1 a) $E(X) = P(X=1) = 0.2 \iff P(X=0) = 0.8$

$0.2 + P(Y=2) + P(Y=4) = 1 \quad \cdots (*)$

$(0)(0.2) + (2)P(Y=2) + 4P(Y=4) = E(Y) = 2 \quad \cdots (**)$

So solving the two equations (*) & (**)

$P(Y=4) = 0.2$ and $P(Y=2) = 0.6$

	0	2	4	
0	0.1	0.6	0.1	0.8
1	0.1	0	0.1	0.2
	0.2	0.6	0.2	1

$E(XY) = (0)(0)(0.1) + (0)(2)(0.6) + \cdots + (1)(4)(0.1)$

$= 0.4$

$\text{cov}(X,Y) = E(XY) - E(X)E(Y) = (0.4) - (0.2)(2) = 0$

b) $P(Y > X \mid X > 0) = P[(Y=2, X=1) \cup (Y=4, X=1) \mid X=1]$

$= (0.1)/0.2 = 0.5$

2) $\text{cov}(X_1, X_2) = \text{Cov}(W+\varepsilon_1, W+\varepsilon_2) = \text{cov}(W,W) + \text{cov}(W,\varepsilon_2) + \text{Cov}(\varepsilon_1, W) + \text{Cov}(\varepsilon_1,\varepsilon_2)$

$= \text{Var}(W) + 0 + 0 + 0$ (independence of W & ε's)

$= 1$

$\text{var}(X_1) = \text{var}(X_2) = \text{Var}(W) + \text{Var}(\varepsilon_1) = \text{Var}(W) + \text{Var}(\varepsilon_2) = 1.01$

$\text{Corr}(X_1, X_2) = \dfrac{\text{cov}(X_1, X_2)}{\sqrt{V(X_1)V(X_2)}} = \dfrac{1}{\sqrt{(1.01)^2}} = \dfrac{1}{1.01}$

4) $f(x,\theta) = \dfrac{1}{(0.5-\theta)}$ for $\theta < x < 0.5$

So Likelihood $L(\theta \mid x_1 \ldots x_n) = \dfrac{1}{(0.5-\theta)^7}$ for $\theta < x_1, \ldots, x_7 < 0.5$

$= \dfrac{1}{(0.5-\theta)^7}$ for $\theta < \min(x_1 \ldots x_7) < \max(x_1 \ldots x_7) < 0.5$

This is increasing in θ but the largest θ can be is $\min(x_1 \ldots x_7)$

So $\hat{\theta}_{MLE} = \min(x_1, \ldots, x_7) = 0.21$

3) $X \sim$ Economy weight $E(X) = 40$, $V(X) = 10^2$

$Y \sim$ Business weight $E(Y) = 30$, $V(Y) = 6^2$

$$E(X_1 + \cdots + X_{50} + Y_1 + \cdots + Y_{12}) = \underbrace{40 + \cdots + 40}_{50 \text{ times}} + \underbrace{30 + \cdots + 30}_{12 \text{ times}}$$

$$= 2360 \text{ Lbs}$$

$$V(X_1 + \cdots + X_{50} + Y_1 + \cdots + Y_{12}) = \underbrace{10^2 + \cdots + 10^2}_{50 \text{ times}} + \underbrace{6^2 + \cdots + 6^2}_{12 \text{ times}}$$

$$= 5432$$

Total weight $= T = X_1 + \cdots + X_{50} + Y_1 + \cdots + Y_{12}$

$$\sim N(2360, 5432)$$

$$P(T \leq 2500) = P\left(Z \leq \frac{2500 - 2360}{\sqrt{5432}}\right)$$

5) Since $\alpha = 3$ is known, only parameter is β

Likelihood for β "

a) $$L(\beta \mid x_1, \ldots, x_{40}) = \prod_{i=1}^{40} \frac{e^{-x_i/\beta} x_i^{\alpha-1}}{\beta^\alpha \sqrt{\alpha}} = \frac{e^{-\sum x_i/\beta} \left(\prod_i^n x_i\right)^{3-1}}{\beta^{3n} (\sqrt{3})^n}$$

[There is a typo: ~~with~~ $e^{x/\beta}$ should be $e^{-x/\beta}$]

log likelihood $$\ell(\beta) = -\frac{\sum x_i}{\beta} - 3n \log\beta + \left(\text{terms that do not have } \beta\right)$$

$$\therefore \frac{d\ell(\beta)}{d\beta} = +\frac{\sum x_i}{\beta^2} - \frac{3n}{\beta} = 0 \Rightarrow \hat{\beta}_{mle} = \frac{\bar{x}}{3}$$

[in general $\bar{x}/\alpha$ if α is known]

So $\hat{\beta} = 15.57/3$

b) The variance is $\alpha\beta^2 \rightarrow$ so estimated to be 75

$$\therefore P(\bar{x} > 15) \approx P\left(\frac{\bar{x} - 15.57}{\sqrt{75}/\sqrt{36}} > \frac{15 - 15.57}{\sqrt{75}/\sqrt{36}}\right)$$

$$\overset{CLT}{\approx} P\left(Z > \frac{(-0.57)(6)}{\sqrt{75}}\right)$$

6) a) $P(\bar{x} > 8) = P\left[\frac{\bar{x} - 7.9}{0.5/\sqrt{60}} > \frac{8 - 7.9}{0.5/\sqrt{60}}\right]$

$$\overset{CLT}{\approx} P\left[Z > \frac{(0.1)(\sqrt{60})}{0.5}\right]$$

b) $P[x_i > 90\text{th percentile}] = 0.1$ (from the definition of percentile)

Let $Y_i = \begin{cases} 1 & \text{if } x_i > 90\text{th percentile} \\ 0 & \text{otherwise} \end{cases}$

Then $K = \sum_{i=1}^{60} Y_i \sim \text{Binomial}(60, 0.1)$

$$P[K \leq 10] = P\left[\frac{K - (60)(0.1)}{\sqrt{60(0.1)(0.9)}} \leq \frac{10 - (60)(0.1)}{\sqrt{60(0.1)(0.9)}}\right]$$

normal approximation to binomial

$$\approx P\left(Z \leq \frac{4}{\sqrt{54}}\right)$$

7) X_1 = # of failures in $[0, T]$ hrs
X_2 = # of failures in $[T, 2T]$ hrs

X_1, X_2 iid Poisson(μ)

Likelihood of $\mu \rightarrow L(\mu / x_1, x_2) = \left(\frac{e^{-\mu}\mu^{x_1}}{x_1!}\right)\left(\frac{e^{-\mu}\mu^{x_2}}{x_2!}\right)$

log likelihood of $\mu = \ell(\mu) = -2\mu + (x_1 + x_2)\log\mu - \log(x_1! x_2!)$

$$\frac{d\ell(\mu)}{d\mu} = -2 + \frac{(x_1 + x_2)}{\mu} = 0 \Rightarrow \hat{\mu}_{mle} = \frac{x_1 + x_2}{2}$$

$$\mu = E(X) \longleftrightarrow \bar{X} = \frac{x_1 + x_2}{2} \Rightarrow \hat{\mu}_{MOM} = \frac{x_1 + x_2}{2}$$

So in this case $\hat{\mu}_{mle} = \hat{\mu}_{mom} \rightarrow$ both are unbiased since $E\left(\frac{x_1 + x_2}{2}\right) = \frac{E(x_1) + E(x_2)}{2} = \frac{\mu + \mu}{2} = \mu$

Last part $\rightarrow Y$ = # of failures in $2T \sim$ Poisson(2μ)

$P(Y = 0) = e^{-2\mu} \rightarrow$ so mle $e^{-2\left(\frac{x_1 + x_2}{2}\right)}$

$$= e^{-(x_1 + x_2)}$$