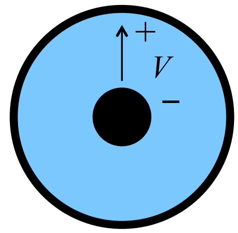
Three Simple Transmission Lines

Parallel Plate

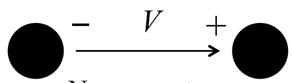
Gauss's Law
+
Planar Symmetry

Coaxial Cable



Gauss's Law
+
Cylindrical Symmetry

Two-Wire Line



No symmetry

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- For the parallel plate and coaxial cable, we use Gauss's Law and symmetry to find C'

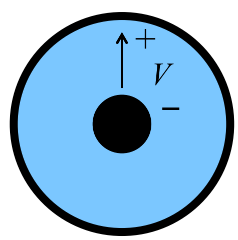
17.1

Three Simple Transmission Lines

Parallel Plate

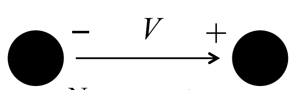
Gauss's Law
+
Planar Symmetry

Coaxial Cable



Gauss's Law
+
Cylindrical Symmetry

Two-Wire Line



No symmetry

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- For the two-wire line, this doesn't work
- ***We have to use another approach!***

17.2

Three Simple Transmission Lines

Two-Wire Line: ***The Key Difficulty***

Charges accumulate unevenly around the wire perimeter

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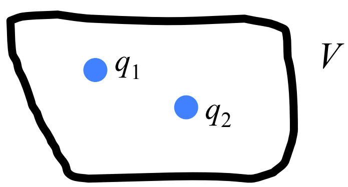
17.3

Dirichlet Boundary Conditions

Dirichlet: If we know the charges inside a closed boundary and the voltage on the boundary, then the voltage is uniquely determined everywhere inside



NOTES:

1. The voltage does not have to be uniform on the boundary
2. The boundary can include infinity
   (where the voltage is usually zero)

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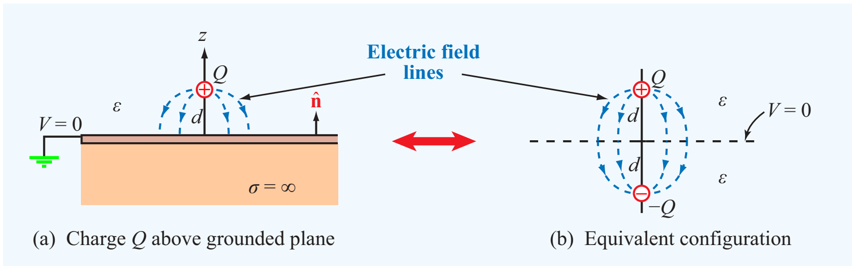
17.4

The Method of Images

Problem: Find the voltage due to a charge Q that is a distance d above a grounded metal plate for the upper half space ($z > 0$).



(a) Charge Q above grounded plane (b) Equivalent configuration

To solve this problem, we consider the problem of a charge Q at $(0,0,d)$ and a charge $-Q$ at $(0,0,-d)$

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17.5

The Method of Images

Problem: Find the voltage due to a charge Q that is a distance d above a grounded metal plate.

Solution: We find

$$V = \frac{Q}{4\pi\varepsilon}\left[\frac{1}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z+d)^2}}\right]$$

On the closed boundary

- $z = 0$ and ∞, we match the boundary conditions

Dirichlet's theorem implies that we have the correct solution for $z > 0$.

NOTE: *The solution for $z < 0$ is irrelevant!*

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17.6

The Method of Images

We can now use Gauss's theorem to obtain the surface charge:

We find

$$\rho_{\mathrm{S}}(x,y) = -\varepsilon\hat{\mathbf{z}}\cdot\mathbf{E}(x,y,z=0) = \varepsilon\left.\frac{\partial V}{\partial z}\right|_{z=0} = -\frac{Qd}{2\pi}\frac{1}{\left(x^2+y^2+d^2\right)^{3/2}}$$

The total surface charge is given by

$$\int_{\mathrm{S}}\rho_{\mathrm{S}}(x,y)\,dx\,dy = -\frac{Qd}{2\pi}\int_0^{2\pi}d\phi\int_0^{\infty}r\,dr\left(r^2+d^2\right)^{-3/2} = -Q$$

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17.7

The Method of Images

Some other problems that can be solved using this method:

- A point inside or outside a metal sphere
- A line charge that is parallel to a metal cylinder
- A point charge inside a 90º metal wedge

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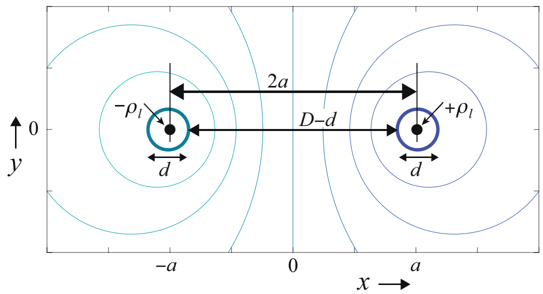
17.8

Two-Wire Line

The key ideas:

- The equipotential surfaces of two equal and opposite line charges are circles (not centered on the line charges)
- We set the equipotential surfaces to coincide with the wire radii and use Dirichlet's theorem
- The integrated surface charge per unit length equals the line charge

Equipotential Surfaces



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17.9

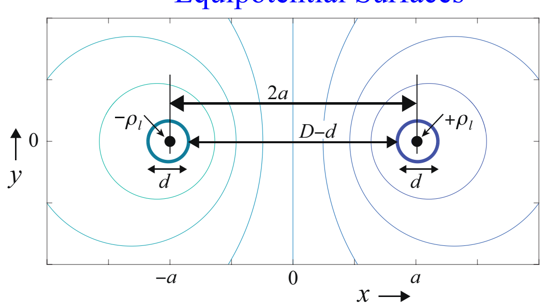
Two-Wire Line

Equipotential Surfaces:

$$\Phi(x,y) = -\frac{\rho_1}{2\pi\varepsilon}\left[\ln\sqrt{(x-a)^2+y^2} - \ln\sqrt{(x+a)^2+y^2}\right]$$

$$= -\frac{\rho_1}{4\pi\varepsilon}\ln\left[\frac{(x-a)^2+y^2}{(x+a)^2+y^2}\right]$$

Equipotential Surfaces



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17.10

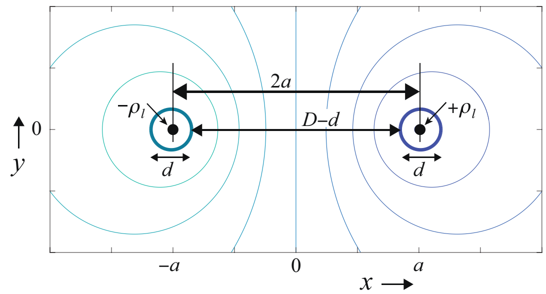
Two-Wire Line

Equation for the equipotential surfaces:

$$(x-a)^2 + y^2 = K^2\left[(x+a)^2 + y^2\right]; \quad K = \exp(-2\pi\varepsilon\Phi / \rho_1)$$

$$\Rightarrow (K^2 - 1)(x^2 + y^2 + a^2) - 2(1+K^2)ax = 0$$

Equipotential Surfaces



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17.11

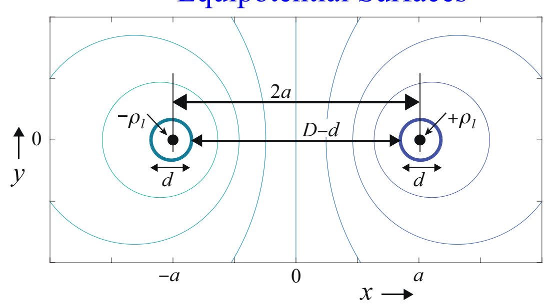
Two-Wire Line

Equation for the equipotential surfaces:

$$\Rightarrow \left(x - \frac{1+K^2}{1-K^2}a\right)^2 + y^2 = \frac{4K^2}{(1-K^2)^2}a^2$$

which is just the equation for a circle

Equipotential Surfaces



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17.12

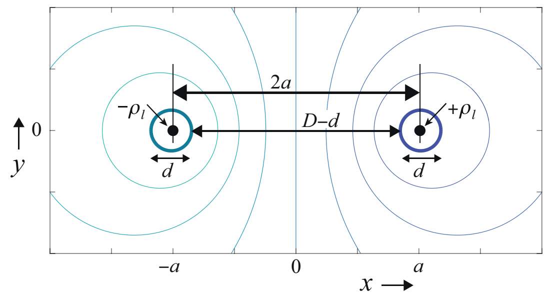
Two-Wire Line

Equation for the equipotential surfaces:

$$D = 2\frac{1+K^2}{1-K^2}a; \quad d = \frac{4K}{1-K^2}a$$

$$\Rightarrow \frac{D}{d} = \frac{1+K^2}{2K}; \quad K = \frac{D}{d} - \left(\frac{D^2}{d^2} - 1\right)^{1/2} \;; \quad K^{-1} = \frac{D}{d} + \left(\frac{D^2}{d^2} - 1\right)^{1/2}$$

Equipotential Surfaces



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17.13

Two-Wire Line

The voltage difference between surfaces is:

$$V = 2\Phi = \frac{\rho_1}{\pi\varepsilon}\ln(K^{-1}) = \frac{\rho_1}{\pi\varepsilon}\ln\left[\frac{D}{d} + \left(\frac{D^2}{d^2} - 1\right)^{1/2}\right]$$

The charge per unit length is:

$$Q' = \rho_1$$

So, the capacitance per unit length is:

$$C' = \pi\varepsilon / \ln\left[\frac{D}{d} + \left(\frac{D^2}{d^2} - 1\right)^{1/2}\right]$$

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17.14

Two-Wire Line

The conductance and inductance per unit length are:

$$G' = \pi\sigma / \ln\left[\frac{D}{d} + \left(\frac{D^2}{d^2} - 1\right)^{1/2}\right]; \quad L' = \frac{\mu}{\pi}\ln\left[\frac{D}{d} + \left(\frac{D^2}{d^2} - 1\right)^{1/2}\right]$$

To calculate the surface charge on the $+V$ surface, we transform coordinates:

$$r\cos\phi = x - \frac{1+K^2}{1-K^2}a; \quad r\sin\phi = y$$

$$\Rightarrow x - a = r\cos\phi + d/2K; \quad x + a = r\cos\phi + dK/2$$

$$\Rightarrow \frac{(x-a)^2 + y^2}{(x+a)^2 + y^2} = \frac{r^2 + (d/K)r\cos\phi + d^2/4K^2}{r^2 + dKr\cos\phi + d^2K^2/4}$$

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17.15

Two-Wire Line

The conductance and inductance per unit length are:

$$G' = \pi\sigma / \ln\left[\frac{D}{d} + \left(\frac{D^2}{d^2} - 1\right)^{1/2}\right]; \quad L' = \frac{\mu}{\pi}\ln\left[\frac{D}{d} + \left(\frac{D^2}{d^2} - 1\right)^{1/2}\right]$$

To calculate the surface charge on the $+V$ surface, we transform coordinates:

$$r\cos\phi = x - \frac{1+K^2}{1-K^2}a; \quad r\sin\phi = y$$

$$\Rightarrow x - a = r\cos\phi + dK/2; \quad x + a = r\cos\phi + d/2K$$

$$\Rightarrow \frac{(x-a)^2 + y^2}{(x+a)^2 + y^2} = \frac{r^2 + dKr\cos\phi + d^2K^2/4}{r^2 + (d/K)r\cos\phi + d^2/4K^2}$$

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17.16

Two-Wire Line

We now find:

$$\mathbf{E} = -\nabla\Phi = \frac{\rho_1}{4\pi\varepsilon}\left[\frac{\hat{\mathbf{r}}\left(2r + dK\cos\phi\right) - \hat{\mathbf{e}}_\phi dK\sin\phi}{r^2 + dKr\cos\phi + \dfrac{d^2K^2}{4}} - \frac{\hat{\mathbf{r}}\left(2r + \dfrac{d}{K}\cos\phi\right) - \hat{\mathbf{e}}_\phi \dfrac{d}{K}\sin\phi}{r^2 + \dfrac{d}{K}r\cos\phi + \dfrac{d^2}{4K^2}}\right]$$

Substituting $r = d / 2$, we find $\mathbf{E} = E_r$, and:

$$\rho_\mathrm{S} = \varepsilon E_r = \frac{\rho_1}{\pi d}\frac{1 - K^2}{1 + K^2 + 2K\cos\phi} = \frac{\rho_1}{\pi d}\frac{(D^2 - d^2)^{1/2}}{D + d\cos\phi}$$

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17.17