Problem Set #4 Solutions

1. We first calculate the distance vectors associated the difference between point 2 and point 1 and between point 3 and point 1. We find $\mathbf{R}_{12} = \hat{\mathbf{x}}(5-1)+\hat{\mathbf{y}}(-3-5)+\hat{\mathbf{z}}(5-3) = \hat{\mathbf{x}}4 - \hat{\mathbf{y}}8$ and $\mathbf{R}_{13} = \hat{\mathbf{x}}2 - \hat{\mathbf{y}}2 - \hat{\mathbf{z}}8$. The area of the parallelogram defined by $\mathbf{R}_{12}$ and $\mathbf{R}_{13}$ is given by $|\mathbf{R}_{12} \times \mathbf{R}_{13}| = |\hat{\mathbf{x}}64 + \hat{\mathbf{y}}32 + \hat{\mathbf{z}}8| = 72$. The area of the triangle is half this area and equals 36.

2. a. When the two vectors are parallel, the cross product equals zero. Calculating the cross product, we find that $\mathbf{A}\times\mathbf{B} = \hat{\mathbf{x}}(-3B_z-8)+\hat{\mathbf{y}}(4B_x-2B_z)+\hat{\mathbf{z}}(4+3B_x) = 0$. We have three relations, but only two are independent and solving for B_x and B_z, we find $B_x = -4/3$ and $B_z = -8/3$.

   b. When the two vectors are perpendicular, the dot product is zero, and we find $2B_x - 6 + 4B_z = 0$ or $B_x + 2B_z = 3$. This relationship defines the plane that is perpendicular to $\mathbf{A}$. Any vector in that plane will be perpendicular to $\mathbf{A}$.

3. We begin by writing $\mathbf{B} = \mathbf{C} + \mathbf{D}$, where $\mathbf{C}$ is parallel to $\mathbf{A}$ and $\mathbf{D}$ is orthogonal to $\mathbf{A}$.

   a. We now write $\mathbf{C} = c\mathbf{A}$. Taking the dot product of of $\mathbf{B} = c\mathbf{A} + \mathbf{D}$ with $\mathbf{A}$, we find $\mathbf{B}\cdot\mathbf{A} = c|\mathbf{A}|^2$, so that $c = (\mathbf{B}\cdot\mathbf{A})/|\mathbf{A}|^2$. We conclude that $\mathbf{C} = c\mathbf{A} = \mathbf{A}(\mathbf{B}\cdot\mathbf{A})/|\mathbf{A}|^2 = \hat{\mathbf{a}}(\mathbf{B}\cdot\hat{\mathbf{a}})$.

   b. Subtraction of $\mathbf{C}$ from $\mathbf{A}$ yields $\mathbf{D}$.

4. a. We have $x = 2\cos(3\pi/4) = -\sqrt{2} \simeq -1.414$, $y = 2\sin(3\pi/4) = \sqrt{2} \simeq 1.414$, which yields $P_1(r = 2, \theta = 3\pi/4, z = -2) = P_1(x = -\sqrt{2}, y = \sqrt{2}, z = -2)$.

   b. We have $x = 3\cos(0) = 3$ and $y = 3\sin(0) = 0$, which yields $P_2(x = 3, y = 0, z = -2)$.

   c. We have $x = 4\cos(\pi/2) = 0$ and $y = 4\sin(\pi/2) = 4$, which yields $P_3(x = 0, y = 4, z = 5)$.

5. We have for the cylindrical to spherical unit vector transformations,

$$\begin{aligned}
\hat{\mathbf{R}} &= \hat{\mathbf{x}}\sin\theta\cos\phi + \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta \\
&\quad (\hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi)\sin\theta\cos\phi + (\hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi)\sin\theta\sin\phi = \hat{\mathbf{z}}\cos\theta \\
&= (\hat{\mathbf{r}}\cos^2\phi + \hat{\mathbf{r}}\sin^2\phi)\sin\theta + \hat{\boldsymbol{\phi}}0 + \hat{\mathbf{z}}\cos\theta \\
&= \hat{\mathbf{r}}\sin\theta + \hat{\mathbf{z}}\cos\theta.
\end{aligned}$$

$$\begin{aligned}
\hat{\boldsymbol{\theta}} &= \hat{\mathbf{x}}\cos\theta\cos\phi + \hat{\mathbf{y}}\cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta \\
&= (\hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi)\cos\theta\cos\phi + (\hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi)\cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta \\
&= \hat{\mathbf{r}}\cos\theta - \hat{\mathbf{z}}\sin\theta.
\end{aligned}$$

$$\begin{aligned}
\hat{\boldsymbol{\phi}} &= -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi \\
&= -(\hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi)\sin\phi + (\hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi)\cos\phi = \hat{\boldsymbol{\phi}}.
\end{aligned}$$

Replacing $\hat{\mathbf{R}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ with A_R, A_θ and A_ϕ, we obtain the vector component transformations. For the spherical to cylindrical unit vector transformations, we have

$$\begin{aligned}
\hat{\mathbf{r}} &= \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi \\
&= (\hat{\mathbf{R}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi)\cos\phi \\
&\qquad + (\hat{\mathbf{R}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\sin\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi)\sin\phi \\
&= \hat{\mathbf{R}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta.
\end{aligned}$$

$$\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi = \hat{\boldsymbol{\phi}}.$$

$$\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta.$$

To obtain the corresponding vector component transformations, we replace $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\mathbf{z}}$ with A_r, A_ϕ, and A_z.

6. a. We have

$$d = \left[(0-1)^2 + (3-1)^2 + (5-2)^2\right]^{1/2} = \sqrt{14} \simeq 3.74.$$

b. We have

$$\begin{aligned}
d &= \left[4^2 + 2^2 - 2\cdot 4\cdot 2\cos(\pi/2 - \pi/3) + (3-1)^2\right]^{1/2} \\
&= \sqrt{16 + 4 - 16\cos(\pi/6) + 4} = \sqrt{16 + 4 - 8\cdot 3^{1/2} + 4} \simeq 3.185
\end{aligned}$$

c. We have

$$\begin{aligned}
d &= \left\{4^2 + 3^2 - 2\cdot 4\cdot 3[\cos(\pi/2)\cos(\pi) + \sin(\pi/2)\sin(\pi)\cos(\pi - \pi/2)]\right\}^{1/2} \\
&= \sqrt{4^2 + 3^2} = 5
\end{aligned}$$

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7 a. We have $\mathbf{A} = \hat{\mathbf{x}}(x+y) = (\hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi)(r\cos\phi + r\sin\phi) = \hat{\mathbf{r}}r\cos\phi(\cos\phi + \sin\phi) - \hat{\boldsymbol{\phi}}r\sin\phi(\cos\phi+\sin\phi)$. With $x = 1$ and $y = 2$, it follows that $r = \sqrt{5}$, $\cos\phi = 1/\sqrt{5}$, $\sin\phi = 2/\sqrt{5}$, and $\mathbf{A} = \hat{\mathbf{r}}(3/\sqrt{5}) - \hat{\boldsymbol{\phi}}(6/\sqrt{5}) \simeq \hat{\mathbf{r}}1.342 - \hat{\boldsymbol{\phi}}2.683$

b. We may write $\mathbf{B} = (\hat{\mathbf{x}} - \hat{\mathbf{y}})(y-x) = \hat{\mathbf{r}}r(2\cos\phi\sin\phi - 1) + \hat{\boldsymbol{\phi}}r(\cos^2\phi - \sin^2\phi) = -\hat{\mathbf{r}}(1-\sin 2\phi) + \hat{\boldsymbol{\phi}}\cos 2\phi$. In this case, $r = 1$ and $\phi = 0$, so that $\mathbf{B} = -\hat{\mathbf{r}} + \hat{\boldsymbol{\phi}}$

c. We have $\mathbf{C} = \hat{\mathbf{r}}\sin\phi\cos\phi(\sin\phi - \cos\phi) - \hat{\boldsymbol{\phi}}(\cos^3\phi + \sin^3\phi) + \hat{\mathbf{z}}4$. We have $\cos\phi = -\sin\phi = 1/\sqrt{2}$. After substitution, we find $\mathbf{C} = \hat{\mathbf{r}}\,(1/\sqrt{2}) + \hat{\mathbf{z}}4 \simeq \hat{\mathbf{r}}\,0.7071 + \hat{\mathbf{z}}4$.

d. We have $\mathbf{D} = (\hat{\mathbf{r}}\sin\theta + \hat{\mathbf{z}}\cos\theta)\sin\theta + (\hat{\mathbf{r}}\cos\theta - \hat{\mathbf{z}}\sin\theta)\cos\theta + \hat{\boldsymbol{\phi}}\cos^2\theta = \hat{\mathbf{r}} + \hat{\boldsymbol{\phi}}\cos^2\theta = \hat{\mathbf{r}} + \hat{\boldsymbol{\phi}}z^2/(r^2+z^2)$. In this case, we have $\cos\theta = z/(z^2+r^2)^{1/2} = 0$. So, we conclude, $\mathbf{D} = \hat{\mathbf{r}} + \hat{\boldsymbol{\phi}}0.5$.

e. We have $\mathbf{E} = (\hat{\mathbf{r}}\sin\theta + \hat{\mathbf{z}}\cos\theta)\cos\phi + (\hat{\mathbf{r}}\cos\theta - \hat{\mathbf{z}}\sin\theta)\sin\phi + \hat{\boldsymbol{\phi}}\sin^2\theta = \hat{\mathbf{r}}(\sin\theta\cos\phi + \cos\theta\sin\phi) + \hat{\boldsymbol{\phi}}\sin^2\theta + \hat{\mathbf{z}}(\cos\theta\cos\phi - \sin\theta\sin\phi) = \hat{\mathbf{r}}\sin(\theta+\phi) + \hat{\boldsymbol{\phi}}\sin^2\theta + \hat{\mathbf{z}}\cos(\theta+\phi) = \hat{\mathbf{r}}[1/(r^2+z^2)^{1/2}](r\cos\phi + z\sin\phi) + \hat{\boldsymbol{\phi}}r^2/(r^2+z^2) + \hat{\mathbf{z}}[1/(r^2+z^2)^{1/2}](z\cos\phi - r\sin\phi)$. When $\theta = \pi/2$ and $\phi = \pi$, we have $\sin(\theta+\phi) = -1$, $\cos(\theta+\phi) = 0$, and $\sin^2\theta = 1$. We conclude, $\mathbf{E} = -\hat{\mathbf{r}} + \hat{\boldsymbol{\phi}}$.

8. a. We begin by recalling that

$$A_x = A_r\cos\phi - A_\phi\sin\phi, \qquad A_y = A_r\sin\phi + A_\phi\cos\phi. \tag{8.1}$$

We must also use the definition that $\partial V(x,y)/\partial x$ is defined as the derivative of V, holding y constant, where V is any vector. It will be useful on occasion to explicitly note the variable that we are holding constant by writing $\partial V/\partial x|_y$ to indicate that y is being held constant. To transform from Cartesian to polar coordinates, we first write

$$\left.\frac{\partial V}{\partial x}\right|_y = \left.\frac{\partial V}{\partial r}\right|_\phi \left.\frac{\partial r}{\partial x}\right|_y + \left.\frac{\partial V}{\partial \phi}\right|_r \left.\frac{\partial \phi}{\partial x}\right|_y . \tag{8.2}$$

We have similarly,

$$\left.\frac{\partial V}{\partial y}\right|_x = \left.\frac{\partial V}{\partial r}\right|_\phi \left.\frac{\partial r}{\partial y}\right|_x + \left.\frac{\partial V}{\partial \phi}\right|_r \left.\frac{\partial \phi}{\partial y}\right|_x . \tag{8.3}$$

Using the relations $r = (x^2+y^2)^{1/2}$ and $\phi = \tan^{-1}(y/x)$, we find

$$\begin{aligned}
\left.\frac{\partial r}{\partial x}\right|_y &= \frac{x}{(x^2+y^2)^{1/2}} = \cos\phi, \qquad & \left.\frac{\partial \phi}{\partial x}\right|_y &= \frac{-y/x^2}{1+(y/x)^2} = -\frac{1}{r}\sin\phi, \\
\left.\frac{\partial r}{\partial y}\right|_x &= \frac{y}{(x^2+y^2)^{1/2}} = \sin\phi, \qquad & \left.\frac{\partial \phi}{\partial y}\right|_x &= \frac{(1/x)}{1+(y/x)^2} = \frac{1}{r}\cos\phi.
\end{aligned} \tag{8.4}$$

Setting $V = A_x$ and using (8.1), (8.2), and (8.4), we have

$$\begin{aligned}
\left.\frac{\partial A_x}{\partial x}\right|_y &= \cos\phi \left.\frac{\partial[A_r\cos\phi - A_\phi\sin\phi]}{\partial r}\right|_\phi - \frac{1}{r}\sin\phi \left.\frac{\partial[A_r\cos\phi - A_\phi\sin\phi]}{\partial\phi}\right|_r \\
&= \cos^2\phi\frac{\partial A_r}{\partial r} - \sin\phi\cos\phi\frac{\partial A_\phi}{\partial r} + \frac{A_r}{r}\sin^2\phi + \frac{A_\phi}{r}\sin\theta\cos\theta \\
&\quad - \frac{1}{r}\sin\phi\cos\phi\frac{\partial A_r}{\partial\theta} + \frac{1}{r}\sin^2\phi\frac{\partial A_\phi}{\partial\phi}.
\end{aligned} \tag{8.5}$$

In a similar way, we find

$$\begin{aligned}
\left.\frac{\partial A_y}{\partial y}\right|_x &= \sin\phi \left.\frac{\partial[A_r\sin\phi + A_\phi\cos\phi]}{\partial r}\right|_\phi + \frac{1}{r}\cos\phi \left.\frac{\partial[A_r\sin\phi + A_\phi\cos\phi]}{\partial\phi}\right|_r \\
&= \sin^2\phi\frac{\partial A_r}{\partial r} + \sin\phi\cos\phi\frac{\partial A_\phi}{\partial r} + \frac{A_r}{r}\cos^2\phi - \frac{A_\phi}{r}\sin\theta\cos\theta \\
&\quad + \frac{1}{r}\sin\phi\cos\phi\frac{\partial A_r}{\partial\theta} + \frac{1}{r}\cos^2\phi\frac{\partial A_\phi}{\partial\phi}.
\end{aligned} \tag{8.6}$$

Note that we suppressed the explicit mention of what is being held constant in most of Eqs. (8.5) and (8.6). Combining Eqs. (8.5) and (8.6), we conclude

$$\begin{aligned}
\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} &= \frac{\partial A_r}{\partial r} + \frac{1}{r}A_r + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} \\
&= \frac{1}{r}\frac{\partial(rA_r)}{\partial r} + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi}.
\end{aligned} \tag{8.7}$$

b. Since there is no dependence on a third variable, our general expression for the divergence becomes

$$\frac{1}{h_1h_2}\left(\frac{\partial}{\partial x_1}h_2A_1 + \frac{\partial}{\partial x_2}h_1A_2\right). \tag{8.8}$$

In the case of polar coordinates, we have $x_1 = r$, $h_1 = 1$, $x_2 = \phi$, and $h_2 = r$. Substituting these quantities into Eq. (8.8), we immediately arrive at Eq. (8.7).

It is clearly much simpler to use the general expression. You can imagine how much more complex this analysis becomes when all three dimensions must be considered and we are trying to calculate the curl instead of the divergence. It is actually much simpler to derive the general formula and use it.

9. a. On the first leg of the contour, we have $y = 0$ and $d\mathbf{l} = \hat{\mathbf{x}}dx$. Hence, we have $\mathbf{E}\cdot d\mathbf{l} = [\hat{\mathbf{x}}x0 - \hat{\mathbf{y}}(x^2+0)]\cdot\hat{\mathbf{x}}dx = 0$. On the second leg, we have $x = 1$ and $d\mathbf{l} = \hat{\mathbf{y}}dy$. Hence, we have $\mathbf{E}\cdot d\mathbf{l} = [\hat{\mathbf{x}}1y - \hat{\mathbf{y}}(1+2y^2)]\cdot\hat{\mathbf{y}}dy = -(1+2y^2)dy$, where y varies between 0 and 1. On the last leg of the contour, we have $x = y$ and

$d\mathbf{l} = (\hat{\mathbf{x}} + \hat{\mathbf{y}})dy$. Hence, we have $\mathbf{E} \cdot d\mathbf{l} = [\hat{\mathbf{x}}y^2 - \hat{\mathbf{y}}(3y^2)] \cdot (\hat{\mathbf{x}} + \hat{\mathbf{y}})dy = -2y^2 dy$, where y varies from 1 to 0. We conclude

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\int_0^1 (1 + 2y^2)dy - \int_1^0 2y^2\, dy = -1 - 2/3 + 2/3 = -1$$

b. We have

$$\nabla \times \mathbf{E} = -\hat{\mathbf{z}} \left[\frac{\partial(x^2 + 2y^2)}{\partial x} + \frac{\partial(xy)}{\partial y} \right] = -3x\hat{\mathbf{z}}.$$

We also have $d\mathbf{s} = \hat{\mathbf{z}}ds$, so that

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{s} = -3 \int_0^1 dy \int_y^1 x\, dx = -\frac{3}{2} \int_0^1 (1 - y^2)dy = -1.$$

We note that the results of parts (a) and (b) are equals as is required by Stokes's theorem.

10 a. $\nabla V = \hat{\mathbf{x}}2y^2z^4 + \hat{\mathbf{y}}4xyz^4 + \hat{\mathbf{z}}8xy^2z^3$; $\nabla^2 V = \nabla \cdot \nabla V = 4xz^4 + 24xy^2z^2$

b. $\nabla V = \hat{\mathbf{x}}(3y + z) + \hat{\mathbf{y}}(3x + 2z) + \hat{\mathbf{z}}(x + 2y)$; $\nabla^2 V = \nabla \cdot \nabla V = 0$

c. $\nabla V = -\hat{\mathbf{x}}[6x/(x^2 + y^2)^2] - \hat{\mathbf{y}}[6y/(x^2 + y^2)^2]$; so that

$$\begin{aligned} \nabla^2 V = \nabla \cdot \nabla V &= -\frac{6}{(x^2 + y^2)^2} + \frac{24x^2}{(x^2 + y^2)^3} - \frac{6}{(x^2 + y^2)^2} - \frac{24y^2}{(x^2 + y^2)^3} \\ &= \frac{12}{(x^2 + y^2)^2} \end{aligned}$$

d. We have

$$\begin{aligned} \nabla V &= \frac{\partial[5\exp(-r)\cos\phi]}{\partial r} + \frac{1}{r}\frac{\partial[5\exp(-r)\cos\phi]}{\partial\phi} \\ &= -\hat{\mathbf{r}}5\exp(-r)\cos\phi - \hat{\boldsymbol{\phi}}\frac{1}{r}5\exp(-r)\sin\phi. \end{aligned}$$

It follows that

$$\begin{aligned} \nabla^2 V = \nabla \cdot \nabla V &= \frac{1}{r}\frac{\partial}{\partial r}[-5r\exp(-r)\cos\phi] - \frac{1}{r^2}\frac{\partial}{\partial\phi}[5\exp(-r)\sin\phi] \\ &= \left(1 - \frac{1}{r} - \frac{1}{r^2}\right) 5\exp(-r)\cos\phi \end{aligned}$$

e. We have

$$\begin{aligned} \nabla V &= \frac{\partial[10\exp(-R)\sin\theta]}{\partial R} + \frac{1}{R}\frac{\partial[10\exp(-R)\sin\theta]}{\partial\theta} \\ &= -\hat{\mathbf{r}}10\exp(-R)\sin\theta + \hat{\boldsymbol{\theta}}\frac{1}{R}10\exp(-R)\cos\theta. \end{aligned}$$

It follows that

$$\begin{aligned}\nabla^2 V = \nabla \cdot \nabla V &= \frac{1}{R^2}\frac{\partial}{\partial R}[-10R^2 \exp(-R)\sin\theta] \\ &\quad + \frac{1}{R^2\sin\theta}\frac{\partial}{\partial\theta}[10\exp(-r)\sin\theta\cos\theta] \\ &= \left(1 - \frac{2}{R} - \frac{1-\cot^2\theta}{R^2}\right) 10\exp(-R)\sin\theta\end{aligned}$$