Problem Set #3 Solutions

1. We have $\gamma = \alpha + j\beta = [(R' + j\omega L')(G' + j\omega C')]^{1/2}$, and we have $Z_0 = [(R' + j\omega L')/(G' + j\omega C')]^{1/2}$, from which we obtain $R' + j\omega L' = \gamma Z_0 = 0.800 + j30.0$. We immediately find $R' = 0.80\ \Omega/\text{m}$, and we have $L' = 30.0/[6.128 \times (1.25 \times 10^8)] = 3.8 \times 10^{-8}$ H/m $= 38$ nH/m. We have $G' + j\omega C' = \gamma / Z_0 = 5.00 \times 10^{-4} + j0.0187$. We immediately find $G' = 5.0 \times 10^{-4}$ S, and we have $C' = 0.0187/[6.128 \times (1.25 \times 10^8)] = 2.4 \times 10^{-11}$ F/m $= 24$ pF/m. The phase velocity is given by $u_{\text{p}} = \omega/\beta = 1.0 \times 10^9$ m/s, which is several times larger than the speed of light in the vacuum. While it is possible to have waveguides that operate at phase velocities that are greater than the speed of light in the vacuum, it is not possible for the TEM guides that we have been discussing. *Caveat emptor*... You cannot trust everything that you read in textbooks.

2. a. We have $\Gamma = (Z_L - Z_0)/(Z_L + Z_0) = 0.10 + j0.56$.

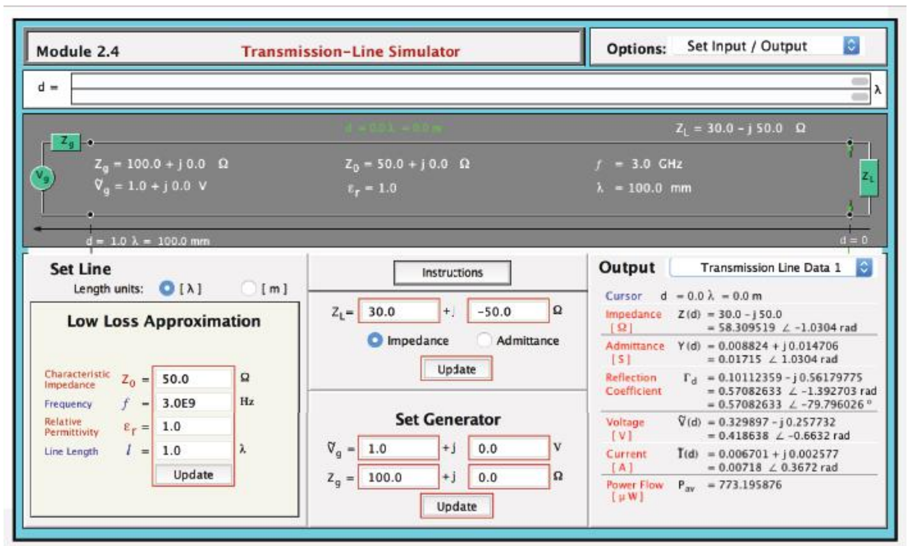
   b. We have $S = (1 + |\Gamma|)/(1 - |\Gamma|) = (1 + 0.571)/(1 - 0.571) = 3.7$.

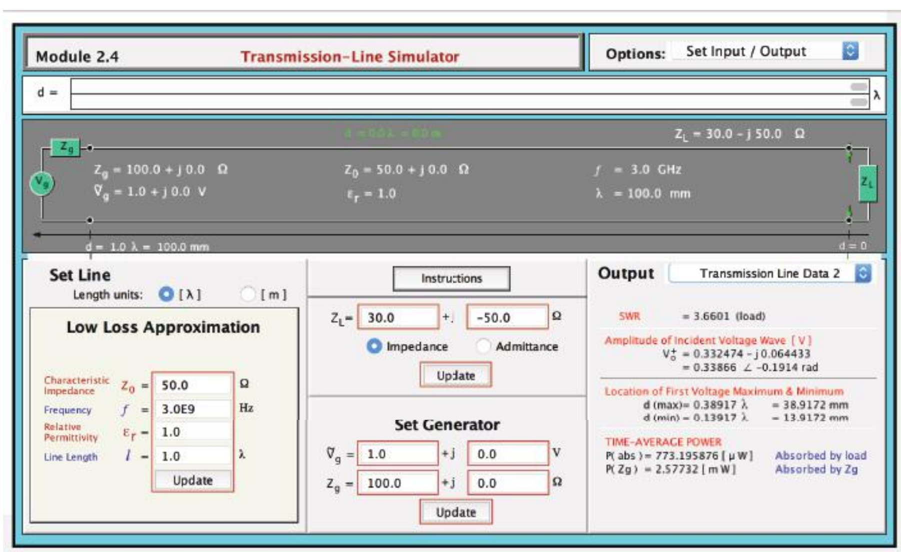
   c. We first note that $\theta_r = -1.39$ rad $= -79.8°$. We also have $\beta = 2\pi/\lambda = 62.8$ m^{-1}. The first maximum is located at $d_{max} = (\theta_r + 2\pi)/2\beta = 3.9$ cm.

   d. The first current maximum is located at the first voltage minimum, which is given by $d_{max} - \lambda/4 = 1.4$ cm.

   e. To obtain a wavelength of $\lambda = 10$ cm in Ulaby et al.'s module 2.4, I set $\epsilon_r = 1$ and the frequency $f = 3.00$ GHz. The screen shots are shown below

ULABY CD MODULE OUTPUT:





3. All the answers can be verified with Ulaby and Ravaioli module 2.4.

   a. We have $\omega = 2\pi f = 3\pi \times 10^9 \ \mathrm{s}^{-1} \ = 9.42 \times 10^9 \ \mathrm{s}^{-1}$. Since we are assuming $\mu = \mu_0$, we have $\epsilon_r = c^2/v_\mathrm{p}^2 = (3 \times 10^8/1.5 \times 10^8)^2 = 4$. We have $z_\mathrm{L} = Z_\mathrm{L}/Z_0 = (100 + j50)/50 = 2.0 + 1.0j$.

   b. We have $\Gamma = (Z_\mathrm{L} - Z_0)/(Z_\mathrm{L} + Z_0) = 0.4 + 0.2j = 0.4472 \exp(j0.4636)$. Hence, we have $|\Gamma| = 0.447$ and $\theta_\mathrm{r} = 0.4636$ rads $= 26.6°$. We also have $S = (1 + |\Gamma|)/(1 - |\Gamma|) = 2.62$.

      To calculate l_max and l_min, we first note that $\lambda = v_\mathrm{p}/f = 1.5 \times 10^8/1.5 \times 10^9 = 0.1$ m $= 10$ cm. We also note, for future reference that $\beta = 2\pi/\lambda = 20\pi \ \mathrm{m}^{-1}$. Maxima occur when $-z = (\theta_\mathrm{r}\lambda/4\pi) + (n\lambda/2) > 0$, or when $(0.4636 + 0.5n)\lambda > 0$. The first maximum occurs when $n = 0$, at which point $l_\mathrm{max} = 0.369$ cm. Since $l_\mathrm{max} < 0.25\lambda = 2.50$ cm, we have $l_\mathrm{min} = l_\mathrm{max} + \lambda/4 = 2.869$ cm.

   c. To determine V_max and V_min, we first find $\beta l = 2\pi l/\lambda = 2\pi \times 24/10 = 15.1$ We must calculate Z_in using Ulaby's Eq. (2.61),

$$Z_\mathrm{in} = Z_0 \frac{\exp(j\beta l) + \Gamma \exp(-j\beta l)}{\exp(j\beta l) - \Gamma \exp(-j\beta l)}.$$

      We find $Z_\mathrm{in} = 30.0 + j33.2$. We then calculate V_0^+ using Ulaby's Eq. (2.66)

$$V_0^+ = \left(\frac{\tilde{V}_g Z_\mathrm{in}}{Z_g + Z_\mathrm{in}}\right)\left(\frac{1}{\exp(j\beta l) + \Gamma \exp(-j\beta l)}\right) = -4.05 - j2.94 \ \mathrm{V}.$$

      We then find that $|V_0^+| = 5.00$ and that $\theta_\mathrm{r} = -144°$. From here, we find $V_\mathrm{max} = |V_0^+|(1 + |\Gamma|) = 7.24$ V and $V_\mathrm{min} = |V_0^+|(1 - |\Gamma|) = 2.74$ V.

   d. The general expression for $v(z, t)$ has the form

$$v(z, t) = A\cos(\omega t - \beta z + \phi_1) + B\cos(\omega t + \beta z + \phi_2),$$

      and we already showed that $\omega = 3\pi \times 10^9 \ \mathrm{s}^{-1}$ and that $\beta = 20\pi \ \mathrm{m}^{-1}$. Since we must have $A = |V_0^+|$ and $\phi_1 = \theta_\mathrm{r}$, we have $A = 5$ V and $\phi_1 = -144°$. We have $V_0^- = \Gamma V_0^+ = -1.030 - j1.985 = 2.24 \exp(-j2.05) = 2.24\angle - 117°$, from which we conclude that $B = 2.24$ and $\phi_2 = -117°$.

   e. The energy in the forward-going wave moves at 2.0×10^8 m/s in the forward direction, and the same is true of the energy in the backward-going wave in the backward direction. The motion of the phase peak carries no energy and hence can move faster than the speed of light in the vacuum. In general, the phase velocity u_p may not equal the velocity at which energy is transferred and can be greater than the speed of light in the vacuum.

4. a. The reflection coefficient at the antenna is given by $\Gamma_{\text{ant}} = (75-50)/(75+50) = 0.2$. Hence, we find

$$Z_{\text{in}i}(i=1,2) = 50 \times \frac{\exp(j0.4\pi) + 0.2 \times \exp(-j0.4\pi)}{\exp(j0.4\pi) - 0.2 \times \exp(-j0.4\pi)}$$
$$= 35.20 - j8.621 \rightarrow 35 - j8.6.$$

b. Since the two antenna lines are exactly the same and they are combined in parallel, half the current goes down each line, and the effective load impedance will be half the impedance of the individual lines. So, we have $Z'_{\text{L}} = 17.60 - j4.311 \rightarrow 18 - j4.3\ \Omega$. Since the imaginary component is positive, the reactance is inductive.

c. The reflection coefficient at the juncture point Γ_{feed} is given by $\Gamma_{\text{feed}} = [(17.60 - 50) - j4.311]/[17.60 + 50) - j4.311] = -0.4733 - j0.09394$. Using this value of Γ_{feed} in the expression,

$$Z_{\text{in}} = 50 \times \frac{\exp(j0.6\pi) + \Gamma_{\text{feed}} \exp(-j0.6\pi)}{\exp(j0.6\pi) - \Gamma_{\text{feed}} \exp(-j0.6\pi)},$$

we obtain $Z_{\text{in}} = 107.6 - j56.7 \rightarrow 110 - j57\ \Omega$.

5. a. The electrical length is defined as the length of the transmission line divided by the wavelength in the medium. That is 0.25 for any quarter-wave transformer. The characteristic impedance of the quarter wave section is given by $Z_0 = (200 \times 73)^{1/2} = 121\ \Omega$.

b. To calculate the physical length of the transformer, we must determine the wavelength in the medium. For that, we need the velocity, which is given by $u_{\text{p}} = c/\sqrt{2.6} = 1.86 \times 10^8$ m/s. We then find that the wavelength in the medium is given by $u_{\text{p}}/f = 1.86$ m. Hence, the quarter-wave transformer must be 46.5 cm in length. To determine the radius of the wires, we must use the expressions in Ulaby, et al.'s Table 2-1 or the table in my slide 3.7. Annoyingly, Ulaby changed his notation, starting in the 2010 edition from his earlier editions, which is why the notation in my table differs from that of the Ulaby, et al. text. I will use the notation in my slide 3.7. We have

$$Z_0 = (\mu/\pi^2\epsilon)^{1/2} \ln\left[(d/2a) + \sqrt{(d/2a)^2 - 1}\right].$$

It follows that

$$\frac{d}{2a} + \left[\left(\frac{d}{2a}\right)^2 - 1\right]^{1/2} = \exp\left(Z_0/Z_{\text{m}}\right),$$

where $Z_{\text{m}} = (\mu/\pi^2\epsilon)^{1/2} = 119.9/\sqrt{\epsilon_r} = 74.37$ (Ω) is the characteristic impedance of the polystyrene medium. We subtract $d/2a$ from both sides, square, and collect terms to find

$$\frac{d}{2a} = (1/2)\exp(Z_0/Z_{\text{m}}) + \exp(Z_0/Z_{\text{m}}).$$

In our case, we have $Z_0 = 120.8$, so that $\exp(Z_0/Z_\mathrm{m}) = 5.077$. It follows that $d/2a = 2.637$. Since we have $d = 2.5$ cm, it follows that $a = 0.4740$ cm or 4.7 mm.

6. The reflection coefficient in this case is given by $\Gamma = (Z_\mathrm{L} - Z_0)/(Z_\mathrm{L} + Z_0) = 1/3 = 0.33$. Because the transmission line is $\lambda/2$ in length, we find that

$$Z_\mathrm{in} = Z_0 \left[\frac{1 + \Gamma \exp(j\beta\lambda)}{1 - \Gamma \exp(j\beta\lambda)}\right] = 2Z_0 = 100\ \Omega.$$

Since we also have $Z_\mathrm{g} = 50\ \Omega$ and $\tilde{V}_\mathrm{g} = 2$ V, we find that $\tilde{V}_\mathrm{i} = (100/150)\tilde{V}_\mathrm{g} = 1.333$ V. The voltage of the positive-going wave is then given by $V_0^+ = \tilde{V}_\mathrm{i}[1/(exp(j\beta\lambda) + \Gamma \exp(-j\beta\lambda)] = -1.00V$. Hence, the average input power is $P_\mathrm{av}^\mathrm{i} = |V_0^+|^2/2Z_0 = 0.0100$ W $= 10$ mW. The reflected power is given by $P_\mathrm{av}^\mathrm{r} = -|\Gamma|^2 P_\mathrm{av}^\mathrm{i} = -(1/9)10.0 = -1.10$ mW, where the sign indicates that the power is reflected. The reflected power is ultimately dissipated at the generator. The remaining power is transmitted into the infinite line, and we find $P_\mathrm{av}^\mathrm{t} = 8.9$ mW. The reason that it is important that $\alpha \neq 0$ is to ensure that there are no back reflections from the second line. The real physical criterion is that the second transmission line must have a length that is much longer than its attenuation length, or it must be impedance -matched at its termination.

7. To solve this problem, we have to trace back the input impedance in two stages. In the first stage, we calculate $Z_\mathrm{in}(-R)$, the input impedance at the position of the resistor. The reflection coefficient from the load is given by $\Gamma_\mathrm{L} = (Z_\mathrm{L} - Z_0)/(Z_\mathrm{L} + Z_0) = 0.0588 + j0.235$. We now note that the resistor is a distance of 0.3λ from the load. So, we find that $\exp(-2j\beta l) = \exp(-j3.770) = -0.809 + j0.588$.

$$Z_\mathrm{in}(-R) = Z_0 \frac{1 + \Gamma_\mathrm{L} \exp(-j3.770)}{1 - \Gamma_\mathrm{L} \exp(-j3.770)} = 32.9 - j10.9.$$

This impedance combines with the 30-Ω resistor to give the effective load impedance that determines the reflection from the resistor. The impedances add in parallel, so that we have $1/Z_\mathrm{eff} = 1/30 + 1/Z_\mathrm{in}(-R)$. We find $Z_\mathrm{eff} = 16.1 - j2.41$. We can now calculate the reflection coefficient Γ_R at the resistor, using this effective impedance. We find $\Gamma_R = -0.511 - j0.0550$. We now find

$$Z_\mathrm{in} = Z_0 \frac{1 + \Gamma_R \exp(-j3.770)}{1 - \Gamma_R \exp(-j3.770)} = 99 - j69$$