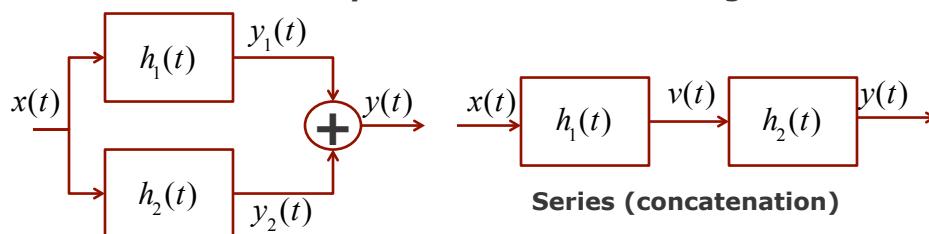


## Lecture 6: LTI systems and convolution

### Combinations of LTI systems

- We can (and do) combine multiple LTI systems into a larger (possibly) LTI system...
- ...so we need some rules.
- There are basically two combination strategies



**Parallel (addition)**

$$h(t) = h_1(t) + h_2(t)$$

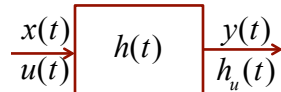
**Series (concatenation)**

$$h(t) = h_1(t) * h_2(t)$$

- ...and the combination equations are the result of the properties of the convolution operator

## The step response

- We have seen that the LTI system  $\mathcal{H}$  is characterized by its impulse response  $h(t)$
- The **step response** is also useful



$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$h(t) = \int_{-\infty}^{\infty} \delta(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)\delta(t-\tau)d\tau$$

$$h_u(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau = \int_{-\infty}^t h(\tau)d\tau$$

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## And we can obtain one from the other

$$\delta(t) = \frac{du(t)}{dt} \Rightarrow h(t) = \frac{dh_u(t)}{dt}$$

- By Liebnitz' Rule for differentiation of integrals

$$h_u(t) = \int_{-\infty}^t h(\tau)d\tau$$

$$\frac{dh_u(t)}{dt} = \frac{dt}{dt} \times h(t) - \frac{d(-\infty)}{dt} h(-\infty) + \int_{-\infty}^t \frac{dh(\tau)}{dt} d\tau$$

$$= 1 \times h(t) - 0 + 0 = h(t)$$

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## LTI Systems and Differential Equations

- LTI systems are often used to describe systems whose behavior is characterized by **linear, constant coefficient differential (or difference) equations**
- ...which appear in a wide range of communications, navigation, control, signal processing, coding, and decoding applications
- ...(which is why we're studying this)
- The general form is

$$\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^M b_k \frac{d^k x(t)}{dt^k}, \quad \frac{d^0 y(t)}{dt^0} \triangleq y(t)$$

- ... which is a function of ***N*** derivatives of the output and ***M*** derivatives of the input.

- From MATH225 we know (or should know) that the solution to this general form consists of

- A **particular solution**  $y_p(t)$  which satisfies the general form, and,
- A **homogeneous solution**  $y_h(t)$  which satisfies

$$\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = 0$$

- ...with the exact form determined by ***N* auxiliary conditions**, which are usually in the form of **initial conditions**.
- The solution is then  $y(t) = y_p(t) + y_h(t)$
- We should also know that  $y(t) = Ae^{st}$ ,  $s = \sigma + j\omega$  are **eigenfunctions** for the LCCDEs under consideration

- If  $x(t)$  is an eigenfunction of an operator  $\mathcal{G}$ , then

$$\mathcal{G}(x(t)) = Bx(t)$$

- ... where  $B$  is a (possibly complex) constant
- For the LCCDE given earlier, it is clear that  $x(t) = Ae^{st}$  is an eigenfunction of

k-th order derivative operator  $\frac{d^k}{dt^k}$

$$\frac{d^k}{dt^k}(Ae^{st}) = As^k e^{st}$$

- And that's why we choose trial solutions of the form

$$y(t) = Ce^{\alpha t}$$

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## Working along these lines

- Consider  $\frac{dy(t)}{dt} + ay(t) = x(t)$ , with  $x(t) = e^{-bt}u(t)$

- It has the right form!
- Our trial particular solution for  $t > 0$  is  $y_p(t) = Ax(t) = Ae^{-bt}$
- Plugging and chugging


$$-bAe^{-bt} + ae^{-bt} = e^{-bt}; t > 0$$

$$A = \frac{1}{a-b}$$

$$y_p(t) = \frac{1}{a-b} e^{-bt}, t > 0$$

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- For the homogeneous solution to  $\frac{dy(t)}{dt} + ay(t) = 0$
- We again guess an eigenfunction  $y_h(t) = Ke^{st}$
- Giving  $\frac{d}{dt}(Ke^{st}) + aKe^{st} = 0$   

$$Kse^{st} + aKe^{st} = 0 \Rightarrow s = -a$$
- The total solution is then  

$$y(t) = y_p(t) + y_h(t)$$

$$= Ke^{-at} + \left(\frac{1}{a-b}\right)e^{-bt}, t > 0$$


$$y(t) = y_p(t) + y_h(t)$$

$$= Ke^{-at} + 0, t < 0$$
- With the value at  $t = 0$  being the initial condition  $Y_I$   

$$Ke^{-at}\Big|_{t=0} = Y_I = K + \left(\frac{1}{a-b}\right) \Rightarrow K = Y_I - \left(\frac{1}{a-b}\right)$$

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## System applications



- In systems application we're much less worried about the particular and homogeneous solutions than we are about the solution with and without an input
- The zero state solution  

$$y_{zs}(t) = y(t)\Big|_{Y_I=0} = \left(\frac{1}{a-b}\right)(e^{-bt} - e^{-at})u(t)$$
- The zero input solution  $y_{zi}(t) = y(t)\Big|_{x(t)=0} = y_h(t) = Y_I e^{-at}$
- And  $y(t) = y_{zs}(t) + y_{zi}(t)$

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## Is the system described linear?

- The linear combinations have to work for any valid inputs, so choose  $x_1(t) = \alpha e^{-bt}u(t)$ ,  $x_2(t) = -x_1(t)$
- Then, from our solution

$$y_1(t) = Y_I e^{-at} + \frac{1}{\alpha} \left( \frac{1}{a-b} \right) (e^{-bt} - e^{-at}) u(t)$$

$$y_2(t) = Y_I e^{-at} - \frac{1}{\alpha} \left( \frac{1}{a-b} \right) (e^{-bt} - e^{-at}) u(t)$$

- If we let  $x(t) = x_1(t) + x_2(t) = 0$ , a linear system will give

$$y(t) = Y_I e^{-at} + \frac{1}{\alpha} \left( \frac{1}{a-b} \right) (e^{-bt} - e^{-at}) u(t) + Y_I e^{-at} - \frac{1}{\alpha} \left( \frac{1}{a-b} \right) (e^{-bt} - e^{-at}) u(t)$$

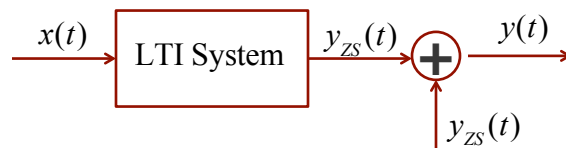
$$y(t) = Y_I e^{-at} + Y_I e^{-at} = 2Y_I e^{-at} = 0 \Leftrightarrow Y_I = 0$$

- So this system is linear iff it is initially at rest!

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## In general



- A system described by a LCCDE is linear if and only if it is initially at rest.
- A causal system that is initially at rest is also Time Invariant

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## Block Diagrams of LCCDE

- How would we implement or synthesize a system if we had a LCCDE?
- Implementing differentiators is problematic because
  - High frequencies in the input are multiplied by a frequency ramp

$$\frac{d \sin(\omega_o t)}{dt} = \omega_o \cos(\omega_o t)$$

- Impulses can result from differentiation
- So we transform the equation into an integral equation
- Here are the steps

$$\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^M b_k \frac{d^k x(t)}{dt^k}, \quad \frac{d^0 y(t)}{dx^0} \triangleq y(t)$$

If  $M < N$ , let  $b_k = 0$ ,  $k = M+1, M+2 \dots N$

If  $N < M$ , let  $a_k = 0$ ,  $k = N+1, N+2 \dots M$

$$\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^N b_k \frac{d^k x(t)}{dt^k}, \quad \frac{d^0 y(t)}{dx^0} \triangleq y(t)$$

Now form a set of auxiliary variables or ,  $y_k(t)$  and  $x_k(t)$

$$y_0(t) = y(t), \quad x_0(t) = x(t)$$

$$y_1(t) = \int_{-\infty}^t y_0(\tau) d\tau, \quad x_1(t) = \int_{-\infty}^t x_0(\tau) d\tau$$

$$y_2(t) = \int_{-\infty}^t y_1(\tau) d\tau, \quad x_2(t) = \int_{-\infty}^t x_1(\tau) d\tau$$

etc...

$$y_N(t) = \int_{-\infty}^t y_{N-1}(\tau) d\tau, \quad x_N(t) = \int_{-\infty}^t x_{N-1}(\tau) d\tau$$

$$\sum_{k=0}^N a_k y_{N-k}(t) = \sum_{k=0}^N b_k x_{N-k}(t) = \sum_{k=0}^M b_k x_{N-k}(t)$$

Where we have integrated both sides  $N$  times...

requiring  $N$  initial conditions (one for each integral)

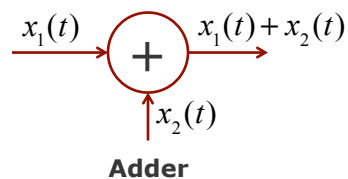
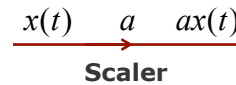
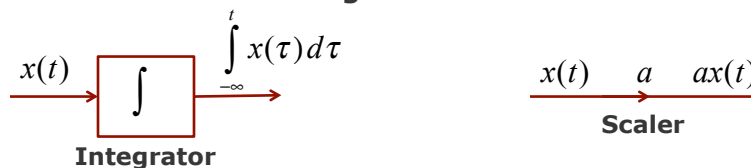
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## Basic Block Diagram Elements

- We'll now use this "integrated" form to *synthesize* an implementation of the system described by the LCCDE

- We need three building blocks



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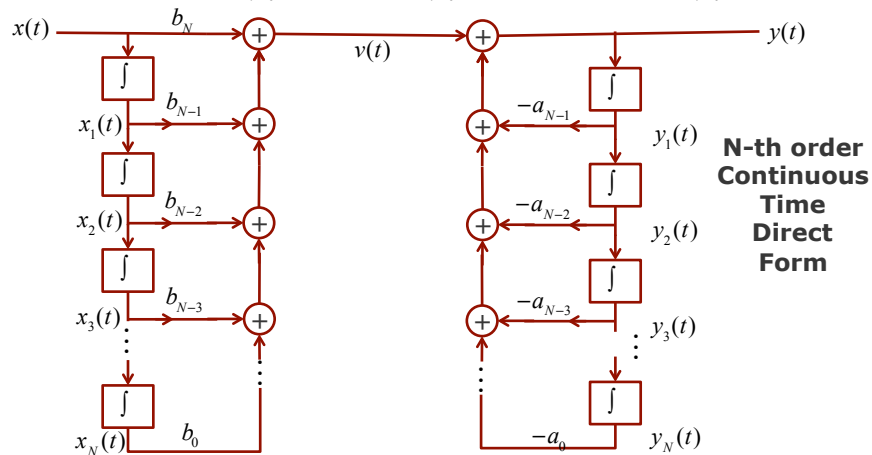


## Synthesis

- Assume  $a_N = 1$  (or divide through to make it so)

- Rewrite

$$y(t) = y_0(t) = \sum_{k=0}^N b_k x_{N-k}(t) - \sum_{k=0}^{N-1} a_k y_{N-k}(t) = v(t) - \sum_{k=0}^{N-1} a_k y_{N-k}(t)$$



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