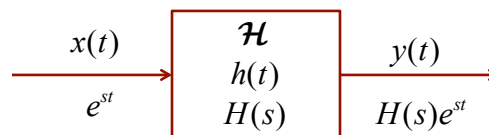


CMPE323: The Laplace Transform

Part I

Back to eigenfunctions

- Remember that the eigenfunctions for LTI systems are complex, not merely imaginary exponentials



- We find the result by convolution

$$y(t) = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau = e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau = H(s) e^{st}$$

$$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau, \text{ where integral exists}$$

- Two-sided Laplace Transform

What does all that mean?

- Consider our causal real exponential $h(t) = e^{-at}u(t)$, with an exponential input

$$H(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st} d\tau = \int_0^{\infty} e^{-(s+a)\tau} d\tau = \frac{1}{s+a}$$

- ...which will exist **EXCEPT** when $s+a < 0$, in which case the integral becomes infinite!
- So to properly express $H(s)$, we need the form and the condition

$$H(s) = \frac{1}{s+a}, s > -a$$

- The condition is called the **Region of Convergence**
- A Laplace Transform is not complete without the RoC

The Region of Convergence

- The range of values for which the Laplace Transform converges is called the **Region of Convergence (RoC)**
- ...and must be specified with each LT we do!
- Why? Doesn't this just make things more complicated?
- Consider the anticausal signal $x(t) = -e^{-at}u(-t)$ for real values of a

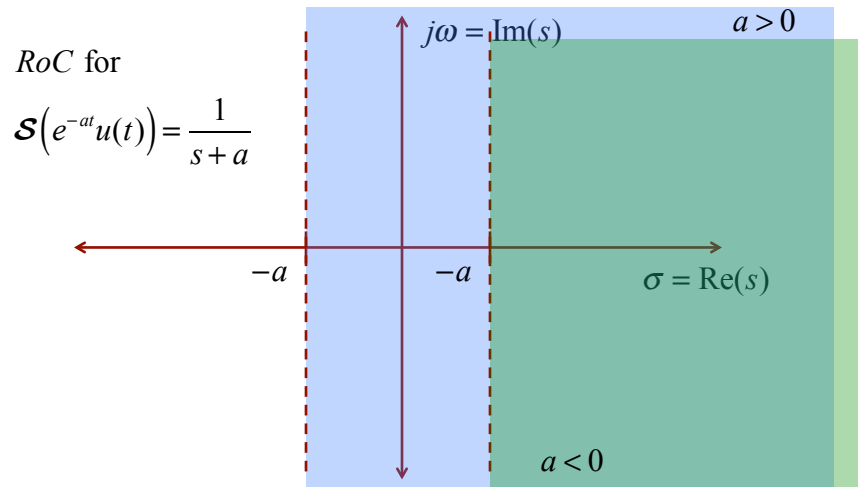
$$X(s) = \int_{-\infty}^{\infty} -e^{-at}u(-t)e^{-st} dt = -\int_{-\infty}^0 e^{-at}e^{-st} dt$$

Converges when $\sigma + a < 0 \Rightarrow \sigma < -a$

$$X(s) = \frac{1}{s+a}, \sigma < -a$$

The s-plane

- It is common to display these conditions graphically, using a Cartesian system known as the *s*-plane

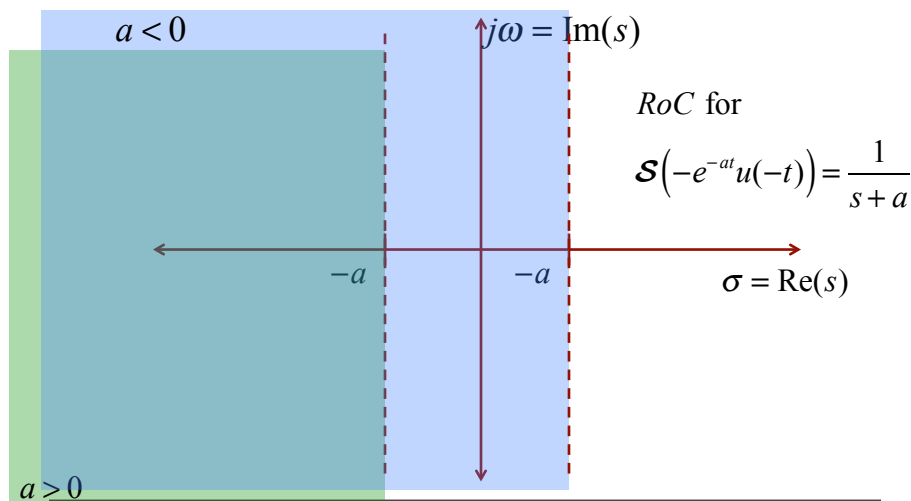


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The s-plane

- The ROC *must* be specified



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LCCDE

Assume we have a LCCDE

$$\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^M b_k \frac{d^k x(t)}{dt^k}, \quad \frac{d^0 y(t)}{dx^0} \triangleq y(t)$$

With $x(t) = e^{st}$ and, therefore, $y(t) = H(s)e^{st}$

$$\sum_{k=0}^N a_k H(s) s^k e^{st} = \sum_{k=0}^M b_k s^k e^{st}, \quad \frac{d^0 y(t)}{dx^0} \triangleq y(t)$$

$$H(s) e^{st} \sum_{k=0}^N a_k s^k = e^{st} \sum_{k=0}^M b_k s^k,$$

$$H(s) = \frac{\sum_{k=0}^M b_k s^k}{\sum_{k=0}^N a_k s^k}$$

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Rational Functions

- The typical case is that the Laplace Transform can be expressed as a rational polynomials

$$X(s) = \frac{B(s)}{A(s)}, \quad \text{for } s \text{ in RoC}$$

$$= \frac{\sum_{k=0}^M b_k s^k}{\sum_{k=0}^N a_k s^k}$$

- $X(s)$ will be of this form whenever $x(t)$ satisfies a LCCDE

$$\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^M b_k \frac{d^k x(t)}{dt^k}, \quad \frac{d^0 y(t)}{dx^0} \triangleq y(t)$$

- ...or is a sum of complex exponentials

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Poles and zeros

- There are M roots of the numerator polynomial

$$B(s) = \sum_{k=0}^M b_k s^k = b_M s^M + b_{M-1} s^{M-1} + \dots + b_1 s + b_0 = 0$$

$$= \underbrace{K}_{\substack{\text{a gain} \\ \text{or scale} \\ \text{factor} \\ \text{not a} \\ \text{function} \\ \text{of } s}} \prod_{k=1}^M (s - z_k) = 0 \Rightarrow s \in \{z_1, z_2, \dots, z_M\}$$

- These are called the zeros of the Laplace function $X(s)$
- ...because $|X(s)| = 0$ at $s = z_k$
- (most of the time, anyway)

Poles and zeros

- Similarly, There are N roots of the denominator polynomial

$$A(s) = \sum_{k=0}^N a_k s^k = a_N s^N + a_{N-1} s^{N-1} + \dots + a_1 s + a_0 = 0$$

$$= \underbrace{G}_{\substack{\text{a gain} \\ \text{or scale} \\ \text{factor} \\ \text{not a} \\ \text{function} \\ \text{of } s}} \prod_{k=1}^N (s - p_k) = 0 \Rightarrow s \in \{p_1, p_2, \dots, p_N\}$$

- These are called the poles of the Laplace function $X(s)$
- ...because $|X(s)| \rightarrow \infty$ as $s \rightarrow p_k$...
- ...except when

$p_k = z_k = s^*$ for some k , in which case, as we approach s^* ,

$$\lim_{s \rightarrow s^*} \frac{(s - s^*)}{(s - s^*)} = \frac{0}{0}; \text{ L'Hopital's Rule: } \lim_{s \rightarrow 0} \frac{1}{1} = 1 \text{ and the zero and pole cancel!}$$

Let's see:

$$x(t) = \frac{1}{2}e^{-3t}u(t) + \frac{1}{2}e^{-t}u(t)$$

Find $X(s)$

Find the poles and zeros

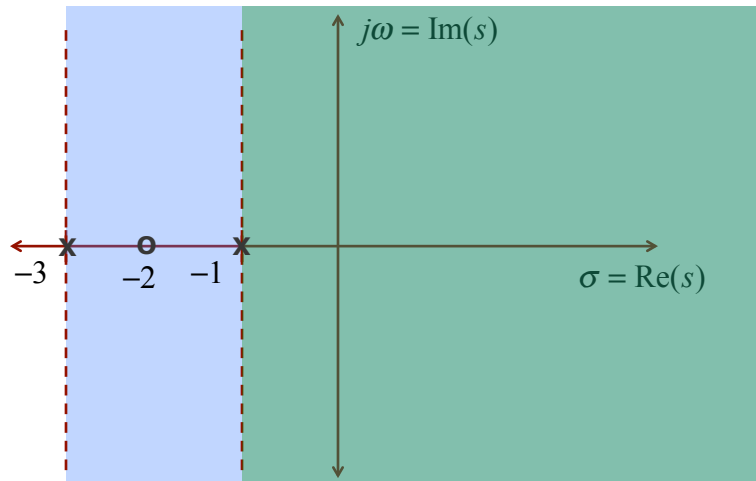
$$X(s) = \frac{s+2}{s^2+4s+3} \Rightarrow B(s) = s+2, A(s) = s^2+4s+3$$

$$\Rightarrow M=1, b_1=1, b_0=2, N=2, a_2=1, a_1=4, a_0=3$$

zeros are the roots of $B(s)$, $s+2=0 \Rightarrow s=-2$

poles are the roots of $A(s)$, $(s+3)(s+1)=0 \Rightarrow s=-3, -1$

- Poles must be outside the RoC
- Zeros may be inside or outside



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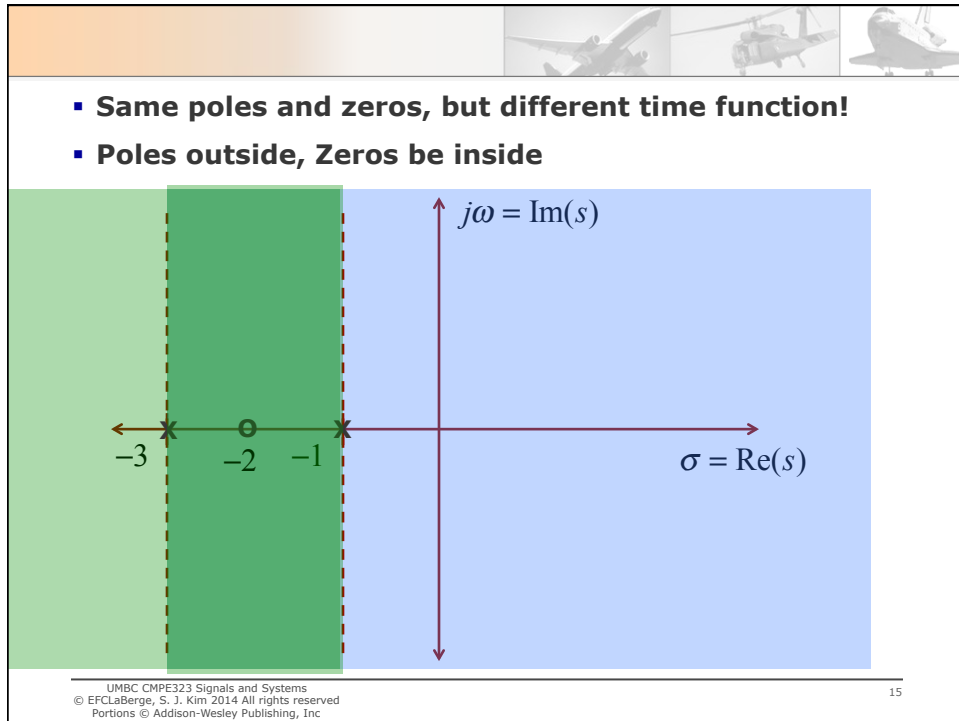
A two-sided signal

$$x(t) = \frac{1}{2}e^{-3t}u(t) - \frac{1}{2}e^{-t}u(-t)$$

Find $X(s)$

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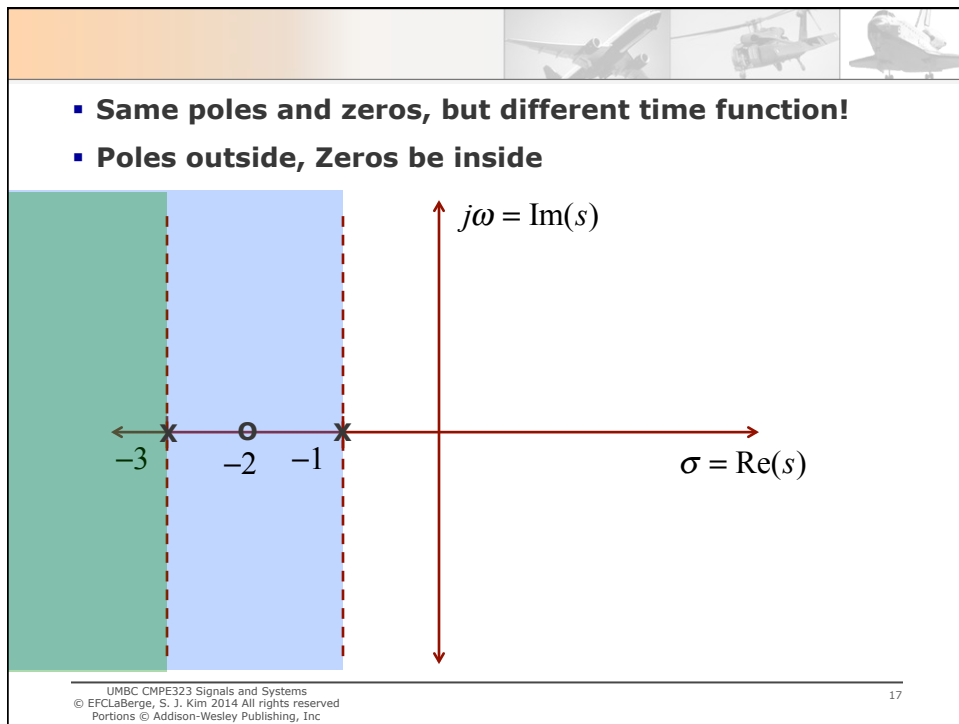
A left-sided signal

$$x(t) = -\frac{1}{2}e^{-3t}u(-t) - \frac{1}{2}e^{-t}u(-t)$$

Find $X(s)$

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Inverse Laplace Transforms

- The equivalent synthesis equation is $x(t) = \int_{-\infty}^{\infty} X(s)e^{st} ds$
- ...and this is often performed via a *Cauchy* integration...
- ...which I'm not going into right now.
- In most cases, the Cauchy integration reduces to the Cauchy Residue Theorem...
- ...and residues reduce to Partial Fraction Expansion (PFE)
- We generally use PFE when $X(s)$ is a rational polynomial, as is usually the case.

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PFE Process

- Factor numerator and denominator polynomials. For the moment, we'll assume that all the roots are distinct and real.

$$X(s) = \frac{K \prod_{k=1}^M (s - z_k)}{\prod_{k=1}^N (s - p_k)}$$

- We want to expand

$$X(s) = \sum_{k=1}^N \frac{R(p_k)}{(s - p_k)} = \frac{R(p_1)}{(s - p_1)} + \frac{R(p_2)}{(s - p_2)} + \dots + \frac{R(p_3)}{(s - p_3)}$$

- ... where $R(p_k)$ are called the *residuals*

See <http://lpsa.swarthmore.edu/BackGround/PartialFraction/PartialFraction.html>

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- For each distinct p_m , form

$$(s - p_m)X(s) = \frac{K \prod_{k=1}^M (s - z_k)}{\prod_{\substack{k=1 \\ k \neq m}}^N (s - p_k)} = R(p_m) + \sum_{\substack{k=1 \\ k \neq m}}^N (s - p_m) \frac{R(p_k)}{(s - p_k)}$$

- ...and evaluate at $s = p_m$

$$\left. \frac{K \prod_{k=1}^M (s - z_k)}{\prod_{\substack{k=1 \\ k \neq m}}^N (s - p_k)} \right|_{s=p_m} = R(p_m) + \sum_{\substack{k=1 \\ k \neq m}}^N 0 \times \frac{R(p_k)}{(s - p_k)} = R(p_m)$$

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PFE Example #1

$$\blacksquare X(s) = \frac{s+2}{(s+1)(s+3)}, \text{ ROC} = \text{Re}(s) > -1$$

$$= \frac{R(-1)}{s - (-1)} + \frac{R(-3)}{s - (-3)}$$

$$R(-1) = \left. \frac{(s+1)(s+2)}{(s+1)(s+3)} \right|_{s=-1} = \frac{(-1+2)}{(-1+3)} = \frac{1}{2}$$

$$R(-3) = \left. \frac{(s+3)(s+2)}{(s+1)(s+3)} \right|_{s=-3} = \frac{(-3+2)}{(-3+1)} = \frac{-1}{-2} = \frac{1}{2}$$

$$X(s) = \frac{\frac{1}{2}}{s+1} + \frac{\frac{1}{2}}{s+3} \Rightarrow x(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

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Difficult Things

- **There are four things that happen that complicate the process**
- **1) $M \geq N$**
- **2) Repeated real roots of the form $(s - p_k)^m$**
- **3) Complex roots**
- **4) Exponentials in $H(s)$**

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M ≥ N

$$X(s) = \frac{6s^2 + 3s + 2}{2s^2 + 14s + 20}$$

Use synthetic division to divide the denominator into the numerator until $M < N$

					3
2	14	20	6	3	2
			6	42	60
			0	-39	-58

$$X(s) = 3 + \frac{-39s - 58}{2s^2 + 14s + 20}$$

$$= 3 + \frac{-39(s - 1.487)}{2(s + 5)(s + 2)} = 3 + \frac{-19.5(s - 1.487)}{(s + 5)(s + 2)}$$

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Multiple real roots

$$X(s) = \frac{s + 3}{s(s + 2)^2(s + 5)}$$

$$= \frac{R(0)}{s} + \frac{R_1(-2)}{s + 2} + \frac{R_2(-2)}{(s + 2)^2} + \frac{R(-5)}{s + 5}$$

$R(0), R_2(-2), R(-5)$ found using standard techniques

$$\begin{aligned} R_1(-2) &= \left[\frac{d}{ds} (s + 2)^2 X(s) \right]_{s=-2} \\ &= \frac{d}{ds} \left(\frac{s + 3}{s(s + 5)} \right)_{s=-2} = \frac{-7}{36} \end{aligned}$$

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