Quiz Questions: Spring 2015

1. Determine the wavelength that corresponds to the frequency $f = 60$ Hz (AC power circuits). Determine the frequency that corresponds to the wavelength $\lambda = 1.5$ μm (light in optical fibers). [Given 02/03/2015]

   Solution: (a) $\lambda = c/f = (3.0 \times 10^8)/(60) = 5 \times 10^6$ m $= 5000$ km; (b) $f = (c/\lambda) = (3.0 \times 10^8)/(1.5 \times 10^{-6}) = 2.0 \times 10^{14}$ Hz.

2. Write θ in radians, $\sin\theta$ and $\cos\theta$ for following angles $\theta = 30°,\ 45°,\ 60°,\ 90°,\ 120°,\ 135°,\ 150°$, and $180°$. Example: $30°$: $\theta = \pi/6 = 0.524$, $\sin\theta = 0.5$, $\cos\theta = \sqrt{3}/2 = 0.866$. [Given 02/03/2015]

   Solution: $\theta = 30°$: $\theta = \pi/6 = 0.52$, $\sin\theta = 0.5$, $\cos\theta = \sqrt{3}/2 = 0.87$;
   $\theta = 45°$: $\theta = \pi/4 = 0.79$, $\sin\theta = \cos\theta = \sqrt{2}/2 = 0.71$;
   $\theta = 60°$: $\theta = \pi/3 = 1.05$, $\sin\theta = \sqrt{3}/2 = 0.87$, $\cos\theta = 0.5$;
   $\theta = 90°$: $\theta = \pi/2 = 1.6$, $\sin\theta = 0$, $\cos\theta = 1$;
   $\theta = 120°$: $\theta = 2\pi/3 = 2.1$, $\sin\theta = \sqrt{3}/2 = 0.87$, $\cos\theta = 0.5$;
   $\theta = 135°$: $\theta = 3\pi/4 = 2.36$, $\sin\theta = -\cos\theta = \sqrt{2}/2 = 0.71$;
   $\theta = 150°$: $\theta = 5\pi/6 = 2.62$, $\sin\theta = 0.5$, $\cos\theta = -0.87$;
   $\theta = 180°$: $\theta = \pi = 3.14$, $\sin\theta = 0$, $\cos\theta = -1$

3. To two significant figures, what are: π, $\pi/2$, $\sqrt{2}/2$, $\sqrt{3}/2$, e, $1/e$. To one significant figure, what is 1 radian in degrees. [Given 02/10/2015]

   Solution: $\pi = 3.1$, $\pi/2 = 1.6$, $\sqrt{2}/2 = 0.71$, $\sqrt{3}/2 = 0.87$, $e = 2.7$, $1/e = 0.37$, 1 radian $= 60°$.

4. Consider the equation that relates the phasors $\tilde{I}$ and $\tilde{V}$ in an RC circuit,

   $$R\,\mathrm{Re}\left[\tilde{I}\exp(j\omega t)\right] + \frac{1}{C}\,\mathrm{Re}\left[\frac{1}{j\omega}\tilde{I}\exp(j\omega t)\right] = \mathrm{Re}\left[V_\mathrm{S}\exp(j\omega t)\right] \tag{4.1}$$

   Show that this equation holds at all points in time if and only if

   $$\left(R + \frac{1}{j\omega C}\right)\tilde{I} = V_\mathrm{S} \tag{4.2}$$

   [Given 02/10/2015]

   Solution: To demonstrate that (4.2) implies (4.1), we multiply (4.2) by $\exp(j\omega t)$, and we take the real part. To demonstrate that (4.1) implies (4.2), we need only consider two special times. The first time is $t = 0$, which yields the real part of (4.2). The second time is $t = \pi/2\omega$, which yields the imaginary part of (4.2).

5. Demonstrate the equation for a geometric series $\sum_{n=0}^{m} x^n = (1-x^{m+1})/(1-x)$. What is the condition for this result to hold in the limit as $m \to \infty$ and what does it become? Use this result to show that $0.3333\ldots = 1/3$. [Given 02/10/2015]

Solution: If we let $S_m = \sum_{n=0}^{m} x^m$, we find that

$$xS_m = \sum_{n=1}^{m+1} x^n = S_m - 1 + x^{m+1}.$$

Solving for S_m, we obtain $S_m = (1 - x^{m+1})/(1 - x)$. In order for this series to have a limit as $m \to \infty$, we must have $|x| < 1$. In this case, we find that the limit is $S_\infty = 1/(1 - x)$. To use this result to find $0.3333\ldots$, we write

$$0.3333\ldots = 3\sum_{n=1}^{\infty}(1/10)^n = 3/(1 - 1/10) - 3 = (30 - 27)/9 = 1/3$$

6. Calculate $(1 + \pi \times 10^{-40})^{1/2} - 1$ to three significant figures. Explain why you cannot obtain this result directly from a standard calculator. [Given 02/24/2015]

   Solution: $(1+\pi\times10^{-40})^{1/2}-1 \simeq [1+(\pi/2)\times10^{-40}]-1 = (\pi/2)\times10^{-40} \simeq 1.57\times10^{-40}$. You cannot obtain this result from a standard calculator because only 64 bits are used to store each digit which corresponds to about 15 digits of accuracy. When the difference between two numbers is smaller than 1 part in 10^{15}, as is the case here, then the calculator will return 0. All digits of accuracy are lost in this subtraction.

7. Show that if the voltage at a load is given by $v(t) = v_L \cos(\omega t + \phi_v)$ and the current is given by $i(t) = i_L \cos(\omega t + \phi_i)$, then the total power dissipated in the load is given by $(1/2)\,\mathrm{Re}(\tilde{V}\tilde{I}^*)$, where $\tilde{V}$ and $\tilde{I}$ are the phasors corresponding to $v(t)$ and $i(t)$. [Given 02/24/2015]

   Solution: In the time domain, we find that the average power dissipated at the load P_L is given by

$$\begin{aligned}
P_L &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} v_L i_L \cos(\omega t + \phi_v)\cos(\omega t + \phi_i)\,dt \\
&= v_L i_L \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \frac{1}{2}\left[\cos(\phi_v - \phi_i) + \cos(2\omega t + \phi_v + \phi_i)\right]\,dt \\
&= \frac{1}{2} v_L i_L \cos(\phi_v - \phi_i).
\end{aligned}$$

   The phasor domain expression may be written

$$\begin{aligned}
\frac{1}{2}\,\mathrm{Re}(\tilde{V}\tilde{I}^*) &= \frac{1}{2}\,\mathrm{Re}\{[v_L \exp(j\phi_v)][i_L \exp(-j\phi_i)]\} \\
&= \frac{1}{2} v_L i_L\,\mathrm{Re}\{\exp[j(\phi_v - \phi_i)]\} = \frac{1}{2} v_L v_i \cos(\phi_v - \phi_i).
\end{aligned}$$

   These two expressions are evidently equal.

8. (a) Show that $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})$. (b) Show that $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\cdot\mathbf{C}$. [Given 03/10/2015]

   Solution: For (a), we have by definition

$$
\begin{aligned}
\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \\
&= A_xB_yC_z + C_xA_yB_z + B_xC_yA_z - A_xC_yB_z - C_xB_yA_z - B_xA_yC_z,
\end{aligned}
$$

   which is manifestly invariant under permutations, $\mathbf{A}\to\mathbf{C}$, $\mathbf{C}\to\mathbf{B}$, $\mathbf{B}\to\mathbf{A}$. For (b), we may write

$$
\mathbf{B}\times\mathbf{C} = \hat{\mathbf{x}}(B_yC_z - B_zC_y) + \hat{\mathbf{y}}(B_zC_x - B_xC_z) + \hat{\mathbf{z}}(B_xC_y - B_yC_x),
$$

   so that

$$
\begin{aligned}
\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) &= \hat{\mathbf{x}}[A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z)] \\
&\quad + \hat{\mathbf{y}}[A_z(B_yC_z - B_zC_y) - A_x(B_xC_y - B_yC_x)] \\
&\quad\quad + \hat{\mathbf{z}}[A_x(B_zC_x - B_xC_z) - A_y(B_yC_z - B_zC_y)] \\
&= \hat{\mathbf{x}}[B_x(A_yC_y + A_zC_z) - C_x(A_yB_y + A_zB_z)] \\
&\quad + \hat{\mathbf{y}}[B_y(A_zC_z + A_xC_x) - C_y(A_zB_z + A_xB_x)] \\
&\quad\quad + \hat{\mathbf{z}}[B_z(A_xC_x + A_yC_y) - C_z(A_xB_x + A_yB_y)] \\
&= \hat{\mathbf{x}}[B_x(A_xC_x + A_yC_y + A_zC_z) - C_x(A_xB_x + A_yB_y + A_zB_z)] \\
&\quad + \hat{\mathbf{y}}[B_y(A_xC_x + A_yC_y + A_zC_z) - C_y(A_xB_x + A_yB_y + A_zB_z)] \\
&\quad\quad + \hat{\mathbf{z}}[B_z(A_xC_x + A_yC_y + A_zC_z) - C_z(A_xB_x + A_yB_y + A_zB_z)] \\
&= (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}.
\end{aligned}
$$

   The use of permutations is necessary to make this calculation efficient. Another approach, which is much faster, is to take advantage of our freedom to choose the x, y, and z-axes. We may first choose the x-axis so that it is in the direction of the vector $\mathbf{B}$ and $\mathbf{B} = \hat{\mathbf{x}}B_x$. We next choose the y-axis so that $\mathbf{C}$ is in the x-y plane and $\mathbf{C} = \hat{\mathbf{x}}C_x + \hat{\mathbf{y}}C_y$. The z-axis is now fixed, and in general $\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$. We now have $\mathbf{B}\times\mathbf{C} = \hat{\mathbf{z}}B_xC_y$. It follows that

$$
\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \hat{\mathbf{x}}(A_yC_y)B_x - \hat{\mathbf{y}}(A_xB_x)C_y.
$$

   We also have

$$
\begin{aligned}
(\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C} &= (A_xC_x + A_yC_y)\hat{\mathbf{x}}B_x - A_xB_x(\hat{\mathbf{x}}C_x + \hat{\mathbf{y}}C_y) \\
&= \hat{\mathbf{x}}(A_yC_y)B_x - \hat{\mathbf{y}}(A_xB_x)C_y,
\end{aligned}
$$

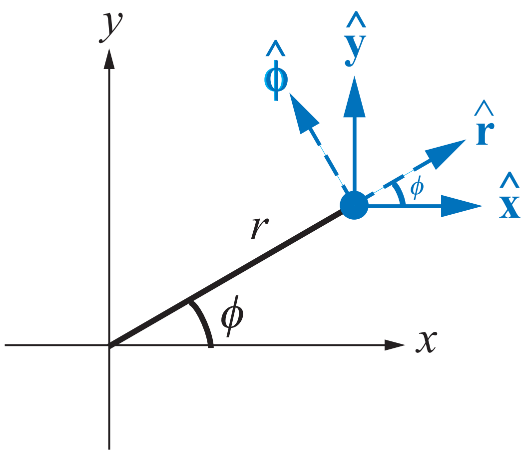
   which proves the identity.

9. In two dimensions, we relate the Cartesian coordinates (x, y) to the polar coordinates (r, ϕ) by the relations $r = (x^2 + y^2)^{1/2}$, $\phi = \tan^{-1}(y/x)$. [Given 03/10/2015]

   a. What are $x(r, \phi)$ and $y(r, \phi)$?

   b. Write $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ as functions of r, ϕ, $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$.

   Solution: We show the geometry in the figure below:



   a. We have $x = r\cos\phi$ and $y = r\sin\phi$.

   b. From the figure, we infer

$$\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi,$$
$$\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi.$$

10. **Slide 7.21:** Show that $\nabla \cdot (\nabla \times \mathbf{A}) = 0$ and $\nabla \times \nabla V = 0$, where $\mathbf{A}$ is an arbitrary vector field and V is an arbitrary scalar. [Given 03/24/2015]

    Solution: We have

$$\nabla \times \mathbf{A} = \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right),$$

from which we obtain

$$\begin{aligned}\nabla \cdot (\nabla \times \mathbf{A}) &= \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \\ &= 0.\end{aligned}$$

We also have

$$\nabla V = \hat{\mathbf{x}}\frac{\partial V}{\partial x} + \hat{\mathbf{y}}\frac{\partial V}{\partial y} + \hat{\mathbf{z}}\frac{\partial V}{\partial z},$$

from which we obtain

$$\nabla \times \nabla V = \hat{\mathbf{x}}\left(\frac{\partial^2 V}{\partial y \partial z} - \frac{\partial^2 V}{\partial z \partial y}\right) + \hat{\mathbf{y}}\left(\frac{\partial^2 V}{\partial z \partial x} - \frac{\partial^2 V}{\partial x \partial z}\right) + \hat{\mathbf{z}}\left(\frac{\partial^2 V}{\partial x \partial y} - \frac{\partial^2 V}{\partial y \partial x}\right) = 0.$$

11. The field from a circular disk of charge along the axis of the disk may be written (Ulaby et al., Example 4-5)

$$\mathbf{E} = \hat{\mathbf{z}} \frac{\rho_{\mathrm{S}} h}{2\epsilon_0} \int_0^a \frac{r\, dr}{(r^2 + h^2)^{3/2}}.$$

Do the integration to obtain an explicit expression. [Given 03/31/2015]

Solution: We have

$$\mathbf{E} = \hat{\mathbf{z}} \frac{\rho_{\mathrm{S}} h}{2\epsilon_0} \left[\frac{-2}{(r^2 + h^2)^{1/2}} \bigg|_0^a \right] = \pm \hat{\mathbf{z}} \frac{\rho_{\mathrm{S}}}{2\epsilon_0} \left[1 - \frac{|h|}{(a^2 + h^2)^{1/2}} \right],$$

where the positive sign applies when $h > 0$ and the negative sign applies when $h < 0$.

12. Show using Taylor expansion that $(1 + x)^\alpha \simeq 1 + \alpha x$ when $x \ll 1$. Use this result to show that

$$\frac{y^2}{(y^2 - a^2)^{3/2}} - \frac{y^2}{(y^2 + a^2)^{3/2}} \simeq \frac{3a^2}{y^3}$$

when $a \ll y$. [Given 03/31/2015]

Solution: To first order, we have that the first two terms in a Taylor expansion at the point $x = 0$ are:

$$(1 + x)^\alpha = (1 + x)^\alpha \big|_{x=0} + \left[\frac{d(1 + x)^\alpha}{dx} \bigg|_{x=0} \right] x = 1 + \alpha x.$$

We may rewrite the terms in the next expression as

$$\frac{y^2}{(y^2 \mp a^2)^{3/2}} = \frac{1}{y} \left(1 \mp \frac{a^2}{y^2} \right)^{-3/2},$$

so that $x = \mp(a^2/y^2)$, $\alpha = -3/2$, and our expression becomes

$$\frac{1}{y} \left[\left(1 + \frac{3}{2} \frac{a^2}{y^2} \right) - \left(1 - \frac{3}{2} \frac{a^2}{y^2} \right) \right] = \frac{3a^2}{y^3}.$$

13. Derive Coulomb's law from Gauss's law. [Given 03/31/2015]

Solution: Gauss's law is $\nabla \cdot \mathbf{D} = \rho_{\mathrm{V}}$, which, using Gauss's theorem becomes,

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{D}\, d\mathcal{V} = \int_S \mathbf{D} \cdot \hat{\mathbf{n}}\, ds = \int_{\mathcal{V}} \rho_{\mathrm{V}}\, d\mathcal{V} = Q,$$

where Q is the total charge in the volume. The charge in Coulomb's law is a point charge, and we take our volume to be the sphere that is centered at the charge and whose surface is a distance R from the center. We then find from symmetry that $\mathbf{D} = \hat{\mathbf{R}}D$ and $\hat{\mathbf{n}} = \hat{\mathbf{R}}$, so that we have

$$Q = \int_S D\,ds = 4\pi R^2 D,$$

where we recall that the surface of sphere has area $4\pi R^2$. We conclude

$$\mathbf{D} = \frac{Q\hat{\mathbf{R}}}{4\pi R^2} \qquad \text{and} \qquad \mathbf{E} = \frac{Q\hat{\mathbf{R}}}{4\pi\epsilon R^2},$$

where we recall that $\mathbf{E} = \mathbf{D}/\epsilon$.

14. Use Maxwell's equations for electrostatics and Stokes theorem to derive Kirchhoff's voltage law. [Given 04/07/2015]

    Solution: The solution to this quiz problem may be found on slide 9.6. We start from the second law of electrostatics, $\nabla \times \mathbf{E} = 0$, from which it follows using Stokes theorem that $\oint \mathbf{E} \cdot d\mathbf{l} = 0$ over any closed path. We now consider any two different paths "path 1" and "path 2" that link two points P_1 and P_2. Since the sum of the integral over path 1 and the *reverse* of the integral over path 2 is a closed path, we conclude

    $$\int_{\text{path 1}} \mathbf{E} \cdot d\mathbf{l} - \int_{\text{path 2}} \mathbf{E} \cdot d\mathbf{l} = 0 \qquad \text{or} \qquad \int_{\text{path 1}} \mathbf{E} \cdot d\mathbf{l} = \int_{\text{path 2}} \mathbf{E} \cdot d\mathbf{l}.$$

    Since the paths are arbitrary, Kirchhoff's voltage law follows.

15. Use Maxwell's equations for magnetostatics and Stokes theorem to derive Kirchhoff's current law. [Given 04/07/2015]

    Solution: The solution to this quiz problem may be found on slide 11.17. We start from the second law of magnetostatics, $\nabla \times \mathbf{H} = \mathbf{J}$, and we find that $\oint \mathbf{H} \cdot d\mathbf{l} = I_{\text{surf}}$, where the magnetic field is integrated over *any* closed path and the current I_{surf} is the current flowing through *any* surface whose edge is on the closed path. We now consider a node in a circuit and enclose that node with a volume. We take any closed path on the surface of the volume. That closed path will split the surface of the volume into two surfaces S_1 and S_2, each of which must have the same current flowing through them with the same orientation. If the net flow is inward for one of the surfaces, the net flow must be outward for the other. We conclude that $I_1 = -I_2$ if we choose the current flow to be inward for both surfaces. Since the choice of the volume enclosing the node and the choice of the closed path on the surface of that volume are both completely arbitrary, we conclude that the sum of the currents flowing into any node must equal zero.

16. Derive the boundary conditions for $\mathbf{E}$ when crossing a boundary with no currents and charges in which the dielectric constant changes, so that in medium 1 the dielectric constant is ϵ_1 and in medium 2 the dielectric constant is ϵ_2. (Given 5/14/2015)

Solution: See slides 9.24 and 9.25 for the derivation. The answer is $E_{1\mathrm{t}} = E_{2\mathrm{t}}$ and $\epsilon_1 E_{1\mathrm{n}} = \epsilon_2 E_{2\mathrm{n}}$, where the subscripts "t" and "n" denote respectively the components tangential and normal to the boundary. Note that because we are going from one dielectric medium to another, i.e., there is no conductor or semi-conductor on either side, there is no free charge. From the statement of the problem, including free charge is a mistake.

17. Derive the boundary condition for **H** when going from medium 2 to medium 1 with magnetic permeabilities μ_2 and μ_1 and with no currents and charges.
(Given 5/14/2015)

Solution: See slides 11.23 and 11.24 and the discussion in the textbook. The answers are $H_{1\mathrm{t}} = H_{2\mathrm{t}}$ and $\mu_1 H_{1\mathrm{n}} = \mu_2 H_{2\mathrm{n}}$.