Problem Set #7 Solutions

1. When the switch is closed, current flows from left to right in the upper branch of the lower loop, creating a magnetic field that goes out of the book through the upper loop. To oppose that, current must flow clockwise. When the switch is opened, current will flow in the opposite direction, which is counterclockwise.

2. We may write the time-varying magnitude of the magnetic flux density as $B = B_{\max}\cos(\omega t + \phi_0)$, where ϕ_0 is some unknown initial phase. The flux is given by $\Phi = B_{\max} A \cos(\omega t + \phi_0)$, and the EMF is given by

$$V_{\text{EMF}} = -\frac{d\Phi}{dt} = B_{\max} A\omega \sin(\omega t + \phi_0),$$

which has its maximum value $V_{\max}$ when $|\sin(\omega t + \phi_0)| = 1$, so that $V_{\max} = B_{\max} A\omega$. We conclude

$$B_{\max} = \frac{V_{\max}}{A\omega} = \frac{20 \times 10^{-3}}{(2\pi \times 300 \times 10^6) \times (2 \times 10^{-2})} = 530 \times 10^{-12}\ \text{T} = 530\ \text{pT}.$$

3. A rotation rate of 3,600 revolutions per minute is the same as 60 revolutions per second or 60 Hz. The amplitude of the induced EMF is given by

$$V_{\text{EMF}} = \omega A B_0 = 2 \times 3.14 \times 60\,(\text{Hz}) \times 0.1\,(\text{m}^2) \times 0.2\,(\text{T}) = 7.54\ \text{V}.$$

With a resistance of 150 Ω, the induced current I_{EMF} is given by $7.54/100 = 0.075\ \text{A} = 75\ \text{mA}$.

4. In solving this problem, we note that the current flow I into the capacitor must account for both the actual current that flows between the plates I_{c}, as well as the charge buildup on the plates I_{d} that leads to a displacement current. Hence, we must have $I = I_{\text{c}} + I_{\text{d}}$.

   a. Taking the z-direction to be the opposite of the direction of the current flow in the figure, we have $E_z = -V/d$, so that $\mathbf{E} = -\hat{\mathbf{z}}(V/d)$. It follows that $\mathbf{J} = \sigma\mathbf{E} + -\hat{\mathbf{z}}(\sigma V/d)$. We thus find $I_{\text{c}} = \int_S \mathbf{J}\cdot d\mathbf{s} = \sigma V A/d$, where we note that the surface of the upper plate is oriented in the $-z$-direction.

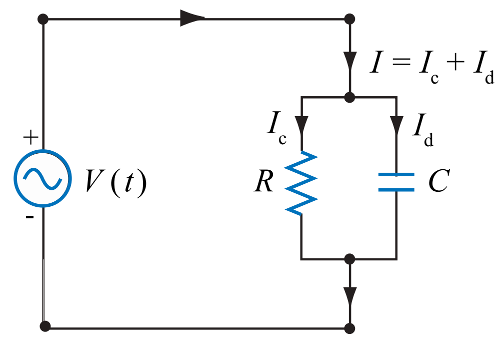
   b. We have $\mathbf{D} = \epsilon\mathbf{E} = -\hat{\mathbf{z}}(\epsilon V/d)$. It follows that

$$\frac{\partial \mathbf{D}}{\partial t} = -\hat{\mathbf{z}}\frac{\epsilon}{d}\frac{dV(t)}{dt},$$

   from which we conclude

$$I_{\text{d}} = \frac{\partial \mathbf{D}}{\partial t}\cdot(-\hat{\mathbf{z}}A) = \frac{\epsilon A}{d}\frac{dV(t)}{dt} = \frac{\epsilon_{\text{r}}\epsilon_0 A}{d}\frac{dV(t)}{dt}.$$

c. The equivalent circuit diagram is shown to the right. The finite conductivity through the medium is equivalent to introducing a parasitic resistance in parallel with the capacitance.



d. Converting to SI units for convenience, we note that $A = 2 \times 10^{-4}$ m^2 and $d = 5 \times 10^{-3}$ m. We have $R = d/\sigma A = (5 \times 10^{-3})/[2.5 \times (2 \times 10^{-4})] = 10.0\ \Omega$, so that $I_c = V(t)/R = 1.0 \cos(3\pi \times 10^3 t)$ A. We have $C = (\epsilon_r \epsilon_0 A/d) = 4 \times (8.85 \times 10^{-12}) \times (2 \times 10^{-4})/(5 \times 10^{-3}) = 2.83 \times 10^{-12}$ F $=$ 1.4 pF, and we also have $dV(t)/dt = 3 \times 3.14 \times 10^3 \times 10 \sin(3\pi \times 10^3 t) = 9.87 \times 10^4$ V/s. As a consequence, $I_d = C\, dV(t)/dt = (1.42 \times 10^{-12}) \times (9.87 \times 10^4) \times \sin(3\pi \times 10^3 t) = (1.39 \times 10^{-7}) \sin(3\pi \times 10^3 t)$ A $=$ 140 nA.

5. We have $\epsilon_r = \epsilon/\epsilon_0 = 4$, so that $u_p = c/\sqrt{\epsilon_r} = 1.50 \times 10^8$ m/s. It follows that $k = \omega/u_p = 2\pi/15.0 = 0.42$ m^{-1}. The direction of propagation is the $-y$-direction. It follows that the electric field is proportional to the magnetic field, but oriented in the $-z$-direction, since $-\hat{\mathbf{z}} \times \hat{\mathbf{x}} = -\hat{\mathbf{y}}$. The magnitude of the electric field will be given by $5 \times (1/u_p \epsilon) = 5/[(1.50 \times 10^8) \times 4 \times (8.85 \times 10^{-12})] = 940$ V/m. The final result for the electric field is

$$\mathbf{E}(y,t) = -\hat{\mathbf{z}} 940 \cos(2\pi \times 10^7 + 0.42y) \text{ V/m}$$

6. Since the line can be considered short, we must have

$$\mathbf{A}(R,t) = \frac{\mu}{4\pi} \int_{v'} \frac{\mathbf{J}(R_i, t - R'/u_p)}{R'}\, dv' = \frac{\mu}{4\pi} \int_{-l/2}^{l/2} \hat{\mathbf{z}} \frac{I_0 \cos(\omega t - kR')}{R'}\, dz$$
$$\simeq \hat{\mathbf{z}}\, \frac{\mu}{4\pi} \frac{l I_0}{R} \cos(\omega t - kR).$$

The corresponding phasor relationship is

$$\tilde{\mathbf{A}}(R) = \hat{\mathbf{z}}\, \frac{\mu}{4\pi} \frac{l I_0}{R} \exp(-jkR).$$

a. To put this relationship in its final form, we must express $\hat{\mathbf{z}}$ in spherical coordinates. To do that, we use the relationship $\hat{\mathbf{z}} = \hat{\mathbf{R}} \cos\theta - \hat{\boldsymbol{\theta}} \sin\theta$, which may be found, for example, in Ulaby's Table 3-2 (p. 117 in both the 2001 and 2004 editions). We thus find,

$$\tilde{\mathbf{A}}(R) = \hat{\mathbf{R}}\,\frac{\mu}{4\pi}\frac{lI_0}{R}\exp(-jkR)\cos\theta - \hat{\boldsymbol{\theta}}\,\frac{\mu}{4\pi}\frac{lI_0}{R}\exp(-jkR)\sin\theta.$$

b. Since $A_\phi = 0$ and both A_R and A_θ are independent of ϕ, it follows that only the ϕ-component of $\nabla \times \tilde{\mathbf{A}}$ will be present. We have

$$\tilde{\mathbf{H}} = \hat{\boldsymbol{\phi}}\,\frac{1}{\mu R}\left[\frac{\partial}{\partial R}(RA_\theta) - \frac{\partial A_R}{\partial \theta}\right] = \hat{\boldsymbol{\phi}}\,\frac{1}{4\pi}\frac{lI_0}{R^2}(1 + jkR)\exp(-jkR)\sin\theta.$$

The behavior changes, depending on whether $kR < 1$ or $kR > 1$. In the former case, referred to as the near field, the field falls off like $1/R^2$, just as in the static case. This behavior is referred to as "near-field" behavior. In the opposite limit, the field only falls off like $1/R$. This behavior is referred to as the "far-field" or "radiation" limit.