Problem Set #1 Solutions

1. In all these problems, we use the formula $\lambda = c/f$, where $c = 2.9979 \times 10^8$ m/s. We also note that there are 2.54 cm/in *exactly*. (Strictly speaking, that is for the international inch. There are other inches defined, like the US survey inch, which is part of the mess with English units.) To four significant figures, which is useful for calculation, we note that there are 0.6214 mi/km, 3.281 ft/m, 0.3937 in/cm, and 39.37 mils/mm. We will be reporting all answers to three significant figures. We kept four digits in the calculation. All calculations were done with MATLAB.

   a. $\lambda = 2.9979 \times 10^8/1.50 = 2.00 \times 10^7$ m = 2000 km = 1240 mi.

   b. $\lambda = 2.9979 \times 10^8/2500 = 1.20 \times 10^5$ m = 120 km = 74.5 mi.

   c. $\lambda = 2.9979 \times 10^8/3.5 \times 10^5 = 856$ m = 0.856 km = 0.532 mi.

   d. $\lambda = 2.9979 \times 10^8/2.0 \times 10^6 = 150$ m = 492 ft.

   e. $\lambda = 2.9979 \times 10^8/5.5 \times 10^7 = 5.45$ m = 17.9 ft

   f. $\lambda = 2.9979 \times 10^8/1.25 \times 10^8 = 2.40$ m = 7.87 ft

   g. $\lambda = 2.9979 \times 10^8/3.0 \times 10^8 = 1.00$ m = 100 cm = 3.28 ft

   h. $\lambda = 2.998 \times 10^8/6.75 \times 10^9 = 4.44 \times 10^{-2}$ m = 4.48 cm = 1.75 in

   i. $\lambda = 2.998 \times 10^8/4.5 \times 10^{10} = 6.66 \times 10^{-3}$ m = 6.66 mm = 262 mils

2. We will convert all quantities to standard SI units throughout. Calculations were done with MATLAB. We will keep three significant figures throughout the calculation and report two at the end.

   a. $u_p = 4\pi \times 10^6/4.5 \times 10^{-2} = 2.79 \times 10^8$ m/s, $\lambda = 2\pi/4.50 \times 10^{-2} = 140$ m; $d =$ 4 km$\times$1000 m/km = 4000 m, so that $T = 4\times10^3/2.79\times10^8 = 1.4\times10^{-5}$ s = 14 μs, $\phi = 180$ rads = 10000°

   b. $u_p = 10\pi \times 10^9/2.00 \times 10^2 = 1.57 \times 10^8$ m/s, $\lambda = 2\pi/2.00 \times 10^2 = 0.031$ m = 3.1 cm; $d = 3$ in $\times$ 0.0254 m/in = 0.0762 m, so that $T = 0.0762/1.57 \times 10^8 = 4.8 \times 10^{-10}$ s = 480 ps, $\phi = 15$ rads = 870°

   c. $u_p = 20\pi \times 10^7/3.50 = 1.8 \times 10^8$ m/s, $\lambda = 2\pi/3.50 = 1.8$ m; $d = 10$ ft $\times$ 0.305 m/ft = 3.05 m, so that $T = 3.05/1.80 \times 10^8 = 1.7 \times 10^{-8}$ s = 17 ns, $\phi = 11$ rads = 610°

   d. $u_p = 3\pi\times10^3/3.20\times10^{-5} = 2.9\times10^8$ m/s, $\lambda = 2\pi/3.20\times10^{-5} = 2.0\times10^5$ m = 200 km; $d = 50$ mi $\times$ 1610 m/mi = 8.05×10^4 m, so that $T = 8.05 \times 10^4/2.95 \times 10^8 = 2.7 \times 10^{-4}$ s = 270 μs, $\phi = 2.6$ rads = 150°

3. a. Since the reflected wave is propagating backwards, it must be the case that

$$y_2(x,t) = B\cos(\omega t + \beta x + \phi_2)$$

We thus find that

$$\begin{aligned} y_{\mathrm{s}}(x,t) &= A\cos(\omega t - \beta x) + B\cos(\omega t + \beta x + \phi_2) \\ &= [A\cos(\beta x) + B\cos(\beta x + \phi_2)]\cos(\omega t) \\ &\qquad + [A\sin(\beta x) - B\sin(\beta x + \phi_2)]\sin(\omega t), \end{aligned}$$

where we have used the cosine addition theorem: $\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$. At $x = 0$, this expression becomes

$$y_{\mathrm{s}}(x=0,t) = [A + B\cos(\phi_2)]\cos(\omega t) - B\sin(\phi_2)\sin(\omega t)$$

In order for $y_{\mathrm{s}}(x = 0, t)$ to equal zero at $x = 0$ at all times, the coefficients of $\cos(\omega t)$ and $\sin(\omega t)$ must both vanish. From the coefficient $\sin(\omega t)$, we find that $\sin(\phi_2) = 0$, which implies that $\phi_2 = 0$ or $\phi_2 = \pi$. If we choose $\phi_2 = 0$, we find that the equation for the coefficient of $\cos(\omega t)$ implies that $B = -A$. If we had chosen $\phi_2 = \pi$, we would have found that $B = A$, which yields an equivalent expression. Using this result, we find that

$$y_{\mathrm{s}}(x,t) = 2A\sin(\omega t)\sin(\beta x).$$

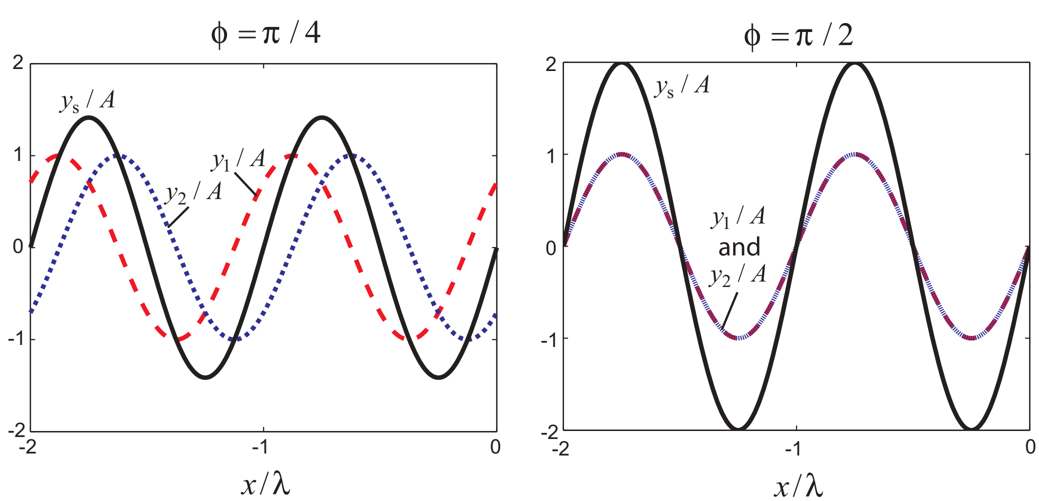
b. When $\omega t = \pi/4$, we have

$$\frac{y_1}{A} = \cos\left(\frac{\pi}{4} - \frac{2\pi x}{\lambda}\right), \qquad \frac{y_2}{A} = -\cos\left(\frac{\pi}{4} + \frac{2\pi x}{\lambda}\right), \qquad \frac{y_{\mathrm{s}}}{A} = \sqrt{2}\sin\left(\frac{2\pi x}{\lambda}\right)$$

When $\omega t = \pi/2$, we have

$$\frac{y_1}{A} = \sin\left(\frac{2\pi x}{\lambda}\right), \qquad \frac{y_2}{A} = -\sin\left(\frac{2\pi x}{\lambda}\right), \qquad \frac{y_{\mathrm{s}}}{A} = 0$$

The plots that are generated by these two cases are shown on the next page:



The MATLAB script that generated these figures is shown next. After the figure was generated, the labeling of the axes was modified using Adobe Illustrator.

```
% Reflection
%
% This file plots the solutions to Ulaby and Ravaioli
% Problem no. 1.7
%
close all
x = -2:0.01:0;     %Set the normalized x-axis values

% Set the values when theta = pi/4
y1_pf = cos((pi/4)-2*pi*x);
y2_pf = -cos((pi/4)+2*pi*x);
ys_pf = y1_pf + y2_pf;
plot(x,y1_pf,'r--',x,y2_pf,'b:',x,ys_pf,'k-.')

figure   %create a new figure

% Set the values when theta = pi/2
y1_ph = cos((pi/2)-2*pi*x);
y2_ph = -cos((pi/2)+2*pi*x);
ys_ph = y1_ph + y2_ph;
plot(x,y1_ph,'r--',x,y2_ph,'b:',x,ys_ph,'k-.')
```

4. Since the difference between the depths at which the two measurements were made is 90 m, we must have $196.04/163.74 = \exp(\alpha \times 90)$, where α is the attenuation coefficient. Keeping four significant figures, we have $\alpha = (1/90)\ln(196.04/163.74) = 2.0004 \times 10^{-3}$ Np/m. You should keep five significant figures because that was the smallest number of figures in the two measured values that were reported.

5. We first convert $z = -2 + j2$ into polar coordinates. We have $r = 2 * \sqrt{2} = 2.828$ and $\theta = \tan^{-1}(-1) = 2.356$ rads $= 135.0°$. Hence in polar coordinates, we write $z = 2.828\exp(j2.828) = 2.828\exp(j135.0°)$.

   a. $1/z = (1/2.828)\exp[(-1)(j2.356)] = 0.357\exp(-j2.36) = 0.357\exp(-j135°)$

   b. $z^3 = (2.828)^3\exp[(3)(j2.356)] = 22.6\exp(j7.068) = 22.6\exp(j0.785)$
   $= 22.6\exp(j45.0°)$

   c. $|z|^2 = 8.00$

   d. $\mathrm{Im}(z) = 2.00$

   e. $\mathrm{Im}(z^*) = -2.00$

   f. $\sqrt{z} = \sqrt{2.828}\exp(j2.356/2) = 1.68\exp(j1.18) = 1.68\exp(j67.5°)$

   Note that items c–e are purely real and are exact integers.

6. First, we carry out the hand calculations:

a. $z = 3 - j5 = \sqrt{34}\exp[-j\tan^{-1}(5/3)] \to \ln(z) = 0.5\ln(34) - j\tan^{-1}(5/3) \simeq 1.76 - j1.03$;

b. $\exp(3 - j4) = \exp(3) \times \exp(-j4) = \exp(3)\cos(4) - j\exp(3)\sin(4) \simeq 20.09 \times (-0.6536) - j20.09 \times (-0.7568) = -13.1 + j15.2$;

c. $z = 3\exp(j\pi/6) = 3\sqrt{3}/2 + j1.5 \to \exp(z) = \exp(3\sqrt{3}/2) \times \exp(j1.5)$
$= \exp(3\sqrt{3}/2)\cos(1.5) + j\exp(3\sqrt{3}/2)\sin(1.5) \simeq 13.44 \times 0.07073 + j13.44 \times 0.9974 = 0.951 + j13.4$

The MATLAB output is below:

```
>> log(3-j*5)

ans =

1.7632 - 1.0304i

>> exp(3-j*4)

ans =

-13.1288 +15.2008i

>> exp(3*exp(j*pi/6))

ans =

0.9506 +13.4042i
```

7. We begin by substituting $i(t) = A\cos(\omega t + \phi_0 - \phi_1)$ into the equation

$$Ri(t) + L\frac{di}{dt} = v_{\mathrm{s}}(t),$$

and we obtain

$$RA\cos(\omega t + \phi_0 - \phi_1) - \omega LA\sin(\omega t + \phi_0 - \phi_1) = V_0\cos(\omega t + \phi_0), \tag{1}$$

with $A = 5$, $V_0 = 5$, and $\omega = 4 \times 10^4$. Since $\sin(\omega t - 30°) = \cos(\omega t - 120°)$, we have $\phi_0 = -120°$. Writing now,

$$\begin{aligned} A\cos(\omega t + \phi_0 - \phi_1) &= A\cos\phi_1\cos(\omega t + \phi_0) + A\sin\phi_1\sin(\omega t + \phi_0), \\ A\sin(\omega t + \phi_0 - \phi_1) &= A\cos\phi_1\sin(\omega t + \phi_0) - A\sin\phi_1\cos(\omega t + \phi_0), \end{aligned} \tag{2}$$

and substituting (2) into (1), we find

$$\begin{aligned} &[RA\cos\phi_1 + \omega LA\sin\phi_1]\cos(\omega t + \phi_0) \\ &\quad + [RA\sin\phi_1 - \omega LA\cos\phi_1]\sin(\omega t + \phi_0) = V_0\cos(\omega t + \phi_0). \end{aligned} \tag{3}$$

Separately equating the coefficients of $\cos(\omega t + \phi_0)$ and $\sin(\omega t + \phi_0)$, we find the two equations

$$\begin{aligned} RA\cos\phi_1 + \omega LA\sin\phi_1 &= V_0, \\ RA\sin\phi_1 - \omega LA\cos\phi_1 &= 0. \end{aligned} \tag{4}$$

Equation (4) is two simultaneous equations for the two unknowns $A\cos\phi_1$ and $A\sin\phi_1$. Solving these equations, we find

$$A\cos\phi_1 = \frac{RV_0}{R^2 + \omega^2 L^2}, \qquad A\sin\phi_1 = \frac{\omega L V_0}{R^2 + \omega^2 L^2}. \tag{5}$$

Dividing $A\sin\phi$ by $A\cos\phi_1$, we find

$$\tan\phi_1 = \omega L/R \qquad \text{or} \qquad \phi_1 = \tan^{-1}(\omega L/R), \tag{6}$$

Squaring and adding $A\cos\phi_1$ and $A\sin\phi_1$, we obtain

$$A^2 = \frac{V_0^2}{R^2 + \omega^2 L^2} \qquad \text{or} \qquad A = \frac{V_0}{(R^2 + \omega^2 L^2)^{1/2}}. \tag{7}$$

Explicitly, we have $\phi_1 = \tan^{-1}[(4 \times 10^4) \times (2 \times 10^{-4})/6] = 0.927 = 53.1°$ and $A = 5/\{6^2 + [(4 \times 10^4) \times (2 \times 10^{-4}]^2\}^{1/2} = 0.5$ (A). We thus find that $i(t) = 0.5\cos(4 \times 10^4 t - 173.1°)$ (A). Hence, we conclude $v_{\mathrm{L}}(t) = L di/dt = 0.5 \times (2 \times 10^{-4}) \times (4 \times 10^4)\sin(4 \times 10^4 t - 173.1°) = 4\cos(4 \times 10^4 t - 83.1°)$ (V).

While not prohibitively complex, this approach is definitely more cumbersome than the phasor approach — aside from the need for the initial guess.

8. a. $v(t) = \mathrm{Re}[9\exp(j\omega t - j\pi/3)$ (V) $\to \tilde{V} = 9\exp(-j\pi/3)$ (V)

   b. $v(t) = 12\cos(\omega t - \pi/4) = \mathrm{Re}[12\exp(j\omega t - j\pi/4)]$ (V) $\to \tilde{V} = 12\exp(-j\pi/4)$ (V)

   c. $i(x,t) = 5\exp(-3x)\,\mathrm{Re}[\exp(j\omega t - j\pi/3)]$ (A) $\to \tilde{I}(x) = 5\exp(-3x)\exp(-j\pi/3)$ (A)

   d. $i(t) = 2\,\mathrm{Re}[\exp(j\omega t - j\pi/4)]$ (A) $\to \tilde{I} = 2\exp(-j\pi/4)$ (A)

   e. Noting that $4\sin(\omega t + \pi/3) = 4\cos(\omega t - \pi/6)$, we have $i(t) = 7\cos(\omega t - \pi/6)$ (A). We then find, $\tilde{I} = 7\exp(-j\pi/6)$ (A)