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MATH 407: HW 06

2.3 **1** Consider the following permutations in S_7

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 5 & 4 & 6 & 1 & 7 \end{pmatrix} \qquad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 5 & 7 & 4 & 6 & 3 \end{pmatrix}$$

Compute the following products:

b $\tau\sigma$

Ans

$$\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 5 & 4 & 6 & 1 & 7 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 5 & 7 & 4 & 6 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 6 & 7 & 4 & 1 & 5 \end{pmatrix}$$

$\square$

f $\tau^{-1}\sigma\tau$

Ans

$$\tau^{-1} = \begin{pmatrix} 2 & 1 & 5 & 7 & 4 & 6 & 3 \\ \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 & 6 & 7 & 4 & 1 & 5 \\ \\ 1 & 7 & 6 & 4 & 5 & 2 & 3 \end{pmatrix}$$

$$\tau^{-1}\sigma = \begin{pmatrix} 2 & 3 & 6 & 7 & 4 & 1 & 5 \\ \\ 1 & 7 & 6 & 4 & 5 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \\ 3 & 2 & 5 & 4 & 6 & 1 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 & 6 & 7 & 4 & 1 & 5 \\ \\ 3 & 7 & 1 & 4 & 6 & 2 & 5 \end{pmatrix}$$

$$\tau^{-1}\sigma\tau = \begin{pmatrix} 2 & 3 & 6 & 7 & 4 & 1 & 5 \\ \\ 3 & 7 & 1 & 4 & 6 & 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \\ 3 & 2 & 5 & 4 & 6 & 1 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 & 6 & 7 & 4 & 1 & 5 \\ \\ 5 & 7 & 3 & 4 & 1 & 2 & 6 \end{pmatrix}$$

$\square$

3 Write $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \\ 3 & 4 & 10 & 5 & 7 & 8 & 2 & 6 & 9 & 1 \end{pmatrix}$ as a product of disjoint cycles and as a product of transpositions. Construct its associated diagram, find its inverse, and find its order.

Ans The product of disjoint cycles:

$$\{(1, 3, 10)(2, 4, 5, 7)(6, 8)\}$$

The product of transpositions:

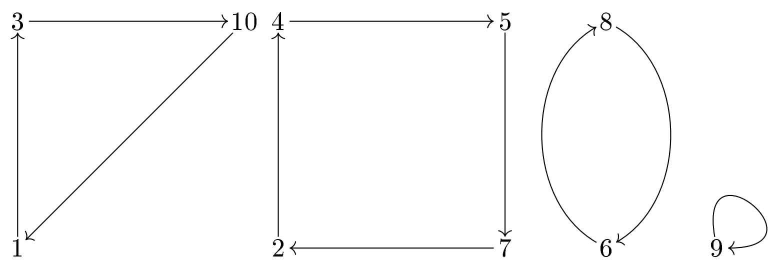
$$\{(1,3,10)(2,4,5,7)(6,8)\} = \{(1,3)(3,10)(2,4)(4,5)(5,7)(6,8)\}$$

Reconstructing the permutation based on the product of transpositions:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 4 & 10 & 5 & 7 & 8 & 2 & 6 & 9 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 & 10 & 2 & 4 & 5 & 7 & 6 & 8 & 9 \\ 3 & 10 & 1 & 4 & 5 & 7 & 2 & 8 & 6 & 9 \end{pmatrix}$$

Constructing the associated diagrams



The inverse of the permutation:

$$\sigma^{-1} = \begin{pmatrix} 3 & 4 & 10 & 5 & 7 & 8 & 2 & 6 & 9 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 10 & 7 & 1 & 2 & 4 & 8 & 5 & 6 & 9 & 3 \end{pmatrix}$$

Since

$$\begin{aligned} o(\sigma) &= lcm(\text{length(cycles)}) \\ &= lcm(\{2,4,3\}) \\ &= 12 \end{aligned}$$

$\square$

5 Let $3 \leq m \leq n$. Calculate $\sigma\tau^{-1}$ for the cycles $\sigma = (1, 2, \ldots, m-1)$ and $\tau = (1, 2, \ldots, m-1, m)$ in S_n.

Ans Given

$$\tau = \begin{pmatrix} 1 & 2 & 3 & \ldots & \text{m-2} & \text{m-1} & \text{m} \\ 2 & 3 & 4 & \ldots & \text{m-1} & \text{m} & 1 \end{pmatrix}$$

$$\tau^{-1} = \begin{pmatrix} 2 & 3 & 4 & \ldots & \text{m-1} & \text{m} & 1 \\ 1 & 2 & 3 & \ldots & \text{m-2} & \text{m-1} & \text{m} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & \text{m-1} & \text{m} \\ \text{m} & 1 & 2 & 3 & \ldots & \text{m-2} & \text{m-1} \end{pmatrix}$$

Therefore the product

$$\sigma\tau^{-1} = \begin{pmatrix} 1 & 2 & 3 & \ldots & \text{m-2} & \text{m-1} \\ 2 & 3 & 4 & \ldots & \text{m-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & \ldots & \text{m-1} & \text{m} \\ \text{m} & 1 & 2 & \ldots & \text{m-2} & \text{m-1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & \ldots & \text{m-1} & \text{m} \\ \text{m} & 2 & 3 & \ldots & \text{m-1} & 1 \end{pmatrix}$$

$$= (1, m)$$ □

11 Prove that in S_n, with $n \geq 3$, any even permutation is a product of cycles of length three.

Hint: $(a, b)(b, c) = (a, b, c)$ and $(a, b)(c, d) = (a, b, c)(b, c, d)$.

Ans □

15 For $\alpha, \beta \in S_n$, let $\alpha \sim \beta$ if there exists $\sigma \in S_n$ such that $\sigma\alpha\sigma^{-1} = \beta$. Show that $\sim$ is an equivalence relation on S_n.

Ans □

16 View S_3 as a subset of S_5, in the obvious way. For $\sigma, \tau \in S_5$, define $\sigma \sim \tau$ if $\sigma\tau^{-1} \in S_3$.

a Show that $\sim$ is an equivalence relation on S_5.

Ans $\square$

b Find the equivalence class of (4, 5).

Ans $\square$

c Find the equivalence class of (1, 2, 3, 4, 5).

Ans $\square$

d Determine the total number of equivalence classes.

Ans $\square$