CMPE 320: Probability, Statistics, and Random Processes

Lecture 15: Normal RV

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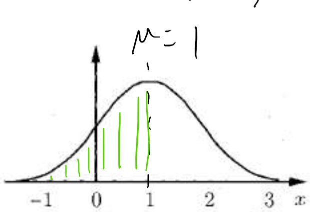
Normal RV

- A continuous RV X is said to be normal or Gaussian if it has a PDF

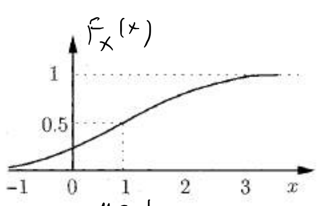
$$\boxed{f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}}$$

The PDF is completely determined by μ and σ^2

$E[X] = \mu$ since $f_X(x)$ is symmetric around $x = \mu$



Normal PDF $f_X(x)$



Normal CDF $F_X(x)$

$$F_X(\infty) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1$$

$$F_X(\mu) = 0.5 = P(X \le \mu)$$

Mean and variance of normal RV

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\,dx$$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

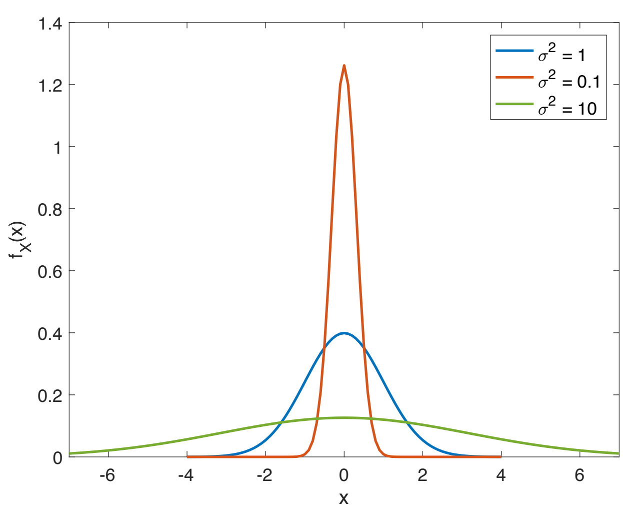
- $E[X] = \mu$, $\text{var}(X) = \sigma^2$

$$\text{var}(X) = E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

Change of variable: $y = \frac{x-\mu}{\sigma} \rightarrow dy = \frac{dx}{\sigma}$

$$\text{var}(X) = \int_{-\infty}^{\infty} \sigma^2 y^2 \frac{1}{\sqrt{2\pi\cancel{\sigma^2}}} e^{-\frac{y^2}{2}} \cdot \cancel{\sigma}\,dy = \sigma^2 \int_{-\infty}^{\infty} \underbrace{y}_{u} \cdot \underbrace{\frac{y}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\,dy}_{dv} \quad \rightarrow v = -\frac{1}{\sqrt{2\pi}} e^{-y^2/2}$$

$$= \underbrace{\cancel{-\sigma^2 y \cdot \frac{1}{\sqrt{2\pi}} e^{-y^2/2} \Big|_{-\infty}^{\infty}}}_{0} + \sigma^2 \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,dy}_{\text{Normal PDF w/ } \mu=0,\ \sigma^2=1} = \sigma^2$$



$\mu = 0$

Normality is preserved by linear transformation

If X is a normal RV, so is $Y = aX + b$

$$E[Y] = aE[X] + b = a\mu_X + b = \mu_Y$$

$$var(Y) = a^2 var(X) = a^2 \sigma_X^2 = \sigma_Y^2$$

$$\Rightarrow f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_Y^2}} e^{-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}}$$

Standard normal RV

- Normal RV Y with zero mean and unit variance is called standard normal

$$\mu_Y = 0, \quad \sigma_Y^2 = 1$$

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$

- CDF of the standard normal RV

$$\Phi(y) = F_Y(y) = \underbrace{\int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt}_{\text{cannot be computed in closed-form}}$$

Use a table

Table of $\Phi(y)$

Only $y \geq 0$ is given.

What about $y < 0$?

$\Phi(-1) = P(Y \leq -1)$

$= P(Y \geq 1)$

$= 1 - P(Y \leq 1)$

$(\{Y \geq 1\} = \{Y < 1\}^c)$

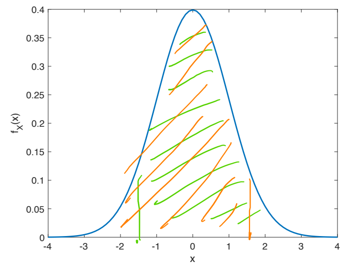
$= 1 - \Phi(1)$

In general,

$\Phi(-y) = 1 - \Phi(y)$

for all y

	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998



Ex1) $P(-0.5 \leq Y \leq 1)$?

$= P(Y \leq 1) - P(Y \leq -0.5)$

$= \Phi(1) - \Phi(-0.5)$

$= \Phi(1) - (1 - \Phi(0.5))$

$= \Phi(1) + \Phi(0.5) - 1$

$= 0.8413 + 0.6915 - 1$

$= 0.5328$

Ex2) $P(Y \geq -1.5)$

$= \Phi(1.5)$

Standardization of a normal RV

- Let X be a normal RV with mean μ and variance σ^2. Can we still use the table of $\Phi(y)$ to compute the CDF for X?

$Y = \frac{X-\mu}{\sigma} \Rightarrow Y$ is normal (linear transformation)

$E[Y] = \frac{E[X]-\mu}{\sigma} = 0$

$var(Y) = \frac{var(X)}{\sigma^2} = 1$

$\Big\}$ Y is standard normal

$F_X(x) = P(X \leq x) = P\left(\frac{X-\mu}{\sigma} \leq \frac{x-\mu}{\sigma}\right) = P\left(Y \leq \frac{x-\mu}{\sigma}\right)$

$= \Phi\left(\frac{x-\mu}{\sigma}\right)$

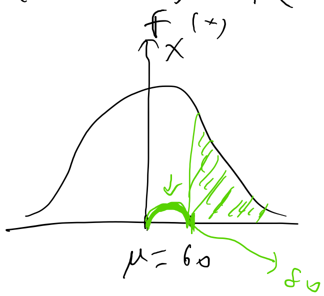
Example 3.7. Using the Normal Table. The annual snowfall at a particular geographic location is modeled as a normal random variable with a mean of $\mu = 60$ inches and a standard deviation of $\sigma = 20$. What is the probability that this year's snowfall will be at least 80 inches?

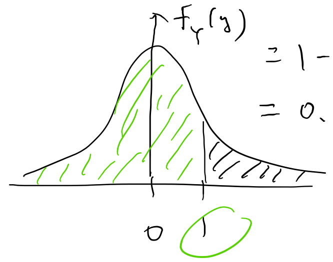
X is normal with $\mu = 60$, $\sigma = 20$

$$P(X \geq 80) = P\Big(Y \geq \underbrace{\frac{80-60}{20}}_{1}\Big) = P(Y \geq 1) = 1 - \Phi(1)$$

$= 1 - 0.8413$

$= 0.1587$





Example 3.8. Signal Detection. A binary message is transmitted as a signal s, which is either -1 or $+1$. The communication channel corrupts the transmission with additive normal noise with mean $\mu = 0$ and variance σ^2. The receiver concludes that the signal -1 (or $+1$) was transmitted if the value received is < 0 (or ≥ 0, respectively) What is the probability of error?

$X = s + N$, N is normal with $\mu = 0$, variance σ^2

Given $s = +1$ transmitted, X is normal with $\mu_X = 1$, variance σ^2

Given $s = -1$ " , X is " " $\mu_X = -1$, " σ^2