HW#4 Solutions

Problem 1. Let X be the number of points the MIT team earns over the weekend. We have

$$\begin{aligned}
\mathbf{P}(X=0) &= 0.6 \cdot 0.3 = 0.18, \\
\mathbf{P}(X=1) &= 0.4 \cdot 0.5 \cdot 0.3 + 0.6 \cdot 0.5 \cdot 0.7 = 0.27, \\
\mathbf{P}(X=2) &= 0.4 \cdot 0.5 \cdot 0.3 + 0.6 \cdot 0.5 \cdot 0.7 + 0.4 \cdot 0.5 \cdot 0.7 \cdot 0.5 = 0.34, \\
\mathbf{P}(X=3) &= 0.4 \cdot 0.5 \cdot 0.7 \cdot 0.5 + 0.4 \cdot 0.5 \cdot 0.7 \cdot 0.5 = 0.14, \\
\mathbf{P}(X=4) &= 0.4 \cdot 0.5 \cdot 0.7 \cdot 0.5 = 0.07, \\
\mathbf{P}(X>4) &= 0.
\end{aligned}$$

Problem 2.

Let X be the number of royal flushes that we get in n hands. We model X as a binomial random variable with parameters n and $p = 1/649740$. Let A be the event of getting at least one royal flush in n hands. Then, A^c is the event of getting no royal flush with a probability $\mathbf{P}(A^c) = \mathbf{P}(X=0) = p_X(0) = \binom{n}{0}p^0(1-p)^{n-0}$. Thus, $\mathbf{P}(A) = 1 - \mathbf{P}(A^c) = 1-(1-p)^n$. Solving the inequality $1-(1-p)^n \geq 1-1/e$, we get $n \geq 649744$. To understand why the threshold value of n is so close to $1/p$, note that for large n, we have

$$(1-1/n)^n \approx 1/e,$$

so that $1-(1-p)^n \approx 1-1/e$ when $n \approx 1/p$ and p is small.

Problem 3.

A claim is first filed in year k with probability $0.05 \cdot (0.9)^{k-1}$, and the corresponding total premium is

$$1000 \cdot \big(1 + 0.9 + \cdots + (0.9)^{k-1}\big) = 1000 \cdot \frac{1-(0.9)^k}{1-0.9} = 10000\big(1-(0.9)^k\big).$$

Thus, the PMF of Y, the total premium paid up to and including the year when the first claim is filed, is

$$p_Y(y) = \begin{cases} 0.05 \cdot (0.9)^{k-1} & \text{if } y = 10000(1-(0.9)^k),\ k = 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

Problem 4. (a) Using the formula $p_Y(y) = \sum_{\{x \mid x \text{ mod}(3)=\text{y}\}} p_X(x)$, we obtain

$$
\begin{aligned}
p_Y(0) &= p_X(0) + p_X(3) + p_X(6) + p_X(9) = 4/10, \\
p_Y(1) &= p_X(1) + p_X(4) + p_X(7) = 3/10, \\
p_Y(2) &= p_X(2) + p_X(5) + p_X(8) = 3/10, \\
p_Y(y) &= 0, \qquad \text{if } y \notin \{0, 1, 2\}.
\end{aligned}
$$

(b) Similarly, using the formula $p_Y(y) = \sum_{\{x \mid 5 \text{ mod}(x+1)=y\}} p_X(x)$, we obtain

$$
p_Y(y) = \begin{cases} 2/10, & \text{if } y = 0, \\ 2/10, & \text{if } y = 1, \\ 1/10, & \text{if } y = 2, \\ 5/10, & \text{if } y = 5, \\ 0, & \text{otherwise.} \end{cases}
$$

Problem 5.

Let $Y = \max\{0, X\}$. By using the formula

$$
p_Y(y) = \sum_{\{x \mid \max\{0,x\}=y\}} p_X(x),
$$

we have

$$
p_Y(y) = \begin{cases} 0 & \text{if } y < 0 \text{ or } b < y, \\ \frac{1-a}{1+b-a} & \text{if } y = 0, \\ \frac{1}{1+b-a} & \text{if } 0 < y \le b. \end{cases}
$$

Let $Y = \min\{0, X\}$. Similarly, we have

$$
p_Y(y) = \begin{cases} 0 & \text{if } 0 < y \text{ or } y < a, \\ \frac{1+b}{1+b-a} & \text{if } y = 0, \\ \frac{1}{1+b-a} & \text{if } a \le y < 0. \end{cases}
$$

Problem 6.

(a) We must have $\sum_{x=-3}^{3} p_X(x) = 1$, so

$$K = \frac{1}{\sum_{x=-3}^{3} x^2} = \frac{1}{28}.$$

(b) Using the formula $p_Y(y) = \sum_{\{x \mid |x|=y\}} p_X(x)$, we obtain

$$p_Y(y) = \begin{cases} 2Kx^2 = \frac{x^2}{14} & \text{if } x = 1, 2, 3, \\ 0 & \text{otherwise.} \end{cases}$$

(c) If $y \geq 0$, $p_Y(y) = \sum_{\{x \mid |x|=y\}} p_X(x) = p_X(y) + p_X(-y)$. Otherwise $p_Y(y) = 0$.

Problem 7.

We have $\cos(k\pi) = 1$ for k: even and $\cos(k\pi) = -1$ for k: odd. Therefore

$$\mathbf{E}[Y] = \sum_{k=1}^{\infty} (-1)^k k \mathbf{P}(X = k) + \sum_{k=1}^{\infty} (-1)^k (-k) \mathbf{P}(X = -k) = 0.$$

where the last equality holds because, by the symmetry assumption, we have $\mathbf{P}(X = k) = \mathbf{P}(X = -k)$.

We have $\sin(k\pi) = 0$ for all integer k, so since X takes only integer values, we have that Y is equal to 0 with probability 1. Therefore, $\mathbf{E}[Y] = 0$.

Problem 8.

(a) Let E be the event that Fischer wins the match. We can express E as

$$E = \bigcup_{n \geq 0} E_n,$$

where E_n is the event that each of the first n games is a draw and the $(n+1)$st game is won by Fischer. Since the E_n's are disjoint, we obtain

$$\mathbf{P}(E) = \sum_{n \geq 0} \mathbf{P}(E_n) = \sum_{n \geq 0} (1 - p - q)^n p = \frac{p}{p+q}.$$

(b) Since the duration D of the match is a geometric random variable with parameter $p+q$, we obtain

$$p_D(d) = (1-p-q)^{d-1}(p+q), \qquad d = 1, 2, \ldots,$$

$$\mathbf{E}[D] = \frac{1}{p+q},$$

and

$$\text{var}(D) = \frac{1-p-q}{(p+q)^2}.$$

Problem 9.

We know that the number of errors in n bits is a binomial random variable with parameters n and $1-p$. Its expected value is $n(1-p)$, so $\mathbf{E}[\text{number of errors}] \leq 10$ if $n(1-p) \leq 10$, or

$$p \geq 1 - \frac{10}{n}$$

Thus for $n = 10,000$, we must have $p \geq 0.999$.

Problem 10.

Using the formula $\text{var}(X) = \mathbf{E}[X^2] - \big(\mathbf{E}[X]\big)^2$, we have

$$\begin{aligned} \mathbf{E}\big[(X_1 + \cdots + X_n)^2\big] &= \text{var}(X_1 + \cdots + X_n) + \big(\mathbf{E}[X_1 + \cdots + X_n]\big)^2 \\ &= n\text{var}(X_1) + (n\mathbf{E}[X_1])^2 \\ &= n\mathbf{E}[X_1^2] - n\big(\mathbf{E}[X_1]\big)^2 + n^2\big(\mathbf{E}[X_1]\big)^2 \\ &= n\mathbf{E}[X_1^2] + n(n-1)\big(\mathbf{E}[X_1]\big)^2. \end{aligned}$$

Thus, $c = n$ and $d = n(n-1)$.