HW#8 Solutions

Problem 1.

(a) We have, for $0 \le x \le 1$,

$$\begin{aligned}
\mathbf{P}(Y \ge X) &= \int_0^1 \int_x^\infty f_{X,Y}(x,y)\, dy\, dx \\
&= \int_0^1 \int_x^\infty 2e^{-2y}\, dy\, dx \\
&= \int_0^1 e^{-2x}\, dx \\
&= \frac{1}{2}(1 - e^{-2}).
\end{aligned}$$

(b) Given that $Y = y$, we have $Z = X + y$, which has the same PDF as X, but shifted to the right by y. Mathematically,

$$f_{Z|Y}(z\,|\,y) = f_X(z-y) = 1, \qquad y \le z \le y+1.$$

(c) Since $Z = X + Y = 3$ and X takes values between 0 and 1, Y can only be between 2 and 3. Thus, $f_{Y|Z}(y\,|\,3) = 0$, for $y \notin [2,3]$. For $y \in [2,3]$, we proceed as follows. We have

$$f_{Y|Z}(y\,|\,3) = \frac{f_{Y,Z}(y,3)}{f_Z(z)} = \frac{f_{Z|Y}(3\,|\,y) f_Y(y)}{f_Z(3)}.$$

By the result of part (b),

$$f_{Z|Y}(3\,|\,y) = 1, \qquad 2 \le y \le 3.$$

Furthermore,

$$f_Z(3) = \int_2^3 f_{Z|Y}(3\,|\,y) f_Y(y)\, dy = \int_2^3 2e^{-2y}\, dy = e^{-4} - e^{-6}.$$

Therefore,

$$f_{Y|Z}(y\,|\,3) = \frac{2e^{-2y}}{e^{-4} - e^{-6}}, \qquad 2 \le y \le 3.$$

Problem 2.

(a) Let W be the event that the machine is functional. Using a version of the total probability theorem, we have

$$\begin{aligned}\mathbf{P}(W) &= \int_0^1 \mathbf{P}(W \mid P = p) f_P(p)\, dp \\ &= \int_0^1 p \cdot 1\, dp \\ &= \frac{1}{2}.\end{aligned}$$

(b) Let A be the event that the machine was functional on m of the last n days. Using Bayes' rule,

$$\begin{aligned}f_{P|A}(p) &= \frac{\mathbf{P}(A \mid p) f_P(p)}{\mathbf{P}(A)} \\ &= \frac{\binom{n}{k} p^k (1-p)^{n-k}}{\binom{n}{k} \int_0^1 p^k (1-p)^{n-k} dp} \\ &= \frac{(n+1)!}{k!(n-k)!} \cdot p^k (1-p)^{n-k}.\end{aligned}$$

(c) We use our solution to part (c) above, and proceed similar to part (b):

$$\begin{aligned}\mathbf{P}(W \mid A) &= \int_0^1 \mathbf{P}(W \mid P = p) f_{P|A}(p)\, dp \\ &= \int_0^1 p f_{P|A}(p)\, dp \\ &= \frac{(n+1)!}{k!(n-k)!} \int_0^1 p^{k+1} (1-p)^{n-k} dp \\ &= \frac{k+1}{n+2}.\end{aligned}$$

Problem 3.

We know from our prior work that

$$f_X(x|B) = \begin{cases} 0, & x \leq a, \\ \frac{f_X(x)}{P[B]}, & a < x \leq b, \\ 0, & x > b. \end{cases}$$

Hence

$$E[X|B] = \int_a^b x f(x) dx / P[B].$$

Here

$$\begin{aligned} P[B] &= F_X(b) - F_X(a) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-1}^{2} \exp(-\frac{1}{2}x^2) dx \\ &= \text{erf}(2) - \text{erf}(-1) \\ &= \text{erf}(2) + \text{erf}(1) \\ &\simeq .82. \end{aligned}$$

Then

$$E[X|B] = \frac{1}{.82} \frac{1}{\sqrt{2\pi}} \int_{-1}^{2} x e^{-\frac{1}{2}x^2} dx = (e^{-\frac{1}{2}} - e^{-2})/2.06 \simeq .22$$

Problem 4.

Let the number of units manufactured at the various sites be denoted n_A, n_B, and n_C, with total number of units simply n. Then from the problem statement we know that

$$n_A = 3n_B \quad \text{and} \quad n_B = 2n_C,$$

and of course $n = n_A + n_B + n_C$. Then from classical probabilities, we get the probability of a unit selected 'at random' as

$$P[A] = \frac{n_A}{n} = \frac{6}{9}, P[B] = \frac{n_B}{n} = \frac{2}{9}, \text{ and } P[C] = \frac{n_C}{n} = \frac{1}{9},$$

where we define event $A \triangleq \{\text{unit comes from plant } A\}$, and so forth for events B and C. Now we can use the concept of conditional expectation to write

$$\begin{aligned} E[X] &= E[X|A]P[A] + E[X|B]P[B] + E[X|C]P[C] \\ &= \frac{1}{5} \int_0^\infty x e^{-x/5} dx \frac{6}{9} + \frac{1}{6.5} \int_0^\infty x e^{-x/6.5} dx \frac{2}{9} + \frac{1}{10} \int_0^\infty x e^{-x/10} dx \frac{1}{9} \\ &= 5\frac{6}{9} + 6.5\frac{2}{9} + 10\frac{1}{9} \approx 5.89 \text{ years.} \end{aligned}$$

Problem 5. Let C denote the event that $X^2 + Y^2 \geq c^2$. The probability $\mathbf{P}(C)$ can be calculated using polar coordinates, as follows:

$$\begin{aligned}\mathbf{P}(C) &= \frac{1}{2\pi\sigma^2}\int_0^{2\pi}\int_c^{\infty} re^{-r^2/2\sigma^2}\,dr\,d\theta \\ &= \frac{1}{\sigma^2}\int_c^{\infty} re^{-r^2/2\sigma^2}\,dr \\ &= e^{-c^2/2\sigma^2}.\end{aligned}$$

Thus, for $(x, y) \in C$,

$$f_{X,Y|C}(x,y) = \frac{f_{X,Y}(x,y)}{\mathbf{P}(C)} = \frac{1}{2\pi\sigma^2}e^{-\frac{1}{2\sigma^2}(x^2+y^2-c^2)}.$$

Problem 6.

(a) The value of a is determined from

$$1 = \int_0^{40} ax\,dx = a\left(\frac{x^2}{2}\right)\Big|_0^{40} = 800a,$$

so that $a = 1/800$. We have

$$f_{X,Y}(x,y) = \begin{cases} 1/1600, & \text{if } 0 \leq x \leq 40 \text{ and } 0 \leq y \leq 2x, \\ 0, & \text{otherwise.}\end{cases}$$

(b) $\mathbf{P}(Y > X) = 1/2$.

(c) Let $Z = Y - X$. We have

$$f_Z(z) = \begin{cases} \dfrac{1}{1600}z + \dfrac{1}{40}, & \text{if } -40 < z \leq 0, \\ -\dfrac{1}{1600}z + \dfrac{1}{40}, & \text{if } 0 < z \leq 40, \\ 0, & \text{otherwise,}\end{cases}$$

$$\mathbf{E}[Z] = 0.$$

(b) Since the conditional PMF $P_{Y|X}(y|x)$ found in part a is just the Poisson PMF $P_N(n)$ shifted right by x, then the conditional mean $E[Y|X=x]$ must be $\mu + x = x + 5$. Alternatively, we can directly use the linearity of conditional expectation, as follows

$$\begin{aligned} E[Y|X=x] &= E[X+N|X=x] \\ &= E[X|X=x] + E[N|X=x] \\ &= x + E[N] \\ &= x + 5, \end{aligned}$$

where the second to last line follows since X and N are independent.

Problem 7.

We use Bayes' formula for pdf's:

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) f_X(x)}{f_Y(y)}.$$

We have

$$f_X(x) = \frac{1}{2}\mathrm{rect}(\frac{x}{2}).$$

Then

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dx = \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_X(x) dx \\ &= \int_{-1}^{1} \frac{1}{2} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(y-x)^2}{2\sigma^2}\right] dx. \end{aligned}$$

Let $\xi = \frac{x-y}{\sigma}$, then $d\xi = \frac{dx}{\sigma}$ and we obtain

$$f_Y(y) = \frac{1}{2}\int_{\frac{-1-y}{\sigma}}^{\frac{1-y}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\xi^2} d\xi = \frac{1}{2}\left[\mathrm{erf}(\frac{1-y}{\sigma}) - \mathrm{erf}(\frac{-1-y}{\sigma})\right].$$

But $erf(x) = -erf(-x)$, hence

$$f_Y(y) = \frac{1}{2}\left[\mathrm{erf}(\frac{1+y}{\sigma}) - \mathrm{erf}(\frac{y-1}{\sigma})\right].$$

Then finally

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x) f_X(x)}{f_Y(y)} = \frac{\frac{1}{\sqrt{2\pi}\,\sigma}\exp[-\frac{(y-x)^2}{2\sigma^2}]\mathrm{rect}(\frac{x}{2})}{\mathrm{erf}(\frac{1+y}{\sigma}) - \mathrm{erf}(\frac{y-1}{\sigma})}.$$

Problem 8.

This is a rather classic problem in detection theory.

$$\begin{aligned}
P[A|M] &= P[X \geq 0.5|M] \\
&= \frac{1}{\sqrt{2\pi}} \int_{0.5}^{\infty} e^{-\frac{1}{2}(x-1)^2} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-0.5}^{\infty} e^{-\frac{1}{2}y^2} dy, \qquad \text{with } y \triangleq x-1, \\
&= \frac{1}{2} + \operatorname{erf}(0.5) \doteq 0.69.
\end{aligned}$$

$$\begin{aligned}
P[A|M^c] &= P[X \geq 0.5|M^c] \\
&= \frac{1}{\sqrt{2\pi}} \int_{0.5}^{\infty} e^{-\frac{1}{2}x^2} dx \\
&= \frac{1}{2} - \operatorname{erf}(0.5) \doteq 0.31,
\end{aligned}$$

$$\begin{aligned}
P[A^c|M^c] &= P[X < 0.5|M^c] \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0.5} e^{-\frac{1}{2}x^2} dx \\
&= \frac{1}{2} + \operatorname{erf}(0.5) \doteq 0.69,
\end{aligned}$$

$$\begin{aligned}
P[A^c|M] &= P[X < 0.5|M] \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0.5} e^{-\frac{1}{2}(x-1)^2} dx \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-0.5} e^{-\frac{1}{2}y^2} dy, \qquad \text{again with } y \triangleq x-1, \\
&= \frac{1}{2} - \operatorname{erf}(0.5) \doteq 0.31.
\end{aligned}$$

From Bayes' Theorem

$$\begin{aligned}
P[M|A] &= \frac{P[A|M]P[M]}{P[A]} = 0.69 \frac{P[M]}{0.69P[M] + 0.31(1-P[M])}, \\
P[M^c|A] &= \frac{P[A|M^c]P[M^c]}{P[A]} = 0.31 \frac{P[M^c]}{0.69P[M^c] + 0.31(1-P[M^c])}, \\
P[M|A^c] &= \frac{P[A^c|M]P[M]}{P[A^c]} = 0.31 \frac{P[M]}{0.31P[M] + 0.69(1-P[M])}, \text{ and} \\
P[M^c|A^c] &= \frac{P[A^c|M^c]P[M^c]}{P[A^c]} = 0.69 \frac{P[M^c]}{0.31P[M] + 0.69(1-P[M])}.
\end{aligned}$$

As a partial check, we note that $P[M|A]+P[M^c|A]=1$ as it must, and likewise for $P[M|A^c]+P[M^c|A^c]$. Then, for $P[M]=10^{-3}$, we get $P[M|A]\simeq 2\times 10^{-3}$, $P[M^c|A]\simeq 0.998$, $P[M|A^c]\simeq 0.45\times 10^{-3}$, and $P[M^c|A^c]\simeq 0.9996$. But, for $P[M]=10^{-6}$, we get $P[M|A]\simeq 2.2\times 10^{-6}$, $P[M^c|A]\simeq 0.999998$, $P[M|A^c]\simeq 0.45\times 10^{-36}$, and $P[M^c|A^c]\simeq 0.999998$. Thus, because of the uncertainty in the prior probability $P[M]$, these calcuated probability numbers have little value for decision making.