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CMPE 320: HW 08

1. Let X have a uniform distribution in the unit interval $[0, 1]$, and let Y have an exponential distribution with parameter $\nu = 2$. Assume that X and Y are independent. Let $Z = X + Y$.

(a) Find $P(Y \geq X)$.

Sol. Since X and Y are independent, $f_{X,Y}(x,y) = f_X(x)g_Y(y)$

Therefore, the joint PDF,

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-2y}, & \text{if } 0 \leq x \leq 1, y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$\begin{aligned} P(Y \geq X) &= 1 - P(X \leq Y) \\ &= \iint_{y \geq x} f_{X,Y}(x,y) \, dx \, dy \\ &= 1 - \int_0^1 \int_0^x 2e^{-2y} \, dx \, dy \\ &= 1 - \int_0^1 1 - e^{-2x} \, dx \\ &= \frac{1}{2} - \frac{e^{-2}}{2} \quad \square \end{aligned}$$

(b) Find the conditional PDF of Z given that $Y = y$.

Sol.

$$\begin{aligned} f_{Z|Y=y}(z) &= f_{X+Y|Y=y}(x+y) \\ &= \begin{cases} 1, & \text{if } y \leq z \leq 1+y \\ 0, & \text{otherwise,} \end{cases} \quad \square \end{aligned}$$

(c) Find the conditional PDF of Y given that $Z = 3$.

Sol. With the laws of conditional probability,

$$f_{Y|3}(y | 3) = \frac{f_{Y,Z}(y, 3)}{f_Z(3)} = \frac{f_{Z|Y=y}(3 | y)f_Y(y)}{f_Z(3)}$$

And,

$$\begin{aligned} F_Z(3) &= \int_0^1 \int_0^{z-x} f_{X,Y}(x, y) dx dy \\ &= \int_0^1 \int_0^{z-x} 2e^{-2y} dx dy \\ &= 1 - \frac{e^{-2(3)+2}}{2} + \frac{e^{-2(3)}}{2} \\ &= e^{-4} - e^{-6} \end{aligned}$$

Therefore,

$$f_{Y|3}(y | 3) = \begin{cases} \frac{2e^{-2y}}{e^{-4} - e^{-6}}, & \text{if } 2 \leq y \leq 3 \\ 0, & \text{otherwise,} \end{cases} \quad \square$$

2. Let P , a random variable which is uniformly distributed between 0 and 1. On any given day, a particular machine is functional with probability P . Furthermore, given the value of P , the status of the machine on different days is independent

- (a) Find the probability that the machine is functional on a particular day.

Sol. Let W represent the event that the machine is functional

Then,

$$\begin{aligned} P(W) &= \int_0^1 P(W | X = x) f_X(x) dx \\ &= \int_0^1 x dx \\ &= \frac{1}{2} \end{aligned} \quad \square$$

- (b) We are told that the machine was functional on m out of the last n days. Find the conditional PDF of P . You may use the identity

$$\int_0^1 p^k (1-p)^{n-k} dp = \frac{k!(n-k)!}{(n+1)!}$$

Sol.

$$\begin{aligned}
 P(W_m) &= \int_0^1 P(W_m | X = x) f_X(x) dx \\
 &= \int_0^1 \binom{n}{m} x^m (1-x)^{n-m} f_X(x) dx \\
 &= \binom{n}{m} \frac{m!(n-m)!}{(n+1)!}
 \end{aligned}$$

Therefore, using Bayes rule,

$$\begin{aligned}
 f_{X|W_m}(x) &= \frac{P(W_m | X = x) f_X(x)}{P(W_m)} \\
 &= \frac{x^m (1-x)^{n-m}}{\frac{m!(n-m)!}{(n+1)!}}, \quad 0 \leq x \leq 1, \quad n \geq m
 \end{aligned}
 \quad \square$$

- (c) Find the conditional probability that the machine is functional today given that it was functional on m out of the last n days.

Sol.

$$\begin{aligned}
 P(W_m) &= \int_0^1 P(W_m | X = x) f_X(x) dx \\
 &= \int_0^1 \binom{n}{m} x^m (1-x)^{n-m} f_X(x) dx \\
 &= \binom{n}{m} \frac{m!(n-m)!}{(n+1)!}
 \end{aligned}
 \quad \square$$

3. Let $B \triangleq \{a < X \leq b\}$. Derive a general expression for $E[X | B]$ if X is a continuous RV. Let $X : N(0, 1)$ with $B = \{-1 < X \leq 2\}$. Compute $E[X | B]$.

Sol. \square

4. A particular model of an HDTV is manufactured in three different plants, say, A , B and C , of the same company. Because the workers at A , B and C are not equally experienced, the quality of the units differs from plant to plant. The PDF's of the time-to-failure X , in years, are

$$\begin{aligned}
 f_X(x) &= \frac{1}{5} \exp(-x/5) u(x) \text{ for } A \\
 f_X(x) &= \frac{1}{6.5} \exp(-x/6.5) u(x) \text{ for } B
 \end{aligned}$$

$$f_X(x) = \frac{1}{10} \exp(-x/10)u(x) \text{ for } C,$$

where $u(x)$ is the unit step. Plant A produces three times as many units as B, which produces twice as many as C. The TVs are all sent to a central warehouse, intermingled, and shipped to retail stores all around the country. What is the expected lifetime of a unit purchased at random?

Sol. The expectations of the exponential distributions, $1/\lambda$,

$$E[A] = 5$$

$$E[B] = 6.5$$

$$E[C] = 10$$

Given the ratio of the units is 6 : 2 : 1,

$$P(A) = \frac{6}{9}$$

$$P(B) = \frac{2}{9}$$

$$P(C) = \frac{1}{9}$$

Therefore, the expected lifetime of a unit purchased at random,

$$\begin{aligned} E &= \frac{5 \times 6}{9} + \frac{6.5 \times 2}{9} + \frac{10 \times 1}{9} \\ &= \frac{53}{6} \end{aligned}$$

□

5. The coordinate X and Y of a point are independent zero mean normal random variables with common variances σ^2 . Given that the point is at a distance of at least c from the origin, find the conditional joint PDF of X and Y .

Sol. Given,

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-x^2}{2\sigma^2}\right)$$

$$f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-y^2}{2\sigma^2}\right)$$

Since X and Y are assumed independent

$$f_{(X,Y)}(x,y) = f_X(x)f_Y(y) = \frac{1}{2\pi\sigma^2} \exp\left(\frac{-(x^2+y^2)}{2\sigma^2}\right)$$

Therefore,

$$\begin{aligned}
 P(x^2 + y^2 \geq c^2) &= \int f_{X,Y}(x,y) \, dx \, dy \\
 &= \frac{1}{2\pi\sigma^2} \int_c^\infty \exp\left(\frac{-r^2}{2\sigma^2}\right) 2\pi r \, dr
 \end{aligned}
 \quad \square$$

6. Alexei is vacationing in Monte Carlo. The amount X (in dollars) he takes to the casino each evening is a random variable with a PDF of the form

$$f_X(x) = \begin{cases} ax, & \text{if } 0 \leq x \leq 40, \\ 0, & \text{otherwise} \end{cases}$$

At the end of each night, the amount Y that he has when leaving the casino is uniformly distributed between zero and twice the amount that he came with.

- (a) Determine the joint PDF $f_{X,Y}(x,y)$.

Sol. Since

$$\begin{aligned}
 \int_0^{40} ax \, dx &= 1 \\
 a \frac{40^2}{2} &= 1 \\
 \implies a &= \frac{1}{800}
 \end{aligned}$$

The uniform distribution,

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{2x}, & \text{if } 0 \leq y \leq 2x, \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$\begin{aligned}
 f_{X,Y}(x,y) &= \frac{f_{Y|X}(y|x)}{f_X(x)} \\
 &= \begin{cases} \frac{ax}{2x}, & \text{if } 0 \leq x \leq 40, 0 \leq y \leq 2x, \\ 0, & \text{otherwise} \end{cases} \\
 &= \begin{cases} \frac{a}{2}, & \text{if } 0 \leq x \leq 40, 0 \leq y \leq 2x, \\ 0, & \text{otherwise} \end{cases} \\
 &= \begin{cases} \frac{1}{1600}, & \text{if } 0 \leq x \leq 40, 0 \leq y \leq 2x, \\ 0, & \text{otherwise} \end{cases}
 \end{aligned}
 \quad \square$$

(b) What is the probability that on a given night Alexei makes a positive profit at the casino?

Sol. Since the probability of loss is $0 \leq y \leq x$, and the probability of profit is $x \leq y \leq 2x$,

$$P(\text{profit}) = P(y - x > 0) = \frac{1}{2}$$

□

(c) Find the PDF of Alexei's profit $Y - X$ on a particular night, and also determine its expected value.

Sol.

$$P(x - y < m) = \begin{cases} \int_m^{40} \frac{x+m}{2x} f(x) dx, & \text{if } t > 0, \\ \int_{-m}^{40} \frac{x-m}{2x} f(x) dx, & \text{otherwise} \end{cases}$$

Since $f(x) = ax$,

$$\begin{aligned} P(x - y < m) &= \begin{cases} \int_m^{40} \frac{x+m}{2} a dx, & \text{if } t > 0, \\ \int_{-m}^{40} \frac{x-m}{2} a dx, & \text{otherwise} \end{cases} \\ &= \begin{cases} \int_m^{40} \frac{x+m}{2} a dx, & \text{if } t > 0, \\ \int_{-m}^{40} \frac{x-m}{2} a dx, & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{1}{1600} \left(\frac{x^2}{2} + mx \right) \Big|_m^{40}, & \text{if } t > 0, \\ \frac{1}{1600} \left(\frac{x^2}{2} - mx \right) \Big|_{-m}^{40}, & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{1}{1600} \left(800 - \frac{3m^2}{2} + 40m \right), & \text{if } t > 0, \\ \frac{1}{1600} \left(800 + \frac{m^2}{2} - 40m \right), & \text{otherwise} \end{cases} \end{aligned}$$

Therefore, the PDF

$$= \begin{cases} \frac{1}{1600} (40 - 3m), & \text{if } t > 0, \\ \frac{1}{1600} (m - 40), & \text{otherwise} \end{cases}$$

And the expected value,

$$\begin{aligned}
&= \frac{1}{1600} \left(\int_0^{40} (40 - 3m)m \, dm + \int_{-40}^0 (m - 40)m \, dm \right) \\
&= \frac{1}{1600} \left((20m^2 - m^3) \Big|_0^{40} + \left(\frac{m^3}{3} - 20m^2 \right) \Big|_{-40}^0 \right) \\
&= \frac{1}{1600} \left(-32000 + \frac{160000}{3} \right) \\
&= \frac{40}{3}
\end{aligned}$$

□

7. Consider a communication channel corrupted by noise. Let X be the value of the transmitted signal and Y be the value of the received signal. Assume that the conditional density of Y given $\{X = x\}$ is Gaussian, that is,

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}\sigma^2} \exp\left(-\frac{(y-x)^2}{2\sigma^2}\right),$$

and X is uniformly distributed on $[-1, 1]$. What is the conditional pdf of X given Y , that is, $f_{X|Y}(x|y)$

Sol. Given,

$$f_X(x) = \begin{cases} \frac{1}{2}, & \text{if } -1 \leq x \leq 1 \\ 0, & \text{otherwise,} \end{cases}$$

And,

$$\begin{aligned}
f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dx \\
&= \int_{-\infty}^{\infty} f_{Y|X}(y|x)f_X(x) \, dx \\
&= \int_{-1}^1 \frac{1}{2} \frac{1}{\sqrt{2\pi}\sigma^2} \exp\left(-\frac{(y-x)^2}{2\sigma^2}\right) \, dx \\
&= \frac{1}{2} \left(\phi\left(\frac{y-1}{\sigma}\right) - \phi\left(\frac{y+1}{\sigma}\right) \right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
f_{X|Y}(x|y) &= \frac{f_{Y|X}(y|x)f_X(x)}{f_Y(y)} \\
&= \frac{\frac{1}{\sqrt{2\pi}\sigma^2} \exp\left(-\frac{(y-x)^2}{2\sigma^2}\right) f_X(x)}{\phi\left(\frac{y-1}{\sigma}\right) - \phi\left(\frac{y+1}{\sigma}\right)}
\end{aligned}$$

□

8. A U.S. defense radar scans the skies for unidentified flying objects (UFOs). Let M be the event that a UFO is present and M^C the event that a UFO is absent. Let $f_{X|M}(x|M) = \frac{1}{\sqrt{2\pi}} \exp(-0.5(x-r)^2)$ be the conditional PDF of the radar return signal X when a UFO is actually there, and let $f_{X|M}(x|M^C) = \frac{1}{\sqrt{2\pi}} \exp(-0.5x^2)$ be the conditional PDF of the radar return signal X when there is no UFO. To be specific, let $r = 1$ and let the *alert level* be $x_A = 0.5$. Let A denote the event of an alert, that is, $\{X > x_A\}$. Compute $P(A|M)$, $P(A^C|M)$, $P(A|M^C)$, $P(A^C|M^C)$. Assume that $P(M) = 10^{-3}$. Compute $P(A|M)$, $P(A^C|M)$, $P(A|M^C)$, $P(A^C|M^C)$. Repeat for $P(M) = 10^{-6}$.

Sol. Given, $r = 1$,

$$f_{X|M}(x|M) = \frac{1}{\sqrt{2\pi}} \exp(-0.5(x-1)^2)$$

$$f_{X|M}(x|M^C) = \frac{1}{\sqrt{2\pi}} \exp(-0.5x^2)$$

□