

## CMPE323 DTFT and DFT

### Chapter 8

## Properties of Fourier Transform

- **Duality:**  $X(f) = \mathcal{F}(x(t)) \Rightarrow x(f) = \mathcal{F}(X(-t))$  and  $x(-f) = \mathcal{F}(X(t))$
- **Linearity:**  $z(t) = ax(t) + by(t) \Rightarrow Y(f) = aX(f) + bY(f)$
- **Time Shift:**  $\mathcal{F}(x(t - t_0)) = e^{-j2\pi f t_0} \mathcal{F}(x(t))$
- **Scaling:** For  $a \neq 0 \in \mathbb{R}$ ,  $\mathcal{F}(x(at)) = \frac{1}{|a|} X\left(\frac{f}{a}\right)$
- **Modulation:**  $\mathcal{F}(x(t)e^{j2\pi f_0 t}) = X(f - f_0)$
- **Conjugation:**  $\mathcal{F}(x^*(t)) = X^*(-f)$
- **Parseval:**  $\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X(f)Y^*(f)df = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)Y^*(\omega)d\omega$
- **Rayleigh:**  $\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)Y^*(\omega)d\omega$

## Advanced Properties of Fourier Transform

- **Integration:**  $\mathcal{F}\left(\int_{-\infty}^t x(t) dt\right) = \frac{X(f)}{j2\pi f} + \frac{1}{2}X(0)\delta(f)$
- **Differentiation:**  $\mathcal{F}\left(\frac{d}{dt}x(t)\right) = j2\pi fX(f)$
- **Moments:**  $\int_{-\infty}^{\infty} t^n x(t) dt = \left(\frac{j}{2\pi}\right)^n \frac{d^n}{df^n} X(f) \Big|_{f=0}$
- **Real Signals:** The Fourier transform of a real signal is **EVEN** in magnitude and **ODD** in phase

## Sampling as multiplication

- We can look at the act of sampling as a multiplication in the time domain

$$x_s(t) = x(t) \times \left( \sum_{k=-\infty}^{\infty} \delta(t - k\Delta t) \right)$$

- So that (assuming we will be doing an integration at some time)

$$x_s(t) = \begin{cases} x(k\Delta t) & t = k\Delta t \\ 0 & \text{elsewhere!} \end{cases}$$

- This series of impulse must have infinite frequency content (like an impulse), despite the frequency content of  $x(t)$

## Fourier Transforms with MATLAB

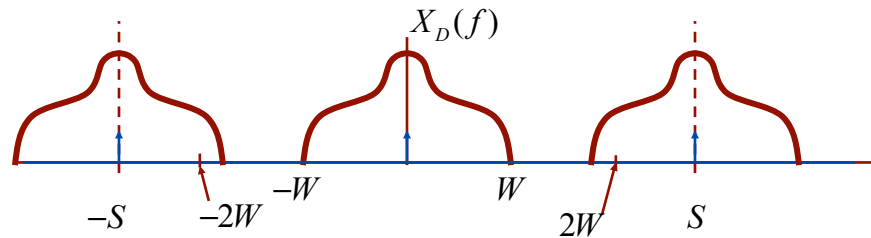
$$\begin{aligned}
 x_s(t) &= x(t) \left( \sum_{k=-\infty}^{\infty} \delta(t - k\Delta t) \right) = \sum_{k=-\infty}^{\infty} x(t) \delta(t - k\Delta t) \\
 X_s(j\omega) &= \int_{-\infty}^{\infty} x_s(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x(k\Delta t) \delta(t - k\Delta t) e^{-j\omega t} dt \\
 &\quad \Omega \triangleq \omega \Delta t \\
 &= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} x(t) \delta(t - k\Delta t) e^{-j\omega t} dt \quad \text{with units } \frac{\text{radians}}{\text{sec}} \times \frac{\text{seconds}}{\text{sample}} \\
 &= \sum_{k=-\infty}^{\infty} x(k\Delta t) e^{-j\omega k\Delta t} = \sum_{k=-\infty}^{\infty} x(k\Delta t) e^{-j\Omega k} = \frac{\text{rad}}{\text{sample}} \\
 &\triangleq \text{Discrete Time Fourier Transform (DTFT)}
 \end{aligned}$$

## Alternatively, we can recognize

$$\begin{aligned}
 x_s(t) &= x(t) \times s(t), \quad s(t) = \sum_{k=-\infty}^{\infty} \delta(t - n\Delta t) \\
 X_s(j\omega) &= X(j\omega) * S(j\omega) \\
 &= \int_{-\infty}^{\infty} X(j(\omega - \alpha)) S(j\alpha) d\alpha \quad S(j\omega) = \sum_{n=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi n}{\Delta t}\right) \\
 X_s(j\omega) &= \int_{-\infty}^{\infty} X(j(\omega - \alpha)) \sum_{n=-\infty}^{\infty} \delta\left(\alpha - \frac{2\pi n}{\Delta t}\right) d\alpha \\
 &= \sum_{n=-\infty}^{\infty} X\left(\omega - \frac{2\pi n}{\Delta t}\right)
 \end{aligned}$$

### The frequency domain view!

- Start with a band limited signal  $x(nt_s) \leftrightarrow X_D(f)$



- Sampling is multiplication in the time domain

$$x(nt_s) = x(t) \times \sum_{n=-\infty}^{\infty} \delta(t - nt_s) = x(t) \times \sum_{n=-\infty}^{\infty} \delta\left(t - \frac{n}{S}\right)$$

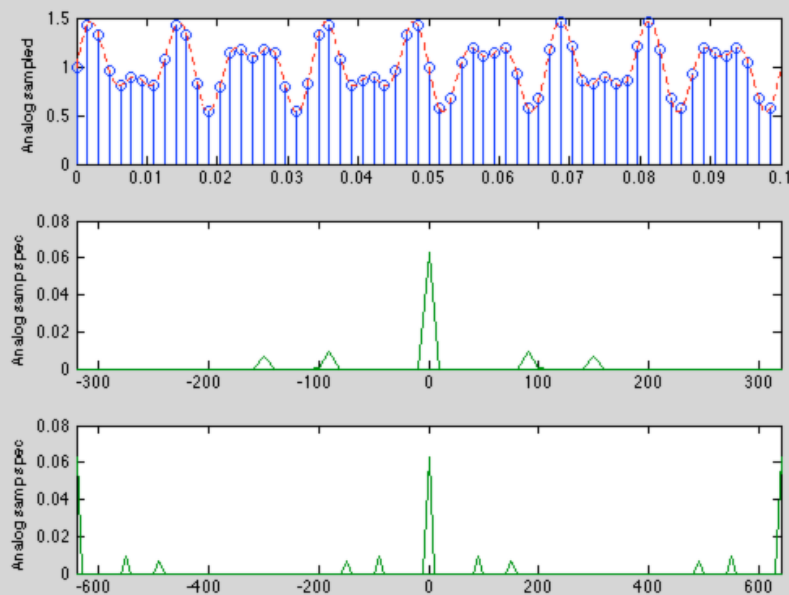
- So it is *convolution* in the frequency domain

$$X_D(f) = X(f) * \mathcal{F}\left(\sum_{n=-\infty}^{\infty} \delta\left(t - \frac{n}{S}\right)\right) = X(f) * \sum_{n=-\infty}^{\infty} \delta(f - nS)$$

$$= \frac{1}{S} X\left(\frac{f}{S}\right) * \sum_{n=-\infty}^{\infty} \delta\left(\frac{f}{S} - n\right) = t_s \sum_{n=-\infty}^{\infty} X\left(\frac{f}{S} - n\right)$$

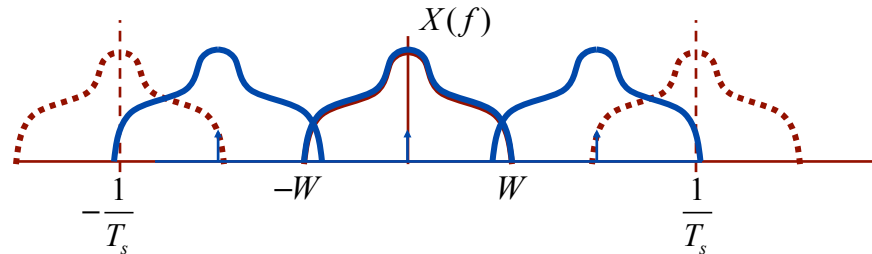
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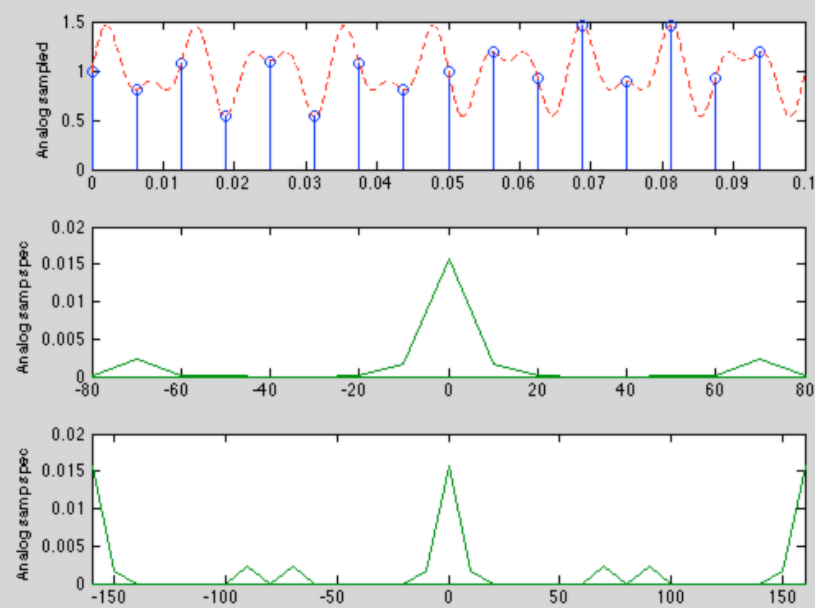


## The aliasing problem

- What if we don't satisfy the Nyquist criteria?



- The sampled spectra overlap...
- ...creating distortion...
- ...that can not be eliminated!
- This is called *aliasing* (acting by another name)



## The DTFT is periodic in frequency!

$$\begin{aligned}
 &= \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} x(t) \delta(t - k\Delta t) e^{-j\omega t} dt \\
 X_s(j\omega) &= \sum_{k=-\infty}^{\infty} x(k\Delta t) e^{-j\omega k\Delta t} \\
 X_s(j(\omega + \omega_s)) &= \sum_{k=-\infty}^{\infty} x(k\Delta t) e^{-j(\omega + 2\pi f_s)k\Delta t} \\
 &= \sum_{k=-\infty}^{\infty} x(k\Delta t) e^{-j\left(\omega + \frac{2\pi}{\Delta t}\right)k\Delta t} = \sum_{k=-\infty}^{\infty} x(k\Delta t) e^{-j\omega k\Delta t} e^{-j2\pi k} \\
 &= \sum_{k=-\infty}^{\infty} x(k\Delta t) e^{-j\omega k\Delta t} = X_s(j\omega)
 \end{aligned}$$

- The periodicity is an artifact of sampling in the time domain...it's actually our aliasing phenomenon!

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## Sampled periodic signals

- We know that a sampled signal has a periodic spectrum...
- ...and a periodic signal has a sampled (discrete spectrum due to the Fourier Series.
- ...so we should get both impulses  $\left(\delta\left[n - \frac{k}{N}\right]\right)$  and periodicity...
- Thus **both** the time sequence and the frequency domain sequence will be periodic
- Let's see how this works

Assume  $x(t)$  is periodic, with period  $T = Nt_s$

$$\begin{aligned}
 c_n &= \frac{1}{T} \int_{0-t_s/2}^{T-t_s/2} x(t) e^{-\frac{j2\pi nt}{T}} dt = \frac{1}{T} \int_{0-t_s/2}^{T-t_s/2} x_1(t) e^{-\frac{j2\pi nt}{T}} dt \\
 x(t) &= \sum_{n=-\infty}^{\infty} c_n e^{j2\pi nt/T} \quad X_p(f) = \sum_{n=-\infty}^{\infty} c_n \delta\left(f - \frac{f}{T}\right)
 \end{aligned}$$

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Now sample  $x(t)$  to get  $x(nt_s) = \sum_{k=-\infty}^{\infty} x[k]\delta(t - kt_s)$

$$\tilde{c}_n = \frac{1}{T} \int_{0-t_s/2}^{T-t_s/2} \sum_{k=-\infty}^{\infty} x[k]\delta(t - kt_s) e^{-\frac{j2\pi nt}{T}} dt$$

Swap the order  $\tilde{c}_n = \frac{1}{T} \sum_{k=-\infty}^{\infty} \int_{0-t_s/2}^{T-t_s/2} x[k]\delta(t - kt_s) e^{-\frac{j2\pi nt}{T}} dt$

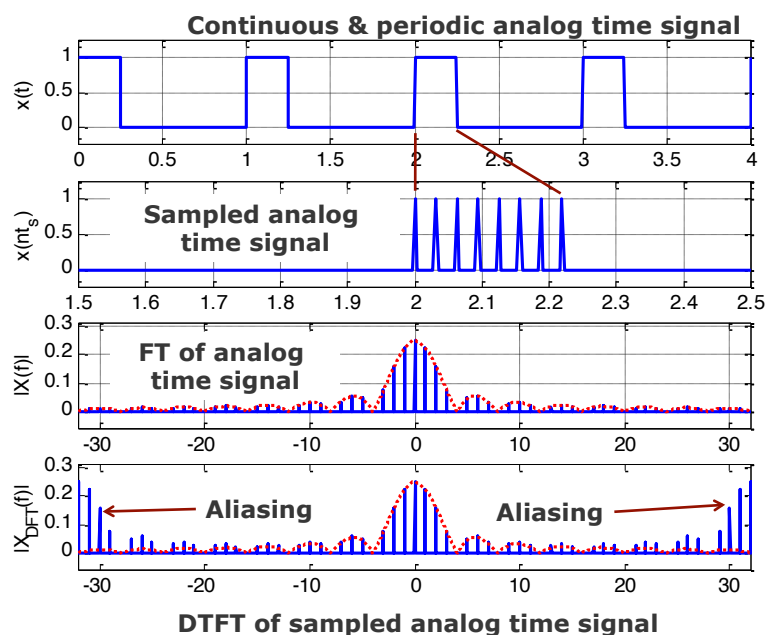
The argument of the  $\delta$  function is only zero for  $k = 0, 1, 2, \dots, N-1$

$$\tilde{c}_n = \frac{1}{T} \sum_{k=0}^{N-1} x[k] e^{-\frac{j2\pi nkt_s}{T}} = \frac{1}{T} X_1(F) \Big|_{F=\frac{n}{Nt_s}} = \frac{1}{T} \sum_{k=0}^{N-1} x_1[k] e^{-\frac{j2\pi nk}{N}}$$

This is still the "continuous time" Fourier series (series of impulses)

To get to discrete time, we have to take the sample time into account and multiply the coefficients by  $t_s$

$$\tilde{c}_n \approx \frac{1}{N} \sum_{k=0}^{N-1} x_1[k] e^{-\frac{j2\pi nk}{N}} \text{ as the coef. of the Discrete Fourier Series (DFS)}$$



## The Discrete Fourier Transform (DFT)

- The DFT is the “unscaled” version

$$X_{DFT}[k] = X_1(F) \Big|_{F=\frac{n}{Nt_s}} = \sum_{k=0}^{N-1} x_1[k] e^{-\frac{j2\pi nk}{N}}, \quad k = 0, 1, 2, \dots, N-1$$

- ...and is defined only over one period of  $F$ , usually  
 $0 \leq F < 1$  or  $-\frac{1}{2} \leq F < \frac{1}{2}$

- The Discrete Fourier Series

$$X_{DFS}[k] = \frac{1}{N} X_1(F) \Big|_{F=\frac{n}{Nt_s}} = \frac{1}{N} \sum_{k=0}^{N-1} x_1[k] e^{-\frac{j2\pi nk}{N}} = \tilde{c}_k$$

- Related to the “discrete approximation” of the analog

$$c_k = X[k] = \frac{1}{T} \int_T x_1(t) e^{-j2\pi kt/T} dt$$

$$\rightarrow \frac{1}{Nt_s} \sum_{n=0}^{N-1} x_1[n] e^{-j2\pi nkt_s/Nt_s} \times t_s = \frac{1}{N} \sum_{n=0}^{N-1} x_1[n] e^{-j2\pi nk/N}$$