

- 3.7** 4 Let G be an abelian group, and let n be any positive integer. Show that the function $\phi : G \rightarrow G$ defined by $\phi(x) = x^n$ is a homomorphism.

Pf. Let $x, y \in G$

Then $\phi(xy) = (xy)^n$

Since G is abelian,

$$\begin{aligned}(xy)^n &= x^n y^n \\ &= \phi(x)\phi(y)\end{aligned}$$

Therefore, $\phi(xy) = \phi(x)\phi(y) \forall x, y \in G$

□

- 6 Define $\phi : \mathbb{C}^\times \rightarrow \mathbb{R}^\times$ by $\phi(a+bi) = a^2+b^2$, for all $a+bi \in \mathbb{C}^\times$. Show that ϕ is a homomorphism.

Pf. Let $a+bi, c+di \in \mathbb{C}^\times$

Then $\phi((a+bi)(c+di)) = (ac-bd)^2 + (ad+bc)^2$

Therefore,

$$\begin{aligned}\phi((a+bi)(c+di)) &= (ac-bd)^2 + (ad+bc)^2 \\ &= (ac-bd)^2 + (ad+bc)^2 \\ &= (ac)^2 - 2acbd + (bd)^2 + (ad)^2 + 2adbc + (bc)^2 \\ &= (ac)^2 + (bd)^2 + (ad)^2 + (bc)^2\end{aligned}$$

Also,

$$\begin{aligned}\phi(a+bi)\phi(c+di) &= (a^2+b^2)(c^2+d^2) \\ &= a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 \\ &= (ac)^2 + (bd)^2 + (ad)^2 + (bc)^2\end{aligned}$$

Therefore $\phi((a+bi)(c+di)) = \phi(a+bi)\phi(c+di)$

□

- 7 Which of the following functions are homomorphisms?

b $\phi : \mathbb{R} \rightarrow \text{GL}_2(\mathbb{R})$ defined by $\phi(a) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$

Pf. Consider,

$$\begin{aligned}\phi(a+b) &= \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \\ &= \phi(a)\phi(b)\end{aligned}$$

Since $\phi(a+b) = \phi(a)\phi(b)$, ϕ is a homomorphism □

d $\phi : \text{GL}_2(\mathbb{R}) \rightarrow \mathbb{R}^\times$ defined by $\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ab$

Pf. For ϕ to be homomorphic, the identity element of $\text{GL}_2(\mathbb{R}) = e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ must

be mapped to the identity element of $\mathbb{R}^\times = e_2 = 1$

But

$$\begin{aligned}\phi\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) &= 1 \times 0 \\ &= 0 \\ &\neq e_2 = 1\end{aligned}$$

Therefore, ϕ is not a homomorphism □

10 Let G be the group of affine functions from \mathbb{R} into \mathbb{R} , as defined in Exercise 10 of Section 3.1.

Define $\phi : G \rightarrow \mathbb{R}^\times$ as follows: for any function $f_{m,b} \in G$, let $\phi(f_{m,b}) = m$. Prove that ϕ is a group homomorphism, and find its kernel and image.

Pf. Given $G = \{f_{m,b} : \mathbb{R} \rightarrow \mathbb{R} \mid m \neq 0 \text{ and } f_{m,b}(x) = mx + b\}$

Consider,

$$\begin{aligned}(f_{n,a} \circ f_{m,b}) &= f_{n,a}(f_{m,b}) \\ &= f_{n,a}(mx + b) \\ &= n(mx + b) + a \\ &= nm x + bn + a \\ &= f_{nm, bn+a}\end{aligned}$$

Therefore,

$$\begin{aligned}\phi(f_{n,a} \circ f_{m,b}) &= \phi(f_{nm,bn+a}) \\ &= nm \\ &= \phi(f_{n,a})\phi(f_{m,b})\end{aligned}$$

Therefore, ϕ is a homomorphism

By definition,

$$\begin{aligned}\ker(\phi) &= \{f_{m,b} \in G \mid \phi(f_{m,b}) = 1\} \\ &= \{f_{m,b} \in G \mid m = 1\}\end{aligned}$$

By definition,

$$\begin{aligned}\text{img}(\phi) &= \{m \in \mathbb{R}^\times \mid \phi(f_{m,b}) = m\} \\ &= \mathbb{R}^\times\end{aligned}$$

□

- 14** Recall that the center of a group G is $\{x \in G \mid xg = gx \text{ for all } g \in G\}$. Prove that the center of any group is a normal subgroup.

Pf. Let H be a subgroup of G , and $x \in H$

Then $xg = gx, \forall g \in G$

Therefore,

$$\begin{aligned}gx &= xg \\ \implies gxg^{-1} &= xgg^{-1} \\ \implies gxg^{-1} &= x \in H\end{aligned}$$

Thus, $\forall x \in H, \forall g \in G, gxg^{-1} \in H$

That is, $\{x \in G \mid xh = gx \text{ for all } g \in G\}$ is a normal subgroup of G

□

- 18** Let the dihedral group D_n be given by elements a of order n and b of order 2, where $ba = a^{-1}b$. Show that any subgroup of $\langle a \rangle$ is normal in D_n .

Pf. Let H be a subgroup of $\langle a \rangle$, such that any elements of the form $(a^m)^d \in H$

Also, since $ba = a^{-1}b \in D_n$,

$$ba^2 = a^{-1}ba$$

$$= a^{-1}a^{-1}b$$

$$= a^{-2}b$$

Thus, $ba^i = a^{-i}b$ for any i

□

3.8 4 For each of the subgroups $\{e, a^2\}$ and $\{e, b\}$ of D_4 , list all left and right cosets.

Pf.

□

9 Let G be a finite group, and let n be a divisor of $|G|$. Show that if H is the only subgroup of G of order n , then H must be normal in G .

Pf. aHa^{-1} is the subgroup of G that is isomorphic to H

Therefore, $aHa^{-1} = H, \forall a \in G$

Which is true iff H is a normal subgroup in G

□

12 Let H and K be normal subgroups of G such that $H \cap K = \langle e \rangle$. Show that $hk = kh$ for all $h \in H$ and $k \in K$.

Pf. Consider $hkh^{-1}k^{-1}$

Since K is a normal subgroup,

$$\begin{aligned} hkh^{-1}k^{-1} &= (hkh^{-1})k^{-1} \\ &= hkh^{-1}, k^{-1} \in K \\ &= hkh^{-1}k^{-1} \in K \end{aligned}$$

Similarly, since H is a normal subgroup,

$$\begin{aligned} hkh^{-1}k^{-1} &= h(kh^{-1}k^{-1}) \\ &= h, kh^{-1}k^{-1} \in H \\ &= hkh^{-1}k^{-1} \in H \end{aligned}$$

Therefore, $hkh^{-1}k^{-1} \in K \cap H$

But, since $hkh^{-1}k^{-1} \in K \cap H = \{e\}$,

$$hkh^{-1}k^{-1} = e$$

$$\implies hk(h^{-1}k^{-1}) = e$$

$$\implies hk = kh$$

□

18 Compute the factor group $(\mathbb{Z}_6 \times \mathbb{Z}_4)/\langle(3, 2)\rangle$.

Pf. Given $\langle(3, 2)\rangle = \{(3, 2), (0, 0)\}$

$\mathbb{Z}_6 \times \mathbb{Z}_4$ has 24 elements, with 24/2 left cosets

The factor groups are,

$$(0, 0) + \langle(3, 2)\rangle = \{(3, 2), (0, 0)\}$$

$$(0, 1) + \langle(3, 2)\rangle = \{(3, 3), (0, 1)\}$$

$$(0, 2) + \langle(3, 2)\rangle = \{(3, 2), (0, 2)\}$$

$$(0, 3) + \langle(3, 2)\rangle = \{(3, 1), (0, 3)\}$$

$$(1, 0) + \langle(3, 2)\rangle = \{(4, 2), (1, 0)\}$$

$$(1, 1) + \langle(3, 2)\rangle = \{(4, 3), (1, 1)\}$$

$$(1, 2) + \langle(3, 2)\rangle = \{(4, 0), (1, 2)\}$$

$$(1, 3) + \langle(3, 2)\rangle = \{(4, 1), (1, 3)\}$$

$$(2, 0) + \langle(3, 2)\rangle = \{(5, 2), (2, 0)\}$$

$$(2, 1) + \langle(3, 2)\rangle = \{(5, 3), (2, 1)\}$$

$$(2, 2) + \langle(3, 2)\rangle = \{(5, 0), (2, 2)\}$$

$$(2, 3) + \langle(3, 2)\rangle = \{(5, 1), (2, 3)\}$$

□

19 Show that $(\mathbb{Z} \times \mathbb{Z})/\langle(0, 1)\rangle$ is an infinite cyclic group.

Pf. Consider $f : \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}/\langle(0, 1)\rangle$ given by $f(n) = (n, 0) + \langle(0, 1)\rangle$

Given, $\langle(0, 1)\rangle = \{(0, n) : n \in \mathbb{Z}\}$ To show that f is a homomorphism,

$$f(a + b) = (a + b, 0) + \langle(0, 1)\rangle$$

$$= (a, 0) + (b, 0) + \langle(0, 1)\rangle$$

$$= (a, 0) + \langle(0, 1)\rangle + (b, 0) + \langle(0, 1)\rangle$$

$$= f(a) + f(b)$$

To show that f is one-to-one,

$$f(a) = f(b)$$

$$\begin{aligned}
&\implies (a, 0) + \langle (0, 1) \rangle \\
&= (b, 0) + \langle (0, 1) \rangle \\
&= (a - b, 0) \in \langle (0, 1) \rangle \\
&\implies a - b = 0 \\
&\implies a = b
\end{aligned}$$

To show that f is onto, suppose $(x, y) + \langle (0, 1) \rangle$ is $\mathbb{Z} \times \mathbb{Z} / \langle (0, 1) \rangle$

Then,

$$f(x) = (x, 0) + \langle (0, 1) \rangle = (x, y) + \langle (0, 1) \rangle$$

Since f is an isomorphism, $\mathbb{Z} \times \mathbb{Z} / \langle (0, 1) \rangle$ is an infinite cyclic group □

23 Let G be the set of all matrices in $\text{GL}_2(\mathbb{Z}_5)$ of the form $\begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix}$

a. Show that G is a subgroup of $\text{GL}_2(\mathbb{Z}_5)$.

Pf. Given G is non-empty and finite with $\mathbb{Z}_5 = \{[0], [1], [2], [3], [4]\}$

For G to be a subgroup, it is enough to show $xy \in G, \forall x, y \in G$

$$\text{Let } x = \begin{bmatrix} n & a \\ 0 & 1 \end{bmatrix}, y = \begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix} \in G$$

Then,

$$\begin{aligned}
xy &= \begin{bmatrix} n & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} nm & nb + a \\ 0 & 1 \end{bmatrix} \\
&\in G
\end{aligned}$$
□

b. Show that the subset N of all matrices in G of the form $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$, with $c \in \mathbb{Z}_5$, is a normal subgroup of G .

Pf. Let $x, y \in G, xy \in N$, with matrices of the form $x = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}, y = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$

Then,

$$xy = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & c+d \\ 0 & 1 \end{bmatrix} \\ \in N$$

Therefore, $N \leq G$

$$\text{Let } a = \begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix} \in N, \text{ and } a^{-1} = \begin{bmatrix} m^{-1} & -m^{-1}b \\ 0 & 1 \end{bmatrix} \in N$$

Then,

$$\begin{aligned} ana^{-1} &= \begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m^{-1} & -m^{-1}b \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} m & mc+b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m^{-1} & -m^{-1}b \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & mc \\ 0 & 1 \end{bmatrix} \\ &\in N \end{aligned}$$

□

c. Show that the factor group G/N is cyclic of order 4.

Pf.

□