CMPE323: The Laplace Transform

Part II

Inverse Laplace Transforms

- **Finding the Laplace (or later, Fourier) Transform is called the *analysis* operation**

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

- **The equivalent *synthesis* equation is** $x(t) = \int_{-\infty}^{\infty} X(s)e^{st}\,ds$
- **…and this is often performed via a *Cauchy* integration…**
- **…which I'm not going into right now.**
- **In most cases, the Cauchy integration reduces to the Cauchy Residue Theorem…**
- **…and residues reduce to Partial Fraction Expansion (PFE)**
- **We generally use PFE when $X(s)$ is a rational polynomial, as is usually the case.**

PFE Process

- **Factor numerator and denominator polynomials. For the moment, we'll assume that all the roots are distinct and real.**

$$X(s) = \frac{K\prod_{k=1}^{M}(s-z_k)}{\prod_{k=1}^{N}(s-p_k)}$$

- **We want to expand**

$$X(s) = \sum_{k=1}^{N}\frac{R(p_k)}{(s-p_k)} = \frac{R(p_1)}{(s-p_1)} + \frac{R(p_2)}{(s-p_2)} + \cdots + \frac{R(p_3)}{(s-p_3)}$$

- **… where $R(p_k)$ are called the *residuals***

See http://lpsa.swarthmore.edu/BackGround/PartialFraction/PartialFraction.html

- **For each distinct p_m, form**

$$(s-p_m)X(s) = \frac{K\prod_{k=1}^{M}(s-z_k)}{\prod_{\substack{k=1\\k\neq m}}^{N}(s-p_k)} = R(p_m) + \sum_{\substack{k=1\\k\neq m}}^{N}(s-p_m)\frac{R(p_k)}{(s-p_k)}$$

- **…and evaluate at $s = p_m$**

$$\left.\frac{K\prod_{k=1}^{M}(s-z_k)}{\prod_{\substack{k=1\\k\neq m}}^{N}(s-p_k)}\right|_{s=p_m} = R(p_m) + \sum_{\substack{k=1\\k\neq m}}^{N} 0\times\frac{R(p_k)}{(s-p_k)} = R(p_m)$$

PFE Example #1

- $X(s) = \dfrac{s+2}{(s+1)(s+3)}$, $\mathrm{ROC} = \mathrm{Re}(s) > -1$

$$= \frac{R(-1)}{s-(-1)} + \frac{R(-3)}{s-(-3)}$$

$$R(-1) = \left.\frac{(s+1)(s+2)}{(s+1)(s+3)}\right|_{s=-1} = \frac{(-1+2)}{(-1+3)} = \frac{1}{2}$$

$$R(-3) = \left.\frac{(s+3)(s+2)}{(s+1)(s+3)}\right|_{s=-3} = \frac{(-3+2)}{(-3+1)} = \frac{-1}{-2} = \frac{1}{2}$$

$$X(s) = \frac{\frac{1}{2}}{s+1} + \frac{\frac{1}{2}}{s+3} \Rightarrow x(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

Difficult Things

- **There are four things that happen that complicate the process**
  - **1)** $M \geq N$
  - **2) Repeated real roots of the form** $(s-p_k)^m$
  - **3) Complex roots**
  - **4) Exponentials in** $H(s)$

M >= N

$$X(s)=\frac{6s^2+3s+2}{2s^2+14s+20},\ RoC>-2$$

Use synthetic division to divide the denominator into the numerator until $M<N$

$$\begin{array}{ccc|ccc} & & & & & 3 \\ \hline 2 & 14 & 20 & 6 & 3 & 2 \\ & & & 6 & 42 & 60 \\ & & & 0 & -39 & -58 \end{array}$$

$$X(s)=3+\frac{-39s-58}{2s^2+14s+20}$$

$$=3+\frac{-39(s+1.487)}{2(s+5)(s+2)}=3+\frac{-19.5(s+1.487)}{(s+5)(s+2)}$$

Let's Digress just a little

- **What is the Laplace Transform of $\delta(t)$?**

$$\Delta(s)=\int_{-\infty}^{\infty}\delta(\tau)e^{-s\tau}\,d\tau=e^{-s\tau}\Big|_{\tau=0}=1\ \ (!!)$$

- **So the Laplace Transform of a delta function is a constant!...**
- **...and vice-versa**
- **So** $X_1(s)=3\Rightarrow 3\delta(t)$

$$X_2(s)=\frac{-19.5(s-1.487)}{(s+5)(s+2)}=-19.5\left(\frac{R(-5)}{s+5}+\frac{R(-2)}{s+2}\right)\quad s>-2$$

$$\frac{-19.5(s+1.487)}{(s+5)(s+2)}$$

$$R(-5) = -19.5\left(\frac{-5+1.487}{-5+2}\right) = -22.83$$

$$R(-2) = -19.5\left(\frac{-2+1.487}{-2+5}\right) = 3.333$$

$$x(t) = 3\delta(t) - 22.83e^{-5t}u(t) + 3.333e^{-2t}u(t)$$

Multiple real roots

$$X(s) = \frac{s+3}{s(s+2)^2(s+5)} = \frac{R(0)}{s} + \frac{R_1(-2)}{s+2} + \frac{R_2(-2)}{(s+2)^2} + \frac{R(-5)}{s+5}$$

$R(0), R_2(-2), R(-5)$ found using standard techniques

$R(0) = 0.15,\ \ R_2(-2) = 0.1667,\ \ R(-5) = 0.0444$

$$R_1(-2) = \left[\frac{d}{ds}(s+2)^2 X(s)\right]_{s=-2}$$

$$= \frac{d}{ds}\left(\frac{s+3}{s(s+5)}\right)_{s=-2} = \frac{d}{ds}\left(\frac{s+3}{s^2+5s}\right) = \frac{1}{s^2+5s} - \frac{(s+3)(2s+5)}{(s^2+5s)^2}$$

$$= \frac{s^2+5s-2s^2-11s-15}{(s^2+5s)^2} = \left(\frac{-s^2-6s-15}{(s^2+5s)^2}\right)_{s=-2}$$

$$= \left(\frac{-(4)-6(-2)-15}{(4-10)^2}\right) = \left(\frac{-3+12-15}{(-6)^2}\right) = \left(\frac{-7}{36}\right) = 0.1944$$

Complex Roots!?

- **No problem…must occur in conjugate pairs**

$$H(s) = \frac{1}{s^2 + 2s + 2} = \frac{1}{(\quad)} \; p_k = -1 \pm j1$$