Math-Phys Quiz 6 Questions:

1. Find the integrals
$$\int_0^\pi \cos^4\theta \sin\theta \, d\theta, \qquad \int_0^\pi \cos^4\theta \, d\theta$$

2. In two dimensions, we relate the Cartesian coordinates (x, y) to the polar coordinates (r, ϕ) by the relations $r = (x^2 + y^2)^{1/2}$, $\phi = \tan^{-1}(y/x)$.

   a. What are $x(r, \phi)$ and $y(r, \phi)$?

   b. Write $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ as functions of r, ϕ, $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$.

Exam Quiz 6 Questions:

1. **Modified from Ulaby et al. Example 3-6, Slide 6.26:**
   A sphere of radius 4 cm contains a charge of density ρ_V given by $\rho_\mathrm{V} = 2\cos^2\theta$. What is the total charge?

2. Consider two points $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$. What is the distance between these points in Cartesian, cylindrical and spherical coordinates? Show the work to derive the distance formulae in cylindrical and spherical coordinates from the distance formula in Cartesian coordinates.

Math-Phys Quiz 6 Solutions:

1. For the first integral, we let $x = \cos\theta$, so that $dx = -\sin\theta\, d\theta$. We then find

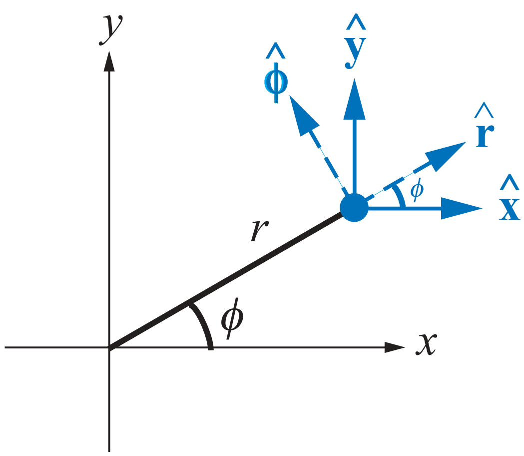
$$\int_0^\pi \cos^4\theta \sin\theta\, d\theta = \int_{-1}^1 x^4\, dx = \left.\frac{x^5}{5}\right|_{-1}^1 = \frac{2}{5}.$$

For the second integral, we write

$$\begin{aligned}\int_0^\pi \cos^4\theta\, d\theta &= \frac{1}{4}\int_0^\pi (1+\cos 2\theta)^2\, d\theta \\ &= \frac{1}{4}\int_0^\pi \left(1 + 2\cos 2\theta + \cos^2 2\theta\right) = \frac{1}{4}\int_0^\pi \left(\frac{3}{2} + 2\cos 2\theta + \frac{1}{2}\cos 4\theta\right) d\theta \\ &= \left[\frac{3\theta}{8} + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta\right]_0^\pi = \frac{3\pi}{8}\end{aligned}$$

.

2. We show the geometry in the figure below:



We have $x = r\cos\phi$ and $y = r\sin\phi$. From the figure, we infer

$$\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi,$$
$$\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi.$$

Exam Quiz 6 Solutions:

1. After converting from cm to m,

$$\begin{aligned}Q &= \int_V \rho_{\mathrm{V}}\, dV = \int_0^{2\pi} d\phi \int_0^\pi d\theta \int_0^{0.04} dR\, (2\cos^2\theta) R^2 \sin\theta \\ &= 2\left.\left(\frac{R^3}{3}\right)\right|_0^{0.04} \int_0^{2\pi} d\phi \int_0^\pi \sin\theta\cos^2\theta\, d\theta = \frac{128\times 10^{-6}}{3} \times 2\pi \times \int_0^\pi \sin\theta \cos^2\theta\, d\theta \\ &= \frac{512\pi}{9}\times 10^{-6} = 1.79\times 10^{-4}\ \mathrm{C} = 179\ \mu\mathrm{C}\end{aligned}$$

2. The distance in Cartesian coordinates is given by

$$d = \left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]^{1/2} .$$

After substituting $x_1 = r_1 \cos\phi_1,\ y_1 = r_1 \sin\phi_1,\ x_2 = r_2 \cos\phi_1,\ y_2 = r_2 \sin\phi_2,$ we obtain for the distance in cylindrical coordinates

$$d = \left[r_2^2 + r_1^2 - 2r_2 r_1 \cos(\phi_2 - \phi_1) + (z_2 - z_1)^2\right]^{1/2} .$$

After substituting $z_1 = R_1 \cos\theta_1,\ r_1 = R_1 \sin\theta_1,\ z_2 = R_2 \cos\theta_2,\ r_2 = R_2 \sin\theta_2,$ we obtain for the distance in spherical coordinates

$$d = \left\{R_2^2 + R_1^2 - 2R_2 R_1 [\cos\theta_2 \cos\theta_1 + \sin\theta_2 \sin\theta_1 \cos(\phi_2 - \phi_1)]\right\}^{1/2} .$$