

1. A stock market trader buys 100 shares of stock A and 200 shares of stock B. Let X and Y be the price changes of A and B, respectively, over a certain time period. And assume that the joint PMF of X and Y is uniform over the set of integers x and y satisfying

$$-2 \leq x \leq 4$$

$$-1 \leq y - x \leq 1.$$

- (a) Find the marginal PMFs and the means of X and Y .

Given:

$$X \in \{x : -2 \leq x \leq 4\}$$

$$Y \in \{y : x - 1 \leq y \leq x + 1, x \in X\}$$

Therefore, the pairs (x, y) consist of:

$$\begin{aligned} (x, y) \in \{ & (-2, -3), (-2, -2), (-2, -1), \\ & (-1, -2), (-1, -1), (-1, 0), \dots, \\ & (4, 3), (4, 4), (4, 5) \} \end{aligned}$$

Totalling in $7 \times 3 = 21$ pairs.

Therefore, the joint PMF is

$$p_{X,Y}(x, y) = \begin{cases} 1/21, & \text{if } -2 \leq x \leq 4, -1 \leq y - x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

The marginal PMF are

$$\begin{aligned} p_X(x) &= \sum_y p_{X,Y}(x, y) \\ &= \begin{cases} 3/21, & \text{if } -2 \leq x \leq 4, \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

$$p_Y(y) = \sum_x p_{X,Y}(x,y)$$

$$= \begin{cases} 1/21, & \text{if } y = -3, 5, \\ 2/21, & \text{if } y = -2, 4, \\ 3/21, & \text{if } -1 \leq x \leq 3, \\ 0, & \text{otherwise} \end{cases}$$

The means,

$$E[X] = \sum_x x \cdot p_X(x)$$

$$= \frac{3}{21}((-2) + (-1) + 0 + 1 + 2 + 3 + 4)$$

$$= \frac{3}{21}(7)$$

$$= 1$$

$$E[Y] = \sum_y y \cdot p_Y(y)$$

$$= \frac{1}{21}((-3) + 5) + \frac{2}{21}((-2) + 4) + \frac{3}{21}((-1) + 0 + 1 + 2 + 3)$$

$$= 1$$

□

(b) Find the mean of the trader's profit.

$$100E[X] + 200E[Y] = 100(1) + 200(1)$$

$$= 300$$

□

2. The MIT football team wins any one game with probability p , and loses it with probability

$1 - p$. Its performance in each game is independent of its performance in other games. Let L_1 be the number of losses before its first win, and let L_2 be the number of losses after its first win and before its second win. Find the joint PMF of L_1 and L_2 .

For $L_1 = 0, L_2 = 0$,

$$\begin{aligned} P(L_1 = 0, L_2 = 0) &= p \cdot p \\ &= p^2 \end{aligned}$$

For $L_1 = 0, L_2 = 1$,

$$\begin{aligned} P(L_1 = 0, L_2 = 1) &= p \cdot ((1 - p) \cdot p) \\ &= p^2(1 - p) \end{aligned}$$

Similarly, for $L_1 = 1, L_2 = 0$,

$$\begin{aligned} P(L_1 = 1, L_2 = 0) &= ((1 - p) \cdot p) \cdot p \\ &= p^2(1 - p) \end{aligned}$$

For $L_1 = 0, L_2 = 2$,

$$\begin{aligned} P(L_1 = 0, L_2 = 2) &= p \cdot ((1 - p) \cdot (1 - p) \cdot p) \\ &= p^2(1 - p)^2 \end{aligned}$$

For $L_1 = 0, L_2 = 3$,

$$\begin{aligned} P(L_1 = 0, L_2 = 3) &= p \cdot ((1 - p) \cdot (1 - p) \cdot (1 - p) \cdot p) \\ &= p^2(1 - p)^3 \end{aligned}$$

And so on. Therefore, the general expression is:

$$p^2(1 - p)^{L_1 + L_2}$$

with the PMF:

$$p_{L_1, L_2}(L_1, L_2) = p^2(1 - p)^{L_1 + L_2}$$

□

3. A class of n students take a test in which each student gets an A with probability p , a B with probability q , and a grade below B with probability $1 - p - q$, independently of any other student. If X and Y are the numbers of students that get an A and a B, respectively, calculate the joint PMF $p_{x,y}$.

Let $r = 1 - p - q$

Then, the multinomial distribution

$$p_{X,Y}(x, y) = \frac{n!}{x!y!(n - x - y)!} \cdot p^x \cdot q^y \cdot r^{(n - x - y)} \text{ for } y = 0, 1, 2, \dots, 0 \leq x + y \leq n$$

□

4. Your probability class has 250 undergraduate students and 50 graduate students. The probability of an undergraduate (or graduate) student getting an A is $1/3$ (or $1/2$, respectively). Let X be the number of students that get an A in your class.

(a) Calculate $E[X]$ by first finding the PMF of X

Let Y and Z represent the number of undergraduate and graduate students who receive an A, respectively.

$$p_Y(y) = \binom{250}{y} \left(\frac{1}{3}\right)^y \left(1 - \frac{1}{3}\right)^{250-y}$$

$$= \binom{250}{y} \left(\frac{1}{3}\right)^y \left(\frac{2}{3}\right)^{250-y}$$

$$p_Z(z) = \binom{50}{z} \left(\frac{1}{2}\right)^z \left(1 - \frac{1}{2}\right)^{50-z}$$

$$= \binom{50}{z} \left(\frac{1}{2}\right)^{50}$$

□

(b) Calculate $E[X]$ by viewing X as a sum of random variables, whose mean is easily calculated.

□

5. A scalper is considering buying tickets for a particular game. The price of the tickets is \$75, and the scalper will sell them at \$150. However, if she can't sell them at \$150, she won't sell them at all. Given that the demand for tickets is a binomial random variable with parameters $n = 10$ and $p = 1/2$, how many tickets should she buy in order to maximize her expected profit?

Let i be the number of tickets,

$$\begin{aligned} i &= (n + 1)p \\ &= (10 + 1)(0.5) \\ &= 5.5 \approx 6 \end{aligned}$$

Therefore, she should buy 6 tickets in order to maximize her expected profit

□

6. Suppose that X and Y are independent discrete random variables with the same geometric PMF:

$$p_X(k) = p_Y(k) = p(1 - p)^{k-1} \quad k = 1, 2, \dots,$$

where p is a scalar with $0, p < 1$. Show that for any integer $n \geq 2$, the conditional PMF

$$P(X = k \mid X + Y = n)$$

is uniform.

$$\begin{aligned} P(X = k \mid X + Y = n) &= \frac{P(X = k \mid X + Y = n)}{P(X + Y = n)} \\ &= \frac{P(X = k \mid Y = n - X)}{P(Y = n - X)} \\ &= \frac{P(X = k \mid Y = n - X)}{P(Y = n - X)} \end{aligned}$$

□

7. Consider four independent rolls of a 6-sided die. Let X be the number of 1s and let Y be the number of 2s obtained. What is the joint PMF of X and Y ?

Let R represent the number yielded after a roll

For a single trial,

$$P(R = 1) = P(R = 2) = \frac{1}{6}$$

and

$$P(R = 3, 4, 5, 6) = \frac{4}{6}$$

Therefore, the multinomial distribution

$$\begin{aligned} p_{X,Y}(x, y) &= \frac{4!}{x!y!(4 - (x + y))!} \cdot P(R = 1)^x \cdot P(R = 2)^y \cdot P(R = 3)^{(4 - (x + y))} \\ &= \frac{4!}{x!y!(4 - (x + y))!} \cdot \left(\frac{1}{6}\right)^x \cdot \left(\frac{1}{6}\right)^y \cdot \left(\frac{4}{6}\right)^{(4 - x - y)} \end{aligned} \quad \square$$

8. Alvin shops for probability books for K hours, where K is a random variable that is equally likely to be 1, 2, 3, or 4. The number of books N that he buys is random and depends on how long he shops according to the conditional PMF

$$p_{N|K}(n | k) = \frac{1}{k}, \text{ for } n = 1, \dots, k$$

(a) Find the joint PMF of K and N

Since K is equally likely to be 1, 2, 3, or 4

$$p_K(k) = \frac{1}{4}, \text{ for } k = 1, 2, 3, 4$$

Therefore

$$\begin{aligned} p_{N,K}(n, k) &= p_{N|K}(n | k) \cdot p_K(k) \\ &= \frac{1}{k} \cdot \frac{1}{4}, \text{ for } k = 1, 2, 3, 4, n = 1, \dots, k \\ &= \begin{cases} \frac{1}{4k}, & \text{if } k = 1, 2, 3, 4, n = 1, \dots, k, \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

□

(b) Find the marginal PMF of N

The marginal PMF is given by

$$p_N(n) = \sum_k p_{N,K}(n, k)$$

Therefore,

$$\begin{aligned} p_N(n=1) &= \frac{1}{4} + \frac{1}{8} + \frac{1}{12} + \frac{1}{16} \\ &= \frac{25}{48} \end{aligned}$$

$$\begin{aligned} p_N(n=2) &= \frac{1}{8} + \frac{1}{12} + \frac{1}{16} \\ &= \frac{13}{48} \end{aligned}$$

$$\begin{aligned} p_N(n=3) &= \frac{1}{12} + \frac{1}{16} \\ &= \frac{7}{48} \end{aligned}$$

$$p_N(n=4) = \frac{1}{16}$$

$$p_N(n > 4) = 0$$

□

(c) Find the conditional PMF of K given that $N = 2$

$$\begin{aligned} p_{K|2}(k | 2) &= \frac{p_{N,K}(2, k)}{p_N(2)} \\ &= \begin{cases} \frac{6}{13}, & \text{if } n = 2, \\ \frac{4}{13}, & \text{if } n = 3, \\ \frac{3}{13}, & \text{if } n = 4, \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

□

- (d) Find the conditional mean and variance of K , given that he bought at least 2 but no more than 3 books.

$$p_{K|(2 \leq n \leq 3)}(k) = \frac{P(K = k, (2 \leq n \leq 3))}{p_N(2) + p_N(3)}$$

$$\text{where } p_N(2) + p_N(3) = \frac{5}{12}$$

$$P(K = k, (2 \leq n \leq 3)) = \begin{cases} \frac{1}{8}, & \text{if } k = 2, \\ \frac{1}{6}, & \text{if } k = 3, \\ \frac{1}{8}, & \text{if } k = 4, \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$p_{K|(2 \leq n \leq 3)}(k) = \begin{cases} \frac{3}{10}, & \text{if } k = 2, \\ \frac{4}{10}, & \text{if } k = 3, \\ \frac{3}{10}, & \text{if } k = 4, \\ 0, & \text{otherwise} \end{cases}$$

Conditional mean

$$E[K | (2 \leq n \leq 3)] = 3$$

Conditional variance

$$\begin{aligned} \text{var}(K | (2 \leq n \leq 3)) &= E[K - E[K | (2 \leq n \leq 3)]^2 | (2 \leq n \leq 3)] \\ &= \frac{3}{10}(2 - 3)^2 + \frac{2}{5}(0) + \frac{3}{10}(4 - 3)^2 \\ &= \frac{3}{5} \end{aligned} \quad \square$$

- (e) The cost of each book is a random variable with mean \$30. What is the expected value of his total expenditure? *Hint:* Condition on the events $\{N = 1\}, \dots, \{N = 4\}$, and use the total expectation theorem.

Let $T = C_1 + \dots + C_N$, where $E[C_i] = 30$ for $i = 1, \dots, N$

$$E[T] = E[E[T \mid N]]$$

$$= E[N \cdot 30]$$

$$= 30E[N]$$

$$= 30\left(1 \cdot \frac{25}{48} + 2 \cdot \frac{13}{48} + 3 \cdot \frac{7}{48} + 4 \cdot \frac{3}{48}\right)$$

$$= 52.5$$

□