Sabbir Ahmed

DATE: April 8, 2018

CMPE 320: HW 06

1. At his workplace, the first thing Oscar does every morning is to go to the supply room and pick up one, two, or three pens with equal probability 1/3. If he picks up three pens, he does not return to the supply room again that day. If he picks up one or two pens, he will make one additional trip to the supply room, where he again will pick up one, two, or three pens with equal probability 1/3. (The number of pens taken in one trip will not affect the number of pens taken in any other trip.)

   a. The probability that Oscar gets a total of three pens on any particular day.

   The probability that he gets 3 pens on his first trip

$$P_1 = \frac{1}{3}$$

   The probability that he gets 2 pens on his first trip and 1 pen on his second

$$P_2 = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$$

   The probability that he gets 1 pen on his first trip and 2 pens on his second

$$P_3 = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$$

   The probability that he gets 3 pens on any particular day

$$P(\text{3 pens}) = \frac{1}{3} + \frac{1}{9} + \frac{1}{9} = \frac{5}{9} \qquad \square$$

   b. The conditional probability that he visited the supply room twice on a given day, given that it is a day in which he got a total of three pens.

   The probability that he makes two trips

$$\begin{aligned} P(\text{two trips} \mid \text{3 pens}) &= \frac{P(\text{two trips} \cap \text{3 pens})}{P(\text{3 pens})} \\ &= \frac{P_2 + P_3}{P(\text{3 pens})} \\ &= \frac{\frac{1}{9} + \frac{1}{9}}{\frac{5}{9}} \end{aligned}$$

$$= \frac{2}{5} \qquad \square$$

c. $\mathbf{E}[N]$ and $\mathbf{E}[N \mid C]$, where $\mathbf{E}[N]$ is the unconditional expectation of N, the total number of pens Oscar gets on any given day, and $\mathbf{E}[N \mid C]$ is the conditional expectation of N given the event $C = \{N > 3\}$.

$$\begin{aligned}
\mathbf{E}[N] &= 2P(\text{2 pens}) + 3P(\text{3 pens}) + 4P(\text{4 pens}) + 5P(\text{5 pens}) \\
&= 2P_2 + 3P(\text{3 pens}) + 4(P_1P_1P_1P_1) + 5(P_1P_1) \\
&= 2 \cdot \frac{1}{9} + 3 \cdot \frac{5}{9} + 4 \cdot \left(\frac{1}{9} \cdot \frac{1}{9}\right) + 5 \cdot \left(\frac{1}{3} \cdot \frac{1}{3}\right) \\
&= \frac{30}{9}
\end{aligned}$$

$$\begin{aligned}
\mathbf{E}[N \mid C] &= \mathbf{E}[N \mid \{\text{4 pens, 5 pens}\}] \\
&= 4 \cdot \frac{P(\text{4 pens})}{P(\text{4 pens}) + P(\text{5 pens})} + 5 \cdot \frac{P(\text{5 pens})}{P(\text{4 pens}) + P(\text{5 pens})} \\
&= 4 \cdot \frac{\frac{1}{9} + \frac{1}{9}}{\frac{1}{9} + \frac{1}{9} + \frac{1}{3} \cdot \frac{1}{3}} + 5 \cdot \frac{\frac{1}{3} \cdot \frac{1}{3}}{\frac{1}{9} + \frac{1}{9} + \frac{1}{3} \cdot \frac{1}{3}} \\
&= \frac{13}{3} \qquad \square
\end{aligned}$$

d. $\sigma_{N|C}$, the conditional standard deviation of the total number of pens Oscar gets on a particular day, where N and C are as in part (c).

Since $\mathbf{E}[N \mid C] = 13/3$,
$\mathbf{E}[N^2 \mid C]$

$$\begin{aligned}
\mathbf{E}[N^2 \mid C] &= \mathbf{E}[N^2 \mid \{\text{4 pens, 5 pens}\}] \\
&= 4^2 \cdot \frac{P(\text{4 pens})}{P(\text{4 pens}) + P(\text{5 pens})} + 5^2 \cdot \frac{P(\text{5 pens})}{P(\text{4 pens}) + P(\text{5 pens})} \\
&= 4^2 \cdot \frac{\frac{1}{9} + \frac{1}{9}}{\frac{1}{9} + \frac{1}{9} + \frac{1}{3} \cdot \frac{1}{3}} + 5^2 \cdot \frac{\frac{1}{3} \cdot \frac{1}{3}}{\frac{1}{9} + \frac{1}{9} + \frac{1}{3} \cdot \frac{1}{3}}
\end{aligned}$$

$$= 19$$

Therefore, $\sigma_{N|C}$

$$\begin{aligned}
\sigma_{N|C} &= \sqrt{\mathbf{E}[N^2 \mid C] - (\mathbf{E}[N^2 \mid C])^2} \\
&= \sqrt{19 - \left(\frac{13}{3}\right)^2} \\
&= \frac{\sqrt{2}}{3}
\end{aligned}$$

□

e. The probability that he gets more than three pens on each of the next 16 days.

$$\begin{aligned}
P(N > 3)^{16} &= \left(\frac{1}{3}\right)^{16} \\
&= \frac{1}{3^{16}}
\end{aligned}$$

□

f. The conditional standard deviation of the total number of pens he gets in the next 16 days given that he gets more than three pens on each of those days.

$$\begin{aligned}
\sigma_{N_1,N_2,\ldots,N_{16}|C} &= \sqrt{16 \cdot (\mathbf{E}[N^2 \mid C] - (\mathbf{E}[N^2 \mid C])^2)} \\
&= \sqrt{16 \cdot \left(19 - \left(\frac{13}{3}\right)^2\right)} \\
&= \frac{4\sqrt{2}}{3}
\end{aligned}$$

□

2. Your computer has been acting very strangely lately, and you suspect that it might have a virus on it. Unfortunately, all 12 of the different virus detection programs you own are outdated. You know that if your computer does have a virus, each of the programs, independently of the others, has a 0.8 chance of believing that your computer as infected, and a 0.2 chance of thinking your computer is fine. On the other hand, if your computer does not have a virus, each program has a 0.9 chance of believing that your computer is fine, and a 0.1 chance of wrongly thinking your computer is infected. Given that your computer has a 0.65 chance of being infected with some virus, and given that you will believe your virus protection programs only if 9 or more of them agree, find the probability that your detection programs will lead you to the right answer.

Let V denote the computer has a virus, and D denote the detection programs detected it Therefore, given

$$\begin{aligned} P(D \mid V) &= 0.8 \\ P(D \mid V^C) &= 0.9 \\ P(V) &= 0.65 \\ P(V^C) &= 0.35 \end{aligned}$$

To find $P(D)$,

$$\begin{aligned} P(V^C \cap D) &= P(D \mid V^C) \cdot P(V^C) \\ &= 0.9 \cdot 0.35 \\ &= 0.315 \\ P(V \cap D) &= P(D \mid V) \cdot P(V) \\ &= 0.8 \cdot 0.65 \\ &= 0.52 \\ P(D) &= P(V \cap D) + P(V^C \cap D) \\ &= 0.835 \end{aligned}$$

Therefore, $P(D) = 0.835$ and $P(D^C) = 0.165$

The probability that 9 or more of the 12 computers detect the virus

$$\begin{aligned} P(X \geq 9) &= \sum_{i=9}^{12} \binom{12}{i} P(D)^i P(D^C)^{12-i} \\ &= 0.87835 \end{aligned}$$

□

3. Joe Lucky plays the lottery on any given week with probability p, independently of whether he played on any other week. Each time he plays, he has a probability q of winning, again independently of everything else. During a fixed time period of n weeks, let X be the number of weeks that he played the lottery and Y be the number of weeks that he won.

a. What is the probability that he played the lottery on any particular week, given that he did not win on that week?

Let L_i represent the event that Joe played the lottery,
and W_i represent the event that he won on week i
Therefore,

$$P(L_i \mid W_i^c) = \frac{P(W_i^c \mid L_i)P(L_i)}{P(W_i^c \mid L_i)P(L_i) + P(W_i^c \mid L_i^c)P(L_i^c)}$$

$$= \frac{(1-q)p}{(1-q)p + (1-p)} \qquad \square$$

b. Find the conditional PMF $p_{Y|X}(y \mid x)$.

The binomial PMF:

$$p_{Y|X}(y \mid x) = \begin{cases} \binom{x}{y} q^y (1-q)^{x-y} & \text{, if } 0 \le y \le x, \\ 0 & \text{, otherwise} \end{cases} \qquad \square$$

c. Find the joint PMF $p_{X,Y}(x, y)$.

Since $p_{X,Y}(x, y) = p_{Y|X}(y \mid x)p_X(x)$

$$p_{Y|X}(y \mid x) = \begin{cases} \binom{x}{y} q^y (1-q)^{x-y} \binom{n}{x} p^x (1-p)^{n-x} & \text{, if } 0 \le y \le x \le n, \\ 0 & \text{, otherwise} \end{cases} \qquad \square$$

d. Find the marginal PMF $p_Y(y)$.

Hint: One possibility is to start with the answer to part (c), but the algebra can be messy. However, if you think intuitively about the procedure that generates Y, you may be able to guess the answer.

Let Y_i be a Bernoulli random variable that takes 1 when Joe wins, and 0 otherwise

$$p_{Y_i}(y) = \begin{cases} pq & \text{, if } y = 1, \\ 1 - pq & \text{, otherwise} \end{cases}$$

Therefore, the marginal PMF

$$p_Y(y) = \begin{cases} \binom{n}{y}(pq)^y(1-(pq))^{n-y} & \text{, if } 0 \le y \le n, \\ 0 & \text{, otherwise} \end{cases} \qquad \square$$

e. Find the conditional PMF $p_{X|Y}(x \mid y)$. Do this algebraically using the preceding answers.

Since $p_{X|Y}(x \mid y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$

$$p_{X|Y}(x \mid y) = \begin{cases} \frac{\binom{x}{y}q^y(1-q)^{x-y}\binom{n}{x}p^x(1-p)^{n-x}}{\binom{n}{y}(pq)^y(1-(pq))^{n-y}} & \text{, if } 0 \le y \le x \le n, \\ 0 & \text{, otherwise} \end{cases} \qquad \square$$

f. Rederive the answer to part (e) by thinking as follows: for each one of the $n-Y$ weeks that he did not win, the answer to part (a) should tell you something.

For each of the $n-Y$ weeks where he did not win, there were $x-y$ weeks where Joe he played
Therefore,

$$P(L_i \mid W_i^C) = \frac{p-pq}{1-pq}$$

with the PMF

$$p_{X|Y}(x \mid y) = \begin{cases} \binom{n-y}{x-y}\left(\frac{p-pq}{1-pq}\right)^{x-y}\left(1-\frac{p-pq}{1-pq}\right)^{n-x} & \text{, if } 0 \le y \le x \le n, \\ 0 & \text{, otherwise} \end{cases}$$

Which is equivalent to the answer in part (e) $\square$

4. The runner-up in a road race is given a reward that depends on the difference between his time and the winner's time. He is given 10 dollars for being one minute behind, 6 dollars for being one to three minutes behind, 2 dollars for being 3 to 6 minutes behind, and nothing otherwise. Given that the difference between his time and the winner's time is uniformly distributed between 0 and 12 minutes, find the mean and variance of the reward of the runner-up.

For being one minute behind,

$$P(X=1) = \frac{1-0}{12-0}$$

$$= \frac{1}{12}$$

For being one to three minutes behind,

$$P(1 < X \leq 3) = \frac{3-0}{12-0} - \frac{1-0}{12-0}$$

$$= \frac{1}{6}$$

For being three to six minutes behind,

$$P(3 < X < 6) = \frac{6-0}{12-0} - \frac{3-0}{12-0}$$

$$= \frac{1}{4}$$

To find the mean and variance,

y	$P(y)$	$yP(y)$	$y^2P(y)$
10	$\frac{1}{12}$	$\frac{5}{6}$	$\frac{25}{3}$
6	$\frac{1}{6}$	1	6
2	$\frac{1}{4}$	$\frac{1}{2}$	1

Therefore,

$$\mathbf{E}[Y] = \sum yP(y)$$

$$= \frac{5}{6} + 1 + \frac{1}{2}$$

$$= \frac{7}{3}$$

And

$$var(Y) = \sum y^2P(y) - \left(\sum yP(y)\right)^2$$

$$= \frac{25}{3} + 6 + 1 - \left(\frac{7}{3}\right)^2$$

$$= \frac{89}{9} \qquad \square$$

5. Let X be a random variable with PDF

$$f_X(x) = \begin{cases} 2x/3 & \text{, if } 1 < x \leq 2, \\ 0 & \text{, otherwise,} \end{cases}$$

and let $Y = X^2$. Calculate $\mathbf{E}[Y]$ and $var(Y)$.

To compute $\mathbf{E}[Y]$

$$\begin{aligned} \mathbf{E}[Y] &= \mathbf{E}[X^2] \\ &= \int_1^2 x^2 \cdot \frac{2x}{3} dx \\ &= \frac{2}{3} \cdot \frac{x^4}{4} \bigg|_1^2 \\ &= \frac{5}{2} \end{aligned}$$

And

$$\begin{aligned} \mathbf{E}[Y^2] &= \mathbf{E}[X^4] \\ &= \int_1^2 x^4 \cdot \frac{2x}{3} dx \\ &= \frac{2}{3} \cdot \frac{x^6}{6} \bigg|_1^2 \\ &= 7 \end{aligned}$$

Therefore, $var(Y)$

$$\begin{aligned} var(Y) &= \mathbf{E}[Y^2] - (\mathbf{E}[Y])^2 \\ &= 7 - \left(\frac{5}{2}\right)^2 \\ &= \frac{3}{4} \qquad \square \end{aligned}$$

6. The pdf of a RV X is shown in Figure 6. The numbers in parentheses indicate area. Compute the value of A.

$$\text{area at } x = 2 \text{ to } x = 3 \text{ + area under } Ae^{-x} \Rightarrow \int_1^4 f_X(x)dx = 1$$

$$\frac{1}{4} + \frac{1}{4} + \int_1^4 Ae^{-x}dx = 1$$

$$\int_1^4 Ae^{-x}dx = \frac{1}{2}$$

$$-Ae^{-x}\Big|_1^4 = \frac{1}{2}$$

$$A(e^{-1} - e^{-4}) = \frac{1}{2}$$

$$\Rightarrow A = \frac{e^4}{2(e^3 - 1)} \qquad \square$$

7. Find the PDF, the mean, and the variance of the random variable X with CDF

$$F_X(x) = \begin{cases} 1 - \frac{a^3}{x^3} & \text{, if } x \geq a, \\ 0 & \text{, if } x < a, \end{cases}$$

where a is a positive constant.

Since the PDF is $\frac{d}{dx}F_X(x)$

$$\begin{aligned} f_X(x) &= \frac{d}{dx}F_X(x) \\ &= \begin{cases} \frac{d}{dx}(1 - \frac{a^3}{x^3}) = \frac{3a^3}{x^4} & \text{, if } x \geq a, \\ 0 & \text{, if } x < a, \end{cases} \end{aligned}$$

Therefore, the mean

$$\begin{aligned} \mathbf{E}[N] &= \int_a^\infty nf(n)dn \\ &= \int_a^\infty \frac{3a^3}{n^3}dn \\ &= 3a^3\left(\frac{-1}{2n^2}\right)\Big|_a^\infty \end{aligned}$$

$$= \frac{3a}{2}$$

And the variance

$$\begin{aligned} var(N) &= \int_a^\infty n^2 f(n) dn - (\mathbf{E}[N])^2 \\ &= \int_a^\infty \frac{3a^3}{n^2} dn - \left(\frac{3a}{2}\right)^2 \\ &= 3a^3 \left(\frac{-1}{n}\right)\Big|_a^\infty - \frac{9a^2}{4} \\ &= \frac{3a^2}{4} \end{aligned}$$ □

8. You are allowed to take a certain test three times, and your final score will be the maximum of the test scores. Your score in test i, where i = 1, 2, 3, takes one of the values from i to 10 with equal probability $1/(11-i)$, independently of the scores in other tests. What is the PMF of the final score?

Let X_i be the score of the ith test, and $m = max\{X_1, X_2, X_3\}$
By independence,

$$\begin{aligned} P(m \le x) &= P(X_1 \le x)P(X_2 \le x)P(X_3 \le x) \\ &= \frac{x-2}{8} \cdot \frac{x-1}{9} \cdot \frac{x}{10}, \quad 3 \le x \le 10 \end{aligned}$$

Therefore, the PMF

$$\begin{aligned} P_X(x) &= P(m \le x) - P(m \le x-1) \\ &= \frac{x-2}{8} \cdot \frac{x-1}{9} \cdot \frac{x}{10} - \frac{x-3}{8} \cdot \frac{x-2}{9} \cdot \frac{x-1}{10} \\ &= \frac{(x-2)(x-1)}{240}, \quad 4 \le x \le 10 \end{aligned}$$ □

9. The **median** of a random variable X is a number μ that satisfies $F_X(\mu) = 1/2$. Find the median of the exponential random variable with parameter λ.

Let m = median

$$P(X < m) = \int_0^m e^{-\lambda x} dx$$

$$= (-\lambda e^x)|_0^m$$

$$\Rightarrow 1 - \lambda e^{-m} = 0.5$$

$$\Rightarrow m = \ln(2)\lambda \qquad \square$$