CMPE323 Intro to Fourier Series

Chapter 7

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We can plot the coefficients
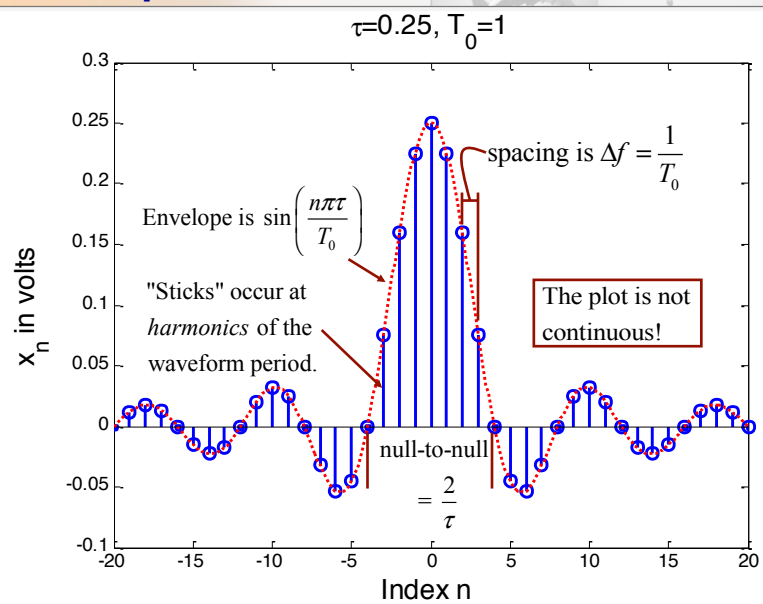
$\tau=0.25$, $T_0=1$

spacing is $\Delta f = \dfrac{1}{T_0}$

Envelope is $\sin\left(\dfrac{n\pi\tau}{T_0}\right)$

"Sticks" occur at *harmonics* of the waveform period.

The plot is not continuous!

null-to-null $= \dfrac{2}{\tau}$

x_n in volts

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So this means

- **We can write the periodic signal**

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi kt}{T_0}} = \frac{A\tau}{T_0} \sum_{k=-\infty}^{\infty} \text{sinc}\left(\frac{n\tau}{T_0}\right) e^{j\frac{2\pi kt}{T_0}}$$

- **...which is defined *only* at integer multiples of**

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{T_0}$$

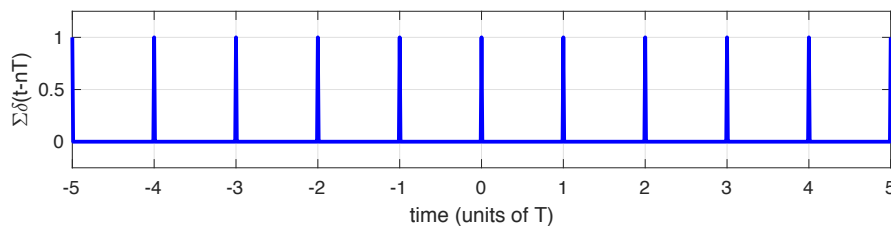- **This expression is known as the Fourier Series, and it relates the time domain $(x(t))$ to the frequency domain**

$$\{c_k\} \text{ or } X(f) = \sum_{k=-\infty}^{\infty} c_k \delta\left(f - \frac{k}{T}\right)$$

- **And the inverse relationship hold as well**

The periodic impulse train

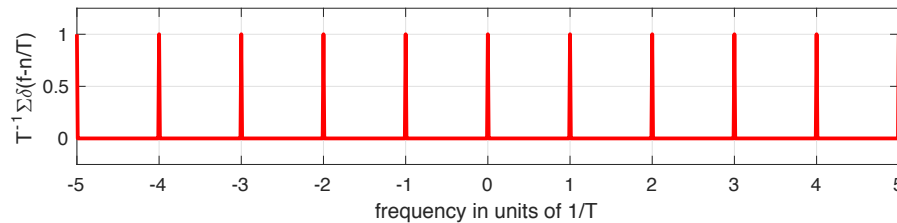- **What about a periodic train of impulses?**

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

- **Can we find its Fourier Series?**

Analysis Equation

$$c_k = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t - 0T) e^{-j\frac{2\pi}{T}kt} \, dt = \frac{1}{T} \; for \; all \; k!!$$

But what if the signal is not periodic?

- **There is an equivalent result for non-periodic signals**

- **A non-periodic signal can be viewed as the *limit* of a periodic signal as** $T_0 \to \infty$

$$c_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-j\frac{2\pi kt}{T_0}} \, dt \qquad \frac{k}{T_0} \to f \qquad \frac{T_0}{2} \to \infty \qquad \frac{1}{T_0} \to df$$

$$X(f) \triangleq \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} \, dt \quad contribution\; between f \; and\; f + df$$

- $X(f)$ **is called the Fourier Transform, or the *voltage spectrum* of** $x(t)$,

- **...and we have** $x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi ft} df = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$

- **Notice** $c_n = \frac{1}{T_0} X(f)_{f=\frac{n}{T_0}} = \frac{1}{T_0} X\left(\frac{n}{T_0}\right)$

But what if the signal is not periodic?
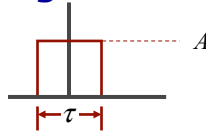
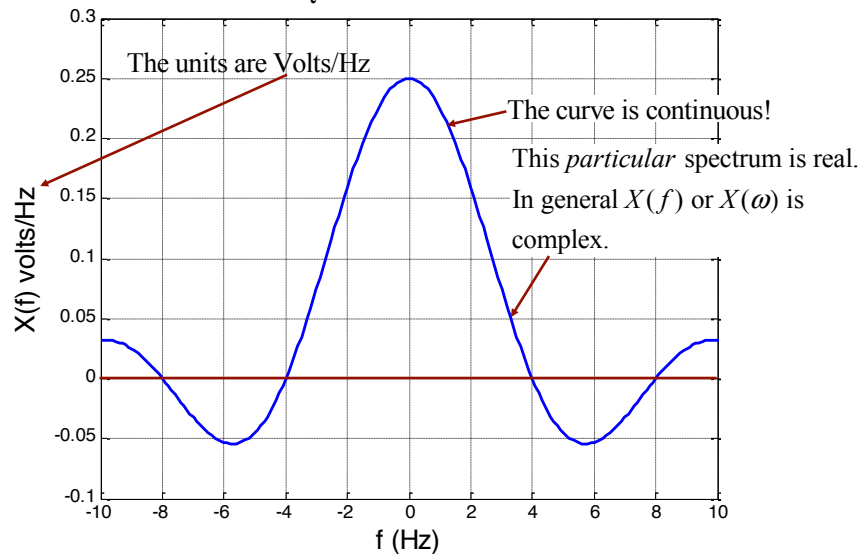- **We define the Fourier Transform of x(t)**

$$X(f) = \int\limits_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt \ \text{ or } \int\limits_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = X(\omega)$$

- **In this case**

$$X(f) = \int\limits_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt = \int\limits_{-\tau/2}^{\tau/2} Ae^{-j2\pi ft}\,dt = \frac{A}{-j2\pi f}e^{-j2\pi ft}\bigg|_{-\tau/2}^{\tau/2}$$

$$= \frac{A}{j2\pi f}\left(e^{j2\pi f\tau2} - e^{-j2\pi f\tau2}\right) = \frac{A}{\pi f}\frac{\left(e^{j\pi f\tau} - e^{-j\pi f\tau}\right)}{j2} = A\tau\frac{\sin(\pi f\tau)}{\pi f\tau} = A\tau\mathrm{sinc}(f\tau)$$

And the plot is the *voltage spectrum*

$\tau = 0.25$ seconds

The units are Volts/Hz

The curve is continuous!

This *particular* spectrum is real.

In general $X(f)$ or $X(\omega)$ is complex.

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Properties of Fourier Transform

- **Duality:** $X(f) = \mathcal{F}(x(t)) \Rightarrow x(f) = \mathcal{F}(X(-t))$ and $x(-f) = \mathcal{F}(X(t))$

- **Linearity:** $z(t) = ax(t) + by(t) \Rightarrow Y(f) = aX(f) + bY(f)$

- **Time Shift:** $\mathcal{F}(x(t - t_0)) = e^{-j2\pi f t_0} \mathcal{F}(x(t))$

- **Scaling:** For $a \neq 0 \in \mathbb{R}$, $\mathcal{F}(x(at)) = \dfrac{1}{|a|} X\left(\dfrac{f}{a}\right)$

- **Modulation:** $\mathcal{F}(x(t)e^{j2\pi f_0 t}) = X(f - f_0)$

- **Conjugation:** $\mathcal{F}(x^*(t)) = X^*(-f)$

- **Parseval:** $\displaystyle\int_{-\infty}^{\infty} x(t)y^*(t)\,dt = \int_{-\infty}^{\infty} X(f)Y^*(f)\,df = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)Y^*(\omega)\,d\omega$

- **Rayleigh** $\displaystyle\int_{-\infty}^{\infty} |x(t)|^2\,dt = \int_{-\infty}^{\infty} |X(f)|^2\,df = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)Y^*(\omega)\,d\omega$

Advanced Properties of Fourier Transform

- **Integration:** $\mathcal{F}\left(\displaystyle\int_{-\infty}^{t} x(t)\,dt\right) = \dfrac{X(f)}{j2\pi f} + \dfrac{1}{2}X(0)\delta(f)$

- **Differentiation:** $\mathcal{F}\left(\dfrac{d}{dt}x(t)\right) = j2\pi f X(f)$

- **Moments:** $\displaystyle\int_{-\infty}^{\infty} t^n x(t)\,dt = \left(\dfrac{j}{2\pi}\right)^n \dfrac{d^n}{df^n} X(f)\Bigg|_{f=0}$

- **Real Signals:** The Fourier transform of a real signal is EVEN in magnitude and ODD in phase

The convolution theorem (very important!)

- **The output of a LTI system with transfer function $H(f)$**

$$Y(f) = X(f)H(f) \qquad y(t) = \int_{-\infty}^{\infty} Y(f)e^{j2\pi ft}df = \int_{-\infty}^{\infty} X(f)H(f)e^{j2\pi ft}df$$

$$\text{Write } H(f) = \int_{-\infty}^{\infty} h(\tau)e^{-j2\pi f\tau}d\tau$$

$$y(t) = \int_{-\infty}^{\infty} X(f)\left(\int_{-\infty}^{\infty} h(\tau)e^{-j2\pi f\tau}d\tau \right)e^{j2\pi ft}df$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} X(f)h(\tau)e^{j2\pi f(t-\tau)}df \right)d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\left(\int_{-\infty}^{\infty} X(f)e^{j2\pi f(t-\tau)}df \right)d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau$$

$$\triangleq x(t) * h(t) \text{ the CONVOLUTION}$$

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Summary

- **We can decompose a periodic signal into the weighted sum of complex exponentials,...**

- **...or, equivalently, to the weighted sum of sines and cosines.**

- **We write the weighted sum as a Fourier Series**

- **The frequency-domain representation consists of a series of harmonically-related terms, with separation equal to the period of the signal,**

- **The coefficients have units of volts (or amps)**

- **We can decompose a non-periodic signal into the weighted integral of complex exponentials,...**

- **...or, equivalently, to the weighted integral of sines and cosines**

- **The frequency-domain representation has a continuous spectrum with units of volts/Hz (or amps/Hz).**

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