On Two Minimax Theorems in Graph

Daniel Blévin - George Apaaboah

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First theorem

Branching

branching (r-arborescence) : In a directed graph $D = (V, A)$ a branching is a set of arcs not containing circuits s.t. each node of D is entered at most by one arc in A'. (So the arc in A' make up a forest).

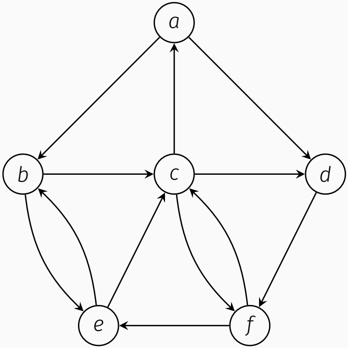

Figure 1: A digraph $D = (A, V)$

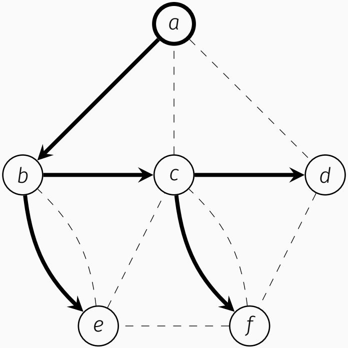

Figure 2: A branching of D routed at a.

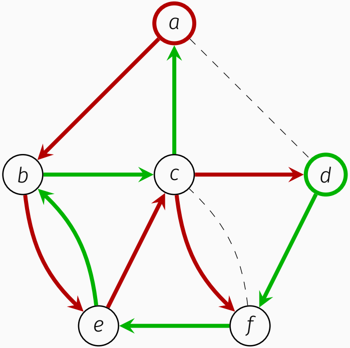

Figure 3: Two edge-disjoint branching routed respectively at a and d.

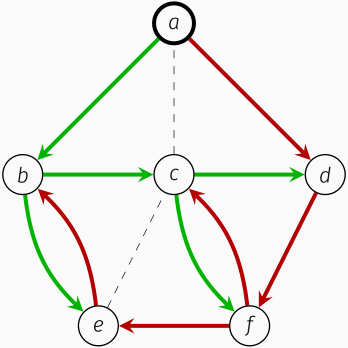

Figure 4: Two edge-disjoint a-routed branching.

a-cut

a-cut : A $a-$cut of G determined by a set $S \subset V(G)$ is the set of edges going from S to $V(G) - S$. It will be denoted by $\mathbf{\Delta}_G(S)$. We also set that $\delta_G(S) = |\mathbf{\Delta}_G(S)|$.

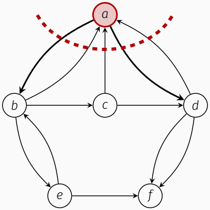

Figure 5: A a-cut of D, $S = \{a\}$, $\mathbf{\Delta}_D(S) = \{ab, ad\}$, $\delta_D(S) = 2$.

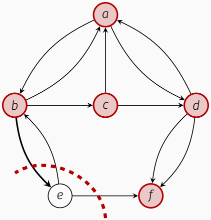

Figure 6: A minimal a-cut of D, $S = \{a, b, c, d, f\}$, $\mathbf{\Delta}_D(S) = \{be\}$, $\delta_D(S) = 1$.

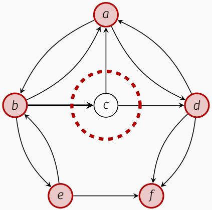

Figure 7: A minimal a-cut of D, $S = \{a, b, d, e, f\}$, $\mathbf{\Delta}_D(S) = \{bc\}$, $\delta_D(S) = 1$.

THEOREM (Edmonds)
The maximum number of edge-disjoint branchings (rooted at a) equals the minimum number of edges in a—cuts.

Proof

Let k be the minimum cardinality of an a-cut.
We prove by induction on k. The case $k = 0$ is trivial.

Let F be a set of edges such that F is an arborescence rooted at a.

$$\Delta_{G-F}(S) \geq k - 1, \ \forall\, S \subset V(G), \ a \in S \tag{1}$$

If (1), we are done by induction. Suppose that (1) holds.

If $F = E(G)$ then it is a branching and we are done ($G - F$ contains $k - 1$ edge-disjoint branchings and F is in the k^{th} edge-disjoint branching).

If $F \neq E(G)$ and only cover a set $T \subset V(G)$ we show we can add an edge $e \in \Delta_G(T)$ to F so that $F + e$ still satisfies (1).

Proof

Consider a maximal set $A \subset V(G)$ s.t :

$$a \in A \tag{2a}$$

$$A \cup T \neq V(G) \tag{2b}$$

$$\delta_{G-F}(S) = k - 1 \tag{2c}$$

If no such A exists, any edge of $\mathbf{\Delta}_G(T)$ can be added to F.

Otherwise let $e = (x, y) \in \mathbf{\Delta}_{G-F}(A \cup T) - \mathbf{\Delta}_{G-F}(A)$ s.t. $x \in T - A$ and $y \in V(G) - T - A$ as shown in **figure 8**.
Such an edge must exist because :

$$\delta_{G-F}(A \cup T) = \delta_G(A \cup T) \geq k > k - 1 = \delta_{G-F}(A)$$

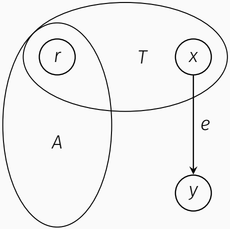

Figure 8

We claim e can be added F s.t. $F + e$ is an arborescence (trivial) and (1) holds.

If $e \notin \mathbf{\Delta}_G(S)$ then

$$\delta_{G-F-e}(S) = \delta_{G-F}(S) \geq k - 1$$

Now, if we have that $e \in \mathbf{\Delta}_G(S)$ then $x \in S$ and $y \in V(G) - S$. That is, x and y are in different set of vertices of G. Next, we will use the inequality;

$$\delta_{G-F}(S \cup A) + \delta_{G-F}(S \cap A) \leq \delta_{G-F}(S) + \delta_{G-F}(A) \tag{3}$$

From the maximality of A : $\delta_{G-F}(A) = k - 1$, $\delta_{G-F}(S \cap A) \geq k - 1$ and $\delta_{G-F}(S \cup A) \geq k$.
Then from (1) we have $\delta_{G-F}(S) \geq k$.
Therefore $\delta_{G-F-e}(S) \geq k - 1$. $\square$

Second theorem

Definitions

directed cut (dicuts) : A directed cut of a weakly connect graph G is the set $D = \Delta_G(S),\ (S \subset V(G),\ S \neq \varnothing)$ provided $\Delta_G(S),\ (V(G) - S) = \varnothing$.

dijoin : A dijoin is a subset $A' \subset A$ which covers all dicuts. Let $\Omega(D)$ denote the maximum number of arc-disjoint dicuts in D and let $\epsilon(D)$ be the minimum cardinality of a dijoin.

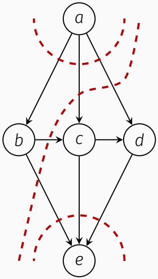

Figure 9: A weakly connected digraph containing 3 dicuts

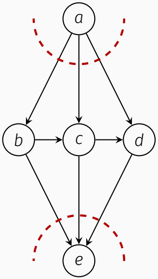

Figure 10: Arc-disjoint dicuts $\Omega(G) = 2$

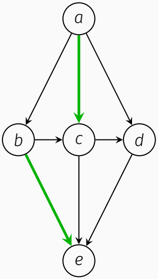

Figure 11: Dijoin of minimal cardinality $\epsilon(G) = 2$

Theorem

THEOREM (Lucchesi and Younger)
The maximum number of disjoint directed cuts equals the minimum number of edges which cover all directed cuts ($\Omega(G) = \epsilon(G)$).

We will use induction on the number of edges.

Case 1: If there are no edges, then the theorem is proved.

Case 2: Let k be the maximum directed cuts in G, G_e'' a digraph resulting from contraction of e. If G_e'' contains at most $k-1$ edge-disjoint directed cuts then, by the induction hypothesis, $\exists$ $k-1$ edges $e_1, \dots, e_{k-1}$ covering all directed cuts of G_e''. From above, we observe that adding e yields $e, e_1, \dots, e_{k-1}$ which is k edges and cover all directed cuts of G. Since we need at least k edges, the assertion is proved and we may assume that G_e'' contains k disjoint directed cuts for each edge e.

If we subdivide all the edges of G by a point, then the graph that we will obtain contains $k+1$ disjoint directed cuts. Hence we can find a subdivision H of G that contains k at most k disjoint directed cuts. If we again subdivide an edge f of H by a point, then it will contain $k+1$ disjoint directed cuts. Hence H contains $k+1$ directed cuts $D_1, \dots, D_{k+1}$ such that only two of them have f as a common edge.

From above, H" f arises from either G or G_f'' by subdivision. Hence from the assumption made above, H contains k disjoint directed cuts $c_1, \dots, c_k$ and f not in c_i. Thus $D_1, \dots, D_{k+1}, c_1, \dots, c_k$ is a collection of directed cuts of G_0 and any edge belongs to at most two of them.

Next, we have the lemma:

LEMMA
If a digraph G contains at most k disjoint directed cuts, and F is any collection of directed cuts in G such that any edge belongs to at most two of them, then $|F| \leq 2k$.

Let $D = \boldsymbol{\Delta}_G(S)$, $(S \subset V(G), S \neq \emptyset)$ provided $\boldsymbol{\Delta}_G(V(G) - S) = \emptyset$. Let S_D be a set uniquely determined by by a directed cut D
First we replace F by a collection of a simple structure. Let $D_1, D_2 \in F$ be called a laminar if $S_{D_1} \cap S_{D_2} = \emptyset$ or $S_{D_1} \subset S_{D_2},$ or $S_{D_2} \subset S_{D_1}$ or $S_{D_1} \cup S_{D_2} = V(G)$. Else D_1, D_2 is a crossing
Let D_1, D_2 be a crossing pair and set the following:

$$D_1' = \boldsymbol{\Delta}_G(S_{D_1} \cup S_{D_2})$$

$$D_2' = \boldsymbol{\Delta}_G(S_{D_1} \cap S_{D_2})$$

$$F' = F \cup \{D_1', D_2'\} - \{D_1, D_2\}$$

From these, we can see that D_1', D_2' are directed cuts and cover any edge the same number of times as D_1, D_2. Hence $|F'| = |F|$.

Also,

$$\sum_{D \in F} |S_D|^2 < \sum_{D \in F'} |S_D|^2$$

since

$$|S_{D_1} \cup S_{D_2}|^2 + |S_{D_1} \cap S_{D_2}| > |S_{D_1}|^2 + |S_{D_2}|^2$$

Hence, we cannot repeat the same process of F for F' and achieve a cycle. We get a collection F_0 of directed cuts such that any two are a laminar and $|F_0| = |F|$.

Next, we prove the *Lemma* for the case when F consist of pairwise laminar cuts.

Let $F = \{D_1, \dots, D_N\}$, then we construct a graph G' as follows:

$V(G') = \{v_1, \dots, v_N\}$ and join v_i to v_j *iff* $D_i \cap D_j \neq \emptyset$. Then G' contains at most k independent points.

Now we will show that G' is bipartite.
Consider a circuit $(v_1, \ldots, v_m)$ in G' and the corresponding sets $S_{D_1}, \ldots, S_{D_m}$. Then $D_1, \ldots, D_m$ must be different. If $D_v = D_\mu$ then each edge of D_v also belongs to D_μ and no other member of F. Hence v has degree 1 and cannot occur in any circuit of G'.

Since $D_i \cap D_{i+1} \neq \emptyset$, we have either $S_{D_i} \subset S_{D_{i+1}}$ or $S_{D_{i+1}} \subset S_{D_i}$

We claim that both possibilities occur alternatively and this will prove that m is even. Suppose not, example, $S_{D_0} \subset S_{D_1} \subset S_{D_2}$.

We say D_i is to the left from D_j if either $S_{D_i} \subset S_{D_j}$ or $V(G) - S_{D_i} \subset S_{D_j}$.

D_i is to the right from D_j if $S_{D_i} \subset V(G) - S_{D_j}$ or $V(G) - S_{D_i} \subset V(G) - S_{D_j}$.

Now, since F consist of laminar cuts, each $D_i \neq D_j$ is either to the left or to the right from D_j.

Since D_2 is to the right from D_1 but $D_0 = D_m$ is to the left from D_1, there is a j, $1 \leq j \leq m-1$ such that D_j is to the right from D_1 but D_{j+1} is to the left from D_1.

Since D_j and D_{j+1} have a common edge e, which must belong to D_1. Hence e belongs to three cuts, which is a contradiction.

$\square$

Definitions : A *hypergraph* H is a finite collection of finite sets. These sets are called edges and the elements of the edges are called vertices. Let $V(H)$ denote the set of vertices. If $E_1, \ldots, E_m$ are the edges and $v_1, \ldots, v_n$ are the vertices of H, then we define

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \in E_j \\ 0, & \text{otherwise} \end{cases}$$

Theorem (Hypergraph)

If any hypergraph H' arising from H by multiplication of the vertices satisfies $v_2(H') = 2v(H')$ then $\tau(H) = v(H)$.

Proof.

First we show that

1. If F is a collection of pairwise laminar directed cut then its incidence matrix A is totally unimodular.

□