

MATH 407

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(1)

$$(1, 2)(1, 2, 3) \neq (1, 2, 3)(1, 2) \text{ [recap]}$$

Def. $\pi_1, \pi_2 \in S_n$ are disjoint
iff $\pi_1(i) \neq i$ implies $\pi_2(i) = i$
and $\pi_2(i) \neq i$ implies $\pi_1(i) = i$

Example:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 8 & 2 & 10 & 11 & 5 & 9 & 4 & 6 & 1 & 3 & 12 & 7 \end{pmatrix}$$

$$\text{cycles} = \{(1, 8, 6, 9), (2), (3, 10), (4, 11, 12, 7), (5)\}$$

$(1, 8, 6, 9)$ $(4, 11, 12, 7)$ are disjoint since they leave each other as fixed points

Thm: If π_1, π_2 are disjoint then $\pi_1 \circ \pi_2 = \pi_2 \circ \pi_1$

Pf. Choose $i \in \{1, \dots, n\}$

a) π_1, π_2 have i as fixed

$$\pi_1(\pi_2(i)) = \pi_1(i) = i$$

$$\pi_2(\pi_1(i)) = \pi_2(i) = i$$

b) $\pi_1(i) \neq i \Rightarrow \pi_2(i) = i$

$$\pi_1(\pi_1(i)) \neq \pi_1(i)$$

$$\pi_2(\pi_1(i)) = \pi_1(i)$$

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$$\pi_1(\pi_2(i)) = \pi_1(i)$$

c) Same vice versa for $\pi_2(i) \neq i \Rightarrow \pi_1(i) = i$

* π_1^k is disjoint from π_2^l , for any k, l

If $\pi_1^k(i) \neq i$, then $\pi_1(i) \neq i$.

Thus $\pi_2(i) = i$

$$\Rightarrow \pi_2^l(i) = i$$

*	i		(1, 8, 6, 9) (2) (3, 10) (4, 11, 12, 7) (5) (i)
1	1		8 ← 1 ← 1 ← 1 ← 1
2	2		2 ← 2 ← 2 ← 2 ← 2
3	3		10 ← 10 ← 10 ← 3 ← 3

Def. Order of a permutation π

$$\pi^{-1}, \pi = (1), \pi^1 = \pi, \dots, \pi^k, \dots \text{ (bi-infinite)}$$

If $\pi \in \text{Sym}(S)$ and S infinite possibly no repeats.

$S_n = \text{Sym}(\mathbb{N}_n)$, there are $i, j = i+k$

$$\pi^i = \pi^{i+k} = \pi^j$$

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$$\pi^0 = \pi^k, \quad (k > 0, \text{ minimum}) \quad \pi^0 = (1) \text{ and } \pi^k = (1) \\ k = \text{order}(\pi) = o(\pi)$$

Def. Order of cyclic permutation is length of permutation.

Thm. If π_1, π_2 disjoint permutations,
 $k_1 = o(\pi_1), k_2 = o(\pi_2)$
 then $k = o(\pi_1, \pi_2) = \text{lcm}(k_1, k_2)$

Pf. $(\pi_1, \pi_2)^{\text{lcm}(k_1, k_2)}$
 Let $L = \text{lcm}(k_1, k_2)$
 Then $(\pi_1, \pi_2)^L = \pi_1^L \pi_2^L = (1)(1) = (1)$

Thus, $k \mid L$.

Conversely, to show $L \mid k$:

Look at $(\pi_1, \pi_2)^k = (1)$

$$\pi_1^k \pi_2^k = (1)$$

Let $\pi_1(i) \neq i$ s.t. $\pi_2(i) = i$

$$\pi_1^k(i) = \pi_1^k(\pi_2^k(i)) = \pi_1^k(i) = i$$

$$\Rightarrow \pi_1^k = (1) = \pi_1^0$$

$k_1 \mid k$ and $k_2 \mid k$

$$\Rightarrow \text{lcm}(k_1, k_2) \mid k = L \mid k$$

Thm. Suppose $\sigma \in S_n$ has cycles $\sigma_1, \sigma_2, \dots, \sigma_r$

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Then $o(\sigma) = \text{lcm}(o(\sigma_1), \dots, o(\sigma_k))$

$$(\pi) \circ (\pi) = \text{id}$$

c) Same as above but $\pi^2 = \text{id}$

Def: Order of a cycle permutation is length of cycle

If $\pi^k = \text{id}$ then k is a multiple of $o(\pi)$

$$\pi^k = \text{id} \Rightarrow k = o(\pi) \cdot m$$

$$\Rightarrow (\pi^k)(i) = i = (\pi^{o(\pi) \cdot m})(i) = i$$

$$\pi = (1, 2, 3, 4) \quad \pi^2 = (1, 3)(2, 4) \quad \pi^3 = (1, 4, 3, 2)$$

$$\pi^4 = \text{id}$$

$$\pi^5 = \pi$$

$$\pi^6 = \pi^2$$

$$\pi^7 = \pi^3$$

Def: Order of a permutation is the least common multiple of the lengths of its cycles

$$\pi = (1, 2, 3, 4) \quad \pi^2 = (1, 3)(2, 4) \quad \pi^3 = (1, 4, 3, 2)$$

$$\pi^4 = \text{id}$$

If $\pi \in S_n$ then $\pi^k = \text{id}$ if and only if k is a multiple of $o(\pi)$

$$\pi^k = \text{id} \Rightarrow k = o(\pi) \cdot m$$

$$\pi^k = \text{id} \Rightarrow k = o(\pi) \cdot m$$

$S_n = \{ \pi \in S_n \mid \pi^k = \text{id} \text{ for some } k \}$

$$\Rightarrow \text{lcm}(k_1, k_2, \dots, k_r) = k$$

Then $\sigma \in S_n$ has order $o(\sigma)$