Problem Set #4 Solutions

1. a. We have $A = \sqrt{1^2+2^2+3^2} = \sqrt{14} \simeq 3.74$. We have $\hat{\mathbf{a}} = \mathbf{A}/A = \hat{\mathbf{x}}(1/\sqrt{14}) + \hat{\mathbf{y}}(2/\sqrt{14}) + \hat{\mathbf{z}}(3/\sqrt{14}) \simeq \hat{\mathbf{x}}0.267 + \hat{\mathbf{y}}0.535 + \hat{\mathbf{z}}0.803$.

   b. The component of $\mathbf{B}$ along $\mathbf{C}$ is given by $\mathbf{B}\cdot\hat{\mathbf{c}} = \mathbf{B}\cdot\mathbf{C}/C = (3)(4)/\sqrt{3^2+4^2} = 12/5 = 2.40$

   c. We first note $\cos\theta_{AC} = \mathbf{A}\cdot\mathbf{C}/AC = [(2)(3)+(3)(-4)]/5\sqrt{14} = -6/5\sqrt{14} \simeq -0.321$, which implies that $\theta_{AC} = \cos^{-1}(-0.321) = 1.90$ rads $= 108°$.

   d. $\mathbf{A}\times\mathbf{C} = \hat{\mathbf{x}}[(2)(-4)-(3)(3)] + \hat{\mathbf{y}}[(3)(0)-(1)(-4)] + \hat{\mathbf{z}}[(1)(3)-(2)(0)] = -\hat{\mathbf{x}}17 + \hat{\mathbf{y}}4 + \hat{\mathbf{z}}3$

   e. We first note $\mathbf{B}\times\mathbf{C} = -\hat{\mathbf{x}}16 + \hat{\mathbf{y}}12 + \hat{\mathbf{z}}9$ We then find $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = (1)(-16)+(2)(12)+(3)(9) = 35$.

   f. $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \hat{\mathbf{x}}[(2)(9)-(3)(12)] + \hat{\mathbf{y}}[(3)(-16)-(1)(9)] + \hat{\mathbf{z}}[(1)(12)-(2)(-16)] = -\hat{\mathbf{x}}18 - \hat{\mathbf{y}}57 + \hat{\mathbf{z}}44$.

   g. $\hat{\mathbf{x}}\times\mathbf{B} = \hat{\mathbf{z}}(1)(4) = \hat{\mathbf{z}}4$.

   h. We have $\mathbf{A}\times\hat{\mathbf{y}} = -\hat{\mathbf{x}}3 + \hat{\mathbf{z}}$. So, it follows that $(\mathbf{A}\times\hat{\mathbf{y}})\cdot\hat{\mathbf{z}} = 1$

2. The volume of a parallelepiped is given by the base $\times$ the height. The base area may be taken to be the area defined by the vectors $\mathbf{B}$ and $\mathbf{C}$. They define a parallelogram, whose area is given by $BC\sin\theta_{BC} = |\mathbf{B}\times\mathbf{C}|$, where $0 \le \theta_{BC} < 180°$, so that $\sin(\theta_{BC}) \ge 0$. The height is given by the component of $\mathbf{A}$ that is orthogonal to the $\mathbf{B}$-$\mathbf{C}$ surface, which is also parallel to the normal to the surface. The magnitude of this component is given by $|A\cos\theta_A|$, where θ_A is the angle that $\mathbf{A}$ makes with respect to the surface normal $\hat{\mathbf{n}}$. We have $\hat{\mathbf{n}} = (\mathbf{B}\times\mathbf{C})/|\mathbf{B}\times\mathbf{C}|$. So, the height is given by $|\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})|/|\mathbf{B}\times\mathbf{C}|$ and multiplying this expression by the expression for the area of the base, we arrive at the expected result.

3. Three non-collinear points that define the plane may be chosen as $P_1(0,0,4)$, $P_2(8,0,0)$, and $P_3(0,4,1)$. Two distance vectors that are parallel to the plane may thus be chosen as $\mathbf{R}_{12} = \mathbf{R}_2 - \mathbf{R}_1 = \hat{\mathbf{x}}8 - \hat{\mathbf{z}}4$ and $\mathbf{R}_{13} = \hat{\mathbf{y}}3 - \hat{\mathbf{z}}3$. We immediately see that the vector defined by the coefficients of x, y, and z, $\mathbf{A} = \hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 + \hat{\mathbf{y}}4$, is orthogonal to both $\mathbf{R}_{12}$ and $\mathbf{R}_{13}$ and is hence normal to the plane. This result is general. The vector whose components are given by the coefficients that define a plane will always be normal to that plane. This vector points away the origin; so, we need merely divide by its magnitude to get the desired normal. We find $\hat{\mathbf{n}} = \mathbf{A}/A = \hat{\mathbf{x}}(2/\sqrt{29}) + \hat{\mathbf{y}}(3/\sqrt{29}) + \hat{\mathbf{z}}(4/\sqrt{29}) \simeq \hat{\mathbf{x}}0.371 + \hat{\mathbf{y}}0.557 + \hat{\mathbf{z}}0.743$.

4. a. Cylindrical coordinates: $r = \sqrt{1^2+2^2} = \sqrt{5}$, $\phi = \tan^{-1}(2/1) \simeq 1.11$ rads $= 63.4°$, $z = 0$. Spherical coordinates: $R = \sqrt{1^2+2^2+0^2} = \sqrt{5}$, $\theta = \pi/2 = 90°$, and $\phi = 63.4°$.

   b. Cylindrical coordinates: $r = 0$, $\phi = 0$ (Actually, ϕ is undefined when the radius is zero, but it is conventional to set it equal to zero.), $z = 3$. Spherical coordinates: $R = |z| = 3$, since $x = y = 0$, $\theta = 0$, corresponding to a vector along the $+z$-direction, and we set $\phi = 0$ like before.

c. Cylindrical coordinates: $r = \sqrt{1^2 + 1^2} = \sqrt{2}$, $\phi = \tan^{-1}(1/1) = \pi/4 = 45°$, $z = 2$. Spherical coordinates: $R = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6}$, $\theta = \tan^{-1}(\sqrt{2}/2) \simeq 0.615$ rads $= 35.3°$, $\phi = \pi/4$.

d. Cylindrical coordinates: $r = \sqrt{(-3)^2 + 3^2} = \sqrt{18}$, $\phi = \tan^{-1}(-1) = 3\pi/4 = 135°$, where we resolve the ambiguity in the angle by noting that x is negative and y is positive so that the angle is in the second quadrant, $z = -3$. Spherical coordinates: $R = \sqrt{18 + (-3)^2} = \sqrt{27}$, $\theta = \tan^{-1}[\sqrt{18}/(-3)] = \tan^{-1}(-\sqrt{2}) = 2.186$ rads $= 125.3°$, $\phi = 3\pi/4$.

5 a. We have

$$S = \int_{z=-2}^{2} \int_{\phi=0}^{\pi/3} (r\, d\phi)(dz) = z\big|_{-2}^{2} \int_{0}^{\pi/3} (3 d\phi) = 4\pi.$$

b. We have

$$S = \int_{\phi=\pi/2}^{\pi} \int_{r=2}^{5} dr\, (r\, d\phi) = \frac{\pi}{2} \int_{2}^{5} r\, dr = \frac{\pi}{2} \cdot \frac{r^2}{2}\bigg|_{2}^{5} = \frac{21\pi}{4}.$$

c. We have

$$S = \int_{z=-2}^{2} \int_{r=2}^{5} dr\, dz = 4 \cdot 3 = 12.$$

d. We have

$$S = \int_{\phi=0}^{\pi} \int_{\theta=0}^{\pi/3} (R\, d\theta)(R \sin\theta\, d\phi) = 2^2\ \phi\big|_0^{\pi} \int_0^{\pi/3} \sin\theta\, d\theta = 4\pi\ (-\cos\theta)\big|_0^{\pi/3}$$
$$= 2\pi.$$

e. We have

$$S = \int_{\phi=0}^{2\pi} \int_{R=0}^{5} dR\, (R \sin\theta\, d\phi) = 2\pi \sin(\pi/3) \int_0^5 R\, dR$$
$$= \pi\sqrt{3}\ \frac{R^2}{2}\bigg|_0^5 = \frac{25\pi\sqrt{3}}{2}.$$

6. a. We have

$$d = \left[(-2-1)^2 + (-3-2)^2 + (2-3)^2\right]^{1/2} = \sqrt{35}.$$

b. In this case, where $\phi_1 = \phi_2$, we have $d = \sqrt{(r_2 - r_1)^2 + (z_2 - z_1)^2} = \sqrt{2^2 + 2^2} = \sqrt{8}$.

c. Because $\phi_1 = \phi_2$, we have $d = \sqrt{R_1^2 + R_2^2 + 2R_1R_2\cos(\theta_2 - \theta_1)}$. Noting that $\cos(\theta_2 - \theta_1) = \cos(\pi/2) = 0$, we have simply $d = \sqrt{R_1^2 + R_2^2} = \sqrt{13}$.

7. a. We have

$$\nabla T = \hat{\mathbf{x}}\frac{\partial T}{\partial x} + \hat{\mathbf{z}}\frac{\partial T}{\partial z} = -4\frac{\hat{\mathbf{x}}x + \hat{\mathbf{z}}z}{(x^2+z^2)^2}.$$

b. Calculating the partial derivatives, we have $\nabla V = \hat{\mathbf{x}}y^2z^3 + \hat{\mathbf{y}}2xyz^3 + \hat{\mathbf{z}}3xy^2z^2$.

c. We have

$$\nabla U = \hat{\mathbf{r}}\frac{\partial U}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial U}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial U}{\partial z} = \frac{1}{(1+r^2)}\left[-\hat{\mathbf{r}}\frac{2rz\cos\phi}{(1+r^2)} - \hat{\boldsymbol{\phi}}\frac{z\sin\phi}{r} + \hat{\mathbf{z}}\cos\phi\right].$$

d. Using the gradient operator in the form

$$\nabla W = \hat{\mathbf{R}}\frac{\partial W}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial W}{\partial \theta},$$

we have $\nabla W = -\hat{\mathbf{R}}\exp(-R)\sin\theta + \hat{\boldsymbol{\theta}}\frac{1}{R}\exp(-R)\cos\theta$.

e. Using the partial derivatives, we have $\nabla = \hat{\mathbf{x}}2x\exp(-z) + \hat{\mathbf{y}}2y - \hat{\mathbf{z}}x^2\exp(-z)$.

f. Here we have $\nabla N = \hat{\mathbf{r}}2r\cos\phi - \hat{\boldsymbol{\phi}}r\sin\phi$.

g. Here, we must use the full gradient operator in spherical coordinates, to obtain

$$\begin{aligned}\nabla M &= \hat{\mathbf{R}}\frac{\partial M}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial M}{\partial \theta} + \frac{1}{R\sin\theta}\frac{\partial M}{\partial \phi} \\ &= \hat{\mathbf{r}}\cos\theta\sin\phi - \hat{\boldsymbol{\theta}}\sin\theta\sin\phi + \hat{\boldsymbol{\phi}}\cot\theta\cos\phi.\end{aligned}$$

8. Since the vector points along $\hat{\mathbf{R}}$, we must $\mathbf{E} = \hat{\mathbf{R}}E_R(R)$. Note that there is no dependence on θ and ϕ in this spherically symmetric problem. It follows that the divergence becomes, using the formula for spherical coordinates,

$$6 = \nabla\cdot\mathbf{E} = \frac{1}{R^2}\frac{d}{dR}(R^2E_R),$$

where I do not use partial derivatives because E_R only depends on R. Integrating this equation, we obtain $(6R^3/3) + C = R^2E_R$, where C is a constant that remains to be determined. We conclude that $E_R = 2R + C/R^2$. From the condition that $E_R = 0$ at $R = 0$, we finally conclude that $E_R(R) = 2R$ and $\mathbf{E}(R) = \hat{\mathbf{R}}2R$.

9. a. We have

$$\oint_S \mathbf{D}\cdot d\mathbf{s} = \int_{z=0}^{5}\int_{\phi=0}^{2\pi} 2^3(2\,d\phi)dz - \int_{z=0}^{5}\int_{\phi=0}^{2\pi} 1^3(d\phi)dz,$$

where I note that when $r=2$, the surface normal is in the direction of $\hat{\mathbf{r}}$ and when $r=1$, it is the opposite direction, which accounts for the minus sign in the second surface integral. So, we have

$$\oint_S \mathbf{D}\cdot d\mathbf{s} = 15\int_{z=0}^{5}\int_{\phi=0}^{2\pi} d\phi\,dz = 150\pi.$$

b. In this cylindrically symmetric system, since $\mathbf{D}$ is only a function of r, we have

$$\nabla\cdot\mathbf{D} = \frac{1}{r}\frac{d}{dr}rD_r = \frac{1}{r}\frac{d}{dr}r^4 = 4r^2.$$

We now have the volume integral

$$\begin{aligned}\int_V \nabla\cdot\mathbf{D}\,dv &= \int_{z=0}^{5}\int_{\phi=0}^{2\pi}\int_{r=1}^{2}(4r^2)\,(dr)(r\,d\phi)(dz)\\ &= 10\pi\int_1^2 4r^3\,dr = 10\pi\; r^4\Big|_1^2 = 150\pi.\end{aligned}$$

This result agrees with the prediction of the divergence theorem.

10 a. This calculation is remarkably intricate. We first write

$$\left.\frac{\partial V}{\partial x}\right|_{y,z} = \left.\frac{\partial V}{\partial R}\right|_{\theta,\phi}\left.\frac{\partial R}{\partial x}\right|_{y,z} + \left.\frac{\partial V}{\partial \theta}\right|_{R,\phi}\left.\frac{\partial \theta}{\partial x}\right|_{y,z} + \left.\frac{\partial V}{\partial \phi}\right|_{R,\theta}\left.\frac{\partial \phi}{\partial x}\right|_{y,z}, \qquad (10.1)$$

where we have used the notation $\partial X/\partial u|_{v,w}$ to indicate explicitly that v and w are being held constant when the derivative of X is taken with respect to u. It is important, as stated in class, to properly track which quantities are being held constant when taking derivatives. From Table 3-2 in Ulaby, we find $R=(x^2+y^2+z^2)^{1/2}$, $\theta=\tan^{-1}[(x^2+y^2)^{1/2}/z]$, and $\phi=\tan^{-1}(y/x)$, as well as the inverse expressions, $x=R\sin\theta\cos\phi$, $y=R\sin\theta\sin\phi$, and $z=R\cos\theta$. Using these expressions, we have

$$\begin{aligned}\left.\frac{\partial R}{\partial x}\right|_{y,z} &= \frac{x}{(x^2+y^2+z^2)^{1/2}} = \sin\theta\cos\phi,\\ \left.\frac{\partial \theta}{\partial x}\right|_{y,z} &= \frac{x}{(x^2+y^2)^{1/2}}\frac{z}{x^2+y^2+z^2} = \frac{1}{R}\cos\theta\cos\phi,\\ \left.\frac{\partial \phi}{\partial x}\right|_{y,z} &= -\frac{y}{x^2+y^2} = -\frac{1}{R\sin\theta}\sin\phi.\end{aligned} \qquad (10.2)$$

Using the expression from Ulaby's Table 3-2, $\hat{\mathbf{x}} = \hat{\mathbf{R}} \sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi - \hat{\boldsymbol{\phi}}\sin\phi$, along with (10.1) and (10.2), we find

$$\begin{aligned}\hat{\mathbf{x}}\frac{\partial V}{\partial x} &= \left(\hat{\mathbf{R}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi\right) \\ &\quad\times\left(\sin\theta\cos\phi\frac{\partial V}{\partial R} + \cos\theta\cos\phi\frac{1}{R}\frac{\partial V}{\partial\theta} - \sin\phi\frac{1}{R\sin\theta}\frac{\partial V}{\partial\phi}\right),\end{aligned} \tag{10.3}$$

where we have suppressed the explicit writing of the variables that are being held constant. In a similar fashion, we find

$$\begin{aligned}\hat{\mathbf{y}}\frac{\partial V}{\partial y} &= \left(\hat{\mathbf{R}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi\right) \\ &\quad\times\left(\sin\theta\sin\phi\frac{\partial V}{\partial R} + \cos\theta\sin\phi\frac{1}{R}\frac{\partial V}{\partial\theta} + \cos\phi\frac{1}{R\sin\theta}\frac{\partial V}{\partial\phi}\right)\end{aligned} \tag{10.4}$$

and

$$\hat{\mathbf{z}}\frac{\partial V}{\partial z} = \left(\hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta\right)\left(\cos\theta\frac{\partial V}{\partial R} - \sin\theta\frac{1}{R}\frac{\partial V}{\partial\theta}\right). \tag{10.5}$$

Combining (10.3)–(10.5) together, we obtain for the $\hat{\mathbf{R}}$ component of ∇V

$$\begin{aligned}\hat{\mathbf{R}}\cdot\nabla V &= \Bigg[\left(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta\right)\frac{\partial V}{\partial R} \\ &+ \left(\sin\theta\cos\theta\cos^2\phi + \sin\theta\cos\theta\sin^2\phi - \sin\theta\cos\theta\right)\frac{1}{R}\frac{\partial V}{\partial\theta} \\ &+ \left(-\sin\theta\sin\phi\cos\phi + \sin\theta\sin\phi\cos\phi + 0\right)\frac{1}{R\sin\theta}\frac{\partial V}{\partial\phi}\Bigg] = \frac{\partial V}{\partial R}.\end{aligned} \tag{10.6}$$

We similarly find

$$\begin{aligned}\hat{\boldsymbol{\theta}}\cdot\nabla V &= \Bigg[\left(\sin\theta\cos\theta\cos^2\phi + \sin\theta\cos\theta\sin^2\phi - \sin\theta\cos\theta\right)\frac{\partial V}{\partial R} \\ &+ \left(\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta\right)\frac{1}{R}\frac{\partial V}{\partial\theta} \\ &+ \left(-\cos\theta\sin\phi\cos\phi + \cos\theta\sin\phi\cos\phi + 0\right)\frac{1}{R\sin\theta}\frac{\partial V}{\partial\phi}\Bigg] = \frac{1}{R}\frac{\partial V}{\partial\theta}\end{aligned} \tag{10.7}$$

and

$$\begin{aligned}\hat{\boldsymbol{\phi}}\cdot\nabla V &= \Bigg[\left(-\sin\theta\sin\phi\cos\phi + \sin\theta\sin\phi\cos\phi + 0\right)\frac{\partial V}{\partial R} \\ &+ \left(-\cos\theta\sin\phi\cos\phi + \cos\theta\sin\phi\cos\phi + 0\right)\frac{1}{R}\frac{\partial V}{\partial\theta} \\ &+ \left(\sin^2\phi + \cos^2\phi + 0\right)\frac{1}{R\sin\theta}\frac{\partial V}{\partial\phi}\Bigg] = \frac{1}{R\sin\theta}\frac{\partial V}{\partial\phi}.\end{aligned} \tag{10.8}$$

From (10.6)–(10.8), we obtain the expression for the gradient in spherical coordinates.

b. By contrast, the derivation using the general result could not be simpler. Using $x_1 = R$, $h_1 = 1$, $x_2 = \theta$, $h_2 = R$, $x_3 = \phi$, and $h_3 = R\sin\theta$, we substitute into the expression on slide 7.22 and immediately obtain the final result.

It is clearly much simpler to use the general expression. You can imagine how much more complex this analysis becomes when we are trying to calculate the divergence or the curl instead of the gradient. It is actually much simpler to derive the general formula and use it.