CMPE 320: Probability, Statistics, and Random Processes

Lecture 12: Independence of RVs

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Independence of a RV and an event

- "Events A and B are independent" means observing event B does not provide any information on A

$P(A|B) = P(A)$

$P(A \cap B) = P(A)P(B)$

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

- "RV X is independent of event A" means observing event A does not provide any information on the value of X

$\equiv$ Event $\{X = x\}$ and A are independent for all x

$P(X = x | A) = P(X = x)$

$$\boxed{P_{X|A}(x) = P_X(x) \quad \text{for all } x}$$

Example 2.19. Consider two independent tosses of a fair coin. Let X be the number of heads and let A be the event that the number of heads is even.

Are X and A independent?

Independence of X and A $\Leftrightarrow$ $P_{X|A}(x) = P_X(x)$

$$P_X(x) = \begin{cases} \frac{1}{4}, & x=0 \quad TT \\ 2\cdot\frac{1}{4}=\frac{1}{2}, & x=1 \quad TH, HT \\ \frac{1}{4} & x=2 \quad HH \end{cases}$$

$$P_{X|A}(x) = \frac{P(X=x \cap A)}{P(A)} = \begin{cases} \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}, & x=0 \\ 0, & x=1 \\ \frac{1}{2}, & x=2 \end{cases}$$

$P(A) = \frac{1}{2}$

$P_X(x) \neq P_{X|A}(x) \Rightarrow X$ and A are <u>not</u> independent.

Independence of RVs

- "RVs X and Y are independent" means that the value of Y provides no information on the value of X

$\equiv$ Events $\{X=x\}$ and $\{Y=y\}$ are independent for all x, y

$P(X=x$ and $Y=y) = P(X=x)\,P(Y=y)$ for all x, y

$\boxed{P_{X,Y}(x,y) = P_X(x)\,P_Y(y)}$ for all x, y

Since $P_{X,Y}(x,y) = P_{X|Y}(x|y)\,P_Y(y)$

$\boxed{P_{X|Y}(x|y) = P_X(x)}$ for all x, and y with $P_Y(y) > 0$

Conditional independence of RVs

- Conditioning with an event A defines a new universe where all probabilities (or PMFs) are replaced by their conditional versions

X and Y are conditionally independent given A if

$P(X=x, Y=y \mid A) = P(X=x \mid A)\, P(Y=y \mid A)$ for all x, y

$\boxed{p_{X,Y|A}(x,y) = p_{X|A}(x)\; p_{Y|A}(y)}$ for all x, y

Since $p_{X,Y|A}(x,y) = p_{X|Y,A}(x|y)\, p_{Y|A}(y)$

$\Rightarrow \boxed{p_{X|Y,A}(x|y) = p_{X|A}(x)}$ Also, $p_{Y|X,A}(y|x) = p_{Y|A}(y)$

Conditional independence may not imply independence, and vice versa

① Are X and Y independent?

Check $p_{X,Y}(x,y) = p_X(x)\, p_Y(y)$ all x, y

$$p_X(x) = \begin{cases} \frac{3}{20}, & x=1 \\ \frac{6}{20}, & x=2 \\ \frac{6}{20}, & x=3 \\ \frac{3}{20}, & x=4 \end{cases} \qquad p_Y(y) = \begin{cases} \frac{1}{20}, & y=1 \\ \frac{5}{20}, & y=2 \\ \frac{9}{20}, & y=3 \\ \frac{5}{20}, & y=4 \end{cases}$$

$p_{X,Y}(x,y)$

y \ x	1	2	3	4
4	1/20	2/20	2/20	0
3	2/20	4/20	1/20	2/20
2	0	1/20	3/20	1/20
1	0	1/20	0	0

$0 = p_{X,Y}(1,1) \neq p_X(1)\, p_Y(1) = \frac{3}{20} \cdot \frac{1}{20}$

$\therefore$ X and Y are not independent

② Are X and Y conditionally independent conditioned on $A = \{X \le 2, Y \ge 3\}$?

Check if $P_{X|Y,A}(x|y) = P_{X|A}(x)$ for all x, y with $P_Y(y) > 0$

$$P_{X|A}(x) = \frac{P(X=x \cap A)}{P(A)} \quad P(A) = \frac{9}{20}$$
$$= \begin{cases} \frac{3}{20}\Big/\frac{9}{20}, & x=1 \\ \frac{6}{20}\Big/\frac{9}{20}, & x=2 \end{cases} = \begin{cases} \frac{1}{3}, & x=1 \\ \frac{2}{3}, & x=2 \end{cases}$$

$$P_{X|Y,A}(x|\underline{y=3}) = \frac{P(X=x, Y=3, A)}{P(Y=3, A)} \quad P(Y=3, A) = \frac{6}{20}$$
$$= \begin{cases} \frac{2}{20}\Big/\frac{6}{20}, & x=1 \\ \frac{4}{20}\Big/\frac{6}{20}, & x=2 \end{cases} = \begin{cases} \frac{1}{3}, & x=1 \\ \frac{2}{3}, & x=2 \end{cases}$$

$$P_{X|Y,A}(x|y=4) = \begin{cases} \frac{1}{3}, & x=1 \\ \frac{2}{3}, & x=2 \end{cases}$$

Expectation of the product of independent RVs

- If X and Y are independent RVs, then $\boxed{E[XY] = E[X]\,E[Y]}$

Why? $E[XY] = \sum_x \sum_y xy\, P_{X,Y}(x,y)$

$= \sum_x \sum_y xy\, P_X(x) P_Y(y)$ (independence of X and Y)

$= \sum_x x P_X(x) \sum_y y P_Y(y) = E[X]\,E[Y]$

- Similarly E[g(X) h(Y)] = $E[g(X)]\,E[h(Y)]$

Why? If X and Y are independent, so are $g(X)$ and $h(Y)$

$\mathrm{var}(X) = E[X^2] - (E[X])^2$

Variance of the sum of independent RVs

- If X and Y are independent RVs, $\boxed{\mathrm{var}(X + Y) = \mathrm{var}(X) + \mathrm{var}(Y)}$

Why? Let $\tilde{X} = X - E[X]$, $\tilde{Y} = Y - E[Y]$

$E[\tilde{X}] = E[X - \underbrace{E[X]}] = E[X] - E[X] = 0$

$\tilde{X}$ and $\tilde{Y}$ are zero-mean and independent.

$\mathrm{var}(X+Y) = \mathrm{var}(\tilde{X} + \tilde{Y})$ [since $\mathrm{var}(X+a) = \mathrm{var}(X)$]

$= E[(\tilde{X}+\tilde{Y})^2] - (\underbrace{E[\tilde{X}+\tilde{Y}]}_{=0})^2$

$= E[\tilde{X}^2 + 2\tilde{X}\tilde{Y} + \tilde{Y}^2] = E[\tilde{X}^2] + 2E[\tilde{X}\tilde{Y}] + E[\tilde{Y}^2]$

$= E[\tilde{X}^2] + 2\underbrace{E[\tilde{X}]}_{=0}\underbrace{E[\tilde{Y}]}_{=0} + E[\tilde{Y}^2] = \mathrm{var}(\tilde{X}) + \mathrm{var}(\tilde{Y})$

$= \mathrm{var}(X) + \mathrm{var}(Y)$

Independence of several RVs

- X, Y, Z are independent RVs if

$$p_{X,Y,Z}(x,y,z) = p_X(x)\,p_Y(y)\,p_Z(z) \quad \text{for all } x, y, z$$

- If X, Y, Z are independent, f(X), g(Y), h(Z) are also independent
  - How about g(X,Y) and h(Z)? Independent
  - How about g(X,Y) and h(Y,Z)? Not necessarily independent

Variance of sum of several independent RVs

- If $X_1, X_2, \ldots, X_n$ are independent RVs,
  var$(X_1 + \underbrace{X_2 + \cdots + X_n}_{Y_2})$ = $var(X_1) + var(X_2) + \cdots + var(X_n)$

Why? $var(X_1 + \underbrace{X_2 + \cdots + X_n}_{Y_2})$ X_1 and Y_2 are indep.

$= var(X_1) + var(X_2 + \cdots + X_n)$

$= \cdots$

$= var(X_1) + var(X_2) + \cdots + var(X_n)$

Example 2.20. Variance of the Binomial and the Poisson. We consider n independent coin tosses, with each toss having probability p of coming up a head. For each i, we let X_i be the Bernoulli random variable which is equal to 1 if the ith toss comes up a head, and is 0 otherwise. Then, $X = X_1 + X_2 + \cdots + X_n$ is a binomial random variable. What are its mean and variance? of X?

$E[X] = E[X_1 + X_2 + \cdots + X_n] = E[X_1] + E[X_2] + \cdots + E[X_n]$

$E[\cdot]$ is linear

$E[X_i] = 1 \cdot p + 0 \cdot (1-p) = p \Rightarrow E[X] = np$

$var(X) = var(X_1 + X_2 + \cdots + X_n) = var(X_1) + var(X_2) + \cdots + var(X_n)$

X_i are independent, not true otherwise

$var(X_i) = E[X_i^2] - E[X_i]^2 = 1^2 \cdot p + 0^2 \cdot (1-p) - p^2 = p - p^2 = p(1-p)$

$\Rightarrow var(X) = np(1-p)$

Example 2.21. Mean and Variance of the Sample Mean. We wish to estimate the approval rating of a president, to be called B. To this end, we ask n persons drawn at random from the voter population, and we let X_i be a random variable that encodes the response of the ith person:

$$X_i = \begin{cases} 1, & \text{if the } i\text{th person approves B's performance,} \\ 0, & \text{if the } i\text{th person disapproves B's performance.} \end{cases}$$

We model $X_1, X_2, \ldots, X_n$ as independent Bernoulli random variables with common mean p and variance $p(1-p)$. Naturally, we view p as the true approval rating of B. We "average" the responses and compute the **sample mean** S_n, defined as

$$S_n = \frac{X_1 + X_2 + \cdots + X_n}{n}.$$

Thus, the random variable S_n is the approval rating of B within our n-person sample. What are the mean and variance of S_n?

$$E[S_n] = E\left[\frac{X_1+X_2+\cdots+X_n}{n}\right] = \frac{1}{n}E[X_1+\cdots+X_n] = \frac{1}{n}\left(E[X_1]+E[X_2]+\cdots+E[X_n]\right)$$

$$= p$$

$$\mathrm{var}(S_n) = \mathrm{var}\left(\frac{X_1+X_2+\cdots+X_n}{n}\right) = \frac{1}{n^2}\mathrm{var}(X_1+\cdots+X_n) \qquad \mathrm{var}(aX) = a^2\mathrm{var}(X)$$

$$= \frac{1}{n^2}\left(\mathrm{var}(X_1)+\cdots+\mathrm{var}(X_n)\right) = \frac{np(1-p)}{n^2}$$

$$= \frac{p(1-p)}{n} \rightarrow 0 \quad \text{as} \quad n\rightarrow\infty$$