Problem Set #5 Solutions

1. a. $Q = \rho_{s0} \int_0^a r\,dr \int_0^{2\pi} \cos\phi\,d\phi = 0$

   b. $Q = \rho_{s0} \int_0^a r\,dr \int_0^{2\pi} \cos^2\phi\,d\phi = \pi\rho_{s0} \int_0^a r\,dr = \pi\rho_{s0}a^2/2.$

   c. $Q = \rho_{s0} \int_0^a r\exp(-2r/b)\,dr \int_0^{2\pi} d\phi = 2\pi\rho_{s0} \int_0^a r\exp(-2r/b)\,dr = 2\pi\rho_{s0}(b^2/4)$ $[1-(1+2a/b)\exp(-2a/b)]$.

   d. $Q = \rho_{s0} \int_0^a r\exp(-r)\,dr \int_0^{2\pi} \sin^2\phi\,d\phi = \pi\rho_{s0} \int_0^a r\exp(-r)\,dr = \pi\rho_{s0}(b^2/4)[1-$ $(1+2a/b)\exp(-2a/b)]$.

2. Noting that $(1/4\pi\epsilon_0) = 8.988\times 10^9$ m/F, we find that the field acting on the charge at the origin is given by

$$\begin{aligned}\mathbf{E} &= (8.988\times 10^9)\times(6\times 10^{-9})\times\left[-\hat{\mathbf{x}}\frac{1}{(0.04)^2}-\hat{\mathbf{y}}\frac{1}{(0.04)^2}\right]\\ &= -\left(\frac{\hat{\mathbf{x}}+\hat{\mathbf{y}}}{\sqrt{2}}\right)\times(4.766\times 10^4)\ \text{V/m}\end{aligned}$$

   I note that $(\hat{\mathbf{x}}+\hat{\mathbf{y}})/\sqrt{2}$ is a unit vector. The corresponding force is given by

$$\begin{aligned}\mathbf{F} = q\mathbf{E} &= -\left(\frac{\hat{\mathbf{x}}+\hat{\mathbf{y}}}{\sqrt{2}}\right)(6\times 10^{-9})\times(4.766\times 10^4)\\ &= -\left(\frac{\hat{\mathbf{x}}+\hat{\mathbf{y}}}{\sqrt{2}}\right)2.860\times 10^{-4} \to -\left(\frac{\hat{\mathbf{x}}+\hat{\mathbf{y}}}{\sqrt{2}}\right)2.9\times 10^{-4}\ \text{N}\end{aligned}$$

3. We have

$$\begin{aligned}\mathbf{E}(r,\phi,0) &= \frac{\rho_l}{4\pi\epsilon_0}\int_{-L/2}^{L/2}\frac{-\hat{\mathbf{z}}z+\hat{\mathbf{r}}r}{(z^2+r^2)^{3/2}}\,dz = \frac{\rho_l}{4\pi\epsilon_0}r\hat{\mathbf{r}}\int_{-L/2}^{L/2}\frac{dz}{(z^2+r^2)^{3/2}}\\ &= \frac{\rho_l}{2\pi\epsilon_0 r}\hat{\mathbf{r}}\frac{(L/2)}{[(L/2)^2+r^2]^{1/2}}\end{aligned}$$

   Noting that $(L/2)/[(L/2)^2+r^2]^{1/2}\to 1$ as $L\to\infty$, we obtain the expected result.

4. $Q = \oint_S \mathbf{D}\cdot d\mathbf{s} = 4\pi\rho_0 a^3$, where we note that $\mathbf{D}\cdot d\mathbf{s} = \rho_0 a\,ds$, which is independent of (θ,ϕ). The entire surface has area $4\pi a^2$.

5. At an arbitrary point (r, ϕ) on the x-y plane, we have

$$V(r,\phi) = \frac{\rho_l}{4\pi\epsilon_0}\int_{-l/2}^{l/2}\frac{1}{(r^2+z^2)^{1/2}}\,dz$$
$$= \frac{\rho_l}{4\pi\epsilon_0}\left\{\ln\left[\frac{l}{2r}+\left(1+\frac{l^2}{4r^2}\right)^{1/2}\right]-\ln\left[-\frac{l}{2r}+\left(1+\frac{l^2}{4r^2}\right)^{1/2}\right]\right\}.$$

Setting $r = b$, we obtain the requested result,

$$V(b,\phi) = \frac{\rho_l}{4\pi\epsilon_0}\left\{\ln\left[\frac{l}{2b}+\left(1+\frac{l^2}{4b^2}\right)^{1/2}\right]-\ln\left[-\frac{l}{2b}+\left(1+\frac{l^2}{4b^2}\right)^{1/2}\right]\right\}.$$

Taking the derivative with respect to r of the voltage, we find

$$\mathbf{E}(r,\phi) = -\hat{\mathbf{r}}\frac{\partial V}{\partial r} = \frac{\rho_l}{2\pi\epsilon_0 r}\hat{\mathbf{r}}\frac{(l/2)}{[(l/2)^2+r^2]^{1/2}},$$

which just equals our previous result if we replace l with L.

6. We have that $\mathbf{E}_{2\mathrm{t}} = \hat{\mathbf{x}}3 - \hat{\mathbf{y}}2$ and $\mathbf{E}_{2\mathrm{n}} = \hat{\mathbf{z}}2$. From the boundary conditions, we have $\mathbf{E}_{1\mathrm{t}} = \mathbf{E}_{2\mathrm{t}} = \hat{\mathbf{x}}3 - \hat{\mathbf{y}}2$ and $\mathbf{E}_{1\mathrm{n}} = (\epsilon_2/\epsilon_1)\mathbf{E}_{2\mathrm{n}} + \hat{\mathbf{z}}(\rho_{\mathrm{S}}/\epsilon_1) = \hat{\mathbf{z}}[4.5 + (3.54\times 10^{-11}/1.771\times 10^{-11})] = \hat{\mathbf{z}}(4.5+2.00) = \hat{\mathbf{z}}6.5$. We conclude that $\mathbf{E}_1 = \hat{\mathbf{x}}3 - \hat{\mathbf{y}}2 + \hat{\mathbf{z}}6.5$. The angle that $\mathbf{E}_2$ makes with respect to the z-axis is given by $\tan^{-1}(E_{2\mathrm{t}}/E_{2\mathrm{n}})$. We have $E_{2\mathrm{t}} = \sqrt{2^2+3^2} = 3.606$ and $E_{2\mathrm{n}} = 2$. Hence, we have $\theta = \tan^{-1}(3.606/2) = 1.0644 = 60.98^\circ \to 61^\circ$.

7. A difference of 5 V over 2 cm implies that the electric field acting on the charge is $E = 250$ V/m. The magnitude of the force acting on the charge is given by $F = |Q_\mathrm{e}|E = (1.6\times 10^{-19})\times(2.50\times 10^2) = 4.0\times 10^{-17}$ N. The acceleration is given by $a = F/m_\mathrm{e} = (4.0\times 10^{-17})/(9.1\times 10^{-31}) = 4.40\times 10^{13}$ m/s. The distance that the electron moves in a given time t is given by $x = (1/2)at^2$, so that the time to move a given distance x is given by $t = (2x/a)^{1/2}$. In our case, we have $x = 0.02$ m, so that

$$t = \sqrt{2\times(2.0\times 10^{-2})/(4.40\times 10^{13})} = 3.02\times 10^{-8} \to 30 \text{ ns}$$

8. a. Since we have $E = V/d$ and the voltage is the same across the entire plate, it follows that the field must be same across the entire plate. Assuming that the plates are transverse to the z-direction and that the upper plate is at the higher voltage, we have $\mathbf{E}_1 = \mathbf{E}_2 = -\hat{\mathbf{z}}(V/d)$.

   b. The energy stored in sections 1 and 2 respectively are given by

$$U_1 = (1/2)\epsilon_1 E_1^2 A_1 d = (1/2)\epsilon_1 V^2 A_1/d,$$
$$U_2 = (1/2)\epsilon_2 E_2^2 A_2 d = (1/2)\epsilon_2 V^2 A_2/d.$$

   We also have $U_1 = (1/2)C_1V^2$ and $U_2 = (1/2)C_2V^2$. Equating the expressions for U_1 and U_2, we find $C_1 = \epsilon_1 A_1/d$ and $C_2 = \epsilon_2 A_2/d$.

   c. Equating $U = (1/2)CV^2$ to $U = U_1 + U_2$, we obtain the result $C = C_1 + C_2$.