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MATH 407: HW 10

3.6 **5** Show that no proper subgroup of S_4 contains both $(1, 2, 3, 4)$ and $(1, 2)$.

Pf. Suppose $H \leq S_4$ with the permutations (1 2 3 4) and (1 2)

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$$

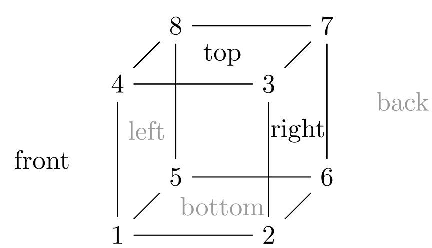
Their product would yield

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} = S_4$$

Therefore, H is not a proper subgroup of S_4 by contradiction $\square$

9 A rigid motion of a cube can be thought of either as a permutation of its eight vertices or as a permutation of its six sides. Find a rigid motion of the cube that has order 3, and express the permutation that represents it in both ways, as a permutation on eight elements and as a permutation on six elements.

Figure 1: Cube of order 3



Pf. Rigid motion of order 3 will be rotating the cube about a vector passing through $(1, 7)$ $120°$

The permutation of the eight vertices are $(2, 4, 5)(3, 8, 6)$

and the sides are (front, left, back)(top, back, right) $\square$

10 Show that the following matrices form a subgroup of $\mathrm{GL}_2(\mathbb{C})$ isomorphic to D_4:

$$\pm\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \pm\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \pm\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

Pf. Let $a = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ and $b = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Then,

$$a^2 = \left(\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\right)^2 = \begin{bmatrix} \text{-1} & 0 \\ 0 & \text{-1} \end{bmatrix}$$

$$a^3 = \left(\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\right)^3 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

$$a^4 = \left(\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\right)^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$b^2 = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$ab = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

$$a^2b = \begin{bmatrix} \text{-1} & 0 \\ 0 & \text{-1} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \text{-1} \\ \text{-1} & 0 \end{bmatrix}$$

$$a^3b = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Which forms the set

$$\begin{aligned} \mathrm{GL}_2(\mathbb{C}) &= \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}, \text{ with } a^4 = b^2 = e,\ ba = a^{-1}b \\ &\cong D_4 \end{aligned}$$ □

17 For any elements $\sigma, \tau \in S_n$, show that $\sigma\tau\sigma^{-1}\tau^{-1} \in A_n$.

Pf. Let $\sigma, \tau \in S_n$ be the products of m and n transpositions respectively
Thus, $\sigma^{-1}, \tau^{-1} \in S_n$ are also the products of m and n transpositions
Therefore, $\sigma\tau\sigma^{-1}\tau^{-1}$ is a product of $m + n + m + n = 2(m + n)$ transpositions
Since $2 \mid 2(m + n)$, $\sigma\tau\sigma^{-1}\tau^{-1} \in A_n$ □

21 Find the center of the dihedral group D_n.

Hint: Consider two cases, depending on whether n is odd or even.

Pf. The center of a group is the subgroup consisting of all the elements that commute with every other element in the group

The group D_1 is $\mathbb{Z}/2\mathbb{Z}$, and D_2 is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

Let $n > 2$, and assume yx^k is in the center,

That is, yx^k must commute with x

Then,

$$xyx^kx^{-1} = yx^{k-2} = yx^k$$

Which implies, $2 \equiv 0 \pmod{n}$, (contradiction)

Therefore, x^k is in the center iff $2k \equiv 0 \pmod{n}$

If n is odd, then the center of D_{2m+1} is $\{e\}$

If n is even, then the center of D_{2m} is $\{e, x^m\}$ □

24 Show that the product of two transpositions is one of (i) the identity; (ii) a 3-cycle; (iii) a product of two (non-disjoint) 3-cycles. Deduce that every element of A_n can be written as a product of 3-cycles.

Pf. Consider $\sigma = (1, 2, 3, 4)$

(i) For identity,

$$(1, 2)(1, 2) = (1)$$

(ii) For a 3-cycle,

$$(1, 2)(2, 3) = (1, 3, 2)$$

(iii) For a product of two (non-disjoint) 3-cycles,

$$(1, 2)(3, 4) = (1, 2, 3)(1, 4, 3)$$

Since A_n is a set of even permutations, is it can be expressed as a product of even number of transpositions □