Midterm Examination Solutions

1. a. $f = 5$ GHz $= 5 \times 10^9$ s^{-1}; $\omega = 2\pi \times f = 3.14 \times 10^{10}$ s^{-1}; $u_\mathrm{p} = c/\sqrt{\epsilon_\mathrm{r}} = [3 \times 10^8$ m/s$]/1.5 = 2.0\times10^8$ m/s; $\lambda = u_\mathrm{p}/f = (2.0\times10^8)/(5\times10^9) = 4.0\times10^{-2}$ m $= 4$ cm; $\beta = 2\pi/\lambda = (6.28/0.04)$ m^{-1} $= 157$ m^{-1}.

   b. Since the decrease is exponential, and the amplitude decreases by a factor of 125 in 300 m, it must decrease by a factor of 5 every 100 m. So, the amplitude is 25 V after 100 m and 5 V after 200 m.

2. Since the circuit consists of a single loop, Kirchhoff's current simply states that the current is equal everywhere, so that $\tilde{I}_\mathrm{R} = \tilde{I}_\mathrm{C} \equiv \tilde{I}$, where the first two currents are respectively the currents flowing through the resistor and the capacitor. From Kirchhoff's voltage law, we then find

$$\tilde{V}_\mathrm{s} = R\tilde{I} + \frac{\tilde{I}}{j\omega C} = R\tilde{I} + \frac{\tilde{I}}{j2\pi f C}.$$

It follows that

$$\tilde{I} = \frac{j2\pi f C}{1 + j2\pi f RC}\tilde{V}_\mathrm{s},$$

so that we obtain for the resistor and capacitor voltage phasors, $\tilde{V}_\mathrm{R}$ and $\tilde{V}_\mathrm{C}$,

$$\tilde{V}_\mathrm{R} = \frac{j2\pi f RC}{1 + j2\pi f RC}\tilde{V}_\mathrm{s}, \qquad \tilde{V}_\mathrm{C} = \frac{1}{1 + j2\pi f RC}\tilde{V}_\mathrm{s}.$$

An ideal capacitor does not dissipate power. So, all the power must be dissipated in the resistor. We can show that mathematically by noting that $\tilde{V}_\mathrm{C}\tilde{I}^*$ is purely imaginary, so that its real part is zero. By contrast, $\tilde{V}_\mathrm{R}\tilde{I}^*$ is purely real.

3 From Kirchoff's voltage law, we find

$$v(z,t) - R'\Delta z i(z,t) - L'\Delta z \frac{\partial i(z,t)}{\partial t} - v(z + \Delta z, t) = 0.$$

Collecting terms and dividing by Δz, we obtain

$$-\left[\frac{v(z + \Delta z, t) - v(z)}{\Delta z}\right] = R' i(z,t) + L' \frac{\partial i(z,t)}{\partial t}.$$

Allow $\Delta z \to 0$, we obtain

$$-\frac{\partial v(z,t)}{\partial z} = R' i(z,t) + L' \frac{\partial i(z,t)}{\partial t}.$$

From Kirchoff's current low, we find

$$i(z,t) - G'\Delta z v(z+\Delta z,t) - C'\Delta z \frac{\partial v(z+\Delta z,t)}{\partial t} - i(z+\Delta z,t) = 0.$$

Collecting terms and dividing by Δz, we obtain

$$-\left[\frac{i(z+\Delta z,t) - i(z,t)}{\Delta z}\right] = G'\Delta z v(z+\Delta z,t) + C'\frac{\partial v(z+\Delta z,t)}{\partial t}.$$

Letting $\Delta z \to 0$, we obtain

$$-\frac{\partial i(z,t)}{\partial z} = G'v(z,t) + C'\frac{\partial v(z,t)}{\partial t}.$$

We may obtain the phasor domain representation by substituting $v(z,t) = \tilde{V}(z)$ $\exp(j\omega t)$ and $i(z,t) = \tilde{I}(z)\exp(j\omega t)$ into the telegrapher's equations. Doing so and dividing by $\exp(j\omega t)$, we obtain

$$-\frac{d\tilde{V}(z)}{dz} = (R' + j\omega L')\tilde{I}(z), \qquad -\frac{d\tilde{I}(z)}{dz} = (G' + j\omega C')\tilde{V}(z).$$

4. We have $Z_0 = \sqrt{L'/C'}$ and $u_{\mathrm{p}} = 1/\sqrt{L'C'}$, from which we find $L' = Z_0 u_p = 100/(2.0\times 10^8) = 5.0\times 10^{-7}$ H $= 500$ nH. We have $C' = L'/Z_0^2 = 5.0\times 10^{-11}$ F $= 50$ pF. From the expressions for L' and C', we have

$$\frac{w}{h} = \frac{\sqrt{\mu/\epsilon}}{\sqrt{L'/C'}} = \frac{Z_{\mathrm{mat}}}{Z_0} = \frac{u_{\mathrm{p}}}{c}\frac{Z_{\mathrm{vac}}}{Z_0},$$

where Z_{mat} is the material impedance and Z_{vac} is the vacuum impedance. We thus have $w/h = (2/3)(377/100) = 754/300 \simeq 750/300 = 2.5$.

5. Since the generator and transmission line impedances are matched, there will be no reflections from the generator, and the voltage and current in the transmission line will reach steady state after one round trip (2 μs). We have

$$V^+ = \frac{Z_0}{Z_0 + Z_L}V_{\mathrm{g}} = \frac{50}{100}\times 100 = 50 \text{ V}.$$

The corresponding current is given by $I_0^+ = V_0^+/Z_0 = 1$ A. For the first two microseconds, we have $V^- = 0$ at the generator. At $t = 1$ μs, the voltage transient reaches the load. The reflection coefficient is given by $\Gamma_{\mathrm{L}} = (100-50)/(100+50) = 1/3$. We thus find that $V^- = (1/3)V_+ = 17$ V after 1 ns. We similarly, find $I^- = -V_-/Z_0 = -0.33$ A after 1 ns. The voltage at the load $V(l,t)$ is thus 67 V after 1 ns and the current at the load $I(l,t)$ is 0.67 A. Note that $V(l,t)/I(l,t) = 100\ \Omega$, as it should. After 2 ns, the backward-propagating voltage and current reach the generator, which is impedance matched. As a consequence, beyond 2 ns, the voltage at the generator is also given by 67 V and the current is given by 0.67 A. To summarize: All quantities are 0 for $t < 0$. We have $V(0,t) = 50$ V for $0 < t < 2$ ns, and we have $V(0,t) = 67$ V for $t > 2$ ns. We have $I(0,t) = 1$ A for $0 < t < 2$ ns, and we have $I(0,t) = 0.67$ A for $t > 2$ ns. We have $V(l,t) = 0$ for $t < 1$ ns, and we have $V(l,t) = 67$ V for $t > 1$ ns. Finally, we have $I(l,t) = 0$ for $t < 1$ ns, and we have $I(l,t) = 0.67$ A for $t > 1$ ns.