Problem Set #9 Solutions

1. a. The phasor for the electric field of a left-circularly-polarized wave propagating in the $+z$-direction is $\widetilde{\mathbf{E}} = a(\hat{\mathbf{x}} + j\hat{\mathbf{y}})\exp(-jkz + \phi)$, where a is real. In this case, we find that the corresponding electric field becomes $\mathbf{E}(z,t) = a[\hat{\mathbf{x}}\cos(\omega t - kz + \phi) - \hat{\mathbf{y}}\sin(\omega t - kz + \phi)]$. In order for the amplitude to equal 5 V/m, we must have $a = 5$ V/m, and in order for the x-polarization to be a positive maximum at $z = 0$ and $t = 0$, we must have $\phi = 0$. In addition, we have that $k = 2\pi f/c = 4.189$ m^{-1}. We finally conclude, $\widetilde{\mathbf{E}} = 5(\hat{\mathbf{x}} + j\hat{\mathbf{y}})\exp(-j4.2z)$ V/m.

   b. We have $\eta'' = \sigma/\omega = [10^{-4}/(6.283 \times 2.00 \times 10^{8}] = 7.96 \times 10^{-13}$ F/m. We now find $\epsilon_{\mathrm{r}2} = (2.25 - j0.0899)$. We now have $\Gamma = (\sqrt{\epsilon_{\mathrm{r}1}} - \sqrt{\epsilon_{\mathrm{r}2}})/(\sqrt{\epsilon_{\mathrm{r}1}} + \sqrt{\epsilon_{\mathrm{r}2}}) = (1 - 1.500 + j0.0300)/(1 + 1.500 - j0.0300) = -0.200 - j0.0096$. We also have $\tau = 1 + \Gamma = 0.8 - j0.0096$.

   c. The reflected field phasor is, $\widetilde{\mathbf{E}}^{\mathrm{r}} = \Gamma\widetilde{\mathbf{E}}^{\mathrm{i}} = -(1.0 + j0.048)(\hat{\mathbf{x}} + j\hat{\mathbf{y}})\exp(j4.2z)$ V/m. We have for the transmitted field phasor $\widetilde{\mathbf{E}}^{\mathrm{t}} = \tau a(\hat{\mathbf{x}} + j\hat{\mathbf{y}})\exp(j\sqrt{\epsilon_{\mathrm{r}2}}kz) = (4.0 - j0.0096)(\hat{\mathbf{x}} + j\hat{\mathbf{y}})\exp[(j6.3 - 0.13)z)$ V/m. We have for the total field, $\widetilde{\mathbf{E}}^{\mathrm{total}} = (\hat{\mathbf{x}} + j\hat{\mathbf{y}})[5\exp(j4.2z) - (1.0 + j0.048)(\hat{\mathbf{x}} + j\hat{\mathbf{y}})\exp(j4.2z)$ V/m. Note that the reflected light is right-circularly polarized.

   d. Since both the reflected electric and magnetic fields have amplitudes that are 1/5 of the incident field, the reflected power must be 1/25 of the incident power, which is 4%. Hence, the transmitted power must be 96% of the incident power. To demonstrate this result directly, we note that the transmitted electric field amplitude is 4/5 of the incident amplitude, but the transmitted magnetic field amplitude is 6/5 of the incident amplitude. Hence, the total transmitted power is 24/25 of the incident power, which is 96%.

2. By analogy with quarter-wave impedance matching in transmission lines, we anticipate that $\epsilon_{\mathrm{r}2}$ will be the geometric mean of $\epsilon_{\mathrm{r}1}$ and $\epsilon_{\mathrm{r}3}$, which will ensure that the characteristic impedance in medium 2 is the geometric mean of the characteristic impedance in medium 3, and that d will be one-quarter of the wavelength in medium 2, to within additional multiples of $n\lambda_2/2$. In other words, we anticipate that $\epsilon_{\mathrm{r}2} = \sqrt{\epsilon_{\mathrm{r}1}\epsilon_{\mathrm{r}3}}$ and $d = [(2n+1)/4]\lambda_2 = [(2n+1)/4](c/f\sqrt{\epsilon_{\mathrm{r}2}})$, where n is any non-negative integer. To demonstrate this result, we note that the field in medium 3 will be purely transmitted. So, we may write the phasor in medium 3 as $\widetilde{\mathbf{E}}_3 = \widetilde{\mathbf{E}}_3^{\mathrm{t}} = (\hat{\mathbf{x}}E_{x3} + \hat{\mathbf{y}}E_{y3})\exp(j2\pi f\sqrt{\epsilon_{\mathrm{r}3}}z/c)$. In medium 2, it follows that $\widetilde{\mathbf{E}}_2 = \widetilde{\mathbf{E}}_2^{\mathrm{i}} + \widetilde{\mathbf{E}}_2^{\mathrm{r}}$, where

$$\widetilde{\mathbf{E}}_2^{\mathrm{i}} = \left(\frac{\sqrt{\epsilon_{\mathrm{r}2}} + \sqrt{\epsilon_{\mathrm{r}3}}}{2\sqrt{\epsilon_{\mathrm{r}2}}}\right)(\hat{\mathbf{x}}E_{x3} + \hat{\mathbf{y}}E_{y3})\exp(j2\pi f\sqrt{\epsilon_{\mathrm{r}2}}z/c),$$

$$\widetilde{\mathbf{E}}_2^{\mathrm{r}} = \left(\frac{\sqrt{\epsilon_{\mathrm{r}2}} - \sqrt{\epsilon_{\mathrm{r}3}}}{2\sqrt{\epsilon_{\mathrm{r}2}}}\right)(\hat{\mathbf{x}}E_{x3} + \hat{\mathbf{y}}E_{y3})\exp(-j2\pi f\sqrt{\epsilon_{\mathrm{r}2}}z/c).$$

The total electric field phasor in medium 2 at $z = -d$ now becomes

$$\widetilde{\mathbf{E}}_2(z=-d) = (\hat{\mathbf{x}}E_{x3} + \hat{\mathbf{y}}E_{y3}) \left[\left(\frac{\sqrt{\epsilon_{\mathrm{r2}}} + \sqrt{\epsilon_{\mathrm{r3}}}}{2\sqrt{\epsilon_{\mathrm{r2}}}} \right) \exp(-j2\pi f \sqrt{\epsilon_{\mathrm{r2}}} d/c) + \left(\frac{\sqrt{\epsilon_{\mathrm{r2}}} - \sqrt{\epsilon_{\mathrm{r3}}}}{2\sqrt{\epsilon_{\mathrm{r2}}}} \right) \exp(j2\pi f \sqrt{\epsilon_{\mathrm{r2}}} d/c) \right].$$

By an analogous calculation, the total magnetic field phasor in medium 2 at $z = -d$ will become

$$\widetilde{\mathbf{H}}_2(z=-d) = \frac{1}{\eta_2}(\hat{\mathbf{x}}E_{x3} + \hat{\mathbf{y}}E_{y3}) \left[\left(\frac{\sqrt{\epsilon_{\mathrm{r2}}} + \sqrt{\epsilon_{\mathrm{r3}}}}{2\sqrt{\epsilon_{\mathrm{r2}}}} \right) \exp(-j2\pi f \sqrt{\epsilon_{\mathrm{r2}}} d/c) - \left(\frac{\sqrt{\epsilon_{\mathrm{r2}}} - \sqrt{\epsilon_{\mathrm{r3}}}}{2\sqrt{\epsilon_{\mathrm{r2}}}} \right) \exp(j2\pi f \sqrt{\epsilon_{\mathrm{r2}}} d/c) \right].$$

In order for the reflected wave in medium 1 to be zero, it must be the case $\widetilde{\mathbf{H}}_2(z = -d) = (1/\eta_1)\widetilde{\mathbf{E}}_2(z = -d)$. Equating terms, we find that this condition requires

$$\begin{aligned}&\sqrt{\epsilon_{\mathrm{r2}}} \left[(\sqrt{\epsilon_{\mathrm{r2}}} + \sqrt{\epsilon_{\mathrm{r3}}}) \exp(-j2\pi d/\lambda_2) - (\sqrt{\epsilon_{\mathrm{r2}}} - \sqrt{\epsilon_{\mathrm{r3}}}) \exp(j2\pi d/\lambda_2) \right] \\ &\quad = \sqrt{\epsilon_{\mathrm{r1}}} \left[(\sqrt{\epsilon_{\mathrm{r2}}} + \sqrt{\epsilon_{\mathrm{r3}}}) \exp(-j2\pi d/\lambda_2) + (\sqrt{\epsilon_{\mathrm{r2}}} - \sqrt{\epsilon_{\mathrm{r3}}}) \exp(j2\pi d/\lambda_2) \right],\end{aligned}$$

where $\lambda_2 = c/f\sqrt{\epsilon_{2\mathrm{r}}}$, which in turn implies

$$\sqrt{\epsilon_{\mathrm{r2}}} (\sqrt{\epsilon_{\mathrm{r3}}} - \sqrt{\epsilon_{\mathrm{r1}}}) \cos(2\pi d/\lambda_2) + j(\epsilon_{\mathrm{r2}} - \sqrt{\epsilon_{\mathrm{r1}}\epsilon_{\mathrm{r3}}}) \sin(2\pi d/\lambda_2).$$

If $\epsilon_{\mathrm{r1}} \neq \epsilon_{\mathrm{r3}}$, these relationships can only be satisfied when $\epsilon_{\mathrm{r2}} = \sqrt{\epsilon_{\mathrm{r1}}\epsilon_{\mathrm{r3}}}$ and $\cos(2\pi d/\lambda_2) = 0$, which implies that $d = [(1 + 2n)/4]\lambda_2$, where n is any non-negative integer.

3. We begin by calculating the ratio $\epsilon''/\epsilon' = \sigma/\omega\epsilon = \sigma/(2\pi f \epsilon_{\mathrm{r}}\epsilon_0) = 4.0/[2\times 3.142 \times (5.0 \times 10^5) \times 72 \times (8.854 \times 10^{-12})] = 1997$, which is large enough that we can assume that we are in the highly conductive limit. In this limit, we have that $\alpha = (\pi f \mu_0 \sigma)^{1/2} = [3.142 \times (5.0 \times 10^5) \times (1.257 \times 10^{-6}) \times 4]^{1/2} = 2.810\ \mathrm{m}^{-1}$. We must next calculate the ratio of the transmitted to incident field in going from air to seawater. We have for seawater $\eta_{\mathrm{sea}} = (1 + j)\alpha/\sigma = (1 + j)(2.810/4) = (1 + j)0.07025\ \Omega$ and for air we have $\eta_{\mathrm{air}} = 376.7\ \Omega$. The magnitude of the field in seawater is given by

$$\begin{aligned}|\widetilde{\mathbf{E}}_{\mathrm{sea}}| &= |\tau \widetilde{\mathbf{E}}_{\mathrm{air}}| \exp(-\alpha z) \\ &= (2 \times 1.414 \times 0.07025/376.7) \exp(-2.810z)|\widetilde{\mathbf{E}}_{\mathrm{air}}| \\ &= 5.404 \times 10^{-4} \exp(-2.810z)|\widetilde{\mathbf{E}}_{\mathrm{air}}|.\end{aligned}$$

We find the maximum ratio $|\widetilde{\mathbf{E}}_{\mathrm{air}}|/|\widetilde{\mathbf{E}}_{\mathrm{sea}}| = 5000/(5.0 \times 10^{-9}) = 10^{12}$. Hence, we find that $\exp(2.810z_{\mathrm{max}}) = 5.404 \times 10^8$, which implies that $z_{\mathrm{max}} = 7.156 \to 7.2$ m, which is not much! The exponential damping is a real killer!!

4. For the red light, we find that $n_{\text{red}} = 1.71 - (4/30) \times 0.7 = 1.6167$. The angle that the red ray makes with respect to the normal to the left face is given by $\sin(\theta_{\text{red,left}}) = (1.0000/1.6167) \sin 50° = 0.47384$, which imples that $\theta_{\text{red,left}} = 0.49365$ rads $= 28.284°$. The ray with respect to the left face, the ray with respect to the right face, and the prism vertex form a triangle, whose angles are respectively $90° - \theta_{\text{red,left}}$, $90° - \theta_{\text{red,right}}$, and $60°$, where all angles are measured in degrees. Since the three angles of a triangle add to $180°$, we conclude that $\theta_{\text{red,right}} = 60° - \theta_{\text{red,left}} = 31.716° = 0.55355$ rads. We finally have that the corresponding outgoing angle is given by $\sin(\theta_{\text{red,out}}) = (1.6167/1.0000) \sin(\theta_{\text{red,right}}) = 0.52571 \times 1.6167 = 0.84990$. So, we find that $\theta_{\text{red,out}} = 1.0158$ rads $= 58.200°$. For the violet light, we find $n_{\text{violet}} = 1.71 - (4/30) \times 0.4 = 1.6567$. A calculation analogous to the one that we carried out for the red ray yields: $\theta_{\text{violet,left}} = 27.542°$, $\theta_{\text{violet,right}} = 32.458°$, $\theta_{\text{violet,out}} = 62.760°$. We conclude that the angular dispersion is $62.760° - 58.200° = 4.560° \to 4.6°$. I carried out this calculation in MATLAB with fifteen significant figures to avoid loss of calculational accuracy. We must be careful with this calculation since we lose over one digit of accuracy due to the subtraction of two nearly equal angles at the end.

5. Since the bubble has an apparent height of 6.81 cm, its horizontal position from the point that a ray at $60°$ would enter the water is given by $\sqrt{3} \times 6.81 = 11.795$ cm. The normal angle that the incoming ray makes in water is given by $\sin(\theta_{\text{water}}) = (1.00/1.33) \sin(60°) = 0.6511$, from which we find that

$$\cos(\theta_{\text{water}}) = [1 - \sin^2(\theta_{\text{water}})]^{1/2} = 0.7590$$

and $\tan(\theta_{\text{water}}) = 0.8580$. Hence, the horizontal distance that the ray has traveled when it has gone a vertical distance of 10 cm in the water is $10 \tan(\theta_{\text{water}}) = 8.580$ cm, leaving a distance of 3.216 cm to reach the horizontal position of the bubble. The angle that the ray makes in the glass is given by $\sin(\theta_{\text{glass}}) = (1.33/1.6) \sin(\theta_{\text{water}}) = 0.5413$, from which we find that $\cos(\theta_{\text{glass}}) = 0.8409$ and $\cot(\theta_{\text{glass}}) = 1.5535$. The vertical distance that the ray travels in the glass is given by $3.216 \cot(\theta_{\text{glass}}) = 3.216 \times 1.5535 = 4.996 \to 5.00$ cm. Thus, the bubble is actually 15 cm below the air-water interface and 5 cm below the water-glass interface. Again, I used MATLAB to be sure to maintain calculational accuracy.

6. We have $\eta_1/\eta_2 = n_2/n_1 = 1.550$. It follows that

$$\begin{aligned} \sin\theta_{\text{i}} &= 0.7071, & \sin\theta_{\text{t}} &= 0.7071/1.550 = 0.4562, \\ \cos\theta_{\text{i}} &= 0.7071, & \cos\theta_{\text{t}} &= 0.8899. \end{aligned}$$

   a. From these expressions, we find

$$\Gamma_\perp = \frac{\cos\theta_{\text{i}} - (\eta_1/\eta_2)\cos\theta_{\text{t}}}{\cos\theta_{\text{i}} + (\eta_1/\eta_2)\cos\theta_{\text{t}}} = \frac{0.7071 - 1.550 \times 0.8899}{0.7071 + 1.550 \times 0.8899} = -0.322$$

and

$$\tau_\perp = \frac{2\cos\theta_\mathrm{i}}{\cos\theta_\mathrm{i} + (\eta_1/\eta_2)\cos\theta_\mathrm{t}} = \frac{2 \times 0.8660}{0.8660 + 1.600 \times 0.9499} = 0.678$$

b. We may choose the direction of propagation into the glass to be the z-direction and the direction of polarization of the electric field to be the y-direction. In this case, we have for the phasors of the transmitted wave that

$$\widetilde{\mathbf{E}}^\mathrm{t}_\perp = \hat{\mathbf{y}}\tau_\perp E^\mathrm{i}_{\perp 0}\exp[-jk_2(x\sin\theta_\mathrm{t} + z\cos\theta_\mathrm{t})],$$
$$\widetilde{\mathbf{H}}^\mathrm{t}_\perp = (-\hat{\mathbf{x}}\cos\theta_\mathrm{t} + \hat{\mathbf{z}}\sin\theta_\mathrm{t})\frac{\tau_\perp E^\mathrm{t}_{\perp 0}}{\eta_2}\exp[-jk_2(x\sin\theta_\mathrm{t} + z\cos\theta_\mathrm{t})].$$

We have $\omega = 2\pi f = 2\pi \times (4.50 \times 10^{14}\ \mathrm{s}^{-1}$, $k_2 = n_2\omega/c = 2\pi \times (2.325 \times 10^6\ \mathrm{m}^{-1}$, $\tau_\perp E^\mathrm{i}_\perp = 33.89$ V/m, $\eta_2 = 376.7/1.55 = 243.1\ \Omega$, and $\tau_\perp E^\mathrm{i}_\perp/\eta_2 = 0.1394$ A/m. Substituting these values, we obtain to two significant figures,

$$\mathbf{E}(x,y,z,t) = \hat{\mathbf{y}}34\cos\bigl[2\pi \times [(4.5 \times 10^{14}t - 2.3\pi(0.46x + 0.89z)\bigr]\ \mathrm{V/m},$$

$$\mathbf{H}(x,y,z,t) = (-\hat{\mathbf{x}}0.89 + \hat{\mathbf{z}}0.46)0.14$$
$$\cos\bigl[2\pi \times [(4.5 \times 10^{14}t - 2.3\pi(0.46x + 0.89z)\bigr]\ \mathrm{A/m}$$