Difficult Things

- There are four things that happen that complicate the process
  - 1) $M \geq N$
  - 2) Repeated real roots of the form $(s-p_k)^m$
  - 3) Complex roots
  - 4) Exponentials in $H(s)$

Let's Digress just a little

- What is the Laplace Transform of $\delta(t)$?

$$\Delta(s) = \int_{-\infty}^{\infty} \delta(\tau) e^{-s\tau}\, d\tau = e^{-s\tau}\Big|_{\tau=0} = 1 \quad (!!)$$

- So the Laplace Transform of a delta function is a constant!...
- ...and vice-versa
- So $X_1(s) = 3 \Rightarrow 3\delta(t)$

$$X_2(s) = \frac{-19.5(s-1.487)}{(s+5)(s+2)} = -19.5\left(\frac{R(-5)}{s+5} + \frac{R(-2)}{s+2}\right) \quad s > -2$$

And a more complicated case

$$H(s)=\frac{A}{s-r}+\frac{C+jD}{s-(\sigma+j\omega)}+\frac{C-jD}{s-\left(\sigma-j\omega\right)},\ \mathrm{Re}[s]>\max(r,\sigma)$$

Causal!

$$h(t)=Ae^{rt}u(t)+\left(C+jD\right)e^{(\sigma+j\omega)t}u(t)+\left(C-jD\right)e^{(\sigma-j\omega)t}u(t)$$

$$=Ae^{rt}u(t)+Ce^{\sigma t}\left(e^{j\omega t}+e^{-j\omega t}\right)u(t)+jDe^{\sigma t}\left(e^{j\omega t}-e^{-j\omega t}\right)u(t)$$

$$=Ae^{rt}u(t)+2Ce^{\sigma t}\cos(\omega t)u(t)+2De^{\sigma t}\sin(\omega t)u(t)$$

$$=\left(Ae^{rt}+e^{\sigma t}\left(E\cos\omega t+F\sin\omega t\right)\right)u(t)$$

$=$ exponential term + second order solution from CMPE306!

Laplace Transforms III: Properties of the Laplace Transform

Summary

- LCCDE → LT: Put in standard form, fine the a, b coefficients, write the rational form, factor & find residuals, convert 1st and 2nd order terms to time domain based on RoC.
- $X(s) \rightarrow x(t)$: Use the synthesis equation

$$x(t) = \frac{1}{2\pi j}\int_{-\infty}^{\infty} X(s)e^{st}\,ds = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds$$

- …which can be complicated because of the integration in the complex plane
- …but there are properties that make things simpler

Key properties

- Linearity: Of course it is linear!

$$\mathcal{L}\left(Ax_1(t) + Bx_2(t)\right) = AX_1(s) + BX_2(s),\ R' \supset R_1 \cap R_2$$

- Time Shift

$$\mathcal{L}\left(x(t-t_0)\right) = e^{-st_0}X(s),\ R' = R$$

- Modulation

$$\mathcal{L}\left(e^{s_0 t}x(t)\right) = X(s-s_0),\quad R' = R + \mathrm{Re}[s_0]$$

- Scaling (remember time scaling!)

$$\mathcal{L}\left(x(at)\right) = \frac{1}{|a|}X\left(\frac{s}{a}\right),\ R' = aR$$

http://lpsa.swarthmore.edu/LaplaceZTable/LaplacePropTable.html

More Properties

- **Differentiation in the time domain**

$$x(t)=\frac{1}{2\pi j}\int_{-\infty}^{\infty} X(s)e^{st}\,ds=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds$$

$$\frac{dx}{dt}=\frac{d}{dt}\left[\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}\,ds\right]=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty}\frac{d}{dt}\left[X(s)e^{st}\right]ds$$

$$=\frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} sX(s)e^{st}\,ds$$

$$\text{Therefore } \mathcal{L}\left(\frac{dx}{dt}\right)=sX(s)\quad R'\supset R$$

- **Differentiation in the transform domain**

$$X(s)=\int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

$$\frac{dX}{ds}=\frac{d}{ds}\left[\int_{-\infty}^{\infty} x(t)e^{-st}\,dt\right]=\int_{-\infty}^{\infty} x(t)\frac{d}{ds}\left[e^{-st}\right]dt=\int_{-\infty}^{\infty}(-t)x(t)e^{-st}\,dt$$

$$\text{Therefore } \mathcal{L}\big((-t)x(t)\big)=\frac{dX}{ds},\quad R'=R$$

- **Integration in the time domain**

$$\int_{-\infty}^{t} x(\tau)\,d\tau=\int_{-\infty}^{t}\frac{1}{2\pi j}\int_{\sigma-\infty}^{\sigma+j\infty} X(s)e^{st}\,ds=\frac{1}{2\pi j}\int_{\sigma-\infty}^{\sigma+j\infty} X(s)\int_{-\infty}^{t} e^{st}\,ds$$

$$=\int_{\sigma-\infty}^{\sigma+j\infty}\frac{1}{s}X(s)e^{st}\,ds$$

$$\text{Therefore } \mathcal{L}\left(\int_{-\infty}^{t} x(\tau)\,d\tau\right)=\frac{1}{s}X(s),\ R'\supset R\cap \mathrm{Re}[s]>0$$

- **Initial Value Theorem**

$$\lim_{s\to\infty}\left(sF(s)\right)=f\left(0^{+}\right)$$

- **Final Value Theorem**

$$\lim_{s\to 0}\left(sF(s)\right)=\lim_{t\to\infty}f(t)$$

The Convolution Theorem

- **One of the primary application of the Laplace (and later Fourier) Domain(s) is the Convolution Theorem**
- **What is the LT of the convolution?**

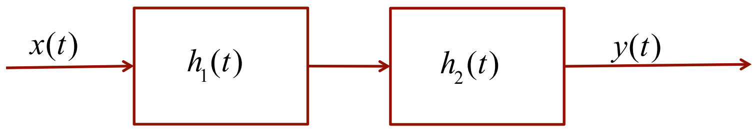
$$\mathcal{L}\left(x*h\right)=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}x(\tau)h(t-\tau)\,d\tau\right)e^{-st}\,dt$$

$$=\int_{-\infty}^{\infty}x(\tau)\underbrace{\left(\int_{-\infty}^{\infty}h(t-\tau)e^{-st}\,dt\right)}_{\text{time shift property!}}d\tau=\int_{-\infty}^{\infty}x(\tau)\left(e^{-s\tau}H(s)\right)d\tau$$

$$=H(s)\int_{-\infty}^{\infty}x(\tau)e^{-s\tau}\,d\tau=H(s)X(s)=X(s)H(s)\quad(!!)$$

So what?

- **Remember our concatenation of systems?**



$$y(t) = x * h = x * (h_1 * h_2)$$
$$H(s) = \mathcal{L}\left(h_1 * h_2\right) = H_1(s)H_2(s)$$
$$Y(s) = \mathcal{L}\left(x * h\right) = X(s)H(s) = X(s)H_1(s)H_2(s)$$

- **A convolution in the time domain is equivalent to a multiplication in the frequency domain…**
- **…and *vice versa***