MATH 407
5/2/18

* Divide $x^4+3x^3+x^2+2$ by $2x^2-3$ (mod 5)

$$
\begin{array}{r l}
x^2+1 \enclose{longdiv}{} & x^4+3x^3+x^2+2 \quad (x^2+3x \\
& \underline{x^4+0+x^2} \\
& \quad 3x^3+0+2 \\
& \quad \underline{3x^3+3x} \\
& \quad\quad -3x+2
\end{array}
$$

$$
\begin{aligned}
(x^4+3x^3+x^2+2) &= (x^2+3x)(x^2+1)+(2x+2) \\
&= (2x^2+x)(2x^2-3)+(2x+2)
\end{aligned}
$$

Thm. If $f, g \in F[x]$, $g \neq 0$, then $f = gq+r$ for unique q, r, $\deg(r) < \deg(g)$

Pf. Uniqueness; if $f = gq_1+r_1$, $\deg(r_1) < \deg(g)$, then $gq_1+r_1 = gq+r$ so $g(q_1-q) = r-r_1$
So, $q-q_1 = 0 \Rightarrow q = q_1$
Thus, $r = r_1$

Pf. Existence; if $f = a_0+a_1x^1+\ldots+a_mx^m$, $a_m \neq 0$
$g = b_0+b_1x^1+\ldots+b_nx^n$, $b_n \neq 0$
$m = \deg(f) < n = \deg(g)$
$q = 0$, $r = f$

Let $f' = f - \frac{a_m}{b_n} g x^{m-n}$, $\deg(f') \leq m-1$

(2)

$f' = gq' + r'$, $\deg(r') < \deg(g)$

$$f = \left(\frac{a_m}{b_m} g x^{m-n} + gq'\right) + r'$$

$$= g\left(\frac{a_m}{b_n} x^{m-n} + q'\right) + r'$$

<u>Thm 4.2.2</u> $I \subseteq F[x]$ w/

ii) I closed under addition

iii) If $f \in I$, $g \in F[x]$, then $fg \in I$

$\langle f \rangle = f F[x] \subseteq I$

Then $I = dF[x]$ for some $d \in F[x]$

If $I \neq \{\emptyset\}$, then $d \neq 0$

Pf. Let $d \in I \setminus \{0\}$ minimum degree

(if $\deg = 0$, $I = 1 \cdot F[x]$)

Let $\deg(d) = k$. Assert $f \in I$ implies $d \mid f$

$r = f - dq \in I$, $f \in I$

$\deg(r) < \deg(d)$

So $\deg(r) = -\infty$

Thus, $r = 0$

If d_1 is another w/ $I = d_1 F[x]$, then

d_1, d non-zero constant multiples

$\therefore$ Unique if monic

(*) Greatest common divisor $\gcd(f,g) = (f,g)$
of f, g in $F[x]$ is d if:
i) $d \mid f$, $d \mid g$
ii) $\delta \mid f, \delta \mid g \Rightarrow \delta \mid d$

* $\gcd(1, f) = 1$
$\gcd(0, f) = f$, any $f \in F[x] \setminus \{0\}$

Thm. 4.2.4 If $f \neq 0$ or $g \neq 0 \in F[x]$, $\exists\, a, b \in F[x]$
s.t. $d = a \cdot f + b \cdot g \in \langle f \rangle + \langle g \rangle$
$= (f, g)$

↳ Take arbitrary
$$\begin{array}{r} af + bg \\ +\alpha f + \beta g \\ \hline (a+\alpha)f + (b+\beta)g \end{array}$$

↳ $\gamma(\alpha f + \beta g) = (\gamma\alpha)f + (\gamma\beta)g$

↳ $\langle f \rangle + \langle g \rangle = dF[x]$, some $d \neq 0$

↳ $d \mid f$, $d \mid g$ Let $\delta \mid f$, $\delta \mid g$, then $\delta \mid (af + bg) = d$

* Euclidean Algorithm for $F[x]$. f, g polys not both 0. Assume $\deg(g) \geq \deg(f)$

(4)

$g = fq + r$, $\deg(r) < \deg(f)$

$d = \gcd(f, g) \Rightarrow d \mid (g - fq) = r$

Thus $d \mid (f, r) = \delta$

$\delta \mid f$, $\delta \mid r$, so $\delta \mid (g = fq + r)$

$\Rightarrow \delta \mid d$, $\delta = d$ if monic

$\gcd(f, g) = \gcd(f, r)$

* $g = fq_1 + r_1$

$f = r_1 q_2 + r_2$

$r_1 = r_2 q_3 + r_3$

$\vdots$

$r_{n-1} = r_n q_{n+1} + 0$

$r_n = (r_i, r_{i+1}) \ \forall i$