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CMPE 320 HW 04

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1. Given,

   probability of not losing the first game: $p_1 = 0.4$

   probability of losing the first game: $p_1^c = 1 - 0.4 = 0.6$

   probability of not losing the second game: $p_2 = 0.7$

   probability of losing the second game: $p_2^c = 1 - 0.7 = 0.3$

   Therefore, the p_X where $X = 0, 1, 2, 4$ represents the number of points earned over the weekend:

$$
\begin{aligned}
P(X=0) &= p_1^c \cdot p_2^c \\
&= 0.6 \cdot 0.3 \\
&= 0.18
\end{aligned}
$$

$$
\begin{aligned}
P(X=1) &= \frac{p_1^c \cdot p_2}{2} + \frac{p_1 \cdot p_2^c}{2} \\
&= \frac{0.6 \cdot 0.7}{2} + \frac{0.4 \cdot 0.3}{2} \\
&= 0.27
\end{aligned}
$$

$$
\begin{aligned}
P(X=2) &= \frac{p_1^c \cdot p_2}{2} + \frac{p_1 \cdot p_2^c}{2} + \frac{p_1}{2} \cdot \frac{p_2}{2} \\
&= \frac{0.6 \cdot 0.7}{2} + \frac{0.4 \cdot 0.3}{2} + \frac{0.7}{2} \cdot \frac{0.4}{2} \\
&= 0.34
\end{aligned}
$$

$$
\begin{aligned}
P(X=3) &= \frac{p_1}{2} \cdot \frac{p_2}{2} + \frac{p_1}{2} \cdot \frac{p_2}{2} \\
&= \frac{0.7}{2} \cdot \frac{0.4}{2} + \frac{0.7}{2} \cdot \frac{0.4}{2}
\end{aligned}
$$

$$= 0.14$$

$$\begin{aligned} P(X = 4) &= \frac{p_1}{2} \cdot \frac{p_2}{2} \\ &= \frac{0.7}{2} \cdot \frac{0.4}{2} \\ &= 0.07 \end{aligned}$$

$$p_X(k) = \begin{cases} 0.18, & \text{if } k = 0, \\ 0.27, & \text{if } k = 1, \\ 0.34, & \text{if } k = 2, \\ 0.14, & \text{if } k = 3, \\ 0.07, & \text{if } k = 4, \\ 0, & \text{otherwise} \end{cases} \qquad \square$$

2. Given $p = 1/649640$. Therefore,

$$\begin{aligned} P(X \geq 1) &= 1 - P(X = 0) \\ &= 1 - \left(\frac{649640 - 1}{649640}\right)^{649640} \\ &= 1 - \left(1 - \frac{1}{649640}\right)^{649640} \end{aligned}$$

If $n = 649640$

$$\begin{aligned} P(X \geq 1) &= 1 - \left(1 - \frac{1}{n}\right)^{n} \\ &= 1 - \lim_{n \to \infty} \left(1 - \frac{1}{n}\right)^{n} \\ &= 1 - \frac{1}{e} \end{aligned} \qquad \square$$

3. A claim is first filed with the geometric probability

$$
\begin{aligned}
p(1-p)^{n-1} &= (0.05)(1-0.05)^{n-1} \\
&= (0.05)(0.095)^{n-1}
\end{aligned}
$$

The total premium is

$$
\begin{aligned}
1000 \cdot \sum_{k=0}^{n-1} (0.9)^k &= 1000 \cdot \frac{1-0.9^n}{1-0.9} \\
&= 10000 \cdot (1-(0.9)^n)
\end{aligned}
$$

Therefore, the PMF is

$$
p_X(k) = \begin{cases} 0.05 \cdot (0.095)^{n-1}, & \text{if } k = 10000 \cdot (1-(0.9)^n), n = 1, 2, \ldots, \\ 0, & \text{otherwise} \end{cases}
$$

□

4. (a) $Y = X\ (mod\ 3)$

$$
\begin{aligned}
P(Y=0) &= P(X = \{0,3,6,9\}) \\
&= \frac{4}{10} \\
&= 0.4
\end{aligned}
$$

$$
\begin{aligned}
P(Y=0) &= P(X = \{0,3,6,9\}) \\
&= \frac{4}{10} \\
&= 0.4
\end{aligned}
$$

$$
\begin{aligned}
P(Y=1) &= P(X = \{1,4,7\}) \\
&= \frac{3}{10} \\
&= 0.3
\end{aligned}
$$

$$
P(Y=2) = P(X = \{2,5,8\})
$$

$$= \frac{3}{10}$$
$$= 0.3$$

(b) $Y = 5\ (mod\ X + 1)$

$$\begin{aligned} P(Y = 0) &= P(X = \{0, 4\}) \\ &= \frac{2}{10} \\ &= 0.2 \end{aligned}$$

$$\begin{aligned} P(Y = 1) &= P(X = \{1, 5\}) \\ &= \frac{2}{10} \\ &= 0.2 \end{aligned}$$

$$\begin{aligned} P(Y = 2) &= P(X = \{2\}) \\ &= \frac{1}{10} \\ &= 0.1 \end{aligned}$$

$$\begin{aligned} P(Y = 5) &= P(X = \{5, 6, 7, 8, 9\}) \\ &= \frac{5}{10} \\ &= 0.5 \end{aligned}$$

□

5. Since X is uniformly distributed over $[a, b]$,

$$p_X(k) = \begin{cases} \frac{1}{b-a+1}, & \text{if } k \in [a, b], \\ 0, & \text{otherwise} \end{cases}$$

and

$$max\{0, X\} = \begin{cases} X, & \text{if } X > 0 \\ 0, & \text{if } X \leq 0 \end{cases}$$

Then,

$$P(max\{0, X\} = 0) = P(X \leq 0)$$
$$= \frac{|a| + 1}{b - a + 1}$$

Similarly, for $min\{0, X\}$

$$P(min\{0, X\} = 0) = P(X \geq 0)$$
$$= \frac{b + 1}{b - a + 1}$$

For $k > 0$,

$$P(max\{0, X\} = k) = P(max\{0, X\} = k)$$
$$= P(X = k)$$
$$= \frac{1}{b - a + 1} \qquad \square$$

6. (a) Find K

$$1 = \sum_{K=-3}^{3} p_X(K)$$
$$1 = K \sum_{x=-3}^{3} x^2$$
$$1 = K(9 + 4 + 1 + 0 + 1 + 4 + 9)$$
$$\Rightarrow K = \frac{1}{28}$$

(b) Find the PMF of Y

Since $Y = |X|$, then $y \in \{0, 1, 2, 3\}$

$$p_Y(0) = p_X(0)$$
$$= 0$$

$$
\begin{aligned}
p_Y(1) &= p_X(-1) + p_X(1) \\
&= \frac{1^2}{28} + \frac{1^2}{28} \\
&= \frac{1}{14}
\end{aligned}
$$

$$
\begin{aligned}
p_Y(2) &= p_X(-2) + p_X(2) \\
&= \frac{2^2}{28} + \frac{2^2}{28} \\
&= \frac{4}{14}
\end{aligned}
$$

$$
\begin{aligned}
p_Y(3) &= p_X(-3) + p_X(3) \\
&= \frac{3^2}{28} + \frac{3^2}{28} \\
&= \frac{9}{14}
\end{aligned}
$$

(c) General formula for p_Y

$$
p_Y = \begin{cases} 2p_X(y), & \text{if } y \in \{0, 1, 2, 3\}, \\ 0, & \text{otherwise} \end{cases} \qquad \square
$$

7. Since $P_x(X) = sin(X\pi) = 0$ for $X \in \mathbb{Z}$:

$$
\begin{aligned}
E[sin(X\pi)] &= \sum_{k \in \mathbb{Z}} kP_x(k) \\
&= 0
\end{aligned}
$$

Since $P_x(X) = cos(X\pi) = 1$ for $X \in \mathbb{Z}$:

$$
\begin{aligned}
E[cos(X\pi)] &= \sum_{k \in \mathbb{Z}} kP_x(k) \\
&= 1
\end{aligned} \qquad \square
$$

8. (a) Since the event where Fischer wins is independent, and a win is determined by a win in the $(n+1)$th until n ties:

$$\sum_{n\geq 0}(1-p-q)^{n-1}(p) = \frac{p}{p+q}$$

(b) The PMF of the geometric probability

$$p_X(k) = (1-p-q)^{k-1}(p+q), \text{ for } k \geq 0$$

The mean of the geometric probability

$$E[X] = \frac{1}{p+q}$$

The variance of the geometric probability

$$var[X] = \frac{1-(p+q)}{(p+q)^2} \qquad \square$$

9. Since the distribution is binomial with $n = 10$

$$\begin{aligned} E[X] = np &\geq 10000 - 10 \\ \Rightarrow np &\geq 9990 \\ \Rightarrow p &\geq 0.999 \end{aligned} \qquad \square$$

10. Since

$$var(X) = E[X^2] - (E[X])^2 \Rightarrow E[X^2] = var(X) + (E[X])^2$$

Then

$$\begin{aligned} E[(X_1 + \ldots + X_n)^2] &= var(X_1 + \ldots + X_n) + (E[(X_1 + \ldots + X_n)])^2 \\ &= n \cdot var(X_1) + (n \cdot E[X_1])^2, \text{ (since the variables are identical)} \\ &= n \cdot (E[X_1^2] - (E[X_1])^2) + n^2 \cdot E[X_1]^2 \\ &= n \cdot E[X_1^2] + (n^2 - n) \cdot (E[X_1])^2 \end{aligned}$$

Let $c = n$ and $d = n^2 - n$, then

$$E[(X_1 + \ldots + X_n)^2] = cE[X_1^2] + d(E[X_1])^2 \qquad \square$$