CMPE 320: Probability, Statistics, and Random Processes

Lecture 12: Independence of RVs

Spring 2018

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Independence of events (Recap)

- "Events A and B are independent" means observing event B does not provide any information on A

$$P(A|B) = P(A) \qquad P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = P(A)P(B)$$

- "RV X is independent of event A" means observing event A does not provide any information on the value of X

$\equiv$ Event $\{X = x\}$ and A are independent for all x

$$P(X = x | A) = P(X = x)$$

$$\boxed{p_{X|A}(x) = p_X(x) \quad \text{for all } x}$$

Example 2.19. Consider two independent tosses of a fair coin. Let X be the number of heads and let A be the event that the number of heads is even.

Are X and A independent?

Independence of X and A $\Leftrightarrow$ $P_{X|A}(x) = P_X(x)$

$$P_X(x) = \begin{cases} \frac{1}{4}, & x = 0 \quad TT \\ 2 \cdot \frac{1}{4} = \frac{1}{2}, & x = 1 \quad TH, HT \\ \frac{1}{4} & x = 2 \quad HH \end{cases}$$

$$P_{X|A}(x) = \frac{P(X = x \cap A)}{P(A)} = \begin{cases} \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}, & x = 0 \\ 0, & x = 1 \\ \frac{1}{2}, & x = 2 \end{cases}$$

$P(A) = \frac{1}{2}$

$P_X(x) \neq P_{X|A}(x) \Rightarrow X$ and A are not independent.

Independence of RVs

- "RVs X and Y are independent" means that the value of Y provides no information on the value of X

$\equiv$ Events $\{X = x\}$ and $\{Y = y\}$ are independent for all x, y

$P(X = x \text{ and } Y = y) = P(X = x)\, P(Y = y)$ for all x, y

$\boxed{P_{X,Y}(x, y) = P_X(x)\, P_Y(y)}$ for all x, y

Since $P_{X,Y}(x, y) = P_{X|Y}(x|y)\, P_Y(y)$

$\boxed{P_{X|Y}(x|y) = P_X(x)}$ for all x, and y with $P_Y(y) > 0$

Conditional independence of RVs

- Conditioning with an event A defines a new universe where all probabilities (or PMFs) are replaced by their conditional versions

X and Y are conditionally independent given A if

$P(X=x, Y=y \mid A) = P(X=x \mid A)\, P(Y=y \mid A)$ for all x, y

$\boxed{P_{X,Y|A}(x,y) = P_{X|A}(x)\; P_{Y|A}(y)}$ for all x, y

Since $P_{X,Y|A}(x,y) = P_{X|Y,A}(x|y)\, P_{Y|A}(y)$

$\Rightarrow \boxed{P_{X|Y,A}(x|y) = P_{X|A}(x)}$ Also, $P_{Y|X,A}(y|x) = P_{Y|A}(y)$

Conditional independence may not imply independence, and vice versa

① Are X and Y independent?

Check $P_{X,Y}(x,y) = P_X(x)\,P_Y(y)$ all x, y

$$P_X(x) = \begin{cases} \frac{3}{20}, & x=1 \\ \frac{6}{20}, & x=2 \\ \frac{6}{20}, & x=3 \\ \frac{3}{20}, & x=4 \end{cases} \qquad P_Y(y) = \begin{cases} \frac{1}{20}, & y=1 \\ \frac{5}{20}, & y=2 \\ \frac{9}{20}, & y=3 \\ \frac{5}{20}, & y=4 \end{cases}$$

$P_{X,Y}(x,y)$

y \ x	1	2	3	4
4	1/20	2/20	2/20	0
3	2/20	4/20	1/20	2/20
2	0	1/20	3/20	1/20
1	0	1/20	0	0

$0 = P_{X,Y}(1,1) \neq P_X(1)\, P_Y(1) = \frac{3}{20} \cdot \frac{1}{20}$

$\therefore$ X and Y are not independent

② Are X and Y conditionally independent conditioned on $A = \{X \le 2, Y \ge 3\}$?

Check if $P_{X|Y,A}(x|y) = P_{X|A}(x)$ for all x, y with $P_Y(y) > 0$

$$P_{X|A}(x) = \frac{P(X=x \cap A)}{P(A)} = \begin{cases} \frac{3}{20} / \frac{9}{20}, & x=1 \\ \frac{6}{20} / \frac{9}{20}, & x=2 \end{cases} = \begin{cases} \frac{1}{3}, & x=1 \\ \frac{2}{3}, & x=2 \end{cases}$$

$P(A) = \frac{9}{20}$

$$P_{X|Y,A}(x|y=3) = \frac{P(X=x, Y=3, A)}{P(Y=3, A)} = \begin{cases} \frac{2}{20} / \frac{6}{20}, & x=1 \\ \frac{4}{20} / \frac{6}{20}, & x=2 \end{cases} = \begin{cases} \frac{1}{3}, & x=1 \\ \frac{2}{3}, & x=2 \end{cases}$$

$P(Y=3, A) = \frac{6}{20}$

$$P_{X|Y,A}(x|y=4) = \begin{cases} \frac{1}{3}, & x=1 \\ \frac{2}{3}, & x=2 \end{cases}$$

Expectation of the product of independent RVs

- If X and Y are independent RVs, then E[XY] = E[X] E[Y]

- Similarly E[g(X) h(Y)] =

Variance of the sum of independent RVs

- If X and Y are independent RVs, var(X + Y) = var(X) + var(Y)

Independence of several RVs

- X, Y, Z are independent RVs if

- If X, Y, Z are independent, f(X), g(Y), h(Z) are also independent
  - How about g(X,Y) and h(Z)?
  - How about g(X,Y) and h(Y,Z)?

Variance of sum of several independent RVs

- If $X_1, X_2, \ldots, X_n$ are independent RVs, var($X_1 + X_2 + \cdots + X_n$) =

Example 2.20. Variance of the Binomial and the Poisson. We consider n independent coin tosses, with each toss having probability p of coming up a head. For each i, we let X_i be the Bernoulli random variable which is equal to 1 if the ith toss comes up a head, and is 0 otherwise. Then, $X = X_1 + X_2 + \cdots + X_n$ is a binomial random variable. What are its mean and variance?

Example 2.21. Mean and Variance of the Sample Mean. We wish to estimate the approval rating of a president, to be called B. To this end, we ask n persons drawn at random from the voter population, and we let X_i be a random variable that encodes the response of the ith person:

$$X_i = \begin{cases} 1, & \text{if the } i\text{th person approves B's performance,} \\ 0, & \text{if the } i\text{th person disapproves B's performance.} \end{cases}$$

We model $X_1, X_2, \ldots, X_n$ as independent Bernoulli random variables with common mean p and variance $p(1-p)$. Naturally, we view p as the true approval rating of B. We "average" the responses and compute the **sample mean** S_n, defined as

$$S_n = \frac{X_1 + X_2 + \cdots + X_n}{n}.$$

Thus, the random variable S_n is the approval rating of B within our n-person sample. What are the mean and variance of S_n?