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**CMPE 320** HW 01

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1. Let $A$ and $B$ be two sets. Under what conditions is the set $A \cap (A \cup B)^c$ empty?

   **Answer** If $A = B$, then $A \cap (A \cup B)^c = A \cap (A \cup A)^c = A \cap A^c = \varnothing$.

2. Prove:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

   **Answer** Let $D = B \cup C$ such that $A \cup B \cup C = A \cup D$. Then:

$$\begin{aligned} P(A \cup B \cup C) &= P(A \cup D) \\ &= P(A) + P(D) - P(A \cap D) \\ &= P(A) + P(B \cup C) - P(A \cap D) \\ &= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap D) \end{aligned}$$

   Since $P(A \cap D) = P(A \cap (B \cup C)) = P((A \cap B) \cup (A \cap C))$ by the distributive law:

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap D) \\ &= P(A) + P(B) + P(C) - P(B \cap C) - P((A \cap B) \cup (A \cap C)) \end{aligned}$$

   Since $P((A \cap B) \cup (A \cap C)) = P(A \cap B) + P(A \cap C) - P((A \cap B) \cap (A \cap C))$:

$$\begin{aligned} P(A \cup B \cup C) &= P(A) + P(B) + P(C) - P(B \cap C) - (P(A \cap B) + P(A \cap C) - P((A \cap B) \cap (A \cap C))) \\ &= P(A) + P(B) + P(C) - P(B \cap C) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C) \end{aligned}$$

$$\therefore P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

3. We are given that $P(A^c) = 0.6$, $P(B) = 0.3$, and $P(A \cap B) = 0.2$. Determine $P(A \cup B)$.

   **Answer**

$$\because P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$\because P(A^c) = 1 - P(A)$$
$$\therefore P(A \cup B) = (1 - 0.6) + 0.3 - 0.2$$
$$= 0.5$$

4. We roll a four-side die once and then we roll it as many times as is necessary to obtain a different face than the one obtained in the first roll. Let the outcome of the experiment be $(r_1, r_2)$ where $r_1$ and $r_2$ are the results of the first and the last rolls, respectively. Assume that all possible outcomes have equal probability. Find the probability that:

   (a) $r_1$ is even.

   **Answer** Since the possible even outcomes $\in \{1, 2, 3, 4\}$ are $\{2, 4\}$, then $P(r_1$ is even$)$ $=$ $2/4 = 1/2$

   (b) Both $r_1$ and $r_2$ are even.

   **Answer** Since the second roll $r_2$ is different from $r_1$, the cardinality of $(r_1, r_2)$ is $4 \cdot 3 = 12$. Therefore, $P(r_1$ and $r_2$ are even$) = 2/12 = 1/6$

   (c) $r_1 + r_2 < 5$.

   **Answer** If $r_1 + r_2 < 5$ then the possible outcomes are only $\{1, 2, 3, 4\}$ with a cardinality of 4. Therefore, $P(r_1 + r_2 < 5) = 4/12 = 1/3$

5. You enter a special kind of chess tournament, whereby you play one game with each of three opponents, but you get to choose the order in which you play your opponents. You win the tournament if you win two games in a row. You know your probability of a win against each of the three opponents. What is your probability of winning the tournament, assuming that you choose the optimal order of playing the opponents?

   **Answer** Let $W_i$ represent winning against player $i = 1, 2, 3$ and $p_i$ represent the probability of winning against player $i = 1, 2, 3$.

   Then the sample space $S$:

$$S = \{W_1^c W_2^c W_3^c,\ W_1^c W_2^c W_3,\ W_1^c W_2 W_3^c,\ W_1^c W_2 W_3,\ W_1 W_2^c W_3^c,\ W_1 W_2^c W_3,\ W_1 W_2 W_3^c,\ W_1 W_2 W_3\}$$

6. Alice and Bob each choose at random a number between zero and two. We assume a uniform probability law under which the probability of an event is proportional to its area. Consider the following events:

   (a) The magnitude of the difference of the two numbers is greater than 1/3.

**Answer** Let $x, y \in [0,2]$ represent the numbers chosen by Alice and Bob.

$$\begin{aligned} x + y &> 3 \\ 3x - 3y &> 1 \end{aligned}$$

$$\begin{aligned} \text{area} &= \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{3} \\ &= \frac{1}{18} \end{aligned}$$

$$\begin{aligned} \text{required area} &= 2 \cdot 2 - \frac{1}{18} \\ &= 4 - \frac{1}{18} \end{aligned}$$

$$\begin{aligned} \text{probability} &= \frac{4 - \frac{1}{18}}{4} \\ &= \frac{71}{72} \end{aligned}$$

(b) At least one of the numbers is greater than 1/3.

**Answer**

(c) The two numbers are equal.

**Answer**

(d) Alice's number is greater than 1/3.

**Answer**

Find the probabilities $P(A)$, $P(B)$, $P(A \cap B)$, $P(C)$, $P(D)$, $P(A \cap D)$,

7. The disc containing the only copy of your term project just got corrupted, and the disc got mixed up with three other corrupted discs that were lying around. It is equally likely that any of the four discs holds the corrupted remains of your term project. Your computer expert friend offers to have a look, and you know from past experience that his probability of finding your term project from any disc is 0.4 (assuming the term project is there). Given that he searches on disc 1 but cannot find your thesis, what is the probability that your thesis is on disc $i$ for $i = 1, 2, 3, 4$?

   **Answer** Let $D_i$ represent the event that the term project in the disc $i$ and $T$ represent the event the term project finds the thesis on disc 1.

$$\therefore P(D_i) = 0.25, \quad \text{for} \;\; i = 1, 2, 3, 4$$

$$P(T \mid D_1) = 0.4$$
$$P(T \mid D_2) = 0$$
$$P(T \mid D_3) = 0$$
$$P(T \mid D_4) = 0$$

$$P(T \mid D_1^c) = 0.6$$
$$P(T \mid D_2^c) = 1$$
$$P(T \mid D_3^c) = 1$$
$$P(T \mid D_4^c) = 1$$

Using Bayes Rule to calculate $P(D_i \mid T^c)$ for $i = 1, 2, 3, 4$:

$$P(D_i \mid T^c) = \frac{P(D_i)P(F^c \mid D_i)}{\sum_{k=1}^{4} P(D_k)P(F^c \mid D_k)}$$

$$\begin{aligned}
P(D_1 \mid T^c) &= \frac{P(D_1)P(F^c \mid D_1)}{\sum_{k=1}^{4} P(D_k)P(F^c \mid D_k)} \\
&= \frac{P(D_1)P(F^c \mid D_1)}{P(D_1)P(F^c \mid D_1) + P(D_2)P(F^c \mid D_2) + P(D_3)P(F^c \mid D_3) + P(D_4)P(F^c \mid D_4)} \\
&= \frac{(0.25 \cdot 0.6)}{(0.25 \cdot 0.6) + (0.25 \cdot 1) + (0.25 \cdot 1) + (0.25 \cdot 1)} \\
&= \frac{0.15}{0.90} \\
&= 0.1667
\end{aligned}$$

Similarly,

$$\begin{aligned}
P(D_2 \mid T^c) &= \frac{P(D_2)P(F^c \mid D_2)}{\sum_{k=1}^{4} P(D_k)P(F^c \mid D_k)} \\
&= \frac{0.25}{0.90} \\
&= 0.2778
\end{aligned}$$

$$
\begin{aligned}
P(D_3 \mid T^c) &= \frac{P(D_3)P(F^c \mid D_3)}{\sum_{k=1}^{4} P(D_k)P(F^c \mid D_k)} \\
&= \frac{0.25}{0.90} \\
&= 0.2778
\end{aligned}
$$

$$
\begin{aligned}
P(D_4 \mid T^c) &= \frac{P(D_4)P(F^c \mid D_4)}{\sum_{k=1}^{4} P(D_k)P(F^c \mid D_k)} \\
&= \frac{0.25}{0.90} \\
&= 0.2778
\end{aligned}
$$

8. A person has forgotten the last digit of a telephone number, so he dials the number with the last digit randomly chosen. How many times does he have to dial (not counting repetitions) in order that the probability of dialing the correct number is more than 0.5?

   **Answer** The probability of correct digit: 1/10. Let $n$ be the number of dials. Then:

$$
\begin{aligned}
n \cdot \frac{1}{10} &> 0.5 \\
n &> 0.5 \cdot 10 \\
n &> 5
\end{aligned}
$$

   Therefore, the minimum number of dials required is $n = 6$.

9. A new test has been developed to determine whether a given student is overstressed. This test is 95% accurate if the student is not overstressed, but only 85% accurate if the student is in fact overstressed. It is known that 99.5% of all students are overstressed. Given that a particular student tests negative for stress, what is the probability that the test results are correct, and that this student is not overstressed?

   **Answer** Let $S$ represent a non-overstressed student, and $N$ represent a negative test. We need to find $P(S \mid N)$.

$$
\begin{aligned}
P(S \mid N) &= \frac{P(S^c)P(N \mid S^c)}{P(N)} \\
&= \frac{0.005 \cdot 0.95}{0.005 \cdot 0.95 + 0.995 \cdot 0.15} \\
&\approx 0.03
\end{aligned}
$$

10. A hiker starts by taking one of $n$ available trails, denoted $1, 2, \ldots, n$. An hour into the hike, trail $i$ subdivides into 1 + $i$ subtrails, only one of which leads to the hiker's destination. The

hiker has no map and makes random choices of trail and subtrail. What is the probability of reaching the destination?

**Answer** Let $T_i$ represent the event of starting with trail $i$.

$$P(T_i) = \frac{1}{n} \text{ for } i = 1, 2, \ldots, n$$

Let $D$ represent reaching the destination.

$$P(D \mid T_i) = \frac{1}{i+1} \text{ for } i = 1, 2, \ldots, n$$

By the total probability theorem, the probability of reaching the destination is:

$$\begin{aligned} P(D) &= P(T_i)P(D \mid T_i) \\ &= \frac{1}{n} \cdot \frac{1}{i+1} \\ &= \frac{1}{n(i+1)} \end{aligned}$$