-disjoint branchings Two branching are edge-disjoint if they do not have any internal edge in common.

A note on orientation A rooted tree is a tree in which one vertex has been designated the root. The edges of a rooted tree can be assigned a natural orientation, either away from or towards the root, in which case the structure becomes a directed rooted tree. When a directed rooted tree has an orientation away from the root, it is called an arborescence, branching, or out-tree; when it has an orientation towards the root, it is called an anti-arborescence or in-tree.

2 Examples

2.1 Branching

2.1.1 Simple branching

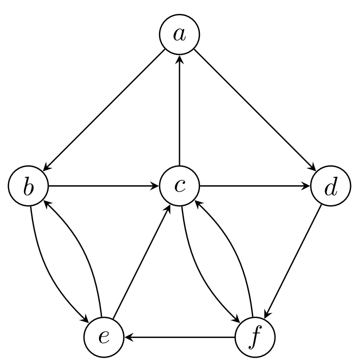

Figure 1: A digraph $D = (A, V)$

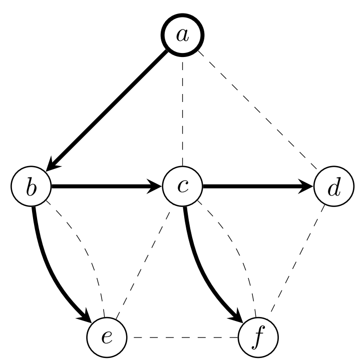

Figure 2: A branching of D routed at a.

2.1.2 Edge disjoint branching

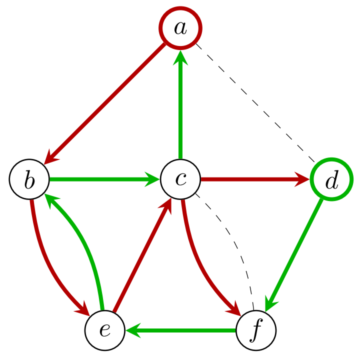

Figure 3: Two edge-disjoint branching routed respectively at a and d.

2.1.3 Edge disjoint a-routed branching

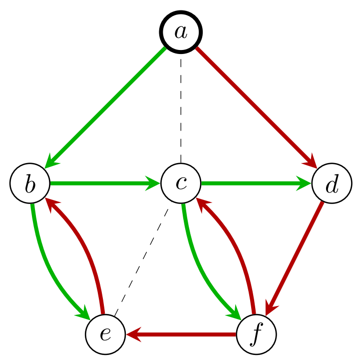

Figure 4: Two edge-disjoint a-routed branching.

2.2 a-cut exemples

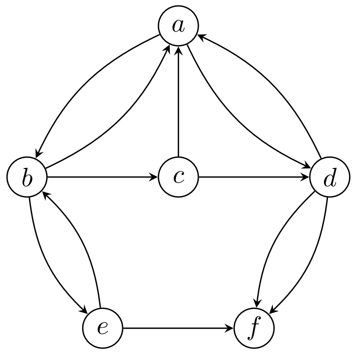

Figure 5: A digraph $D = (A, V)$.

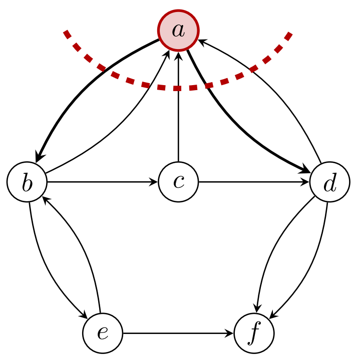

Figure 6: An a-cut of D,
$S = \{a\}$,
$\Delta_D(S) = \{ab, ad\}$,
$\delta_D(S) = 2$.

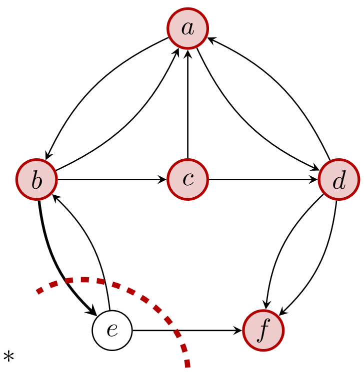

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Figure 7: An minimal a-cut of D,
$S = \{a, b, c, d, f\}$,
$\Delta_D(S) = \{be\}$,
$\delta_D(S) = 1$.

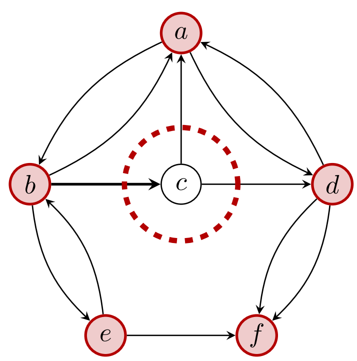

Figure 8: An minimal a-cut of D,
$S = \{a, b, d, e, f\}$,
$\Delta_D(S) = \{bc\}$,
$\delta_D(S) = 1$.

3 Theorems

THEOREM 1 (Edmonds)**.** *The maximum number of edge-disjoint branchings (rooted at a) equals the minimum number of edges in $a-$cuts.*

Proof. Let k represent the number of edges. If $\Delta_G(S) \geq k \; \forall \; S \subset V(G), a \in S$ then there are k edge-disjoint branchings.

Let F be a set of edges such that F is an arborescence rooted at a. Then we want to show that $\Delta_{G-F}(S) \geq k - 1 \; \forall \; S \subset V(G)$, $a \in S$.

If $F = E(G)$ then it is a branching and we are done, else $G-F$ contains $k-1$ edge-disjoint branchings and F is in the kth edge-disjoint branching. Following from this, we can see that F only covers a set of vertices $T \subset V(G)$. Therefore, there exist a vertex v that is not connected to F and we can add an edge $e \in \Delta_G(T)$. This will yield an arborescence $F+e$ such that F covers all points of G, satisfying $\Delta_{G-F}(S) \geq k-1 \ \forall \ S \subset V(G)$, $a \in S$.

If we have a maximal set $A \subset V(G)$ such that A contains the root a, $A \cup T \neq V(G)$ and $\delta_{G-F}(S) = k-1$. If no such A exist then we have set A which is not maximal. Then we have $\delta_{G-F}(A \cup T) > \delta_{G-F}(A)$ that we can add a new edge $e = (x, y)$ which belongs to $\Delta_{G-F}(A \cup T) - \Delta_{G-F}(A)$ such that $x \in T - A$ and $y \in V(G) - T - A$.

Now we want to show that that we can add e to F and $F+e$ will be an arborescence and satisfy $\Delta_{G-F}(S) \geq k-1$. For sure, we can see that F is an arborescence. If $e \notin \Delta_G(S)$ then

$$\delta_{G-F-e}(S) = \delta_{G-F}(S) \geq k-1$$

Now, if we have that $e \in \Delta_G(S)$ then $x \in S$ and $y \in V(G) - S$. That is, x and y are in different set of vertices of G. Next, we will use the inequality;

$$\delta_{G-F}(S \cup A) + \delta_{G-F}(S \cap A) \leq \delta_{G-F}(S) + \delta_{G-F}(A) \tag{1}$$

We already know from the maximality of A that $\delta_{G-F}(A) = k-1$, $\delta_{G-F}(S \cap A) \geq k-1$ and $\delta_{G-F}(S \cup A) \geq k$. Then from (1) we have $\delta_{G-F}(S) \geq k$. Therefore $\delta_{G-F-e}(S) \geq k-1$.

□