More practice

- **More practice 1**

$$x(t)=\begin{cases} 1 & 0\le t<3 \\ 0 & otherwise \end{cases}$$

$$h(t)=\begin{cases} 1 & 0\le t<2 \\ 0 & otherwise \end{cases}$$

- **More practice 2**

$$x(t)=\begin{cases} 1 & 0\le t<3 \\ 0 & otherwise \end{cases}$$

$$h(t)=\begin{cases} 2e^{-2t} & 0\le t<\infty \\ 0 & otherwise \end{cases}$$

- **More practice 3**

$$x(t)=\begin{cases} \sin(2\pi t) & 0\le t<\infty \\ 0 & otherwise \end{cases}$$

$$h(t)=\begin{cases} 1 & 0\le t<1 \\ 0 & otherwise \end{cases}$$

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Lecture 4: Properties of the Convolution Operator

Properties

- **Convolution is such a common operation that it's useful to know (memorize) some common properties**
- **Property 1: Combination of delays**

If $h(t)=0$ for $t<t_1$, and $x(t)=0$ for $t<t_2$, then $y(t)=x*h=0$ for $t<t_1+t_2$

- **Property 2: Combination of durations**

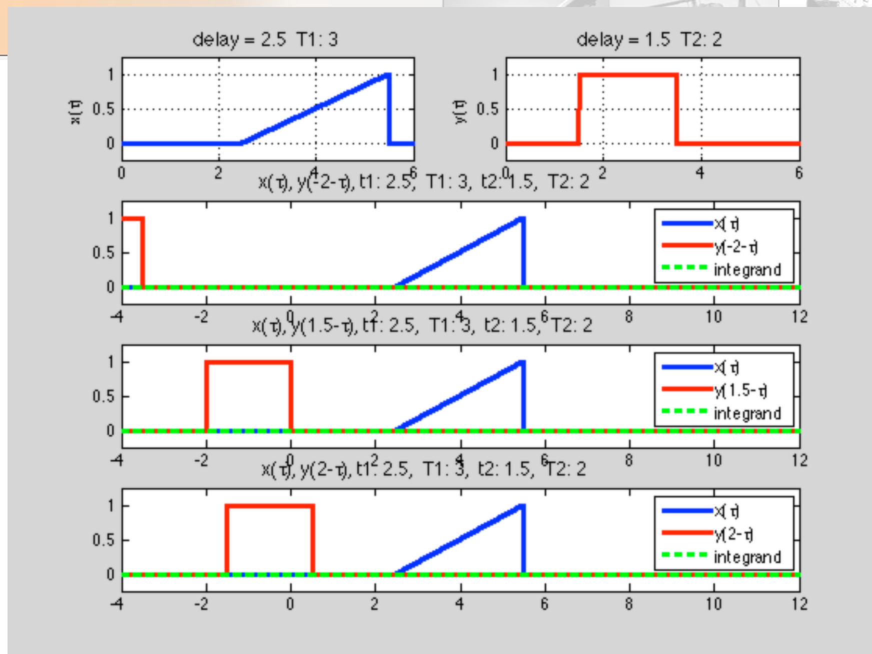
If $h(t)\neq 0$ for $T_1=t_{12}-t_{11}$, and $x(t)\neq 0$ for $T_2=t_{22}-t_{21}$,

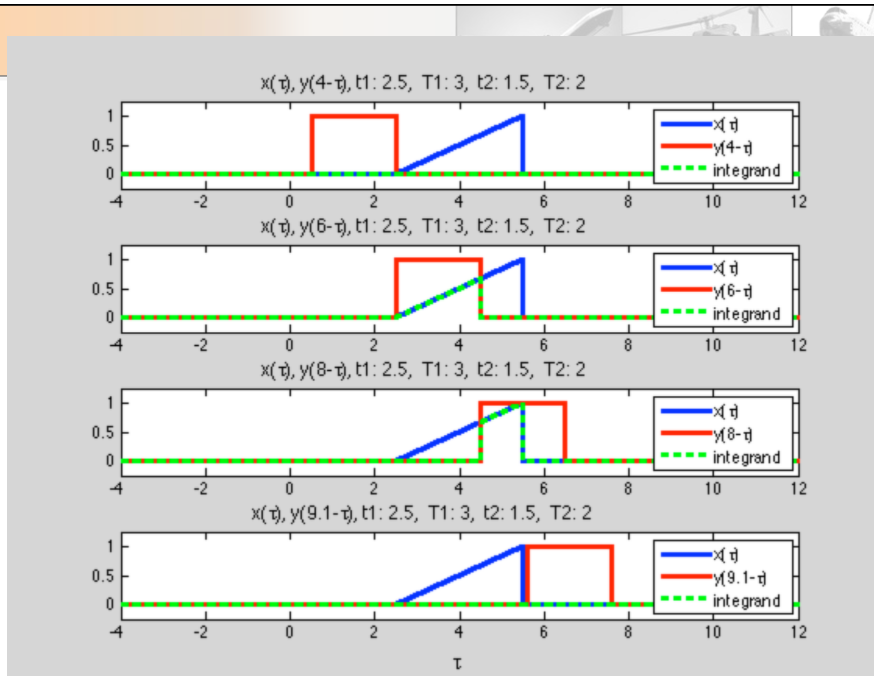
then $y(t)=x*h\neq 0$ for T_2+T_1

- **To see these consider the convolution of these signals**

$$x(\tau)=\begin{cases}\dfrac{t-t_1}{T_1} & t_1\leq t<t_1+T_1\\ 0 & elsewhere\end{cases},\quad y(\tau)=\begin{cases}1 & t_2\leq t<t_2+T_2\\ 0 & elsewhere\end{cases}$$

Note that we're computing $w(t)=\int_{-\infty}^{\infty}x(\tau)y(t-\tau)\,d\tau$


delay = 2.5 T1: 3
delay = 1.5 T2: 2
x(τ)
y(τ)
x(τ), y(-2-τ), t1: 2.5, T1: 3, t2: 1.5, T2: 2
x(τ)
y(-2-τ)
integrand
x(τ), y(1.5-τ), t1: 2.5, T1: 3, t2: 1.5, T2: 2
x(τ)
y(1.5-τ)
integrand
x(τ), y(2-τ), t1: 2.5, T1: 3, t2: 1.5, T2: 2
x(τ)
y(2-τ)
integrand


x(τ), y(4-τ), t1: 2.5, T1: 3, t2: 1.5, T2: 2
x(τ)
y(4-τ)
integrand
x(τ), y(6-τ), t1: 2.5, T1: 3, t2: 1.5, T2: 2
x(τ)
y(6-τ)
integrand
x(τ), y(8-τ), t1: 2.5, T1: 3, t2: 1.5, T2: 2
x(τ)
y(8-τ)
integrand
x(τ), y(9.1-τ), t1: 2.5, T1: 3, t2: 1.5, T2: 2
x(τ)
y(9.1-τ)
integrand
τ

More Properties

- **Property 3: Even/Odds**

  If $x(t)$ is even and $h(t)$ is odd, or $x(t)$ is odd and $h(t)$ is even,

  then $y(t) = x * h = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$ is odd.

  If $x(t)$ and $h(t)$ are both odd or both even, then $y(t)$ is even.

Or, the convolution of an odd function with an even function is odd, and the convoluton of two even function or two odd functions is even.

Remembering proofs

- **In the real world, a fair amount of manipulation of signals is done symbolically, that is, by analysis…**
- **…so it's useful to actually prove these things to get used to manipulating the convolution operation**
- **Remembering proofs from CMSC203, all forms of this "even/odd" statement are logical implications**

  $$\left(x(t)\text{ even}\right) \wedge \left(h(t)\text{ even}\right) \rightarrow x * h \text{ even}$$

- **The proof is itself a design problem:**
  - **What are the requirements: prove (or disprove) the implication**
  - **What do we know?**

    $$x(t)\text{ even } \equiv x(t) = x(-t);\;\; x(t)\text{ odd } \equiv x(t) = -x(-t)$$

    $$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

- **What do we need to find out? Not much**
- **Find the engineering solution, in this case, the proof**
  - **Hold one side of the convolution equation unchanged**

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

Let $\lambda = -\tau,\ \ d\lambda = -d\tau,\ \ \tau = \pm\infty \rightarrow \lambda = \mp\infty,$ changing variables

$$y(t) = -\int_{\infty}^{-\infty} x(-\lambda)h(t+\lambda)\,d\lambda = \int_{-\infty}^{\infty} x(-\lambda)h(t+\lambda)\,d\lambda$$

x is even, so $x(-\lambda) = x(\lambda)$

h is even, so $h(t+\lambda) = h(-(t+\lambda)) = h(-t-\lambda)$

Substituting $y(t) = \int_{-\infty}^{\infty} x(\lambda)h(-t-\lambda)\,d\lambda = y(-t)$

Other Properties

- **Property 4 Periodicity** $x(t)$ periodic $\rightarrow y(t)$ periodic
- **Property 5 Time Scaling** $y(ct) = cx(ct) * h(ct)$ for $c > 0$
- **Property 6 Time Reversal** $y(-t) = x(-t) * h(-t)$
- **Property 7 Area Property**

  Let $A_x = \int_{-\infty}^{\infty} x(t)\,dt,$ similarly A_h and A_y. If $y = x * h,\ A_y = A_x A_y$

- **Property 8 Centroid/Center of Gravity Property**

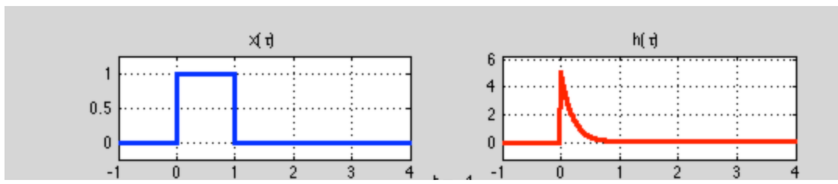
  Let $A_{tx} = \int_{-\infty}^{\infty} \tau x(\tau)\,d\tau,$ and $D_x = \dfrac{A_{tx}}{A_x}$

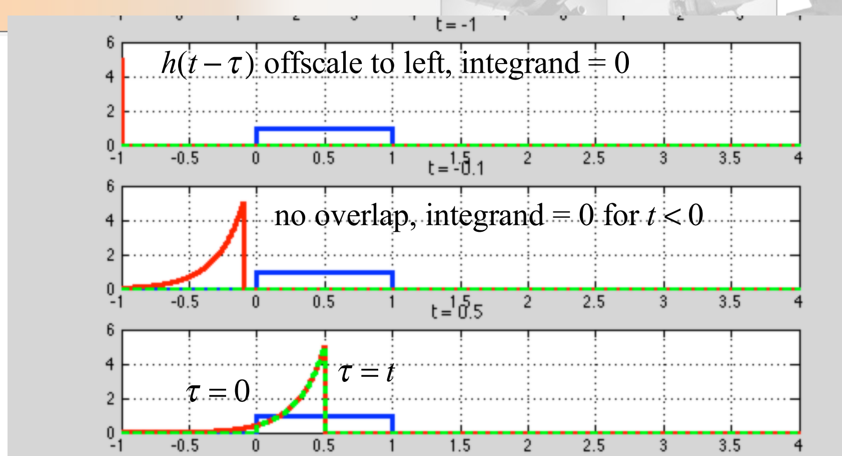
  Then $y = x * h \rightarrow D_y = D_x + D_h$

- **Property 9 Associativity:** $\left(x * y\right) * z = x * \left(y * z\right)$

A simple problem

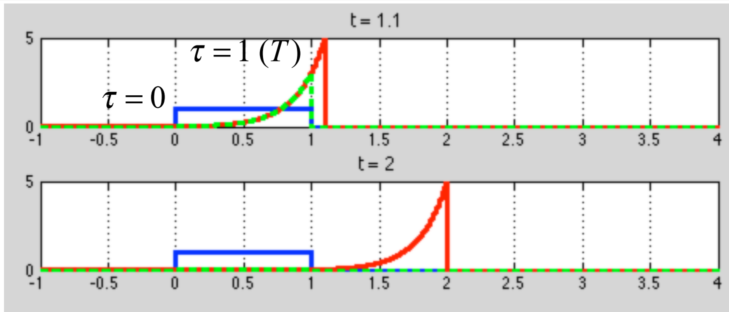
$$x(t)=\begin{cases}1 & 0\le t<t\\ 0 & otherwise\end{cases}\quad,\quad h(t)=\begin{cases}5e^{-5t} & t\ge 0\\ 0 & otherwise\end{cases}$$





$0\le t<1\ (T)$

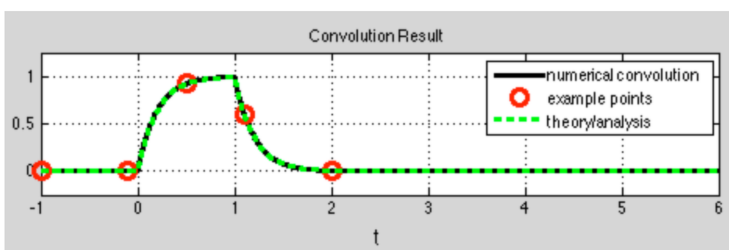
$$y(t)=\int_0^t (1)\left(5e^{-5(t-\tau)}\right)d\tau=5e^{-5t}\times\frac{1}{5}e^{5\tau}=e^{-5t}\left(e^{5\tau}\right)_0^t=1-e^{-5t}$$



For $t \geq 1\ (T)$

$$y(t) = \int_0^1 (1)\left(5e^{-5(t-\tau)}\right) d\tau = 5e^{-5t} \times \frac{1}{5} e^{5\tau} = e^{-5t}\left(e^{5\tau}\right)_0^1 = e^{-5t}(e^5 - 1)$$

$$= e^{-5(t-1)} - e^{-5t} \approx e^{-5(t-1)}$$



$$y(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-5t} & 0 \leq t < 1\ (T) \\ e^{-5(t-1)} - e^{-5} & t \geq 1 \end{cases}$$

We recognize (!!) this as the output of a single-pole low-pass filter from CMPE306!

More work

- **Let's walk through a more complicated example**

$$x(t)=\begin{cases} 0 & t<-2 \\ 1 & -2\le t<-1 \\ 2 & -1\le t<1 \\ 1 & 1\le t<2 \\ 0 & t>2 \end{cases} \quad , \quad h(t)=\begin{cases} 5e^{-5t} & t\ge 0 \\ 0 & elsewhere \end{cases}$$

Find $y(t)=x*h$

Solution done on board! (or in homework, as time permits)