Problem Set #3 Solutions

1. We first note from Ulaby et al.'s equation (2.29),

$$Z_0 = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} = \sqrt{\frac{L'}{C'}}\sqrt{\frac{R'/L' + j\omega}{G'/C' + j\omega}} = \sqrt{L'/C'},$$

where we note that for a distortionless line, we have $R'/L' = G'/C'$. We also have the general expression for γ,

$$\gamma = [(R' + j\omega L')(G' + j\omega C')]^{1/2} = [R'G' + j\omega(L'G' + R'C') - \omega^2 L'C']^{1/2}.$$

For a distortionless line, we have $L'G' + R'C' = 2R'C' = 2L'G' = 2\sqrt{L'G'}\sqrt{R'C'}$. Substituting this last relation into the expression for γ, we have

$$\alpha + j\beta = \gamma = \left[R'G' + 2j\omega\sqrt{R'G'}\sqrt{L'C'} + L'C'\right]^{1/2} = \sqrt{R'G'} + j\omega\sqrt{L'C'}.$$

We now conclude $\alpha = \sqrt{R'G'} = R'\sqrt{C'/L'}$ and $\beta = \omega\sqrt{L'C'}$.

2. According to Ulaby et al.'s Eq. 2.73, we have

$$S = \frac{1 + |\Gamma|}{1 - |\Gamma|}, \quad \text{which implies} \quad |\Gamma| = \frac{S - 1}{S + 1}.$$

In our case, $S = 1.5/0.8 = 1.875$, so that $|\Gamma| = 0.7/2.3 = 0.304$.

3. All the answers, with the exception of part (e) can be found in Ulaby's module 2.1B – 2.4B in the 2001–2007 editions. They can also be obtained using module 2.4 in the 2010 edition.

   a. We have $\omega = 2\pi f = 3\pi \times 10^9\ \mathrm{s}^{-1} = 9.42 \times 10^9\ \mathrm{s}^{-1}$. Since we are assuming $\mu = \mu_0$, we have $\epsilon_r = c^2/v_\mathrm{p}^2 = (3 \times 10^8/1.5 \times 10^8)^2 = 4$. We have $z_\mathrm{L} = Z_\mathrm{L}/Z_0 = (25 - 25j)/50 = 0.5 - 0.5j$.

   b. We have $\Gamma = (Z_\mathrm{L} - Z_0)/(Z_\mathrm{L} + Z_0) = -0.2 - 0.4j = 0.447\exp(-j2.034)$. Hence, we have $|\Gamma| = 0.447$ and $\theta_\mathrm{r} = -2.034$ rads $= -117°$. We also have $S = (1 + |\Gamma|)/(1 - |\Gamma|) = 2.62$.

   To calculate l_max and l_min, we first note that $\lambda = v_\mathrm{p}/f = 1.5\times 10^8/1.5\times 10^9 = 0.1$ m $= 10$ cm. We also note, for future reference that $\beta = 2\pi/\lambda = 20\pi\ \mathrm{m}^{-1}$. Maxima occur when $-z = (\theta_\mathrm{r}\lambda/4\pi) + (n\lambda/2) > 0$, or when $(-0.162 + 0.5n)\lambda > 0$. The first maximum occurs when $n = 1$, at which point $l_\mathrm{max} = 3.38$ cm. Since $l_\mathrm{max} > 0.25\lambda = 2.50$ cm, we have $l_\mathrm{min} = l_\mathrm{max} - \lambda/4 = 0.88$ cm.

c. To determine V_{max} and V_{min}, we first find $\beta l = 2\pi/\lambda = 2\pi \times 24/10 = 15.1$ We must calculate Z_{in} using Ulaby's Eq. (2.61),

$$Z_{\text{in}} = Z_0 \frac{\exp(j\beta l) + \Gamma \exp(-j\beta l)}{\exp(j\beta l) - \Gamma \exp(-j\beta l)}.$$

We find $Z_{\text{in}} = 71.1 - j55.8$. We then calculate V_0^+ using Ulaby's Eq. (2.66)

$$V_0^+ = \left(\frac{\tilde{V}_g Z_{\text{in}}}{Z_g + Z_{\text{in}}} \right) \left(\frac{1}{\exp(j\beta l) + \Gamma \exp(-j\beta l)} \right) = -4.05 - j2.94 \text{ V}.$$

We then find that $|V_0^+| = 5.00$ and that $\theta_{\text{r}} = -144^\circ$. From here, we find $V_{\text{max}} = |V_0^+|(1 + |\Gamma|) = 7.24$ V and $V_{\text{min}} = |V_0^+|(1 - |\Gamma|) = 2.74$ V.

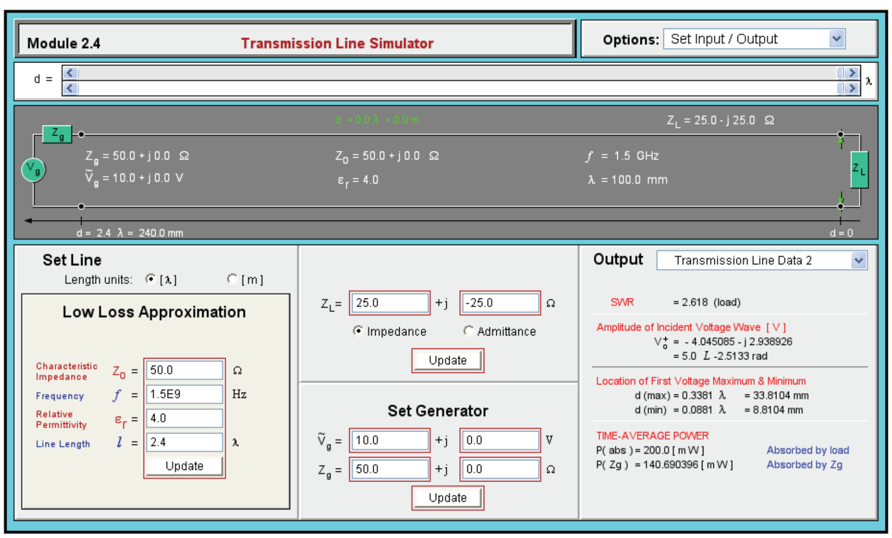
d. The general expression for $v(z,t)$ has the form

$$v(z,t) = A\cos(\omega t - \beta z + \pi_1) + B\cos(\omega_t + \beta z + \phi_2),$$

and we already showed that $\omega = 3\pi \times 10^9 \text{ s}^{-1}$ and that $\beta = 20\pi \text{ m}^{-1}$. Since we must have $A = |V_0^+|$ and $\phi_1 = \theta_{\text{r}}$, we have $A = 5$ V and $\phi_1 = -144^\circ$. We have $V_0^- = \Gamma V_0^+ = -0.367 + j2.21 = 2.24\exp(j1.74) = 2.24\angle 99.4^\circ$, from which we conclude that $B = 2.24$ and $\phi_2 = 99.4^\circ$.

e. The output from Ulaby et al.'s module 2.4 follows:

ULABY CD MODULE OUTPUT:



f. Writing $V_0^+ = |V_0^+|\exp(j\phi^+)$, we find

$$\tilde{V}(z) = |V_0^+| \left[\exp(-j\beta z + \phi^+) + |\Gamma| \exp(j\beta z + \theta_{\text{r}} + \phi^+) \right].$$

Writing $\tilde{V}(z) = |\tilde{V}(z)| \exp[j\phi(z)]$, we find first

$$
\begin{aligned}
|V(z)| &= \Big\{ \big[\exp(-j\beta z + \phi^+) + |\Gamma| \exp(j\beta z + \theta_\mathrm{r} + \phi^+)\big] \\
&\qquad \times \big[\exp(+j\beta z - \phi^+) + |\Gamma| \exp(-j\beta z - \theta_\mathrm{r} - \phi^+)\big] \Big\}^{1/2} \\
&= \big[1 + |\Gamma|^2 + \exp(2j\beta z + \theta_\mathrm{r}) + \exp(-2j\beta z - \theta_\mathrm{r})\big]^{1/2} \\
&= \big[1 + |\Gamma|^2 + 2|\Gamma| \cos(2\beta z + \theta_\mathrm{r})\big]^{1/2}
\end{aligned}
$$

We also have

$$\phi(z) = \tan^{-1}\left\{ \frac{\mathrm{Im}[\tilde{V}(z)]}{\mathrm{Re}[\tilde{V}(z)]} \right\} = \tan^{-1}\left\{ -\frac{\sin(\beta z - \phi^+) - |\Gamma| \sin(\beta z + \theta_\mathrm{r} + \phi^+)}{\cos(\beta z - \phi^+) + |\Gamma| \cos(\beta z + \theta_\mathrm{r} + \phi^+)} \right\}.$$

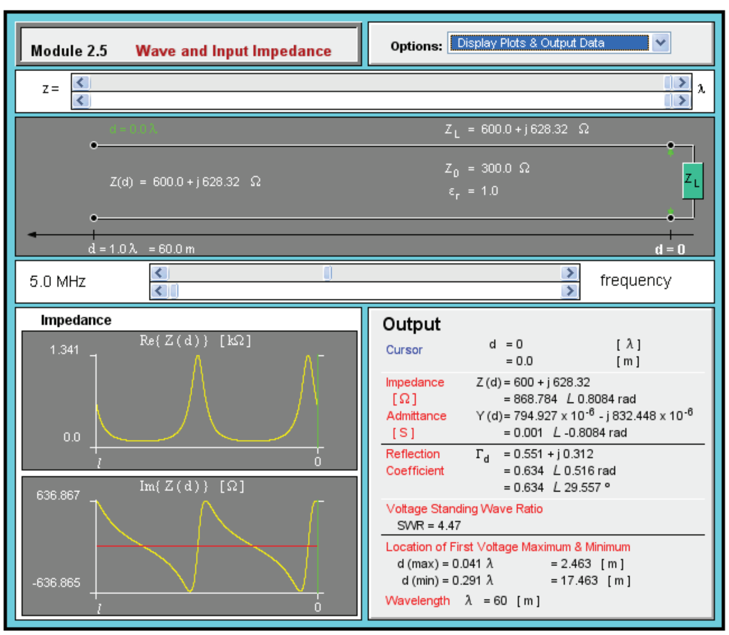
We finally have

$$v(z,t) = \mathrm{Re}\left\{ |\tilde{V}(z)| \exp[j\phi(z)] \exp(j\omega t) \right\} = |\tilde{V}(z)| \cos[j\omega t + \phi(z)],$$

which is the result on slide 4.14.

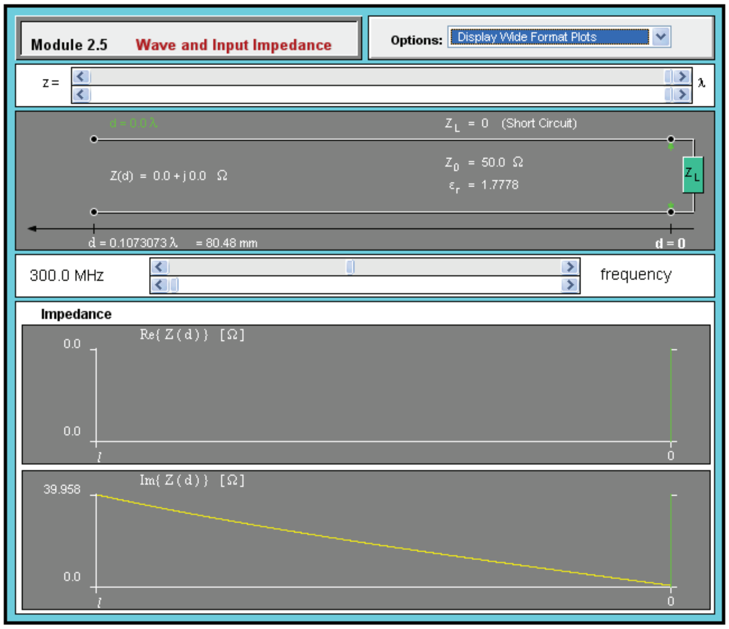
4. The load impedance is given by $Z_\mathrm{L} = R + j\omega L = 600 + j6.28 \times (5.00 \times 10^6) \times (2.00 \times 10^{-5}) = 600 + j628\ \Omega$. We also have $\lambda = c/f = (3.00 \times 10^8)/(5.00 \times 10^6) = 60.0$ m. We now find (a) $\Gamma = (Z_\mathrm{L} - Z_0)/(Z_\mathrm{L} + Z_0) = 0.55 + j0.31 = 0.63 \exp(j0.52)$ and (b) $S = (1 + |\Gamma|)/(1 - |\Gamma|) = 4.5$. (c) Since $\theta_\mathrm{r} > 0$, the first voltage maximum is given by $d_\mathrm{max} = \theta_\mathrm{r}\lambda/4\pi = (0.516 \times 60.0)/12.6 = 2.5$ m. (d) The first current maximum is given by the first voltage minimum, which is given by $d_\mathrm{min} = d_\mathrm{max} + \lambda/4 = 17$ m. The output from Ulaby et al.'s module 2.5 follows:

ULABY CD MODULE OUTPUT:



5. For a short-circuited line, we have $Z_{\rm in}^{\rm sc} = jZ_0 \tan \beta l$. (See Ulaby et al., Eq. 2.84.) Since $Z_0 = 50\ \Omega$, to obtain a reactance of $40\ \Omega$, we must have $\tan \beta l = 0.800$, which implies that $\beta l = 0.675$. With 300 MHz and a velocity of 2.25×10^8 m/s, we find $\beta = 2\pi f / u_{\rm p} = 6.28 \times (3.00 \times 10^8)/(2.25 \times 10^8) = 8.38\ {\rm m}^{-1}$. It follows that $l = 0.675/8.38 = 0.080$ m or 8.0 cm. The output from Ulaby's module 2.5 follows:

ULABY CD MODULE OUTPUT:



6. We begin by recalling that

$$V_i(t) = \mathrm{Re}[\tilde{V}_i \exp(j\omega t)] = \frac{1}{2}\left[\tilde{V}_i \exp(j\omega t) + \tilde{V}_i^* \exp(-j\omega t)\right]$$

and

$$I_i(t) = \mathrm{Re}[\tilde{I}_i \exp(j\omega t)] = \frac{1}{2}\left[\tilde{I}_i \exp(j\omega t) + \tilde{I}_i^* \exp(-j\omega t)\right]$$

It follows that

$$V_{\rm i}(t)I_{\rm i}(t) = \frac{1}{4}[\tilde{V}_{\rm i}\tilde{I}_{\rm i} \exp(2j\omega t) + \tilde{V}_{\rm i}\tilde{I}_{\rm i}^* + \tilde{V}_{\rm i}^*\tilde{I}_{\rm i} + \tilde{V}_{\rm i}^*\tilde{I}_{\rm i}^* \exp(-2j\omega t)].$$

We now calculate

$$\begin{aligned} P_{\rm av}^{\rm i} &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} V_{\rm i}(t)I_{\rm i}(t)\,dt \\ &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \frac{1}{4}[\tilde{V}_{\rm i}\tilde{I}_{\rm i} \exp(2j\omega t) + \tilde{V}_{\rm i}\tilde{I}_{\rm i}^* + \tilde{V}_{\rm i}^*\tilde{I}_{\rm i} + \tilde{V}_{\rm i}^*\tilde{I}_{\rm i}^* \exp(-2j\omega t)]\,dt \\ &= \frac{1}{4}[\tilde{V}_{\rm i}\tilde{I}_{\rm i}^* + \tilde{V}_{\rm i}^*\tilde{I}_{\rm i}] = \frac{1}{2}\,\mathrm{Re}(V_{\rm i}I_{\rm i}^*). \end{aligned}$$

The proof for the reflected power is identical.

7. We begin by calculating the voltage reflection coefficient, $\Gamma = (Z_\mathrm{L} - Z_0)/(Z_\mathrm{L} + Z_0) = 25/125 = 0.200$. We also have $\exp(j\beta l) = \exp(j2\pi \times 0.15) = 0.588 + j0.809$.

   a. We now find

   $$Z_\mathrm{in} = Z_0 \frac{\exp(j\beta l) + \Gamma \exp(-j\beta l)}{\exp(j\beta l) - \Gamma \exp(-j\beta l)} = 41.3 - j16.3 = 44.4\angle - 21.2$$

   b. We have

   $$\tilde{V}_\mathrm{i} = \left(\frac{\tilde{V}_\mathrm{g} Z_\mathrm{in}}{Z_\mathrm{g} + Z_\mathrm{in}} \right) = 46.9 - j9.51 = 47.8 \exp(-j0.200) = 47.8\angle - 11.5^\circ \text{ V},$$

   and $\tilde{I}_\mathrm{i} = \tilde{V}_\mathrm{i}/Z_\mathrm{in} = 1.06 + j0.190 = 1.08 \exp(j0.177) = 1.08\angle 10.2^\circ$ A.

   c. We have $P_\mathrm{in} = 0.5\,\mathrm{Re}(V_\mathrm{i} I_\mathrm{i}^*) = 24.0$ W.

   d. We have

   $$\begin{aligned} V_0^+ &= V_\mathrm{i} \left(\frac{1}{\exp(j\beta l) + \Gamma \exp(-j\beta l)} \right) \\ &= 29.4 - j40.5 = 50.0 \exp(-j0.942) = 50.0\angle - 54.0^\circ \end{aligned}$$

   and $V_0^- = \Gamma V_0^+ = 0.200 V_0^+ = 10.0 \exp(-j0.942)$, so that $\tilde{V}_\mathrm{L} = 60.0 \exp(-j0.942)$. We have $\tilde{I}_\mathrm{L} = \tilde{V}_\mathrm{L}/Z_\mathrm{L} = 0.800 \exp(-j0.942)$. It follows that $P_\mathrm{L} = 0.5\,\mathrm{Re}(\tilde{V}_\mathrm{L} \tilde{I}_\mathrm{L}) = 24.0$ W. This value is the same as P_in and indicates that all the incoming power is dissipated in the load.

   e. The power delivered by the generator is $P_\mathrm{g} = 0.5\,\mathrm{Re}(\tilde{V}_\mathrm{g} \tilde{I}_\mathrm{i}^*) = 53.1$ W. The power that is dissipated in Z_g equals $0.5\,\mathrm{Re}(Z_\mathrm{g} I_\mathrm{i} I_\mathrm{i}^*) = 29.1$ W. The power that is dissipated in Z_g and the power that is dissipated in the load equals the power that is supplied by the generator.