CMPE323 Properties of Fourier Transforms

Chapter 8

Properties of Fourier Transform

- **Duality:** $X(f) = \mathcal{F}\big(x(t)\big) \Rightarrow x(f) = \mathcal{F}\big(X(-t)\big)$ and $x(-f) = \mathcal{F}\big(X(t)\big)$

- **Linearity:** $z(t) = ax(t) + by(t) \Rightarrow Y(f) = aX(f) + bY(f)$

- **Time Shift:** $\mathcal{F}\big(x(t - t_0)\big) = e^{-j2\pi f t_0} \mathcal{F}\big(x(t)\big)$

- **Scaling:**   For $a \neq 0 \in \mathbb{R}$, $\mathcal{F}\big(x(at)\big) = \dfrac{1}{|a|} X\left(\dfrac{f}{a}\right)$

- **Modulation:** $\mathcal{F}\big(x(t)e^{j2\pi f_0 t}\big) = X(f - f_0)$

- **Conjugation:** $\mathcal{F}\big(x^*(t)\big) = X^*(-f)$

- **Parseval:** $\displaystyle \int_{-\infty}^{\infty} x(t)y^*(t)\, dt = \int_{-\infty}^{\infty} X(f)Y^*(f)\, df = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)Y^*(\omega)\, d\omega$

- **Rayleigh** $\displaystyle \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |X(f)|^2 \, df = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)Y^*(\omega)\, d\omega$

Advanced Properties of Fourier Transform

- **Integration:** $\mathcal{F}\left(\displaystyle\int_{-\infty}^{t} x(t)\,dt\right) = \dfrac{X(f)}{j2\pi f} + \dfrac{1}{2}X(0)\delta(f)$

- **Differentiation:** $\mathcal{F}\left(\dfrac{d}{dt}x(t)\right) = j2\pi f X(f)$

- **Moments:** $\displaystyle\int_{-\infty}^{\infty} t^{n}x(t)\,dt = \left(\dfrac{j}{2\pi}\right)^{n}\dfrac{d^{n}}{df^{n}}X(f)\Bigg|_{f=0}$

- **Real Signals: The Fourier transform of a real signal is EVEN in magnitude and ODD in phase**

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The convolution theorem (very important!)

- **The output of a LTI system with transfer function** $H(f)$

$$Y(f) = X(f)H(f) \qquad y(t) = \int_{-\infty}^{\infty} Y(f)e^{j2\pi ft}\,df = \int_{-\infty}^{\infty} X(f)H(f)e^{j2\pi ft}\,df$$

$$\text{Write } H(f) = \int_{-\infty}^{\infty} h(\tau)e^{-j2\pi f\tau}\,d\tau$$

$$y(t) = \int_{-\infty}^{\infty} X(f)\left(\int_{-\infty}^{\infty} h(\tau)e^{-j2\pi f\tau}\,d\tau\right)e^{j2\pi ft}\,df$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} X(f)h(\tau)e^{j2\pi f(t-\tau)}\,df\right)d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\left(\int_{-\infty}^{\infty} X(f)e^{j2\pi f(t-\tau)}\,df\right)d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)\,d\tau$$

$$\triangleq x(t) * h(t) \text{ the CONVOLUTION}$$

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Summary

- **We can decompose a periodic signal into the weighted sum of complex exponentials,...**

- **...or, equivalently, to the weighted sum of sines and cosines.**

- **We write the weighted sum as a Fourier Series**

- **The frequency-domain representation consists of a series of harmonically-related terms, with separation equal to the period of the signal,**

- **The coefficients have units of volts (or amps)**

- **We can decompose a non-periodic signal into the weighted integral of complex exponentials,...**

- **...or, equivalently, to the weighted integral of sines and cosines**

- **The frequency-domain representation has a continuous spectrum with units of volts/Hz (or amps/Hz).**

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Examples (done on board)

$$p(t) = \begin{cases} 0 & t < -\dfrac{\tau}{2} \\[2mm] 1 & -\dfrac{\tau}{2} \le t < \dfrac{\tau}{2} \\[2mm] 0 & t \ge \dfrac{\tau}{2} \end{cases}$$

$$v(t) = \begin{cases} 0 & t < -\tau \\ t+\tau & -\tau \le t < 0 \\ t-t & 0 \le t < \tau \\ 0 & t \ge \tau \end{cases}$$

$$g(t) = \text{Gibbs phenomenon, } p(t), \text{ perfect rectangular filter}$$

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