HW#9 Solutions

Problem 1.

Let $Y = |X|$.

(a) Since X takes nonnegative values only, the PDF of Y is the same as that of X:

$$f_Y(y) = \begin{cases} \lambda e^{-\lambda y}, & y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

(b) Values of Y in the interval $[0, 1]$ are twice as likely as values in the interval $[1, 2]$. Thus, we have

$$f_Y(y) = \begin{cases} 2/3, & \text{if } 0 < y \leq 1, \\ 1/3, & 1 < y \leq 2. \end{cases}$$

(c) We first compute the CDF and then differentiate to obtain the PDF of Y. We have, for $y \geq 0$,

$$F_Y(y) = \mathbf{P}(Y \leq y) = \mathbf{P}\big(|X| \leq y\big) = \mathbf{P}(-y \leq X \leq y) = F_X(y) - F_X(-y),$$

and taking the derivative we obtain

$$f_Y(y) = \begin{cases} f_X(y) + f_X(-y), & \text{if } y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Problem 2. Let $Z = |X - Y|$. We have

$$F_Z(z) = P\big(|X - Y| \leq z\big) = 1 - (1 - z)^2.$$

(To see this, draw the event of interest as a subset of the unit square and calculate its area.) Taking derivatives, the desired PDF is

$$f_Z(z) = \begin{cases} 2(1 - z), & \text{if } 0 \leq z \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Problem 3.

(a) Let S and R be the events that the day is sunny and rainy, respectively. We are given that $f_{X|S}$ and $f_{X|R}$ are uniform PDFs over the intervals $[30, 45]$ and $[40, 60]$, respectively. Moreover we know that $\mathbf{P}(S) = 2/3$ and $\mathbf{P}(R) = 1/3$. If follows that

$$\begin{aligned} f_X(x) &= \mathbf{P}(S) f_{X|S}(x) + \mathbf{P}(R) f_{X|R}(x) \\ &= \begin{cases} (2/3)(1/15) = 0.0444, & \text{if } 30 \le x < 40, \\ (2/3)(1/15) + (1/3)(1/20) = 0.0611, & \text{if } 40 \le x \le 45, \\ (1/3)(1/20) = 0.0167, & \text{if } 45 < x \le 60\ . \end{cases} \end{aligned}$$

As for the expected value and the variance of X, we have

$$\mathbf{E}[X] = \mathbf{E}[X \mid S]\mathbf{P}(S) + \mathbf{E}[X \mid R]\mathbf{P}(R) = \frac{30+45}{2} \cdot \frac{2}{3} + \frac{40+60}{2} \cdot \frac{1}{3} = \frac{125}{3} = 41.67,$$

$$\begin{aligned} \mathbf{E}[X^2] &= \mathbf{E}[X^2 \mid S]\mathbf{P}(S) + \mathbf{E}[X^2 \mid R]\mathbf{P}(R) \\ &= \frac{2}{3} \cdot \frac{1}{15} \int_{30}^{45} x^2\, dx + \frac{1}{3} \cdot \frac{1}{20} \int_{40}^{60} x^2\, dx \\ &= 1794.4, \end{aligned}$$

and

$$\mathrm{var}(X) = 1794.4 - (41.67)^2 = 58.01.$$

(b)

$$\mathbf{P}(R \mid X = 45) = \frac{\mathbf{P}(R) f_X(45 \mid R)}{f_X(45)} = \frac{(1/3)(1/20)}{0.0611} = 0.2733.$$

(c) Let V be the average speed. We have $V = 20/X$, so that

$$\begin{aligned} f_V(v) &= f_X\left(\frac{20}{v}\right) \left| \frac{d}{dv} \left(\frac{20}{v}\right) \right| \\ &= \frac{20}{v^2} f_X\left(\frac{20}{v}\right) \\ &= \begin{cases} 1/(3v^2), & \text{if } 20/60 \le v < 20/45, \\ 55/(45v^2), & \text{if } 20/45 \le v \le 20/40, \\ 8/(9v^2), & \text{if } 20/40 < v \le 20/30, \end{cases} \end{aligned}$$

$$\mathbf{E}[V] = \int_{20/60}^{20/45} \frac{v}{2v^2} dv + \int_{20/45}^{20/40} \frac{25v}{18v^2} dv + \int_{20/40}^{20/30} \frac{8v}{9v^2} dv = 0.496,$$

$$\mathbf{E}[V^2] = \int_{20/60}^{20/45} \frac{v^2}{2v^2}dv + \int_{20/45}^{20/40} \frac{25v^2}{18v^2}dv + \int_{20/40}^{20/30} \frac{8v^2}{9v^2}dv = 0.253,$$

and

$$\mathrm{var}(V) = 0.253 - (0.496)^2 = 6.984 \cdot 10^{-3}.$$

Problem 4.

(a) We have

$$\mathbf{P}(B \,|\, A) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A)} = \frac{\mathbf{P}(X < Y \le 0.5)}{1 - \mathbf{P}(Y \ge 0.5)} = \frac{0.25}{1 - 0.25} = \frac{1}{3}.$$

(b) We have

$$f_{X|Y}(x \,|\, 0.5) = \frac{f_{X,Y}(x, 0.5)}{f_Y(0.5)} = \begin{cases} 2, & \text{if } 0 < x \le 0.5, \\ 0, & \text{otherwise.} \end{cases}$$

The conditional expectation is $1/4$ and the conditional variance is $(0.5)^2/12 = 1/3$.

(c) We have

$$\begin{aligned} f_{X,Y|B}(x,y) &= \begin{cases} f_{X,Y}(x,y)/\mathbf{P}(B), & \text{if } (x,y) \in B, \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} 4, & \text{if } (x,y) \in B, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore,

$$\begin{aligned} f_{X|B}(x) &= \int_0^1 f_{X,Y|B}(x,y)\,dy \\ &= \begin{cases} \displaystyle\int_x^{1-x} 4\,dx = 4(1-2x), & \text{if } 0 < x \le 0.5, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

(d) We have

$$\mathbf{E}[XY] = \int_0^1 \int_0^{1-x} 2xy\,dy\,dx = \int_0^1 x(1-x)^2\,dx = \frac{1}{12}.$$

(e) Let $Z = Y/X$. If $z \ge 0$,

$$f_Z(z) = \frac{d}{dz}\mathbf{P}\left(\frac{Y}{X} \le z\right) = \frac{d}{dz}\left(\frac{z}{z+1}\right) = \frac{1}{(z+1)^2}.$$

Problem 5. The random variables $X_1, \ldots, X_n$ have common mean μ, common variance σ^2 and, furthermore, $\mathbf{E}[X_i X_j] = c$ for every pair of distinct i and j. Derive a formula for the variance of $X_1 + \cdots + X_n$, in terms of μ, σ^2, c, and n.

Solution: We first note that, for distinct i and j, we have

$$\text{cov}(X_i, X_j) = \mathbf{E}[X_i X_j] - \mathbf{E}[X_i]\,\mathbf{E}[X_j] = c - \mu^2.$$

Thus, using the formula for the variance of the sum of random variables,

$$\text{var}(X_1 + \cdots + X_n) = n\sigma^2 + (n^2 - n)(c - \mu^2).$$

Problem 6. Consider n independent tosses of a die. Each toss has probability p_i of resulting in i. Let X_i be the number of tosses that result in i. Show that X_1 and X_2 are negatively correlated (i.e., a large number of ones suggests a smaller number of twos).

Solution: Let A_t (respectively, B_t) be a Bernoulli random variable which is equal to 1 if and only if the tth toss resulted in 1 (respectively, 2). We have $\mathbf{E}[A_t B_t] = 0$ and $\mathbf{E}[A_t B_s] = \mathbf{E}[A_t]\mathbf{E}[B_s] = p_1 p_2$ for $s \neq t$. We have

$$\mathbf{E}[X_1 X_2] = \mathbf{E}\big[(A_1 + \cdots + A_n)(B_1 + \cdots + B_n)\big] = n\mathbf{E}\big[A_1(B_1 + \cdots + B_n)\big] = n(n-1)p_1 p_2,$$

and

$$\text{cov}(X_1, X_2) = \mathbf{E}[X_1 X_2] - \mathbf{E}[X_1]\mathbf{E}[X_2] = n(n-1)p_1 p_2 - np_1 np_2 = -np_1 p_2.$$

Problem 7. Let $X = Y - Z$ where Y and Z are nonnegative random variables such that $YZ = 0$.

(a) Show that $\mathrm{cov}(Y, Z) \leq 0$.

(b) Show that $\mathrm{var}(X) \geq \mathrm{var}(Y) + \mathrm{var}(Z)$.

(c) Use the result of part (b) to show that

$$\mathrm{var}(X) \geq \mathrm{var}\big(\max\{0, X\}\big) + \mathrm{var}\big(\max\{0, -X\}\big).$$

Solution: (a) Let m_Y and m_Z be the means of Y and Z, respectively. Note that these means are nonnegative. We have

$$\mathrm{cov}(Y, Z) = \mathbf{E}[YZ] - m_Y m_Z = -m_Y m_Z \leq 0.$$

(b) We have

$$\mathrm{var}(X) = \mathrm{var}(Y) + \mathrm{var}(Z) - 2\mathrm{cov}(Y, Z) \geq \mathrm{var}(Y) + \mathrm{var}(Z).$$

(c) Let $Y = \max\{0, X\} \geq 0$ and $Z = \max\{0, -X\} \geq 0$. Note that $X = Y - Z$ and $YZ = 0$. The result follows from part (b).

Problem 8. Consider two random variables X and Y. Assume for simplicity that they both have zero mean.

(a) Show that X and $\mathbf{E}[X \,|\, Y]$ are positively correlated.

(b) Show that the correlation coefficient of Y and $\mathbf{E}[X \,|\, Y]$ has the same sign as the correlation coefficient of X and Y.

Solution: (a) We have

$$\mathrm{cov}\big(X, \mathbf{E}[X \,|\, Y]\big) = \mathbf{E}\big[X\mathbf{E}[X \,|\, Y]\big] = \mathbf{E}\big[\mathbf{E}\big[X\mathbf{E}[X \,|\, Y] \,|\, Y\big]\big] = \mathbf{E}\big[\big(\mathbf{E}[X \,|\, Y]\big)^2\big] \geq 0.$$

(b) We have

$$\mathrm{cov}\big(Y, \mathbf{E}[X \,|\, Y]\big) = \mathbf{E}\big[Y\mathbf{E}[X \,|\, Y]\big] = \mathbf{E}\big[\mathbf{E}[XY \,|\, Y]\big] = \mathbf{E}[XY] = \mathrm{cov}(X, Y).$$