Lecture 3: Math Refresher





## CMPE323 is math intensive

- Complex numbers
- Differential Equations
- Complex Calculus

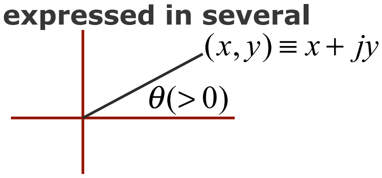
- You should know all of this, but we'll review it anyway

- Weaknesses will soon be apparent!

## Complex Numbers

- **Mathematicians refer to the complex numbers as "the extension" of real numbers.**
- **A complex number consists of an "ordered pair" of real numbers, which can be expressed in several ways**

$$z = \underbrace{(x, y)}_{\text{vector}} = \underbrace{x + jy}_{\text{rectangular}} = \underbrace{re^{j\theta}}_{\text{polar}}$$




- **The "tag"** $j$ or $i$ **has the following properties**

$$j^2 = j \times j = -1, \quad \frac{1}{j} = -j$$

- **The magnitude** $|z| = \sqrt{x^2 + y^2}$
- **The phase**

$$\angle z = \tan^{-1}\left(\frac{y}{z}\right) \text{ where } \tan^{-1} \text{ is the 4 quadrant arctangent}$$





## Euler's Identity

- **Nobel-winning physicist Richard Feynman called this "the most amazing equation in mathematics"**

$$e^{j\theta} = \cos\theta + j\sin\theta$$

- **From which we can see**

$$re^{j\theta} = r\cos\theta + jr\sin\theta$$

$$\left|re^{j\theta}\right| = \sqrt{\left(r^2\cos^2\theta + r^2\sin^2\theta\right)} = \sqrt{r^2\left(\cos^2\theta + \sin^2\theta\right)} = r^2$$

$$\angle(re^{j\theta}) = \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta}\right) = \tan^{-1}\left(\tan\theta\right) = \theta$$

- **And** $z_1 = r_1 e^{j\theta_1}, z_2 = r_2 e^{j\theta_2} \Rightarrow z = z_1 z_2$

$$= r_1 r_2 e^{j\theta_1} e^{j\theta_2} = r_1 r_2 e^{j(\theta_1 + \theta_2)}$$

$$= r_1 r_2 \left(\cos(\theta_1 + \theta_2) + j\sin(\theta_1 + \theta_2)\right)$$

## Euler's Identity and Phasors

$$e^{(\sigma+j\omega)t} = e^{\sigma t}e^{j\omega t} = e^{\sigma t}\left(\cos\omega t + j\sin\omega t\right)$$

$$e^{-(\sigma+j\omega)t} = e^{-\sigma t}e^{-j\omega t} = e^{-\sigma t}\left(\cos(-\omega t) + j\sin(-\omega t)\right) = e^{-\sigma t}\left(\cos\omega t - j\sin\omega t\right)$$

For sinusoid $v(t) = V_m\cos(\omega t + \phi)$

$$\Rightarrow v(t) = V_m\cos(\omega t + \phi) = \mathrm{Re}\left[V_m e^{j(\omega t+\phi)}\right]$$

If $\mathbf{V} = V_m e^{j\phi} = V_m\angle\phi$

[phasor representation of sinusoid $v(t)$]

$$\Rightarrow v(t) = \mathrm{Re}\left(\mathbf{V}e^{j\omega t}\right)$$





## Complex Number Operations

$$z_1 = x_1 + jy_1 = \sqrt{x_1^2 + y_1^2}\angle\tan^{-1}\frac{y_1}{x_1} = r_1\angle\phi_1 \text{ and } z_2 = x_2 + jy_2 = r_2\angle\phi_2$$

Addition: $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$

Subtraction: $z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$

Multiplication: $z_1 z_2 = (x_1x_2 - y_1y_2) + j(x_1y_2 + x_2y_1)$

$$= r_1 r_2\angle(\phi_1 + \phi_2)$$

Division: $\dfrac{z_1}{z_2} = \dfrac{r_1}{r_1}\angle\phi_1 - \phi_2$

Reciprocal: $\dfrac{1}{z} = \dfrac{1}{r}\angle -\phi$

Square Root: $\sqrt{z} = \sqrt{r}\angle\phi/2$

Complex Conjugate: $z^* = x - jy = re^{-j\phi} = r\angle -\phi$

**Add and subtract using the rectangular form**

**Multiply and divide using the polar form**

## Differentiation with complex numbers

$$z(t) = A(t)e^{j(\omega t+\theta(t))}$$

$$\frac{dz}{dt} = \frac{dA}{dt}e^{j(\omega t+\theta(t))} + A(t)j\left(\frac{d}{dt}(\omega t+\theta(t))\right)e^{j(\omega t+\theta(t))}$$

$$= \frac{dA}{dt}e^{j(\omega t+\theta(t))} + A(t)j\left(\omega+\frac{d\theta}{dt}\right)e^{j(\omega t+\theta(t))}$$

$$z'(t) = \left[A'(t) + jA(t)\big(\omega+\theta'(t)\big)\right]e^{j(\omega t+\theta(t))}$$

$$\text{If } A(t) = A_0,\ \theta(t) = \theta_0,\ z'(t) = j\omega A_0 e^{j(\omega t+\theta_0)}$$

- **Apply your usual product and chain rule process**





## Integration and complex numbers

$$z(t) = A_0 e^{j(\omega t+\theta_0)}$$

$$\int z(t)\,dt = \int A_0 e^{j(\omega t+\theta_0)}\,dt = A_0 e^{j\theta_0}\int e^{j\omega t}\,dt = \frac{A_0}{j\omega}e^{j(\omega t+\theta_0)}$$

and, similarly

$$\int z(t)\,d\omega = \int A_0 e^{j(\omega t+\theta_0)}\,d\omega = A_0 e^{j\theta_0}\int e^{j\omega t}\,d\omega = \frac{A_0}{jt}e^{j(\omega t+\theta_0)}$$

## Liebnitz Integration Rule in 1 dimension

Let $f(x,y)$ have a partial derivative wrt $x$, $\dfrac{\delta f(x,y)}{\delta x}$ exists

Then

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,y)\,dy = \frac{db}{dx} f(x,b(x)) - \frac{da}{dx} f(x,a(x)) + \int_{a(x)}^{b(x)} \frac{\delta f(x,y)}{\delta x}\,dy$$

- **Example**

$$\text{Let } x(t) = \int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

$$\frac{dx}{dt} = \frac{d}{dt}\left(\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega\right)$$

$$= \frac{d\infty}{dt} X(\infty)e^{j\infty t} - \frac{d(-\infty)}{dt} X(-\infty)e^{-j\infty t} + \int_{-\infty}^{\infty} \frac{\delta}{\delta t}\left(X(\omega)e^{j\omega t}\right)d\omega$$

$$= 0 - 0 + \int_{-\infty}^{\infty} \frac{X(\omega)}{j\omega} e^{j\omega t}\,d\omega = \int_{-\infty}^{\infty} \frac{X(\omega)}{j\omega} e^{j\omega t}\,d\omega$$





- **Example #2**

$$y(t) = \int_{-\infty}^{\infty} x(\tau)x(t-\tau)\,d\tau,\;\; x(\infty) = x(-\infty) = 0$$

$$\frac{dy}{dt} = \frac{d\infty}{dt}\big(x(\infty)x(t-\infty)\big) - \frac{d(-\infty)}{dt}\big(x(-\infty)x(t+\infty)\big) + \int_{-\infty}^{\infty} \frac{\delta}{\delta t}\big(x(\tau)x(t-\tau)\big)\,d\tau$$

$$= 0 - 0 + \int_{-\infty}^{\infty} \big(x(\tau)x'(t-\tau)\big)\,d\tau = \int_{-\infty}^{\infty} \big(x(\tau)x'(t-\tau)\big)\,d\tau$$

## Linear, Constant Coefficient Differential Eqs.

- **A system can be described by an $n^{th}$-order LCCDE with $x(t)$ as the input and $y(t)$ as the output**

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = b_n\frac{d^n x}{dt^n} + b_{n-1}\frac{d^{n-1} x}{dt^{n-1}} + \cdots + b_1\frac{dx}{dt} + b_0 x$$

- **We can solve such equations in the time domain**
  - **Homogeneous and particular solutions (306, 225)**
  - **Zero-input and Zero-state solutions (306, 225)**
  - **Convolution (this course)**
  - **Numerical (MATLAB) solutions (this course)**
- **…Or we can use "transform" techniques**
  - **Fourier Transforms (this course)**
  - **Laplace Transforms (this course and 225)**





## Time Domain

- **Homogeneous and Particular solution**
  - **The particular solution satisfies the LCCDE without necessarily meeting the initial conditions**
  - **The homogenous solution is the solution to the LCCDE with zero input**
  - **Example** $y'(t) + y(t) = 1,\ t \ge 0, y(0) = 0$

$y_p(t) = 1,\ t \ge 0$ (a constant) satisfies the LCCDE

$0 + 1 = 1$ for $t > 0$, but $y_p(0) = 1$

$y_p(t) = 1,\ t \ge 0$ is the *particular* sol'n

Find $y_h(t)$ such that $y(t) = y_p(t) + y_h(t)$ satisfies the init. cond

$y_h'(t) + y_h(t) = 0$ for $t \ge 0$ Choose $y_h(t) = Ae^{-at}$

$-Aae^{-at} + Ae^{-at} = 0$

$A(-a+1) = 0 \Rightarrow -a = 1.$

$y(t) = y_p(t) + y_h(t)$

$y(0) = 0 = 1 + Ae^{-t} \Rightarrow A = -1$

$y(t) = 1 - e^{-t},\ t \ge 0$

## Zero-input and Zero-state

- In this case the solution is the sum of the solution with zero input and the solution with zero initial conditions (zero state) $y(t) = y_{ZI}(t) + y_{ZS}(t)$
- We already have a form for the zero input, that was our particular solution…
- …and, in fact, we also have the zero state solution, because our initial condition is $y(t) = 0.$
- In general, this is not the case.
- For most of the systems in this course, Laplace methods are most effective!
- There's a very good summary of solutions of LCCDE in Chapter 5.1-5.15, 5.16, and 5.17 of your text
- When we get to that subject matter, I'll assume you know how to find the solutions…
- …then we can concentrate on insights, properties, etc..



## Some Practice

$$\frac{d^2y}{dt^2} + 200\frac{dy}{dt} + 10^8 y(t) = 200\frac{dx}{dt},\ y(0^+) = 0,\ \left.\frac{dy}{dt}\right|_{t=0^+} = 200\quad x(t) = 1,\ t > 0$$

Characteristic equation:

$s^2 + 200s + 10^8 = 0,\ s \approx -100 \pm j10^4$

$y_h(t)e^{-100t}(A\cos 10^4 t + B\sin 10^4 t),\ y_p(t) = 0,$ because $\dfrac{dx}{dt} = 0$ for $t > 0.$

$y(0^+) = 0 = 1\times\left(A\times 1 + B\times 0\right) \Rightarrow A = 0$

$\left.\dfrac{dy}{dt}\right|_{t=0^+} = 200 = -100\times 1\times(A\times 1 + B\times 0) + 1\times\left(-10^4\times A\times 0 + 10^4\times B\times 1\right)$

$200 = 10^4 B,\quad 200 = 10^4 B \Rightarrow B = 0.02$

$y(t) \approx 0.02e^{-100t}\sin 10^4 t$ for $t > 0$

Check?

$y(0) = 0,$ as required

$y'(0) = 200,$ as required

Verification that $y(t)$ satisfies the LCCDE left for homework