CMPE 330 Spring 2017 Second Midterm Examination

(Each problem is worth 10 points)

1. In two dimensions, we relate the Cartesian coordinates (x, y) to the polar coordinates (r, ϕ) by the relations $r = (x^2 + y^2)^{1/2}$, $\phi = \tan^{-1}(y/x)$.

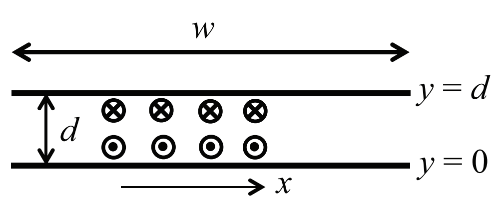
   a. What are $x(r, \phi)$ and $y(r, \phi)$?

   b. Write $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ as functions of r, ϕ, $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$.

2. Give the fundamental definition of the gradient operator and use the definition to give the form of the gradient operator in Cartesian, cylindrical, and spherical coordinates.

3. Consider a coaxial cable with inner radius a and outer radius b. Between the two conductors, the material is characterized by permittivity ϵ, permeability μ, and conductivity σ. Determine the capacitance per unit length C'. What are the inductance per unit length L' and the conductance per unit length G'?

4. Consider a parallel-plate microstrip line of width w and a plate separation d with $d \ll w$. Suppose that the width is in the x-direction, the first plate is at $y = 0$, and the second plate is at $y = d$. A current I flows in the $+z$-direction on the plate at $y = 0$ and in the $-z$-direction in the plate at $y = d$. The geometry is shown below. Use Ampere's law to show that the magnetic flux between the plates is given by $\mathbf{B} = -\hat{\mathbf{x}}(\mu I/w)$. Show that the energy per unit length that is contained between the plates is given by $W'_{\mathrm{m}} = (\mu/2)(d/w)I^2$.

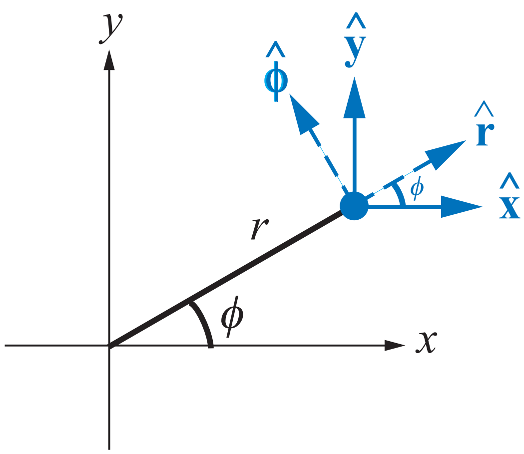


5. Consider an infinite planar interface at $z = 0$ that separates a medium with permittivity and permeability ϵ_1 and μ_1 when $z > 0$ and with permittivity and permeability ϵ_2 and μ_2 when $z < 0$. Both media are non-conducting. Derive the relations between $\mathbf{E}$ and $\mathbf{D}$ and between $\mathbf{H}$ and $\mathbf{B}$ in the upper and lower layers.

Second Midterm Examination Solutions

1. The characteristic impedance of the impedance-matching line (Z_{im} equals the geometric mean of the two impedances, so that $Z_{\text{im}} = (Z_1 Z_2)^{1/2} = \sqrt{5000} = 100/\sqrt{2} = 0.707 \times 100 = 71\ \Omega$. The length of the line L_{im} is given by $L_{\text{im}} = \lambda/4 = u_{\text{p}}/(4f) = 2 \times 10^8/(400 \times 10^9) = 5 \times 10^{-4}$ m $= 500\ \mu$m.

2. We show the geometry in the figure below:



   a. We have $x = r\cos\phi$ and $y = r\sin\phi$.

   b. From the figure, we infer

$$\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi,$$
$$\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi.$$

3. From Gauss's law, we have that the field due a surface charge ρ_{S} on a long cylinder of radius a and length L is given by $\mathbf{E} = \hat{\mathbf{r}}(\rho_S/\epsilon)(a/r)$, so that the voltage difference between the inner and outer radius is $V = (\rho_{\text{S}}/\epsilon)a\ln(b/a)$. The charge Q is given by $Q = 2\pi\rho_{\text{S}}aL$. We thus have that the capacitance per unit length is given by $C' = C/L = Q/VL = (2\pi\epsilon)/\ln(b/a)$. Recalling that $C'L' = \mu\epsilon$, we find that $L' = (\mu/2\pi)\ln(b/a)$. Recalling that $G' = (\sigma/\epsilon)C'$, we find $G' = (2\pi\sigma)/\ln(b/a)$.

4. Ampere's law states that $\oint \mathbf{B}\cdot d\mathbf{l} = \mu I$, where I is the current flowing through any closed loop. Taking a loop that encloses a portion $\Delta x = l$ of the lower plate, recalling that the field below the lower plate is zero, and noting that the current flowing through that portion is given by Il/w, we have $Bl = \mu Il/w$. From the right-hand rule, the field is oriented in the $-x$ direction, so that the field is given by $\mathbf{B} = -\hat{\mathbf{x}}\mu(I/w)$. It follows that $\mathbf{H} = -\hat{\mathbf{x}}(I/w)$, so that the energy density is given by $w_{\text{m}} = (1/2)\mathbf{B}\cdot\mathbf{H} = (\mu/2)(I^2/w^2)$. Since the area between the plates is given by wd, we conclude that the energy per unit length is given by $W'_{\text{m}} = (\mu/2)(d/w)I^2$.

5. We first take a small rectangular loop in the x-z plane of length Δl in the x-direction and length Δh in the z-direction, where $\Delta h \ll \Delta l$. Tracing the field around the loop, we find $\oint \mathbf{E} \cdot d\mathbf{l} = (E_{2x} - E_{1x})\Delta l = 0$, which implies that $E_{2x} = E_{1x}$. We find in a similar way that $E_{2y} = E_{1y}$. Next we consider a cylindrical volume of height Δh in the z-direction and radius Δr. Since there is no charge, we find $\int \mathbf{D} \cdot d\mathbf{s} = \hat{\mathbf{z}} \cdot (\mathbf{D}_1 - \mathbf{D}_2) = 0$, which implies $D_{2z} = D_{1z}$. These relations in turn imply $D_{1x}/\epsilon_1 = D_{2x}/\epsilon_2$, $D_{1y}/\epsilon_1 = D_{2y}/\epsilon_2$, and $\epsilon_1 E_{1z} = \epsilon_2 E_{2z}$.

   The relations between $\mathbf{B}$ and $\mathbf{H}$ are similarly obtained. Using a small rectangular loop and the relation $\oint \mathbf{H} \cdot d\mathbf{l} = 0$, we find $H_{2x} = H_{1x}$ and $H_{2y} = H_{1y}$. Using a small cylinder and the relation $\int \mathbf{B} \cdot \mathbf{s} = 0$, we find $B_{2z} = B_{1z}$. From these relationships, we find in turn, $B_{2x}/\mu_2 = B_{1x}/\mu_1$, $B_{2y}/\mu_2 = B_{1y}/\mu_2$, and $\mu_2 H_{2z} = \mu_1 H_{1z}$.