Problem Set #2 Solutions

1. We first note that the Shmitt criterion becomes $l/\lambda \simeq 20$ since $R = f$ in this case, which differs from the Ulaby, et al. criterion by a factor of 5. Any criterion of this sort is a "rule of thumb." What really determines whether transmission line effects are negligible is whether the finite delay has an impact on the operation of the electronics, which must be determined experimentally. Both the Ulaby, et al. and Schmitt criteria give you an indication of when you are likely to run into trouble.

   a. In this case, we have $l = 0.20$ m and $\lambda = 3.00 \times 10^8/2.00 \times 10^4 = 1.5 \times 10^4$ m, so that $l/\lambda = 1.3 \times 10^{-5}$ and transmission line effects are negligible according to both criteria.

   b. In this case, we have $l = 5.0\times10^4$ m and $\lambda = 3.00\times10^8/60.0 = 5.0\times10^6$ m, so that $l/\lambda = 10^{-2}$. According to the Shmitt criterion, we may neglect transmission line effects, but we are on the edge of where transmission line effects may be neglected according to Ulaby, et al. Depending on the application, we may probably neglect transmission line effects, but we should watch out for them.

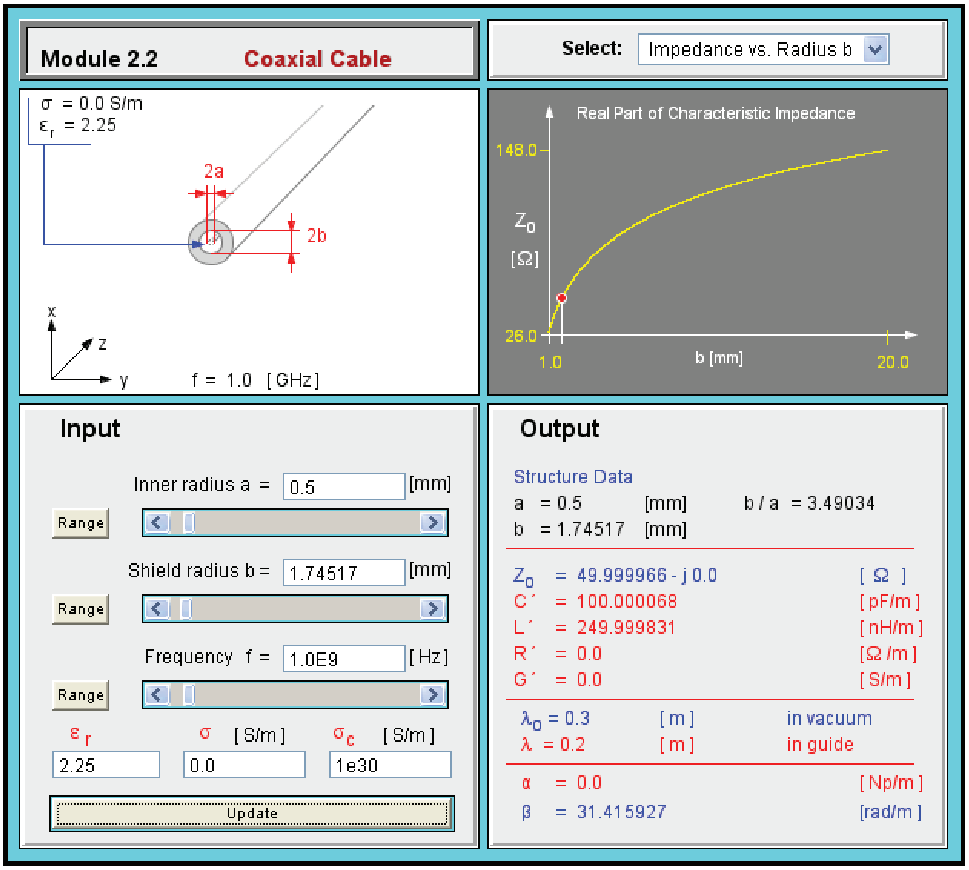
   c. In this case, we have $l = 0.20$ m and $\lambda = 3.00 \times 10^8/6.00 \times 10^8 = 0.5$ m. We find $l/\lambda = 0.4$ and transmission line effects cannot be neglected according to both criteria.

   d. In this case, we have $l = 1.0\times10^{-3}$ m and $\lambda = 3.00\times10^8/1.00\times10^{11} = 3.0\times10^{-3}$ m, so that $l/\lambda = 0.33$. We find once again that transmission line effects cannot be neglected.

2. We first calculate $R_S = (\pi f \mu_c/\sigma_c)^{1/2} = [\pi \times (5 \times 10^8) \times (4\pi \times 10^{-7})/(5.8 \times 10^7)]^{1/2} = 5.83 \times 10^{-3}$ Ω. We now find $R' = 2R_S/w = (2 \times 5.83 \times 10^{-3}/1.5 \times 10^{-2}) = 0.78$ Ω/m. In our case, $w/d = 10.0$ and $\epsilon = \epsilon_0\epsilon_r = 8.85 \times 10^{-12} \times 2.6 = 2.30 \times 10^{-11}$ F/m. We now find $L' = 4 \times \pi \times 10^{-7}/10.0 = 1.3 \times 10^{-7}$ H/m $= 130$ nH/m; $G' = 0$; and $C' = 2.30 \times 10^{-11} \times 10.0 = 2.3 \times 10^{-10}$ F/m $= 230$ pF/m. We next have $\alpha + j\beta = \sqrt{(R' + j\omega L')(j\omega C')} = 0.0001248 + j16.89$, so that $\alpha = 1.2 \times 10^{-4}$ m^{-1} and $\beta = 17$ m^{-1}. We have $Z_0 = \sqrt{(R' + j\omega L')/(j\omega C')} = 23 - j0.00017$ Ω.

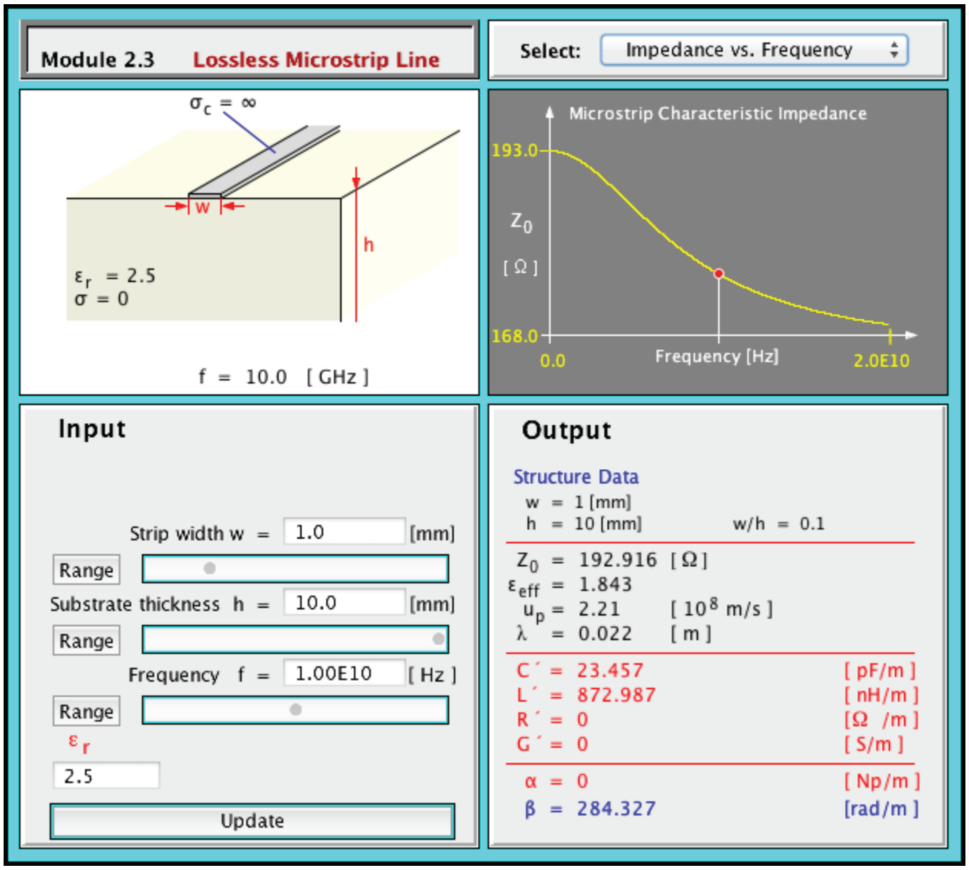
3. From Ulaby, et al.'s Table 2-1 (reproduced on my slide 3.7), we have $Z_0 = \sqrt{L'/C'} = (1/2\pi)\sqrt{\mu_0/\epsilon}\ln(b/a)$, where a is the inner radius and b is the outer radius. We note that the normalizing impedance is given by $Z_N = (1/\pi)\sqrt{\mu_0/\epsilon} = (1/2\pi)\sqrt{\mu_0/(\epsilon_r\epsilon_0)} = 40.0$ Ω to three significant figures. We now have $b/a = \exp(Z_0/Z_N) = 3.49$. Since the inner radius is given by $a = 0.500$ mm, we conclude that the outer radius is given by $b = 1.75$ mm and the outer diameter is given by $2b = 3.49$ mm. We have $L' = (\mu_0\epsilon)^{1/2}Z_0 = 2.50 \times 10^{-7}$ H/m $= 250$ nH/m. We have $C' = L'/Z_0^2 = 1.00 \times 10^{-10}$ F/m $= 100$ pF/m. The output from Ulaby's module 2.2 follows. Note that I chose a large, but non-zero conductivity for the outer radius, which is what the module requires.

ULABY CD MODULE OUTPUT:



4. Our basic inputs to the empirical formulae in Ulaby, et al.'s equations (2.36)–(2.41) are $\epsilon_r = 2.5$ and $s = w/h = 0.100$. I used MATLAB to calculate all quantities; so, there will be no accumulation of numerical inaccuracy. I am reporting the answers to three significant figures for comparison to the results of Ulaby, et al. From Eq. (2.38), we obtain $x = 0.526$ and $y = 0.833$. Using Eq. (2.36), we then find $\epsilon_{\rm eff} = 1.85$. From Eq. (2.40), we obtain $t = 73.3$. Using this result in Eq. (2.39), we obtain $Z_0 = 193\ \Omega$. At 10 GHz, we find that $\omega = 6.28 \times 10^{10}\ \mathrm{s}^{-1}$, so that, using Eq. (2.41f), $\beta = 2.85 \times 10^2\ \mathrm{m}^{-1}$. These answers agree with the answers at the back of the book. (However, the book also report five significant figures — an absurd number, given the imprecision evident in their heights, widths, etc.) I show the screen shot on the next page.

ULABY CD MODULE OUTPUT:



5. Our starting point is the equations on slide 3.23, which in Ulaby's notation becomes

$$V(0,t) = \frac{Z_0}{R_g + Z_0}\big[V_g(t) + (1+\Gamma_g)\Gamma_L V_g(t-2T) + (1+\Gamma_g)(\Gamma_g\Gamma_L)\Gamma_L V_S(t-4T) \\ + (1+\Gamma_g)(\Gamma_g\Gamma_L)^2\Gamma_L V_g(t-6T) + \cdots\big],$$

$$V(l,t) = \frac{Z_0}{R_g + Z_0}(1+\Gamma_L)\big[V_g(t-T) + \Gamma_g\Gamma_L V_g(t-3T) \\ + (\Gamma_g\Gamma_L)^2 V_g(t-5T) + \cdots\big].$$

This result is equivalent to Ulaby's Eq. (2.156), where the voltage is evaluated at different points along the transmission line.

a. We begin with the equation for $V(0,t)$. We write

$$V(0,t) = \frac{Z_0 V_g}{R_g + Z_0}\left\{1 + (1+\Gamma_g)\Gamma_L \sum_{m=0}^{m_g-1} (\Gamma_g\Gamma_L)^m\right\},$$

where the sum is defined as zero when $m_g - 1 < 0$. Using the definition of the floor function, we find $m_g = 0$ when $0 \le t < 2T$, $m_g = 1$ when $2T \le t < 4T$, and so on. We now find that the sum is zero when $0 \le t < 2T$; it equals 1 when $2T \le t < 4T$; it equals $1 + \Gamma_{\rm L}\Gamma_{\rm g}$ when $4T \le t < 6T$; it equals $1 + \Gamma_{\rm L}\Gamma_{\rm g} + (\Gamma_{\rm L}\Gamma_{\rm g})^2$ when $6T \le t < 8T$; and so on. Hence, this expression for $V(0,t)$ is equivalent to the original expression. Using the expression for the sum of a geometric series, we now find

$$V(0,t) = \frac{Z_0 V_{\rm g}}{R_{\rm g} + Z_0}\left[1 + (1+\Gamma_{\rm g})\Gamma_{\rm L}\frac{1 - (\Gamma_{\rm g}\Gamma_{\rm L})^{m_{\rm g}}}{1 - \Gamma_{\rm g}\Gamma_{\rm L}}\right].$$

Rewriting this expression, we have

$$V(0,t) = \frac{Z_0 V_{\rm g}}{R_{\rm g} + Z_0}\left[\frac{(1 - \Gamma_{\rm g}\Gamma_{\rm L}) + (\Gamma_{\rm L} + \Gamma_{\rm g}\Gamma_{\rm L})\big[1 - (\Gamma_{\rm g}\Gamma_{\rm L})^{m_{\rm g}}\big]}{1 - \Gamma_{\rm g}\Gamma_{\rm L}}\right].$$

Finally, collecting terms, we obtain

$$V(0,t) = \frac{Z_0 V_{\rm g}}{R_{\rm g} + Z_0}\left[\frac{1 - (\Gamma_{\rm g}\Gamma_{\rm L})^{m_{\rm g}+1} + \Gamma_{\rm L}\big[1 - (\Gamma_{\rm g}\Gamma_{\rm L})^{m_{\rm g}}\big]}{1 - \Gamma_{\rm g}\Gamma_{\rm L}}\right],$$

which is the desired expression. It is even more straightforward to obtain the expression for $V(l,t)$, since we may write

$$V(l,t) = \frac{Z_0 V_{\rm g}}{R_{\rm g} + Z_0}(1 + \Gamma_{\rm L})\sum_{m=0}^{m_{\rm L}-1}(\Gamma_{\rm g}\Gamma_{\rm L})^m,$$

which, using the expression for the sum of a geometric series, becomes

$$V(l,t) = \frac{Z_0 V_{\rm g}}{R_{\rm g} + Z_0}\left[\frac{(1 + \Gamma_{\rm L})\big[1 - (\Gamma_{\rm g}\Gamma_{\rm L})^{m_{\rm L}}\big]}{1 - \Gamma_{\rm g}\Gamma_{\rm L}}\right],$$

which is the desired expression. Since the reflection coefficients for the current are the same as for the voltage, except that the sign changes, we can obtain the expressions for $I(0,t)$ and $I(l,t)$ by taking the expressions for $V(0,t)$ and $V(l,t)$, replacing $\Gamma_{\rm L}$ and $\Gamma_{\rm g}$ by $-\Gamma_{\rm L}$ and $-\Gamma_{\rm g}$ and dividing through by Z_0.

b. In this system, we have $\Gamma_{\rm L} = 1/3 = 0.333$, $\Gamma_{\rm g} = -1$, $V_{\rm g} = 30$ V, $Z_0 = 50\ \Omega$, $R_{\rm L} = 0$, and $T = 2\ \mu$s. We are interested in times up to 16 μs. After substitution into the expression for $V(l,t)$, we find that

$$V(l,t) = 30\big[1 - (-1/3)^{m_{\rm L}}\big]\ ({\rm V})$$

We note that $(-1/3)^{m_{\rm L}} = 1$ $(0 \le t < 2\ \mu{\rm s})$, $(-1/3)^{m_{\rm L}} = -1/3 = 0.333$ $(2 \le t < 6\ \mu{\rm s})$, $(-1/3)^{m_{\rm L}} = 1/9 = 0.111$ $(6 \le t < 10\ \mu{\rm s})$, $(-1/3)^{m_{\rm L}} = -1/27 = 0.0370$ $(10 \le t < 14\ \mu{\rm s})$, and, finally, $(-1/3)^{m_{\rm L}} = 1/81 = 0.0123$ $(14 \le t < 18\ \mu{\rm s})$, which takes us outside the range of interest. In the limit as t and hence $m_{\rm L} \to \infty$, the expression for $V(0,t)$ tends to 30 V. At times up to 16 μs, we find, using the values for $(-1/3)^{m_\mathrm{L}}$ that $V(0,t) = 0$ V ($0 \le t < 2$ μs), $V(0,t) = 40$ V ($2 \le t < 6$ μs), $V(0,t) = 26.7$ V ($6 \le t < 10$ μs), $V(0,t) = 31.1$ V ($10 \le t < 14$ μs), and $V(0,t) = 29.6$ V ($14 \le t < 18$ μs). Substituting into the expression for the current, we find

$$I(0,t) = 0.6\left[\frac{1}{2} + \frac{1}{2}(-1/3)^{m_\mathrm{g}}\right] \text{ (A)}.$$

We find that $I(0,t) \to 0.3$ A as $t \to \infty$. Substituting the appropriate values for m_g, we find that the current in the example on slide 3.29 is reproduced.

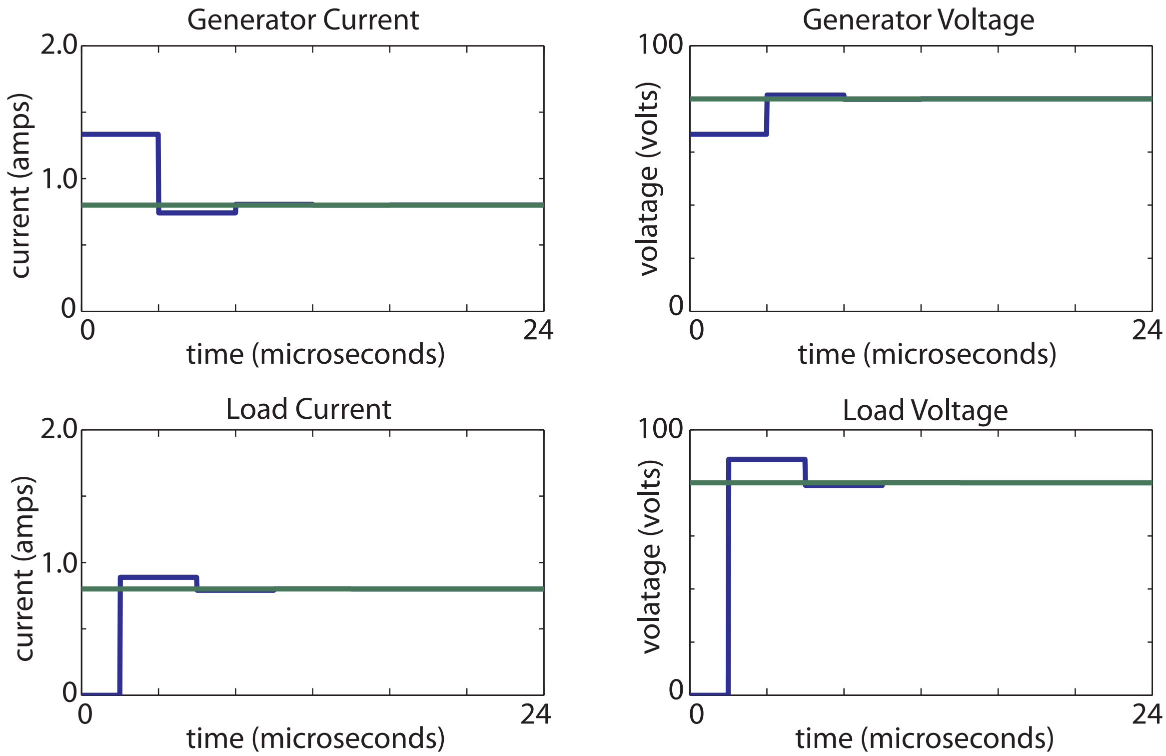
6. This problem is close, but not identical to the example on slides 3.29 and 3.30. We have $T = 400/2 \times 10^8 = 2$ μs. Hence, we have $12T = 24$ μs. Note that we are going out to a longer time than in the related example.

   a. Since all the impedances are real, we have $R' = G' = 0$. We have $L' = Z_0/u_\mathrm{p} = 250$ nH and $C' = 1/Z_0 u_\mathrm{p} = 100$ pF.

   b. We have $V_\mathrm{fac} = V_\mathrm{g} Z_0/(R_\mathrm{g}+Z_0) = 66.7$ V, and we have $I_\mathrm{fac} = V_C/Z_0 = 1.33$ A. We also have $\Gamma_\mathrm{g} = (R_\mathrm{g} - Z_0)/(R_\mathrm{g} + Z_0) = -0.333$ and $\Gamma_\mathrm{L} = (R_\mathrm{L} - Z_0)/(R_\mathrm{L} + Z_0) = 0.500$. It follows that $\Gamma_\mathrm{g}\Gamma_\mathrm{L} = -0.333$. We have $0 \le m_\mathrm{g} \le 6$ and $0 \le m_\mathrm{L} \le 6$. We then find $(\Gamma_\mathrm{g}\Gamma_\mathrm{L})^m$, $m = 0$–6 is given by [1.00, -0.111, 1.23×10^{-2}, -1.37×10^{-3}, 1.52×10^{-4}, -1.639×10^{-5}, 1.88×10^{-6}]. After substitution into the expressions for $V(0,t)$, $I(0,t)$, $V(l,t)$, and $I(l,t)$, we obtain $V(0,t) = [67, 81, 80, 80, 80, 80]$ V and $I(0,t) = [1.3, 0.74, 0.81, 0.80, 0.80, 0.80]$ A, where the vector elements ($m = 1$–7) refer respectively to the times $(m-1)(4\ \mu\mathrm{s}) \le t < m(4\ \mu\mathrm{s})$. We similarly, obtain $V(l,t) = [0.0, 89, 79, 80, 80.0, 80.0, 80.0]$ V and $I(l,t) = [0.0, 0.89, 0.79, 0.80, 0.80, 0.80, 0.80]$ A, where the first vector element applies to the time $0 \le t < 2$ μs, and the remaining vector elements ($m = 2$–7) apply to times $(2m-3)(2\ \mu\mathrm{s}) \le t < (2m-1)(2\ \mu\mathrm{s})$. Note the very rapid convergence of all quantities to their final values. Substituting into the expressions for the asymptotic values, we find $V(l,t) \to 80$ V and $I(l,t) \to 0.80$ A, which is consistent with the time evolution that we found.

   c. The MATLAB output and code follow on the next two pages.

MATLAB OUTPUT PLOTS:



MATLAB CODE:

```
% Transmission_Line_1_mod
%
% This routine calculates the transient response of a transmission
% line.  SI units are used.  It has been modified from
% Transmission_Line_1 for use in solving Problem 2.6.

% LINE PARAMETERS
length = 400;              %transmission line length
Z_0 = 50;                  %characteristic impedance
velocity = 2e8;            %propagation velocity

% GENERATOR AND LOAD PARAMETERS
Z_g = 25;                  %input impedance
V_g = 100;                 %input voltage
Z_L = 100;                 %load impedance

Total_Time = 24e-6;        %total time for the plot

Delta = 0:0.001:1.0;
Cmult = ones(size(Delta));
time = Delta*Total_Time;   %the time axis is cut into increments
T = length/velocity;       %the transit time is calculated

% Time-independent coefficients
Gamma_g = (Z_g - Z_0)/(Z_g + Z_0);   % generator reflection coefficient
Gamma_L = (Z_L - Z_0)/(Z_L + Z_0);   % load reflection coefficient
I_fac = V_g/(Z_g + Z_0);             % current factor
V_fac = Z_0*I_fac;                   % voltage factor
C_f = Gamma_g*Gamma_L;               % concatenation coefficent

% Time-dependent coefficients
M_g = floor(0.5*time/T);             % bounce number at the generator
M_L = floor(0.5*(time+T)/T);         % bounce number at the load
C_vec_g = C_f.^M_g;                  % generator concatenation vector
C_vec_L = C_f.^M_L;                  % load concatenation vector

% Time-dependent load and generator currents and voltages
I_vec_g = I_fac*(1.0 - C_f*C_vec_g ...
    - Gamma_L*(1.0 - C_vec_g))./(1.0 - C_f);  % generator current
V_vec_g = V_fac*(1.0 - C_f*C_vec_g ...
    + Gamma_L*(1.0 - C_vec_g))./(1.0 - C_f);  % generator voltage
I_vec_L = I_fac*((1.0 - Gamma_L)*(1.0 - C_vec_L))./(1.0 - C_f);
                                              % load current
V_vec_L = V_fac*((1.0 + Gamma_L)*(1.0 - C_vec_L))./(1.0 - C_f);
                                              % load voltage
% Asymptotic values
I_asym_g = I_fac*Cmult*(1.0 - Gamma_L)/(1.0 - C_f);
V_asym_g = V_fac*Cmult*(1.0 + Gamma_L)/(1.0 - C_f);
I_asym_L = I_fac*Cmult*(1.0 - Gamma_L)/(1.0 - C_f);
V_asym_L = V_fac*Cmult*(1.0 + Gamma_L)/(1.0 - C_f);

% Plotting
subplot(2,2,1), plot(time,I_vec_g,time,I_asym_g,'LineWidth',2);
    title('Generator Current'), xlabel('time (sec)'),
    ylabel ('current (amps)'), axis([0.0, Total_Time, 0.0, 2.0])
subplot(2,2,2), plot(time,V_vec_g,time,V_asym_g,'LineWidth',2);
    title('Generator Voltage'), xlabel('time (sec)'),
    ylabel ('voltage (volts)'), axis([0.0, Total_Time, 0.0, 100.0])
subplot(2,2,3), plot(time,I_vec_L,time,I_asym_L,'LineWidth',2);
    title('Load Current'), xlabel('time (sec)'),
    ylabel ('current (amps)'), axis([0.0, Total_Time, 0.0, 2.0])
subplot(2,2,4), plot(time,V_vec_L,time,V_asym_L,'LineWidth',2);
    title('Load Voltage'), xlabel('time (sec)'),
    ylabel ('voltage (volts)'), axis([0.0, Total_Time, 0.0, 100.0])
```