



CMPE323 FT, FS, DTFT, DFT: Making Sense of it all

I'm confused!

- We have these things that sound similar, but aren't the same.
- How do I keep it straight?
- Let's start with a piecewise continuous in time waveform that is not periodic $x(t)$
- To analyze this in the frequency domain, we use the Fourier Transform, which essentially gives us the "voltage spectral density"

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft} dt = X(f)$$

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft} df = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)e^{j\omega t} d\omega$$

- Piecewise continuous in time but not periodic → Fourier Transform

I'm confused (part 2)

- Now let our function be piecewise continuous in time, but periodic with period T
- In this case, the analysis is the Fourier Series
- In the frequency domain, we talk about the Fourier Coefficients, c_k , which are related to *harmonics* of the period of the time waveform

$$c_k = \frac{1}{T} \int_{\alpha}^{\alpha+T} x(t) e^{-j\omega_0 k t} dt = \frac{1}{T} \int_{\alpha}^{\alpha+T} x(t) e^{-j \frac{2\pi}{T} k t} dt$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{j \frac{2\pi}{T} k t}$$

- Periodic in time results in discrete in frequencies, with spacing of $\Delta f = 1/T$

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I'm confused, part 3

- Now let's assume that we have a waveform that we will assume is not periodic in time, but is sampled in time at constant intervals Δt
- In the frequency domain, this gives us the Discrete Time Fourier Transform (DTFT), which is a (piecewise) continuous, periodic waveform as a function of frequency

$$\begin{aligned} X_s(\omega) &= \int_{-\infty}^{\infty} \left(\sum_{n=-\infty}^{\infty} x(t) \delta(t - n\Delta t) \right) e^{-j\omega t} dt \\ &= \sum_{n=-\infty}^{\infty} \left[\int_{-\infty}^{\infty} (x(t) e^{-j\omega t}) \delta(t - n\Delta t) dt \right] = \sum_{n=-\infty}^{\infty} x(n\Delta t) e^{-j\omega n\Delta t} = DTFT \end{aligned}$$

$$x_s(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_s(\omega) e^{j\omega t} d\omega$$

- Not periodic in time, sampled in time \rightarrow periodic and continuous in frequency

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I'm confused (part 4)

- Now let's consider a periodic, piecewise continuous time waveform that is sampled at Δt
- We start by computing the coefficients of the Fourier Series, using \hat{c}_k to remind ourselves that these are the coefficients from the sampled waveform
- Furthermore, assume that Δt is chosen such that

$$N = \frac{T}{\Delta t} > 2 \in \mathbb{N}$$

$$\hat{c}_k = \frac{1}{T} \int_{\alpha}^{\alpha+T} x(t) \sum_{n=-\infty}^{\infty} \delta(t - n\Delta t) e^{-j\frac{2\pi}{T}kt} dt = \frac{1}{T} \sum_{n=-\infty}^{\infty} \int_{\alpha}^{\alpha+T} x(t) e^{-j\frac{2\pi}{T}kt} \delta(t - n\Delta t) dt$$

Let $\alpha = 0^-$, then $\int_{0^-}^{T^-}$ picks out $n = 0, 1, 2, \dots, N-1$, all other terms are 0

$$\hat{c}_k = \frac{1}{N\Delta t} \sum_{n=0}^{N-1} x(n\Delta t) e^{-j\frac{2\pi}{N\Delta t}kn\Delta t} = \frac{1}{N\Delta t} \sum_{n=0}^{N-1} x(n\Delta t) e^{-j\frac{2\pi kn}{N}}$$

- This expression is the Discrete Fourier Series (DFS)

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continued

- By analogy with

$$c_k = \frac{1}{T} X\left(\frac{2\pi k}{T}\right) \text{ (frequency in rad/s) or } \frac{1}{T} X\left(\frac{k}{T}\right) \text{ (frequency in Hz)}$$

$$\hat{c}_k = \frac{1}{T} X_s\left(\frac{k}{T}\right) \Rightarrow X_s\left(\frac{k}{T}\right) = \sum_{n=0}^{N-1} x(n\Delta t) e^{-j\frac{2\pi kn}{N}}$$

- Where $X_s\left(\frac{k}{T}\right)$ is the Discrete Fourier Transform (DFT)