

More practice

▪ More practice 1

$$x(t) = \begin{cases} 1 & 0 \leq t < 3 \\ 0 & \text{otherwise} \end{cases}$$

$$h(t) = \begin{cases} 1 & 0 \leq t < 2 \\ 0 & \text{otherwise} \end{cases}$$

▪ More practice 2

$$x(t) = \begin{cases} 1 & 0 \leq t < 3 \\ 0 & \text{otherwise} \end{cases}$$

$$h(t) = \begin{cases} 2e^{-2t} & 0 \leq t < \infty \\ 0 & \text{otherwise} \end{cases}$$

▪ More practice 3

$$x(t) = \begin{cases} \sin(2\pi t) & 0 \leq t < \infty \\ 0 & \text{otherwise} \end{cases}$$

$$h(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases}$$

Lecture 4: Properties of the Convolution Operator

Properties

- Convolution is such a common operation that it's useful to know (memorize) some common properties

- Property 1: Combination of delays**

If $h(t) = 0$ for $t < t_1$, and $x(t) = 0$ for $t < t_2$, then $y(t) = x * h = 0$ for $t < t_1 + t_2$

- Property 2: Combination of durations**

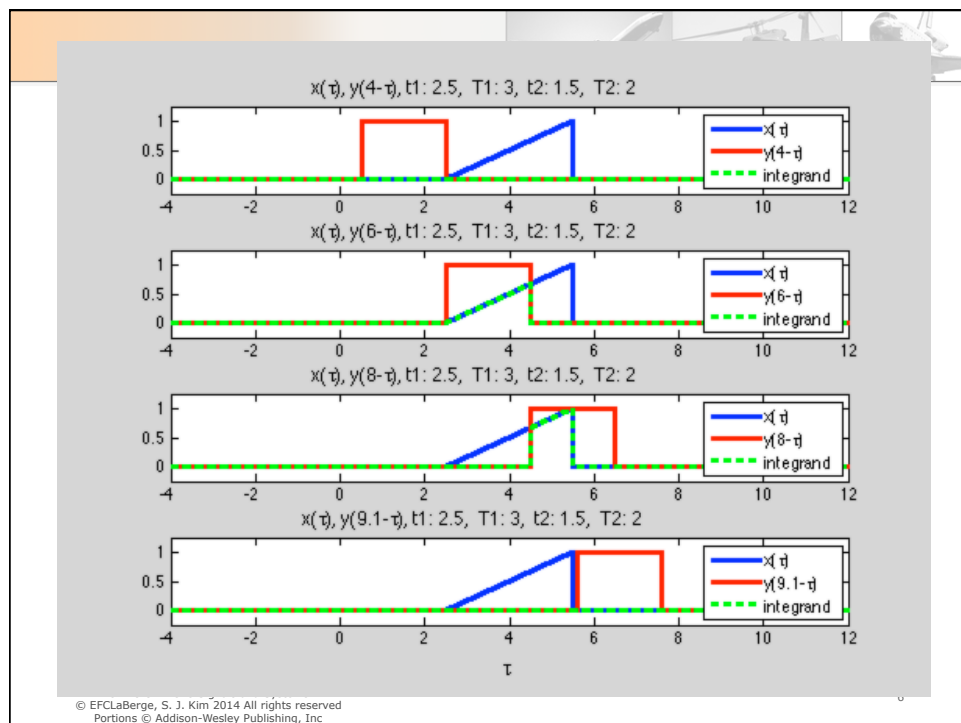
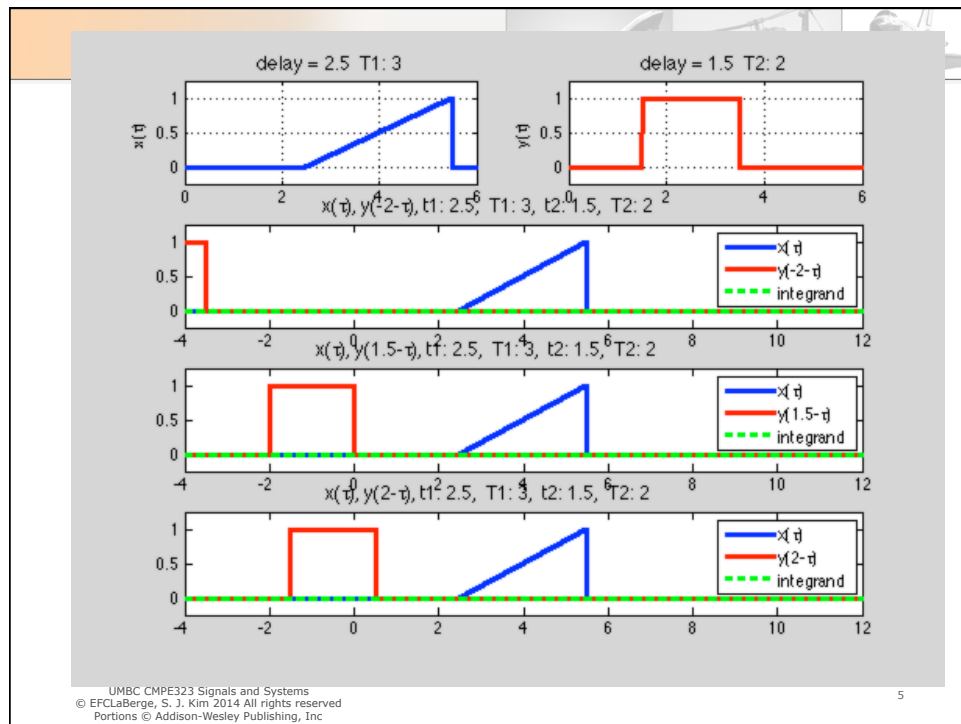
If $h(t) \neq 0$ for $T_1 = t_{12} - t_{11}$, and $x(t) \neq 0$ for $T_2 = t_{22} - t_{21}$,

then $y(t) = x * h \neq 0$ for $T_2 + T_1$

- To see these consider the convolution of these signals**

$$x(\tau) = \begin{cases} \frac{t-t_1}{T_1} & t_1 \leq \tau < t_1 + T_1 \\ 0 & \text{elsewhere} \end{cases}, \quad y(\tau) = \begin{cases} 1 & t_2 \leq \tau < t_2 + T_2 \\ 0 & \text{elsewhere} \end{cases}$$

Note that we're computing $w(t) = \int_{-\infty}^{\infty} x(\tau)y(t-\tau)d\tau$



More Properties

- **Property 3: Even/Odds**

If $x(t)$ is even and $h(t)$ is odd, or $x(t)$ is odd and $h(t)$ is even,

then $y(t) = x * h = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$ is odd.

If $x(t)$ and $h(t)$ are both odd or both even, then $y(t)$ is even.

Or, the convolution of an odd function with an even function is odd,
and the convolution of two even functions or two odd functions is even.

Remembering proofs

- **In the real world, a fair amount of manipulation of signals is done symbolically, that is, by analysis...**
- **...so it's useful to actually prove these things to get used to manipulating the convolution operation**
- **Remembering proofs from CMSC203, all forms of this "even/odd" statement are logical implications**

$$(x(t) \text{ even}) \wedge (h(t) \text{ even}) \rightarrow x * h \text{ even}$$

- **The proof is itself a design problem:**
 - **What are the requirements: prove (or disprove) the implication**
 - **What do we know?**

$$x(t) \text{ even} \equiv x(t) = x(-t); \quad x(t) \text{ odd} \equiv x(t) = -x(-t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

- **What do we need to find out? Not much**
- **Find the engineering solution, in this case, the proof**
 - **Hold one side of the convolution equation unchanged**

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Let $\lambda = -\tau$, $d\lambda = -d\tau$, $\tau = \pm\infty \rightarrow \lambda = \mp\infty$, changing variables

$$y(t) = -\int_{\infty}^{-\infty} x(-\lambda)h(t+\lambda)d\lambda = \int_{-\infty}^{\infty} x(-\lambda)h(t+\lambda)d\lambda$$

x is even, so $x(-\lambda) = x(\lambda)$

h is even, so $h(t+\lambda) = h(-(t+\lambda)) = h(-t-\lambda)$

$$\text{Substituting } y(t) = \int_{-\infty}^{\infty} x(\lambda)h(-t-\lambda)d\lambda = y(-t)$$

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Other Properties

- **Property 4 Periodicity** $x(t)$ periodic $\rightarrow y(t)$ periodic
- **Property 5 Time Scaling** $y(ct) = cx(ct) * h(ct)$ for $c > 0$
- **Property 6 Time Reversal** $y(-t) = x(-t) * h(-t)$
- **Property 7 Area Property**

Let $A_x = \int_{-\infty}^{\infty} x(t)dt$, similarly A_h and A_y . If $y = x * h$, $A_y = A_x A_h$

- **Property 8 Centroid/Center of Gravity Property**

$$\text{Let } A_{tx} = \int_{-\infty}^{\infty} \tau x(\tau) d\tau, \text{ and } D_x = \frac{A_{tx}}{A_x}$$

Then $y = x * h \rightarrow D_y = D_x + D_h$

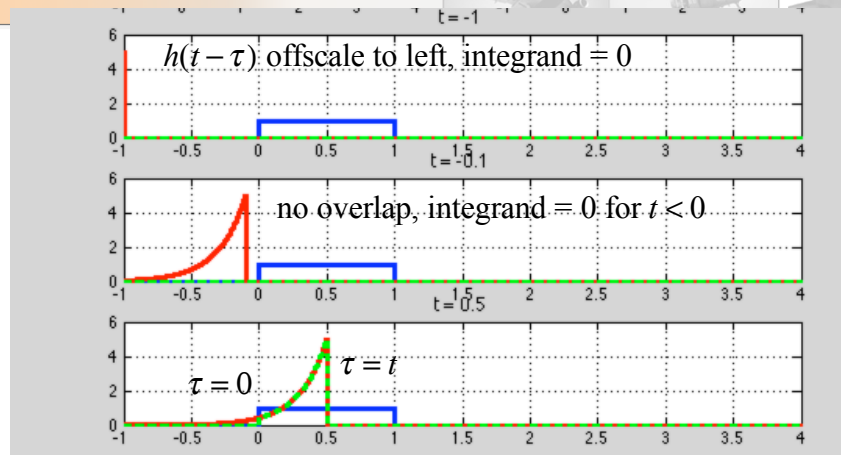
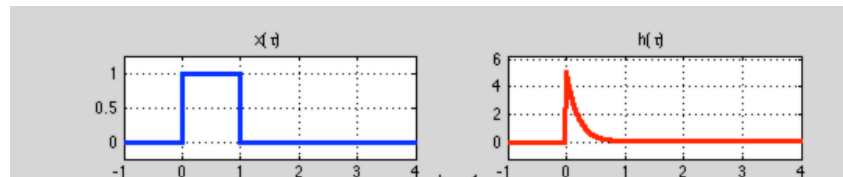
- **Property 9 Associativity:** $(x * y) * z = x * (y * z)$

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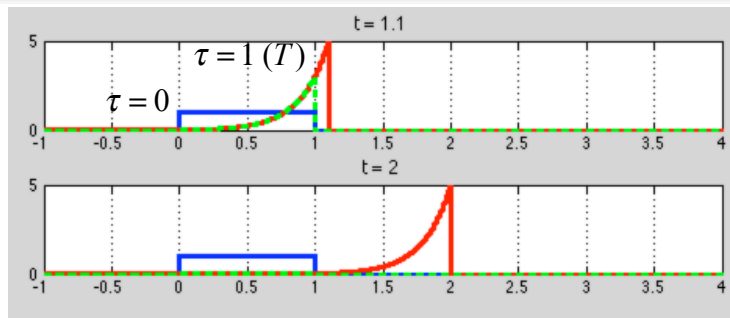
A simple problem

$$x(t) = \begin{cases} 1 & 0 \leq t < 1 \\ 0 & \text{otherwise} \end{cases}, \quad h(t) = \begin{cases} 5e^{-5t} & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$



$$0 \leq t < 1 (T)$$

$$y(t) = \int_0^t (1)(5e^{-5(t-\tau)})d\tau = 5e^{-5t} \times \frac{1}{5}e^{5\tau} = e^{-5t} \left(e^{5\tau} \right)_0^t = 1 - e^{-5t}$$



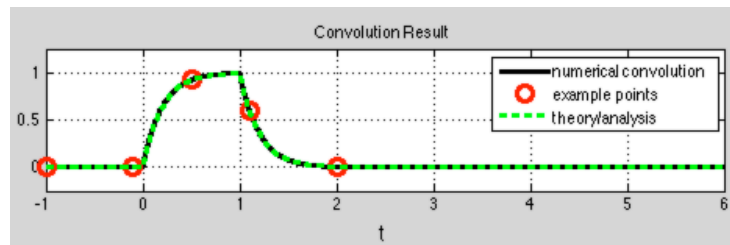
For $t \geq 1 (T)$

$$y(t) = \int_0^1 (1) (5e^{-5(t-\tau)}) d\tau = 5e^{-5t} \times \frac{1}{5} e^{5\tau} = e^{-5t} (e^{5\tau})_0^1 = e^{-5t} (e^5 - 1)$$

$$= e^{-5(t-1)} - e^{-5t} \approx e^{-5(t-1)}$$

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$$y(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-5t} & 0 \leq t < 1 (T) \\ e^{-5(t-1)} - e^{-5t} & t \geq 1 \end{cases}$$

We recognize (!!) this as the output of a single-pole low-pass filter from CMPE306!

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More work

- Let's walk through a more complicated example

$$x(t) = \begin{cases} 0 & t < -2 \\ 1 & -2 \leq t < -1 \\ 2 & -1 \leq t < 1 \\ 1 & 1 \leq t < 2 \\ 0 & t > 2 \end{cases}, \quad h(t) = \begin{cases} 5e^{-5t} & t \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find $y(t) = x * h$

Solution done on board! (or in homework, as time permits)