CMPE 320: Probability, Statistics, and Random Processes

Lecture 12: Independence of RVs

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Independence of events (Recap)

• "Events A and B are independent" means observing event B does not provide any information on A

le any information on A
$$P(A|B) = P(A)$$

$$P(A \cap B) = P(A) P(B)$$

• "RV X is independent of event A" means observing event A does not provide any information on the value of X

$$E = Event \{X = x\}$$
 and A are independent for all x
$$P(X = x \mid A) = P(X = x)$$

$$P(X \mid A) = P(X \mid x)$$

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> Example 2.19. Consider two independent tosses of a fair coin. Let X be the number of heads and let A be the event that the number of heads is even.

Are X and A independent?

In Appartence of X and A
$$\Rightarrow$$
 $P_{X|A}(x) = P_{X}(x)$

$$P_{X(A}(x) = \frac{P(X=x \cap A)}{P(A)} = \begin{cases} \frac{1}{2} = \frac{1}{2}, x=0 \\ \frac{1}{2} = \frac{1}{2}, x=0 \end{cases}$$

Independence of RVs

• "RVs X and Y are independent" means that the value of Y provides no information on the value of X

Events {
$$X=xy$$
 and { $Y=7y$ are independent for all x,y $P(X=x)$ and $Y=y) = P(X=x) P(Y=y)$ for all x,y $P_{X,Y}(x,y) = P_{X}(x) P_{Y}(y)$ for all x,y

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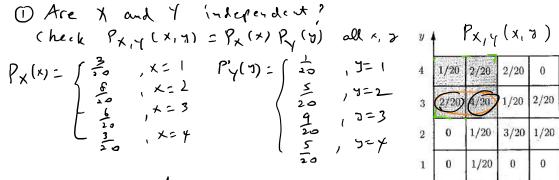
Conditional independence of RVs

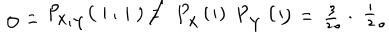
 Conditioning with an event A defines a new universe where all probabilities (or PMFs) are replaced by their conditional versions

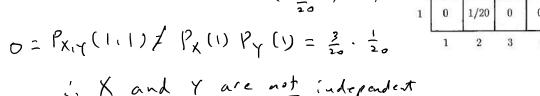
X and Y are conditionally independent given A if
$$P(X=\times, Y=\Im|A) = P(X=\times|A) P(Y=\Im|A) \text{ for all } x,y$$

$$P_{X,Y|A}(x,y) = P_{X|A}(x) P_{Y|A}(y) \text{ for all } x,y$$

Conditional independence may not imply independence, and vice versa







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Expectation of the product of independent RVs

If X and Y are independent RVs, then E[XY] = E[X] E[Y]

• Similarly E[g(X) h(Y)] =

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Variance of the sum of independent RVs

• If X and Y are independent RVs, var(X + Y) = var(X) + var(Y)

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Independence of several RVs

- X, Y, Z are independent RVs if
- If X, Y, Z are independent, f(X), g(Y), h(Z) are also independent
 - How about g(X,Y) and h(Z)?
 - How about g(X,Y) and h(Y,Z)?

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Variance of sum of several independent RVs

• If X_1, X_2, \dots, X_n are independent RVs, $\text{var}(X_1 + X_2 + \dots + X_n) =$

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Example 2.20. Variance of the Binomial and the Poisson. We consider n independent coin tosses, with each toss having probability p of coming up a head. For each i, we let X_i be the Bernoulli random variable which is equal to 1 if the ith toss comes up a head, and is 0 otherwise. Then, $X = X_1 + X_2 + \cdots + X_n$ is a binomial random variable. What are its mean and variance?

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Example 2.21. Mean and Variance of the Sample Mean. We wish to estimate the approval rating of a president, to be called B. To this end, we ask n persons drawn at random from the voter population, and we let X_i be a random variable that encodes the response of the ith person:

$$X_i = \left\{ \begin{array}{ll} 1, & \text{if the ith person approves B's performance,} \\ 0, & \text{if the ith person disapproves B's performance.} \end{array} \right.$$

We model X_1, X_2, \ldots, X_n as independent Bernoulli random variables with common mean p and variance p(1-p). Naturally, we view p as the true approval rating of B. We "average" the responses and compute the **sample mean** S_n , defined as

$$S_n = \frac{X_1 + X_2 + \dots + X_n}{n}.$$

Thus, the random variable S_n is the approval rating of B within our n-person sample. What are the mean and variance of S_n ?