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MATH 407 HW 02

A.1 Let A, B, C be subsets of a given set S. Prove the following statements.

10 $(A-B)\cup(B-A)=(A\cup B)-(A\cap B)$

Ans Let $x\in(A-B)\cup(B-A)$.

Therefore, $x\in(A-B)$ or $x\in(B-A)$.

If $x\in(A-B)$, then $x\in A, x\notin B$.

Therefore:

$$x\in(A\cup B),\ x\notin(A\cap B)$$
$$\Rightarrow x\in(A\cup B)-(A\cap B)$$

Similarly, if $x\in(B-A)$, then $x\in(A\cup B)-(A\cap B)$.

Therefore, $(A-B)\cup(B-A)\subseteq(A\cup B)-(A\cap B)$

Conversely, if $x\in(A\cup B)-(A\cap B)$, then $x\in(A\cup B),\ x\notin(A\cap B)$

If $x\in(A\cup B)$, then $x\in A$ or $x\in B$.

If $x\in A$, then $x\notin B$.

Therefore:

$$x\in A,\ x\notin B\Rightarrow x\in(A-B)$$
$$\Rightarrow x\in(A-B)\cup(B-A)$$

Similary, if $x\in B$, then $x\in(A-B)\cup(B-A)$.

Therefore, $(A\cup B)-(A\cap B)\subseteq(A-B)\cup(B-A)$

$\therefore (A-B)\cup(B-A)=(A\cup B)-(A\cap B)$ $\square$

11 $(A\cup B)\times C=(A\times C)\cup(B\times C)$

Ans Let $(x,y)\in(A\cup B)\times C$.

Then:

$$\begin{aligned}
&x \in (A \cup B),\ y \in C \\
&\Rightarrow (x \in A,\ y \in C) \text{ or } (x \in B,\ y \in C) \\
&\Leftrightarrow ((x,y) \in A \times C) \text{ or } ((x,y) \in B \times C) \\
&\Leftrightarrow (x,y) \in (A \times C) \cup (B \times C)
\end{aligned}$$

$\therefore (A \cup B) \times C = (A \times C) \cup (B \times C)$ □

A.4 **9** Let $a_1, \ldots, a_n$ be positive real numbers, $G_n = \sqrt[n]{a_1 a_2 \ldots a_n}$, and $A_n = \frac{1}{n}\sum_{i=1}^{n} a_i$. Then G_n is called the geometric mean and A_n is called the arithmetic mean. We wish to show that $G_n \leq A_n$.

1. Show that $G_2 \leq A_2$.

Ans Substituting $n = 2$:

$$\begin{aligned}
G_2 &= \sqrt[2]{a_1 a_2 \ldots a_2} \\
&= \sqrt[2]{a_1 a_2} \\
&= \sqrt{a_1 a_2} \\
A_2 &= \frac{1}{2}\sum_{i=1}^{2} a_i \\
&= \frac{1}{2}(a_1 + a_2)
\end{aligned}$$

If $a_1 = a_2$, then $G_2 = a_1 = A_2 = a_1$.

If $a_1 \neq a_2$, then let $a_1 = k, a_2 = nk + 1$, $k \in \mathbb{R}^+$.
Then, $G_2 = \sqrt{(k)(k+1)}$, $G_2^2 = k^2 + k$
and $A_2 = \frac{1}{2}(2k+1)$, $A_2^2 = k^2 + k + \frac{1}{4}$

$\therefore G_2 \leq A_2$ □

2. Show that $G_{2^n} \leq A_{2^n}$ by using induction on n.

Ans It was proven the proposition held for $n = 2^k$ for $k = 1 \Rightarrow 2$.

Suppose the proposition holds for $n = 2^k,$ for $k > 1$. Therefore:

$$\begin{aligned}
A_{2^k} &= \frac{1}{2^k}\sum_{i=1}^{2^k} a_i \\
&= \frac{1}{2^k}(a_1 + a_2 + \ldots + a_{2^k}) \\
&= \frac{\frac{1}{2^{k-1}}(a_1 + a_2 + \ldots + a_{2^{k-1}}) + \frac{1}{2^{k-1}}(a_{2^{k-1}+1} + a_{2^{k-1}+2} + \ldots + a_{2^k})}{2} \\
&\geq \frac{\sqrt[2^{k-1}]{a_1 + a_2 + \ldots + a_{2^{k-1}}} + \sqrt[2^{k-1}]{a_{2^{k-1}+1} + a_{2^{k-1}+2} + \ldots + a_{2^k}}}{2} \\
&\geq \sqrt{\sqrt[2^{k-1}]{a_1 + a_2 + \ldots + a_{2^{k-1}}} + \sqrt[2^{k-1}]{a_{2^{k-1}+1} + a_{2^{k-1}+2} + \ldots + a_{2^k}}} \\
&\geq \sqrt[2^k]{a_1 a_2 \ldots a_{2^k}} \\
&= G_{2^k}
\end{aligned}$$

$\therefore G_{2^k} \leq A_{2^k}$ $\square$

10 Let a and b be real numbers. Prove the binomial theorem , which states that

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i} \quad \text{where} \quad \binom{n}{i} = \frac{n!}{i!(n-i)!}$$

and $n!$ = $n(n-1)\ldots 2 \cdot 1$ for $n \geq 1$ and $0! = 1$.

Hint: $\binom{m+1}{k} = \binom{m}{k} + \binom{m}{k-1}$.

Ans Base case: For $n = k = 0$:

$$\begin{aligned}
(a+b)^0 &= \sum_{i=0}^{0} \binom{n}{i} a^i b^{n-i} \\
&= \binom{0}{0} a^0 b^0 \\
&= 1
\end{aligned}$$

Assume the proposition holds for $n = k + 1$. Then:

$$\begin{aligned}
(a+b)^{k+1} &= (a+b)(a+b)^k \\
&= (a+b)\sum_{i=0}^{k}\binom{k}{i}a^i b^{k-i} \\
&= a\sum_{i=0}^{k}\binom{k}{i}a^i b^{k-i} + b\sum_{i=0}^{k}\binom{k}{i}a^i b^{k-i} \\
&= \sum_{i=0}^{k}\binom{k}{i}a^{i+1} b^{k-i} + \sum_{i=0}^{k}\binom{k}{i}a^i b^{k+1-i} \\
&= \sum_{i=1}^{k+1}\binom{k}{i-1}a^{(i-1)+1} b^{k-(i-1)} + \sum_{i=0}^{k}\binom{k}{i}a^i b^{k+1-i} \\
&= \sum_{i=1}^{k+1}\binom{k}{i-1}a^{i} b^{k+1-i} + \sum_{i=0}^{k}\binom{k}{i}a^i b^{k+1-i} \\
&= \binom{0}{0}a^0 b^{k+1} + \sum_{i=1}^{k+1}\binom{k}{i-1}a^{i} b^{k+1-i} + \binom{0}{0}a^{k+1}b^0 + \sum_{i=1}^{k}\binom{k}{i}a^i b^{k+1-i} \\
&= \binom{0}{0}a^0 b^{k+1} + \binom{k+1}{k+1}a^{k+1}b^0 + \sum_{i=1}^{k}\left(\binom{k}{i} + \binom{k}{i-1}\right)a^i b^{k+1-i} \\
&= b^{k+1} + a^{k+1} + \sum_{i=1}^{k}\binom{k+1}{i}a^i b^{k+1-i} \\
\therefore (a+b)^{k+1} &= \sum_{i=0}^{k+1}\binom{k+1}{i}a^i b^{k+1-i} \qquad \square
\end{aligned}$$

11 Find a formula for the derivative of the product of n functions, and give a detailed proof by induction (assuming the product rule for the derivative of two functions).

Ans Let $h = \prod_{i=1}^{n} f_i$ be the product of n functions.
We need to show:

$$h' = f_1'(f_2 \cdot f_3 \cdot \ldots \cdot f_n) + f_2'(f_1 \cdot f_3 \cdot \ldots \cdot f_n) + \ldots + f_n'(f_1 \cdot f_2 \cdot f_3 \cdot \ldots \cdot f_{n-1}) \text{ (product rule)}$$

Base case: For $n = 2$, $h = \prod_{i=1}^{2} f_i$

$$h' = f_1' \cdot f_2 + f_2' \cdot f_1 \text{ (product rule)}$$

Suppose the proposition holds for $n = k \geq 2$, with $h = f_1 \cdot f_2 \cdot f_3 \cdot \ldots \cdot f_{k+1}$. Then:

$$\begin{aligned} h' &= (f_1 \cdot f_2 \cdot f_3 \cdot \ldots \cdot f_k)' f_{k+1} + f'_{k+1}(f_1 \cdot f_2 \cdot f_3 \cdot \ldots \cdot f_k) \text{ (product rule)} \\ &= (f'_1(f_2 \cdot f_3 \cdot \ldots \cdot f_k) + f'_2(f_1 \cdot f_3 \cdot \ldots \cdot f_k) + \ldots \\ &\qquad + f'_k(f_1 \cdot f_2 \cdot f_3 \cdot \ldots \cdot f_{k-1}))f_{k+1} + f'_{k+1}(f_1 \cdot f_2 \cdot f_3 \cdot \ldots \cdot f_k) \\ \therefore h' &= f'_1(f_2 \cdot f_3 \cdot \ldots \cdot f_{k+1}) + f'_2(f_1 \cdot f_3 \cdot \ldots \cdot f_{k+1}) + \ldots + f'_{k+1}(f_1 \cdot f_2 \cdot f_3 \cdot \ldots \cdot f_k) \qquad \square \end{aligned}$$

12 Find a formula for the nth derivative of the product of two functions, and give a detailed proof by induction.

Ans Let f, g be two functions.

Using the general Leibniz rule, the nth derivative of a product of two functions is given by:

$$(fg)^{(n)} = \sum_{i=0}^{n} \binom{n}{i} f^{(n-i)} g^{(i)}$$

Base case: For $n = k = 0$,

$$\begin{aligned} (fg)^{(0)} &= fg \\ &= \binom{0}{0} f^{(0)} g^{(0)} \end{aligned}$$

Assume the proposition holds for $n = k + 1$, such that $(fg)^{(k+1)} = \big((fg)^{(k)}\big)'$.

Then:

$$
\begin{aligned}
(fg)^{(k+1)} &= \left((fg)^{(k)}\right)' \\
&= \left(\sum_{i=0}^{k}\binom{k}{i} f^{(k-i)} g^{(i)}\right)' \\
&= \sum_{i=0}^{k}\binom{k}{i}\left(f^{(k-i)} g^{(i)}\right)' \\
&= \sum_{i=0}^{k}\binom{k}{i}\left(f^{(k+1-i)} g^{(i)} + f^{(k-i)} g^{(i+1)}\right) \\
&= \sum_{i=0}^{k}\binom{k}{i} f^{(k+1-i)} g^{(i)} + \sum_{i=0}^{k}\binom{k}{i} f^{(k-i)} g^{(i+1)} \\
&= \sum_{i=0}^{k}\binom{k}{i} f^{(k+1-i)} g^{(i)} + \sum_{i=1}^{k+1}\binom{k}{i-1} f^{(k-(i-1))} g^{((i-1)+1)} \\
&= \sum_{i=0}^{k}\binom{k}{i} f^{(k+1-i)} g^{(i)} + \sum_{i=1}^{k+1}\binom{k}{i-1} f^{(k+1-i)} g^{(i)} \\
&= \binom{0}{0} f^{(k+1)} g^{(0)} + \sum_{i=1}^{k}\binom{k}{i} f^{(k+1-i)} g^{(i)} + \binom{k}{k} f^{(0)} g^{(k+1)} + \sum_{i=1}^{k}\binom{k}{i-1} f^{(k+1-i)} g^{(i)} \\
&= \binom{0}{0} f^{(k+1)} g^{(0)} + \binom{k+1}{k+1} f^{(0)} g^{(k+1)} + \sum_{i=1}^{k}\left(\binom{k}{i} + \binom{k}{i-1}\right) f^{(k+1-i)} g^{(i)} \\
&= f^{(k+1)} + g^{(k+1)} + \sum_{i=1}^{k}\left(\binom{k}{i} + \binom{k}{i-1}\right) f^{(k+1-i)} g^{(i)}
\end{aligned}
$$

Since $\binom{m+1}{k} = \binom{m}{k} + \binom{m}{k-1}$,

$$
\begin{aligned}
(fg)^{(k+1)} &= f^{(k+1)} + g^{(k+1)} + \sum_{i=1}^{k}\left(\binom{k+1}{i} + \binom{k}{i-1}\right) f^{(k+1-i)} g^{(i)} \\
\therefore (fg)^{(k+1)} &= \sum_{i=0}^{k+1}\binom{k+1}{i} f^{(k+1-i)} g^{(i)} \qquad \square
\end{aligned}
$$