CMPE 320: Probability, Statistics, and Random Processes

Lecture 12: Independence of **RVs**

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Independence of a RV and an event

• "Events A and B are independent" means observing event B does not provide any information on A P(AIB)= P(B)

$$P(A|B) = P(A)$$

$$P(A|B) = P(A)P(B)$$

• "RV X is independent of event A" means observing event A does not provide any information on the value of X

$$E = Event \{X = x\}$$
 and A are independent for all x
$$P(X = x \mid A) = P(X = x)$$

$$P(X \mid A) = P(X \mid x)$$

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Example 2.19. Consider two independent tosses of a fair coin. Let X be the number of heads and let A be the event that the number of heads is even.

Are X and A independent?

In Apparlament of X and A
$$\Rightarrow$$
 $P_{X|A}(x) = P_{X}(x)$

$$P_{X}(x) = \begin{cases} \frac{1}{4}, & x = 0 \\ \frac{1}{2}, & x = 1 \end{cases}$$

$$TH, HT$$

$$P_{X(A}(x) = \frac{P(X=x \cap A)}{P(A)} = \begin{cases} \frac{1}{2} = \frac{1}{2}, x=0 \\ \frac{1}{2} = \frac{1}{2}, x=0 \end{cases}$$

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Independence of RVs

 "RVs X and Y are independent" means that the value of Y provides no information on the value of X

Events {
$$X=xy$$
 and { $Y=7y$ are independent for all x,y $P(X=x)$ and $Y=y) = P(X=x) P(Y=y)$ for all x,y $P_{X,Y}(x,y) = P_{X}(x) P_{Y}(y)$ for all x,y

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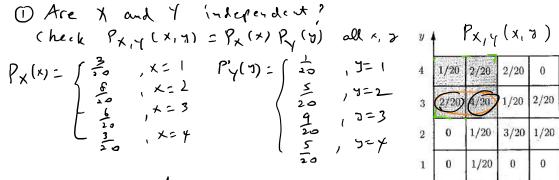
Conditional independence of RVs

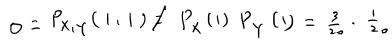
 Conditioning with an event A defines a new universe where all probabilities (or PMFs) are replaced by their conditional versions

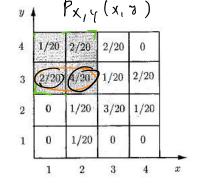
X and Y are conditionally independent given A if
$$P(X=\times, Y=\Im|A) = P(X=\times|A) P(Y=\Im|A) \text{ for all } x,y$$

$$P_{X,Y|A}(x,y) = P_{X|A}(x) P_{Y|A}(y) \text{ for all } x,y$$

Conditional independence may not imply independence, and vice versa







i. X and Y are not independent

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Expectation of the product of independent RVs

* If X and Y are independent RVs, then E[XY] = E[X] E[Y]

Who? E[XY] = \(\Sigma\) \(\text{X} \text{Y} \)

= \(\Sigma\) \(\text{X} \) \(\Sigma\) \(\text{Independence} \) \(\text{And Y} \)

= \(\Sigma\) \

• Similarly
$$E[g(X) h(Y)] = E[g(X)] E[h(Y)]$$

why? If X and Y are independent, so are $g(X)$ and $h(Y)$

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VAY $(X) = E[X^2] - E[X]^2$

Variance of the sum of independent RVs

If X and Y are independent RVs, var(X + Y) = var(X) + var(Y)

Why? Let
$$\tilde{X} = X - E[X]$$

$$\tilde{Y} = Y - E[Y]$$

$$= E[X] - E[X]$$

$$= E[X] - E[X] = 0$$

$$X \text{ and } \tilde{Y} \text{ are } 2ero-mean \text{ and } independent.$$

$$Var(X+Y) = Var(X+\tilde{Y}) \quad [Since var(X+a) = Var(X)]$$

$$= E[(\tilde{X}+\tilde{Y})^2] - (E[\tilde{X}+\tilde{Y}])^2$$

$$= E[(\tilde{X}^2 + 2\tilde{X}\tilde{Y} + \tilde{Y}^2)] = E[\tilde{X}^2] + 2E[\tilde{X}\tilde{Y}] + E[\tilde{Y}^2]$$

$$= E[\tilde{X}^2] + 2E[\tilde{X}]E[\tilde{Y}] + E[\tilde{Y}^2] = Var(X) + Var(\tilde{Y})$$

$$= Var(X) + Var(Y)$$

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Independence of several RVs

• X, Y, Z are independent RVs if

- If X, Y, Z are independent, f(X), g(Y), h(Z) are also independent
 - How about g(X,Y) and h(Z)? In he pendet
 - How about g(X,Y) and h(Y,Z)? Not necessarily independent

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Variance of sum of several independent RVs

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Example 2.20. Variance of the Binomial and the Poisson. We consider n independent coin tosses, with each toss having probability p of coming up a head. For each i, we let X_i be the Bernoulli random variable which is equal to 1 if the ith toss comes up a head, and is 0 otherwise. Then, $X = X_1 + X_2 + \cdots + X_n$ is a binomial random variable. What are its mean and variance? $o \notin X$?

$$E[X] = E[X_1 + X_2 + \dots + X_n] = E[X_1] + E[X_2] + \dots + F[X_n]$$

$$f[X_1] = 1 \cdot p + 0 \cdot (1-p) = p \Rightarrow E[X] = np$$

$$Var(X) = Var(X_1 + X_2 + \dots + X_n) = Var(X_1) + Var(X_2) + \dots + Var(X_n)$$

$$X_1 \text{ are inherently, unit true otherwise}$$

$$Var(X_1') = E[X_1'^2] - E[X_1]^2 = 1^2 \cdot p + 0^2 \cdot (1-p) - p^2 = p - p^2 = p(1-p)$$

$$\Rightarrow Var(X) = n \cdot p(1-p)$$

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Example 2.21. Mean and Variance of the Sample Mean. We wish to estimate the approval rating of a president, to be called B. To this end, we ask n persons drawn at random from the voter population, and we let X_i be a random variable that encodes the response of the ith person:

$$X_i = \left\{ \begin{array}{ll} 1, & \text{if the ith person approves B's performance,} \\ 0, & \text{if the ith person disapproves B's performance.} \end{array} \right.$$

We model X_1, X_2, \ldots, X_n as independent Bernoulli random variables with common mean p and variance p(1-p). Naturally, we view p as the true approval rating of B. We "average" the responses and compute the **sample mean** S_n , defined as

$$S_n = \frac{X_1 + X_2 + \dots + X_n}{n}.$$

Thus, the random variable S_n is the approval rating of B within our n-person sample. What are the mean and variance of S_n ?

$$E[S_n] = E\left[\frac{x_1 + x_2 + \dots + x_n}{n}\right] = \frac{1}{n} E[x_1 + \dots + x_n] = \frac{1}{n} (E[x_1] + E[x_1] + \dots + E[x_n])$$

$$= P$$

$$Var(S_n) = Var\left(\frac{X_1 + X_2 + \dots + X_n}{n}\right) = \frac{1}{n^2} Var(X_1 + \dots + X_n) \qquad Var(aX) = a^2 car(X)$$

$$= \frac{1}{n^2} \left(Var(X_1) + \dots + Var(X_n)\right) = \frac{n p(1-p)}{n^2}$$

$$= \frac{f(1-p)}{n} \rightarrow 0 \quad \text{as} \quad n \rightarrow \infty$$