Problem Set #5 Solutions

1. Our starting point is

$$Q = \int_V \rho_{\mathrm{V}}\, r dr\, d\phi\, dz.$$

   a. We have

$$Q = \int_V \frac{-\rho_0 a^2}{a^2 + r^2}\, r dr\, d\phi\, dz = -2\pi L\rho_0 \int_0^{r_0} \frac{a^2 r}{a^2 + r^2}\, dr$$
$$= -\pi L\rho_0 a^2 \ln\left(a^2 + r^2\right)\Big|_0^{r_0} = -\pi L\rho_0 a^2 \ln\left(1 + \frac{r_0^2}{a^2}\right)$$

   b. The current in the $+z$-direction is given by

$$I = (Q/L)\mathbf{u}\cdot d\hat{\mathbf{z}} = -\pi u\rho_0 a^2 \ln\left(1 + \frac{r_0^2}{a^2}\right).$$

   Since the electrons are negatively charged and are flowing in the $+z$-direction, the current is flowing in the $-z$-direction. Its magnitude is $\pi u\rho_0 a^2 \ln[1+(r_0/a)^2]$.

2. We begin by recalling that in free space,

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_{l'} \hat{\mathbf{R}}' \frac{\rho_l\, dl'}{\hat{\mathbf{R}}'^2},$$

where $\hat{\mathbf{R}}' = \mathbf{R}(P) - \mathbf{R}(P')$, which becomes in our case, $\hat{\mathbf{R}}' = (\hat{\mathbf{z}}z) - (\hat{\mathbf{r}}r) = -\hat{\mathbf{x}}r\cos\phi - \hat{\mathbf{y}}r\sin\phi + \hat{\mathbf{z}}z$. Noting that $|\hat{\mathbf{R}}'| = (r^2 + z^2)^{1/2}$, we may write

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\int_0^{\pi/4} \frac{-\hat{\mathbf{x}}r\cos\phi - \hat{\mathbf{y}}r\sin\phi + \hat{\mathbf{z}}z}{(r^2 + z^2)^{3/2}}\, r d\phi.$$

Doing the ϕ-integrations, we obtain

$$\mathbf{E} = \frac{\rho_{\mathrm{V}}}{4\pi\epsilon_0}\frac{r}{(r^2 + z^2)^{3/2}}\left[-\hat{\mathbf{x}}\frac{\sqrt{2}}{2}r - \hat{\mathbf{y}}\left(1 - \frac{\sqrt{2}}{2}\right)r + \hat{\mathbf{z}}\frac{\pi}{4}z\right].$$

For numerical evaluation, it is useful to rewrite the field as

$$\mathbf{E} = \frac{\rho_{\mathrm{V}}}{4\pi\epsilon_0 r}\frac{1}{[1 + (z/r)^2]^{3/2}}\left[-\hat{\mathbf{x}}\frac{\sqrt{2}}{2} - \hat{\mathbf{y}}\left(1 - \frac{\sqrt{2}}{2}\right) + \hat{\mathbf{z}}\frac{\pi}{4}\frac{z}{r}\right].$$

Given our parameter set, we have $\rho_{\mathrm{V}}/(4\pi\epsilon_0 r) = 2.25\times 10^6$ V/m. When $z = \pm 5$ cm, we have $1/\left[1 + (z/r)^2\right]^{3/2} = 0.0512$. We thus find

a. When $z = 0$,

$$\mathbf{E} = 2.25 \times 10^6 \times (-\hat{\mathbf{x}}0.707 - \hat{\mathbf{y}}0.293)\ \text{V/m} = -\hat{\mathbf{x}}1.6 - \hat{\mathbf{y}}0.66\ \text{MV/m}.$$

b. When $z = 5$ cm,

$$\begin{aligned}\mathbf{E} &= 2.25 \times 10^6 \times 5.12 \times 10^{-2} \times (-\hat{\mathbf{x}}0.707 - \hat{\mathbf{y}}0.293 + \hat{\mathbf{z}}1.96)\ \text{V/m} \\ &= -\hat{\mathbf{x}}81 - \hat{\mathbf{y}}34 + \hat{\mathbf{z}}230\ \text{kV/m}\end{aligned}.$$

c. When $z = -5$ cm,

$$\mathbf{E} = -\hat{\mathbf{x}}81 - \hat{\mathbf{y}}34 - \hat{\mathbf{z}}230\ \text{kV/m}.$$

3. From the symmetry of the problem statement, it is evident that the charge must be located on the positive x-axis, that it is positively charged, and that its x-coordinate, which we will denote l must satisfy $0 < l < d$. To find l, we note that the total field that the charge at $x = 0$ experiences is given by

$$\mathbf{E}_0 = \hat{\mathbf{x}}\frac{1}{4\pi\epsilon}\left(\frac{q}{l^2} - \frac{36e}{d^2}\right)$$

and the total field experienced by the charge at $x = d$ is given by

$$\mathbf{E}_d = \hat{\mathbf{x}}\frac{1}{4\pi\epsilon}\left[-\frac{q}{(d-l)^2} + \frac{9e}{d^2}\right].$$

We thus infer,

$$\frac{q}{l^2} = \frac{36e}{d^2}, \qquad \frac{q}{(d-l)^2} = \frac{9e}{d^2}.$$

From these two equations, we find $(d-l)^2/l^2 = 36/9$, which implies $(d-l)/d = 2$ or $l = d/3$. Substituting this value into either of the equations above, we find $q = 4e$. Finally, we note that the field on the charge at $x = l = d/3$ is given by

$$\mathbf{E}_l = \hat{\mathbf{x}}\frac{1}{4\pi\epsilon}\left[-\frac{9e}{(d/3)^2} + \frac{36e}{(2d/3)^2}\right] = 0.$$

There are three forces that must equal zero and only two quantities that we are free to change (q and l). Generally, three equations with two unknowns do not have a solution unless one of the equations is redundant. You should think about where the redundancy comes from in this case.

4. We start by writing $\nabla \cdot \mathbf{D} = \rho_\text{V}$ in Cartesian coordinates. Noting that $\mathbf{D}$ has no dependence on z, we have

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} = \rho_\text{V}.$$

a. Since we have $\partial D_x/\partial x = 2$ and $\partial D_y/\partial y = -2$, we conclude that $\rho_{\mathrm{V}} = 0$.

b. Since $\rho_{\mathrm{V}} = 0$, we must have $Q = 0$ when we integrate over the volume.

c. In principle, we must integrate $\mathbf{D} \cdot d\mathbf{s}$ over all six faces of the cube. However, since $\mathbf{D}$ is independent of z, the two faces at $z = 2$ and $z = 0$ must cancel. So, we will not bother with them. Thus, we must calculate

$$\begin{aligned}\int_S \mathbf{D} \cdot d\mathbf{s} &= \int_0^2 dz \int_0^2 dx \, [D_y(x, y = 2, z) - D_y(x, y = 0, z)] \\ &\quad + \int_0^2 dz \int_0^2 dy \, [D_x(x = 2, y, z) - D_x(x = 0, y, z)] \\ &= 2 \int_0^2 dx(-4) + 2 \int_0^2 dy(4) = -16 + 16 = 0.\end{aligned}$$

5. At any point (r, ϕ, z), we have

$$\begin{aligned}V(r, \phi, z) &= \frac{\rho_{\mathrm{S}}}{4\pi\epsilon_0} \int_0^{2\pi} d\phi' \int_0^a r' dr' \, \frac{1}{[r^2 + r'^2 + 2rr' \cos(\phi' - \phi) + (z - z')^2]^{1/2}} \\ &= \frac{\rho_{\mathrm{S}}}{4\pi\epsilon_0} \int_0^{2\pi} d\phi' \int_0^a r' dr' \, \frac{1}{[r^2 + r'^2 + 2rr' \cos\phi + (z - z')^2]^{1/2}},\end{aligned}$$

where we let $\phi'' = \phi' - \phi$ and then just rename this new variable ϕ' since it is a dummy variable. The key point is that this expression is independent of ϕ. As a consequence, $\partial V/\partial x$ and $\partial V/\partial y$ must both equal zero at $r = 0$. {How would we "prove" this result? From the independence with respect to ϕ, it follows that $V(\epsilon, 0, z) = V(-\epsilon, 0, z)$ for any ϵ, so that $[V(\epsilon, 0, z) - V(-\epsilon, 0, z)]/(2\epsilon) = 0$. Letting ϵ go to zero, we find that $\partial V/\partial x$ is zero. An analogous proof holds for $\partial V/\partial y$. Of course, a proof is hardly necessary; the geometry makes it fairly obvious.} We thus have $\nabla V = \hat{\mathbf{z}} \, \partial V/\partial z$, a fact that we will use shortly.

a. Along the z-axis, the expression for the voltage becomes

$$\begin{aligned}V(0, 0, z) &= \frac{\rho_{\mathrm{S}}}{4\pi\epsilon_0} \int_0^{2\pi} d\phi' \int_0^a r' dr' \, \frac{1}{[r'^2 + (z - z')^2]^{1/2}} \\ &= \frac{\rho_{\mathrm{S}}}{2\epsilon_0} \int_0^a r' dr' \, \frac{1}{[r'^2 + (z - z')^2]^{1/2}} = \frac{\rho_{\mathrm{S}}}{2\epsilon_0} \left[(a^2 + z^2)^{1/2} - z \right].\end{aligned}$$

b. We have $E = -\nabla V$, which along the z-axis becomes

$$E = -\hat{\mathbf{z}} \frac{\partial V}{\partial z} = \frac{\rho_{\mathrm{S}}}{2\epsilon_0} \left[1 - \frac{z}{(a^2 + z^2)^{1/2}} \right]$$

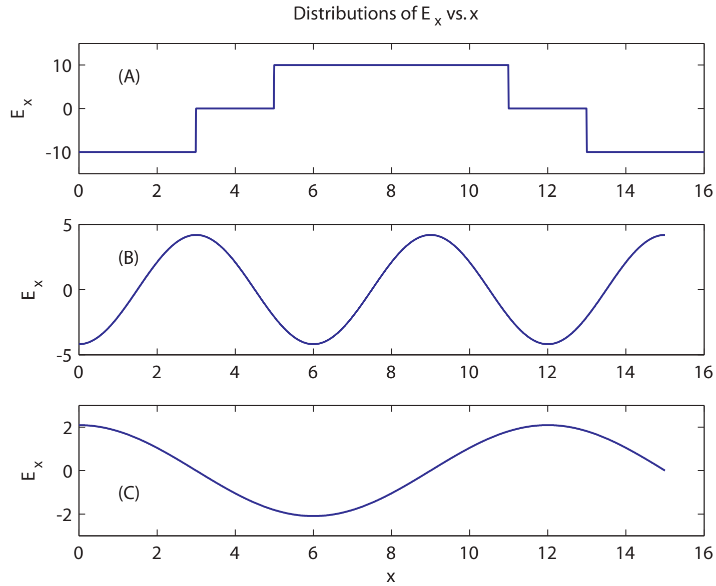
when $z > 0$, which is the same result that we found earlier using Coulomb's law. When $z < 0$, we get just the opposite sign

6. a. The functional forms are as follows: (a) $V(x) = 10x$ V $(0 \le x \le 3)$, $V(x) = 30$ V $(3 \le x \le 5)$, $V(x) = 30 - 10(x - 5) = 80 - 10x$ V $(5 \le x \le 11)$, $V(x) = -30$ V $(11 \le x \le 13)$, $V(x) = -30 + 10(x - 13) = -160 + 10x$ V $(13 \le x \le 16)$; (b) $V(x) = 4\sin(2\pi x/6)$ V $(0 \le x \le 15)$; (c) $V(x) = -4\sin(2\pi x/12)$ V $(0 \le x \le 15)$.

   To determine the electric field, we take the negative of the derivative with respect to x. The only non-zero component is the x-component: (a) $E_x = -10$ V/m $(0 \le x \le 3)$, $E_x = 0$ V/m $(3 \le x \le 5)$, $E_x = 10$ V/m $(5 \le x \le 11)$, $E_x = 0$ V/m $(11 \le x \le 13)$, $E_x = -10$ V/m $(13 \le x \le 16)$; (b) $E_x = -(4\pi/3)\cos(2\pi x/6)$ V/m $(0 \le x \le 15)$, (c) $E_x = (2\pi/3)\cos(2\pi x/12)$ V/m

   b. The MATLAB output plots and listing are on the next page:

MATLAB PLOT OUTPUT



MATLAB LISTING

```
% Distributions_C330_PS5_no6:  CMPE 330 Problem Set 5 no. 6

% Set x-values for plot A
x=0:0.01:16;
Nmax = length(x);    %length of vector x
None = (Nmax-1)/16;  %number of values when Delta-x = 1

% Calculate values for plot A
A(1:3*None)=-10;           %0 < x < 3
A(1+3*None:5*None)=0;      %3 < x < 5
A(1+5*None:11*None)=10;    %5 < x < 11
A(1+11*None:13*None)=0;   %11 < x < 13
A(1+13*None:Nmax)=-10;    %13 < x < 16

%Set x-values for plots B and C
xtrunc = x(1:1+15*None);

%Calculate values for plots B and C
B = -(4*pi/3)*cos(2*pi*xtrunc/6);
C = (2*pi/3)*cos(2*pi*xtrunc/12);

%Plot results
subplot(3,1,1), plot(x,A)
title('Distributions of E_x vs. x')
ylabel('E_x')
text(1,7.5,'(A)')
axis([0 16 -15 15])
subplot(3,1,2), plot(xtrunc,B)
ylabel('E_x')
text(1,2.5,'(B)')
axis([0 16 -5 5])
subplot(3,1,3), plot(xtrunc,C)
axis([0 16 -3 3])
ylabel('E_x')
text(1,-1,'(C)')
xlabel('x')
```

7. Each of the two concentric cylinders acts as a separate resistor, and the two resistors are in parallel because they are both capped by the same conducting plates. Hence, the conductances add. The material with conductivity σ_1 has an area πa^2, so that its conductance is given by $G_1 = 2\pi a^2 \sigma_1 / l$, and the material with conductivity σ_2 has an area $\pi(b^2 - a^2)$, so that its conductance is given by $G_2 = 2\pi(b^2 - a^2)\sigma_2 / l$. The total conductance is given by

$$G = G_1 + G_2 = \frac{2\pi}{l}\left[\sigma_1 a^2 + \sigma_2 (b^2 - a^2)\right].$$

Using the relation $R = 1/G$, we obtain the final result.

8. a. We first note that $\mathbf{D}_2 = 9\epsilon_0 \mathbf{E}_2 = -27\epsilon_0 \hat{\mathbf{R}} \cos\theta$. Since $\hat{\mathbf{n}} = \hat{\mathbf{R}}$ on the surface of the sphere, we have $\rho_{\mathrm{S}} = \mathbf{D}_2 \cdot \hat{\mathbf{R}} = -27\epsilon_0 \cos\theta$, which agrees with the solution in the back of Ulaby's book.

   b. The existence of a $\hat{\boldsymbol{\theta}}$ component in the original formulation implies that there is a non-zero tangential electric field on the conductor, which is not possible.

9. Since both $\mathbf{D}$ and $\mathbf{E}$ are normal to the interface between the two dielectric materials, we have $\mathbf{D}_1 = \mathbf{D}_2$ and $\epsilon_1 \mathbf{E}_1 = \epsilon_2 \mathbf{E}_2$. If we take the z-direction to be the direction that is normal to the capacitor plates and goes from the negative plate to the positive plate, we have $\mathbf{D}_1 = \mathbf{D}_2 = -\hat{\mathbf{z}}\rho_{\mathrm{S}}$, where ρ_{S} is the surface charge. It follows that $\mathbf{E}_1 = -\hat{\mathbf{z}}\rho_{\mathrm{S}}/\epsilon_1$ and $\mathbf{E}_2 = -\hat{\mathbf{z}}\rho_{\mathrm{S}}/\epsilon_2$.

   a. Taking the point $z = 0$ to correspond to the negative capacitor plate, setting $V = 0$ at that plate, and integrating the electric field up to $z = d_2 + d_1$, we have that $V = V(z = d_2 + d_1) = -\int_0^{d_2} \mathbf{E}_2 \cdot d\mathbf{z} - \int_{d_2}^{d_2+d_1} \mathbf{E}_1 \cdot d\mathbf{z} = \rho_{\mathrm{S}}[(d_2/\epsilon_2) + (d_1/\epsilon_1)]$. We thus find

$$\rho_{\mathrm{S}} = \frac{V}{(d_2/\epsilon_2) + (d_1/\epsilon_1)}.$$

   Writing $E_1 = |\mathbf{E}_1|$ and $E_2 = |\mathbf{E}_2|$, we now find

$$E_1 = \frac{V/\epsilon_1}{(d_2/\epsilon_2) + (d_1/\epsilon_1)}, \qquad E_2 = \frac{V/\epsilon_2}{(d_2/\epsilon_2) + (d_1/\epsilon_1)}.$$

   b. The energy U_1 stored in dielectric medium 1 is

$$U_1 = \frac{1}{2}\epsilon_1 E_1^2\, d_1 A = \frac{1}{2}\frac{V^2 (d_1/\epsilon_1) A}{\left[(d_2/\epsilon_2) + (d_1/\epsilon_1)\right]^2},$$

   and

$$U_2 = \frac{1}{2}\epsilon_2 E_2^2\, d_2 A = \frac{1}{2}\frac{V^2 (d_2/\epsilon_2) A}{\left[(d_2/\epsilon_2) + (d_1/\epsilon_1)\right]^2}.$$

Adding these two contributions to obtain the total energy U, we find

$$U = U_1 + U_2 = \frac{1}{2} \frac{V^2 A}{(d_2/\epsilon_2) + (d_1/\epsilon_1)}.$$

c. Since we must have $U = (1/2)CV^2$, we conclude

$$C = \frac{A}{(d_2/\epsilon_2) + (d_1/\epsilon_1)}.$$

Noting that $C_1 = \epsilon_1 A/d_1$ and $C_2 = \epsilon_2 A/d_2$, we find

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2},$$

which is the desired expression.