HW#3 Solutions

Problem 1.

The probability that persons 1 and 2 both roll a particular face is $1/n^2$. Therefore,

$$\mathbf{P}(A_{12}) = \mathbf{P}(A_{13}) = \mathbf{P}(A_{23}) = \frac{n}{n^2} = \frac{1}{n}.$$

Similarly, we also have

$$\mathbf{P}(A_{12} \cap A_{13}) = \mathbf{P}(\text{all players roll the same face}) = \frac{n}{n^3} = \frac{1}{n^2},$$

so $\mathbf{P}(A_{12} \cap A_{13}) = \mathbf{P}(A_{12}) \cdot \mathbf{P}(A_{13})$. Hence A_{12} and A_{13} are independent, and the same is true of any other pair from the events A_{12}, A_{13}, and A_{23}. However, A_{12}, A_{13}, and A_{23} are not independent. In particular, if A_{12} and A_{13} occur, then A_{13} also occurs.

Problem 2.

The answer is no. Consider two tosses of a fair coin. The events $A = \{HH, TT\}$, $B = \{HH, HT\}$, $C = \{HH, TH\}$ satisfy the independence assumptions in the problem statement. On the other hand,

$$\mathbf{P}(A \mid B \cup C) = \frac{1}{3} \neq \frac{1}{2} = \mathbf{P}(A),$$

and A is not independent of $B \cup C$.

Problem 3. (Here, $A_1 \cap A_2$ is short-handed to $A_1 A_2$)

(a) A closed circuit can occur as

$$(A_2 A_4 \cup A_3 A_5)\, A_1 A_6 = A_1 A_2 A_4 A_6 \cup A_1 A_3 A_5 A_6.$$

(b) Now in general $P[A \cup B] = P[A] + P[B] - PAB]$, thus

$$\begin{aligned} P[\{\text{at least one closed path}\}] &= P[A_1 A_2 A_4 A_6] + P[A_1 A_3 A_5 A_6] - P[A_1 A_2 A_3 A_4 A_5 A_6] \\ &= 2p^4 - p^6 \\ &= 2p^4(1 - \frac{1}{2}p^2). \end{aligned}$$

Problem 4.

Let $E = \{$event of successful transmission on short path$\}$;$F = \{$event of successful transmission on a long path$\}$. Then $P[\{\text{successful transmission}\}] = 1 - P[E^cF^c]$ and $P[E^c] = 1 - p^3$, while $P[F^c] = 1 - p^5$, where $p \triangleq 1 - q$. Therefore

$$\begin{aligned} P[\{\text{successful transmission}\}] &= 1 - P[E^cF^c] \\ &= 1 - (1-p^3)(1-p^5) \\ &= p^3 + p^5 - p^8. \end{aligned}$$

Problem 5.

The sample space for the compound experiment is

$$\Omega = \{(x_1, x_2, ..., x_{100}) : 2 \le x_i \le 12, 1 \le i \le 100\}.$$

For the individual experiment with the two die, we can write the sample space as the locations (ξ_1, ξ_2) in the 6×6 table

(ξ_1, ξ_2)	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

,

where we have entered in each cell the sum of the die's upward faces. Now we set the event $A \triangleq \{$the sum is 7$\}$ and find $P[A] = 6/36 = 1/6 \triangleq p$. As for the compound experiment consisting of $N = 100$ tries, it is seen to be Bernoulli trials with $n = 100$ and $p = 1/6$. So the answer for '10 seven's in 100 tries' is $b(7; 100, 1/6) = \binom{100}{10} p^{10}(1-p)^{90}$.

Problem 6.

(a) The probability that a BM gets destroyed is

$$\begin{aligned} 1 - P[\{\text{both AMM miss}\}] &= 1 - (0.2)(0.2) \\ &= 0.96. \end{aligned}$$

Hence for all BMs to get destroyed, we need six wins in six tries:

$$\binom{6}{6}(0.96)^6(0.04)^0 = (0.96)^6 \simeq 0.783.$$

(b) $P[\{\text{at least one BM gets through }\}] = 1 - P[\{\text{all are destroyed}\}] \simeq 1 - 0.783 = 0.217.$

(c)

$$\begin{aligned} P[\{\text{exactly one gets through}\}] &= \binom{6}{5}(0.96)^5(0.04)^1 \\ &= 6(0.96)^5(0.04) \\ &\simeq 0.196. \end{aligned}$$

Problem 7.

Let Q be the event that someone is qualified, Q^c the event that someone is unqualified, and A the event that the 20 questions correctly determine whether the candidate is qualified or not. Using the total probability theorem, we have

$$\begin{aligned} \mathbf{P}(A) &= \mathbf{P}(Q)\mathbf{P}(A \mid Q) + \mathbf{P}(Q^c)\mathbf{P}(A \mid Q^c) \\ &= q\sum_{i=15}^{20}\binom{20}{i}p^i(1-p)^{20-i} + (1-q)\sum_{i=6}^{20}\binom{20}{i}p^i(1-p)^{20-i}. \end{aligned}$$

Problem 8.

(a) Let A be the event that all 20 cars tested are good. We are asked to find $\mathbf{P}(K = 0 \mid A)$. Using Bayes' rule, we have

$$\mathbf{P}(K = 0 \mid A) = \frac{\mathbf{P}(K = 0)\mathbf{P}(A \mid K = 0)}{\sum_{i=0}^{9}\mathbf{P}(K = i)\mathbf{P}(A \mid K = i)}.$$

It is given that $\mathbf{P}(K = i) = 1/10$ for all i. To compute $\mathbf{P}(A \mid K = i)$, we condition on the event of exactly i lemons, and reason as follows. The first selected car has probability $(100 - i)/100$ of being good. Having succeeded in the first selection, we are left with 99 cars out of which i are lemons; thus, the second selected car has probability $(99 - i)/99$ of being good. Continuing similarly, and using the multiplication rule, we obtain

$$\mathbf{P}(A \mid K = i) = \frac{(100-i)(99-i)\cdots(81-i)}{100 \cdot 99 \cdot 81},$$

from which we can then obtain $\mathbf{P}(K=0 \mid A)$.

(b) We use the exact same argument as in part (a), except that we need to recalculate $\mathbf{P}(A \mid K=i)$. Since the cars are chosen with replacement, we are dealing with 20 independent Bernoulli trials. The probability of finding a good car in any one trial is $(100-i)/100$. The probability of finding good cars in all 20 trials is

$$\mathbf{P}(A \mid k=i) = \left(\frac{100-i}{100}\right)^{20},$$

from which we can then obtain $\mathbf{P}(K=0 \mid A)$.

Problem 9.

Think of lining up the jelly beans, by first placing the red ones, then the orange ones, etc. We also place 5 dividers to indicate where one color ends and another starts. (Note that two dividers can be adjacent if there are no jelly beans of some color.) By considering both jelly beans and dividers, we see that there is a total of 105 positions. Choosing the number of jelly beans of each color is the same as choosing the positions of the dividers. Thus, there are $\binom{105}{5}$ possibilities, and this is the number of possible jars.

Problem 10.

(a) The word "children" consists of 8 distinct letters, so the number of arrangements is the same as the number of possible permutations, namely 8!.

(b) In the word "bookkeeper", the letters "b", "o", "k", "e", "p", and "r" appear 1, 2, 2, 3, 1, and 1 times respectively. Arguing exactly as in the text the number of distinguishable rearrangements is

$$\frac{10!}{3!\,2!\,2!}.$$