Math-Phys Quiz 7 Question:

1. **Slide 7.21:** Show that $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ and $\nabla \times \nabla V = 0$, where $\mathbf{A}$ is an arbitrary vector field and V is an arbitrary scalar.

2. **Slides 7.16 and 7.20:**

   a. Use Gauss's theorem to show that if $\nabla \cdot \mathbf{D}(\mathbf{R}) = \rho(\mathbf{R})$, and we consider a volume V with a surface S and surface normal $\mathbf{s}$, then $\oint_S \mathbf{D} \cdot d\mathbf{s} = \int_V \rho \, dv$.

   b. Use Stokes's theorem to show if $\nabla \times \mathbf{H}(\mathbf{R}) = \mathbf{J}(\mathbf{R})$, and we consider a surface S that has a border C, then $\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{s}$.

Exam Quiz 7 Questions:

1. **Slides 5A.1–4:** The speed of light transmission in an optical fiber is $u_{\mathrm{p}} = 2.0 \times 10^8$ m/s, and its wavelength is $\lambda = 1.5$ μm. The signal bandwidth is 30 nm, and the birefringence $\Delta n/n = \Delta\epsilon_{\mathrm{r}}/2\epsilon_{\mathrm{r}} = 10^{-6}$.

   a. Show that for any quantity for which $y(x) = ax^r$, where a is a positive constant and r is a constant power, then we have $\Delta y/y = |r|\Delta x/x$, where Δx is the absolute value of a small change in x, and Δy is the absolute value of the small change in y.

   b. Over what distance is there a 2π phase slip of the light in each of the two polarizations due to the birefringence.

2. **Slides 7.26–7.28:**
   Show using an appropriate picture and symmetry that the general expression for the curl in an orthogonal coordinate system may be written

$$\begin{aligned}\nabla \times \mathbf{A} =& \hat{\mathbf{x}}_1 \left[\frac{1}{h_2 h_3}\left(\frac{\partial}{\partial x_2} h_3 A_3 - \frac{\partial}{\partial x_3} h_2 A_2\right)\right] + \hat{\mathbf{x}}_2 \left[\frac{1}{h_3 h_1}\left(\frac{\partial}{\partial x_3} h_1 A_1 - \frac{\partial}{\partial x_1} h_3 A_3\right)\right] \\ &+ \hat{\mathbf{x}}_3 \left[\frac{1}{h_1 h_2}\left(\frac{\partial}{\partial x_1} h_2 A_2 - \frac{\partial}{\partial x_2} h_1 A_1\right)\right],\end{aligned}$$

   where $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, and $\hat{\mathbf{x}}_3$ are unit vectors in the 1- 2- and 3-directions, and h_1, h_2, h_3 are the corresponding differential length coefficients. Use this result to obtain the expression in cylindrical coordinates.

Math-Physics Quiz 7 Solutions:

1. We have

$$\nabla \times \mathbf{A} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right),$$

from which we obtain

$$\begin{aligned}\nabla \cdot (\nabla \times \mathbf{A}) &= \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \\ &= 0.\end{aligned}$$

We also have

$$\nabla V = \hat{\mathbf{x}}\frac{\partial V}{\partial x} + \hat{\mathbf{y}}\frac{\partial V}{\partial y} + \hat{\mathbf{z}}\frac{\partial V}{\partial z},$$

from which we obtain

$$\nabla \times \nabla V = \hat{\mathbf{x}}\left(\frac{\partial^2 V}{\partial y \partial z} - \frac{\partial^2 V}{\partial z \partial y}\right) + \hat{\mathbf{y}}\left(\frac{\partial^2 V}{\partial z \partial x} - \frac{\partial^2 V}{\partial x \partial z}\right) + \hat{\mathbf{z}}\left(\frac{\partial^2 V}{\partial x \partial y} - \frac{\partial^2 V}{\partial y \partial x}\right) = 0.$$

2. From Gauss's theorem, we have $\int_V (\nabla \cdot \mathbf{D}) dv = \oint_S \mathbf{D} \cdot d\mathbf{s}$, and, from Stokes theorem, we have $\int_S (\nabla \times \mathbf{H}) \cdot \mathbf{s} = \oint_C \mathbf{H} \cdot d\mathbf{l}$.

   a. Using Gauss's theorem, we obtain $\int_V \rho\, dv = \int_V (\nabla \cdot \mathbf{D}) dv = \oint_C \mathbf{D} \cdot d\mathbf{s}$.

   b. Using Stokes's theorem, we obtain $\int_S \mathbf{J} \cdot d\mathbf{s} = \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{l} = \oint_C \mathbf{H} \cdot d\mathbf{l}$.

Exam Quiz 7 Solutions:

1. a. Making a Taylor expansion of the relation $y = ax^r$, we obtain

$$\begin{aligned}y_2 &= ax_1^r + ax_1^{r-1}(x_2 - x_1) + \text{higher order terms} \\ &\simeq y_1 + r\frac{x_1}{y_1}(x_2 - x_1)\end{aligned}$$

We conclude that

$$\frac{y_2 - y_1}{y_1} = r\frac{x_2 - x_1}{x_1},$$

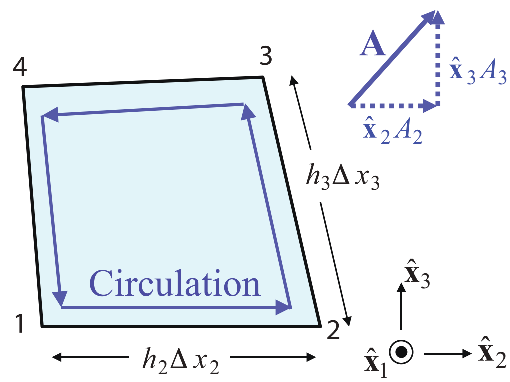
which implies

$$\frac{|y_2 - y_1|}{y_1} = |r|\frac{|x_2 - x_1|}{x_1}.$$

We obtain the same result if we replace x_1 with x_2 and y_1 with y_2. So, to this order, we can just write $x_1 = x_2 \equiv x$ and $y_1 = y_2 \equiv y$ in the denominators.

b. We have $z = u_p t = (c/n)t$, where t is a fixed propagation time. It follows that $\Delta z/z = \Delta n/n$. A slip of one period occurs when $\Delta z = \lambda/n = 1.0\ \mu$m. It follows that $z = \Delta z(n/\Delta n) = 1$ m.

2. A picture that illustrates the circulation orthogonal to the 1-direction is shown below.



Corner 1 is at (x_1, x_2, x_3); corner 2 is at $(x_1, x_2 + \Delta x_2, x_3)$; corner 3 is at $(x_1, x_2 + \Delta x_2, x_3 + \Delta x_3)$; corner 4 is at $(x_1, x_2, x_3 + \Delta x_3)$. Integrating from 1 to 2, we find $I_{12} = \int_{12} \mathbf{A} \cdot d\mathbf{l} \simeq A_2(x_1, x_2, x_3)h_2(x_1, x_2, x_3)\Delta x_2$. We similarly find $I_{23} \simeq A_3(x_1, x_2 + \Delta x_2, x_3)h_3(x_1, x_2 + \Delta x_2, x_3)\Delta x_3$, $I_{34} \simeq -A_2(x_1, x_2, x_3 + \Delta x_3)h_2(x_1, x_2, x_3 + \Delta x_3)\Delta x_2$, and $I_{41} \simeq -A_3(x_1, x_2, x_3)h_3(x_1, x_2, x_3)\Delta x_3$. We note the minus signs in front of I_{34} and I_{41}. As a consequence, we have

$$\begin{aligned} I_{12} + I_{34} &= [A_2(x_1, x_2, x_3)h_2(x_1, x_2, x_3) \\ &\qquad - A_2(x_1, x_2, x_3 + \Delta x_3)h_2(x_1, x_2, x_3 + \Delta x_3)]\Delta x_2 \\ &\simeq -\left.\frac{\partial h_2 A_2}{\partial x_3}\right|_{x_1, x_2, x_3} \Delta x_2 \Delta x_3 \end{aligned}$$

We similarly have

$$I_{23} + I_{41} \simeq \left.\frac{\partial h_3 A_3}{\partial x_2}\right|_{x_1, x_2, x_3} \Delta x_2 \Delta x_3$$

Putting these pieces together and dividing by $h_2 \Delta x_2 h_3 \Delta x_3$, we find that the x_1-component of the curl is given by

$$\left[\frac{1}{h_2 h_3}\left(\frac{\partial}{\partial x_2} h_3 A_3 - \frac{\partial}{\partial x_3} h_2 A_2\right)\right]$$

Rotating 1 to 2, 2 to 3, and 3 to 1 gives the x_2-component, and one more rotation gives the x_3-component.

To get the expression in cylindrical coordinates, we recall that for cylindrical coordinates, we have $h_1 = 1$, $h_2 = r$, and $h_3 = 1$. So, our expression becomes

$$\nabla \times \mathbf{A} = \hat{\mathbf{r}}\left[\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right] + \hat{\boldsymbol{\phi}}\left[\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right] + \hat{\mathbf{z}}\left[\frac{1}{r}\frac{\partial r A_\phi}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \phi}\right]$$