Hw 7 Key answer

2 $X_1, \ldots, X_n \sim \text{uniform}(0, \theta)$

we know $\hat{\theta}_{MLE} = \max\{X_1, \ldots, X_n\}$

(a) $P(Y \le y) = P(\max(X_1, \ldots X_n) \le y) = P(X_1 \le y, X_2 \le y, \ldots, X_n \le y)$

$= P(X_1 \le y)\, P(X_2 \le y) \cdots P(X_n \le y) = \left[P(X \le y)\right]^n = \left[\frac{y}{\theta}\right]^n$

(b) $f_{\max\{X_1 \ldots X_n\}}(y) = \frac{d}{dy}\left[\frac{y}{\theta}\right]^n = \frac{n}{\theta}\left[\frac{y}{\theta}\right]^{n-1} = \frac{n\, y^{n-1}}{\theta^n} \quad 0 \le y \le \theta$

(c) $E(\hat{\theta}_{MLE}) = E(\max\{X_1, \ldots, X_n\}) = \int_0^\theta \max\{X_1 \ldots X_n\}\, f_{\max\{X_1, \ldots, X_n\}}(y)\, dy$

$= \int_0^\theta y \frac{n}{\theta^n} y^{n-1}\, dy = \frac{n}{\theta^n} \frac{y^{n+1}}{n+1}\Big|_0^\theta = \frac{n}{n+1}\theta$

$\Rightarrow E(\hat{\theta}_{MLE}) = \frac{n}{n+1}\theta \Rightarrow E(\hat{\theta}_{MLE}) \ne \theta \Rightarrow \hat{\theta}_{MLE}$ is biased

we want to find c to make $\hat{\theta}_{MLE}$ unbiased $\Rightarrow E\left[\frac{n+1}{n}\hat{\theta}_{MLE}\right)$

$= \frac{n+1}{n} E(\hat{\theta}_{MLE}) = \frac{n+1}{n}\left(\frac{n}{n+1}\theta\right) = \theta$

$\Rightarrow c = \frac{n+1}{n}$ will make $\hat{\theta}_{MLE}$ unbiased

(d) To get 90th percentile for MLE we ~~[illegible]~~
inverse the cdf

$$P(Y \le y) = \left(\frac{y}{\theta}\right)^n$$

~~is~~ the value y that will give $P(Y \le y) = .9$

$$(.9)^{\frac{1}{n}} = \frac{y}{\theta} \Rightarrow y = \theta (.9)^{\frac{1}{n}}$$

(e) $P(X < r)$, $0 \le r \le \theta$, $X \sim \text{uniform}(0, \theta)$

$$P(X \le r) = \frac{r}{\theta} = \frac{r}{\hat{\theta}_{MLE}}$$

$\boxed{1}$ Please check the worksheet