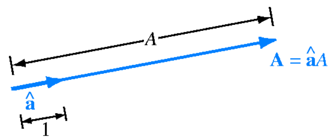
Vector Algebra

Definition

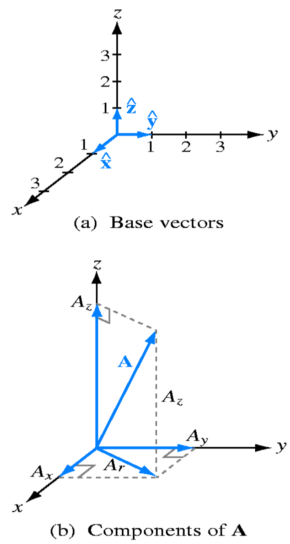
A vector $\mathbf{A}$ has a magnitude $A = |\mathbf{A}|$ and a direction given by the unit vector $\hat{\mathbf{a}} = \mathbf{A} / |\mathbf{A}|$.

In Cartesian coordinates (x, y, z), we find

$$\mathbf{A} = \hat{\mathbf{a}}A = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$$
$$= (A_x, A_y, A_z)$$



Ulaby Figure 3-1



Ulaby Figure 3-2

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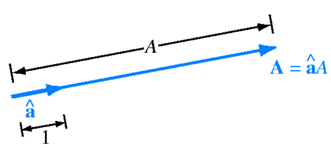
Vector Algebra

Definition

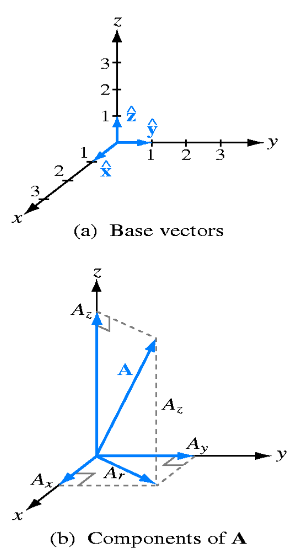
We also have

$$A = \left(A_x^2 + A_y^2 + A_z^2\right)^{1/2}$$

$$\hat{\mathbf{a}} = \frac{\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z}{\left(A_x^2 + A_y^2 + A_z^2\right)^{1/2}}$$



Ulaby Figure 3-1



Ulaby Figure 3-2

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Vector Algebra

Equality

Two vectors **A** and **B** are equal if

$$A_x = B_x,\ A_y = B_y,\ \text{and } A_z = B_z$$

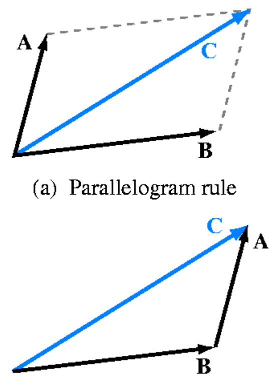
Addition and Subtraction

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \quad \text{(addition is commutative)}$$

$$= \hat{\mathbf{x}}(A_x + B_x) + \hat{\mathbf{y}}(A_y + B_y) + \hat{\mathbf{z}}(A_z + B_z)$$

$$\mathbf{D} = \mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$$

$$= \hat{\mathbf{x}}(A_x - B_x) + \hat{\mathbf{y}}(A_y - B_y) + \hat{\mathbf{z}}(A_z - B_z)$$



(a) Parallelogram rule

(b) Head-to-tail rule

Ulaby Figure 3-3

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Vector Algebra

Position and Distance vectors

For a point $P(x, y, z)$, the *position vector* **R** goes from the origin to the point

$$\mathbf{R} = \hat{\mathbf{x}}\, x + \hat{\mathbf{y}}\, y + \hat{\mathbf{z}}\, z$$

The distance vector between two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is defined as

$$\mathbf{R}_{12} = \mathbf{R}_2 - \mathbf{R}_1$$

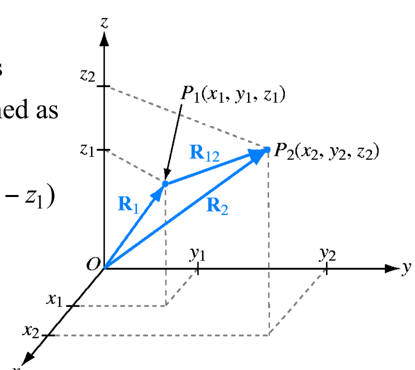
$$= \hat{\mathbf{x}}(x_2 - x_1) + \hat{\mathbf{y}}(y_2 - y_1) + \hat{\mathbf{z}}(z_2 - z_1)$$

where $\mathbf{R}_i = \hat{\mathbf{x}}\, x_i + \hat{\mathbf{y}}\, y_i + \hat{\mathbf{z}}\, z_i,\ \ i = 1, 2$

The distance d between two points equals $|\mathbf{R}_{12}|$

$$d = |\mathbf{R}_{12}|$$

$$= \left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]^{1/2}$$



Ulaby Figure 3-4

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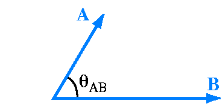
Vector Algebra

Multiplication — Simple Product

When a vector **A** is multiplied by a scalar k, the magnitude is multiplied by k and the direction is unchanged

$$\mathbf{B} = k\mathbf{A} = \hat{\mathbf{a}}(kA)$$
$$= \hat{\mathbf{x}}(kA_x) + \hat{\mathbf{y}}(kA_y) + \hat{\mathbf{z}}(kA_z)$$



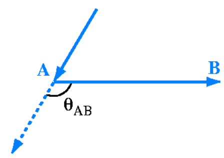
(a)

Multiplication — Scalar or Dot Product

The scalar or dot product of two vectors **A** and **B** is defined by

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta_{AB}$$

where θ_{AB} is the angle between **A** and **B**



(b)

Ulaby Figure 3-5

When $\theta_{AB} < 90^{\circ}$, $\mathbf{A}\cdot\mathbf{B} > 0$
When $\theta_{AB} > 90^{\circ}$, $\mathbf{A}\cdot\mathbf{B} < 0$
When $\theta_{AB} = 90^{\circ}$, $\mathbf{A}\cdot\mathbf{B} = 0$, and the vectors are called *orthogonal*

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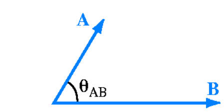
Vector Algebra

Multiplication — Scalar or Dot Product

We have

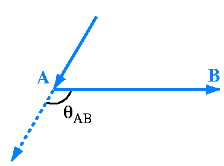
$$\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$$
$$\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{x}}\cdot\hat{\mathbf{z}} = 0$$



(a)

so that

$$\mathbf{A}\cdot\mathbf{B} = \left(\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z\right)\cdot\left(\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z\right)$$
$$= A_xB_x + A_yB_y + A_zB_z$$



(b)

Other properties:

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A} \quad \text{(commutative)}$$
$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C} \quad \text{(distributive)}$$

Ulaby Figure 3-5

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Vector Algebra

Multiplication — Vector or Cross Product

The vector or cross product of two vectors **A** and **B** is defined by

$$\mathbf{A}\times\mathbf{B} = \hat{\mathbf{n}}\, AB\sin\theta_{AB}$$

where θ_{AB} is the angle *from* **A** *to* **B** and $\hat{\mathbf{n}}$ is a unit vector determined by the right-hand rule

The vector product is *anti-commutative*

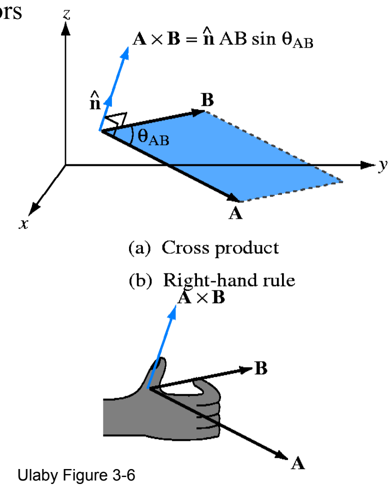
$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$$

and distributive

$$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B}+\mathbf{A}\times\mathbf{C}$$

$$\hat{\mathbf{x}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}\times\hat{\mathbf{z}} = 0$$

$$\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}},\quad \hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}},\quad \hat{\mathbf{z}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}$$



(a) Cross product

(b) Right-hand rule

Ulaby Figure 3-6

Vector Algebra

Multiplication — Vector or Cross Product

We have

$$\hat{\mathbf{x}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}\times\hat{\mathbf{z}} = 0$$

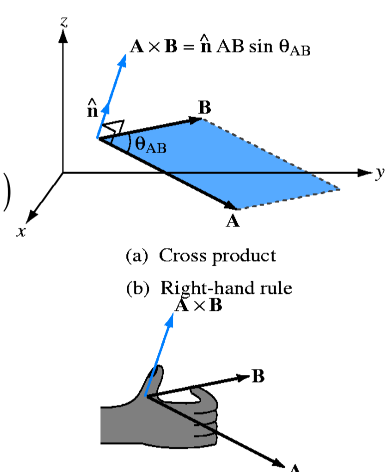
$$\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}},\quad \hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}},\quad \hat{\mathbf{z}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}$$

so that

$$\begin{aligned}\mathbf{A}\times\mathbf{B} &= \left(\hat{\mathbf{x}}A_x+\hat{\mathbf{y}}A_y+\hat{\mathbf{z}}A_z\right)\times\left(\hat{\mathbf{x}}B_x+\hat{\mathbf{y}}B_y+\hat{\mathbf{z}}B_z\right)\\ &= \hat{\mathbf{x}}(A_yB_z-A_zB_y)+\hat{\mathbf{y}}(A_zB_x-A_xB_z)\\ &\quad+\hat{\mathbf{z}}(A_xB_y-A_yB_x)\end{aligned}$$

We may also write

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}$$



(a) Cross product

(b) Right-hand rule

Ulaby Figure 3-6

This determinant form is very useful!

Vector Algebra

Summary of vector products

Product type	Product elements	Representation
Simple product	(Scalar) x (Vector) → Vector	$\mathbf{C} = k\mathbf{A}$
Scalar product or Dot product	(Vector) x (Vector) → Scalar	$\mathbf{C} = \mathbf{A} \bullet \mathbf{B}$
Vector product or Cross product	(Vector) x (Vector) → Vector	$\mathbf{C} = \mathbf{A} \times \mathbf{B}$

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6.9

Vector Algebra

Vectors and Angles: Ulaby and Ravaioli Example 3-1

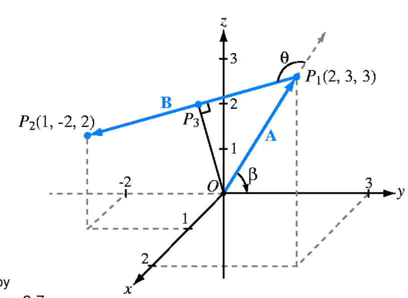
Question: In Cartesian coordinates, vector **A** is directed from the origin to the point $P_1(2, 3, 3)$, and vector **B** is directed from P_1 to $P_2(1, -2, 2)$. Find (a) the vector **A**, its magnitude A, and its unit vector $\hat{\mathbf{a}}$, (b) the angle that **A** makes with the y-axis, (c) vector **B**, (d) the angle between **A** and **B**, and (e) the perpendicular distance from the origin to **B**.

Answer: (a)

$$\mathbf{A} = \hat{\mathbf{x}}\,2 + \hat{\mathbf{y}}3 + \hat{\mathbf{z}}3$$

$$A = \sqrt{4+9+9} = \sqrt{22}$$

$$\hat{\mathbf{a}} = \mathbf{A} / A = \left(\hat{\mathbf{x}}\,2 + \hat{\mathbf{y}}3 + \hat{\mathbf{z}}3\right) \big/ \sqrt{22}$$



Ulaby
Figure 3-7

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6.10

Vector Algebra

Vectors and Angles: Ulaby and Ravaioli Example 3-1

Answer (continued): (b) The angle β between **A** and the y-axis may be found from the expression $\mathbf{A}\cdot\hat{\mathbf{y}} = A\cos\beta$, which implies

$$\beta = \cos^{-1}\left(\frac{\mathbf{A}\cdot\hat{\mathbf{y}}}{A}\right) = \cos^{-1}\left(\frac{3}{\sqrt{22}}\right) = 0.879 \text{ rads} = 50.2°$$

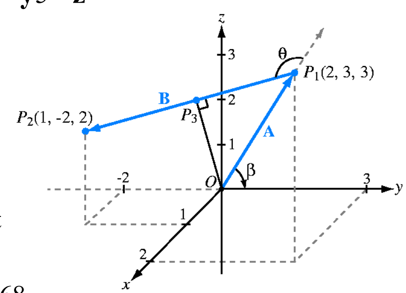
(c) $\mathbf{B} = \hat{\mathbf{x}}(1-2) + \hat{\mathbf{y}}(-2-3) + \hat{\mathbf{z}}(2-3) = -\hat{\mathbf{x}} - \hat{\mathbf{y}}5 - \hat{\mathbf{z}}$

(d) $$\theta = \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{AB}\right) = \cos^{-1}\left(\frac{-20}{\sqrt{22}\sqrt{27}}\right)$$

$$= 2.533 \text{ rads} = 145.1°$$

(e) The points OP_1P_3 form a right triangle. The magnitude of the line segment OP_3 is given by

$$A\sin(\pi-\theta) = \sqrt{22}\sin(0.609) = 2.68$$



Ulaby Figure 3-7

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Vector Algebra

Triple Scalar Product

This product can be written in the equivalent forms

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

The equivalence can be demonstrated from the determinant representation

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

The absolute value of this scalar product is the volume of the parallelepiped whose sides are the vectors **A**, **B**, and **C.**

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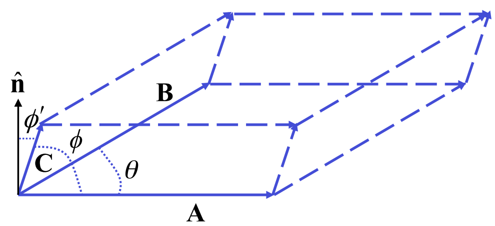
Vector Algebra

Triple Scalar Product

This product can be written in the equivalent forms

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

It corresponds to the volume of a parallelepiped



$$\hat{\mathbf{n}} = \frac{\mathbf{A}\times\mathbf{B}}{|\mathbf{A}\times\mathbf{B}|}$$

$$\phi' = \frac{\pi}{2} - \phi$$

Surface area $= |\mathbf{A}\times\mathbf{B}| = |\mathbf{A}||\mathbf{B}|\sin\theta$

Volume $= |\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})| = |\mathbf{A}||\mathbf{B}||\mathbf{C}|\sin\theta\cos\phi' = |\mathbf{A}||\mathbf{B}||\mathbf{C}|\sin\theta\sin\phi$

Vector Algebra

Triple Vector Product

This product is defined as

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) \quad \text{and we note} \quad \mathbf{A}\times(\mathbf{B}\times\mathbf{C}) \neq (\mathbf{A}\times\mathbf{B})\times\mathbf{C}$$

This relationship can also be written in the form

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

Orthogonal Coordinate Systems

Three coordinate systems

Coordinate systems (with three coordinates) specify points in space

Orthogonal systems have coordinates that are mutually perpendicular

— at least locally

There are three coordinate systems that are often used

- Cartesian coordinates (x, y, z): Simplest and orthogonal everywhere
- Cylindrical coordinates (r, ϕ, z)
  — used with optical fibers, coaxial cables, cylindrical waveguides
- Spherical coordinates (R, θ, ϕ)
  — used with antenna radiation, radar, earth-ionosphere waveguide

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6.15

Three Coordinate Systems

Summary of vector and differential relations

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, θ, z	R, θ, ϕ
Vector, $\mathbf{A} =$	$\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$
Magnitude, $\lvert\mathbf{A}\rvert =$	$\left(A_x^2 + A_y^2 + A_z^2\right)^{1/2}$	$\left(A_r^2 + A_\phi^2 + A_z^2\right)^{1/2}$	$\left(A_R^2 + A_\theta^2 + A_\phi^2\right)^{1/2}$
Position vector	$\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$ for $P(x_1, y_1, z_1)$	$\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1$ for $P(r_1, \phi_1, z_1)$	$\hat{\mathbf{R}}R_1$ for $P(R_1, \theta_1, \phi_1)$
Dot product, $\mathbf{A}\cdot\mathbf{B}$	$A_xB_x + A_yB_y + A_zB_z$	$A_rB_r + A_\phi B_\phi + A_zB_z$	$A_RB_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product, $\mathbf{A}\times\mathbf{B}$	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length, $d\mathbf{l}$	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}\,r d\phi + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}\,R d\theta + \hat{\boldsymbol{\phi}}\,R\sin\theta dz$
Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\,dydz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\,dzdx$ $d\mathbf{s}_z = \hat{\mathbf{z}}\,dxdy$	$d\mathbf{s}_r = \hat{\mathbf{r}}\,rd\phi dz$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dzdr$ $d\mathbf{s}_z = \hat{\mathbf{z}}\,rdrd\phi$	$d\mathbf{s}_R = \hat{\mathbf{R}}\,R^2\sin\theta d\theta d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}\,R\sin\theta d\phi dR$ $d\mathbf{s}_z = \hat{\boldsymbol{\phi}}\,R dR d\theta$
Differential volume	$dv = dx\,dy\,dz$	$dv = r dr\,d\phi\,dz$	$dv = R^2\sin\theta\,dR\,d\theta\,d\phi$

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6.16

Cartesian Coordinates

Differential Relations

Length: $d\mathbf{l} = \hat{\mathbf{x}}\,dl_x + \hat{\mathbf{y}}\,dl_y + \hat{\mathbf{z}}\,dl_z = \hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$

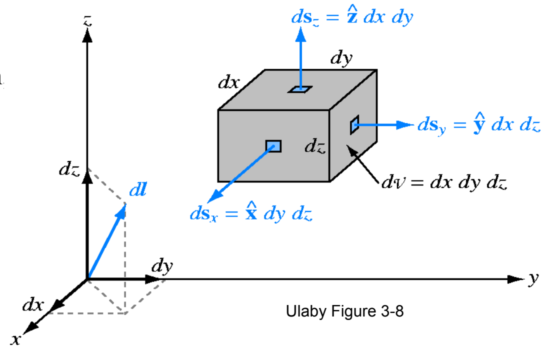
Surface area: $d\mathbf{s}_x = \hat{\mathbf{x}}\,dl_y\,dl_z = \hat{\mathbf{x}}\,dy\,dz$ (y-z plane)

$d\mathbf{s}_y = \hat{\mathbf{y}}\,dz\,dx = \hat{\mathbf{y}}\,dx\,dz$ (x-z plane)

$d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$ (x-y plane)

Volume: $dv = dx\,dy\,dz$

For each coordinate system there are three differential lengths that also determine the differential areas and volumes



Ulaby Figure 3-8

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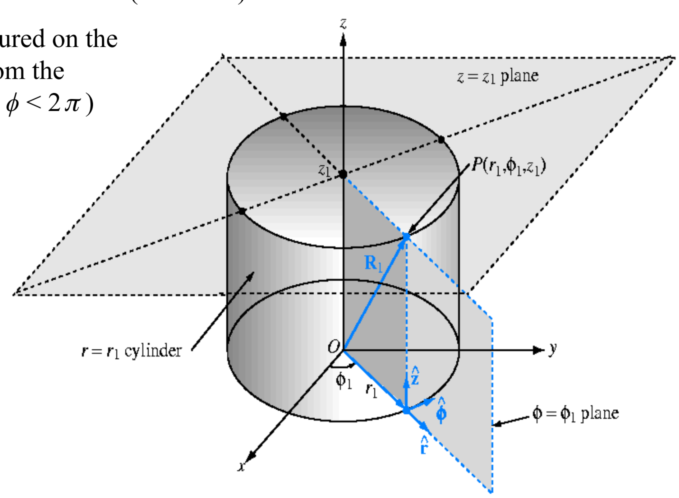
Cylindrical Coordinates

r = distance from the z-axis ($0 < r < \infty$)

ϕ = angle measured on the x-y plane from the $+x$-axis ($0 < \phi < 2\pi$)



Ulaby Figure 3-9

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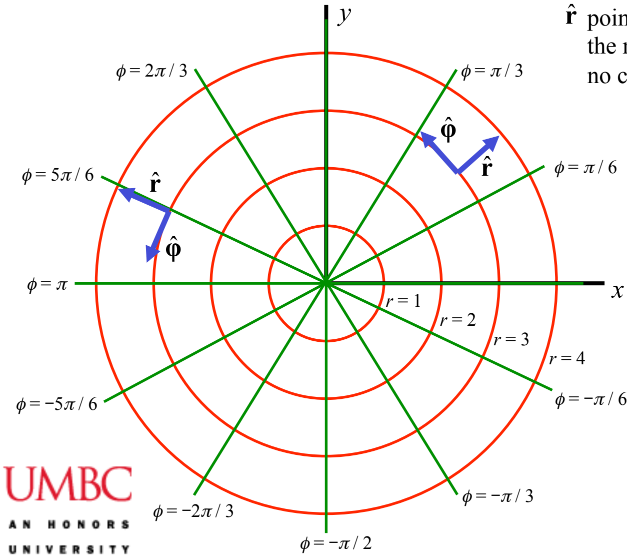
6.18

Cylindrical Coordinates



$\hat{\mathbf{r}}$ points in the direction of the maximum change in r; no change in ϕ, z

$\hat{\boldsymbol{\varphi}}$ points in the direction of the maximum change in ϕ; no change in r, z

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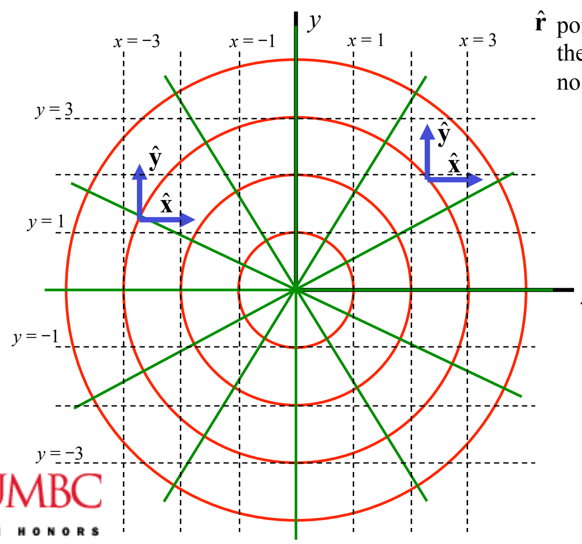
6.19

Cylindrical Coordinates



$\hat{\mathbf{r}}$ points in the direction of the maximum change in r; no change in ϕ, z

$\hat{\boldsymbol{\varphi}}$ points in the direction of the maximum change in ϕ; no change in r, z

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$$
$$\hat{\boldsymbol{\varphi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$$

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6.20

Cylindrical Coordinates

Base vectors

$$\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}}=\hat{\mathbf{z}},\quad \hat{\boldsymbol{\phi}}\times\hat{\mathbf{z}}=\hat{\mathbf{r}},\quad \hat{\mathbf{z}}\times\hat{\mathbf{r}}=\hat{\boldsymbol{\phi}}$$

$$\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}=\hat{\boldsymbol{\phi}}\cdot\hat{\boldsymbol{\phi}}=\hat{\mathbf{z}}\cdot\hat{\mathbf{z}}=1,\quad \hat{\mathbf{r}}\times\hat{\mathbf{r}}=\hat{\boldsymbol{\phi}}\times\hat{\boldsymbol{\phi}}=\hat{\mathbf{z}}\times\hat{\mathbf{z}}=0$$

$$\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\phi}}=\hat{\mathbf{r}}\cdot\hat{\mathbf{z}}=\hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{z}}=0$$

Vector relations

$$\mathbf{A}=\hat{\mathbf{a}}A=\hat{\mathbf{r}}A_r+\hat{\boldsymbol{\phi}}A_\phi+\hat{\mathbf{z}}A_z$$

$$A=|\mathbf{A}|=\sqrt{\mathbf{A}\cdot\mathbf{A}}=\left(A_r^2+A_\phi^2+A_z^2\right)^{1/2}$$

Letting $P=P(r_1,\phi_1,z_1)$, the position vector $\mathbf{R}_1=OP=\hat{\mathbf{r}}\,r_1+\hat{\mathbf{z}}\,z_1$

We note that the direction of $\hat{\mathbf{r}}$ depends on ϕ

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Cylindrical Coordinates

Differential relations

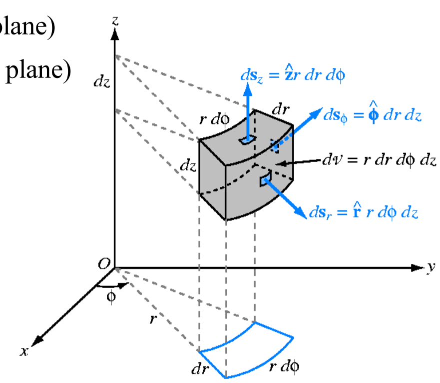
Length: $d\mathbf{l}=\hat{\mathbf{r}}\,dl_r+\hat{\boldsymbol{\phi}}\,dl_\phi+\hat{\mathbf{z}}\,dl_z=\hat{\mathbf{r}}\,dr+\hat{\boldsymbol{\phi}}\,r d\phi+\hat{\mathbf{z}}\,dz$

Surface area: $d\mathbf{s}_r=\hat{\mathbf{r}}\,r d\phi\,dz$ (ϕ-z plane)

$d\mathbf{s}_\phi=\hat{\boldsymbol{\phi}}\,dr\,dz$ (r-z plane)

$d\mathbf{s}_z=\hat{\mathbf{z}}\,r dr\,d\phi$ (r-ϕ plane)

Volume: $dv=r dr\,d\phi\,dz$



Ulaby Figure 3-10

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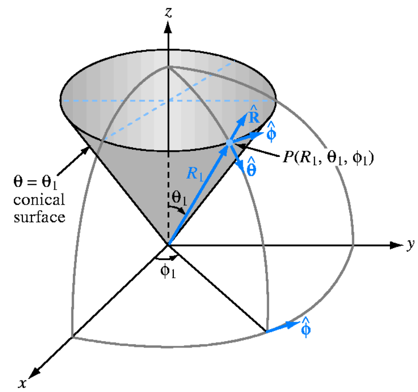
Divergence

Spherical Coordinates

R = distance from the origin ($0 < R < \infty$)

θ = angle measured from the +z-axis ($0 < \theta < \pi$) = *zenith angle*

ϕ = angle measured from the +x-axis on the x-y plane ($0 < \phi < 2\pi$)
= *azimuthal angle*



Ulaby Figure 3-13

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Spherical Coordinates

Base vectors

$$\hat{\mathbf{R}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\varphi}}, \quad \hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\varphi}} = \hat{\mathbf{R}}, \quad \hat{\boldsymbol{\varphi}} \times \hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$$

$$\hat{\mathbf{R}} \cdot \hat{\mathbf{R}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\varphi}} \cdot \hat{\boldsymbol{\varphi}} = 1, \quad \hat{\mathbf{R}} \times \hat{\mathbf{R}} = \hat{\boldsymbol{\varphi}} \times \hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\theta}} = 0$$

$$\hat{\mathbf{R}} \cdot \hat{\boldsymbol{\theta}} = \hat{\mathbf{R}} \cdot \hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\varphi}} = 0$$

Vector relations

$$\mathbf{A} = \hat{\mathbf{a}}A = \hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\varphi}}A_\phi$$

$$A = |\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}} = \left(A_R^2 + A_\theta^2 + A_\phi^2\right)^{1/2}$$

Letting $P = P(R_1, \theta_1, \phi_1)$, the position vector $\mathbf{R}_1 = OP = \hat{\mathbf{R}} R_1$

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Spherical Coordinates

Differential relations

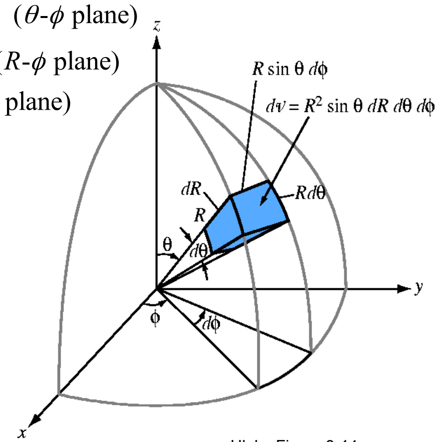
Length: $d\mathbf{l} = \hat{\mathbf{R}}\,dl_R + \hat{\boldsymbol{\theta}}\,dl_\theta + \hat{\boldsymbol{\phi}}\,dl_\phi = \hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}\,R\,d\theta + \hat{\boldsymbol{\phi}}\,R\sin\theta\,d\phi$

Surface area: $d\mathbf{s}_R = \hat{\mathbf{R}}\,R^2 \sin\theta\,d\theta\,d\phi$ (θ-ϕ plane)

$d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}\,R\sin\theta\,dR\,d\phi$ (R-ϕ plane)

$d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,R\,dR\,d\theta$ (R-θ plane)

Volume: $dv = R^2 \sin\theta\,dR\,d\theta\,d\phi$



Ulaby Figure 3-14

Spherical Coordinates

Charge in a Sphere: Ulaby and Ravaioli Example 3-6

Question: A sphere of radius 2 cm contains a charge of density ρ_V given by

$$\rho_V = 4\cos^2\theta$$

What is the total charge?

Answer: After converting from cm to m,

$$Q = \int_v \rho_V\,dv$$

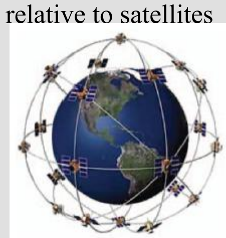
$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{R=0}^{2\times10^{-2}} \left(4\cos^2\theta\right) R^2 \sin\theta\,dR\,d\theta\,d\phi$$

$$= 4\int_0^{2\pi}\int_0^{\pi} \left(\frac{R^3}{3}\right)\Bigg|_0^{2\times10^{-2}} \sin\theta\cos^2\theta\,d\theta\,d\phi$$

$$= \frac{64}{9}\times10^{-6}\int_0^{2\pi} d\phi = \frac{128\pi}{9}\times10^{-6} = 44.68\ \mu\text{C}$$

Tech Brief 5: GPS

- •Originally developed by DOD
- •Originally 24 satellites, now 31
- •Operation principles
  - •Satellites constantly broadcast a message containing
    - •Their location
    - •Message time
    - •System health and rough orbit details
  - •GPS receivers use triangulation to determine location relative to satellites
    - •Determine distances to each satellite by solving an equation including the sat. position and time, multiplying by the speed of light.



$$d_1^2 = (x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2 = c\,[(t_1 + t_0)]^2\,,$$
$$d_2^2 = (x_2 - x_0)^2 + (y_2 - y_0)^2 + (z_2 - z_0)^2 = c\,[(t_2 + t_0)]^2\,,$$
$$d_3^2 = (x_3 - x_0)^2 + (y_3 - y_0)^2 + (z_3 - z_0)^2 = c\,[(t_3 + t_0)]^2\,,$$
$$d_4^2 = (x_4 - x_0)^2 + (y_4 - y_0)^2 + (z_4 - z_0)^2 = c\,[(t_4 + t_0)]^2\,.$$

Four satellites are needed to correct for the imprecise receiver clock (quartz).

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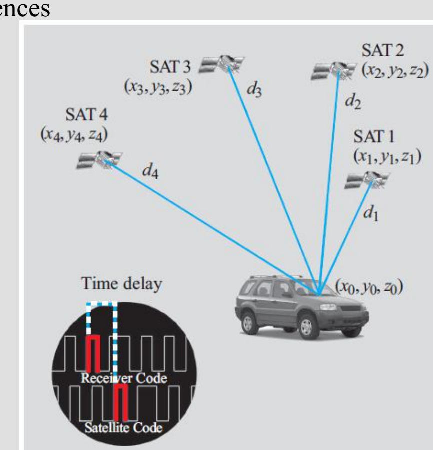
Tech Brief 5: Differential GPS

- •GPS accuracy: 20-30m

- •Differential GPS (DGPS) uses a static reference of known location in the receiver's area to correct for inaccuracy factors
  - •Time-delay errors (speed of light differences
  - •Multipath interference
  - •Satellite location errors

- •Reference receiver calculates correction factors and transmits to DGPS receivers in the area



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