STATS 244 Homework 3

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Section 6.6: Exercise 1

Compare the optimal execution rule N^* of Bertsimas and Lo with that of Obizhaeva and Wang (OW) under the model (6.1) by using the performance measures VWAP and TWAP.

Part I

The discrete-time OW rule can be obtained by replacing $(N-1)(1-e^{-\rho\tau})$ in the continuous-time solution in Section 6.3.4 by $\rho(T-2)$. Show that

$$\mathrm{OW}_1^* = \mathrm{OW}_T^* = \frac{\tilde{N}}{\rho(T-2)+2} \text{ and } \mathrm{OW}_t^* = \frac{\rho\tilde{N}}{\rho(T-2)+2}$$

for $t = 2, \ldots, T-1$.

Proof. From (6.3.4),

$$\xi_0^* = \xi_N^* = \frac{x_0}{(N-1)(1-e^{-\rho\tau})+2}.$$

Substituting $(N-1)(1-e^{-\rho\tau})$ by $\rho(T-2)$, we obtain

$$\xi_0^* = \xi_N^* = \frac{x_0}{\rho(T-2)+2}.$$

Then we have by Obizhaeva and Wang (OW),

$$\mathrm{OW}_1^* = \mathrm{OW}_T^* = \frac{\tilde{N}}{\rho(T-2)+2}.$$

For $i = 1, \ldots, N-1$, from (6.3.4),

$$\xi_i^* = \frac{x_0 - 2\xi_0^*}{N-1}.$$

Thus, for OW, we have for $t = 2, \dots, T-1$,

$$
\begin{aligned}
OW_t^* &= \frac{\tilde{N} - \frac{2\tilde{N}}{\rho(T-2)+2}}{T-2} \\
&= \frac{\tilde{N}\rho(T-2) + 2\tilde{N} - 2\tilde{N}}{[\rho(T-2)+2](T-2)} \\
&= \frac{\tilde{N}\rho(T-2)}{[\rho(T-2)+2](T-2)} \\
&= \frac{\rho\tilde{N}}{\rho(T-2)+2}
\end{aligned}
$$

□

Part II

Show also that $VWAP_{OW} \neq TWAP_{OW}$ in general.

Claim: $\mathrm{VWAP_{OW}} = \mathrm{TWAP_{OW}}$ if $\rho = 1$.

Proof. We have that

$$
\begin{aligned}
\mathrm{VWAP_{OW}} &= \frac{\sum_{i=1}^{T} N_i P_i}{\sum_{i=1}^{T} N_i} \\
&= \frac{N_1 P_1 + \sum_{i=2}^{T-1} N_i P_i + N_T P_T}{\tilde{N}} \\
&= \frac{\frac{\tilde{N}}{\rho(T-2)+2}\left(P_1 + \rho\sum_{i=2}^{T-1} P_i + P_T\right)}{\tilde{N}} \\
&= \frac{P_1 + \rho\sum_{i=2}^{T-1} P_i + P_T}{\rho(T-2)+2}
\end{aligned}
$$

If $\rho = 1$,

$$
\begin{aligned}
\mathrm{VWAP_{OW}} &= \frac{P_1 + \rho\sum_{i=2}^{T-1} P_i + P_T}{\rho(T-2)+2} \\
&= \frac{P_1 + \sum_{i=2}^{T-1} P_i + P_T}{(T-2)+2} \\
&= \frac{\sum_{i=1}^{T} P_i}{T} \\
&= \mathrm{TWAP_{OW}}
\end{aligned}
$$

Otherwise,

$$\text{VWAP}_{\text{OW}} = \frac{P_1 + \rho \sum_{i=2}^{T-1} P_i + P_T}{\rho(T-2)+2} \neq \frac{P_1 + \sum_{i=2}^{T-1} P_i + P_T}{(T-2)+2} = \text{TWAP}_{\text{OW}}$$

□

Part III

Moreover, investigate the relationship between $\mathbb{E}[VWAP_{OW}]$ and $\mathbb{E}[VWAP_{N^*}]$ under different values of ρ.

From Part II,

$$\text{VWAP}_{\text{OW}} = \frac{P_1 + \rho \sum_{i=2}^{T-1} P_i + P_T}{\rho(T-2)+2}$$

Thus,

$$\begin{aligned}\mathbb{E}\Big[\text{VWAP}_{\text{OW}}\Big] &= \mathbb{E}\left[\frac{P_1 + \rho \sum_{i=2}^{T-1} P_i + P_T}{\rho(T-2)+2}\right] \\ &= \frac{\mathbb{E}[P_1] + \rho \sum_{i=2}^{T-1} \mathbb{E}[P_i] + \mathbb{E}[P_T]}{\rho(T-2)+2}\end{aligned}$$

Section 6.6: Exercise 4

Show that the optimal trading strategy is again deterministic in the sense that it is independent of P_t. Provide details to derive the optimal solution $N^*_{T-k} = \delta_{Y,k} Y_{T-k} + \delta_{X,k} X_{T-k},$, where

$$\delta_{Y,k} = \frac{1}{1+k},\ \ \delta_{X,k} = \frac{\rho b_{k-1}}{2a_{k-1}},\ \ a_k = \frac{\theta}{2}\left(1 + \frac{1}{k+1}\right),$$

$$b_k = \gamma + \frac{\theta \rho b_{k-1}}{2a_{k-1}}\ \ (> 0,\ \text{if } \theta, \rho, \gamma > 0),\ \ b_0 = \gamma.$$

Section 7.9: Exercise 2

(a)

Let X be a random variable on $\mathbb{Z}_+$ with $\mathbb{E}[X] < \infty$. Show that $\mathbb{E}[X] = \sum_{s=1}^{\infty} P(X \geq s)$.

Proof. We begin with the definition of probability,

$$P(X \geq s) = \sum_{i=s}^{\infty} p_X(i).$$

Summing these probabilities and adjusting summations, we obtain

$$
\begin{aligned}
\sum_{s=1}^{\infty} P(X \geq s) &= \sum_{s=1}^{\infty} \sum_{i=s}^{\infty} p_X(i). \\
&= \sum_{i=1}^{\infty} \sum_{s=1}^{i} p_X(i) \\
&= \sum_{i=1}^{\infty} i p_X(i) \\
&= \mathbb{E}[x].
\end{aligned}
$$

□

(b)

Using (a) or otherwise, show that $\mathbb{E}[\min(n, U_t^i)] = \sum_{s=1}^{n} T_i(s)$ *in Section 7.6.1*

Proof. For an arbitrary venue, i, we can perform this proof algebraically. That is, if we redefine the probability measure as

$$\mathbb{E}[\min(n, U_t^i)] = \mathbb{E}_s[\min(n, s)],$$

we have,

$$
\begin{aligned}
\mathbb{E}_s[\min(n, U_t^i)] &= \sum_{s=1}^{\infty} P_i(s) \min(s, n) \\
&= \sum_{s=1}^{n-1} s P_i(s) + n T_i(n) \\
\text{Noting that for any } s, & \\
P_i(s-1) + T_i(s) = T_i(s-1), \text{ we have} & \\
(s-1)P_i(s-1) + sT_i(s) = (s-1)T_i(s-1) + T_i(s), & \\
&= T_i(1) + \ldots + T_i(n-1) + T_i(n) \\
&= \sum_{s=1}^{n} T_i(s).
\end{aligned}
$$

□

(c)

Prove by induction that the optimal solution of (7.20) is provided by the greedy algorithm in Section 7.6.1. (Hint: mathematical induction)

Proof. We begin with the initial case. If $v_t = 1$, then the optimal venue for placing the share is

$$j_1 = \text{arg max}_{i \in 1:K} T_i(1).$$

Thus, optimizing n_t results in a solution that satisfies $n_t^{j_1} = 1$ and $n_t^i = 0$ for $i \neq j$.
For the arbitrary case, we assume the k case is true.

Then it follows that if $v_t = k + 1$, then using $j_1 = \text{arg max}_{i \in 1:K} T_i(1)$ as the venue for placing the first share, $j_1 = \text{arg max}_{i \in 1:K} T_i(1)$, for the second share, $\ldots$, and $j_k = \text{arg max}_{i \in 1:K} T_i(k)$ as the venue for placing the k-th share is optimal, where $v_i = 1$ if $i = j_1$ and $v_i = 0$ otherwise.

We can see that proceeding in this way gives us the optimal allocation rule $n_{t,\text{opt}}$ that maximizes equation (7.20).

Thus, we have shown that the greedy algorithm stays ahead by showing that the partial solutions for each segments are no worse than the initial segments of the previous solution based on the probability measure we used. In addition, by design, we have shown optimality of the solution. Thus, the optimal solution of (7.20) is provided by the greedy algorithm in section 7.6.1. $\square$