HW#5 Solutions

Problem 1. (a) There are 21 integer pairs (x, y) in the region

$$R = \{(x, y) \mid -2 \le x \le 4, \ -1 \le y - x \le 1\},$$

so that the joint PMF of X and Y is

$$p_{X,Y}(x, y) = \begin{cases} 1/21, & \text{if } (x, y) \text{ is in } R, \\ 0, & \text{otherwise.} \end{cases}$$

For each x in the range $[-2, 4]$, there are three possible values of Y. Thus, we have

$$p_X(x) = \begin{cases} 3/21, & \text{if } x = -2, -1, 0, 1, 2, 3, 4, \\ 0, & \text{otherwise.} \end{cases}$$

The mean of X is the midpoint of the range $[-2, 4]$:

$$\mathbf{E}[X] = 1.$$

The marginal PMF of Y is obtained by using the tabular method. We have

$$p_Y(y) = \begin{cases} 1/21, & \text{if } y = -3, \\ 2/21, & \text{if } y = -2, \\ 3/21, & \text{if } y = -1, 0, 1, 2, 3, \\ 2/21, & \text{if } y = 4, \\ 1/21, & \text{if } y = 5, \\ 0, & \text{otherwise.} \end{cases}$$

The mean of Y is

$$\mathbf{E}[Y] = \frac{1}{21} \cdot (-3 + 5) + \frac{2}{21} \cdot (-2 + 4) + \frac{3}{21} \cdot (-1 + 1 + 2 + 3) = 1.$$

(b) The profit is given by

$$P = 100X + 200Y,$$

so that

$$\mathbf{E}[P] = 100 \cdot \mathbf{E}[X] + 200 \cdot \mathbf{E}[Y] = 100 \cdot 1 + 200 \cdot 1 = 300.$$

Problem 2.

Since the outcomes of the games are independent, the joint PMF of L_1 and L_2 satisfies

$$p_{L_1,L_2}(m,n) = p_{L_1}(m) \cdot p_{L_2}(n).$$

The random variables L_1 and L_2 are identically distributed, and they have a geometric distribution shifted by 1:

$$\mathbf{P}(L_1 = m) = \mathbf{P}(L_2 = m) = (1-p)^m \cdot p.$$

Therefore

$$p_{L_1,L_2}(m,n) = p_{L_1}(m) \cdot p_{L_2}(n) = (1-p)^{n+m} \cdot p^2.$$

Problem 3.

The probability of any set of class grades where x students get an A and y students get a B is $p^x q^y (1-p-q)^{n-x-y}$. The number of possible such sets of class grades is equal to the number of partitions of the class in three groups of x, y, and $n-x-y$ students, and is given by the multinomial coefficient

$$\binom{n}{x,\, y,\, n-x-y} = \frac{n!}{x!y!(n-x-y)!}.$$

Thus,

$$p_{X,Y}(x,y) = \begin{cases} \frac{n!}{x!y!(n-x-y)!} p^x q^y (1-p-q)^{n-x-y} & \text{if } x \geq 0,\ y \geq 0,\ x+y \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Problem 4.

(a) For $i = 1, \ldots, 250$, let U_i be the random variable taking the value 1 if the ith undergraduate student get an A, and 0 otherwise. Similarly, for $i = 1, \ldots, 50$, let G_i be the random variable taking the value 1 if the ith graduate student gets an A, and 0 otherwise. Let

$$U = \sum_{i=1}^{250} U_i, \qquad G = \sum_{i=1}^{50} G_i.$$

We have $X = U + G$. The random variables U and G are binomial with PMFs

$$p_U(u) = \binom{250}{u}(1/3)^u(2/3)^{250-u}, \text{ for } u = 0, 1, \ldots, 250,$$

and

$$p_G(g) = \binom{50}{g}(1/2)^g(1/2)^{50-g}, \text{ for } g = 0, 1, \ldots, 50.$$

If follows that

$$\begin{aligned} p_X(x) &= \mathbf{P}(U + G = x) \\ &= \sum_{u=0}^{250} \mathbf{P}(U = u)\mathbf{P}(U + G = x \,|\, U = u) \\ &= \sum_{u=0}^{250} \mathbf{P}(U = u)\mathbf{P}(G = x - u). \end{aligned}$$

Therefore,

$$p_X(x) = \sum_{u=\min\{0, x-50\}}^{x} \binom{250}{u}(1/3)^u(2/3)^{250-u}\binom{50}{x-u}(1/2)^{x-u}(1/2)^{50-x+u},$$

for $x = 1, \ldots, 300$ and $p_X(x) = 0$ otherwise. If we evaluate $\sum_{x=0}^{300} x p_X(x)$ numerically, we end up with $\mathbf{E}[X] \approx 108.34$.

(b) We have

$$X = \sum_{i=1}^{250} U_i + \sum_{i=1}^{50} G_i,$$

and hence

$$\begin{aligned} \mathbf{E}[X] &= \sum_{i=1}^{250} \mathbf{E}[U_i] + \sum_{i=1}^{50} \mathbf{E}[G_i] \\ &= 250 \cdot \mathbf{P}(U_i = 1) + 50 \cdot \mathbf{P}(G_i = 1) \\ &= 250 \cdot (1/3) + 50 \cdot (1/2) \\ &\approx 108.34 \end{aligned}$$

Problem 5.

Let D and b be the numbers of tickets demanded and bought, respectively. If S is the number of tickets sold, then $S = \min\{D, b\}$. The scalper's expected profit is

$$r(b) = \mathbf{E}[150S - 75b] = 150\mathbf{E}[S] - 75b.$$

We first find $\mathbf{E}[S]$. We assume that $b \leq 10$, since clearly buying more than the maximum number of demanded tickets, which is 10, cannot be optimal. We have

$$\begin{aligned}
\mathbf{E}[S] &= \mathbf{E}[S \mid D \leq b]\mathbf{P}(D \leq b) + \mathbf{E}[S \mid D > b]\mathbf{P}(D > b) \\
&= \sum_{i=0}^{b} i \binom{10}{i} \left(\frac{1}{2}\right)^{10} + b \sum_{i=b+1}^{10} \binom{10}{i} \left(\frac{1}{2}\right)^{10} \\
&= \left(\frac{1}{2}\right)^{10} \left(\sum_{i=0}^{b} i \binom{10}{i} + b \sum_{i=b+1}^{10} \binom{10}{i} \right).
\end{aligned}$$

Thus

$$r(b) = 150 \left(\frac{1}{2}\right)^{10} \left(\sum_{i=0}^{b} i \binom{10}{i} + b \sum_{i=b+1}^{10} \binom{10}{i} \right) - 75b.$$

A computer solution is now required to maximize the above expression over the range $0 \leq b \leq 10$.

Problem 6.

We first note that

$$\mathbf{P}(X = k \,|\, X + Y = n) = \frac{\mathbf{P}(X = k, X + Y = n)}{\mathbf{P}(X + Y = n)}.$$

We have

$$\begin{aligned}
\mathbf{P}(X = k, X + Y = n) &= \mathbf{P}(X = k, Y = n - k) \\
&= \mathbf{P}(X = k)\mathbf{P}(Y = n - k) \\
&= \begin{cases} p(1-p)^{k-1}p(1-p)^{n-k-1} & \text{if } k = 1, 2, \ldots, n-1,\ n \geq 2, \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} p^2(1-p)^{n-2} & \text{if } k = 1, 2, \ldots, n-1,\ n \geq 2, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

We also have

$$\begin{aligned}
\mathbf{P}(X + Y = n) &= \sum_{\{(x,y)\,|\,x+y=n\}} \mathbf{P}(X = x, Y = y) \\
&= \sum_{x=1}^{n-1} \mathbf{P}(X = x, Y = n - x) \\
&= \begin{cases} \sum_{x=1}^{n-1} p(1-p)^{x-1}p(1-p)^{n-x-1} & \text{if } n \geq 2, \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} (n-1)p^2(1-p)^{n-2} & \text{if } n \geq 2, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

The preceding equations yield

$$\begin{aligned}
\mathbf{P}(X = k \,|\, X + Y = n) &= \begin{cases} \frac{p^2(1-p)^{n-2}}{(n-1)p^2(1-p)^{n-2}} & \text{if } n \geq 2 \text{ and } k = 1, 2, \ldots, n-1, \\ 0 & \text{otherwise,} \end{cases} \\
&= \begin{cases} \frac{1}{n-1} & \text{if } n \geq 2 \text{ and } k = 1, 2, \ldots, n-1, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

For a more intuitive line of reasoning, consider the experiment in which we toss a biased coin with probability p of getting heads until we get the second head. Let X be the number of tosses up to and including the first head, and let Y be the number of coin tosses starting with the toss after the first head and up to and including the second head. Then $X + Y$ is the number of coin tosses until we get exactly two heads, and $\mathbf{P}(X = k \,|\, X + Y = n)$ is the probability of getting a head on the kth toss given that it took exactly n tosses to get exactly two heads. This implies that the nth toss was a head and that the first through $(n-1)$st tosses contained exactly one head and the rest tails. Each of these tosses is equally likely to be the head. So the events $X = k$ given that $X + Y = n$ are equally likely as we vary k from 1 through $n-1$. Therefore

$$\mathbf{P}(X = k \,|\, X + Y = n) = \begin{cases} \frac{1}{n-1} & \text{if } n \geq 2 \text{ and } k = 1, 2, \ldots, n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Problem 7. The marginal PMF p_Y is given by the binomial formula

$$p_Y(y) = \binom{4}{y} \left(\frac{1}{6}\right)^y \left(\frac{5}{6}\right)^{4-y}, \qquad y = 0, 1, \ldots, 4.$$

To compute the conditional PMF $p_{X|Y}$, note that given that $Y = y$, X is the number of 1's in the remaining $4 - y$ rolls, each of which can take the 5 values $1, 3, 4, 5, 6$ with equal probability $1/5$. Thus, the conditional PMF $p_{X|Y}$ is binomial with parameters $4 - y$ and $p = 1/5$:

$$p_{X|Y}(x \,|\, y) = \binom{4-y}{x} \left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{4-y-x},$$

for all nonnegative integers x and y such that $0 \leq x + y \leq 4$. The joint PMF is now given by

$$\begin{aligned} p_{X,Y}(x, y) &= p_Y(y) p_{X|Y}(x \,|\, y) \\ &= \binom{4}{y} \left(\frac{1}{6}\right)^y \left(\frac{5}{6}\right)^{4-y} \binom{4-y}{x} \left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{4-y-x}, \end{aligned}$$

for all nonnegative integers x and y such that $0 \leq x + y \leq 4$. For other values of x and y, we have $p_{X,Y}(x, y) = 0$.

Problem 8.

We are given that

$$p_K(k) = \begin{cases} 1/4 & \text{if } k = 1, 2, 3, 4, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$p_{N\,|\,K}(n\,|\,k) = \begin{cases} 1/k & \text{if } n = 1, \ldots, k, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Applying the equation

$$p_{N,K}(n,k) = p_{N\,|\,K}(n\,|\,k)p_K(k),$$

we obtain

$$p_{N,K}(n,k) = \begin{cases} 1/4k & \text{if } k = 1, 2, 3, 4 \text{ and } n = 1, \ldots, k, \\ 0 & \text{otherwise.} \end{cases}$$

(b) The marginal PMF $p_N(n)$ is given by

$$p_N(n) = \sum_k p_{N,K}(n,k) = \sum_{k=n}^{4} 1/4k,$$

or

$$p_N(n) = \begin{cases} 1/4 + 1/8 + 1/12 + 1/16 = 25/48 & \text{if } n = 1, \\ 1/8 + 1/12 + 1/16 = 13/48 & \text{if } n = 2, \\ 1/12 + 1/16 = 7/48 & \text{if } n = 3, \\ 1/16 = 3/48 & \text{if } n = 4, \\ 0 & \text{otherwise.} \end{cases}$$

(c) We have

$$p_{K\,|\,N}(k\,|\,2) = \frac{p_{N,K}(2,k)}{p_N(2)} = \begin{cases} 6/13 & \text{if } k = 2, \\ 4/13 & \text{if } k = 3, \\ 3/13 & \text{if } k = 4, \\ 0 & \text{otherwise.} \end{cases}$$

(d) Let A be the event $2 \leq$N≤ 3. We first find the conditional PMF of K given A. We have

$$p_{K\,|\,A}(k) = \frac{\mathbf{P}(K = k, A)}{\mathbf{P}(A)},$$

$$\mathbf{P}(A) = p_N(2) + p_N(3) = \frac{5}{12},$$

$$\mathbf{P}(K = k, A) = \begin{cases} \frac{1}{8} & \text{if } k = 2, \\ \frac{1}{12} + \frac{1}{12} & \text{if } k = 3, \\ \frac{1}{16} + \frac{1}{16} & \text{if } k = 4, \\ 0 & \text{otherwise,} \end{cases}$$

and finally

$$p_{K \mid A}(k) = \begin{cases} \frac{3}{10} & \text{if } k = 2, \\ \frac{2}{5} & \text{if } k = 3, \\ \frac{3}{10} & \text{if } k = 4, \\ 0 & \text{otherwise.} \end{cases}$$

The conditional PMF of K given A is symmetric around $k = 3$, so

$$\mathbf{E}[K \mid A] = 3.$$

The conditional variance of K given A is given by

$$\text{var}(K \mid A) = \mathbf{E}\big[(K - \mathbf{E}[K \mid A])^2 \mid A\big] = \frac{3}{10} \cdot (2-3)^2 + \frac{2}{5} \cdot 0 + \frac{3}{10} \cdot (4-3)^2 = \frac{3}{5}.$$

(e) We are given that $\mathbf{E}[C_i] = 30$, where C_i is the cost of book i. Let T be the total cost, so that $T = C_1 + \ldots + C_N$. We find $\mathbf{E}[T]$ by using the total expectation theorem:

$$\begin{aligned} \mathbf{E}[T] &= \mathbf{E}[T \mid N = 1]p_N(1) + \mathbf{E}[T \mid N = 2]p_N(2) + \mathbf{E}[T \mid N = 3]p_N(3) \\ &\quad + \mathbf{E}[T \mid N = 4]p_N(4) \\ &= \mathbf{E}[C_1]p_N(1) + \mathbf{E}[C_1 + C_2]p_N(2) + \mathbf{E}[C_1 + C_2 + C_3]p_N(3) \\ &\quad + \mathbf{E}[C_1 + C_2 + C_3 + C_4]p_N(4) \\ &= \mathbf{E}[C_i]p_N(1) + 2\mathbf{E}[C_i]p_N(2) + 3\mathbf{E}[C_i]p_N(3) + 4\mathbf{E}[C_i]p_N(4) \\ &= 30 \cdot \frac{25}{48} + 60 \cdot \frac{13}{48} + 90 \cdot \frac{7}{48} + 120 \cdot \frac{1}{16} \\ &= 52.5. \end{aligned}$$