MATH 407

3/7/18

* Permutations:

Let S be a set.
A permutation of S is bijection $f: S \to S$

$Sym(S)$ = all permutations

1) 1_S is permutation

2) f, g in $Sym(S)$ then so is $g \circ f$

3) $f \in Sym(S)$ implies $f^{-1} \in Sym(S)$

If $k \in \mathbb{Z}$, $f \in Sym(S)$
$f^k \in Sym(S)$

Let S be finite $|S| = n$. TFAE

a) $f \in Sym(S)$

b) f is 1-1

c) f is onto

$S = \mathbb{N}_n = \{1, \dots, n\}$
$S_n = Sym(S)$

* Permutation Notation:

$\sigma \in S_n$
$\sigma \{(1, \sigma(1)), (2, \sigma(2)), \dots, (n, \sigma(n))\}$

(2)

Or:
$$\begin{pmatrix} (1, \sigma(1)) \\ \vdots \\ (n, \sigma(n)) \end{pmatrix} = \begin{pmatrix} 1 & \sigma(1) \\ \vdots & \vdots \\ n & \sigma(n) \end{pmatrix}$$
(removed parenthesis to construct a $2 \times n$ matrix)

Transposed:
$$\begin{pmatrix} 1 & \cdots & n \\ \sigma(1) & & \sigma(n) \end{pmatrix}$$

Ex/
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$$

* arrows represent construction of $\tau \circ \sigma$

$$\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

(*) $\tau \circ \sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix}$

(*) $\sigma^{-1} = \begin{pmatrix} 4 & 3 & 1 & 2 \\ 1 & 2 & 3 & 4 \end{pmatrix}$

(*) $\tau \circ \sigma(1) = 1$, 1 is a fixed point
No othe fixed points

$(\rightarrow)$

* Let $\pi = \tau \circ \sigma$

Then $\pi(2) = 4$
$\pi(4) = \pi^2(2) = 3$
$\pi(3) = \pi^3(2) = 2 = \pi^0(2)$

* $\{f^{-1}(x) = f^0(x), f^1(x), \ldots, f^k(x)\}$ = cycle/orbit of f starting at x

Let $y = f(x)$ (1st iterate)

Then $\{f^{-1}(y), y, f^1(y), f^2(y), \ldots\}$

$y = f^j(x)$ orbit of f starting at y has same elements

* Def. Let $f \in Sym(S)$
Say $x \sim_f y$ (x, y in S)

iff $\exists k \in \mathbb{Z}$ s.t. $f^k(x) = y$
$\sim_f$ is an equivalence relation on S

Symmetry of $\sim_f$: $f^k(x) = y \Rightarrow x = f^{-k}(y)$

Transitivity of $\sim_f$: $f^k(x) = y$,
$f^j(y) = z$ implies $f^{k+j}(x) = z$

Ex. 2.3.4: $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 8 & 2 & 10 & 11 & 5 & 9 & 4 & 6 & 1 & 3 & 12 & 7 \end{pmatrix}$

According to $\sim_f$: $(1, 8, 6, 9)$ = cycle of π started at 1
$\equiv (8, 6, 9, 1)$ = cycle of π started at 8
$\not\equiv (1, 6, 8, 9)$ = cycle of π started at 1

④

* $\{(1,8,6,9),(2),(3,10),(4,11,12,7),(5)\}$:= 5 different cycles of π

Elements of this set forms an equivalence class.

* $1_{\mathbb{N}} = \{(1),(2),\ldots,(n)\}$ \ opposite ends of the spectrum

$n_{\mathbb{N}} = \{(1,2,\ldots,n)\}$ /

(back to example)

* Cycles of len > 1 (ignore fixed points)

$\{(1,8,6,9),(3,10),(4,11,12,7)\}$

* if $\{(1,8,6,9)\}$ (everything else fixed)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 8 & 2 & 3 & 4 & 5 & 9 & 7 & 6 & 1 & 10 & 11 & 12 \end{pmatrix}$$

* Example to show not-commutative:

$\{(1,2)\}$ $\{(1,2,3)\}$ (cycles)

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

Multiplying the 2 matrices:

$[(1,2)\circ(1,2,3)](1)$

$(1,2)(2) = (1)$

$(\longrightarrow)$

$[(1,2)\circ(1,2,3)](2)$

$(1,2)(3) = (3)$

$[(1,2)\circ(1,2,3)](3)$

$(1,2)(1) = (2)$

$\therefore (1,2)\circ(1,2,3)$ is the transposition of $(2,3)$

$[(1,2)\circ(1,2,3)] = (2,3)$

$[(1,2,3)\circ(1,2)] = (1,3)(2) = (1,3)$