CMPE 320: Probability, Statistics, and Random Processes

Lecture 16: Joint PDFs of multiple RVs

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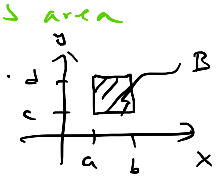
Joint PDF

- The notion of PDF can be extended to multiple RVs (just like PMF)
- If two continuous RVs X and Y are associated with the same experiment, joint PDF $f_{X,Y}$ is a nonnegative function that satisfies

$$P((X,Y) \in B) = \iint_{(x,y)\in B} f_{X,Y}(x,y)\,dx\,dy$$

– integrate over this area

for all subsets B of 2-dimensional plane.



- In particular, if B={$(x, y) | a \le x \le b, c \le y \le d$}
  P($a \le X \le b, c \le Y \le d$) = $\int_a^b \int_c^d f_{X,Y}(x,y)\,dy\,dx$
- B = $\mathbb{R}^2$ (entire 2-dimensional plane) $P(\mathbb{R}^2) = 1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy$

Interpretation of joint PDF

- For small positive δ,
  P($a \leq X \leq a+\delta, c \leq Y \leq c+\delta$) = $\int_a^{a+\delta}\int_c^{c+\delta} f_{X,Y}(x,y)\,dy\,dx$

$$\approx f_{X,Y}(a,c)\,\delta^2$$

$\Rightarrow f_{X,Y}(x,y)$ is the probability per unit area around (a,c)

Marginal PDF from joint PDF

- Probability of $\{X \in A\}$ from joint PDF $f_{X,Y}$

$$P(X\in A) = P(X \in A \cap Y \in (-\infty,\infty))$$

$$= \int_A \boxed{\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy}\,dx$$

Recall that $f_X(x)$ is a PDF of X if for any A

$$P(X\in A) = \int_A f_X(x)\,dx$$

Comparing: $\boxed{f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy}$

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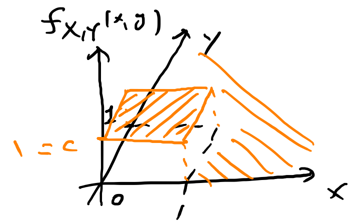
Example 3.9. Two-Dimensional Uniform PDF. Romeo and Juliet have a date at a given time, and each will arrive at the meeting place with a delay between 0 and 1 hour. Let X and Y denote the delays of Romeo and Juliet, respectively. Assuming that no pairs (x, y) in the unit square are more likely than others, determine $f_{X,Y}(x,y)$.

$$f_{X,Y}(x,y) = \begin{cases} c, & \text{if } 0 \le x \le 1,\ 0 \le y \le 1 \\ 0, & \text{otherwise} \end{cases}$$



How to determine c?

$$1 = \underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy}_{\text{volume under the surface } f_{X,Y}(x,y)} = \int_0^1\int_0^1 c\,dx\,dy = c \Rightarrow c = 1$$

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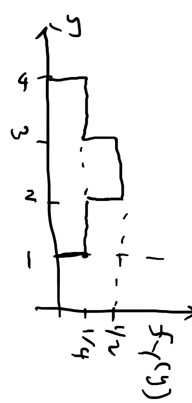
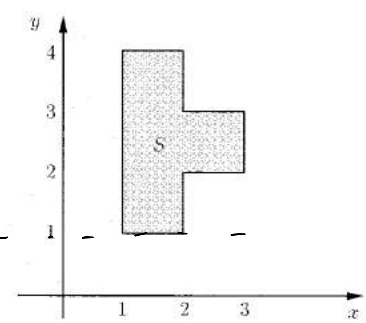
Example 3.10. We are told that the joint PDF of the random variables X and Y is a constant c on the set S shown in Fig. and is zero outside. We wish to determine the value of c and the marginal PDFs of X and Y.

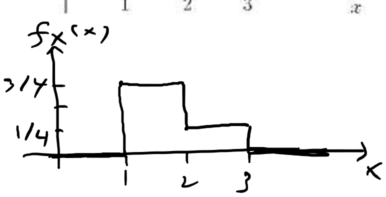
$$f_{X,Y}(x,y) = \begin{cases} c & \text{if } (x,y) \in S \\ 0 & \text{otherwise} \end{cases}$$

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = \int_S c\,dx\,dy = c\,(\text{area of } S) = 4c \Rightarrow c = 1/4$$





$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \begin{cases} 3/4 & \text{if } 1 \le x \le 2 \\ 1/4 & \text{if } 2 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$



$$f_Y(y) = \begin{cases} 1/4 & \text{if } 1 \le y \le 2 \text{ or } 3 \le y \le 4 \\ 1/2 & \text{if } 2 \le y \le 3 \\ 0 & \text{otherwise} \end{cases}$$

Joint CDF

- If X and Y are two RVs associated with the same experiment, their joint CDF is defined as

$$F_{X,Y}(x,y) = P(X \le x \text{ and } Y \le y)$$

- From joint PDF to joint CDF

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(s,t)\,dt\,ds$$

- From joint CDF to joint PDF

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y)$$

Example 3.12.
The joint CDF of RVs X and Y is given by

$$F_{X,Y}(x,y) = \mathbf{P}(X \le x,\, Y \le y) = xy, \qquad \text{for } 0 \le x, y \le 1.$$

Compute the joint PDF $f_{X,Y}(x,y)$.

$$f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y) = 1 \quad \text{if } 0 \le x, y \le 1$$

$$\frac{\partial^2}{\partial x \partial y}(xy) = \underbrace{\frac{\partial}{\partial x} \underbrace{\frac{\partial}{\partial y}(xy)}_{}}_{= x} = 1$$

Expectation of a function of X and Y

- Z = g(X,Y) is a RV. Its expectation can be computed using $f_{X,Y}$

$$E[Z] = E[g(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{X,Y}(x,y)\,dx\,dy$$

If $g(X,Y) = aX + bY + c$,

$$E[aX + bY + c] = aE[X] + bE[Y] + c$$

More than two RVs

For example: 3 RVs $X, Y,$ and Z, $f_{X,Y,Z}(x,y,z)$ is a PDF

if $P((X,Y,Z) \in B) = \iiint_{(x,y)\in B} f_{X,Y,Z}(x,y,z)\,dx\,dy\,dz$

for any subset B

Marginalization: $f_{XY}(x,y) = \int_{-\infty}^{\infty} f_{X,Y,Z}(x,y,z)\,dz$

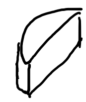
$$f_X(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y,Z}(x,y,z)\,dy\,dz$$

Expectation: $E[g(X,Y,Z)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y,z)\, f_{X,Y,Z}(x,y,z)\,dx\,dy\,dz$

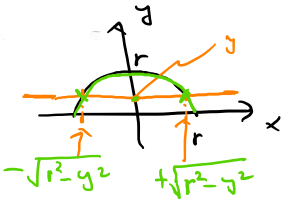
Problem 15. A point is chosen at random (according to a uniform PDF) within a semicircle of the form $\{(x, y) \mid \boxed{x^2 + y^2 \leq r^2}, \ y \geq 0\}$, for some given $r > 0$.

(a) Find the joint PDF of the coordinates X and Y of the chosen point.

(b) Find the marginal PDF of Y and use it to find $\mathbf{E}[Y]$.



(a) $f_{X,Y}(x,y) = \begin{cases} c & \text{if } (x,y) \in \text{semi-disc} \\ 0 & \text{otherwise} \end{cases}$



$c = ?$ $\qquad c = \dfrac{2}{\pi r^2}$

(b) $f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\,dx = \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} c\,dx = \dfrac{2}{\pi r^2} \cdot 2\sqrt{r^2-y^2} = \dfrac{4\sqrt{r^2-y^2}}{\pi r^2}$, if $0 \leq y \leq r$

$E[Y] = \int_{-\infty}^{\infty} y\, f_Y(y)\,dy = \int_0^r y \cdot \dfrac{4\sqrt{r^2-y^2}}{\pi r^2}\,dy =$

$= \dfrac{4r}{3\pi}$