

3.6 5 Show that no proper subgroup of S_4 contains both $(1, 2, 3, 4)$ and $(1, 2)$.

Pf. Suppose $H \leq S_4$ with the permutations $(1\ 2\ 3\ 4)$ and $(1\ 2)$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$$

Their product would yield

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} = S_4$$

Therefore, H is not a proper subgroup of S_4 by contradiction □

9 A rigid motion of a cube can be thought of either as a permutation of its eight vertices or as a permutation of its six sides. Find a rigid motion of the cube that has order 3, and express the permutation that represents it in both ways, as a permutation on eight elements and as a permutation on six elements.

Pf. □

10 Show that the following matrices form a subgroup of $GL_2(\mathbb{C})$ isomorphic to D_4 :

$$\pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \pm \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

Pf. □

15 (a) Show that $A_4 = \{\sigma \in S_4 \mid \sigma = \tau^2 \text{ for some } \tau \in S_4\}$

Pf. □

(b) Show that $A_5 = \{\sigma \in S_5 \mid \sigma = \tau^2 \text{ for some } \tau \in S_5\}$

Pf. □

(c) Show that $A_6 = \{\sigma \in S_6 \mid \sigma = \tau^2 \text{ for some } \tau \in S_6\}$

Pf. □

(d) What can you say about A_n if $n > 6$?

Pf. □

17 For any elements $\sigma, \tau \in S_n$, show that $\sigma\tau\sigma^{-1}\tau^{-1} \in A_n$.

Pf. □

21 Find the center of the dihedral group D_n .

Hint: Consider two cases, depending on whether n is odd or even.

Pf. □

24 Show that the product of two transpositions is one of (i) the identity; (ii) a 3-cycle; (iii) a product of two (nondisjoint) 3-cycles. Deduce that every element of A_n can be written as a product of 3-cycles.

Pf. □

3.7 4 Let G be an abelian group, and let n be any positive integer. Show that the function $\phi : G \rightarrow G$ defined by $\phi(x) = x^n$ is a homomorphism.

Pf. □

6 Define $\phi : \mathbb{C}^\times \rightarrow \mathbb{R}^\times$ by $\phi(a+bi) = a^2+bi$, for all $a+bi \in \mathbb{C}^\times$. Show that ϕ is a homomorphism.

Pf. □

7 b $\phi : \mathbb{R} \rightarrow \text{GL}_2(\mathbb{R})$ defined by $\phi(a) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$

Pf. □

d $\phi : \text{GL}_2(\mathbb{R}) \rightarrow \mathbb{R}^\times$ defined by $\phi \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = ad - bc$

Pf. □

10 Let G be the group of affine functions from \mathbb{R} into \mathbb{R} , as defined in Exercise 10 of Section 3.1. Define $\phi : G \rightarrow \mathbb{R}^\times$ as follows: for any function $f_{m,b} \in G$, let $\phi(f_{m,b}) = m$. Prove that ϕ is a group homomorphism, and find its kernel and image.

Pf. □

- 14 Recall that the center of a group G is $\{x \in G \mid xh = gx \text{ for all } g \in G\}$. Prove that the center of any group is a normal subgroup.

Pf. □

- 18 Let the dihedral group D_n be given by elements a of order n and b of order 2, where $ba = a^{-1}b$. Show that any subgroup of $\langle a \rangle$ is normal in D_n .

Pf. □

- 3.8 4 For each of the subgroups $\{e, a^2\}$ and $\{e, b\}$ of D_4 , list all left and right cosets.

Pf. □

- 9 Let G be a finite group, and let n be a divisor of $|G|$. Show that if H is the only subgroup of G of order n , then H must be normal in G .

Pf. □

- 12 Let H and K be normal subgroups of G such that $H \cap K = \langle e \rangle$. Show that $hk = kh$ for all $h \in H$ and $k \in K$.

Pf. □

- 18 Compute the factor group $(\mathbb{Z}_6 \times \mathbb{Z}_4)/\langle(3, 2)\rangle$.

Pf. □

- 19 Show that $(\mathbb{Z} \times \mathbb{Z})/\langle(0, 1)\rangle$ is an infinite cyclic group.

Pf. □

- 23 a. Show that G is a subgroup of $\text{GL}_2(\mathbb{Z}_5)$.

Pf. □

- b. Show that the subset N of all matrices in G of the form $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$, with $c \in \mathbb{Z}_5$, is a normal subgroup of G .

Pf.

□

- c. Show that the factor group G/N is cyclic of order 4.

Pf.

□