MATH 407

5/14/18

$f(x) = a_0 + a_1 x^1 + \dots + a_n x^n$ [RECAP]

$0 = f(\frac{r}{s})$, $r | a_0$, $s | a_n$

<u>Cor.</u> If $a_n = \pm 1$, only integer roots.

$$x^n - c^n = (x-c)(x^{n-1} + x^{n-2}c + \dots x c^{n-2} + c^{n-2})$$
$$= (x-c) q_n(x)$$

$$f(x) - f(c) = \sum_{i=0}^{n} a_i (x^i - c^i)$$
$$= (x-c) \sum_{i=0}^{n} a_i q_i(x)$$
$$= (x-c) Q(x)$$

If n is an integer, $f(n) - f(c) = (n-c) Q(n)$

so $(n-c) | (f(n) - f(c))$

Thus $f(c) = 0 \Rightarrow (n-c) | f(n)$

<u>Ex.</u> $x^3 + 15x - 3x - 6$

Possible roots $c = \pm 1, \pm 2, \pm 3, \pm 6$

$f(1) = 7$, so $c \neq 1$

$\Rightarrow c = 2, -6$ are the only possibilities

$2-1 | 7$, $f(2) = 56$

$-6-1 | 7$, $f(-6) = 336$

* $\mathbb{R}[x]$, we know if $\deg(f)$ is odd, then f has a root. If f is non-linear then reducible.
Irreducibility $\Rightarrow$ even deg.

* In $\mathbb{C}[x]$,
  a) all non-const polynomials have roots
  b) all non-const polynomials factor completely into linear factors

Let $f \in \mathbb{R}[x] \subseteq \mathbb{C}[x]$

* Complex conjugate: if $z = a+bi$, then $\bar{z} = a-bi$

↳ $z \to \bar{z}$ is real linear bijection (in fact, isometry)
↳ $|z| = \sqrt{a^2+b^2} = |\bar{z}|$

↳ $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$, ↳ $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$
↳ $\bar{\bar{z}} = z$ (conjugation is idempotent)

↳ $z + \bar{z} = 2a$	↳ $z - \bar{z} = 2bi$
$= 2\operatorname{Re}\{z\}$	$= 2\operatorname{Im}\{z\}i$
$= 2\operatorname{Re}\{z\}$	$z = \bar{z}$ iff z is real

↳ $z \cdot \bar{z} = |z|^2 = a^2+b^2$, so $z\left(\frac{\bar{z}}{a^2+b^2}\right) = 1$

so $z^{-1} = \frac{\bar{z}}{a^2+b^2}$

$\hookrightarrow \bar{z}^{-1} = \overline{(z^{-1})} = \overline{\frac{z}{a^2+b^2}} = \frac{\bar{z}}{a^2+b^2}$

$\hookrightarrow (\bar{z})^n = \overline{z^n}$

$$* \overline{p(z)} = \overline{\left(\sum_{i=1}^{n} a_i z^i\right)}$$
$$= \sum_{i=1}^{n} \bar{a}_i \overline{z^i}$$
$$= \sum_{i=1}^{n} \bar{a}_i (\bar{z})^i$$

If $p \in \mathbb{R}[z]$ then $p(\bar{z}) = \sum_{i=1}^{n} a_i (\bar{z})^i$

$$= \overline{\left(\sum_{i=1}^{n} a_i z^i\right)}$$
$$= \overline{p(z)}$$

Note: $\mathbb{R}[z]$: polys in indeterminant z

Thm. If $p \in \mathbb{R}[z]$ then all roots of p occur in conjugate pairs

$p(z) = (z - r_1) \dots (z - r_k)(z - z_{k+1})(z - \bar{z}_{k+1})$
w/ $r_i \in \mathbb{R}$

$z_{k+1} = a + bi$, $(z - z_{k+1})(z - \bar{z}_{k+1})$
$= (z - (a+bi))(z - (a+bi))$

$\Longrightarrow$

$= \left(z^2 - (a+bi)z - (a+bi)z + (a^2+b^2)\right)$
$= \left(z^2 - 2az + (a^2+b^2)\right)$

$\therefore$ In real polynomials, irreducible ones are quadratic, negative discriminant. (Refinement of irreducibility criteria)

Thm. 4.4.5 A polynomial in $\mathbb{Z}[x]$ which may be written as $a(x)\cdot b(x)$ w/ $a,b \in \mathbb{Q}[x]$ may be written as $\alpha(x)\cdot\beta(x)$ w/ $\alpha, \beta \in \mathbb{Z}[x]$

Cor. $p \in \mathbb{Z}[x]$ is irreducible over $\mathbb{Z}$ iff it is over $\mathbb{Q}$

Thm 4.4.6 : Eisenstein irreducibility criterion
$f(x) = a_n x^n + \dots + a_0$. Let p be prime.
$a_{n-1} \equiv a_{n-2} \equiv \dots \equiv a_0 \equiv 0 \pmod{p}$, $a_i \in \mathbb{Z}[x]$
$a_n \not\equiv 0 \pmod{p}$
$a_0 \not\equiv 0 \pmod{p^2}$

Then f is irreducible in $\mathbb{Z}[x]$

Ex. $\Phi_p(x) = x^{p-1} + x^{p-2} + \dots + x + 1$
p prime $= \dfrac{x^p - 1}{x-1}$

Let $\Psi_p = \Phi_p(x+1)$.

$T_c : f \to f(x+c)$

Then $T_c : F[x] \to F[x]$ is isomorphism.

$$\Psi = \frac{(x+1)^p - 1}{x} = \sum_{k=1}^{p} \binom{p}{k} x^{k-1}$$