CMPE 320: Probability, Statistics, and Random Processes

Lecture 13: Continuous RVs and PDFs

Spring 2018

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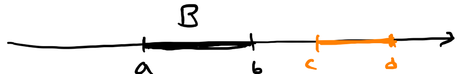
Continuous RVs

- So far we dealt with RVs that can take only discrete values
- RVs with continuous range of possible values are also common
  - Velocity of a vehicle traveling in a highway
  - Weight of a college student
  - Delay of a packet transmitted over a computer network
- Concepts and methods developed for discrete RVs have counterparts for continuous RVs



Continuous RV and its PDF

Probability density function or PDF or pdf

- A RV X is a continuous RV if there is a nonnegative function f_X such that

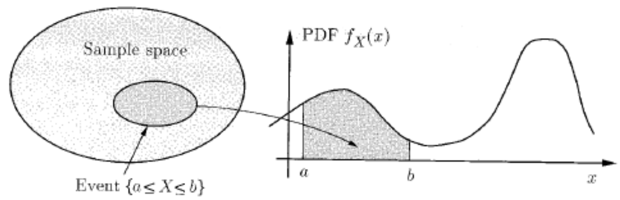
$f_X(x) \geq 0$ for all x

$$P(X \in B) = \int_B f_X(x)\,dx$$

for every subset of $\mathbb{R}$

For example, subset $B = \{a \leq x \leq b\}$

$$P(X \in B) = \boxed{P(a \leq X \leq b) = \int_a^b f_X(x)\,dx}$$



[The area under the curve]

$B = \{X = a\}$

$\Rightarrow P(X = a) = \int_a^a f_X(x)\,dx = 0$ [The probability of a single point is 0]

Properties of PDF

- Non-negativity

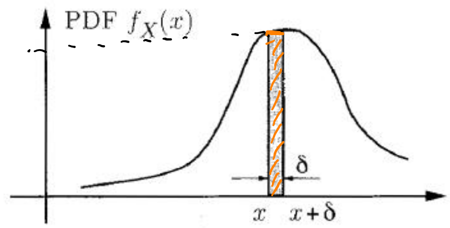
$$f_X(x) \geq 0 \quad \text{for all } x$$

- Normalization property

$B = \mathbb{R}$ (entire real line)

$$P(X \in \mathbb{R}) = P(-\infty < X < \infty) = \boxed{\int_{-\infty}^{\infty} f_X(x)\,dx = 1}$$

Interpretation of PDF



Consider $B = [x, x+\delta)$ for a very small $\delta > 0$

$$P(X \in [x, x+\delta]) = \int_x^{x+\delta} f_X(x)\,dx \simeq f_X(x)\,\delta$$

$f_X(x)$ can be interpreted as the probability mass per unit length

$f_X(x)$ itself is not probability [It is very possible that $f_X(x) > 1$]

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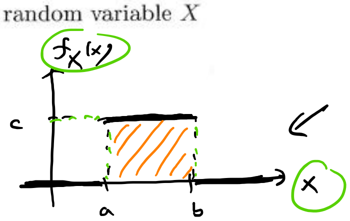
Example 3.1. Continuous Uniform Random Variable. A gambler spins a wheel of fortune, continuously calibrated between a and b, and observes the resulting number. Assuming that any two subintervals of $[a,b]$ of the same length have the same probability, this experiment can be modeled in terms of a random variable X with PDF

$$f_X(x) = \begin{cases} c, & \text{if } a \le x \le b, \\ 0, & \text{otherwise.} \end{cases}$$

Determine c and plot $f_X(x)$.



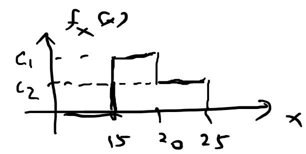
Use normalization property

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_a^b c\,dx = c(b-a) \Rightarrow c = \frac{1}{b-a}$$

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{o.w} \end{cases} \quad : \text{uniform PDF}$$

Example 3.2. Piecewise Constant PDF. Alvin's driving time to work is between 15 and 20 minutes if the day is sunny, and between 20 and 25 minutes if the day is rainy, with all times being equally likely in each case. Assume that a day is sunny with probability 2/3 and rainy with probability 1/3. What is the PDF of the driving time, viewed as a random variable X?

All times equally likely $\equiv$ PDF of X is constant in $[15,20)$ and in $[20,25]$



$$P(\text{sunny}) = P(15 \le X \le 20) = \int_{15}^{20} f_X(x)\,dx = 5c_1 = \frac{2}{3} \Rightarrow c_1 = \frac{2}{15}$$

$$P(\text{rainy}) = P(20 \le X \le 25) = \int_{20}^{25} f_X(x)\,dx = 5c_2 = \frac{1}{3} \Rightarrow c_2 = \frac{1}{15}$$

Expectation

- Recall the expectation of a discrete RV X

$$E[X] = \sum_x x\, p_X(x)$$

- The expectation of a continuous RV is defined as

$$E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\,dx$$

- Expectation of a function of a RV

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$

In case $g(x) = ax + b$

$$E[aX+b] = \int_{-\infty}^{\infty} (ax+b) f_X(x)\,dx = \int_{-\infty}^{\infty} [a x f_X(x) + b f_X(x)]\,dx$$

$$= a\underbrace{\int_{-\infty}^{\infty} x f_X(x)\,dx} + b\underbrace{\int_{-\infty}^{\infty} f_X(x)\,dx}_{1}$$

$$= aE[X] + b$$

Moment and variance

- n-th moment of X $E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)\,dx$
- Variance of X $var(X) = E[(X - E[X])^2] = E[X^2] - (E[X])^2$

Let $E[X] = \mu$

$$var(X) = E((X-\mu)^2) = E(X^2 - 2\mu X + \mu^2)$$
$$= E[X^2] - E[2\mu X] + E[\mu^2]$$
$$= E[X^2] - \underbrace{2\mu \underbrace{E[X]}_{\mu}}_{2\mu^2} + \mu^2 = E[X^2] - \mu^2 =$$

var(aX + b)

$$= \cancel{a\,var(X) + b}$$
$$= a^2 var(X)$$

$$var(\underbrace{aX+b}_{Y}) = E[(aX + b - \underbrace{E[aX+b]}_{aE[X]+b})^2]$$

$$var(Y) = E[(Y - E[Y])^2] = E[\{a(X - E[X])\}^2]$$
$$= E[a^2 (X - E[X])^2]$$
$$= a^2 \underbrace{E[(X - E[X])^2]}_{var(X)}$$

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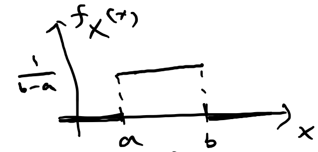
Expectation and variance of a uniform RV

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & \text{if } a \le x \le b, \\ 0, & \text{otherwise} \end{cases}$$



$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_a^b x \cdot \frac{1}{b-a}\,dx = \left[\frac{x^2}{2}\cdot\frac{1}{b-a}\right]_a^b = \frac{b^2-a^2}{2(b-a)} = \frac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}}$$

$$= \boxed{\frac{b+a}{2}}$$

$$\mathrm{var}(X) = ?\quad E[(X-E[X])^2] \quad \text{or} \quad E[X^2] - (E[X])^2$$

$$E[X^2] = \int x^2 f_X(x)\,dx = \int_a^b x^2 \frac{1}{b-a}\,dx = \left(\frac{1}{3}x^3\cdot\frac{1}{b-a}\right)\Big|_a^b = \frac{b^3-a^3}{3(b-a)} = \frac{\cancel{(b-a)}(a^2+ab+b^2)}{3\cancel{(b-a)}} = \frac{a^2+ab+b^2}{3}$$

$$\mathrm{var}(X) = \frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{4a^2+4ab+4b^2-3(a^2+2ab+b^2)}{12}$$

$$= \frac{a^2-2ab+b^2}{12} = \boxed{\frac{(a-b)^2}{12}}$$

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$\int u\,dv = uv - \int v\,du$ $\quad \infty\cdot e^{-\lambda\infty} \quad 0\cdot e^{-\lambda 0}$

Expectation and variance of an exponential RV

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \ge 0, \\ 0, & \text{otherwise} \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^\infty \underbrace{x}_{u}\,\underbrace{\lambda e^{-\lambda x}}_{dv}\,dx = -\underbrace{x}_{u}\,\underbrace{e^{-\lambda x}}_{v}\Big|_0^\infty + \int_0^\infty \underbrace{1}_{du}\cdot\underbrace{e^{-\lambda x}}_{v}\,dx$$

$$= -(0-0) + \int_0^\infty e^{-\lambda x}\,dx = -\frac{1}{\lambda}e^{-\lambda x}\Big|_0^\infty$$

$$= -\left(0-\frac{1}{\lambda}\right) = \frac{1}{\lambda}$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_0^\infty \underbrace{x^2}_{u}\,\underbrace{\lambda e^{-\lambda x}}_{dv}\,dx = x^2(-e^{-\lambda x})\Big|_0^\infty + \underbrace{\int_0^\infty 2x e^{-\lambda x}\,dx}_{\frac{2}{\lambda}E[X]}$$

$$= \frac{2}{\lambda}\cdot\frac{1}{\lambda} = \frac{2}{\lambda^2} \Rightarrow \mathrm{var}[X] = E[X^2] - (E[X])^2$$

$$= \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

Example 3.5. The time until a small meteorite first lands anywhere in the Sahara desert is modeled as an exponential random variable with a mean of 10 days. The time is currently midnight. What is the probability that a meteorite first lands some time between 6 a.m. and 6 p.m. of the first day?