Sabbir Ahmed
DATE: April 26, 2018
MATH 407: HW 11

3.7 **4** Let G be an abelian group, and let n be any positive integer. Show that the function $\phi : G \to G$ defined by $\phi(x) = x^n$ is a homomorphism.

Pf. Let $x, y \in G$

Then $\phi(xy) = (xy)^n$

Since G is abelian,

$$\begin{aligned}(xy)^n &= x^n y^n \\ &= \phi(x)\phi(y)\end{aligned}$$

Therefore, $\phi(xy) = \phi(x)\phi(y)\ \forall x, y \in G$ $\square$

6 Define $\phi : \mathbb{C}^\times \to \mathbb{R}^\times$ by $\phi(a+bi) = a^2+b^2$, for all $a+bi \in \mathbb{C}^\times$. Show that ϕ is a homomorphism.

Pf. Let $a + bi, c + di \in \mathbb{C}^\times$

Then $\phi((a + bi)(c + di)) = (ac - bd) + i(ad + bc)$

Therefore,

$$\begin{aligned}\phi((a + bi)(c + di)) &= (ac - bd) + i(ad + bc) \\ &= (ac - bd)^2 + (ad + bc)^2 \\ &= (ac)^2 - 2acbd + (bd)^2 + (ad)^2 + 2adbc + (bc)^2 \\ &= (ac)^2 + (bd)^2 + (ad)^2 + (bc)^2\end{aligned}$$

Also,

$$\begin{aligned}\phi(a + bi)\phi(c + di) &= (a^2 + b^2)(c^2 + d^2) \\ &= a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 \\ &= (ac)^2 + (bd)^2 + (ad)^2 + (bc)^2\end{aligned}$$

Therefore $\phi((a + bi)(c + di)) = \phi(a + bi)\phi(c + di)$ $\square$

7 Which of the following functions are homomorphisms?

b $\phi : \mathbb{R} \to \mathrm{GL}_2(\mathbb{R})$ defined by $\phi(a) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$

Pf. Consider,

$$\begin{aligned}
\phi(a+b) &= \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \\
&= \phi(a)\phi(b)
\end{aligned}$$

Since $\phi(a+b) = \phi(a)\phi(b)$, ϕ is a homomorphism $\square$

d $\phi : \mathrm{GL}_2(\mathbb{R}) \to \mathbb{R}^\times$ defined by $\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ab$

Pf. For ϕ to be homomorphic, the identity element of $\mathrm{GL}_2(\mathbb{R}) = e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ must be mapped to the identity element of $\mathbb{R}^\times = e_2 = 1$

But

$$\begin{aligned}
\phi\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) &= 1 \times 0 \\
&= 0 \\
&\neq e_2 = 1
\end{aligned}$$

Therefore, ϕ is not a homomorphism $\square$

10 Let G be the group of affine functions from $\mathbb{R}$ into $\mathbb{R}$, as defined in Exercise 10 of Section 3.1. Define $\phi : G \to \mathbb{R}^\times$ as follows: for any function $f_{m,b} \in G$, let $\phi(f_{m,b}) = m$. Prove that ϕ is a group homomorphism, and find its kernel and image.

Pf. Given $G = \{f_{m,b} : \mathbb{R} \to \mathbb{R} \mid m \neq 0 \text{ and } f_{m,b}(x) = mx + b\}$

Consider,

$$\begin{aligned}
(f_{n,a} \circ f_{m,b}) &= f_{n,a}(f_{m,b}) \\
&= f_{n,a}(mx+b) \\
&= n(mx+b) + a \\
&= nmx + bn + a \\
&= f_{nm,bn+a}
\end{aligned}$$

Therefore,

$$\begin{aligned}\phi(f_{n,a} \circ f_{m,b}) &= \phi(f_{nm,bn+a}) \\ &= nm \\ &= \phi(f_{n,a})\phi(f_{m,b})\end{aligned}$$

Therefore, ϕ is a homomorphism

By definition,

$$\begin{aligned}\ker(\phi) &= \{f_{m,b} \in G \mid \phi(f_{m,b}) = 1\} \\ &= \{f_{m,b} \in G \mid m = 1\}\end{aligned}$$

By definition,

$$\begin{aligned}\mathrm{img}(\phi) &= \{m \in \mathbb{R}^{\times} \mid \phi(f_{m,b}) = m\} \\ &= \mathbb{R}^{\times}\end{aligned}$$

□

14 Recall that the center of a group G is $\{x \in G \mid xg = gx \text{ for all } g \in G\}$. Prove that the center of any group is a normal subgroup.

Pf. Let H be a subgroup of G, and $x \in H$

Then $xg = gx$, $\forall g \in G$

Therefore,

$$\begin{aligned}gx &= xg \\ \implies gxg^{-1} &= xgg^{-1} \\ \implies gxg^{-1} &= x \in H\end{aligned}$$

Thus, $\forall x \in H$, $\forall g \in G$, $gxg^{-1} \in H$

That is, $\{x \in G \mid xh = gx \text{ for all } g \in G\}$ is a normal subgroup of G □

18 Let the dihedral group D_n be given by elements a of order n and b of order 2, where $ba = a^{-1}b$. Show that any subgroup of $\langle a \rangle$ is normal in D_n.

Pf. Let H be a subgroup of $\langle a \rangle$, such that any elements of the form $(a^m)^d \in H$

Also, since $ba = a^{-1}b \in D_n$,

$$ba^2 = a^{-1}ba$$

$$= a^{-1}a^{-1}b$$
$$= a^{-2}b$$

Thus, $ba^i = a^{-i}b$ for any i

$\square$

3.8 **4** For each of the subgroups $\{e, a^2\}$ and $\{e, b\}$ of D_4, list all left and right cosets.

Pf. Given $D = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$, with $a^4 = e$, $b^2 = e$, $ba = a^3b$

The elements $\{a, b, ab\} \cap \{e, a^2\} = \varnothing$

Therefore,

$$\begin{aligned}
a\{e, a^2\} &= \{a, a^3\} \\
b\{e, a^2\} &= \{b, ba^2\} \\
&= \{b, baa\} \\
&= \{b, a^3ba\} \\
&= \{b, a^3a^3b\} \\
&= \{b, a^2b\} \\
ab\{e, a^2\} &= \{ab, aba^2\} \\
&= \{ab, abaa\} \\
&= \{ab, aa^3ba\} \\
&= \{ab, ba\} \\
&= \{ab, a^3b\}
\end{aligned}$$

Since D_4 consists of 8 elements, there are $8/2$ left cosets and $8/2$ right cosets

Left cosets: $\{\{e, a^2\}, a\{e, a^2\}, b\{e, a^2\}, ab\{e, a^2\}\}$

Right cosets: $\{\{e, a^2\}, \{e, a^2\}a, \{e, a^2\}b, \{e, a^2\}ab\}$

For $\{e, b\}$, the elements $\{a, a^2, a^3\} \cap \{e, b\} = \varnothing$

Therefore,

$$\begin{aligned}
a\{e, b\} &= \{a, ab\} \\
a^2\{e, b\} &= \{a^2, a^2b\} \\
a^3\{e, b\} &= \{a^3, a^3b\}
\end{aligned}$$

and,

$$\begin{aligned}
\{e,b\}a &= \{a, ba\} \\
&= \{a, a^3b\} \\
\{e,b\}a^2 &= \{a^2, ba^2\} \\
&= \{a^2, ba^2\} \\
&= \{a^2, baa\} \\
&= \{a^2, a^3ba\} \\
&= \{a^2, a^3a^3b\} \\
&= \{a^2, a^2b\} \\
\{e,b\}a^3 &= \{a^3, ba^3\} \\
&= \{a^3, baaa\} \\
&= \{a^3, a^3baa\} \\
&= \{a^3, a^3a^3ba\} \\
&= \{a^3, a^2a^3b\} \\
&= \{a^3, ab\}
\end{aligned}$$

Since D_4 consists of 8 elements, there are $8/2$ left cosets and $8/2$ right cosets

Left cosets: $\{\{e,b\}, a\{e,b\}, b\{e,b\}, ab\{e,b\}\}$

Right cosets: $\{\{e,b\}, \{e,b\}a, \{e,b\}b, \{e,b\}ab\}$ □

9 Let G be a finite group, and let n be a divisor of $|G|$ Show that if H is the only subgroup of G of order n, then H must be normal in G.

Pf. aHa^{-1} is the subgroup of G that is isomorphic to H

Therefore, $aHa^{-1} = H$, $\forall a \in G$

Which is true iff H is a normal subgroup in G □

12 Let H and K be normal subgroups of G such that $H \cap K = \langle e \rangle$. Show that $hk = kh$ for all $h \in H$ and $k \in K$.

Pf. Consider $hkh^{-1}k^{-1}$

Since K is a normal subgroup,

$$\begin{aligned}
hkh^{-1}k^{-1} &= (hkh^{-1})k^{-1} \\
&= hkh^{-1}, k^{-1} \in K
\end{aligned}$$

$$= hkh^{-1}k^{-1} \in K$$

Similarly, since H is a normal subgroup,

$$\begin{aligned} hkh^{-1}k^{-1} &= h(kh^{-1}k^{-1}) \\ &= h, kh^{-1}k^{-1} \in H \\ &= hkh^{-1}k^{-1} \in H \end{aligned}$$

Therefore, $hkh^{-1}k^{-1} \in K \cap H$

But, since $hkh^{-1}k^{-1} \in K \cap H = \{e\}$,

$$\begin{aligned} hkh^{-1}k^{-1} &= e \\ \implies hk(h^{-1}k^{-1}) &= e \\ \implies hk &= kh \end{aligned}$$

□

18 Compute the factor group $(\mathbb{Z}_6 \times \mathbb{Z}_4)/\langle(3,2)\rangle$.

Pf. Given $\langle(3,2)\rangle = \{(3,2),(0,0)\}$

$\mathbb{Z}_6 \times \mathbb{Z}_4$ has 24 elements, with 24/2 left cosets

The factor groups are,

$$\begin{aligned} (0,0) + \langle(3,2)\rangle &= \{(3,2),(0,0)\} \\ (0,1) + \langle(3,2)\rangle &= \{(3,3),(0,1)\} \\ (0,2) + \langle(3,2)\rangle &= \{(3,2),(0,2)\} \\ (0,3) + \langle(3,2)\rangle &= \{(3,1),(0,3)\} \\ (1,0) + \langle(3,2)\rangle &= \{(4,2),(1,0)\} \\ (1,1) + \langle(3,2)\rangle &= \{(4,3),(1,1)\} \\ (1,2) + \langle(3,2)\rangle &= \{(4,0),(1,2)\} \\ (1,3) + \langle(3,2)\rangle &= \{(4,1),(1,3)\} \\ (2,0) + \langle(3,2)\rangle &= \{(5,2),(2,0)\} \\ (2,1) + \langle(3,2)\rangle &= \{(5,3),(2,1)\} \\ (2,2) + \langle(3,2)\rangle &= \{(5,0),(2,2)\} \\ (2,3) + \langle(3,2)\rangle &= \{(5,1),(2,3)\} \end{aligned}$$

□

19 Show that $(\mathbb{Z} \times \mathbb{Z})/\langle(0,1)\rangle$ is an infinite cyclic group.

Pf. Consider $f : \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}/\langle(0,1)\rangle$ given by $f(n) = (n,0) + \langle(0,1)\rangle$

Given, $\langle(0,1)\rangle = \{(0,n) : n \in \mathbb{Z}\}$ To show that f is a homomorphism,

$$\begin{aligned} f(a+b) = (a+b,0) + \langle(0,1)\rangle \\ &= (a,0) + (b,0) + \langle(0,1)\rangle \\ &= (a,0) + \langle(0,1)\rangle + (b,0) + \langle(0,1)\rangle \\ &= f(a) + f(b) \end{aligned}$$

To show that f is one-to-one,

$$\begin{aligned} & f(a) = f(b) \\ & \implies (a,0) + \langle(0,1)\rangle \\ & \qquad = (b,0) + \langle(0,1)\rangle \\ & \qquad = (a-b,0) \in \langle(0,1)\rangle \\ & \implies a - b = 0 \\ & \implies a = b \end{aligned}$$

To show that f is onto, suppose $(x,y) + \langle(0,1)\rangle$ is $\mathbb{Z} \times \mathbb{Z}/\langle(0,1)\rangle$

Then,

$$f(x) = (x,0) + \langle(0,1)\rangle = (x,y) + \langle(0,1)\rangle$$

Since f is an isomorphism, $\mathbb{Z} \times \mathbb{Z}/\langle(0,1)\rangle$ is an infinite cyclic group $\square$

23 Let G be the set of all matrices in $\mathrm{GL}_2(\mathbb{Z}_5)$ of the form $\begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix}$

a. Show that G is a subgroup of $\mathrm{GL}_2(\mathbb{Z}_5)$.

Pf. Given G is non-empty and finite with $\mathbb{Z}_5 = \{[0],[1],[2],[3],[4]\}$

For G to be a subgroup, it is enough to show $xy \in G$, $\forall x, y \in G$

Let $x = \begin{bmatrix} n & a \\ 0 & 1 \end{bmatrix}, y = \begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix} \in G$

Then,

$$xy = \begin{bmatrix} n & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} nm & nb+a \\ 0 & 1 \end{bmatrix}$$

$$\in G$$ □

b. Show that the subset N of all matrices in G of the form $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$, with $c \in \mathbb{Z}_5$, is a normal subgroup of G.

Pf. Let $x, y \in G$, $xy \in N$, with matrices of the form $x = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$, $y = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$

Then,

$$\begin{aligned} xy &= \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & c+d \\ 0 & 1 \end{bmatrix} \\ &\in N \end{aligned}$$

Therefore, $N \leq G$

Let $a = \begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix} \in N$, and $a^{-1} = \begin{bmatrix} m^{-1} & -m^{-1}b \\ 0 & 1 \end{bmatrix} \in N$

Then,

$$\begin{aligned} ana^{-1} &= \begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m^{-1} & -m^{-1}b \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} m & mc+b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m^{-1} & -m^{-1}b \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & mc \\ 0 & 1 \end{bmatrix} \\ &\in N \end{aligned}$$ □

c. Show that the factor group G/N is cyclic of order 4.

Pf. Let G be the matrices in the form $\begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix}$ in $\mathrm{GL}_2(\mathbb{Z}_5)$, with $m \neq 0$

Then, G has matrices of the form $\begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix}$, for $m = \{1, 2, 3, 4\}$ and $b \in \mathbb{Z}_5 = \{[0], [1], [2], [3], [4]\}$

Therefore, G has 20 matrices

Consider N as the matrices in G in the form $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$

Since $c \in \mathbb{Z}_5$, N has 5 matrices

Therefore, G/N has $20/5 = 4$ matrices

To show G/N is cyclic, pick generator [1] of $\mathbb{Z}_5$

Then, any element $gN = [1]^m N = ([1]N)^m$ $\square$