Problem Set #2 Solutions

1. Kirchoff's voltage law becomes,

$$\begin{aligned} v(z,t) - \frac{R'}{2}\Delta z\, i(z,t) - \frac{L'}{2}\Delta z \frac{\partial i(z,t)}{\partial z} &= v_{\text{node}}(z,t) \\ &= \frac{R'}{2}\Delta z\, i(z+\Delta z,t) + \frac{L'}{2}\Delta z \frac{\partial i(z+\Delta z,t)}{\partial z} + v(z+\Delta z,t), \end{aligned} \tag{2.1}$$

and the current law becomes,

$$i(z,t) - G'\Delta z\, v_{\text{node}}(z,t) - C'\Delta z \frac{\partial v_{\text{node}}(z,t)}{\partial t} = i(z+\Delta z,t), \tag{2.2}$$

where $v_{\text{node}}(z,t)$ is the voltage at the node point above the capacitor and the resistor connecting the two conductors. Equation (2.1) becomes

$$\begin{aligned} -\left[\frac{v(z+\Delta z,t) - v(z,t)}{\Delta z}\right] = R' &\left[\frac{i(z,t) + i(z+\Delta z,t)}{2}\right] \\ &+ L'\frac{\partial}{\partial t}\left[\frac{i(z,t) + i(z+\Delta z,t)}{2}\right], \end{aligned} \tag{2.3}$$

while equation (2.2) becomes

$$-\left[\frac{i(z+\Delta z,t) - i(z,t)}{\Delta z}\right] = G' v_{\text{node}}(z,t) + C'\frac{\partial v_{\text{node}}(z,t)}{\partial t}, \tag{2.4}$$

where

$$\begin{aligned} v_{\text{node}}(z,t) = &\frac{1}{2}\left[v(z,t) + v(z+\Delta z,t)\right] \\ &+ \frac{R'}{4}\Delta z\left[i(z+\Delta z,t) - i(z,t)\right] + \frac{L'}{4}\Delta z\frac{\partial}{\partial z}\left[i(z+\Delta z,t) - i(z,t)\right], \end{aligned} \tag{2.5}$$

Since $(1/2)[i(z,t) + i(z+\Delta z,t)] \to i(z,t)$ as $\Delta z \to 0$, we find that when $\Delta z \to 0$, equation (2.3) becomes

$$-\frac{\partial v(z,t)}{\partial z} = R' i(z,t) + L'\frac{\partial i(z,t)}{\partial t},$$

as expected. As $\Delta z \to 0$, Eq. (2.5) becomes $v_{\text{node}}(z,t) = v(z,t)$, so that we obtain

$$-\frac{\partial i(z,t)}{\partial z} = G' v(z,t) + C'\frac{\partial v(z,t)}{\partial t}.$$

These last two equations are just the telegrapher's equations.

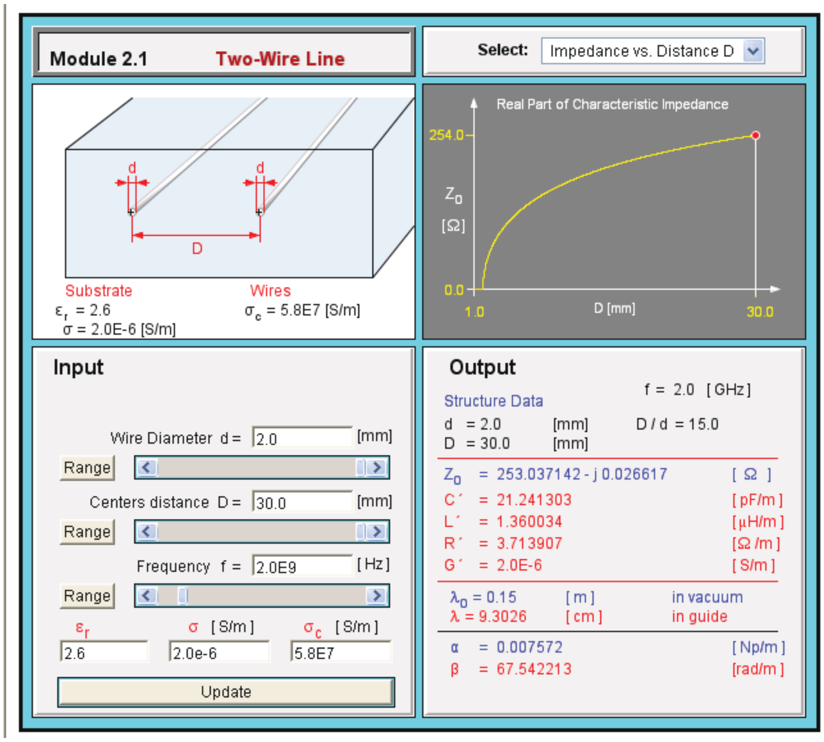
This exercise demonstrates the important point that the telegrapher's equations do not depend on the details of the physical model that is used to derive them.

2. We have the geometrical constant $\ln\{(D/d) + [(D/d)^2 - 1]^{1/2}\} = 3.40$. We also have $\mu_\mathrm{c} = \mu_0 = 1.257 \times 10^{-6}$ H/m, and we have $\sigma_\mathrm{c} = 5.80 \times 10^7$ S/m. See Ulaby et al., Tables B-2 and B-3, for μ_c and σ_c.

   a. We now have $R_\mathrm{S} = [(3.14)(2.00 \times 10^9)(4 \times 3.14 \times 10^{-7})/(5.80 \times 10^7)]^{1/2} = 1.17 \times 10^{-2}\ \Omega$, from which we obtain $R' = (0.0117)/[(3.14)(0.001)] = 3.7\ \Omega$/m. We also have $L' = (1.257 \times 10^{-6}/3.14)3.40 = 1.4 \times 10^{-6}$ H/m $= 1.4\ \mu$H/m, $G' = (3.14)(2.00 \times 10^{-6})/3.40 = 1.8 \times 10^{-6}$ S/m, $C' = (3.14)(2.60)(8.85 \times 10^{-12})/3.40 = 2.1 \times 10^{-11}$ F/m $= 21$ pF/m. We have $R' + j\omega L' = R' + j2\pi f L' = 3.72 + j1.71 \times 10^4$ and $G' + j\omega C' = 1.85 \times 10^{-7} + j0.267$. We can determine α, β, and Z_0 by hand, but it is sufficiently complicated that the solution here was obtained using MATLAB. Since MATLAB has 15 significant figures, we do not have to worry about roundoff in this case for a solution to two significant figures. We obtain $\gamma = \alpha + j\beta = [(3.72 + j1.71 \times 10^4)(1.85 \times 10^{-7} + j0.267)]^{1/2} = 0.0074 + j68$ m^{-1}, so that $\alpha = 0.0074$ Np/m and $\beta = 68$ rad/m. We also obtain $Z_0 = [(R' + j\omega L')/(G' + j\omega C')]^{1/2} = 250 - j0.027\ \Omega$. Finally, we have $u_\mathrm{p} = \omega/\beta = 1.9 \times 10^8$ m/s, which is lower than the speed of light in the vacuum by 2/3 and is consistent with what we expect.

   b. The screen shot is on the next page. Note that the answer for α differs in the second digit. The answer reported here has been verified by hand. A repeated problem in the Ulaby, et al. book is that they report more significant digits than they have.

ULABY CD MODULE OUTPUT (Problem 2):

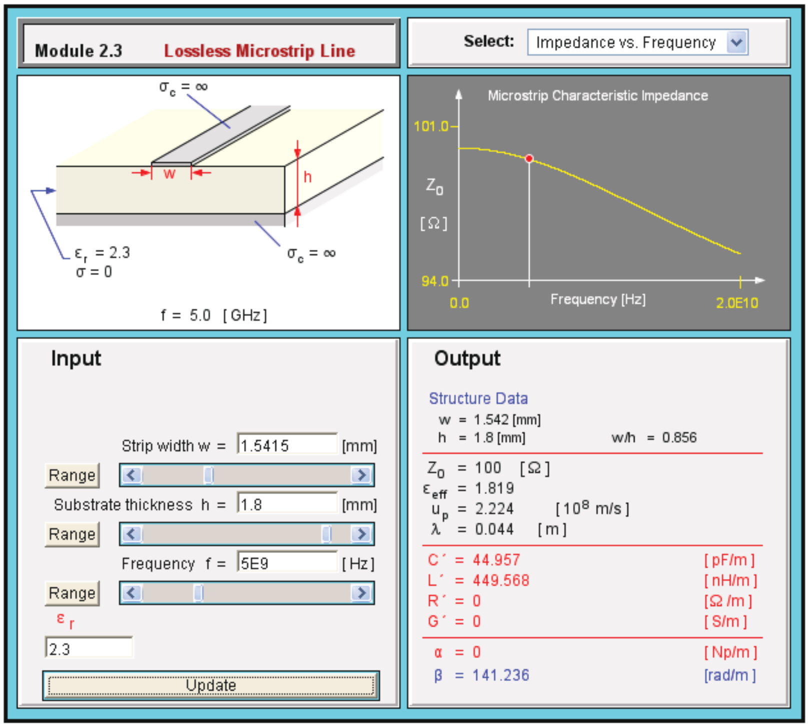


3. In this case, the geometrical constant is $\ln(b/a) = \ln(2b/2a) = \ln(2) = 0.693$. We have, as before $\mu_c = \mu_0 = 1.257 \times 10^{-6}$ H/m and $\sigma_c = 5.8 \times 10^7$ S/m.

   a. We have $R_S = [3.14(1 \times 10^9)(1.257 \times 10^{-6})/(5.80 \times 10^7)]^{1/2} = 0.00825\ \Omega$, from which we obtain $R' = (0.00825/6.28)[(1/0.0025) + (1/0.005)] = 0.79\ \Omega$/m. We also have $L' = [(1.257 \times 10^{-6})/6.28](0.693) = 1.40 \times 10^{-7}$ H/m $= 140$ nH/m, $G' = (6.28)(1 \times 10^{-3}/(0.693) = 0.0091$ S/m, and $C' = (6.28)(4.50)(8.85 \times 10^{-12})/(0.693) = 3.6 \times 10^{-10}$ F/m $= 360$ pF/m. We have $F_1 = R' + j\omega L' = 0.787 + j871\ \Omega$/m, and $F_2 = G' + j\omega C' = 0.00907 + j2.27$ S. We now find $\gamma = \alpha + j\beta = (F_1 F_2)^{1/2} = 0.109 + j44.46 = (0.11 + j44)$ m^{-1}, so that $\alpha = 0.11$ Np/m and $\beta = 44$ rad/m. We have $Z_0 = (F_1/F_2)^{1/2} = 20 + j0.030\ \Omega$. Finally, we have $u_p = \omega/\beta = 1.4 \times 10^8$ m/s. The phase velocity is reasonable.

   b. The screen shot is on the next page

ULABY CD MODULE OUTPUT (Problem 3):



4. The calculations presented here were done using MATLAB. Three significant figures are presented for the intermediate calculations, but MATLAB keeps approximately 15 digits. Since $Z_0 > 44 - 2 \times 2.3 = 39.4$, we use Eq. 2.43 in Ulaby, et al. We find $p = 2.25$ and $s = 0.861$. Hence, we obtain $w = sh = 1.54 \times 10^{-3}$ m $= 1.54$ mm. To obtain the wavelength λ, we must first calculate ϵ_{eff}, for which we need $x = 0.524$ and $y = 0.988$, using Ulaby, et al., Eq. 2.38. Using Ulaby, et al., Eq. 2.36, we now find $\epsilon_{\text{eff}} = 1.83$. The wavelength in the waveguide is given by $\lambda = c/f * \sqrt{\epsilon_{\text{eff}}} = 4.44 \times 10^{-2}$ m $= 4.44$ cm, where we use $f = 5$ GHz $= 5 \times 10^{-9}$ s^{-1}. To two significant figures, our design results are $w = 1.5$ mm and $\lambda = 4.4$ cm. The screen shot verifying these results is below. It should be noted that to obtain exactly $Z_0 = 100\ \Omega$, we used $w = 1.5415$ mm, since the module presents 5 or 6 significant figures for Z_0. However, since the original formulae are only good to within about 2%, the last two digits for w are completely meaningless.

5. Our starting point is the equations on slide 3.23, which in Ulaby's notation becomes

$$V(0,t) = \frac{Z_0}{R_{\text{g}} + Z_0}\big[V_{\text{g}}(t) + (1+\Gamma_{\text{g}})\Gamma_{\text{L}}V_{\text{g}}(t-2T) + (1+\Gamma_{\text{g}})(\Gamma_{\text{g}}\Gamma_{\text{L}})\Gamma_{\text{L}}V_{\text{g}}(t-4T)$$
$$+ (1+\Gamma_{\text{g}})(\Gamma_{\text{g}}\Gamma_{\text{L}})^2\Gamma_{\text{L}}V_{\text{g}}(t-6T) + \cdots\big],$$

$$V(l,t) = \frac{Z_0}{R_{\text{g}} + Z_0}(1+\Gamma_{\text{L}})\big[V_{\text{g}}(t-T) + \Gamma_{\text{g}}\Gamma_{\text{L}}V_{\text{g}}(t-3T)$$
$$+ (\Gamma_{\text{g}}\Gamma_{\text{L}})^2 V_{\text{g}}(t-5T) + \cdots\big].$$

This result is equivalent to Ulaby's Eq. (2.156), where the voltage is evaluated at different points along the transmission line.

a. We will first derive the first half of each equation. We begin with the equation for $V(0,t)$. We write

$$V(0,t) = \frac{Z_0 V_{\mathrm{g}}}{R_{\mathrm{g}} + Z_0} \left\{ 1 + (1+\Gamma_{\mathrm{g}})\Gamma_{\mathrm{L}} \sum_{m=0}^{m_{\mathrm{g}}-1} (\Gamma_{\mathrm{g}}\Gamma_{\mathrm{L}})^m \right\},$$

where the sum is defined as zero when $m_{\mathrm{g}} - 1 < 0$. Using the definition of the floor function, we find $m_{\mathrm{g}} = 0$ when $0 \le t < 2T$, $m_{\mathrm{g}} = 1$ when $2T \le t < 4T$, and so on. We now find that the sum is zero when $0 \le t < 2T$; it equals 1 when $2T \le t < 4T$; it equals $1 + \Gamma_{\mathrm{L}}\Gamma_{\mathrm{g}}$ when $4T \le t < 6T$; it equals $1 + \Gamma_{\mathrm{L}}\Gamma_{\mathrm{g}} + (\Gamma_{\mathrm{L}}\Gamma_{\mathrm{g}})^2$ when $6T \le t < 8T$; and so on. Hence, this expression for $V(0,t)$ is equivalent to the original expression. Using the expression for the sum of a geometric series, we now find

$$V(0,t) = \frac{Z_0 V_{\mathrm{g}}}{R_{\mathrm{g}} + Z_0} \left[1 + (1+\Gamma_{\mathrm{g}})\Gamma_{\mathrm{L}} \frac{1 - (\Gamma_{\mathrm{g}}\Gamma_{\mathrm{L}})^{m_{\mathrm{g}}}}{1 - \Gamma_{\mathrm{g}}\Gamma_{\mathrm{L}}} \right].$$

Rewriting this expression, we have

$$V(0,t) = \frac{Z_0 V_{\mathrm{g}}}{R_{\mathrm{g}} + Z_0} \left[\frac{(1 - \Gamma_{\mathrm{g}}\Gamma_{\mathrm{L}}) + (\Gamma_{\mathrm{L}} + \Gamma_{\mathrm{g}}\Gamma_{\mathrm{L}})\left[1 - (\Gamma_{\mathrm{g}}\Gamma_{\mathrm{L}})^{m_{\mathrm{g}}}\right]}{1 - \Gamma_{\mathrm{g}}\Gamma_{\mathrm{L}}} \right].$$

Finally, collecting terms, we obtain

$$V(0,t) = \frac{Z_0 V_{\mathrm{g}}}{R_{\mathrm{g}} + Z_0} \left[\frac{1 - (\Gamma_{\mathrm{g}}\Gamma_{\mathrm{L}})^{m_{\mathrm{g}}+1} + \Gamma_{\mathrm{L}}\left[1 - (\Gamma_{\mathrm{g}}\Gamma_{\mathrm{L}})^{m_{\mathrm{g}}}\right]}{1 - \Gamma_{\mathrm{g}}\Gamma_{\mathrm{L}}} \right],$$

which is the desired expression. It is even more straightforward to obtain the expression for $V(l,t)$, since we may write

$$V(l,t) = \frac{Z_0 V_{\mathrm{g}}}{R_{\mathrm{g}} + Z_0} (1+\Gamma_{\mathrm{L}}) \sum_{m=0}^{m_{\mathrm{L}}-1} (\Gamma_{\mathrm{g}}\Gamma_{\mathrm{L}})^m,$$

which, using the expression for the sum of a geometric series, becomes

$$V(l,t) = \frac{Z_0 V_{\mathrm{g}}}{R_{\mathrm{g}} + Z_0} \left[\frac{(1+\Gamma_{\mathrm{L}})\left[1 - (\Gamma_{\mathrm{g}}\Gamma_{\mathrm{L}})^{m_{\mathrm{L}}}\right]}{1 - \Gamma_{\mathrm{g}}\Gamma_{\mathrm{L}}} \right],$$

which is the desired expression. Since the reflection coefficients for the current are the same as for the voltage, except that the sign changes, we can obtain the expressions for $I(0,t)$ and $I(l,t)$ by taking the expressions for $V(0,t)$ and $V(l,t)$, replacing $\Gamma_{\rm L}$ and $\Gamma_{\rm g}$ by $-\Gamma_{\rm L}$ and $-\Gamma_{\rm g}$ and dividing through by Z_0.

To obtain the complete expressions, we now subtract the similar expressions with the delay time $t_{\rm D}$.

b. In this example, we have $\Gamma_{\rm g} = (300-100)/(300+100) = 1/2$ and $\Gamma_{\rm L} = 1$, $V_{\rm g} = 20$ V, $Z_0 = 100\ \Omega$, $R_{\rm g} = 300\ \Omega$, $T = 1$ ns, and $t_{\rm D} = 1$ ns. We are interested in times up to 10 ns. Since the load is an open circuit, we find immediately that $I(l,t) = 0$. We have $V_{\rm g}/(R_{\rm g} + Z_0) = 0.05$ A, and we have $Z_0 V_{\rm g}/(R_{\rm g} + Z_0) = 5$ V. We have $\Gamma_{\rm g}\Gamma_{\rm L} = 1/2$, so that $1 - \Gamma_{\rm g}\Gamma_{\rm L} = 1/2$. We have $1 + \Gamma_{\rm L} = 2$. For $0 < t < 1$ ns, we have $V(l,t) = 0$ since the transient has not arrived. Between $1 < t < 2$ ns, we have $V(l,t) = 5 \times [2 \times (1/2)/(1/2)] = 10$ V. Between 2 ns and 3 ns, the back half of the transient has passed, and $V(l,t) = 0$. For $2m < t < (2m+1)$ ns, where m is any integer, we find in a similar way that $V(l,t)$ must equal zero. In the interval between 3 ns and 4 ns, we have $V(l,t) = 5 \times \{2 \times [(3/4) - (1/2)]/(1/2)\} = 5$ V. The second term in the sum for $V(l,t)$ leads to a decrease by a factor of two in the voltage. The voltage continues to decrease by a factor of two in each of the intervals in which it is non-zero. So, we find $V(l,t) = 2.5$ between 5 and 6 ns, it equals 1.25 between 7 and 8 ns, and, finally, it equals 0.625 between 9 and 10 ns.

For $0 < t < 1$ ns, we have $V(0,t) = 5$ V and $I(0,t) = 0.05$ A, since these quantities are unaffected by the load. We also have $V(0,t) = 0$ and $I(0,t) = 0$ for $(2m+1) < t < (2m+2)$ ns, m is any non-negative integer, since at those times the 1 ns transient, which is bouncing back and forth in the transmission line, is not present at the generator. Between 3 ns and 4 ns, we have $V(0,t) = 5 \times \{[(3/4) + (1/2)]/(1/2)\} - 5 = 7.5$ V, and we have $I(0,t) = 0.05 \times \{[(3/4) - (1/2)/(1/2)\} - 0.05 = -0.025$ A $= -25$ mA. From those values, the voltage and current at the generator continue to diminish by a factor of two on each round trip, so that $V(0,t)$ and $I(0,t)$ are respectively, 3.75 V and -12.5 mA between 5 and 6 ns, 1.875 V and -6.25 mA between 7 and 8 ns, and 0.9375 V and -3.125 mA between 9 and 10 ns. As usual, we note that the higher digits are only meaningful to the extent that *all* the transmission line parameters, the input resistance, and the input voltage are known to that number of digits. For all the digits in our solution to be meaningful, we would have to know all these quantities to at least four digits of accuracy.

6. a. The requested MATLAB M-file and output follow. The size of the labels in the output was changed to make it easier to read. Otherwise, it is unchanged from the raw output.

MATLAB M-FILE AND OUTPUT:

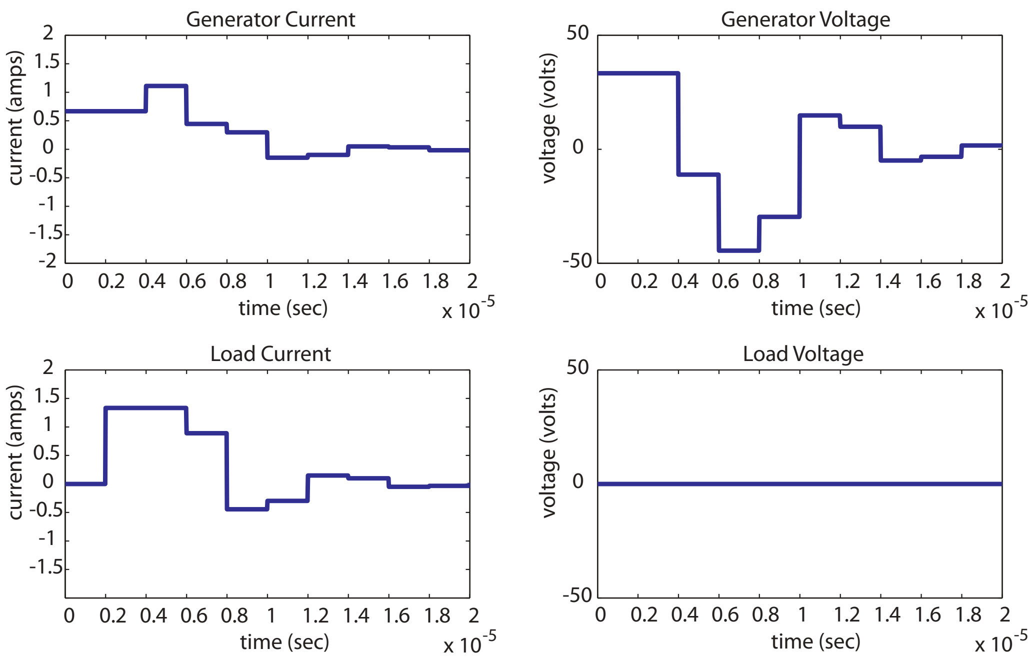
MATLAB M-FILE (Part 1)

```
% Transmission_Line_3
%
% This routine calculates the transient response of a pulsed
% transmission line.  SI units are used.  (Paul, Example 6.2)

% LINE PARAMETERS
length = 400;              %transmission line length
Z_0 = 50;                  %characteristic impedance
velocity = 2e8;            %propagation velocity

% GENERATOR AND LOAD PARAMETERS
Z_g = 100;                 %input impedance
V_g = 100;                 %input voltage
t_D = 6e-6;                %pulse duration
Z_L = 0.0;                 %load impedance

Total_Time = 20e-6;        %total time for the plot

Delta = 0:0.001:1.0;  tinc = 0.001*Total_Time;
Ndelay = round(t_D/tinc);
Ccomp = zeros(1,Ndelay);
time = Delta*Total_Time;   %the time axis is cut into increments
T = length/velocity;       %the transit time is calculated

% Time-independent coefficients
Gamma_g = (Z_g - Z_0)/(Z_g + Z_0);   % generator reflection coefficient
Gamma_L = (Z_L - Z_0)/(Z_L + Z_0);   % load reflection coefficient
I_fac = V_g/(Z_g + Z_0);             % current factor
V_fac = Z_0*I_fac;                   % voltage factor
C_f = Gamma_g*Gamma_L;               % concatenation coefficent

% Time-independent coefficients
Gamma_g = (Z_g - Z_0)/(Z_g + Z_0);   % generator reflection coefficient
Gamma_L = (Z_L - Z_0)/(Z_L + Z_0);   % load reflection coefficient
I_fac = V_g/(Z_g + Z_0);             % current factor
V_fac = Z_0*I_fac;                   % voltage factor
C_f = Gamma_g*Gamma_L;               % concatenation coefficent

% Time-dependent coefficients
M_g = floor(0.5*time/T);             % bounce number at the generator
M_L = floor(0.5*(time+T)/T);         % bounce number at the load
C_vec_g = C_f.^M_g;                  % generator concatenation vector
C_vec_L = C_f.^M_L;                  % load concatenation vector
```

MATLAB M-FILE (Part 2)

```
% Time-dependent load and generator currents and voltages
I_vec_g = I_fac*(1.0 - C_f*C_vec_g ...
    - Gamma_L*(1.0 - C_vec_g))./(1.0 - C_f);  % generator current
I_vec_g = I_vec_g - [Ccomp, I_vec_g(1:1001-Ndelay)]; % back subtraction

V_vec_g = V_fac*(1.0 - C_f*C_vec_g ...
    + Gamma_L*(1.0 - C_vec_g))./(1.0 - C_f);  % generator voltage
V_vec_g = V_vec_g - [Ccomp, V_vec_g(1:1001-Ndelay)];  % back subtraction

I_vec_L = I_fac*((1.0 - Gamma_L)*(1.0 - C_vec_L))./(1.0 - C_f);
I_vec_L = I_vec_L - [Ccomp, I_vec_L(1:1001-Ndelay)];  %load current

V_vec_L = V_fac*((1.0 + Gamma_L)*(1.0 - C_vec_L))./(1.0 - C_f);
V_vec_L = V_vec_L - [Ccomp, V_vec_L(1:1001-Ndelay)];  %load voltage

% Plotting
subplot(2,2,1), plot(time,I_vec_g,'LineWidth',2);
    title('Generator Current'), xlabel('time (sec)'),
    ylabel ('current (amps)'), axis([0.0, Total_Time, -2.0, 2.0])
subplot(2,2,2), plot(time,V_vec_g,'LineWidth',2);
    title('Generator Voltage'), xlabel('time (sec)'),
    ylabel ('voltage (volts)'), axis([0.0, Total_Time, -50.0, 50.0])
subplot(2,2,3), plot(time,I_vec_L,'LineWidth',2);
    title('Load Current'), xlabel('time (sec)'),
    ylabel ('current (amps)'), axis([0.0, Total_Time, -2.0, 2.0])
subplot(2,2,4), plot(time,V_vec_L,'LineWidth',2);
    title('Load Voltage'), xlabel('time (sec)'),
    ylabel ('voltage (volts)'), axis([0.0, Total_Time, -50.0, 50.0])
```

MATLAB OUTPUT



6. b. Using the same notation as in slide 3.32, as well as Ulaby's notation, we find that $T = 2\ \mu$s and $Z_C = Z_0 = 50\ \Omega$, $R_S = Z_g = 100\ \Omega$, $R_L = 0\ \Omega$, so that

$$\Gamma_S = \frac{R_S - Z_C}{R_S + Z_C} = \frac{100 - 50}{100 + 50} = \frac{1}{3}, \qquad \Gamma_R = \frac{R_L - Z_C}{R_L + Z_C} = \frac{0 - 100}{0 + 100} = -1.$$

The voltage at the source can be written as

$$V_S(t) = V_S[H(t) - H(t - t_D)],$$

where $V_S = 20$ V, H(t) is the Heaviside function that equals 0 when $t < 0$ and equals 1 when $t > 0$, and $t_D = 6\ \mu$s. Using the formulae on slide 3.32, we have

$$\begin{aligned} V(0,t) =& 100 \times \frac{50}{100+50}\left[\frac{1 - (-1/3)^{m_{S1}+1} - 1 \times [1 - (-1/3)^{m_{S1}}]}{[1 + (1/3)]}\right] \\ & - 100 \times \frac{50}{100+50}\left[\frac{1 - (-1/3)^{m_{S2}+1} - 1 \times [1 - (1/3)^{m_{S2}}]}{[1 + (1/3)]}\right] \text{ (V)} \end{aligned}$$

at the load, where $m_{S1} = \lfloor t/2(6\ \mu\text{s})\rfloor$ is for the rising portion of the response and $m_{S2} = \lfloor (t - 1\text{ ns})/2(6\ \mu\text{s})\rfloor$ is for the falling portion of the response, and it is understood that the second term equals zero when $t < 6\ \mu$s. We then find that $V(0,t)$ is initially 33.3 V; it changes to -11.1 V at $t = 4\ \mu$s, to -44.4 V at $t = 6\ \mu$s, to -29.6 V at $t = 8\ \mu$s, to 14.8 V at $t = 10\ \mu$s, to 9.88 V at $t = 12\ \mu$s, to -4.94 V at $t = 14\ \mu$s, to -3.29 V at $t = 16$, to -3.29 V at $t = 18\ \mu$s, and finally to 1.65 V at $t = 20\ \mu$s.

The voltage must tend toward zero because only a finite amount of energy is injected and as long as the voltage is non-zero, there will be energy dissipated in the load. From a mathematical standpoint, the geometric series generated by both the rise and by the follow tend toward the same asymptotic value and hence cancel as $t \to \infty$.