Exam I

Math 407

Spring 2015

Part I

Do 4 problems (13 points each)

1. Determine the prime factorization of 2015.
2. Find all ordered pairs (a, b) of positive integers so that $a^2 - b^2 = 2015$.
3. Determine the first two (ones and tens) digits of 7^{42} (base 10).
4. Find the general solution x to the following system: (*Hint*: Problem 8)
   x$\equiv$ 3 (mod 14)
   x$\equiv$ 4 (mod 15)
   (Extra Credit: Determine all positive solutions)
5. Give an example of a pair of functions f and g on $\mathbb{N}$ having $g \circ f$ onto, yet f is not onto.

Part II

Do 3 problems (16 points each)

6. Let $f : \mathbb{X} \to \mathbb{X}$ be a function so that for any $x \in \mathbb{X}$ there is $k \in \mathbb{N}$ so that the k-th iterate $f^k(x)$ is x. Show that f is onto.

7. Verify by induction that $1 + x^{n+1} = (1+x)(1 - x + x^2 + \ldots + (-x)^n)$ for all even $n \in \mathbb{N}$ (x is a variable).

8. Let n_1 and n_2 be relatively prime and larger than 1. Let $n = n_1 n_2$. Show that if x_1 is a solution of $x \equiv a \ mod \ n_1$ and $x \equiv b \ mod \ n_2$, then x_2 is another solution iff $n | (x_1 - x_2)$.

9. If $n \in \mathbb{N}$ has prime factorization $p_1^{r_1} \ldots p_k^{r_k}$ with $p_1 < p_2 < \ldots < p_k$ and all $r_i > 0$, set $f(n) = \cancel{p_k^{r_k}}$. Show that $lim_{n \to \infty} f(n) = \infty$. (Hint: Show that if $M \in \mathbb{N}$ only finitely many n have $f(n) \leq M$).
$\max(p_i^{r_i})$

10. Let $fac(n)$ denote the number of factors of an $n \in \mathbb{N}$. Characterize, using the prime factorization of n, all n so that $fac(n)$ is even.

11. Define the relation $m \sim n$ on $\mathbb{N}$ to hold iff there are i and j in $\mathbb{N}$ with $n | m^i$ and $m | n^j$.

    (a) Show that $\sim$ is an equivalence relation

    (b) Describe $[n]_\sim$ using the prime factorization of n.

12. (Bonus) In Problem 6, show that f is also 1-1.