Problem Set #1 Solutions

1. In all these problems, we use the formula $\lambda = c/f$, where $c = 2.998 \times 10^8$ m/s. We also note that there are 2.54 cm/in *exactly*. (Strictly speaking, that is for the international inch. There are other inches defined, like the US survey inch, which is part of the mess with English units.) To four significant figures, which is useful for calculation, we note that there are 0.6214 mi/km, 3.281 ft/m, 0.3937 in/cm, and 39.37 mils/mm. We will be reporting all answers to three significant figures. We kept four digits in the calculation.

   a. $\lambda = 2.998 \times 10^8/100 = 3.00 \times 10^6$ m $= 3000$ km $= 3.00$ Mm. We also have $\lambda = 0.6214$ mi/km $\times$ 3000 km $= 1860$ mi.

   b. $\lambda = 2.998 \times 10^8/1500 = 2.00 \times 10^5$ m $= 200$ km. We also have $\lambda = 0.6214$ mi/km$\times$ 200 km $= 124$ mi.

   c. $\lambda = 2.9979 \times 10^8/3.3 \times 10^5 = 908$ m. We also have $\lambda = 3.281$ ft/m $\times$ 908.46 m $=$ 2980.5 ft $=$ (2980.5/5280) $=$ 0.564 mi. In this case, the answer to five significant figures is 0.56449. As a consequence, one must keep at least five digits in this case to know whether to round up or down!

   d. $\lambda = 2.998 \times 10^8/1.0 \times 10^6 = 300$ m. We also have $\lambda = 3.281$ ft/m$\times$299.8 m $= 984$ ft

   e. $\lambda = 2.998 \times 10^8/3.0 \times 10^7 = 9.99$ m. We also have $\lambda = 3.281$ ft/m $\times$ 9.993 m $=$ 32.8 ft

   f. $\lambda = 2.998 \times 10^8/1.25 \times 10^8 = 2.40$ m. We also have $\lambda = 3.281$ ft/m $\times$ 2.398 m $=$ 7.87 ft

   g. $\lambda = 2.998 \times 10^8/3.10 \times 10^8 = 0.967$ m $= 96.7$ cm. We also have $\lambda = 3.281$ ft/m $\times$ 0.9671 m $= 3.17$ ft

   h. $\lambda = 2.998 \times 10^8/6.50 \times 10^9 = 4.61 \times 10^{-2}$ m $= 4.61$ cm. We also have $\lambda =$ 0.3937 in/cm $\times$ 4.612 cm $= 1.81$ in

   i. $\lambda = 2.9979 \times 10^8/4.3 \times 10^{10} = 6.97 \times 10^{-3}$ m $= 6.97$ mm. We also have $\lambda =$ 39.370 mils/mm $\times$ 6.9719 mm $= 274$ mils. In this case, one must once again keep two extra digits to avoid rounding errors.

2. In this case, we must determine λ as before and then use it to divide the distance. We will again report results to three significant figures except in part (d) and use the conversion factors in problem 1. We only report two significant figures in part (d) because the speed of light is only given to two significant digits and reporting more digits would be meaningless.

   a. $\lambda = 2.998 \times 10^8/50 = 5.996 \times 10^6$ m; $d = 10^3$ m/km $\times$ 10^3km $= 1.000 \times 10^5$ m; $d/\lambda = 0.0167$

   b. $\lambda = 2.998 \times 10^8/4 \times 10^5 = 749.5$ m; $d = 0.3048$ m/ft $\times$ 500 ft $= 152.4$ m; $d/\lambda = 0.203$

c. $\lambda = 2.998\times10^8/1.10\times10^8 = 2.725$ m; $d = 0.3048$ m/ft $\times$ 5 ft $= 1.524$ m; $d/\lambda = 0.559$

d. $\lambda = 1.5\times10^8/3.0\times10^9 = .0500$ m; $d = 0.0254$ m/in$\times$2.0 in $= .0508$ m; $d/\lambda = 0.98$

3. We will convert all quantities to standard SI units throughout. We will show up to three significant figures throughout the calculation and report two at the end. The actual calculations were done with MATLAB, which has close to 15 digits of precision.

   a. $u_p = 2\pi\times10^6/2.2\times10^{-2} = 2.9\times10^8$ m/s, $\lambda = 2\pi/2.2\times10^{-2} = 290$ m; $d =$ 5 km$\times$1000 m/km $=$ 5000 m, so that $T = 5\times10^3/2.86\times10^8 = 1.0\times10^{-5}$ s $= 18$ μs, $\phi = 110$ rads $= 6300°$

   b. $u_p = 6\pi\times10^9/7.54\times10^1 = 2.5\times10^8$ m/s, $\lambda = 2\pi/7.54\times10^1 = 0.083$ m $= 8.3$ cm; $d = 3$ in $\times$ 0.0254 m/in $= 0.0833$ m, so that $T = 0.0833/2.50\times10^8 = 3.0\times10^{-10}$ s $= 300$ ps, $\phi = 5.7$ rads $= 330°$

   c. $u_p = 30\pi\times10^7/3.15 = 3.0\times10^8$ m/s, $\lambda = 2\pi/3.15 = 2.0$ m; $d = 15$ ft $\times$ 0.305 m/ft $= 4.57$ m, so that $T = 4.57/2.99\times10^8 = 1.5\times10^{-8}$ s $= 15$ ns, $\phi = 14$ rads $= 830°$

   d. $u_p = 6\pi\times10^3/1.26\times10^{-4} = 1.5\times10^8$ m/s, $\lambda = 2\pi/1.26\times10^{-3} = 5.0\times10^3$ m $= 5.0$ km; $d = 40$ mi $\times$ 1610 m/mi $= 6.44\times10^4$ m, so that $T = 6.44\times10^4/1.50\times10^8 = 4.3\times10^{-4}$ s $= 430$ μs, $\phi = 8.11$ rads $= 460°$

4. a. We have $f = \omega/2\pi = \pi\times10^9/2\pi = 5\times10^8$ Hz $= 500$ MHz. We have $\lambda = 2\pi/\beta = 2\pi/5\pi = 0.4$ m $= 40.0$ cm. We have $u_p = f\lambda = 0.2\times10^9 = 2.00\times10^8$ m/s.

   b. We have $4\exp(-\alpha\times2) = 1$, which implies that $\alpha = 0.5\ln(4) = 0.361$ Np.

5 a. We have that $f = c/\lambda = 2.998\times10^8/1.5\times10^{-6} = 2.00\times10^{14}$ Hz. In fiber optics, unlike almost all other areas of electromagnetics, one rarely quotes the frequency. One always quotes the vacuum wavelength. Nonetheless, it is still the frequency that is fundamental since the wavelength in the optical fiber is not equal to the vacuum wavelength, but the frequency does not change inside the fiber.

   b. $u_p = \omega/\beta = c/n$, where n is the index of refraction. It is common in many areas of electromagnetics to quote the index of refraction, rather than the phase velocity, because the index of refraction typically changes slowly with frequency. In our case, $u_p = 2.998\times10^8/1.5 = 2.00\times10^8$ m/s.

   c. The time separation is given by $1/R$, where R is the data rate. In our case $T_{\text{bit}} = 1/(1.00\times10^{10}) = 1.0\times10^{-10}$ s $= 100$ ps.

   d. Since the frequency is the same for the two polarizations, we have that $\Delta u_p = (\omega/\beta_1) - (\omega/\beta_2) = \omega\Delta\beta/\beta_1\beta_2$, where β_2 is the wavenumber of the faster of the two polarizations and β_1 is the wavenumber of the slower of the two polarizations. Since they only differ to one part in 10^5, it follows that to four significant figures, they are the same. To this approximation, we may write $\Delta u_p = \omega \Delta\beta/\beta^2 = u_p(\Delta\beta/\beta) = 2.00 \times 10^8 \times 10^{-5} = 2.00 \times 10^3$ m/s.

e. To calculate this value, we must calculate the signal delay in reaching a point z. We have that $T_1 = z/u_{p1}$ and $T_2 = z/u_{p2}$, where u_{p1} is the faster speed. Strictly speaking, we should use the group velocity instead of the phase velocity, but they are nearly the same in optical fibers. We then have $\Delta T = T_2 - T_1 \simeq z\Delta u_p/u_p^2 = (z/u_p)(\Delta n/n)$, where we are making the same approximation as in part (d). We want to find out at what z-value, $\Delta T = 100$ ps. Solving for z, we have

$$z = \frac{u_p \Delta T}{\Delta n/n} = \frac{(1.999 \times 10^8) \times (1.000 \times 10 \times 10^{-11}}{1.000 \times 10^{-5}} = 2.0 \times 10^3 \text{ m} = 2.00 \text{ km}$$

This is a very short distance in optical fiber communication systems. So, one must keep the birefringence very low in practice.

6. a. $z_1 = 4\cos(\pi/3) + j4\sin(\pi/3) = 2.00 + j3.46$

b. $z_2 = 1.732\cos(3\pi/4) + 1.732j\sin(3\pi/4) = -1.22 + j1.22$

c. $z_3 = 2\cos(\pi/2) + j2\sin(\pi/2) = j2$

d. $z_4 = j^2 j = -j$

e. $z_5 = 1/j^2 = -1$

f. $z_6 = \sqrt{2}[\exp(-j\pi/4]^5 = \sqrt{2^5}\exp(-j5\pi/4) = -4\sqrt{2}(1-j)$

g. $z_7 = \left[\sqrt{2}\exp(-j\pi/4)\right]^{1/2} = \sqrt[4]{2}\exp(-j\pi/8) = 1.189\cos(\pi/8) - 1.189\sin(\pi/8) = 1.10 - j0.455$

7. We begin by substituting $i(t) = A\cos(\omega t + \phi_0 - \phi_1)$ into the equation

$$Ri(t) + \frac{1}{C}\int i(t)\,dt = v_{\mathrm{s}}(t)$$

we obtain

$$RA\cos(\omega t + \phi_0 - \phi_1) + \frac{1}{\omega C}A\sin(\omega t + \phi_0 - \phi_1) = V_0\sin(\omega t + \phi_0). \tag{1}$$

Writing now,

$$\begin{aligned} A\cos(\omega t + \phi_0 - \phi_1) &= A\cos\phi_1\cos(\omega t + \phi_0) + A\sin\phi_1\sin(\omega t + \phi_0), \\ A\sin(\omega t + \phi_0 - \phi_1) &= A\cos\phi_1\sin(\omega t + \phi_0) - A\sin\phi_1\cos(\omega t + \phi_0), \end{aligned} \tag{2}$$

and substituting (2) into (1), we find

$$\begin{aligned} &\left[RA\cos\phi_1 - \frac{A}{\omega C}\sin\phi_1\right]\cos(\omega t + \phi_0) \\ &\quad + \left[RA\sin\phi_1 + \frac{A}{\omega C}\cos\phi_1\right]\sin(\omega t + \phi_0) = V_0\sin(\omega t + \phi_0). \end{aligned} \tag{3}$$

Separately equating the coefficients of $\cos(\omega t + \phi_0)$ and $\sin(\omega t + \phi_0)$, we find the two equations

$$\begin{aligned} RA\cos\phi_1 - \frac{A}{\omega C}\sin\phi_1 &= 0, \\ RA\sin\phi_1 + \frac{A}{\omega C}\cos\phi_1 &= V_0. \end{aligned} \tag{4}$$

Equation (4) is two simultaneous equations for the two unknowns $A\cos\phi_1$ and $A\sin\phi_1$. Solving these equations, we find

$$A\cos\phi_1 = \frac{\omega C V_0}{1+\omega^2 R^2 C^2}, \qquad A\sin\phi_1 = \frac{\omega^2 R C^2 V_0}{1+\omega^2 R^2 C^2}. \tag{5}$$

Dividing $A\sin\phi_1$ by $A\cos\phi_1$, we find

$$\tan\phi_1 = \omega RC \qquad \text{or} \qquad \phi_1 = \tan^{-1}(\omega RC), \tag{6}$$

Squaring and adding $A\cos\phi_1$ and $A\sin\phi_1$, we obtain

$$A^2 = \frac{\omega^2 C^2 V_0^2}{1+\omega^2 R^2 C^2} \qquad \text{or} \qquad A = \frac{\omega C V_0}{(1+\omega^2 R^2 C^2)^{1/2}}. \tag{7}$$

While not prohibitively complex, this approach is definitely more cumbersome than the phasor approach — aside from the need for the initial guess.

8. a. The basic equation in this case is

$$Ri(t) + \frac{1}{C}\int i(t)\,dt + L\frac{di(t)}{dt} = v_\mathrm{s}(t).$$

b. The phasor substitution yields

$$R\tilde{I} + \frac{\tilde{I}}{j\omega C} + j\omega L\tilde{I} = V_0\exp(j\pi/3).$$

c. Solving this equation for $\tilde{I}$ yields

$$\tilde{I} = \frac{V_0\exp(j\pi/3)}{R + (1/j\omega C) + j\omega L} = \frac{j\omega C V_0\exp(j\pi/3)}{1-\omega^2 LC + j\omega RC}$$