MATH407

4/23/18

[Review] $K \trianglelefteq G$ normal subgroup

$$gKg^{-1} = K, \forall g$$

$$gK = Kg, \forall g$$

* If $g \in G$, $k \in K$ there are k', k'' s.t. $gk = k'g$, $kg = gk''$

* If $\Phi: G \to G^{-1}$ homomorphism, then $\Phi^{-1}(\{e'\}) = \ker(\Phi)$ is normal

* If K' normal in G, then $\Phi^{-1}(K')$ normal in G
* If K normal in G, then $\Phi(K)$ normal in $\Phi(G)$

Pf. (If K' normal in G, then $\Phi^{-1}(K')$ normal in G)

Let K be normal in G.

Let $g' \in \Phi(G)$, so there is $g \in G$

$\Phi(g) = g^{-1}$

Look at $g'k'(g')^{-1}$, where $k' = \Phi(k)$, $k \in K$

$$\begin{aligned} g'k'(g')^{-1} &= \Phi(g)\Phi(k)\Phi(g^{-1}) \\ &= \Phi(gkg^{-1}) \\ &= \Phi(k_0), \text{ some } k_0 \in K \\ &\in \Phi(K) \end{aligned}$$

$(\longrightarrow)$

* Let K be normal subgroup of G.

$G/K = \{gK : g \in G\}$

$= \{[g]_K : g \in G\}$, $g \sim g'$ iff $g' \in gK$

$g^{-1}g' \in K$

* $[a]_K [b]_K = [ab]_K$

* $[e]_K$ is identity of G/K
* $[g^{-1}]_K$ is inverse $[g]_K^{-1}$

* Showed $\pi_K : G \to G/K$

$\pi_K(g) = [g]_K = gK$ is homomorphism

$\ker(\pi_K) = [e]_K = K$

$\pi_K : \mathbb{Z} \to \mathbb{Z}_n$, $\ker(\pi_n) = n\mathbb{Z}$

* Look at $\tilde{\Phi}([g]_K) = \Phi(g)$, where $\Phi : G \to G'$ is homomorphism w/ $\ker(\Phi) = K$

If $g' \sim g$, $g'g^{-1} \in K = eK$

Thus, $\Phi(g'g^{-1}) = \Phi(e) = e'$

$\Rightarrow \Phi(g')\Phi(g^{-1}) = e'$

$\Rightarrow \Phi(g')\Phi^{-1}(g) = e'$

$\Rightarrow \Phi(g') = e'\Phi(g)$

$= \Phi(g)$

* $\pi_K : g \to [g]_K$

$\tilde{\Phi} : [g]_K \to \Phi(g)$

so, $\tilde{\Phi}(\pi_K(g)) = \Phi(g)$

surjection → π_K ; $\tilde{\Phi}$ ← injection

$$\begin{tikzcd} G \arrow[r, "\pi_K"] \arrow[rr, bend right, "\Phi"'] & G/K \arrow[r, "\tilde{\Phi}"] & \Phi(G) \subseteq G' \end{tikzcd}$$

($\tilde{\Phi}$ is also isomorphic)

Thm. $\tilde{\Phi}$ is an isomorphism from G/K to $\Phi(G)$

$\tilde{\Phi}([x]_K) = \tilde{\Phi}([e]_K)$ iff $\Phi(x) = \Phi(e) = e'$
iff $x \in K$

(*) Above is the Fundamental Homomorphism Thm.

* $G_1 \oplus G_2$

$H_1 = \{(g_1, e_2) : g_1 \in G_1\}$

$H_2 = \{(e_1, g_2) : g_2 \in G_2\}$

$G_1 \simeq H_1$, $G_2 \simeq H_2$, $(g_1 e_2)(e_1, g_2) = (e_1, g_2)(g_1, e_2) = (g_1, g_2)$
(commutative)

$H_1 \cap H_2 = \{(e_1, e_2)\}$

$H_1 \oplus H_2 \simeq G_1 \oplus G_2$

Look at G w/ two subgroups H, K. When is $H \oplus K \simeq G$

Natural map $\Phi : (h, k) \to hk$, $\forall h \in H$, $\forall k \in K$

$\left(\longrightarrow \right)$

④

<u>Thm.</u> If H, K are normal subgroups of G s.t. $H \cap K = \{e\}$ then $H \oplus K \simeq HK \subseteq G$

Show $hk = kh$, $\forall h \in H$, $\forall k \in K$.

$(hk)(h^{-1}k^{-1}) = e$

$\Rightarrow h(kh^{-1}k^{-1}) = e$, $h'' = kh^{-1}k^{-1} \in H$

$hh'' = e$, $hh'' \in H$

$\Rightarrow (hkh^{-1})k^{-1} = e$, $k'' = hkh^{-1} \in K$

$k''k^{-1} = e$, $k''k^{-1} \in K$

$* (h,k) \to hk$ is isomorphic