1. (30 pts.) True/False. Give a brief explanation

   (a) Let $\exp(x, y)$ be the binary operation on $\mathbb{R}^+$ defined by $\exp(x, y) = x^y$. Exp is neither associative nor commutative.

   (b) $T_n = \{$products of an even number of cycles of length 3 in $S_n\}$ is a subgroup of S_n.

   (c) If the order of a group is 125, then the group contains an element of order 5.

   (d) If p and q are prime then $\mathbb{Z}_{pq}$ has $pq - 1$ generators.

   (e) $\mathbb{Z}_{10} \times \mathbb{Z}_8$ is isomorphic to $\mathbb{Z}_{40} \times \mathbb{Z}_2$.

   (f) For a permutation to be odd, the number of cycles of even length in the cycle decomposition must be odd.

2. (20 pts.)

   (a) If $\sigma \in S_n$ and $\tau = \sigma^k$ is a cycle of length n for some $k \geq 1$, why must σ also be a cycle of length n?

   (b) Let $\tau = \sigma^3 = (1\,2\,3\,5\,7\,8\,4\,6)$. Determine $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_8)$.

3. (20 pts.)

   (a) Let $\alpha = (1\,4\,2\,3\,5)(1\,2) \in S_5$. What is α^{2015}?

   (b) How many elements in S_5 have order 3?

   (c) How many subgroups does $\mathbb{Z}_{30}$ have?

   (d) Let G be a cyclic group of order 11 with generator g. What is the order of g^{500}?

Choose **two** from questions 4, 5, 6, 7, [redacted] and do all parts (15 pts. each)

4. Let G be an Abelian group, and let $S \subset G$ be the subset of elements of finite order. Prove that S is a subgroup of G.

5. Let g and h be non-commuting elements of the a group G of odd order. Suppose that g has order 3 and that $ghg^{-1} = h^3$. Determine the order of h.

6. Let $A \subseteq GL_2(\mathbb{R})$ be defined as $A := \left\{ \begin{pmatrix} \mathrm{a} & \mathrm{b} \\ 0 & 1 \end{pmatrix} : a, b \in \mathbb{R},\, a \neq 0 \right\}$.
   Prove that A is a subgroup of $GL_2(\mathbb{R})$.
   Determine all the elements of A which have order 2.

7. For any subset S of a group G, define $N(S) = \{h : h \in G, hSh^{-1} = S\}$.
   (a) Show that $N(S)$ is a subgroup of G.
   (b) Show that if S is a subgroup of G, then S is a normal subgroup of $N(S)$.