CMPE 320: Probability, Statistics, and Random Processes

Lecture 6: Independence (2)

Spring 2018

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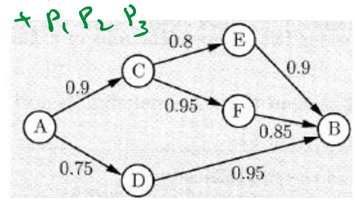
Reliability analysis

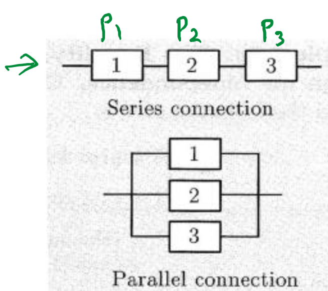
$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_2 \cap A_3) - P(A_3 \cap A_1) + P(A_1 \cap A_2 \cap A_3)$$

- In probabilistic models involving a complex system, it is often useful to assume the components are independent

$$= p_1 + p_2 + p_3 - p_1 p_2 - p_2 p_3 - p_3 p_1 + p_1 p_2 p_3$$

Example 1.24. Network Connectivity. A computer network connects two nodes A and B through intermediate nodes C, D, E, F, as shown in Fig. 1.15(a). For every pair of directly connected nodes, say i and j, there is a given probability p_{ij} that the link from i to j is up. We assume that link failures are independent of each other. What is the probability that there is a path connecting A and B in which all links are up?





p_i : Prob. that the i-th component is "up"

For the series connection,

$$P(\text{success}) = p_1 \, p_2 \, p_3$$

For the parallel connection, all components fail

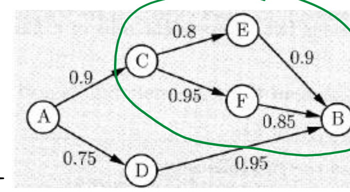
$$P(\text{failure}) = (1-p_1)(1-p_2)(1-p_3)$$

$$P(\text{success}) = 1 - (1-p_1)(1-p_2)(1-p_3)$$

Parallel connection of

$A \to C \to B$

$A \to D \to B$



$P_{succ}(ADB) = 0.75 \times 0.95 = 0.712$

Now $A \to C \to B$ is a serial connection of

$A \to C$ and $C \to B$ parallel connection of

$C \to E \to B$: 0.8×0.9

$C \to F \to B$: 0.95×0.85

$$P_{succ}(C \to B) = 1 - (1 - 0.8 \times 0.9)(1 - 0.95 \times 0.85)$$

$$= 0.9461$$

$$P(A \to C \to B) = 0.9 \times 0.9461 = 0.8515$$

$$P_{succ}(A \to B) = 1 - (1 - 0.712)(1 - 0.8515) = 0.957$$

Announcement

- HW#2 is due on 2/21, Wednesday
- HW#3 is due on 2/26, Monday
- My office hours will be back to 12pm Tuesdays.
- The TA's office hours will be 12pm Thursdays.

Problem 36. A power utility can supply electricity to a city from n different power plants. Power plant i fails with probability p_i, independent of the others.

(a) Suppose that any one plant can produce enough electricity to supply the entire city. What is the probability that the city will experience a black-out?

(b) Suppose that two power plants are necessary to keep the city from a black-out. Find the probability that the city will experience a black-out.

(a) $P(\text{Blackout}) = P(\text{all } n \text{ power plants failing}) = p_1 p_2 \cdots p_n$

(b) $P(\text{Blackout}) = P(\text{all } n \text{ power plants failing})$
$+ P(n-1 \text{ power plants fail})$

$$= p_1 p_2 \cdots p_n + p_1 \cdots p_{n-1}(1-p_n) + p_1 \cdots p_{n-2}(1-p_{n-1}) p_n + \cdots + (1-p_1) p_2 p_3 \cdots p_n$$

$$= p_1 p_2 \cdots p_n + \sum_{i=1}^{n} \underline{(1-p_i)}\, p_1 p_2 \cdots p_{i-1} p_{i+1} \cdots p_n$$

Independent Bernoulli trials

- Independent trials = an experiment involving independent but identical stages

  5 rolls of a die

  = 5 independent rolling of a die

- Independent Bernoulli trials = Each stage can have only 2 outcomes

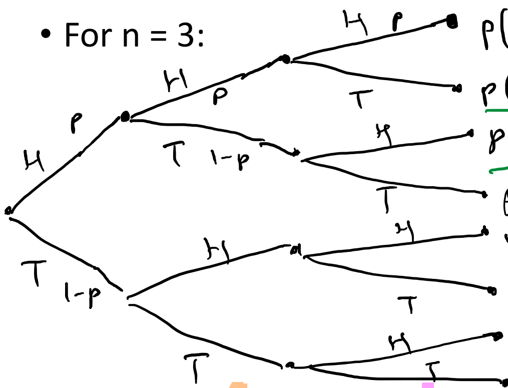
  6 coin tosses

  Each toss yields $\{H, T\}$

n independent coin tosses

- This experiment is an independent Bernoulli trial
- For n = 3:

p = prob. of getting a head from a single trial



$P(HHH) = p^3$
$P(HHT) = p^2(1-p)$
$P(HTH) = p^2(1-p)$
$P(HTT) = p(1-p)^2$
$P(THH) = p^2(1-p)$
$P(THT) = p(1-p)^2$
$P(TTH) = p(1-p)^2$
$P(TTT) = (1-p)^3$

- P(k=2) = P(2 heads in a 3-toss sequence) = $3\, p^2 (1-p)^1$, $1 = 3 - 2$

\# of ways H, H, T can be arranged

$n! = n(n-1)(n-2)\cdots 1$

Binomial probabilities

- P(k) = P(k heads in an n-toss sequence) = $\boxed{\binom{n}{k} p^k (1-p)^{n-k}}$
- $\binom{n}{k}$ = number of distinct n-toss sequence that contain k heads

"n choose k"

"n combination k"

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} \qquad \binom{3}{2} = \frac{3!}{2!\,1!} = \frac{3\cdot 2\cdot 1}{2\cdot 1\cdot 1} = 3$$

- Binomial formula

Since $P(\Omega) = 1$ $\quad \sum_{k=0}^{n} P(k) = \boxed{1 = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k}}$

Example: You want to transmit a binary message (either “0” or “1”) over a computer network. The network introduces errors to the message (flips the bit) with probability 0.1. In order to make the transmission more secure, you send the same messages 5 times. The receiver will decide on the message that is the majority. What is the probability that an incorrect message is received?

$P(\text{error}) = P(\text{more flipped bits than correct bits})$

$= P(K=3) + P(K=4) + P(K=5)$ $\quad K$: # of bit flips

$= \binom{5}{3} 0.1^3 (1-0.1)^{5-3} + \binom{5}{4} 0.1^4\, 0.9^1 + \binom{5}{5} 0.1^5\, 0.9^0$

Example 1.25. Grade of Service. An internet service provider has installed c modems to serve the needs of a population of n dialup customers. It is estimated that at a given time, each customer will need a connection with probability p, independent of the others. What is the probability that there are more customers needing a connection than there are modems?