Maxwell's Equations: Differential and Integral Forms

Name of Law	Differential Form	Integral Form
Gauss's Law	$\nabla \cdot \mathbf{D} = \rho_V$	$\oint_S \mathbf{D} \cdot d\mathbf{s} = \int_v \rho_V \, dv = Q$
Faraday's Law	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$
Gauss's Law of Magnetics	$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$
Ampere's Law	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{s}$

This version of the integral forms is the most useful for implementation of numerical methods

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12.1

Maxwell's Equations: Alternative Integral Forms

Faraday's Law:

$$\oint_C \mathbf{E}_{\text{EMF}} \cdot d\mathbf{l} = \oint_C \left(\mathbf{E} + \mathbf{u} \times \mathbf{B} \right) \cdot d\mathbf{l} = \frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{s} = \frac{d\Phi}{dt}$$

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} = \text{magnetic flux}$$

Ampere's Law:

$$\oint_C \mathbf{H} \cdot d\mathbf{l} = I = I_c + I_d$$

I_c = conduction current; I_d = displacement current

This version of the integral forms

- ***has historical and conceptual importance***
- ***is the basis for understanding motors and generators***

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12.2

Faraday's Law

Electromotive Force:

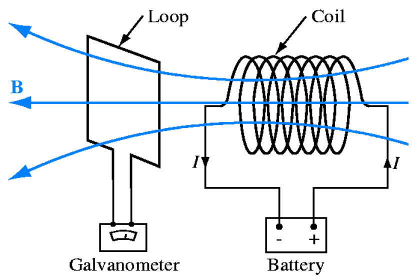
When a changing magnetic flux passes through a wire loop, it induces a loop voltage V_{EMF}, which is called ***the electromotive force***.

There are two possible sources of change:

- The magnetic flux density varies in time, leading to a transformer EMF, $V_{\text{EMF}}^{\text{tr}}$
- The loop area normal to the flux density varies in time, leading to a motional EMF, $V_{\text{EMF}}^{\text{m}}$

We have

$$V_{\text{EMF}} = V_{\text{EMF}}^{\text{tr}} + V_{\text{EMF}}^{\text{m}}$$



Ulaby Figure 6-1

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12.3

Faraday's Law

Electromotive Force:

When the loop has N turns, the effect of the induced EMF is multiplied N times, so we have:

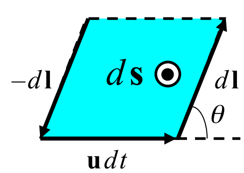
$$V_{\text{EMF}} = -N\frac{d\Phi}{dt} = -N\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{s}$$

$$V_{\text{EMF}}^{\text{tr}} = -N\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{s}$$

$$V_{\text{EMF}}^{\text{m}} = -N\int_S \mathbf{B}\cdot\frac{d\mathbf{s}}{dt} = -N\oint_C \mathbf{B}\cdot(\mathbf{u}\times d\mathbf{l}) = N\oint_C (\mathbf{u}\times\mathbf{B})\cdot d\mathbf{l}$$

$$|d\mathbf{s}| = |d\mathbf{l}||\mathbf{u}\,dt|\sin\theta$$

$$\hat{\mathbf{n}} = \frac{(\mathbf{u}\times d\mathbf{l})}{|\mathbf{u}\times d\mathbf{l}|}, \quad d\mathbf{s} = \hat{\mathbf{n}}\,|d\mathbf{s}|$$



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12.4

Faraday’s Law

Lenz’s Law:

The current in a loop is always in such a direction as to oppose the change of the magnetic flux $\Phi(t)$

This law allows us to rapidly determine the direction of the current that is induced by an EMF

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12.5

Tech Brief 12: EMF Sensors

EMF Sensors

Generate an induced voltage in response to an external stimulus

Piezoelectric transducers

Certain crystals, such as quartz, become electrically polarized when subjected to mechanical pressure, thereby exhibiting a voltage difference. Under no applied pressure, the polar domains are randomly oriented, but under compressive or tensile stress, the domains align along a principal axis of the crystal.

Compression and stretching generate opposite voltages.

Discovered in 1880 by Curie brothers. In 1881, Lippman predicted converse property (that electrical stimulus would change the shape of the crystal).

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12.6

Tech Brief 12: EMF Sensors

Piezoelectric transducers

Used in microphones, loudspeakers, positioning sensors for scanning tunneling microscopes (can measure deformations as small as nanometers), accelerometers (can measure from 10^{-4}g to 100g), spark generators, clocks and electronic circuits (precision oscillators), medical ultrasound transducers, and numerous other applications

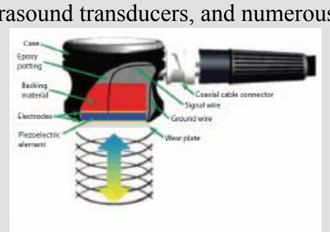
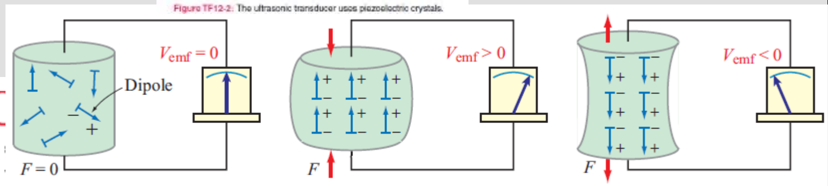


Figure TF12-2: The ultrasonic transducer uses piezoelectric crystals.



(a) No force (b) Compressed crystal (c) Stretched crystal

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Tech Brief 12: EMF Sensors

Faraday Magnetic Flux Sensor

According to Faraday's law, the emf voltage induced across the terminals of a conducting loop is directly proportional to the rate of change of magnetic flux passing through the loop. In the configuration below, Vemf and its derivative directly indicate the velocity and acceleration of the loop.

$$V_{\text{EMF}} = -uB_0 l$$

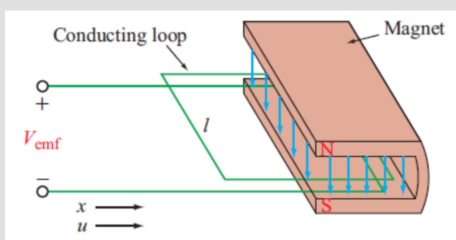

Figure TF12-3: In a Faraday accelerometer, the induced emf is directly proportional to the velocity of the loop (into and out of the magnet's cavity).

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Tech Brief 12: EMF Sensors

Thermocouple

Seebeck discovered in 1821 that a junction of two conducting materials will generate a thermally induced emf (called the Seebeck potential) when heated.

Becquerel in 1826 used this concept to measure an unknown temperature by relative to a cold reference junction. Traditionally, the cold reference is an ice bath, but in modern thermocouples, an electric circuit generates the reference potential.

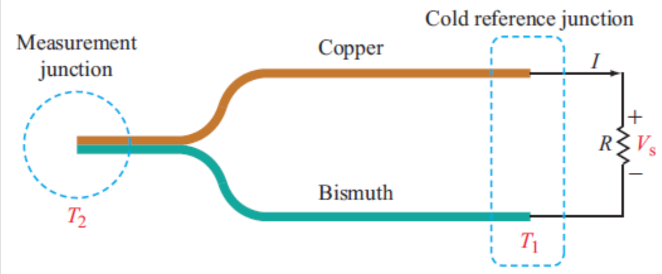

Figure TF12-4: Principle of the thermocouple.

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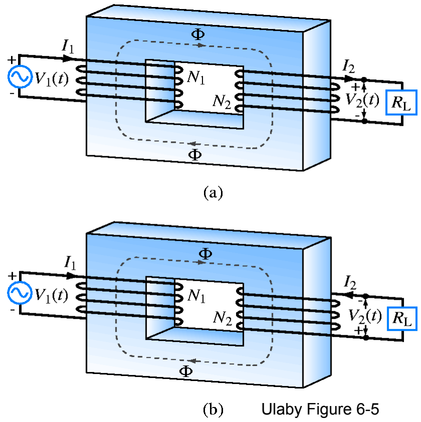
Faraday's Law

Transformers:

These are used to transform voltages, currents, and hence impedances.

In practice, they are made by winding current loops with different numbers of turns around a common magnetic core, which fixes the magnetic flux Φ.

NOTE: The direction of the winding determines the polarity of the output.



Ulaby Figure 6-5

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Faraday’s Law

Transformers:

Voltage Transformation:

In the primary: The oscillating voltage creates an oscillating flux, found by integrating

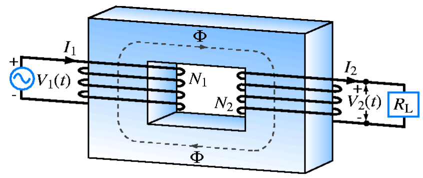
$$V_1 = -N_1 \frac{d\Phi}{dt}$$

In the secondary:

$$V_2 = -N_2 \frac{d\Phi}{dt}$$

We conclude

$$\frac{V_1}{V_2} = \frac{N_1}{N_2}$$



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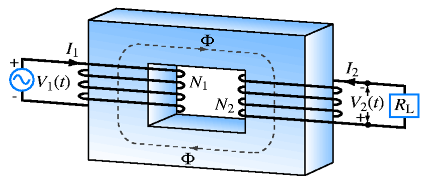


(b) Ulaby Figure 6-5

Faraday’s Law

Motors and Generators:

Current Transformation:

In an ideal transformer, the power is conserved, so we have $P_1 = P_2$. Since $P_1 = I_1 V_1$ and $P_2 = I_2 V_2$, we have

$$\frac{I_1}{I_2} = \frac{V_2}{V_1} = \frac{N_2}{N_1}$$

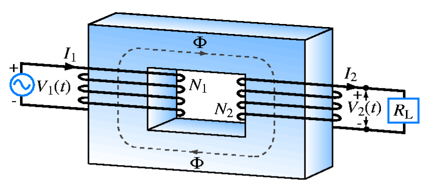
Impedance Transformation:

Ouput resistance, $R_L = V_2 / I_2$

Equivalent input resistance, $R_{in} = V_1 / I_1$

We conclude

$$Z_{in} = \left(\frac{N_1}{N_2}\right)^2 Z_L$$



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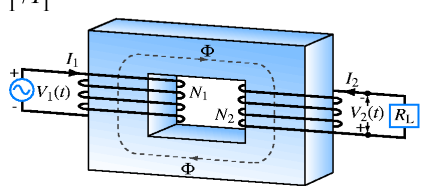


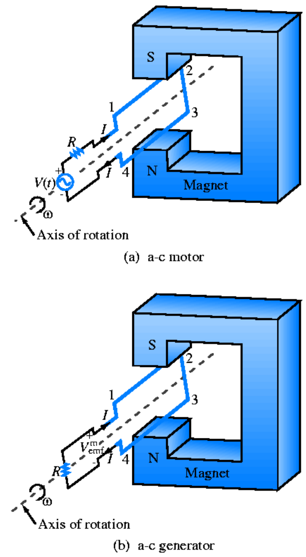
(b) Ulaby Figure 6-5

Magnetic Forces

Motors and Generators

We previously discussed motors and generators as an application of magnetostatics. That is possible because the magnetic field is fixed and the currents are moving, so that V_{EMF} is purely motional.

We can instead directly use Faraday's law!



Ulaby Figure 6-11 12.13

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Magnetic Forces

Motors and Generators

We can instead directly use Faraday's law!

We first calculate the magnetic flux

$$\Phi = \int_S \mathbf{B}\cdot d\mathbf{s} = \int_S \hat{\mathbf{z}} B_0 \cdot \hat{\mathbf{n}}\, ds$$

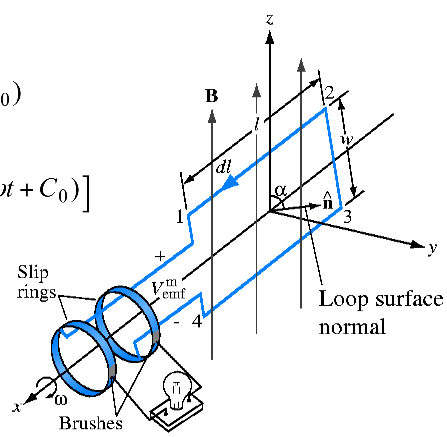
$$= B_0 A \cos\alpha = B_0 A \cos(\omega t + C_0)$$

We thus find

$$V_{\mathrm{EMF}} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\left[B_0 A \cos(\omega t + C_0)\right]$$

$$= \omega B_0 A \sin(\omega t + C_0)$$

which is the same result that is obtained from the motional EMF



Ulaby Figure 6-12 12.14

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Displacement Current

… or Maxwell's great insight!

Original (static) version of Ampere's law: $\oint_C \mathbf{H}\cdot d\mathbf{l} = I_c = \int_S \mathbf{J}\cdot d\mathbf{s}$

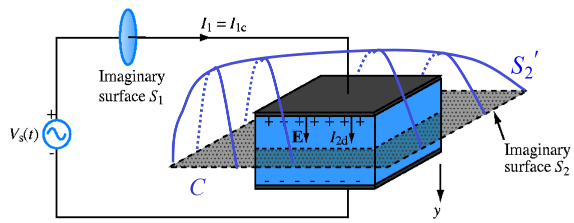
This version of Ampere's law is inconsistent! ***Why?***

Different surfaces attached to same closed curve C yield different results!

— Compare S_2 and S_2'

Maxwell assumed that adding $I_d = \int_S \frac{\partial \mathbf{D}}{\partial t}\cdot d\mathbf{s}$ to Ampere's law would fix things

…and all experimental evidence indicates that it has.



Modified from Ulaby Figure 6-12

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12.15

Displacement Current

Example: Current flow onto a parallel plate capacitor

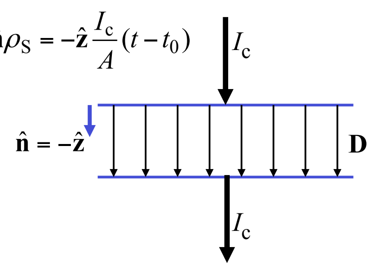
Question: During a time interval τ, a steady current I_c flows onto a parallel plate capacitor with area A, separation d, and dielectric constant ε, show that the displacement current I_d equals the conduction current.

Answer: The charge that accumulates on the upper plate of the capacitor is given by $Q = I_c(t - t_0)$, where t_0 is the initial time at which current starts to flow. Neglecting fringing fields, we have

$$\rho_S = \frac{Q}{A} = \frac{I_c}{A}(t - t_0), \text{ which implies } \mathbf{D} = \hat{\mathbf{n}}\rho_S = -\hat{\mathbf{z}}\frac{I_c}{A}(t - t_0)$$

We conclude

$$I_d = \int_S \frac{\partial \mathbf{D}}{\partial t}\cdot d\mathbf{s} = \left(-\hat{\mathbf{z}}\frac{I_c}{A}\right)\cdot\left(-\hat{\mathbf{z}}A\right) = I_c$$



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12.16

Charge-Current Continuity

Charge conservation

$$0 = \nabla \cdot (\nabla \times \mathbf{H}) = \nabla \cdot \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) = \nabla \cdot \mathbf{J} + \frac{\partial \rho_{\mathrm{V}}}{\partial t} \quad \Rightarrow \quad \nabla \cdot \mathbf{J} = -\frac{\partial \rho_{\mathrm{V}}}{\partial t}$$

Charge cannot be created or destroyed!

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12.17

Charge-Current Continuity

Current Dissipation in Conductors

In real conductors with finite conductivity, excess charge dissipates in a finite time. Using the relations $\mathbf{J} = \sigma \mathbf{E}$, $\nabla \cdot \mathbf{E} = \rho_{\mathrm{V}} / \varepsilon$, we obtain

$$\frac{\partial \rho_{\mathrm{V}}}{\partial t} + \frac{\sigma}{\varepsilon} \rho_{\mathrm{V}} = 0 \quad \Rightarrow \quad \rho_{\mathrm{V}}(t) = \rho_{\mathrm{V}0} \exp\left(-t / \tau_{\mathrm{r}}\right)$$

$$\tau_{\mathrm{r}} = \varepsilon / \sigma = \text{material relation time}$$

A good conductor

— copper: $\varepsilon = \varepsilon_0 = 8.9 \times 10^{-12}$ F/m, $\sigma = 5.8 \times 10^{8}$ S/m

$$\Rightarrow \quad \tau_{\mathrm{r}} = 1.5 \times 10^{-19} \text{ s}$$

A good insulator

— mica: $\varepsilon = 6 \varepsilon_0 = 5 \times 10^{-11}$ F/m, $\sigma = 10^{-15}$ S/m

$$\Rightarrow \quad \tau_{\mathrm{r}} = 5 \times 10^{4} \text{ s} = 15 \text{ hours}$$

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12.18

Electromagnetic Potentials

Dynamic Potentials

As before: $\nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \mathbf{B} = \nabla \times \mathbf{A}$

From Faraday's law:

$$0 = \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E} + \frac{\partial (\nabla \times \mathbf{A})}{\partial t} = \nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right)$$

Hence, we must have

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V \quad \Rightarrow \quad \mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$$

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12.19

Electromagnetic Potentials

Retarded Potentials

In the static case

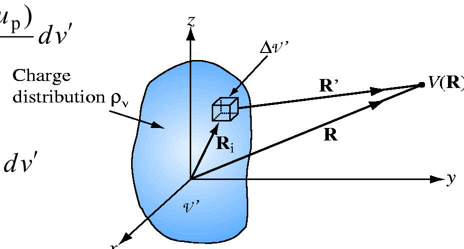
$$V(\mathbf{R}) = \frac{1}{4\pi\varepsilon} \int_{v'} \frac{\rho_V(\mathbf{R}_i)}{R'} dv', \quad R' = |\mathbf{R} - \mathbf{R}_i|$$

In the dynamic case, we must take into account the finite time delay:

$$V(\mathbf{R}, t) = \frac{1}{4\pi\varepsilon} \int_{v'} \frac{\rho_V(\mathbf{R}_i, t - R'/u_p)}{R'} dv'$$

For the vector potential, we have

$$\mathbf{A}(\mathbf{R}, t) = \frac{\mu}{4\pi} \int_{v'} \frac{\mathbf{J}(\mathbf{R}_i, t - R'/u_p)}{R'} dv'$$



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12.20

Phasor Fields

Time Harmonic Potentials

The building blocks for understanding dynamical systems are sinusoidally varying signals (phasors).

- In a linear system, *any* time variation can be found by adding up the phasors
- Maxwell's equations are linear, assuming linear material relations

We thus write $\rho_{\mathrm{V}}(\mathbf{R}_{\mathrm{i}},t) = \rho_{\mathrm{V}}(\mathbf{R}_{\mathrm{i}})\cos\omega t = \mathrm{Re}\left[\tilde{\rho}_{\mathrm{V}}(\mathbf{R}_{\mathrm{i}})\exp(j\omega t)\right]$

For the retarded charge density, we have

$$\rho_{\mathrm{V}}(\mathbf{R}_{\mathrm{i}},t-R'/u_{\mathrm{p}}) = \mathrm{Re}\left[\tilde{\rho}_{\mathrm{V}}(\mathbf{R}_{\mathrm{i}})\exp\left(j\omega t - j\frac{\omega R'}{u_{\mathrm{p}}}\right)\right]$$

$$= \mathrm{Re}\left[\tilde{\rho}_{\mathrm{V}}(\mathbf{R}_{\mathrm{i}})\exp\left(-jkR'\right)\exp(j\omega t)\right], \quad \text{with } k = \omega / u_{\mathrm{p}}$$

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NOTE: In this section of his book, Ulaby et al. use k and the designation *wavenumber*, instead of β and the designation *phase constant*.

12.21

Phasor Fields

Time Harmonic Potentials

For the voltage field, we have

$$V(\mathbf{R},t) = \mathrm{Re}\left[\tilde{V}(\mathbf{R})\exp(j\omega t)\right]$$

$$= \mathrm{Re}\left[\frac{1}{4\pi\varepsilon}\int_{v'}\frac{\tilde{\rho}_{\mathrm{V}}(\mathbf{R}_{\mathrm{i}})\exp\left(-jkR'\right)}{R'}\exp(j\omega t)\,dv'\right]$$

which implies

$$\tilde{V}(\mathbf{R}) = \frac{1}{4\pi\varepsilon}\int_{v'}\frac{\tilde{\rho}_{\mathrm{V}}(\mathbf{R}_{\mathrm{i}})\exp\left(-jkR'\right)}{R'}\,dv'$$

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12.22

Phasor Fields

Time Harmonic Potentials

For the vector potential and current density fields, we have similarly

$$\mathbf{A}(\mathbf{R},t) = \mathrm{Re}\left[\tilde{\mathbf{A}}(\mathbf{R})\exp(j\omega t)\right],$$

$$\mathbf{J}(\mathbf{R}_\mathrm{i}, t - R'/u_\mathrm{p}) = \mathrm{Re}\left[\tilde{\mathbf{J}}(\mathbf{R}_\mathrm{i})\exp(-jkR')\exp(j\omega t)\right]$$

which imply

$$\tilde{\mathbf{A}}(\mathbf{R}_\mathrm{i}) = \frac{\mu}{4\pi}\int_{v'} \frac{\tilde{\mathbf{J}}(\mathbf{R}_\mathrm{i})\exp(-jkR')}{R'}\,dv'$$

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12.23

Phasor Fields

Electric and Magnetic Fields

— in a non-conducting medium ($\mathbf{J} = 0$)

From the definition of the vector potential, we have

$$\mathbf{H} = \frac{1}{\mu}\nabla\times\mathbf{A} \quad \Rightarrow \quad \tilde{\mathbf{H}} = \frac{1}{\mu}\nabla\times\tilde{\mathbf{A}}$$

From Ampere's law, we have

$$\nabla\times\mathbf{H} = \varepsilon\frac{\partial\mathbf{E}}{\partial t} \quad \Rightarrow \quad \nabla\times\tilde{\mathbf{H}} = j\omega\varepsilon\tilde{\mathbf{E}} \;\text{ or }\; \tilde{\mathbf{E}} = \frac{1}{j\omega\varepsilon}\nabla\times\tilde{\mathbf{H}}$$

From Faraday's law, we have

$$\nabla\times\mathbf{E} = -\mu\frac{\partial\mathbf{H}}{\partial t} \quad \Rightarrow \quad \nabla\times\tilde{\mathbf{E}} = -j\omega\mu\tilde{\mathbf{H}} \;\text{ or }\; \tilde{\mathbf{H}} = -\frac{1}{j\omega\mu}\nabla\times\tilde{\mathbf{E}}$$

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12.24

Phasor Fields

Fields and Dispersion Relations: Ulaby and Ravaioli Example 6-8

Question: In a nonconducting medium with $\varepsilon = 16\,\varepsilon_0$ and $\mu = \mu_0$, the electric field intensity is given by

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}10\sin\left(10^{10}t - kz\right)\ \text{V/m}.$$

Determine the associated magnetic field intensity **H** and the value of *k*.

Answer: We have

$$\mathbf{E}(z,t) = \hat{\mathbf{x}}10\cos\left(\omega t - kz - \pi/2\right)\ \text{V/m, with } \omega = 10^{10}\ \text{s}^{-1}$$

so that $\tilde{\mathbf{E}}(z) = \hat{\mathbf{x}}10\exp[-j(kz+\pi/2)] = -\hat{\mathbf{x}}\,j10\exp(-jkz)$

From Faraday's law: $\tilde{\mathbf{H}} = -\dfrac{1}{j\omega\mu}\nabla\times\tilde{\mathbf{E}} = -\hat{\mathbf{y}}\,j\dfrac{10k}{\omega\mu}\exp(-jkz)$

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From Ampere's law: $\tilde{\mathbf{E}} = \dfrac{1}{j\omega\varepsilon}\nabla\times\tilde{\mathbf{H}} = -\hat{\mathbf{x}}\,j\dfrac{10k^2}{\omega^2\mu\varepsilon}\exp(-jkz)$

12.25

Phasor Fields

Fields and Dispersion Relations: Ulaby Example 6-8

Answer (continued): Equating the two expressions for $\tilde{\mathbf{E}}(z)$, we obtain the dispersion relation

$$k^2 = \omega^2\mu\varepsilon$$

This is the same dispersion relation that we found with transmission lines!
This is no coincidence!!

Explicitly, we have

$$k = \omega\sqrt{\mu\varepsilon} = 4\omega\sqrt{\mu_0\varepsilon_0} = \frac{4\omega}{c} = \frac{4\times10^{10}}{3\times10^{8}} = 130\ \text{rad/m.}$$

We now find for the magnetic field

$$\mathbf{H}(z,t) = \text{Re}\left[\tilde{\mathbf{H}}(z)\exp(j\omega t)\right]$$

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$$= \text{Re}\left[-\hat{\mathbf{y}}\,j\frac{10k}{\omega\mu}\exp(-jkz)\exp(j\omega t)\right] = \hat{\mathbf{y}}\,0.11\sin\left(10^{10}t - 130z\right)\ \text{A/m}$$

12.26

Assignment

Reading: Ulaby, Chapter 7

Problem Set 7: Some notes.

- There are 7 problems. As always, YOU MUST SHOW YOUR WORK TO GET FULL CREDIT!
- All problems come from Ulaby et al.
- Please watch significant digits.
- Get started early!

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12.27