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CMPE 320: HW 07

1. A radar tends to overestimate the distance of an aircraft, and the error is a normal random variable with a mean of 50 meters and a standard deviation 100 meters. What is the probability that the measured distance will be smaller than the true distance?

$$\begin{aligned}P(X < 0) &= P\left(Y < \frac{x - \mu}{\sigma}\right) \\&= P\left(Y < \frac{0 - 50}{100}\right) \\&= P(Y < -0.5) \\&= \Phi(-0.5) \\&= 1 - \Phi(0.5) \\&= 0.3085\end{aligned}$$

□

2. Let X be normal with mean 1 and variance 4. Let $Y = 2X + 3$.

(a) Calculate the PDF of Y .

$$\begin{aligned}E[Y] &= E[2X + 3] \\&= 2E[X] + 3 \\&= 2(1) + 3 \\&= 5\end{aligned}$$

$$\begin{aligned}\text{var}(Y) &= \text{var}(2X + 3) \\&= (2)^2 \text{var}(X) \\&= (2)^2(4)\end{aligned}$$

$$= 16$$

The PDF of the normal random variable is

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-5)^2}{32}}, & \text{if } -\infty < y \leq \infty, \\ 0, & \text{otherwise,} \end{cases} \quad \square$$

(b) Find $P(Y \geq 0)$.

$$\begin{aligned} P(Y \geq 0) &= P\left(Z \geq \frac{y - \mu}{\sigma}\right) \\ &= P\left(Z \geq \frac{0 - 5}{\sqrt{16}}\right) \\ &= P(Z \geq -1.25) \\ &= \Phi(-1.25) \\ &= 1 - \Phi(1.25) \\ &= 0.1056 \end{aligned} \quad \square$$

3. A signal of amplitude $s = 2$ is transmitted from a satellite but is corrupted by noise, and the received signal is $X = s + W$, where W is noise. When the weather is good, W is normal with zero mean and variance 1. When the weather is bad, W is normal with zero mean and variance 4. In the absence of any weather information:

(a) Calculate the PDF of X .

Let G represent good weather, and B represent bad weather

where $P(G) = P(B) = \frac{1}{2}$

We are given $W \sim N(0, 1)$ given G , and $W \sim N(0, 4)$ given B

Therefore, the unconditional PDF pf W ,

$$\begin{aligned} f_W(w) &= P(G) \cdot f_{W|G}(w | G) + P(B) \cdot f_{W|B}(w | B) \\ &= \frac{1}{2} \cdot \frac{1}{\sqrt{(1)2\pi}} e^{-\frac{w^2}{(1)^2}} + \frac{1}{2} \cdot \frac{1}{\sqrt{(4)2\pi}} e^{-\frac{w^2}{(4)^2}} \\ &= \frac{1}{2\sqrt{2\pi}} \left(e^{-\frac{w^2}{2}} + \frac{1}{2} e^{-\frac{w^2}{(4)^2}} \right) \end{aligned}$$

$$= \frac{1}{2\sqrt{2\pi}} \left(e^{-\frac{w^2}{2}} + \frac{1}{2} e^{-\frac{w^2}{8}} \right)$$

Since $X = W + 2$,

$$f_X(x) = \frac{1}{2\sqrt{2\pi}} \left(e^{-\frac{(x-2)^2}{2}} + \frac{1}{2} e^{-\frac{(x-2)^2}{8}} \right)$$

□

(b) Calculate the probability that X is between 1 and 3.

$$P(1 \leq X \leq 3) = P(1 \leq W + 2 \leq 3) = P(-1 \leq W \leq 1)$$

Using the total probability theorem,

$$\begin{aligned} P(-1 \leq W \leq 1) &= P(G)P(-1 \leq W \leq 1 \mid G) + P(B)P(-1 \leq W \leq 1 \mid B) \\ &= P(G)(N(0, 1)) + P(B)(N(0, 4)) \\ &= P(G)(\Phi(1) - \Phi(-1)) + P(B)\left(\Phi\left(\frac{1}{2}\right) - \Phi\left(-\frac{1}{2}\right)\right) \\ &= P(G)(\Phi(1) - (1 - \Phi(1))) + P(B)\left(\Phi\left(\frac{1}{2}\right) - \left(1 - \Phi\left(\frac{1}{2}\right)\right)\right) \\ &= P(G)(2\Phi(1) - 1) + P(B)\left(2\Phi\left(\frac{1}{2}\right) - 1\right) \\ &= \frac{1}{2}\left(2\Phi(1) - 1\right) + \frac{1}{2}\left(2\Phi\left(\frac{1}{2}\right) - 1\right) \\ &= \Phi(1) - \frac{1}{2} + \Phi\left(\frac{1}{2}\right) - \frac{1}{2} \\ &= \Phi(1) + \Phi\left(\frac{1}{2}\right) - 1 \\ &= 0.8413 + 0.6915 - 1 \\ &= 0.5328 \end{aligned}$$

□

4. Oscar uses his high-speed modem to connect to the Internet. The modem transmits zeros and ones by sending signals -1 and $+1$, respectively. We assume that any given bit has probability p of being a zero. The network cable introduces additive zero-mean Gaussian noise with variance σ^2 (so, the receiver at the other end receives a signal which is the sum of the transmitted signal and the channel noise). The value of

the noise is assumed to be independent of the encoded signal value.

- (a) Let a be a constant between -1 and 1 . The receiver at the other end decides that the signal -1 (respectively, $+1$) was transmitted if the value it receives is less (respectively, more) than a . Find a formula for the probability of making an error.

□

- (b) Find a numerical answer for the question of part (a) assuming that $p = 2/5$, $a = 1/2$ and $\sigma^2 = 1/4$.

□

5. An old modem can take anywhere from 0 to 30 seconds to establish a connection, with all times between 0 and 30 being equally likely.

- (a) What is the probability that if you use this modem you will have to wait more than 15 seconds to connect?

For the uniformly distributed normal variable

$$\int_{15}^{30} \frac{1}{30} dx = \frac{1}{30} x \Big|_{15}^{30}$$

$$= 0.5$$

□

- (b) Given that you have already waited 10 seconds, what is the probability of having to wait at least 10 more seconds?

$$\int_{20}^{30} \frac{1}{20} dx = \frac{1}{20} x \Big|_{20}^{30}$$

$$= 0.5$$

□

6. Consider a random variable X with PDF

$$f_X(x) = \begin{cases} 2x/3, & \text{if } 1 < x \leq 2, \\ 0, & \text{otherwise,} \end{cases}$$

and let A be the event $\{X \geq 1.5\}$. Calculate $E[X]$, $P(A)$ and $E[X | A]$.

$$\begin{aligned}
E[x] &= \int xP(x) \, dx \\
&= \int_1^2 x \frac{2x}{3} \, dx \\
&= \frac{2}{3} \frac{x^3}{3} \Big|_1^2 \\
&= \frac{14}{9}
\end{aligned}$$

$$\begin{aligned}
P(x) &= \int_{1.5}^2 \frac{2x}{3} \, dx \\
&= \frac{2}{3} \frac{x^2}{2} \Big|_{1.5}^2 \\
&= \frac{7}{12}
\end{aligned}$$

$$P(X \mid A) = P(X \mid X \geq 1.5) = 1$$

$$\begin{aligned}
E[X \mid A] &= \int_1^2 1x \, dx \\
&= \frac{x^2}{2} \Big|_1^2 \\
&= \frac{3}{2}
\end{aligned}$$

□

7. Dino, the cook, has good days and bad days with equal frequency. On a good day, the time (in hours) it takes Dino to cook a souffle is described by the PDF

$$f_G(g) = \begin{cases} 2, & \text{if } 1/2 < g \leq 1, \\ 0, & \text{otherwise,} \end{cases}$$

but on a bad day, the time it takes is described by the PDF

$$f_B(b) = \begin{cases} 1, & \text{if } 1/2 < b \leq 3/2, \\ 0, & \text{otherwise,} \end{cases}$$

Find the conditional probability that today was a bad day, given that it took Dine less than three quarters of an hour to cook a souffle.

Let X represent the condition that it took Dine less than $3/4$ of an hour to cook a souffle

Then

$$P(B | X) = \frac{P(X | B)P(B)}{P(X | B)P(B) + P(X | G)P(G)}$$

Since

$$\begin{aligned} P(X | B) &= \int_{1/2}^{3/4} 1 \, db \\ &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned} P(X | G) &= \int_{1/2}^{3/4} 2 \, dg \\ &= \frac{1}{2} \end{aligned}$$

Therefore,

$$\begin{aligned} P(B | X) &= \frac{\frac{1}{4} \frac{1}{2}}{\frac{1}{4} \frac{1}{2} + \frac{1}{2} \frac{1}{2}} \\ &= \frac{1}{3} \end{aligned}$$

□

8. One of the two wheels of fortune, A and B , is selected by the toss of a fair coin, and the wheel chosen is spun once to determine the value of a random variable X . If wheel A is selected, the PDF of X is

$$f_{X|A}(x | A) = \begin{cases} 1, & \text{if } 0 < x \leq 1, \\ 0, & \text{otherwise,} \end{cases}$$

If wheel B is selected, the PDF of X is

$$f_{X|B}(x|B) = \begin{cases} 3, & \text{if } 0 < x \leq 1/3, \\ 0, & \text{otherwise,} \end{cases}$$

If we are told that the value of X was less than $1/4$, what is the conditional probability that wheel A was the one selected.

$$\begin{aligned} P\left(x \leq \frac{1}{4} \mid A\right) &= \int_0^{1/4} f_{X|A}(x|A) \, dx \\ &= \int_0^{1/4} 1 \, dx \\ &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned} P\left(x \leq \frac{1}{4} \mid B\right) &= \int_0^{1/4} f_{X|B}(x|B) \, dx \\ &= \int_0^{1/4} 3 \, dx \\ &= \frac{3}{4} \end{aligned}$$

$$\begin{aligned} P\left(A \mid x \leq \frac{1}{4}\right) &= \frac{P\left(x \leq \frac{1}{4} \mid A\right)P(A)}{P\left(x \leq \frac{1}{4} \mid A\right)P(A) + P\left(x \leq \frac{1}{4} \mid B\right)P(B)} \\ &= \frac{\frac{1}{4} \cdot \frac{1}{2}}{\frac{1}{4} \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2}} \\ &= \frac{1}{4} \end{aligned}$$

□