Maxwell's Equations

The complete equations in differential form:

$$\nabla \cdot \mathbf{D}(\mathbf{R},t) = \rho_V(\mathbf{R},t), \qquad \nabla \times \mathbf{E}(\mathbf{R},t) = -\frac{\partial \mathbf{B}(\mathbf{R},t)}{\partial t},$$

$$\nabla \cdot \mathbf{B}(\mathbf{R},t) = 0, \qquad \nabla \times \mathbf{H}(\mathbf{R},t) = \mathbf{J}(\mathbf{R},t) + \frac{\partial \mathbf{D}(\mathbf{R},t)}{\partial t}$$

with the constitutive relations:

$$\mathbf{D}(\mathbf{R},t) = \varepsilon(\mathbf{R})\mathbf{E}(\mathbf{R},t), \qquad \mathbf{B}(\mathbf{R},t) = \mu(\mathbf{R})\mathbf{H}(\mathbf{R},t)$$

Other important forms that we will see:

(1) Integral form (using Stokes and Gauss's theorems)

(2) Phasor/Frequency domain form

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8.1

Maxwell's Equations

Time-independent (Static) Forms:

$$\nabla \cdot \mathbf{D}(\mathbf{R},t) = \rho_V(\mathbf{R},t), \qquad \nabla \times \mathbf{E}(\mathbf{R},t) = 0,$$

$$\nabla \cdot \mathbf{B}(\mathbf{R},t) = 0, \qquad \nabla \times \mathbf{H}(\mathbf{R},t) = \mathbf{J}(\mathbf{R},t)$$

The electric and magnetic fields decouple; they can be treated independently!

This observation is the starting point for electrostatics and magnetostatics

When can we neglect the time variations?

In the same limit that circuit theory holds

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8.2

Maxwell's Equations

Time-independent (Static) Forms:

$$\nabla \cdot \mathbf{D}(\mathbf{R},t) = \rho_V(\mathbf{R},t), \qquad \nabla \times \mathbf{E}(\mathbf{R},t) = 0,$$
$$\nabla \cdot \mathbf{B}(\mathbf{R},t) = 0, \qquad \nabla \times \mathbf{H}(\mathbf{R},t) = \mathbf{J}(\mathbf{R},t)$$

The electric and magnetic fields decouple; they can be treated independently!

This observation is the starting point for electrostatics and magnetostatics

So, why bother with statics?

(1) **Important applications:** near fields of radiating systems; inductors and capacitors; electrostatic discharge

(2) **Visualizing fields:** The full system is complex; it contains radiative contributions; charge contributions; current contributions. It is important to learn about each of them separately.

Constitutive Relations

$$\mathbf{D}(\mathbf{R},t) = \varepsilon(\mathbf{R})\mathbf{E}(\mathbf{R},t), \qquad \mathbf{B}(\mathbf{R},t) = \mu(\mathbf{R})\mathbf{H}(\mathbf{R},t)$$

Where do they come from?

There are two kinds of charge:

free (these flow in conductors) [This charge is included in ρ_V]

bound (these are dipole charges in dielectrics)

[This charge is what determines ε!]

In statics, the bound charges always tend to cancel the free charges.

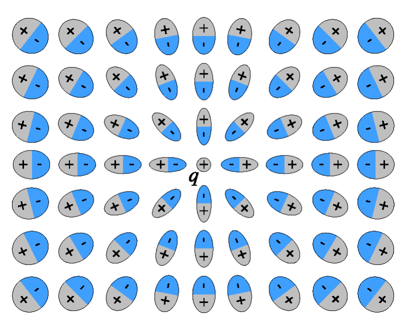
Thus, we have:

$$\varepsilon > \varepsilon_0, \quad |\mathbf{D}| > |\mathbf{E}|$$

NOTE:

D = field response to free charge

E = field response to total charge



Ulaby Figure 1-6

Constitutive Relations

$$\mathbf{D}(\mathbf{R},t) = \varepsilon(\mathbf{R})\mathbf{E}(\mathbf{R},t), \qquad \mathbf{B}(\mathbf{R},t) = \mu(\mathbf{R})\mathbf{H}(\mathbf{R},t)$$

Where do they come from?

There are two kinds of charge:

free (these flow in conductors) [This charge is included in ρ_V]

bound (these are dipole charges in dielectrics)

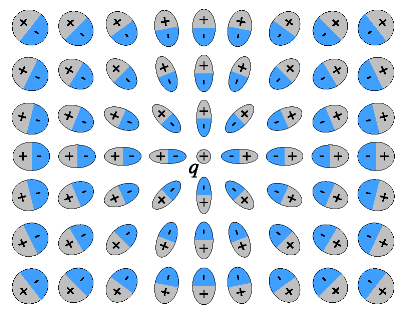
[This charge is what determines ε!]

- Approximate calculations of ε are possible semi-classically

BUT

- Exact calculations usually require quantum mechanics

An important exception is plasmas, which can be treated classically



Ulaby Figure 1-6

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8.5

Constitutive Relations

$$\mathbf{D}(\mathbf{R},t) = \varepsilon(\mathbf{R})\mathbf{E}(\mathbf{R},t), \qquad \mathbf{B}(\mathbf{R},t) = \mu(\mathbf{R})\mathbf{H}(\mathbf{R},t)$$

Where do they come from?

There are also two kinds of current:

free (these flow in conductors) [This current is included in **J**]

bound (in magnetic materials)

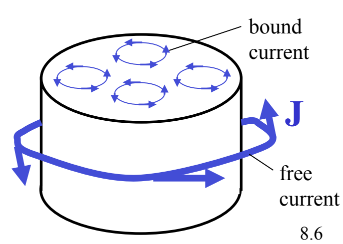
Important note: Static current always flows in loops!

That is a consequence of $\nabla \cdot \mathbf{B} = 0$

NOTE:

H = field response to free current

B = field response to total current



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8.6

Constitutive Relations

$$\mathbf{D}(\mathbf{R},t) = \varepsilon(\mathbf{R})\mathbf{E}(\mathbf{R},t), \qquad \mathbf{B}(\mathbf{R},t) = \mu(\mathbf{R})\mathbf{H}(\mathbf{R},t)$$

Where do they come from?

There are also two kinds of current:

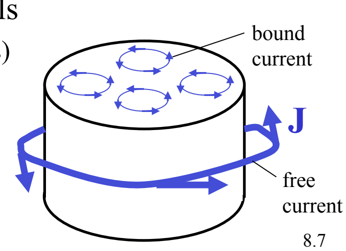
free (these flow in conductors) [This current is included in **J**]

bound (in magnetic materials)

BUT: the behavior of magnetic materials is complicated even in the static limit!

There are three kinds of magnetic materials

- Ferromagnetic (permanent magnets)
- Paramagnetic
- Diamagnetic



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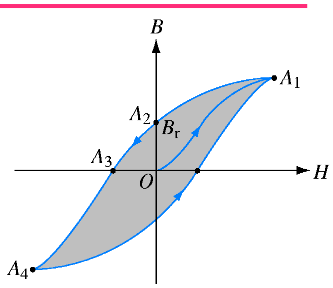
Constitutive Relations

Ferromagnetism: $\mathbf{B} \neq \mu \mathbf{H}$

These materials are highly nonlinear and have hysteresis

Quantum mechanics must be used to explain this phenomenon; no semi-classical explanation is possible



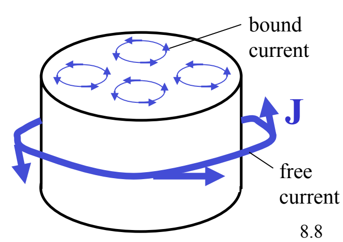
Ulaby
Figure 5-22

Paramagnetism: $\mathbf{B} \approx \mu \mathbf{H}$

A small amount of hysteresis may be present

The bound flow is in the same direction as the free flow and enhances it ($\mu > \mu_0$)

Semi-classical explanation is not possible



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Constitutive Relations

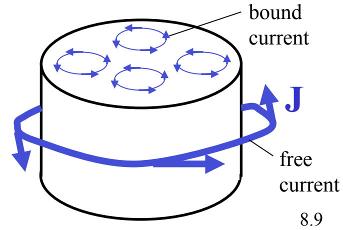
Diamagnetism: $\mathbf{B} = \mu \mathbf{H}$

No hysteresis is present

The bound flow is in the opposite direction from the free flow and decreases it ($\mu_r < 1$)

Semi-classical explanation is possible

Fortunately, in almost all dielectric materials: $\mathbf{B} = \mu_0 \mathbf{H}$



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8.9

Charge and Current Distributions

Volume charge density

$$\rho_V = \lim_{\Delta v \to 0} \frac{\Delta q}{\Delta v} = \frac{dq}{dv}, \quad \text{where } \Delta q \text{ is the charge is a small volume } \Delta v$$

Conversely, we have in a finite volume v:

$$Q = \int_v \rho_V \, dv$$

It is useful to define analogous surface and line charge densities

Surface and line charge densities

$$\rho_S = \lim_{\Delta s \to 0} \frac{\Delta q}{\Delta s} = \frac{dq}{ds}, \quad \rho_l = \lim_{\Delta l \to 0} \frac{\Delta q}{\Delta l} = \frac{dq}{dl}$$

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The converses are:

$$Q = \int_s \rho_S \, ds, \quad Q = \int_l \rho_l \, dl$$

8.10

Charge and Current Distributions

Surface Charge Distribution: Ulaby et al. Example 4-2

Question: A circular disk of electric charge is azimuthally symmetric and increases linearly with r from 0 to 6 $\mathrm{C/m^2}$ at $r = 3$ cm. What is the total charge on the surface?

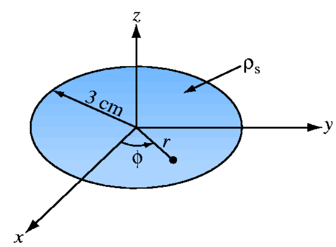
Answer: We have

$$\rho_S = \frac{6r}{3\times10^{-2}} = 2\times10^2 r$$

so that

$$Q = \int_s \rho_S \, ds = \int_{\phi=0}^{2\pi}\int_{r=0}^{3\times10^{-2}} \left(2\times10^2 r\right) r\,dr\,d\phi$$

$$= 2\pi\times2\times10^2 \left.\frac{r^3}{3}\right|_0^{3\times10^{-2}} = 11.3 \text{ mC}$$



Surface charge distribution
Ulaby Figure 4-1(b)

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Charge and Current Distributions

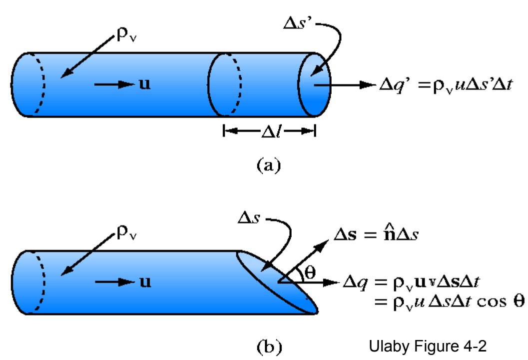
Current density

The increment of current that flows through a surface $\Delta\mathbf{s}$ in a time Δt is given by:

$$\Delta I = \frac{\Delta q}{\Delta t} = \rho_V \mathbf{u}\cdot\Delta\mathbf{s} = \mathbf{J}\cdot\Delta\mathbf{s}, \quad \text{where} \quad \mathbf{J} = \rho_V\mathbf{u}$$

The converse is:

$$I = \int_s \mathbf{J}\cdot d\mathbf{s}$$



Ulaby Figure 4-2

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Charge and Current Distributions

Coulomb's Law:

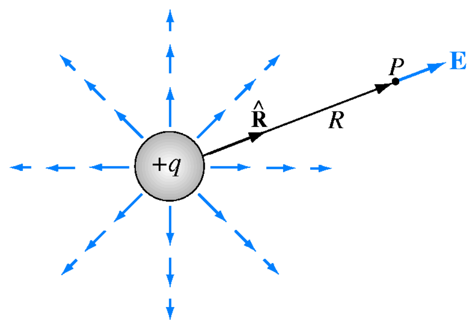
1) An isolated charge q induces an electric field $\mathbf{E}$ at every point in space and at the point P, the field $\mathbf{E}$ is given by

$$\mathbf{E} = \hat{\mathbf{R}} \frac{q}{4\pi\varepsilon R^2}, \quad \text{where}$$

$\hat{\mathbf{R}}$ = unit vector pointing from q to P.

R = distance between q and P.

ε = dielectric constant



Ulaby Figure 4-2

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Charge and Current Distributions

Coulomb's Law:

2) An electric field $\mathbf{E}$ at a point P in space, which may be due to one charge or many charges, induces a force on a charge q' that is given by

$$\mathbf{F} = q'\mathbf{E}$$

NOTE: The only way to detect the presence of a field (electric or magnetic) is by the force that it exerts on charges.

— In this sense, the fields are an abstraction, albeit a very useful one

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Charge and Current Distributions

Multiple Point Charges:

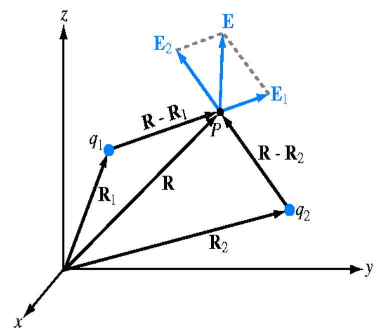
When we have multiple point charges, we add the field contributions from each of them vectorially

$$\mathbf{E} = \frac{1}{4\pi\varepsilon}\sum_{i=1}^{N}\frac{q_i\left(\mathbf{R}-\mathbf{R}_i\right)}{\left|\mathbf{R}-\mathbf{R}_i\right|^3}, \quad \text{where}$$

N = the total number of charges

q_i = amount of the i-th charge

$\mathbf{R}_i$ = position vector of the i-th charge



IMPORTANT NOTE: A charge does not contribute to the electric field at its own location!

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Ulaby Figure 4-4

8.15

Charge and Current Distributions

Electric Field Due to Two Point Charges: Ulaby et al. Example 4-3

Question: Two point charges with $q_1 = 2 \times 10^{-5}$ C and $q_2 = -4 \times 10^{-5}$ C are located at (1, 3, –1) and (–3, 1, –2), respectively, in a Cartesian coordinate system. Find (a) the electric field $\mathbf{E}$ at (3, 1, –2) and (b) the force on a charge $q_3 = 8 \times 10^{-5}$ C located at that point.

Answer: (a) Since $\varepsilon = \varepsilon_0$ and there are two charges, we have

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1\left(\mathbf{R}-\mathbf{R}_1\right)}{\left|\mathbf{R}-\mathbf{R}_1\right|^3} + \frac{q_2\left(\mathbf{R}-\mathbf{R}_2\right)}{\left|\mathbf{R}-\mathbf{R}_2\right|^3}\right]$$

with

$$\mathbf{R}_1 = \hat{\mathbf{x}} + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}$$

$$\mathbf{R}_2 = -\hat{\mathbf{x}}3 + \hat{\mathbf{y}} - \hat{\mathbf{z}}2$$

$$\mathbf{R} = \hat{\mathbf{x}}3 + \hat{\mathbf{y}} - \hat{\mathbf{z}}2$$

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8.16

Charge and Current Distributions

Electric Field Due to Two Point Charges: Ulaby et al. Example 4-3

Answer: (a) [continued] After substitution, we find

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\left[\frac{2(\hat{\mathbf{x}}2 - \hat{\mathbf{y}}2 - \hat{\mathbf{z}})}{27} - \frac{4(\hat{\mathbf{x}}6)}{216}\right]\times 10^{-5} = \frac{\hat{\mathbf{x}} - \hat{\mathbf{y}}4 - \hat{\mathbf{z}}2}{108\pi\varepsilon_0}\times 10^{-5} \text{ V/m}$$

(b) Using the force equation, we have

$$\mathbf{F} = q_3\mathbf{E} = 8\times 10^{-5}\times\frac{\hat{\mathbf{x}} - \hat{\mathbf{y}}4 - \hat{\mathbf{z}}2}{108\pi\varepsilon_0}\times 10^{-5} = \frac{\hat{\mathbf{x}}2 - \hat{\mathbf{y}}8 - \hat{\mathbf{z}}4}{27\pi\varepsilon_0}\times 10^{-10} \text{ N}$$

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8.17

Charge and Current Distributions

Continuous Charge Distributions:

When we have a continuous charge density, each increment of charge $dq = \rho_V\, dv'$ contributes

$$d\,\mathbf{E} = \hat{\mathbf{R}}'\frac{dq}{4\pi\varepsilon R'^2} = \hat{\mathbf{R}}'\frac{\rho_V\, dv'}{4\pi\varepsilon R'^2}, \quad \text{where}$$

$\mathbf{R}'$ = vector from differential volume dv' to point P

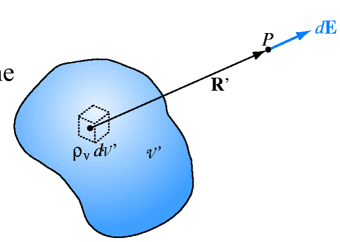
Integrating over a complete volume, we obtain:

$$\mathbf{E} = \int_{v'} d\,\mathbf{E} = \frac{1}{4\pi\varepsilon}\int_{v'}\hat{\mathbf{R}}'\frac{\rho_V\, dv'}{R'^2}$$

For surface and line distributions, these become

$$\mathbf{E} = \frac{1}{4\pi\varepsilon}\int_{s'}\hat{\mathbf{R}}'\frac{\rho_S\, ds'}{R'^2} \quad \text{(surface)}$$

$$\mathbf{E} = \frac{1}{4\pi\varepsilon}\int_{l'}\hat{\mathbf{R}}'\frac{\rho_l\, dl'}{R'^2} \quad \text{(line)}$$



Ulaby Figure 4-5

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8.18

Charge and Current Distributions

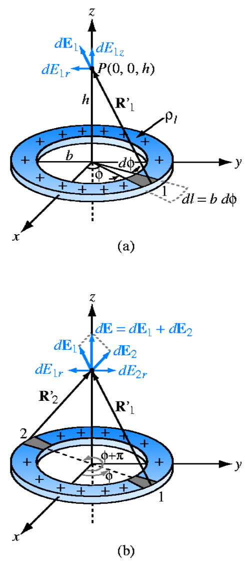
Example: An Infinite Charge Sheet

Question: An infinite plane of charge is located on the *x*-*y* plane. What is the electric field at the point $P(0, 0, h)$?

Answer: We will build up the answer in two parts. The first part is to integrate over a ring of charge (Ulaby et al., Example 4-4) and then integrate over many rings.

Integration over the ring:

$$\mathbf{R}'_1 = -\hat{\mathbf{r}}b + \hat{\mathbf{z}}h = -\hat{\mathbf{x}}b\cos\phi - \hat{\mathbf{y}}b\sin\phi + \hat{\mathbf{z}}h$$



Ulaby Figure 4-6

Charge and Current Distributions

Example: An Infinite Charge Sheet

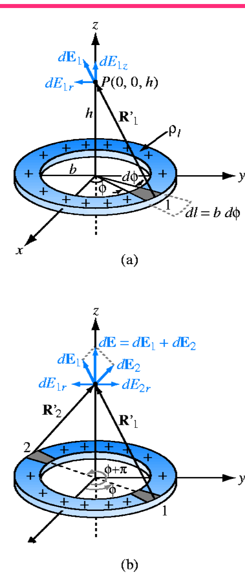
Answer (continued): *Integration over the ring:*

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int_{l'} \hat{\mathbf{R}}' \frac{\rho_l\, dl'}{R'^2}$$

$$= \frac{\rho_l\, b}{4\pi\varepsilon_0}\int_{\phi=0}^{2\pi} \frac{-\hat{\mathbf{x}}b\cos\phi - \hat{\mathbf{y}}b\sin\phi + \hat{\mathbf{z}}h}{\left(b^2+h^2\right)^{3/2}}\, d\phi$$

$$= \hat{\mathbf{z}}\frac{\rho_l\, bh}{2\varepsilon_0\left(b^2+h^2\right)^{3/2}}$$

Due to the symmetry, only the z-component appears!



Ulaby Figure 4-6

Charge and Current Distributions

Example: An Infinite Charge Sheet

Answer (continued): *Integration over a circle:*
Making the replacements, $b \to r$ and $\rho_l \to \rho_S \, dr$ and integrating from $r = 0$ to $r = a$, we find

$$\mathbf{E} = \hat{\mathbf{z}} \frac{\rho_S h}{2\varepsilon_0} \int_{r=0}^{a} \frac{r}{\left(r^2 + h^2\right)^{3/2}} dr$$

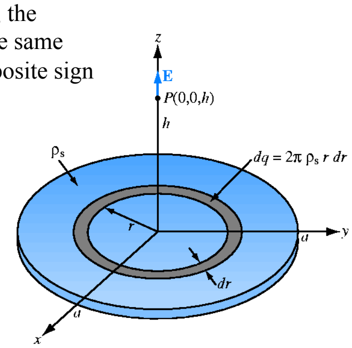
$$= \hat{\mathbf{z}} \frac{\rho_S}{2\varepsilon_0} \left[1 - \frac{h}{\sqrt{a^2 + h^2}} \right] \quad (h > 0)$$

When $h < 0$, the answer is the same with the opposite sign

Integration over the plane: When $a \to \infty$, we find

$$\mathbf{E} = \hat{\mathbf{z}} \frac{\rho_S}{2\varepsilon_0}$$

NOTE: This simple, yet important result can also be obtained directly from Gauss's law



Ulaby Figure 4-7

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