CMPE323 Intro to Fourier Series

Chapter 7

Orthogonal Series

- **We can expand a periodic time waveform in an orthogonal series of periodic functions with period *T***

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \phi_k(t)$$

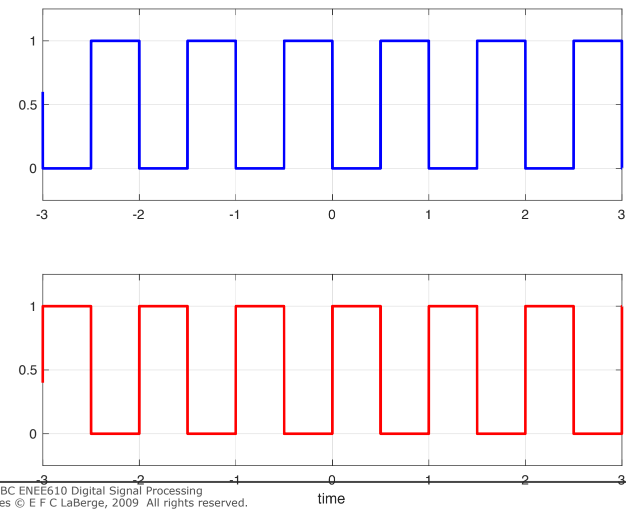
- **…where $\{\phi_k(t)\}$ are orthogonal…**

$$\int_{\alpha}^{\alpha+T} \phi_k(t)\phi_m^*(t)\,dt = \begin{cases} 0 & k \neq m \\ C & k = m \end{cases}$$

- **…and, ideally, orthonormal, where $C = 1$ so that**

$$\int_{\alpha}^{\alpha+T} \phi_k(t)\phi_k^*(t)\,dt = \int_{\alpha}^{\alpha+T} |\phi_k|^2\,dt = 1$$

- **For example, these are orthogonal, but not orthonormal**



Complex exponentials as eigenfunctions

- **We know the complex exponentials are eigenfunctions of LTI systems**
- **If we can decompose any generic input signal into the sum of complex exponentials…**
- **…then we can write the output as the sum or superposition of *scaled versions* of the complex exponentials.**
- **This process of decomposition is called Fourier Analysis**

LTI systems and eigenfunctions

The input signal $x(t) = Ae^{st}$ → The LTI system $H(s)$ → $y(t) = H(s)Ae^{st} = H(s)e^{st}$ The output signal

- ...and even more importantly, because the system is both linear and time invariant

$x(t) = \sum \alpha_n e^{s_n(t-\tau_n)}$ → The LTI system $H(s)$ → $y(t) = \sum H(s_n)\alpha_n e^{s_n(t-\tau_n)}$

- Now what happens if we have an arbitrary (but well-behaved) input signal?

Fourier analysis of periodic signals

- Let be $x(t)$ be "well-behaved" and periodic with period T
- We want to write $x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi f_k t} = \sum_{k=-\infty}^{\infty} c_k e^{\frac{j2\pi kt}{T}}$
- Let's look at the following integral

$$I_m = \frac{1}{T}\int_{\alpha}^{\alpha+T} x(t) e^{-j\frac{2\pi mt}{T}}\,dt = \frac{1}{T}\int_{\alpha}^{\alpha+T}\left(\sum_{k=-\infty}^{\infty} c_k e^{\frac{j2\pi mt}{T}}\right) e^{-j\frac{2\pi kt}{T}}\,dt$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} c_k \left(\int_{\alpha}^{\alpha+T} e^{\frac{j2\pi mt}{T}} e^{-j\frac{2\pi kt}{T}}\,dt\right)$$

$= 0$ if $m \neq k$ because the complex exponentials are orthogonal over one period!

$$= \begin{cases} c_k & \text{if } m = k \\ 0 & \text{if } m \neq k \end{cases}$$

Complex exponentials are an orthonormal set

$$\left|\phi_n\right|^2 = \frac{1}{T_0}\int_0^{0+T_0} \underbrace{e^{jn\omega_0 t}}_{\phi_n(t)}\underbrace{e^{-jn\omega_0 t}}_{\phi_n^*(t)}\,dt = \frac{1}{T_0}\int_0^{0+T_0} 1\,dt = 1,\ \omega_0 \triangleq \frac{2\pi}{T_0}$$

And similarly for the normal equation $\frac{1}{T_0}\int_\alpha^{\alpha+T_0} \phi_n(t)\phi_m^*(t) = 0,\ n \neq m$

$$\phi_n \cdot \phi_k = \frac{1}{T_0}\int_0^{0+T_0} e^{jn\omega_0 t}e^{-jk\omega_0 t}\,dt = \frac{1}{T_0}\int_0^{0+T_0} e^{j(n-k)\omega_0 t}\,dt$$

$$= \frac{1}{T_0}\int_0^{0+T_0}\left(\underbrace{\cos((n-k)\omega_0 t)}_{\substack{n\neq k,\ n-k\in\mathbb{Z},\\ \text{so this is an}\\ \text{integer number}\\ \text{of cycles}}} + j\underbrace{\sin((n-k)\omega_0 t)}_{\substack{n\neq k,\ n-k\in\mathbb{Z},\\ \text{so this is an}\\ \text{integer number}\\ \text{of cycles}}}\right)dt = 0$$

The Fourier series

- **So, we can compute this interval for each k, and write**

$$c_k = x(t)\cdot\phi_n(t) = \frac{1}{T}\int_\alpha^{\alpha+T} x(t)e^{-j\omega_0 kt}\,dt, \omega_0 = \frac{2\pi}{T}$$

Fourier Coefficients

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega_0 kt}$$

Fourier Series

or

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos\left(n\omega_0 t\right) + b_n\sin(n\omega_0 t)\right)$$

$$a_n = \frac{2}{T}\int_\alpha^{\alpha+T} x(t)\cos(n\omega_0 t)dt,\ b_n = \frac{2}{T}\int_\alpha^{\alpha+T} x(t)\sin(n\omega_0 t)dt$$

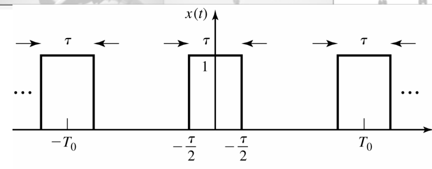
- **…and, under reasonable conditions, the infinite sum is *not* an approximation, but an equality**

Let's try it

- **This particular waveform is both *real* and *even***

real: $\mathrm{Re}[x(t)] = x(t)$

even: $x(t) = x(-t)$



$$c_k = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)e^{-j\frac{2\pi kt}{T}}\,dt = \frac{A}{T_0}\int_{-\tau/2}^{\tau/2} e^{-j\frac{2\pi kt}{T}}\,dt$$

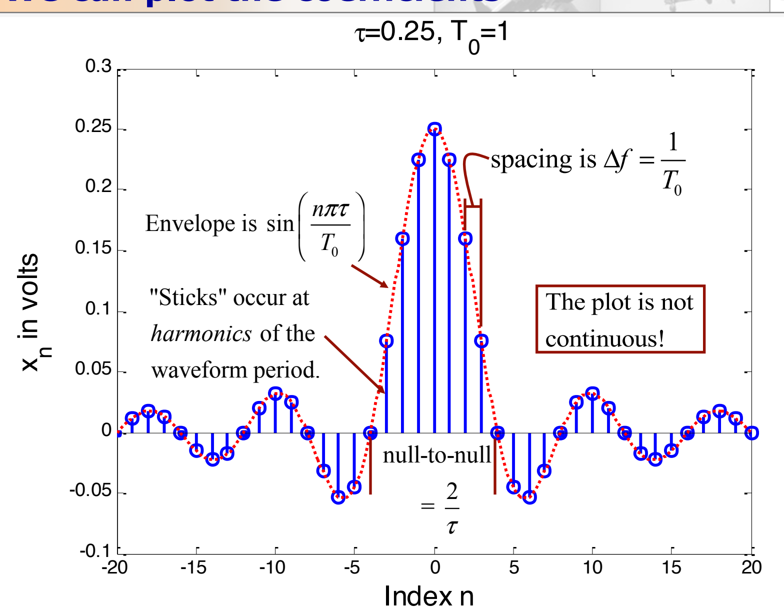
$$c_k = \frac{A}{T_0}\left(\frac{1}{-j\omega_0 k}e^{-j\omega_0 kt}\right)\Bigg|_{-\tau/2}^{\tau/2} = \frac{A}{T_0}\left(\frac{e^{j\omega_0 k\tau/2} - e^{-j\omega_0 k\tau/2}}{j\omega_0 k}\right)$$

$$= \frac{A}{T_0}\left(\frac{2j\sin(\omega_0 k\tau/2)}{j\omega_0 k}\right)\left(\frac{\tau/2}{\tau/2}\right)$$

$$= \frac{A\tau}{T_0}\frac{\sin(n\pi\tau/T_0)}{(n\pi\tau/T_0)} @ \frac{A\tau}{T_0}\mathrm{sinc}\left(\frac{n\tau}{T_0}\right)$$

notice the π in the definition

We can plot the coefficients



So this means

- We can write the periodic signal

$$x(t)=\sum_{k=-\infty}^{\infty}c_k e^{j\frac{2\pi kt}{T_0}}=\frac{A\tau}{T_0}\sum_{k=-\infty}^{\infty}\operatorname{sinc}\left(\frac{n\tau}{T_0}\right)e^{j\frac{2\pi kt}{T_0}}$$

- …which is defined *only* at integer multiples of

$$f_0=\frac{\omega_0}{2\pi}=\frac{1}{T_0}$$

- This expression is known as the Fourier Series, and it relates the time domain $\left(x(t)\right)$ to the frequency domain

$$\{c_k\}\text{ or }X(f)=\sum_{k=-\infty}^{\infty}c_k\delta\left(f-\frac{k}{T}\right)$$

- And the inverse relationship hold as well

But what if the signal is not periodic?

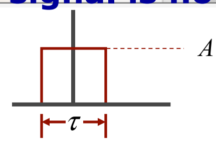
- There is an equivalent result for non-periodic signals
- A non-periodic signal can be viewed as the *limit* of a periodic signal as $T_0\rightarrow\infty$

$$c_k=\frac{1}{T_0}\int_{-T_0/2}^{T_0/2}x(t)e^{-j\frac{2\pi kt}{T_0}}dt\qquad\frac{k}{T_0}\rightarrow f\qquad\frac{T_0}{2}\rightarrow\infty\qquad\frac{1}{T_0}\rightarrow df$$

$$X(f)\triangleq\int_{-\infty}^{\infty}x(t)e^{-j2\pi ft}dt\qquad\text{contribution between } f \text{ and } f+df$$

- $X(f)$ is called the Fourier Transform, or the *voltage spectrum* of $x(t)$,
- …and we have $x(t)=\int_{-\infty}^{\infty}X(f)e^{j2\pi ft}df=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)e^{j\omega t}d\omega$
- Notice $c_n=\frac{1}{T_0}X(f)\Big|_{f=\frac{n}{T_0}}=\frac{1}{T_0}X\left(\frac{n}{T_0}\right)$

But what if the signal is not periodic?



- **We define the Fourier Transform of x(t)**

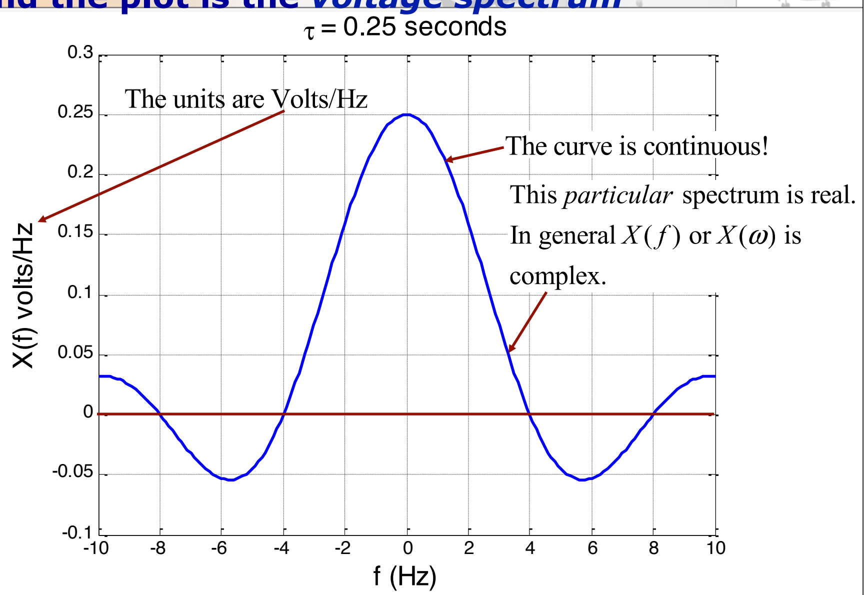
$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \text{ or } \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = X(\omega)$$

- **In this case**

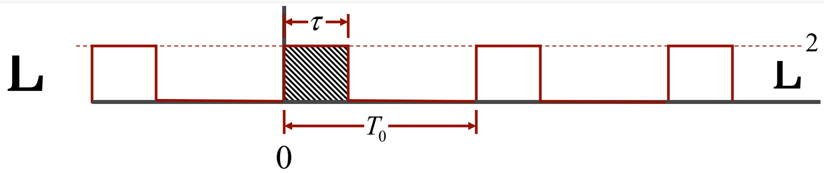
$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt = \int_{-\tau/2}^{\tau/2} Ae^{-j2\pi ft}dt = \frac{A}{-j2\pi f}e^{-j2\pi ft}\Big|_{-\tau/2}^{\tau/2}$$

$$= \frac{A}{j2\pi f}\left(e^{j2\pi f\tau 2} - e^{-j2\pi f\tau 2}\right) = \frac{A}{\pi f}\frac{\left(e^{j\pi f\tau} - e^{-j\pi f\tau}\right)}{j2} = A\tau\frac{\sin(\pi f\tau)}{\pi f\tau} = A\tau\mathrm{sinc}(f\tau)$$

And the plot is the *voltage spectrum*



Now you try (in class example) (10 minutes)



Group 1A: Compute a_n

Group 1B: Compute b_n

Group 2: Compute x_n **(Hint factor out** $e^{-j\frac{\pi n}{T_0}}$ **and simplify)**

Group 3: Compute the Fourier Transform, $X(f)$ **of the cross-hatched waveform**

Detailed solutions will be posted on Blackboard tonight!

Properties of Fourier Transform

- **Duality:** $X(f) = \mathbf{F}\left(x(t)\right) \Rightarrow x(f) = \mathbf{F}\left(X(-t)\right) \text{ and } x(-f) = \mathbf{F}\left(X(t)\right)$
- **Linearity:** $z(t) = ax(t) + by(t) \Rightarrow Y(f) = aX(f) + bY(f)$
- **Time Shift:** $\mathbf{F}\left(x(t-t_0)\right) = e^{-j2\pi f t_0}\mathbf{F}\left(x(t)\right)$
- **Scaling:** $\text{For } a \neq 0 \in {}^\circ, \ \mathbf{F}\left(x(at)\right) = \frac{1}{|a|}X\left(\frac{f}{a}\right)$
- **Modulation:** $\mathbf{F}\left(x(t)e^{j2\pi f_0 t}\right) = X(f - f_0)$
- **Conjugation:** $\mathbf{F}\left(x^*(t)\right) = X^*(-f)$
- **Parseval:** $\int_{-\infty}^{\infty} x(t)y^*(t)dt = \int_{-\infty}^{\infty} X(f)Y^*(f)df$
- **Rayleigh** $\int_{-\infty}^{\infty} |x(t)|^2\, dt = \int_{-\infty}^{\infty} |X(f)|^2\, df$

Advanced Properties of Fourier Transform

- **Integration:** $\mathbf{F}\left(\int_{-\infty}^{t} x(t)dt\right) = \frac{X(f)}{j2\pi f} + \frac{1}{2}X(0)\delta(f)$
- **Differentiation:** $\mathbf{F}\left(\frac{d}{dt}x(t)\right) = j2\pi f X(f)$
- **Moments:** $\int_{-\infty}^{\infty} t^n x(t)dt = \left(\frac{j}{2\pi}\right)^n \frac{d^n}{df^n}X(f)\Big|_{f=0}$

The convolution theorem (very important!)

- **The output of a LTI system with transfer function** $H(f)$

$$Y(f) = X(f)H(f) \qquad y(t) = \int_{-\infty}^{\infty} Y(f)e^{j2\pi ft}df = \int_{-\infty}^{\infty} X(f)H(f)e^{j2\pi ft}df$$

$$\text{Write } H(f) = \int_{-\infty}^{\infty} h(\tau)e^{-j2\pi f\tau}d\tau$$

$$y(t) = \int_{-\infty}^{\infty} X(f)\left(\int_{-\infty}^{\infty} h(\tau)e^{-j2\pi f\tau}d\tau\right)e^{j2\pi ft}df$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} X(f)h(\tau)e^{j2\pi f(t-\tau)}df\right)d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\left(\int_{-\infty}^{\infty} X(f)e^{j2\pi f(t-\tau)}df\right)d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau$$

$@x(t) * h(t)$ the CONVOLUTION

Summary

- We can decompose a periodic signal into the weighted sum of complex exponentials,…
- …or, equivalently, to the weighted sum of sines and cosines.
- We write the weighted sum as a Fourier Series
- The frequency-domain representation consists of a series of harmonically-related terms, with separation equal to the period of the signal,
- The coefficients have units of volts (or amps)
- We can decompose a non-periodic signal into the weighted integral of complex exponentials,…
- …or, equivalently, to the weighted integral of sines and cosines
- The frequency-domain representation has a continuous spectrum with units of volts/Hz (or amps/Hz).