

A.1 Let A, B, C be subsets of a given set S . Prove the following statements.

10 $(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$

Ans Let $x \in (A - B) \cup (B - A)$.

Therefore, $x \in (A - B)$ or $x \in (B - A)$.

If $x \in (A - B)$, then $x \in A, x \notin B$.

Therefore:

$$x \in (A \cup B), x \notin (A \cap B)$$

$$\Rightarrow x \in (A \cup B) - (A \cap B)$$

Similarly, if $x \in (B - A)$, then $x \in (A \cup B) - (A \cap B)$.

Therefore, $(A - B) \cup (B - A) \subseteq (A \cup B) - (A \cap B)$

Conversely, if $x \in (A \cup B) - (A \cap B)$, then $x \in (A \cup B), x \notin (A \cap B)$

If $x \in (A \cup B)$, then $x \in A$ or $x \in B$.

If $x \in A$, then $x \notin B$.

Therefore:

$$x \in A, x \notin B \Rightarrow x \in (A - B)$$

$$\Rightarrow x \in (A - B) \cup (B - A)$$

Similarly, if $x \in B$, then $x \in (A - B) \cup (B - A)$.

Therefore, $(A \cup B) - (A \cap B) \subseteq (A - B) \cup (B - A)$

$$\therefore (A - B) \cup (B - A) = (A \cup B) - (A \cap B)$$

□

11 $(A \cup B) \times C = (A \times C) \cup (B \times C)$

Ans Let $(x, y) \in (A \cup B) \times C$.

Then:

$$\begin{aligned}
 x &\in (A \cup B), y \in C \\
 &\Rightarrow (x \in A, y \in C) \text{ or } (x \in B, y \in C) \\
 &\Leftrightarrow ((x, y) \in A \times C) \text{ or } ((x, y) \in B \times C) \\
 &\Leftrightarrow (x, y) \in (A \times C) \cup (B \times C)
 \end{aligned}$$

$$\therefore (A \cup B) \times C = (A \times C) \cup (B \times C) \quad \square$$

A.4 9 Let a_1, \dots, a_n be positive real numbers, $G_n = \sqrt[n]{a_1 a_2 \dots a_n}$, and $A_n = \frac{1}{n} \sum_{i=1}^n a_i$. Then G_n is called the geometric mean and A_n is called the arithmetic mean. We wish to show that $G_n \leq A_n$.

1. Show that $G_2 \leq A_2$.

Ans Substituting $n = 2$:

$$\begin{aligned}
 G_2 &= \sqrt[2]{a_1 a_2 \dots a_2} \\
 &= \sqrt[2]{a_1 a_2} \\
 &= \sqrt{a_1 a_2} \\
 A_2 &= \frac{1}{2} \sum_{i=1}^2 a_i \\
 &= \frac{1}{2}(a_1 + a_2)
 \end{aligned}$$

If $a_1 = a_2$, then $G_2 = a_1 = A_2 = a_1$.

If $a_1 \neq a_2$, then let $a_1 = k, a_2 = nk + 1, k \in \mathbb{R}^+$.

Then, $G_2 = \sqrt{(k)(k+1)}, G_2^2 = k^2 + k$

and $A_2 = \frac{1}{2}(2k+1), A_2^2 = k^2 + k + \frac{1}{4}$

$$\therefore G_2 \leq A_2 \quad \square$$

2. Show that $G_{2^n} \leq A_{2^n}$ by using induction on n .

Ans It was proven the proposition held for $n = 2^k$ for $k = 1 \Rightarrow 2$.

Suppose the proposition holds for $n = 2^k$, for $k > 1$. Therefore:

$$\begin{aligned}
 A_{2^k} &= \frac{1}{2^k} \sum_{i=1}^{2^k} a_i \\
 &= \frac{1}{2^k} (a_1 + a_2 + \dots + a_{2^k}) \\
 &= \frac{\frac{1}{2^{k-1}} (a_1 + a_2 + \dots + a_{2^{k-1}}) + \frac{1}{2^{k-1}} (a_{2^{k-1}+1} + a_{2^{k-1}+2} + \dots + a_{2^k})}{2} \\
 &\geq \frac{\sqrt[2^{k-1}]{a_1 + a_2 + \dots + a_{2^{k-1}}} + \sqrt[2^{k-1}]{a_{2^{k-1}+1} + a_{2^{k-1}+2} + \dots + a_{2^k}}}{2} \\
 &\geq \sqrt[2^{k-1}]{\sqrt[2^{k-1}]{a_1 + a_2 + \dots + a_{2^{k-1}}} + \sqrt[2^{k-1}]{a_{2^{k-1}+1} + a_{2^{k-1}+2} + \dots + a_{2^k}}} \\
 &\geq \sqrt[2^k]{a_1 a_2 \dots a_{2^k}} \\
 &= G_{2^k}
 \end{aligned}$$

$$\therefore G_{2^k} \leq A_{2^k}$$

□

3. Show that $G_n \leq A_n$.

Hint: Let m be such that $2^m \geq n$, and set $a_{n+1} = a_{n+2} = \dots = a_{2^m} = A_n$ and apply part (2).

Ans

□

10 Let a and b be real numbers. Prove the binomial theorem, which states that

$$(a + b)^n = \sum_{i=0}^n \binom{n}{i} a^i b^{n-i} \text{ where } \binom{n}{i} = \frac{n!}{i!(n-i)!}$$

and $n! = n(n-1) \dots 2 \cdot 1$ for $n \geq 1$ and $0! = 1$.

Hint: $\binom{m+1}{k} = \binom{m}{k} + \binom{m}{k-1}$.

Ans Base case: For $n = k = 0$:

$$\begin{aligned}
 (a + b)^0 &= \sum_{i=0}^0 \binom{0}{i} a^i b^{0-i} \\
 &= \binom{0}{0} a^0 b^0 \\
 &= 1
 \end{aligned}$$

Assume the proposition holds for $n = k + 1$. Then:

$$\begin{aligned}
(a+b)^{k+1} &= (a+b)(a+b)^k \\
&= (a+b) \sum_{i=0}^k \binom{k}{i} a^i b^{k-i} \\
&= a \sum_{i=0}^k \binom{k}{i} a^i b^{k-i} + b \sum_{i=0}^k \binom{k}{i} a^i b^{k-i} \\
&= \sum_{i=0}^k \binom{k}{i} a^{i+1} b^{k-i} + \sum_{i=0}^k \binom{k}{i} a^i b^{k+1-i} \\
&= \sum_{i=1}^{k+1} \binom{k}{i-1} a^{(i-1)+1} b^{k-(i-1)} + \sum_{i=0}^k \binom{k}{i} a^i b^{k+1-i} \\
&= \sum_{i=1}^{k+1} \binom{k}{i-1} a^i b^{k+1-i} + \sum_{i=0}^k \binom{k}{i} a^i b^{k+1-i} \\
&= \binom{0}{0} a^0 b^{k+1} + \sum_{i=1}^{k+1} \binom{k}{i-1} a^i b^{k+1-i} + \binom{0}{0} a^{k+1} b^0 + \sum_{i=1}^k \binom{k}{i} a^i b^{k+1-i} \\
&= \binom{0}{0} a^0 b^{k+1} + \binom{k+1}{k+1} a^{k+1} b^0 + \sum_{i=1}^k \left(\binom{k}{i} + \binom{k}{i-1} \right) a^i b^{k+1-i} \\
&= b^{k+1} + a^{k+1} + \sum_{i=1}^k \binom{k+1}{i} a^i b^{k+1-i} \\
&= \sum_{i=0}^{k+1} \binom{k+1}{i} a^i b^{k+1-i}
\end{aligned}$$

□

11 Find a formula for the derivative of the product of n functions, and give a detailed proof by induction (assuming the product rule for the derivative of two functions).

Ans Let $h = \prod_{i=1}^n f_i$ be the product of n functions.

We need to show:

$$h' = f_1'(f_2 \cdot f_3 \cdot \dots \cdot f_n) + f_2'(f_1 \cdot f_3 \cdot \dots \cdot f_n) + \dots + f_n'(f_2 \cdot f_3 \cdot \dots \cdot f_{n-1}) \quad (\text{product rule})$$

Base case: For $n = 2$, $h = \prod_{i=1}^2 f_i$

$$h' = f_1' \cdot f_2 + f_2' \cdot f_1 \quad (\text{product rule})$$

Suppose the proposition holds for $n = k \geq 2$, with $h = f_1 \cdot f_2 \cdot f_3 \cdot \dots \cdot f_{k+1}$. Then:

$$\begin{aligned}
 h' &= (f_1 \cdot f_2 \cdot f_3 \cdot \dots \cdot f_k)' f_{k+1} + f'_{k+1} (f_1 \cdot f_2 \cdot f_3 \cdot \dots \cdot f_k) \text{ (product rule)} \\
 &= (f'_1(f_2 \cdot f_3 \cdot \dots \cdot f_k) + f'_2(f_1 \cdot f_3 \cdot \dots \cdot f_k) + \dots \\
 &\quad + f'_k(f_2 \cdot f_3 \cdot \dots \cdot f_{k-1})) f_{k+1} + f'_{k+1} (f_1 \cdot f_2 \cdot f_3 \cdot \dots \cdot f_k) \\
 &= f'_1(f_2 \cdot f_3 \cdot \dots \cdot f_{k+1}) + f'_2(f_1 \cdot f_3 \cdot \dots \cdot f_{k+1}) + \dots + f'_{k+1} (f_2 \cdot f_3 \cdot \dots \cdot f_k) \quad \square
 \end{aligned}$$

12 Find a formula for the n th derivative of the product of two functions, and give a detailed proof by induction.

Ans

□