CMPE 320: Probability, Statistics, and Random Processes

Lecture 22: Conditional expectation and variance revisited

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Announcement

- Don't forget to put in your course evaluation!
- Final exam: 3:30-5:30pm on Monday, 5/21
- Next class will a review session. Bring your questions.
- Special review session: Monday 11am-1pm. The T.A. review Midterm #2 problems.

Conditional expectation as a RV

$\sum_x x P_{X|Y}(x|y)$ or $\int_{-\infty}^{\infty} x f_{X|Y}(x|y)\,dx$

- Conditional expectation E[X | Y = y] is a function of y

$$E[X|Y=y] = g(y)$$

- In the same token, E[X|Y] is a function of Y

$$E[X|Y] = g(Y) \Rightarrow E[X|Y] \text{ is a RV}$$

Example 4.16. We are given a biased coin and we are told that because of manufacturing defects, the probability of heads, denoted by Y, is itself random, with a known distribution over the interval $[0,1]$. We toss the coin a fixed number n of times, and we let X be the number of heads obtained. Compute E[X|Y = y].

$$E[X|Y=y] = ny$$
$$E[X|Y] = nY$$

Law of iterated expectation

- Since E[X|Y] is a RV, E[E[X|Y]] can be computed

$$E[\underbrace{E[X|Y]}_{g(Y)}] = \begin{cases} \sum_y g(y) P_Y(y) = \sum_y E[X|Y=y] P_Y(y) & (Y: \text{discrete}) \\ \int_{-\infty}^{\infty} g(y) f_Y(y)\,dy = \int_{-\infty}^{\infty} E[X|Y=y] f_Y(y)\,dy & (Y: \text{continuous}) \end{cases}$$

- But this is nothing but the total expectation (See Lectures 11 and 19)

$$\boxed{E_Y[E_{X|Y}[X|Y]] = E_X[X]}$$

Example 4.16 (continued). Suppose that Y, the probability of heads for our coin is uniformly distributed over the interval $[0,1]$. Let X be the number of heads obtained from n tosses. Compute E[X].

$$E[X] = E[E[X|Y]] = E[nY] = nE[Y] = \frac{n}{2}$$

Example 4.17. We start with a stick of length ℓ. We break it at a point which is chosen randomly and uniformly over its length, and keep the piece that contains the left end of the stick. We then repeat the same process on the piece that we were left with. What is the expected length of the piece that we are left with after breaking twice?

Y: the length of the stick after first break
X: " " 2nd "

$$E[X] = E[E[X|Y]] = E\left[\frac{Y}{2}\right] = \frac{1}{2}E[Y] = \frac{1}{2}\cdot\frac{1}{2}\ell = \frac{\ell}{4}$$

A useful property

- For any function g, $E[X\, \underbrace{g(Y)}_{\text{constant given } Y} \mid Y] = g(Y)\, E[X|Y]$

$$E[X g(Y) \mid Y=y] = E[X\, g(y) \mid Y=y] = g(y)\, E[X \mid Y=y]$$

Conditional expectation as an estimator

- If observing a RV Y provides information on RV X, a sensible estimator for X is
- Estimation error

Orthogonality principle

Conditional variance

- var(X|Y) =

Total variance

- $\mathrm{var}(\tilde{X}) =$

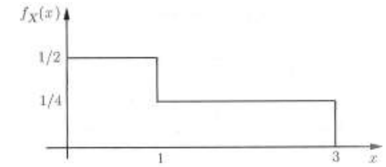
- $\mathrm{var}(X) =$

Example 4.16 (continued). We consider n independent tosses of a biased coin whose probability of heads, Y, is uniformly distributed over the interval $[0, 1]$. With X being the number of heads obtained, compute var(X).

Example 4.17 (continued). Consider again the problem where we break twice a stick of length ℓ at randomly chosen points. Here Y is the length of the piece left after the first break and X is the length after the second break. Calculate var(X).

Example 4.21. Computing Variances by Conditioning. Consider a continuous random variable X with the PDF given in Fig. 4.13. We define an auxiliary random variable Y as follows:

$$Y = \begin{cases} 1, & \text{if } x < 1, \\ 2, & \text{if } x \geq 1. \end{cases}$$



Compute var(X) using total variation.