Homework #8

Problem 1. Let X have a uniform distribution in the unit interval $[0,1]$, and let Y have an exponential distribution with parameter $\nu = 2$. Assume that X and Y are independent. Let $Z = X + Y$.

(a) Find $\mathbf{P}(Y \geq X)$.

(b) Find the conditional PDF of Z given that $Y = y$.

(c) Find the conditional PDF of Y given that $Z = 3$.

Problem 2. Let P a random variable which is uniformly distributed between 0 and 1. On any given day, a particular machine is functional with probability P. Furthermore, given the value of P, the status of the machine on different days is independent

(a) Find the probability that the machine is functional on a particular day.

(b) We are told that the machine was functional on m out of the last n days. Find the conditional PDF of P. You may use the identity

$$\int_0^1 p^k (1-p)^{n-k} dp = \frac{k!(n-k)!}{(n+1)!}.$$

(c) Find the conditional probability that the machine is functional today given that it was functional on m out of the last n days.

Problem 3.

Let $B \triangleq \{a < X \leq b\}$. Derive a general expression for $E[X|B]$ if X is a continuous RV. Let $X : N(0,1)$ with $B = \{-1 < X \leq 2\}$. Compute $E[X|B]$.

Problem 4.

A particular model of an HDTV is manufactured in three different plants, say, A, B, and C, of the same company. Because the workers at A, B, and C are not equally experienced, the quality of the units differs from plant to plant. The pdf's of the time-to-failure X, in years, are

$$f_X(x) = \frac{1}{5}\exp(-x/5)u(x) \text{ for } A$$

$$f_X(x) = \frac{1}{6.5}\exp(-x/6.5)u(x) \text{ for } B$$

$$f_X(x) = \frac{1}{10}\exp(-x/10)u(x) \text{ for } C,$$

where $u(x)$ is the unit step. Plant A produces three times as many units as B, which produces twice as many as C. The TVs are all sent to a central warehouse, intermingled, and shipped to retail stores all around the country. What is the expected lifetime of a unit purchased at random?

Problem 5. **The coordinates X and Y of a point are independent zero mean normal random variables with common variance σ^2. Given that the point is at a distance of at least c from the origin. find the conditional joint PDF of X and Y.**

Problem 6. Alexei is vacationing in Monte Carlo. The amount X (in dollars) he takes to the casino each evening is a random variable with a PDF of the form

$$f_X(x) = \begin{cases} ax, & \text{if } 0 \le x \le 40, \\ 0, & \text{otherwise.} \end{cases}$$

At the end of each night, the amount Y that he has when leaving the casino is uniformly distributed between zero and twice the amount that he came with.

(a) Determine the joint PDF $f_{X,Y}(x, y)$.

(b) What is the probability that on a given night Alexei makes a positive profit at the casino?

(c) Find the PDF of Alexei's profit $Y - X$ on a particular night, and also determine its expected value.

Problem 7.

Consider a communication channel corrupted by noise. Let X be the value of the transmitted signal and Y be the value of the received signal. Assume that the conditional density of Y given $\{X = x\}$ is Gaussian, that is,

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-(y-x)^2}{2\sigma^2}\right),$$

and X is uniformly distributed on $[-1, 1]$. What is the conditional pdf of X given Y, that is, $f_{X|Y}(x|y)$?

Problem 8.

A U.S. defense radar scans the skies for unidentified flying objects (UFOs). Let M be the event that a UFO is present and M^c the event that a UFO is absent. Let $f_{X/M}(x|M) = \frac{1}{\sqrt{2\pi}} \exp(-0.5[x-r]^2)$ be the conditional pdf of the radar return signal X when a UFO is actually there, and let $f_{X/M}(x/M^c) = \frac{1}{\sqrt{2\pi}} \exp(-0.5[x]^2)$ be the conditional pdf of the radar return signal X when there is no UFO. To be specific, let $r = 1$ and let the *alert level* be $x_A = 0.5$. Let A denote the event of an alert, that is, $\{X > x_A\}$. Compute $P[A|M]$, $P[A^c|M]$, $P[A|M^c]$, $P[A^c|M^c]$.
Assume that $P[M] = 10^{-3}$. Compute

$$P[M|A], P[M|A^c], P[M^c|A], P[M^c|A^c].$$ Repeat for $P[M] = 10^{-6}$.

Note: By assigning drastically different numbers to $P[M]$, this problem attempts to illustrate the difficulty of using probability in some types of problems. Because a UFO appearance is so rare (except in Roswell, New Mexico), it may be considered a *one-time event* for which accurate knowledge of the prior probability $P[M]$ is near impossible. Thus, in the surprise attack by the Japanese on Pearl Harbor in 1941, while the radar clearly indicated a massive cloud of incoming objects, the signals were ignored by the commanding officer (CO). Possibly the CO assumed that the prior probability of an attack was so small that a radar failure was more likely.