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MATH 407: HW 13

4.2 **1** Use the division algorithm to find the quotient and remainder when $f(x)$ is divided by $g(x)$ over the field of rational numbers $\mathbb{Q}$.

c $f(x) = x^5 + 1, \quad g(x) = x + 1$

Pf.

$$
\begin{array}{r|l}
 & x^4 - x^3 + x^2 - x + 1 \\ \hline
x+1 & x^5 \qquad\qquad\qquad\qquad + 1 \\
 & \underline{- x^5 - x^4} \\
 & - x^4 \\
 & \underline{x^4 + x^3} \\
 & x^3 \\
 & \underline{- x^3 - x^2} \\
 & - x^2 \\
 & \underline{x^2 + x} \\
 & x + 1 \\
 & \underline{- x - 1} \\
 & 0
\end{array}
$$

Therefore,

$$
\begin{aligned}
f(x) &= g(x)(x^4 - x^3 + x^2 - x + 1) + (0) \\
&= (x + 1)(x^4 - x^3 + x^2 - x + 1) + (0) \pmod{\mathbb{Q}}
\end{aligned}
$$
□

2 Use the division algorithm to find the quotient and remainder when $f(x)$ is divided by $g(x)$ over the indicated field.

c $f(x) = x^5 + 2x^3 + 3x^2 + x - 1, \quad g(x) = x^2 + 5$ over $\mathbb{Z}_5$

Pf.

$$
\begin{aligned}
f(x) &= x^5 + 2x^3 + 3x^2 + x - 1 \\
&\equiv x^5 + 2x^3 + 3x^2 + x + 4 \pmod{\mathbb{Z}_5} \\
g(x) &= x^2 + 5 \\
&\equiv x^2 \pmod{\mathbb{Z}_5}
\end{aligned}
$$

$$
\begin{array}{rlllll}
 & x^3 & & & +2x & +3 \\
\cline{2-6}
x^2 \;\big) & x^5 & +2x^3 & +3x^2 & +x & +4 \\
 & x^5 & & & & \\
\cline{2-3}
 & & 2x^3 & & & \\
 & & 2x^3 & & & \\
\cline{3-4}
 & & & 3x^2 & & \\
 & & & 3x^2 & +x & +4 \\
\cline{4-6}
 & & & & +x & +4
\end{array}
$$

Therefore,

$$
\begin{aligned}
f(x) &= g(x)(x^3+2x+3) + (x+4) \\
&= (x^2)(x^3+2x+3) + (x+4) \pmod{\mathbb{Z}_5}
\end{aligned}
\qquad \square
$$

3 Find the greatest common divisor of $f(x)$ and f', over $\mathbb{Q}$.

d $f(x) = x^4 + 2x^3 + 3x^2 + 2x + 1$

Pf. Given $f(x) = x^4 + 2x^3 + 3x^2 + 2x + 1$

and $f' = 4x^3 + 6x^2 + 6x + 2$

And, $\frac{f(x)}{f'}$,

$$
\begin{array}{r|l}
 & \hfill \tfrac{1}{4}x + \tfrac{1}{8} \\
\cline{2-2}
4x^3+6x^2+6x+2 & \; x^4 + 2x^3 + 3x^2 + 2x + 1 \\
 & - x^4 - \tfrac{3}{2}x^3 - \tfrac{3}{2}x^2 - \tfrac{1}{2}x \\
\cline{2-2}
 & \hfill \tfrac{1}{2}x^3 + \tfrac{3}{2}x^2 + \tfrac{3}{2}x + 1 \\
 & \hfill - \tfrac{1}{2}x^3 - \tfrac{3}{4}x^2 - \tfrac{3}{4}x - \tfrac{1}{4} \\
\cline{2-2}
 & \hfill \tfrac{3}{4}x^2 + \tfrac{3}{4}x + \tfrac{3}{4}
\end{array}
$$

Multiplying the remainder with a non-zero constant keeps it unchanged, and therefore,

$$
\begin{aligned}
\text{remainder} &= \left(\frac{3}{4}x^2 + \frac{3}{4}x + \frac{3}{4}\right)\frac{4}{3} \\
&= x^2 + x + 1
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\gcd(x^4 + 2x^3 + 3x^2 + 2x + 1, 4x^3 + 6x^2 + 6x + 2) \\
&\quad = \gcd(4x^3 + 6x^2 + 6x + 2, x^2 + x + 1)
\end{aligned}
$$

Dividing as before,

$$
\begin{array}{r}
4x+2 \\
x^2+x+1 \overline{\big) \; 4x^3+6x^2+6x+2} \\
\underline{-4x^3-4x^2-4x} \\
2x^2+2x+2 \\
\underline{-2x^2-2x-2} \\
0
\end{array}
$$

Therefore,

$$\gcd(x^4+2x^3+3x^2+2x+1, 4x^3+6x^2+6x+2) = x^2+x+1 \qquad \square$$

5 Find the greatest common divisor of the given polynomials, over the given field.

c $x^5+4x^4+6x^3+6x^2+5x+2, \quad x^4+3x^2+3x+6$ over $\mathbb{Z}_7$

Pf. Doing long division until remainder is 0,

$$
\begin{array}{r|rrrrrr}
 & x & +4 & & & & \\
\hline
x^4+3x^2+3x+6 & +x^5 & +4x^4 & +6x^3 & +6x^2 & +5x & +2 \\
 & -(x^5 & & +3x^3 & +3x^2 & +6x) & \\
\hline
 & & +4x^4 & +3x^3 & +3x^2 & +x & \\
 & & -(4x^4 & & +5x^2 & +5x & +3) \\
\hline
 & & & +3x^3 & +5x^2 & +3x & +6
\end{array}
$$

$$
\begin{array}{r|rrrrr}
 & 5x & +1 & & & \\
\hline
3x^3+5x^2+3x+6 & +x^4 & & +3x^2 & +3x & +6 \\
 & -(x^4 & +4x^3 & +x^2 & +2x) & \\
\hline
 & & +3x^3 & +2x^2 & +x & +6 \\
 & & -(3x^3 & +5x^2 & +3x & +6) \\
\hline
 & & & +4x^2 & +5x &
\end{array}
$$

$$
\begin{array}{r|rrrr}
 & 6x & +6 & & \\
\hline
4x^2+5x & +3x^3 & +5x^2 & +3x & +6 \\
 & -(3x^3 & +2x^2) & & \\
\hline
 & & +3x^2 & +3x & +6 \\
 & & -(3x^2 & +2x) & \\
\hline
 & & & +x & +6
\end{array}
$$

$$
\begin{array}{rrrrr}
 & 4x & +2 & & \\
\cline{2-3}
x+6 \;\big) & +4x^2 & +5x & & \\
 & -(4x^2 & +3x) & & \\
\cline{2-3}
 & & +2x & & \\
 & & -(2x & +5) & \\
\cline{3-4}
 & & & +2 &
\end{array}
$$

$$
\begin{array}{rrr}
 & 4x & +3 \\
\cline{2-3}
2 \;\big) & +x & +6 \\
 & -(x) & \\
\cline{2-3}
 & & +6 \\
 & & -(6) \\
\cline{2-3}
 & & 0
\end{array}
$$

Therefore, $\gcd(x^5 + 4x^4 + 6x^3 + 6x^2 + 5x + 2, x^4 + 3x^2 + 3x + 6) = 2 \in \mathbb{Z}_7$ $\square$

9 Let $a \in \mathbb{R}$, and let $f(x) \in \mathbb{R}[x]$, with derivative $f'(x)$. Show that the remainder when $f(x)$ is divided by $(x-a)^2$ is $f'(a)(x-a) + f(a)$.

Pf. By the division algorithm, there exists unique polynomials $q(x), r(x) \in F[x]$, such that $f(x) = q(x)(x-a)^2 + r(x)$, where $\deg(x) < 2$

Let $r(x) = bx + c$

Then, $f(a) = 0 + r(a) = ba + c$

Deriving,

$$f'(x) = (q'(x)(x-a) + 2q(x))(x-a) + b$$

So, $f'(a) = b$

Also,

$$
\begin{aligned}
c &= f(a) - ba \\
&= f(a) - f'(a)a
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
r(x) &= f'(a)x + f(a) - f(a)a \\
&= f'(a)(x-a) + f(a)
\end{aligned}
$$

$\square$

11 Find the irreducible factors of $x^6 - 1$ over $\mathbb{R}$.

Pf. Factoring $x^6 - 1$,

$$\begin{aligned} x^6 - 1 &= (x^3)^2 - 1^2 \\ &= (x^3 - 1)(x^3 + 1) \\ &= (x^3 - 1^3)(x^3 + 1^3) \\ &= (x^3 - 1^3)(x + 1)(x^2 - x + 1) \\ &= (x^3 - 1^3)(x + 1)(x^2 - x + 1) \\ &= (x - 1)(x^2 + x + 1)(x + 1)(x^2 - x + 1) \end{aligned}$$

Factors of degree 1, $(x - 1)$ and $(x + 1)$ are irreducible

Also, both factors $(x^2 + x + 1)$ and $(x^2 - x + 1)$ have no roots in $\mathbb{R}$ since their discriminant $(b^2 - 4ac)$ are less than zero.

Therefore, all factors of $x^6 - 1$; $(x - 1)$,$(x^2 + x + 1)$,$(x + 1)$, and $(x^2 - x + 1)$ are irreducible over $\mathbb{R}$ $\square$

18 Compute the following products.

b $(a + bx)(c + dx) \equiv ???\ (\text{mod } x^2 - 2)$ over $\mathbb{Q}$.

Pf. Since $x^2 \equiv 2\ (\text{mod } x^2 - 2)$

$$\begin{aligned} (a + bx)(c + dx) &= ac + adx + cbx + bdx^2 \\ &= ac + adx + cbx + 2bd\ (\text{mod } x^2 - 2) \\ &= (ac + 2bd) + (ad + cb)x\ (\text{mod } x^2 - 2) \end{aligned}$$ $\square$