1 MATH 407

4/16/18

Rigid Motions (contd)

$T : \mathbb{R}^n \to \mathbb{R}^n$

Length preserving $\|\bar{X} - \bar{Y}\| = \|T(\bar{X}) - T(\bar{Y})\|$

$\|(x_1, x_2, \dots, x_n)\| = \sqrt{x_1^2 + \dots + x_n^2}$

Thm. TFAE

i) T is rigid motion

ii) $\|\bar{X} - \bar{Y}\|^2 = \|T(\bar{X}) - T(\bar{Y})\|^2$

iii) $\bar{X} \cdot \bar{Y} = T(\bar{X}) \cdot T(\bar{Y})$

Thus $\bar{X} \perp \bar{Y}$ iff $T(\bar{X}) \perp T(\bar{Y})$

$\bar{X} \cdot \bar{X} = T(\bar{X}) \cdot T(\bar{X})$

$\Rightarrow \bar{X}^2 = \|T(\bar{X})\|^2$

Thm. Rigid Motions are bijections

Thm. $T : \mathbb{R}^n \to \mathbb{R}^n$, $S : \mathbb{R}^n \to \mathbb{R}^n$

Rigid motions, so are $S \circ T$, T^{-1}

$= \|S(T(\bar{X})) - S(T(\bar{Y}))\|$

$= \|T(\bar{X}) - T(\bar{Y})\|$

$= \|\bar{X} - \bar{Y}\|$

* Ex. i) Translations: $T_{\overline{X}_0}(\overline{X}) = \overline{X} - \overline{X}_0$

ii) Rotations: $T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

iii) Reflections: $(x,y) \to (y,x)$
$(x,y) \to (x,-y)$
$(x,y) \to (-x,y)$
$(x,y) \to (-x,-y)$

* If $T(0) = \overline{X}_0$ then $S(\overline{X}) = T(\overline{X}) - T(0)$
has $S(0) = 0$

Thm. If S is a rigid motion of $\mathbb{R}^n$ w/ $S(0)=0$,
then $S \in GL_n(\mathbb{R})$.
If S is linear rigid motion of $\mathbb{R}^n$ w/ matrix A,
then columns = orthonormal basis

Thm Any linear rigid motion in $\mathbb{R}^n$ is a composition of
a rotation w/ a reflection
Taking the $\det(A)$ and getting $+$ implies rotation,
and $+1$ implies reflection.

Def. If $A, B \subseteq \mathbb{R}^n$ then $A \simeq B$ iff there is a rigid
motion $B = T(A)$. (Congruence)
$(\longrightarrow)$

(3)

* Congruence is an equivalence relation.

* $A \xrightarrow{T} B \xrightarrow{S} C$ then $S \circ T : A \to C$

* Similarity: $\| T(\bar{x}) - T(\bar{y}) \|$
$= k \| \bar{x} - \bar{y} \|$, some $k > 0$
iff $T = kI_n \circ S$, where S is rigid motion

$A \simeq_s B$, A is similar to B iff $B = TA$,
where T is a similarity relation
$= \frac{1}{k} I_n T = S$

<u>Def.</u> $A \subseteq \mathbb{R}^n$. A rigid motion T of $\mathbb{R}^n$ is a geometric symmetry of A iff $T(A) = A$, $T^{-1}(A) = A$

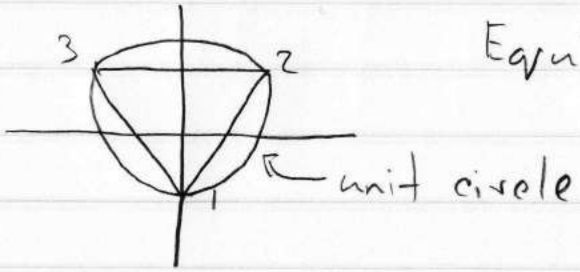
* If T' is another symmetry of A,
then $T' \simeq T$ iff $T'(a) = T(a)$, $\forall a \in A$

* Symmetries of A form a group

* $S \circ T(A) = S(A) = A$

$(\longrightarrow)$

* Symmetries of regular polygons:



Equilateral triangles

$\{a_1, a_2\} \in A, \; \|T(a_1) - T(a_2)\| = \|a_1 - a_2\|$