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**MATH 407** HW 02

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**A.1** Let A, B, C be subsets of a given set S. Prove the following statements.

**10** $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

**Ans** Let $x \in (A - B) \cup (B - A)$.
Therefore, $x \in (A - B)$ or $x \in (B - A)$.

If $x \in (A - B)$, then $x \in A, x \notin B$.
Therefore:

$$x \in (A \cup B),\ x \notin (A \cap B)$$
$$\Rightarrow x \in (A \cup B) - (A \cap B)$$

Similarly, if $x \in (B - A)$, then $x \in (A \cup B) - (A \cap B)$.
Therefore, $(A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B)$

Conversely, if $x \in (A \cup B) - (A \cap B)$, then $x \in (A \cup B),\ x \notin (A \cap B)$
If $x \in (A \cup B)$, then $x \in A$ or $x \in B$.
If $x \in A$, then $x \notin B$.
Therefore:

$$x \in A,\ x \notin B \Rightarrow x \in (A - B)$$
$$\Rightarrow x \in (A - B) \cup (B - A)$$

Similary, if $x \in B$, then $x \in (A - B) \cup (B - A)$.
Therefore, $(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A)$

$\therefore (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$ $\square$

**11** $(A \cup B) \times C = (A \times C) \cup (B \times C)$

**Ans** Let $(x, y) \in (A \cup B) \times C$.

Then:

$$
\begin{aligned}
& x \in (A \cup B),\ y \in C \\
& \Rightarrow (x \in A,\ y \in C) \ \text{ or } \ (x \in B,\ y \in C) \\
& \Leftrightarrow ((x,y) \in A \times C) \ \text{ or } \ ((x,y) \in B \times C) \\
& \Leftrightarrow (x,y) \in (A \times C) \cup (B \times C)
\end{aligned}
$$

$\therefore (A \cup B) \times C = (A \times C) \cup (B \times C)$ $\square$

**A.4** **9** Let $a_1, \ldots, a_n$ be positive real numbers, $G_n = \sqrt[n]{a_1 a_2 \ldots a_n}$, and $A_n = \frac{1}{n}\sum_{i=1}^{n} a_i$. Then $G_n$ is called the geometric mean and $A_n$ is called the arithmetic mean. We wish to show that $G_n \leq A_n$.

1. Show that $G_2 \leq A_2$.

**Ans** Substituting $n = 2$:

$$
\begin{aligned}
G_2 &= \sqrt[2]{a_1 a_2 \ldots a_2} \\
&= \sqrt[2]{a_1 a_2} \\
&= \sqrt{a_1 a_2} \\
A_2 &= \frac{1}{2}\sum_{i=1}^{2} a_i \\
&= \frac{1}{2}(a_1 + a_2)
\end{aligned}
$$

If $a_1 = a_2$, then $G_2 = a_1 = A_2 = a_1$.

If $a_1 \neq a_2$, then let $a_1 = k, a_2 = nk + 1$, $k \in \mathbb{R}^+$.
Then, $G_2 = \sqrt{(k)(k+1)}$, $G_2^2 = k^2 + k$
and $A_2 = \frac{1}{2}(2k+1)$, $A_2^2 = k^2 + k + \frac{1}{4}$

$\therefore G_2 \leq A_2$ $\square$

2. Show that $G_{2^n} \leq A_{2^n}$ by using induction on $n$.

**Ans** It was proven the proposition held for $n = 2^k$ for $k = 1 \Rightarrow 2$.

Suppose the proposition holds for $n = 2^k,$ for $k > 1$. Therefore:

$$
\begin{aligned}
A_{2^k} &= \frac{1}{2^k}\sum_{i=1}^{2^k} a_i \\
&= \frac{1}{2^k}(a_1 + a_2 + \ldots + a_{2^k}) \\
&= \frac{\frac{1}{2^{k-1}}(a_1 + a_2 + \ldots + a_{2^{k-1}}) + \frac{1}{2^{k-1}}(a_{2^{k-1}+1} + a_{2^{k-1}+2} + \ldots + a_{2^k})}{2} \\
&\geq \frac{\sqrt[2^{k-1}]{a_1 + a_2 + \ldots + a_{2^{k-1}}} + \sqrt[2^{k-1}]{a_{2^{k-1}+1} + a_{2^{k-1}+2} + \ldots + a_{2^k}}}{2} \\
&\geq \sqrt{\sqrt[2^{k-1}]{a_1 + a_2 + \ldots + a_{2^{k-1}}} + \sqrt[2^{k-1}]{a_{2^{k-1}+1} + a_{2^{k-1}+2} + \ldots + a_{2^k}}} \\
&\geq \sqrt[2^k]{a_1 a_2 \ldots a_{2^k}} \\
&= G_{2^k}
\end{aligned}
$$

$\therefore G_{2^k} \leq A_{2^k}$ □

3. Show that $G_n \leq A_n$. *Hint:* Let $m$ be such that $2^m \geq n$, and set $a_{n+1}$ = $a_{n+2}$ = $a_{2^m}$ = $A_n$ and apply part (2).

**Ans** □

**10** Let $a$ and $b$ be real numbers. Prove the binomial theorem , which states that

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i} \text{ where } \binom{n}{i} = \frac{n!}{i!(n-i)!}$$

and $n!$ = $n(n-1)\ldots 2 \cdot 1$ for $n \geq 1$ and $0! = 1$.

*Hint:* $\binom{m+1}{k} = \binom{m}{k} + \binom{m}{k-1}$.

**Ans** □

**11** Find a formula for the derivative of the product of $n$ functions, and give a detailed proof by induction (assuming the product rule for the derivative of two functions).

**Ans** □

**12** Find a formula for the $n$th derivative of the product of two functions, and give a detailed proof by induction.

**Ans** □