Time Averages of Time Waveforms

- We are usually interested in the energy contained in a waveform or signal…
- …as energy is the key to overcoming noise in the system.

$$\mathcal{E} = \int_{-\infty}^{\infty} x(t)x^*(t)\,dt = \int_{-\infty}^{\infty} |x(t)|^2\,dt \underbrace{=}_{\substack{\text{real}\\\text{valued}\\\text{signal}}} \int_{-\infty}^{\infty} x^2(t)\,dt$$

- If our signal has units of volts, then $\mathcal{E}$ has units of volt^2-sec, which is not joules!
- When computing the mathematical energy (as above) we assume there is a 1Ω resistor as a scale factor, giving volt^2-sec/ohm = watt-sec = joules!
- Using this definition, the energy in a periodic signal is infinite!! (Why?)

- For periodic signals, we compute the "average energy per period"

$$\frac{\mathcal{E}}{T} = \frac{1}{T}\int_{\alpha}^{T+\alpha} x(t)x^*(t)\,dt = \frac{1}{T}\int_{\alpha}^{T+\alpha} |x(t)|^2\,dt \underbrace{=}_{\substack{\text{real}\\\text{valued}\\\text{signal}}} \frac{1}{T}\int_{0}^{T} x^2(t)\,dt$$

- Again using a 1Ω resistor as a scale factor, we wind up with volt^2-sec/ohm-sec=watt-sec/sec=watt…
- …so this is power!
- A signal with finite energy is called an energy signal
- A signal with meaningful power is called a power signal.
- Some signals are neither power or energy signals, but these are rare.

Examples

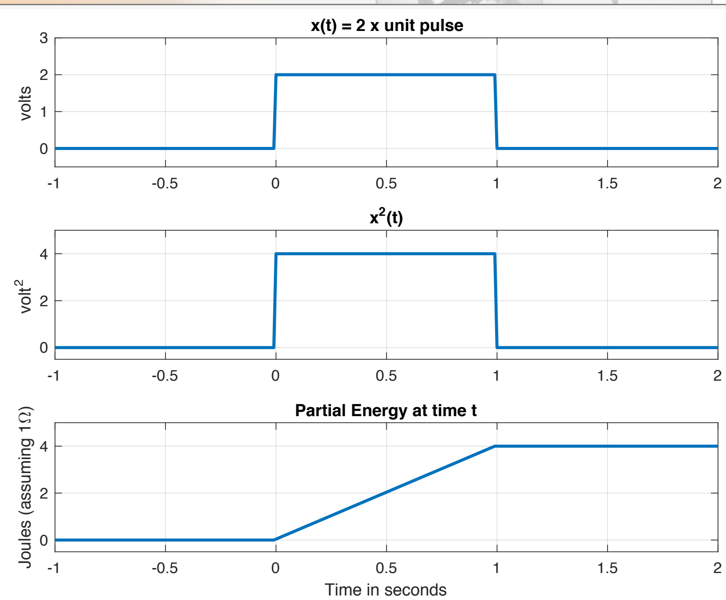
- **Unit pulse**

$$p(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$x(t) = 2p(t)$$

$$x^2(t) = \begin{cases} 4 & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{E} = \int_{-\infty}^{\infty} x^2(t)\,dt = \int_0^1 4dt = 4 \text{ joules}$$



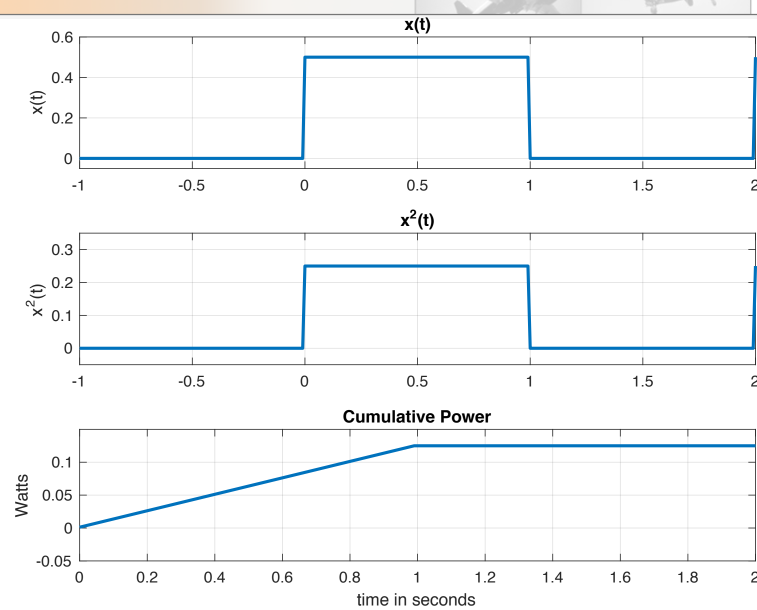
- **Pulse train**

$$p(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$x(t) = \sum_{-\infty}^{\infty} 0.5\, p(t-2n)$$

$$x^2(t) = \sum_{-\infty}^{\infty} 0.25\, p(t-2n)$$

$$P = \frac{1}{2}\int_0^2 0.25\, dt = 0.125 \text{ watts}$$



You try

- **Compute the power or energy, as appropriate**

$$a(t)=\begin{cases} e^{-0.5t} & 0\le t<2 \\ 0 & \text{o/w} \end{cases} \quad \text{(exponential pulse)}$$

$$b(t)=\sum_{k=-\infty}^{\infty} a(t-4k)$$

$$c(t)=120\cos(120\pi t+\pi/3)$$

$$d(t)=\begin{cases} 120\cos(120\pi t) & -\frac{1}{240}\le t<\frac{1}{240} \\ 0 & \text{o/w} \end{cases}$$

$$f(t)=\begin{cases} 0.25t & 0\le t<4 \\ 0 & \text{o/w} \end{cases}$$

$$g(t)=\sum_{k=-\infty}^{\infty} f(t-16k)$$

$a(t)$ is an energy signal

$$\mathcal{E}=\int_0^2 (e^{-0.5t})^2\,dt=\int_0^2 (e^{-1t})\,dt=1-e^{-2}$$

$b(t)$ is periodic with period 4, therefore it is a power signal

$$P=\frac{1}{4}\int_0^4 (b(t))^2\,dt=\frac{1}{4}\left(\int_0^2 (e^{-0.5t})^2\,dt+\int_2^4 (0)^2\,dt\right)=\frac{1-e^{-2}}{4}$$

$c(t)$ is periodic with period $\frac{2\pi}{120\pi}=\frac{1}{60}$. The phase does not affect the period, $c(t)$ is a power signal

Let $120\pi t_0=\frac{\pi}{3}\Rightarrow t_0=\frac{1}{360}$, $c(t)=120\cos(120\pi(t+t_0))$

$$P=\left(\frac{1}{60}\right)120^2\int_{-1/360}^{-1/360+1/60}\cos^2(120\pi(t+t_0))\,dt=\left(\frac{1}{60}\right)120^2\int_0^{1/60}\cos^2 120\pi\tau\,d\tau$$

$$=\left(\frac{1}{60}\right)120^2\int_0^{1/60}\left(\frac{1}{2}+\frac{1}{2}\cos 240\pi\tau\right)d\tau$$

$$=120^2\frac{((1/60)-0)}{2\times 60}+120^2\frac{\left(\sin\frac{240\pi}{60}-\sin 0\right)}{2\times 60\times 240\pi}=\frac{120^2}{2}=7200\text{ watts}$$

Manipulation of the Time Axis

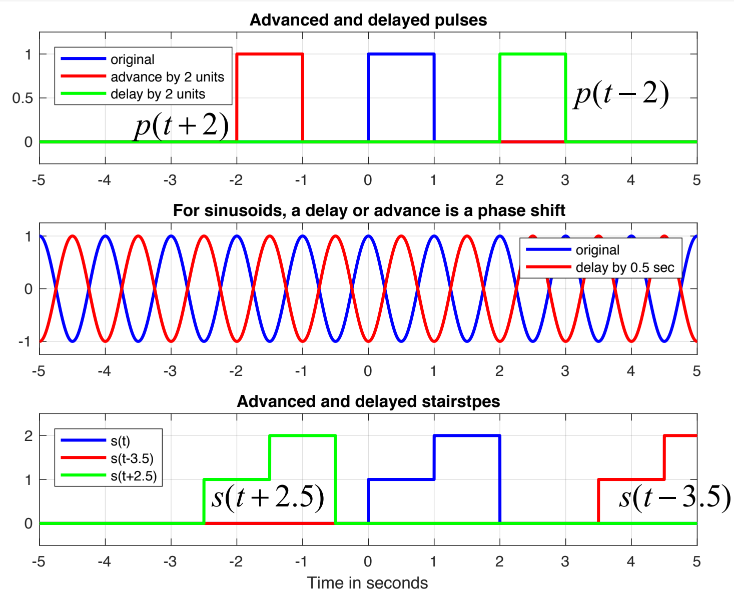
CMPE323

We've been talking about functions of time

- **Unit Step**
- **Unit Impulse**
- **Sinusoid**
- **Complex Exponential**
- **Unit pulse of duration T** $p(t;T) = \begin{cases} 1 & 0 \le t < T \\ 0 & \text{otherwise} \end{cases}$
- **Now lets examine what happens as we manipulate the time axis…**
- **…that is, as we change the *argument* of the function**
- **We'll use the unit pulse for examples**

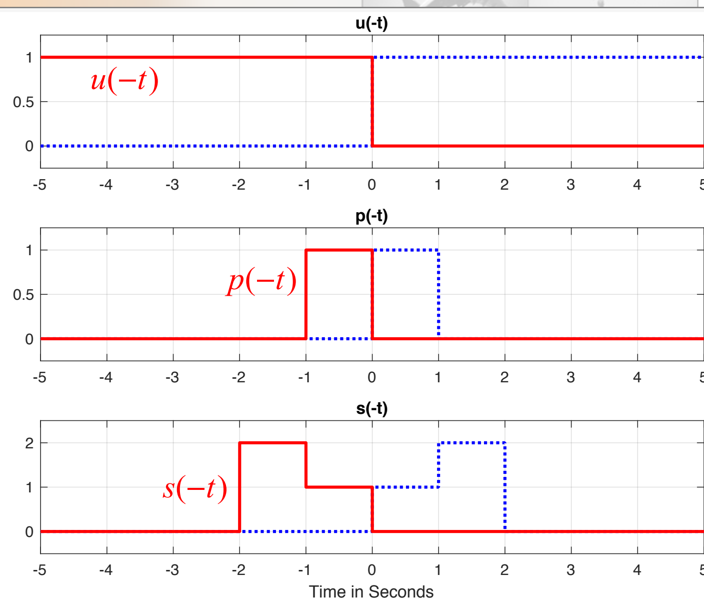
Basic manipulations

- **Advance – we advance a waveform when we move all of its features to occur at an *earlier* time (shift to the left), this shifts in the direction of *decreasing* time**
- **Delay – we delay a waveform when we move all of its features to occur at a later time, shift in direction of *increasing time***
- **Reversal – we reverse a waveform when we change the sign of the time variable (and only the time variable!)**
- **Expansion – we expand a waveform if we increase its duration, that is, if we move its features farther apart in time**
- **Contraction – we contract a waveform if we decrease its duration, that is, if we move its features closer together in time.**
- **Linear adjustment – a linear adjustment in time is a combination of advance/delay and expand/contract**



A reversal does just that

- By changing t to $-t$ in the waveform argument, you make time "increase" from right to left…
- …that is, you reflect the waveform around the line $t = 0$
- We saw this in CMPE306, when we used $u(-t)$ to turn a voltage source off at $t = 0$
- When we make the $t \to -t$ swap, it affects *only* t, *not the entire argument*



Combined reversal and shift

- **Of course we can combine things, but we have to be careful in how we define what we're doing!!**
- **There are three ways to look at it (I like way 1)**
- **"Shift then flip"**

  $p(-t-2) = p(\tau-2)_{\tau=-t}$ shift $p(\tau)$ to the right

  and then reverse, reflecting about $\tau = 0$.
- **"Flip then shift"**

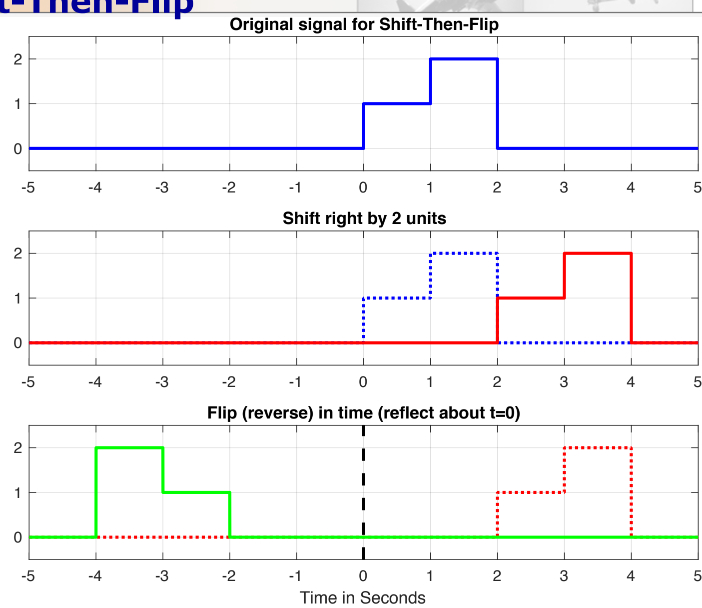
  $p(-t-2) =$ reverse to get $p(-t)$, then shift toward

  increasing values of $-t$ (effectively "left"!)
- **"Flip shifted"**

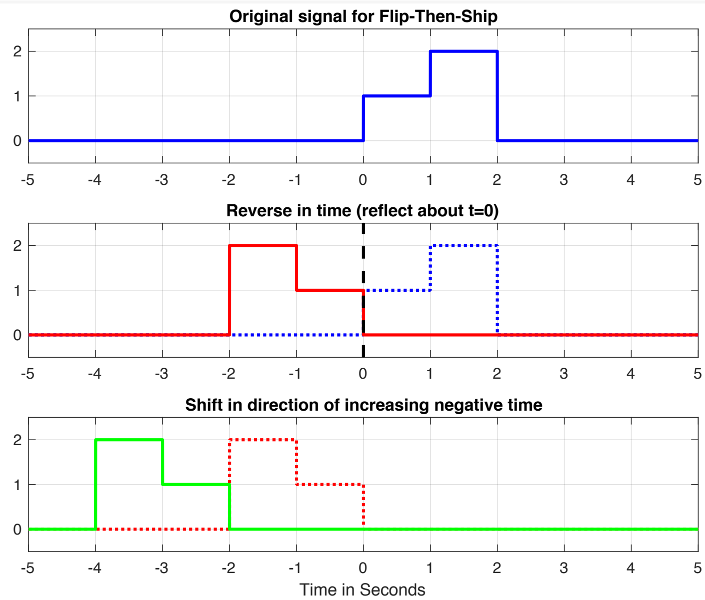
  $p(-t-2) = p(-(t+2))$ shift to left by 2, then

  reverse around the $t = -2$ axis $(t = -2 \Rightarrow t+2 = 0)$

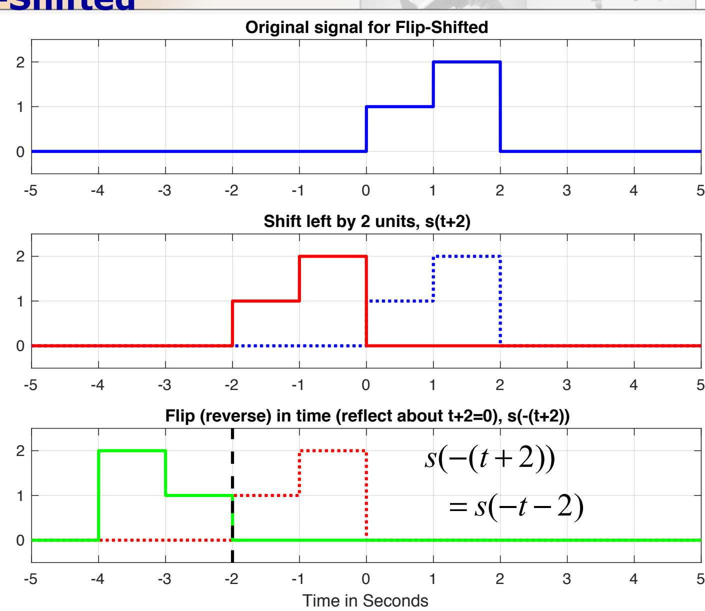
Shift-Then-Flip



Flip-then-Shift

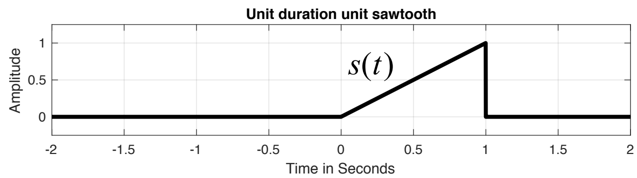


Flip-Shifted



Your turn

- **In your groups, try the following. Use the unit duration unit sawtooth $s(t)$ shown below.**



- **Use different techniques to verify your answers**

$$a(t) = s(-t+2)$$

$$b(t) = s(-(t-2))$$

$$c(\tau) = s(\tau - 3)$$

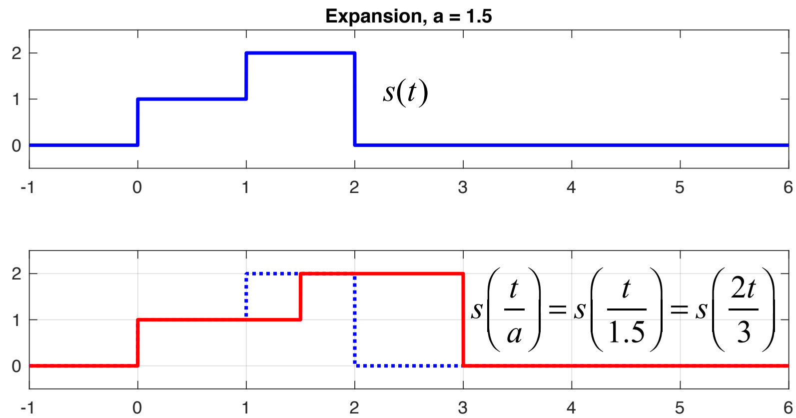
$$d(\tau) = s(3-\tau)$$

$$e(\tau) = s(t-\tau), \text{ for some fixed value of } t.$$

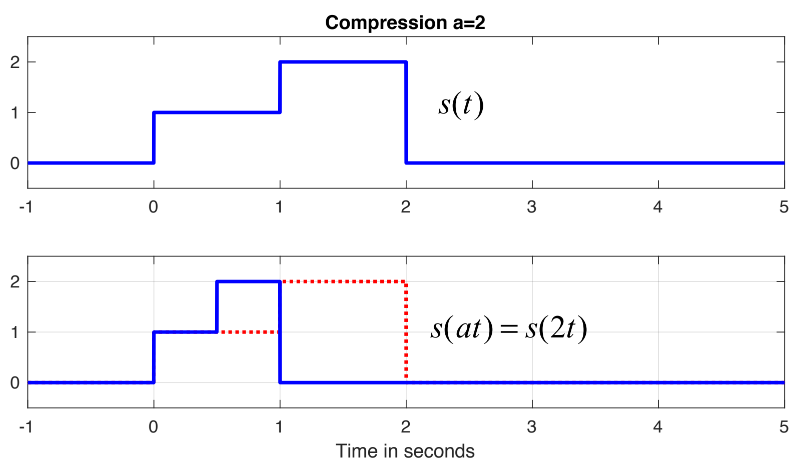
Compression and expansion

- **We now *scale* the time axis by a multiplicative constant, creating** $x(at)$
- **If** $0 < |a| < 1$**, this creates an expansion of the waveform**
- **If** $|a| > 1$ **this creates a compression of the waveform**
- **If** $a < 0$ **we get both a reversal and an expansion/ contraction**
- $a = 1$ **does nothing (obviously!)**
- $a = -1$ **is a simple reversal.**

Expansion example



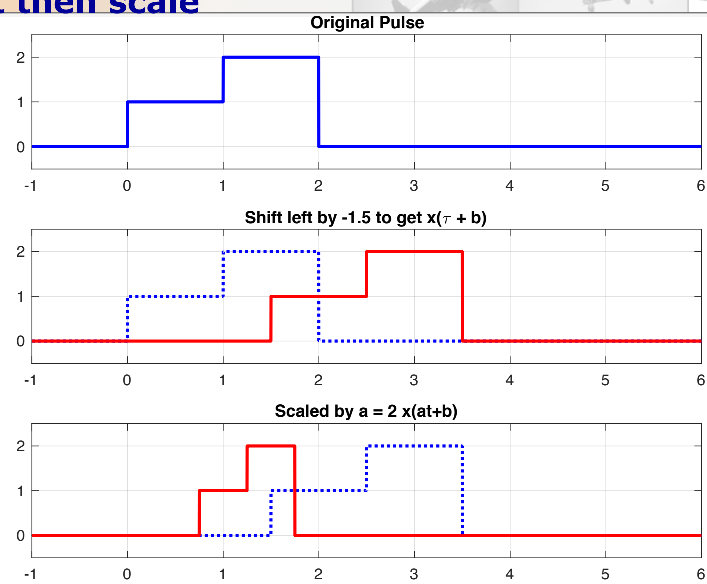
Compression Example



Linear transformation of the time

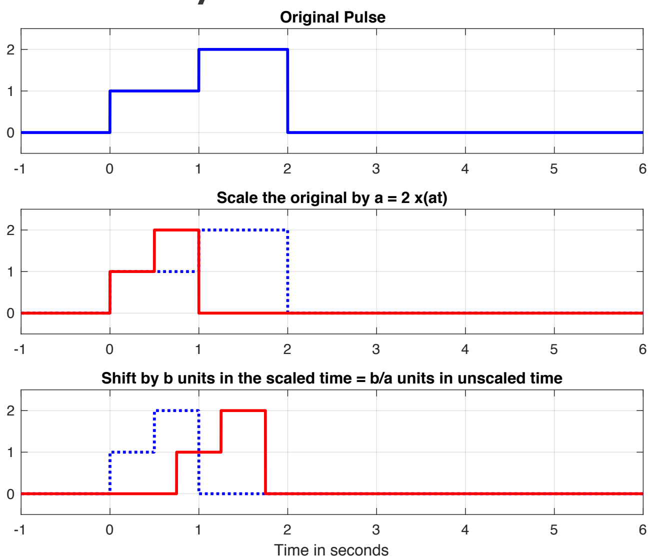
- Now we combine expansion/compression/reversal with offset $y(t) = x(at + b)$
- Use the similar techniques as with reversal + offset
  - Shift then scale = shift by b unscaled units then scale the result
  - Scale then shift = scale by a, then shift by b scaled units
- You must be very careful about what unit to use for the shift!

Shift then scale



Scale then shift

- **You must shift by units in the scaled time axis!!!!**



You try

- **Use the unit sawtooth to find the following**

$$f(t) = s(0.5t - 3)$$
$$g(t) = s(2t + 5)$$
$$h(t) = s(-2t + 5)$$

Even and Odd

- **Finally, let's discuss even and odd functions**
- **For an even function** $f(t) = f(-t)$
  - **Examples** $x(t) = t^2,\ x(t) = \cos(\omega t),\ x(t) = A$
- **For an odd function** $f(t) = -f(-t)$
  - **Examples** $x(t) = t^3, x(t) = at,\ x(t) = \sin(\omega t)$
- **But what about a general function, like** $u(t)$**?**
- **Any function can be decomposed into the sum of an even function and an odd function**

$$f_E(t) = \frac{f(t) + f(-t)}{2},\ f_O(t) = \frac{f(t) - f(-t)}{2}$$

$$f(t) = f_E(t) + f_O(t)$$