CMPE323 Properties of Fourier Transforms

Chapter 8

Properties of Fourier Transform

- **Duality:** $X(f) = \mathcal{F}\left(x(t)\right) \Rightarrow x(f) = \mathcal{F}\left(X(-t)\right)$ and $x(-f) = \mathcal{F}\left(X(t)\right)$
- **Linearity:** $z(t) = ax(t) + by(t) \Rightarrow Y(f) = aX(f) + bY(f)$
- **Time Shift:** $\mathcal{F}\left(x(t-t_0)\right) = e^{-j2\pi f t_0}\mathcal{F}\left(x(t)\right)$
- **Scaling:** For $a \neq 0 \in \mathbb{R}$, $\mathcal{F}\left(x(at)\right) = \frac{1}{|a|}X\left(\frac{f}{a}\right)$
- **Modulation:** $\mathcal{F}\left(x(t)e^{j2\pi f_0 t}\right) = X(f - f_0)$
- **Conjugation:** $\mathcal{F}\left(x^*(t)\right) = X^*(-f)$
- **Parseval:** $\int_{-\infty}^{\infty} x(t)y^*(t)\,dt = \int_{-\infty}^{\infty} X(f)Y^*(f)\,df = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)Y^*(\omega)\,d\omega$
- **Rayleigh** $\int_{-\infty}^{\infty} |x(t)|^2\,dt = \int_{-\infty}^{\infty} |X(f)|^2\,df = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)Y^*(\omega)\,d\omega$

Advanced Properties of Fourier Transform

- **Integration:** $\mathcal{F}\left(\int_{-\infty}^{t} x(t)\,dt\right) = \frac{X(f)}{j2\pi f} + \frac{1}{2}X(0)\delta(f)$
- **Differentiation:** $\mathcal{F}\left(\frac{d}{dt}x(t)\right) = j2\pi f X(f)$
- **Moments:** $\int_{-\infty}^{\infty} t^n x(t)\,dt = \left(\frac{j}{2\pi}\right)^n \frac{d^n}{df^n} X(f)\Big|_{f=0}$
- **Real Signals: The Fourier transform of a real signal is EVEN in magnitude and ODD in phase**

The convolution theorem (very important!)

- **The output of a LTI system with transfer function** $H(f)$

$$Y(f) = X(f)H(f) \qquad y(t) = \int_{-\infty}^{\infty} Y(f)e^{j2\pi ft}df = \int_{-\infty}^{\infty} X(f)H(f)e^{j2\pi ft}df$$

$$\text{Write } H(f) = \int_{-\infty}^{\infty} h(\tau)e^{-j2\pi f\tau}d\tau$$

$$y(t) = \int_{-\infty}^{\infty} X(f)\left(\int_{-\infty}^{\infty} h(\tau)e^{-j2\pi f\tau}d\tau\right)e^{j2\pi ft}df$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} X(f)h(\tau)e^{j2\pi f(t-\tau)}df\right)d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\left(\int_{-\infty}^{\infty} X(f)e^{j2\pi f(t-\tau)}df\right)d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau$$

$$\triangleq x(t) * h(t) \text{ the CONVOLUTION}$$

Summary

- We can decompose a periodic signal into the weighted sum of complex exponentials,…
- …or, equivalently, to the weighted sum of sines and cosines.
- We write the weighted sum as a Fourier Series
- The frequency-domain representation consists of a series of harmonically-related terms, with separation equal to the period of the signal,
- The coefficients have units of volts (or amps)
- We can decompose a non-periodic signal into the weighted integral of complex exponentials,…
- …or, equivalently, to the weighted integral of sines and cosines
- The frequency-domain representation has a continuous spectrum with units of volts/Hz (or amps/Hz).

Examples (done on board)

$$p(t) = \begin{cases} 0 & t < -\dfrac{\tau}{2} \\ 1 & -\dfrac{\tau}{2} \le t < \dfrac{\tau}{2} \\ 0 & t \ge \dfrac{\tau}{2} \end{cases}$$

$$v(t) = \begin{cases} 0 & t < -\tau \\ t + \tau & -\tau \le t < 0 \\ t - t & 0 \le t < \tau \\ 0 & t \ge \tau \end{cases}$$

$$g(t) = \text{Gibbs phenomenon, } p(t), \text{ perfect rectangular filter}$$