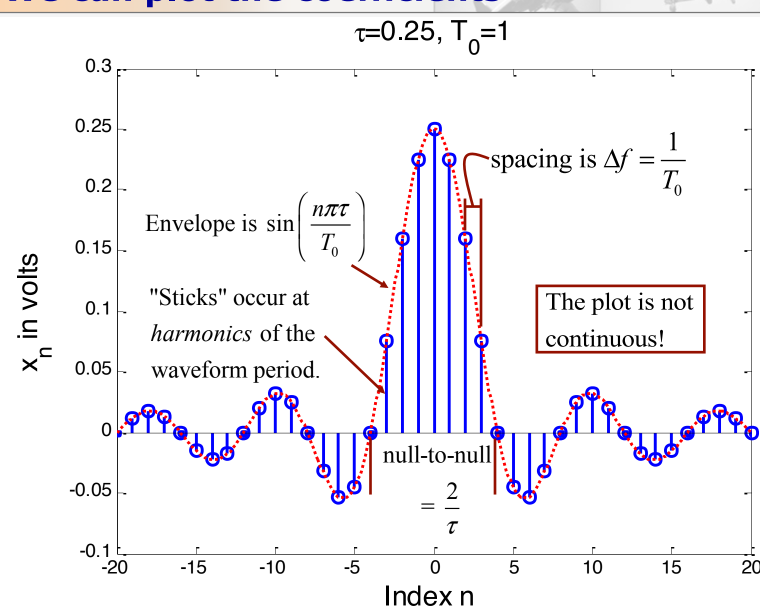
CMPE323 Intro to Fourier Series

Chapter 7

We can plot the coefficients



So this means

- **We can write the periodic signal**

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi kt}{T_0}} = \frac{A\tau}{T_0} \sum_{k=-\infty}^{\infty} \operatorname{sinc}\left(\frac{n\tau}{T_0}\right) e^{j\frac{2\pi kt}{T_0}}$$

- **…which is defined *only* at integer multiples of**

$$f_0 = \frac{\omega_0}{2\pi} = \frac{1}{T_0}$$

- **This expression is known as the Fourier Series, and it relates the time domain $\left(x(t)\right)$ to the frequency domain**

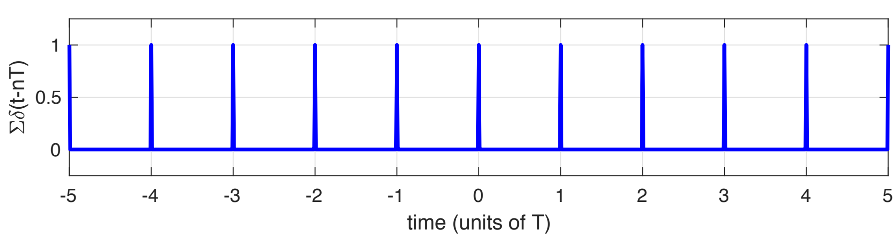
$$\{c_k\} \text{ or } X(f) = \sum_{k=-\infty}^{\infty} c_k \delta\left(f - \frac{k}{T}\right)$$

- **And the inverse relationship hold as well**

The periodic impulse train

- **What about a periodic train of impulses?**

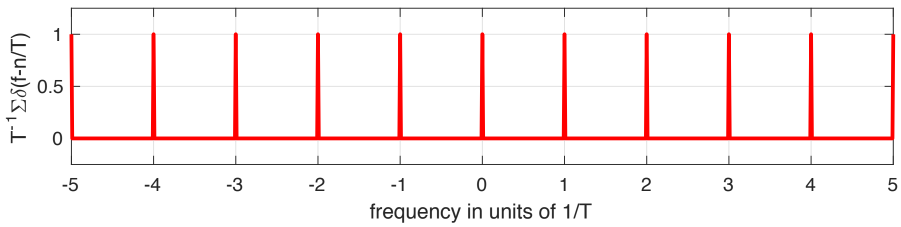
$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$



- **Can we find its Fourier Series?**

Analysis Equation

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t-0T)e^{-j\frac{2\pi}{T}kt}\,dt = \frac{1}{T}\ for\ all\ k!!$$



But what if the signal is not periodic?

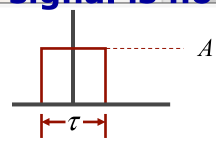
- **There is an equivalent result for non-periodic signals**
- **A non-periodic signal can be viewed as the *limit* of a periodic signal as** $T_0 \to \infty$

$$c_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-j\frac{2\pi kt}{T_0}}\,dt \qquad \frac{k}{T_0}\to f \qquad \frac{T_0}{2}\to\infty \qquad \frac{1}{T_0}\to df$$

$$X(f) \triangleq \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt \quad \text{contribution between } f \text{ and } f+df$$

- $X(f)$ **is called the Fourier Transform, or the *voltage spectrum* of** $x(t)$**,**
- **…and we have** $x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$
- **Notice** $c_n = \frac{1}{T_0}X(f)_{f=\frac{n}{T_0}} = \frac{1}{T_0}X\left(\frac{n}{T_0}\right)$

But what if the signal is not periodic?



- **We define the Fourier Transform of x(t)**

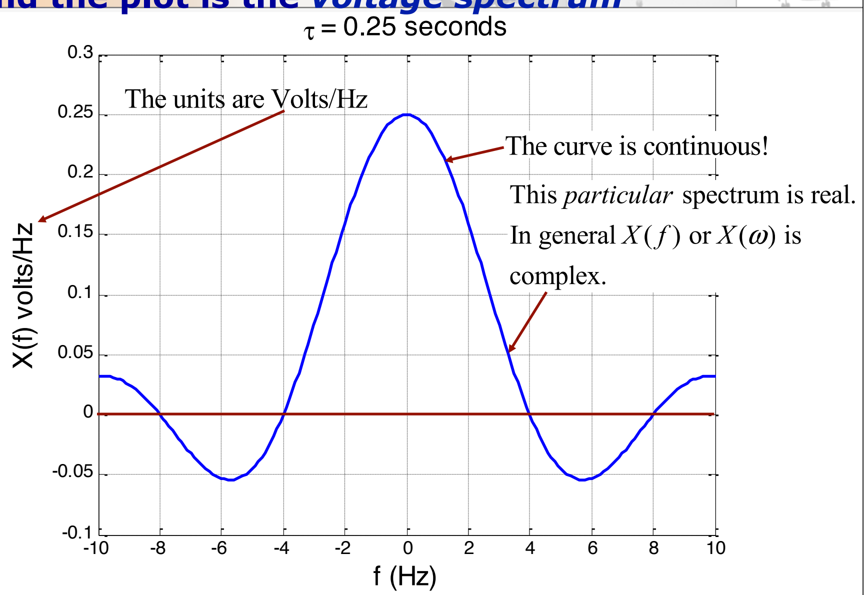
$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt \text{ or } \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = X(\omega)$$

- **In this case**

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt = \int_{-\tau/2}^{\tau/2} A e^{-j2\pi ft} dt = \frac{A}{-j2\pi f} e^{-j2\pi ft} \Big|_{-\tau/2}^{\tau/2}$$

$$= \frac{A}{j2\pi f}\left(e^{j2\pi f\tau 2} - e^{-j2\pi f\tau 2}\right) = \frac{A}{\pi f}\frac{\left(e^{j\pi f\tau} - e^{-j\pi f\tau}\right)}{j2} = A\tau\frac{\sin(\pi f\tau)}{\pi f\tau} = A\tau \operatorname{sinc}(f\tau)$$

And the plot is the *voltage spectrum*



Properties of Fourier Transform

- **Duality:** $X(f)=\mathcal{F}\big(x(t)\big)\Rightarrow x(f)=\mathcal{F}\big(X(-t)\big)$ and $x(-f)=\mathcal{F}\big(X(t)\big)$
- **Linearity:** $z(t)=ax(t)+by(t)\Rightarrow Y(f)=aX(f)+bY(f)$
- **Time Shift:** $\mathcal{F}\big(x(t-t_0)\big)=e^{-j2\pi f t_0}\mathcal{F}\big(x(t)\big)$
- **Scaling:** For $a\neq 0\in\mathbb{R}$, $\mathcal{F}\big(x(at)\big)=\dfrac{1}{|a|}X\left(\dfrac{f}{a}\right)$
- **Modulation:** $\mathcal{F}\big(x(t)e^{j2\pi f_0 t}\big)=X(f-f_0)$
- **Conjugation:** $\mathcal{F}\big(x^*(t)\big)=X^*(-f)$
- **Parseval:** $\displaystyle\int_{-\infty}^{\infty} x(t)y^*(t)\,dt=\int_{-\infty}^{\infty} X(f)Y^*(f)\,df=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)Y^*(\omega)\,d\omega$
- **Rayleigh** $\displaystyle\int_{-\infty}^{\infty}\left|x(t)\right|^2 dt=\int_{-\infty}^{\infty}\left|X(f)\right|^2 df=\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)Y^*(\omega)\,d\omega$

Advanced Properties of Fourier Transform

- **Integration:** $\mathcal{F}\left(\displaystyle\int_{-\infty}^{t} x(t)\,dt\right)=\dfrac{X(f)}{j2\pi f}+\dfrac{1}{2}X(0)\delta(f)$
- **Differentiation:** $\mathcal{F}\left(\dfrac{d}{dt}x(t)\right)=j2\pi fX(f)$
- **Moments:** $\displaystyle\int_{-\infty}^{\infty} t^n x(t)\,dt=\left(\frac{j}{2\pi}\right)^n\frac{d^n}{df^n}X(f)\bigg|_{f=0}$
- **Real Signals: The Fourier transform of a real signal is EVEN in magnitude and ODD in phase**

The convolution theorem (very important!)

- The output of a LTI system with transfer function $H(f)$

$$Y(f) = X(f)H(f) \qquad y(t) = \int_{-\infty}^{\infty} Y(f)e^{j2\pi ft}df = \int_{-\infty}^{\infty} X(f)H(f)e^{j2\pi ft}df$$

Write $H(f) = \int_{-\infty}^{\infty} h(\tau)e^{-j2\pi f\tau}d\tau$

$$y(t) = \int_{-\infty}^{\infty} X(f)\left(\int_{-\infty}^{\infty} h(\tau)e^{-j2\pi f\tau}d\tau\right)e^{j2\pi ft}df$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} X(f)h(\tau)e^{j2\pi f(t-\tau)}df\right)d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\left(\int_{-\infty}^{\infty} X(f)e^{j2\pi f(t-\tau)}df\right)d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau$$

$$\triangleq x(t) * h(t) \text{ the CONVOLUTION}$$

Summary

- We can decompose a periodic signal into the weighted sum of complex exponentials,…
- …or, equivalently, to the weighted sum of sines and cosines.
- We write the weighted sum as a Fourier Series
- The frequency-domain representation consists of a series of harmonically-related terms, with separation equal to the period of the signal,
- The coefficients have units of volts (or amps)
- We can decompose a non-periodic signal into the weighted integral of complex exponentials,…
- …or, equivalently, to the weighted integral of sines and cosines
- The frequency-domain representation has a continuous spectrum with units of volts/Hz (or amps/Hz).