

- 3.7** 4 Let  $G$  be an abelian group, and let  $n$  be any positive integer. Show that the function  $\phi : G \rightarrow G$  defined by  $\phi(x) = x^n$  is a homomorphism.

**Pf.** Let  $x, y \in G$

Then  $\phi(xy) = (xy)^n$

Since  $G$  is abelian,

$$\begin{aligned}(xy)^n &= x^n y^n \\ &= \phi(x)\phi(y)\end{aligned}$$

Therefore,  $\phi(xy) = \phi(x)\phi(y) \forall x, y \in G$

□

- 6** Define  $\phi : \mathbb{C}^\times \rightarrow \mathbb{R}^\times$  by  $\phi(a+bi) = a^2+b^2$ , for all  $a+bi \in \mathbb{C}^\times$ . Show that  $\phi$  is a homomorphism.

**Pf.** Let  $a+bi, c+di \in \mathbb{C}^\times$

Then  $\phi((a+bi)(c+di)) = (ac-bd)^2 + (ad+bc)^2$

Therefore,

$$\begin{aligned}\phi((a+bi)(c+di)) &= (ac-bd)^2 + (ad+bc)^2 \\ &= (ac-bd)^2 + (ad+bc)^2 \\ &= (ac)^2 - 2acbd + (bd)^2 + (ad)^2 + 2adbc + (bc)^2 \\ &= (ac)^2 + (bd)^2 + (ad)^2 + (bc)^2\end{aligned}$$

Also,

$$\begin{aligned}\phi(a+bi)\phi(c+di) &= (a^2+b^2)(c^2+d^2) \\ &= a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 \\ &= (ac)^2 + (bd)^2 + (ad)^2 + (bc)^2\end{aligned}$$

Therefore  $\phi((a+bi)(c+di)) = \phi(a+bi)\phi(c+di)$

□

- 7** Which of the following functions are homomorphisms?

**b**  $\phi : \mathbb{R} \rightarrow \text{GL}_2(\mathbb{R})$  defined by  $\phi(a) = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$

**Pf.** Consider,

$$\begin{aligned}\phi(a+b) &= \begin{bmatrix} 1 & a+b \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \\ &= \phi(a)\phi(b)\end{aligned}$$

Since  $\phi(a+b) = \phi(a)\phi(b)$ ,  $\phi$  is a homomorphism □

**d**  $\phi : \text{GL}_2(\mathbb{R}) \rightarrow \mathbb{R}^\times$  defined by  $\phi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ab$

**Pf.** For  $\phi$  to be homomorphic, the identity element of  $\text{GL}_2(\mathbb{R}) = e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  must

be mapped to the identity element of  $\mathbb{R}^\times = e_2 = 1$

But

$$\begin{aligned}\phi\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) &= 1 \times 0 \\ &= 0 \\ &\neq e_2 = 1\end{aligned}$$

Therefore,  $\phi$  is not a homomorphism □

**10** Let  $G$  be the group of affine functions from  $\mathbb{R}$  into  $\mathbb{R}$ , as defined in Exercise 10 of Section 3.1.

Define  $\phi : G \rightarrow \mathbb{R}^\times$  as follows: for any function  $f_{m,b} \in G$ , let  $\phi(f_{m,b}) = m$ . Prove that  $\phi$  is a group homomorphism, and find its kernel and image.

**Pf.** Given  $G = \{f_{m,b} : \mathbb{R} \rightarrow \mathbb{R} \mid m \neq 0 \text{ and } f_{m,b}(x) = mx + b\}$

Consider,

$$\begin{aligned}(f_{n,a} \circ f_{m,b}) &= f_{n,a}(f_{m,b}) \\ &= f_{n,a}(mx + b) \\ &= n(mx + b) + a \\ &= nm x + bn + a \\ &= f_{nm, bn+a}\end{aligned}$$

Therefore,

$$\begin{aligned}\phi(f_{n,a} \circ f_{m,b}) &= \phi(f_{nm,bn+a}) \\ &= nm \\ &= \phi(f_{n,a})\phi(f_{m,b})\end{aligned}$$

Therefore,  $\phi$  is a homomorphism

By definition,

$$\begin{aligned}\ker(\phi) &= \{f_{m,b} \in G \mid \phi(f_{m,b}) = 1\} \\ &= \{f_{m,b} \in G \mid m = 1\}\end{aligned}$$

By definition,

$$\begin{aligned}\text{img}(\phi) &= \{m \in \mathbb{R}^\times \mid \phi(f_{m,b}) = m\} \\ &= \mathbb{R}^\times\end{aligned}$$

□

- 14** Recall that the center of a group  $G$  is  $\{x \in G \mid xg = gx \text{ for all } g \in G\}$ . Prove that the center of any group is a normal subgroup.

**Pf.** Let  $H$  be a subgroup of  $G$ , and  $x \in H$

Then  $xg = gx, \forall g \in G$

Therefore,

$$\begin{aligned}gx &= xg \\ \implies gxg^{-1} &= xgg^{-1} \\ \implies gxg^{-1} &= x \in H\end{aligned}$$

Thus,  $\forall x \in H, \forall g \in G, gxg^{-1} \in H$

That is,  $\{x \in G \mid xh = gx \text{ for all } g \in G\}$  is a normal subgroup of  $G$

□

- 18** Let the dihedral group  $D_n$  be given by elements  $a$  of order  $n$  and  $b$  of order 2, where  $ba = a^{-1}b$ . Show that any subgroup of  $\langle a \rangle$  is normal in  $D_n$ .

**Pf.** Let  $H$  be a subgroup of  $\langle a \rangle$ , such that any elements of the form  $(a^m)^d \in H$

Also, since  $ba = a^{-1}b \in D_n$ ,

$$ba^2 = a^{-1}ba$$

$$\begin{aligned}
&= a^{-1}a^{-1}b \\
&= a^{-2}b
\end{aligned}$$

Thus,  $ba^i = a^{-i}b$  for any  $i$

□

**3.8**    **4** For each of the subgroups  $\{e, a^2\}$  and  $\{e, b\}$  of  $D_4$ , list all left and right cosets.

**Pf.** Given  $D = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$ , with  $a^4 = e$ ,  $b^2 = e$ ,  $ba = a^3b$

The elements  $\{a, b, ab\} \cap \{e, a^2\} = \emptyset$

Therefore,

$$\begin{aligned}
a\{e, a^2\} &= \{a, a^3\} \\
b\{e, a^2\} &= \{b, ba^2\} \\
&= \{b, baa\} \\
&= \{b, a^3ba\} \\
&= \{b, a^3a^3b\} \\
&= \{b, a^2b\} \\
ab\{e, a^2\} &= \{ab, aba^2\} \\
&= \{ab, abaa\} \\
&= \{ab, aa^3ba\} \\
&= \{ab, ba\} \\
&= \{ab, a^3b\}
\end{aligned}$$

Since  $D_4$  consists of 8 elements, there are 8/2 left cosets and 8/2 right cosets

Left cosets:  $\{\{e, a^2\}, a\{e, a^2\}, b\{e, a^2\}, ab\{e, a^2\}\}$

Right cosets:  $\{\{e, a^2\}, \{e, a^2\}a, \{e, a^2\}b, \{e, a^2\}ab\}$

For  $\{e, b\}$ , the elements  $\{a, a^2, a^3\} \cap \{e, b\} = \emptyset$

Therefore,

$$\begin{aligned}
a\{e, b\} &= \{a, ab\} \\
a^2\{e, b\} &= \{a^2, a^2b\} \\
a^3\{e, b\} &= \{a^3, a^3b\}
\end{aligned}$$

and,

$$\begin{aligned}
\{e, b\}a &= \{a, ba\} \\
&= \{a, a^3b\} \\
\{e, b\}a^2 &= \{a^2, ba^2\} \\
&= \{a^2, ba^2\} \\
&= \{a^2, baa\} \\
&= \{a^2, a^3ba\} \\
&= \{a^2, a^3a^3b\} \\
&= \{a^2, a^2b\} \\
\{e, b\}a^3 &= \{a^3, ba^3\} \\
&= \{a^3, baaa\} \\
&= \{a^3, a^3baa\} \\
&= \{a^3, a^3a^3ba\} \\
&= \{a^3, a^2a^3b\} \\
&= \{a^3, ab\}
\end{aligned}$$

Since  $D_4$  consists of 8 elements, there are 8/2 left cosets and 8/2 right cosets

Left cosets:  $\{\{e, b\}, a\{e, b\}, b\{e, b\}, ab\{e, b\}\}$

Right cosets:  $\{\{e, b\}, \{e, b\}a, \{e, b\}b, \{e, b\}ab\}$

□

- 9** Let  $G$  be a finite group, and let  $n$  be a divisor of  $|G|$ . Show that if  $H$  is the only subgroup of  $G$  of order  $n$ , then  $H$  must be normal in  $G$ .

**Pf.**  $aHa^{-1}$  is the subgroup of  $G$  that is isomorphic to  $H$

Therefore,  $aHa^{-1} = H, \forall a \in G$

Which is true iff  $H$  is a normal subgroup in  $G$

□

- 12** Let  $H$  and  $K$  be normal subgroups of  $G$  such that  $H \cap K = \langle e \rangle$ . Show that  $hk = kh$  for all  $h \in H$  and  $k \in K$ .

**Pf.** Consider  $hkh^{-1}k^{-1}$

Since  $K$  is a normal subgroup,

$$\begin{aligned}
hkh^{-1}k^{-1} &= (hkh^{-1})k^{-1} \\
&= hkh^{-1}, k^{-1} \in K
\end{aligned}$$

$$= hkh^{-1}k^{-1} \in K$$

Similarly, since  $H$  is a normal subgroup,

$$\begin{aligned} hkh^{-1}k^{-1} &= h(kh^{-1}k^{-1}) \\ &= h, kh^{-1}k^{-1} \in H \\ &= hkh^{-1}k^{-1} \in H \end{aligned}$$

Therefore,  $hkh^{-1}k^{-1} \in K \cap H$

But, since  $hkh^{-1}k^{-1} \in K \cap H = \{e\}$ ,

$$\begin{aligned} hkh^{-1}k^{-1} &= e \\ \implies hk(h^{-1}k^{-1}) &= e \\ \implies hk &= kh \end{aligned}$$

□

**18** Compute the factor group  $(\mathbb{Z}_6 \times \mathbb{Z}_4)/\langle(3, 2)\rangle$ .

**Pf.** Given  $\langle(3, 2)\rangle = \{(3, 2), (0, 0)\}$

$\mathbb{Z}_6 \times \mathbb{Z}_4$  has 24 elements, with 24/2 left cosets

The factor groups are,

$$\begin{aligned} (0, 0) + \langle(3, 2)\rangle &= \{(3, 2), (0, 0)\} \\ (0, 1) + \langle(3, 2)\rangle &= \{(3, 3), (0, 1)\} \\ (0, 2) + \langle(3, 2)\rangle &= \{(3, 2), (0, 2)\} \\ (0, 3) + \langle(3, 2)\rangle &= \{(3, 1), (0, 3)\} \\ (1, 0) + \langle(3, 2)\rangle &= \{(4, 2), (1, 0)\} \\ (1, 1) + \langle(3, 2)\rangle &= \{(4, 3), (1, 1)\} \\ (1, 2) + \langle(3, 2)\rangle &= \{(4, 0), (1, 2)\} \\ (1, 3) + \langle(3, 2)\rangle &= \{(4, 1), (1, 3)\} \\ (2, 0) + \langle(3, 2)\rangle &= \{(5, 2), (2, 0)\} \\ (2, 1) + \langle(3, 2)\rangle &= \{(5, 3), (2, 1)\} \\ (2, 2) + \langle(3, 2)\rangle &= \{(5, 0), (2, 2)\} \\ (2, 3) + \langle(3, 2)\rangle &= \{(5, 1), (2, 3)\} \end{aligned}$$

□

**19** Show that  $(\mathbb{Z} \times \mathbb{Z})/\langle(0, 1)\rangle$  is an infinite cyclic group.

**Pf.** Consider  $f : \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z} / \langle (0, 1) \rangle$  given by  $f(n) = (n, 0) + \langle (0, 1) \rangle$   
 Given,  $\langle (0, 1) \rangle = \{(0, n) : n \in \mathbb{Z}\}$  To show that  $f$  is a homomorphism,

$$\begin{aligned} f(a+b) &= (a+b, 0) + \langle (0, 1) \rangle \\ &= (a, 0) + (b, 0) + \langle (0, 1) \rangle \\ &= (a, 0) + \langle (0, 1) \rangle + (b, 0) + \langle (0, 1) \rangle \\ &= f(a) + f(b) \end{aligned}$$

To show that  $f$  is one-to-one,

$$\begin{aligned} f(a) &= f(b) \\ \implies (a, 0) + \langle (0, 1) \rangle &= (b, 0) + \langle (0, 1) \rangle \\ &= (a-b, 0) \in \langle (0, 1) \rangle \\ \implies a-b &= 0 \\ \implies a &= b \end{aligned}$$

To show that  $f$  is onto, suppose  $(x, y) + \langle (0, 1) \rangle$  is  $\mathbb{Z} \times \mathbb{Z} / \langle (0, 1) \rangle$   
 Then,

$$f(x) = (x, 0) + \langle (0, 1) \rangle = (x, y) + \langle (0, 1) \rangle$$

Since  $f$  is an isomorphism,  $\mathbb{Z} \times \mathbb{Z} / \langle (0, 1) \rangle$  is an infinite cyclic group □

**23** Let  $G$  be the set of all matrices in  $\text{GL}_2(\mathbb{Z}_5)$  of the form  $\begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix}$

a. Show that  $G$  is a subgroup of  $\text{GL}_2(\mathbb{Z}_5)$ .

**Pf.** Given  $G$  is non-empty and finite with  $\mathbb{Z}_5 = \{[0], [1], [2], [3], [4]\}$

For  $G$  to be a subgroup, it is enough to show  $xy \in G, \forall x, y \in G$

$$\text{Let } x = \begin{bmatrix} n & a \\ 0 & 1 \end{bmatrix}, y = \begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix} \in G$$

Then,

$$xy = \begin{bmatrix} n & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} nm & nb+a \\ 0 & 1 \end{bmatrix} \\ \in G \quad \square$$

- b. Show that the subset  $N$  of all matrices in  $G$  of the form  $\begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$ , with  $c \in \mathbb{Z}_5$ , is a normal subgroup of  $G$ .

**Pf.** Let  $x, y \in G$ ,  $xy \in N$ , with matrices of the form  $x = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix}$ ,  $y = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$

Then,

$$xy = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \\ = \begin{bmatrix} 1 & c+d \\ 0 & 1 \end{bmatrix} \\ \in N$$

Therefore,  $N \leq G$

$$\text{Let } a = \begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix} \in N, \text{ and } a^{-1} = \begin{bmatrix} m^{-1} & -m^{-1}b \\ 0 & 1 \end{bmatrix} \in N$$

Then,

$$ana^{-1} = \begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m^{-1} & -m^{-1}b \\ 0 & 1 \end{bmatrix} \\ = \begin{bmatrix} m & mc+b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m^{-1} & -m^{-1}b \\ 0 & 1 \end{bmatrix} \\ = \begin{bmatrix} 1 & mc \\ 0 & 1 \end{bmatrix} \\ \in N \quad \square$$

- c. Show that the factor group  $G/N$  is cyclic of order 4.

**Pf.** □