CMPE 320: Probability, Statistics, and Random Processes

Lecture 8: Random Variable; Probability Mass Function

Spring 2018

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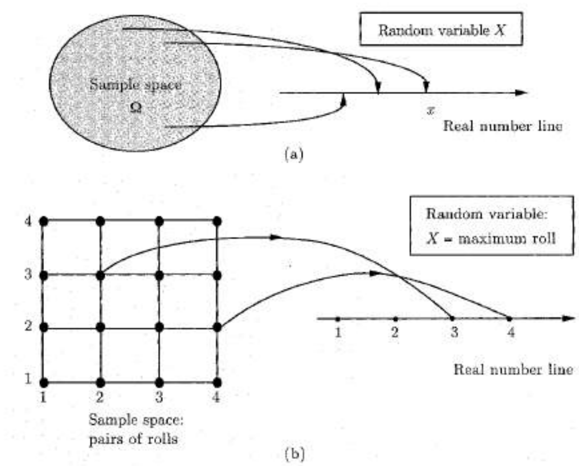
Announcements

- HW#4 has been posted on Blackboard (Due: 3/5)
  - See the TA if you haven't got your HW#1 back
- Midterm exam will be on Wednesday 3/7 during the class
  - Will cover everything up to Expectation (Sec. 2.4)
  - Closed book, closed note.
  - Calculators are allowed (No smartphones, tablets, laptop PCs allowed)
- Review session on 3/5. Can ask questions on HW problems
- TA Office Hours: Thursdays at 12pm @ ITE 353
  Instructor Office Hours: Tuesdays at 12pm @ ITE 312

Random variable

- RV is a function that maps the outcomes in Ω to numeric values



Some concepts we will learn about RVs

- A RV is a real-valued function of the outcome of an experiment
- A function of a RV defines another RV
- We can associate with each RV certain “averages” of interest, such as the mean and variance
- There is a notion of independence of a RV from an event or another RV

Discrete RV

- A discrete RV has its range (set of values the RV can take) either finite or countably infinite

capital letters to denote a RV

  - Number of heads from 5 coin tosses $X \in \{0, 1, 2, 3, 4, 5\}$
  - Sum of two rolls of a die $X \in \{2, 3, \ldots, 12\}$

Probability mass function (PMF)

- The RVs are characterized by the probabilities of the values it can take
- For discrete RVs, PMF is what does this

$$\underbrace{P_X}_{\substack{\text{RV} \\ \text{(upper-case)}}}(\underbrace{x}_{\substack{\text{value} \\ \text{(lower-case)}}}) = P(\underbrace{\{X = x\}}_{\text{RV value}})$$

all outcomes that give rise to value of X equal to x

$$P(\Omega) = \sum_{\text{all } x} P(X = x) = \boxed{1 = \sum_{\text{all } x} P_X(x)}$$

PMF example

- For 2 rolls of a 4-sided die, X = maximum roll

$X \in \{1, 2, 3, 4\}$

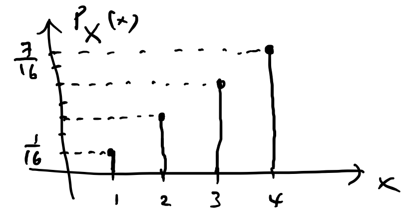
$P(X=1) = P(\{(1,1)\}) = \frac{1}{16}$

$P(X=2) = P(\{(2,2), (1,2), (2,1)\}) = \frac{3}{16}$

$P(X=3) = P(\{(3,3), (1,3), (3,1), (2,3), (3,2)\}) = \frac{5}{16}$

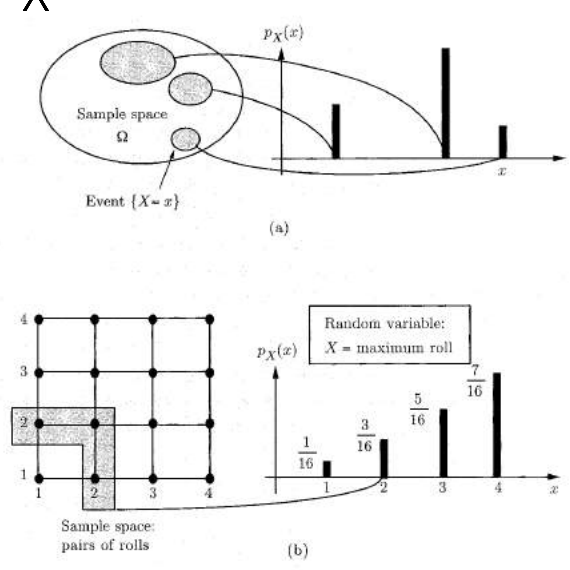
$P(X=4) = P(\{(4,4), (1,4), (4,1), (2,4), (4,2), (3,4), (4,3)\}) = \frac{7}{16}$

$$p_X(x) = \begin{cases} \frac{1}{16} & \text{if } x = 1 \\ \frac{3}{16} & \text{if } x = 2 \\ \frac{5}{16} & \text{if } x = 3 \\ \frac{7}{16} & \text{if } x = 4 \end{cases}$$



Calculation of PMF of a RV X

- For each possible value x of X:
    1. Collect all possible outcomes that give rise to event {X = x}
    2. Add their probabilities to obtain $p_X(x)$



- For two independent coin tosses of a fair coin, X = number of heads obtained. Calculate the PMF of X.

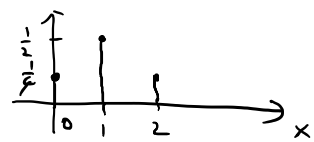
$$X \in \{0, 1, 2\}$$

$$P_X(0) = P(X=0) = P(\{TT\}) = \left(\tfrac{1}{2}\right)^2 = \tfrac{1}{4}$$

$$P_X(1) = P(X=1) = P(\{HT, TH\}) = \tfrac{1}{4} \times 2 = \tfrac{1}{2}$$

$$P_X(2) = P(X=2) = P(\{HH\}) = \tfrac{1}{4}$$

$$P_X(x) = \begin{cases} \tfrac{1}{4} & \text{if } x = 0 \text{ or } 2 \\ \tfrac{1}{2} & \text{if } x = 1 \end{cases}$$



Bernoulli RV

$$X = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$

PMF $P_X(x) = \begin{cases} p & \text{if } x = 1 \\ 1-p & \text{if } x = 0 \end{cases}$

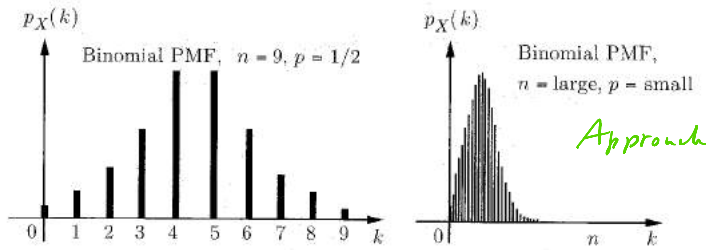
Useful for modeling
- coin toss
- state of phone line that's either busy / free
- person who is either sick / healthy

Binomial RV

X = number of heads in n coin tosses where a head comes up with prob. p

= binomial RV with parameters n and p

PMF $p_X(k) = P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$



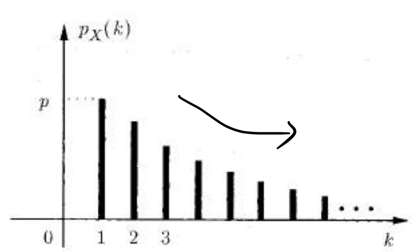
Approaches a Poisson RV

Geometric RV

For independent coin tosses with $P(\text{head}) = p$,

X = number of tosses needed for a head to come up for the first time

$$p_X(k) = P(X = k) = (1-p)^{k-1} p \qquad k = 1, 2, 3, \ldots$$



Poisson RV

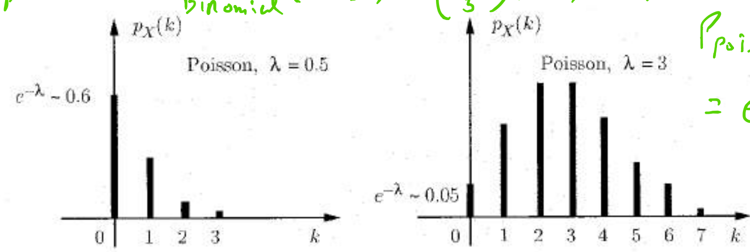
An event occurring at a constant rate λ per unit time

X = number of event occurrences in unit time

PMF $\quad P_X(k) = e^{-\lambda}\dfrac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$

Binomial RV with large n and small p can be well approximated by a Poisson RV with $\lambda = np$

$n = 100, \quad p = 0.01 \quad P_{Binomial}(k=5) = \binom{100}{5}(0.01)^5\, 0.99^{95} = 0.00290$

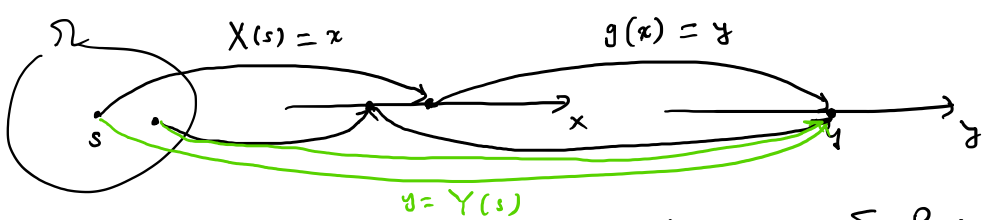


$P_{Poisson}(k=5) \quad \lambda = 1$

$= e^{-1}\dfrac{1^5}{5!} = 0.00306$

Functions of a RV

- X = today's temperature in Celsius
  Y = $\underbrace{1.8X + 32}_{g(X)}$ = today's temperature in Fahrenheit
- Y is also a RV defined in the same sample space Ω



PMF $\quad P_Y(y) = P(Y = y) = \displaystyle\sum_{\{x:\, g(x) = y\}} P(X = x) = \sum_{\{x:\, g(x)=y\}} P_X(x)$

Example 2.1. Let $Y = |X|$. Calculate $P_Y(y)$ when $P_X(x)$ is given by

$$p_X(x) = \begin{cases} 1/9, & \text{if } x \text{ is an integer in the range } [-4, 4], \\ 0, & \text{otherwise} \end{cases}$$

Possible values of X : $-4, -3, -2, -1, 0, 1, 2, 3, 4$

Possible values of Y : $0, 1, 2, 3, 4$

$P(Y=0) = P(X=0) = \frac{1}{9}$

$P(Y=1) = P(X=1 \text{ or } X=-1) = P(X=1) + P(X=-1) = \frac{1}{9} + \frac{1}{9} = \frac{2}{9}$

$P(Y=2) = P(X=2) + P(X=-2) = \frac{2}{9}$

$\vdots$

$P(Y=4) = \frac{2}{9}$

$$P_Y(y) = \begin{cases} \frac{1}{9} & \text{if } y = 0 \\ \frac{2}{9} & \text{if } y = 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}$$