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DATE: May 13, 2018

CMPE 320: HW 09

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1. Let X be a random variable with PDF f_X. Find the PDF of the random variable $|X|$ in the following three cases.

(a) X is exponentially distributed with parameter λ.

Sol. Given, $f_X = \lambda e^{-\lambda x}$

$$\begin{aligned}\int_{-\infty}^{\infty} f_X(x)\,\mathrm{d}x &= \int_{-\infty}^{0} \lambda e^{-\lambda x}\,\mathrm{d}x + \int_{0}^{\infty} \lambda e^{-\lambda x}\,\mathrm{d}x \\ &= \int_{0}^{\infty} \lambda e^{-\lambda x}\,\mathrm{d}x + \int_{0}^{\infty} \lambda e^{-\lambda x}\,\mathrm{d}x \\ &= 2\end{aligned}$$

Therefore, $f_X(x) = \frac{1}{2}\lambda e^{-\lambda|x|}$ □

(b) X is uniformly distributed in the interval $[-1,2]$.

Sol. Given, f_X is uniform between $[-1,2]$,

$$\begin{aligned}f_X(x) &= \frac{1}{(b-a)} \\ &= \frac{1}{(2-(-1))} \\ &= \frac{1}{3}\end{aligned}$$

□

(c) f_X is a general PDF.

Sol.

$$\begin{aligned}\int_{\infty}^{\infty} f_X(x)\,\mathrm{d}x &= 1 \\ \implies 2\int_{0}^{\infty} f_X(x)\,\mathrm{d}x &= 1 \\ \implies \int_{0}^{\infty} f_X(x)\,\mathrm{d}x &= \frac{1}{2}\end{aligned}$$

□

2. Let X and Y be independent random variables, uniformly distributed in the interval $[0,1]$. Find the CDF and the PDF of $|X-Y|$.

Sol. Given, X and Y be independent and uniformly distributed

Therefore, the PDF

$$f_X(x) = \frac{1}{b-a},\ a \leq X \leq b$$

and CDF

$$F_X(x) = \frac{x-a}{b-a},\ a \leq X \leq b$$

Let $Z = |X-Y|$. Then, the CDF,

$$\begin{aligned}
F_Z(z) &= P(Z \leq z) \\
&= P(|X-Y| \leq z) \\
&= P(-z \leq (X-Y) \leq z) \\
&= F_Z(z) - F_Z(-z) \\
&= \frac{z-0}{1-0} - \frac{-z-0}{1-0} \\
&= 2z
\end{aligned}$$

Since $f_Z(z) = f'_Z(z)$,

$$\begin{aligned}
f_Z(z) &= F_Z(z) + F_Z(-z) \\
&= \frac{z-0}{1-0} - \frac{-z-0}{1-0} \\
&= 0
\end{aligned}$$

□

3. Your driving time to work is between 30 and 45 minutes if the day is sunny, and between 40 and 60 minutes if the day is rainy, with all times being equally likely in each case. Assume that a day is sunny with probability 2/3 and rainy with probability 1/3.

(a) Find the PDF, the mean, and the variance of your driving time.

Sol. Given, $P(\text{sunny}) = \frac{2}{3}$, $P(\text{rainy}) = \frac{1}{3}$

PDF of uniform distribution

$$
f_X(x) = \begin{cases} c_1, & \text{if } 30 \leq x \leq 40, \\ c_2, & \text{if } 40 \leq x \leq 45, \\ c_3, & \text{if } 45 \leq x \leq 60, \\ 0, & \text{otherwise} \end{cases}
$$

$$
= \begin{cases} \frac{1}{15} \cdot P(\text{sunny}), & \text{if } 30 \leq x \leq 40, \\ \frac{1}{15} \cdot P(\text{sunny}) + \frac{1}{20} \cdot P(\text{rainy}), & \text{if } 40 \leq x \leq 45, \\ \frac{1}{20} \cdot P(\text{rainy}), & \text{if } 45 \leq x \leq 60, \\ 0, & \text{otherwise} \end{cases}
$$

$$
= \begin{cases} \frac{2}{45}, & \text{if } 30 \leq x \leq 40, \\ \frac{11}{180}, & \text{if } 40 \leq x \leq 45, \\ \frac{1}{80}, & \text{if } 45 \leq x \leq 60, \\ 0, & \text{otherwise} \end{cases}
$$

Mean of uniform distribution

$$
\begin{aligned} E[X] &= \frac{30+45}{2} \cdot \frac{2}{3} + \frac{40+60}{2} \cdot \frac{1}{3} \\ &= \frac{125}{3} \end{aligned}
$$

Variance of uniform distribution

$$
\begin{aligned} E[X^2] &= \frac{2}{45} \int_{30}^{45} x^2 \, \mathrm{d}x + \frac{1}{60} \int_{40}^{60} x^2 \, \mathrm{d}x \\ &= \frac{16150}{9} \end{aligned}
$$

$$
var(X) = E[X^2] - (E[X])^2
$$

$$= \frac{175}{3} \qquad \square$$

(b) On a given day your driving time was 45 minutes. What is the probability that this particular day was rainy?

Sol.

$$\begin{aligned} P(\text{rainy} \mid X = 45) &= \frac{P(\text{rainy}) f(X = 45 \mid \text{rainy})}{f(X = 45)} \\ &= \frac{\frac{1}{3} \cdot \frac{1}{20}}{\frac{2}{45} + \frac{1}{60}} \\ &= \frac{3}{11} \end{aligned} \qquad \square$$

(c) Your distance to work is 20 miles. What is the PDF, the mean, and the variance of your average speed (driving distance over driving time)?

Sol. PDF of speed

$$f_X = \begin{cases} \frac{2}{3} \cdot 40 - 40 \text{ mph}, & \text{if sunny}, \\ 20 - 30 \text{ mph}, & \text{if rainy}, \end{cases}$$

The mean

$$E[X] = \frac{26.67 + 40}{2} \cdot \frac{2}{3} + \frac{20 + 30}{2} \cdot \frac{1}{3}$$

$$= 30.56$$

Variance of uniform distribution

$$E[X^2] = \frac{26.67 + 40}{2} \int_{26.67}^{40} x^2 \, \mathrm{d}x + \frac{20 + 30}{2} \int_{20}^{30} x^2 \, \mathrm{d}x$$

$$= 65691$$

$$var(X) = E[X^2] - (E[X])^2$$

$$= 65691 - (30.56)^2 \qquad \square$$

4. The random variables X and Y have the join PDF

$$f_{X,Y}(x,y) = \begin{cases} 2, & \text{if } x > 0 \text{ and } y > 0 \text{ and } x + y \leq 1, \\ 0, & \text{otherwise} \end{cases}$$

Let A be the event $\{Y \leq 0.5\}$ and let B be the event $\{Y > X\}$.

(a) Calculate $P(B \mid A)$.

Sol. Since $P(B \mid A) = \frac{P(A \cap B)}{P(A)}$,

$$P(A \cap B) = \int_0^{0.5} \int_0^{1-y} 2 \, \mathrm{d}x \, \mathrm{d}y$$

$$= \frac{3}{4}$$

$$P(A) = \int_0^{0.5} \int_x^{0.5} 2 \, \mathrm{d}y \, \mathrm{d}x$$

$$= \frac{1}{4}$$

Therefore,

$$P(B \mid A) = \frac{1}{3}$$

□

(b) Calculate $f_{X|Y}(x \mid 0.5)$. Calculate also the conditional expectation and the conditional variance of X, given that $Y = 0.5$.

Sol. Since $f_{X|Y}(x \mid y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$,

$$f(y) = \int_0^{1-y} 2 \, \mathrm{d}x$$

$$= 2 - 2y$$

$$f_{X|Y}(x \mid 0.5) = \frac{2}{(2 - 2(0.5))}$$

$$= 2$$

And,

$$E[X \mid Y = 0.5] = \int_0^{0.5} 2x \, \mathrm{d}x$$

$$= \frac{1}{4}$$

$$E[X^2 \mid Y = 0.5] = \int_0^{0.5} 2x^2 \,\mathrm{d}x$$

$$= \frac{1}{12}$$

$$\mathrm{var}(X \mid Y = 0.5) = \frac{1}{12} - \left(\frac{1}{4}\right)^2$$

$$= \frac{1}{48} \qquad \square$$

(c) Calculate $f_{X|B}(x)$.

Sol. Since $f_{X|B}(x) = \frac{f(X \cap B)}{f(B)}$,

$$f(X \cap B) = \int_x^{1-x} 2 \,\mathrm{d}y$$

$$= 2 - 4x$$

$$f(B) = \int_0^{1/2} \int_x^{1-x} 2 \,\mathrm{d}x \,\mathrm{d}y$$

$$= \frac{1}{2}$$

$$f_{X|B}(x) = \frac{2-4x}{\frac{1}{2}}$$

$$= 4 - 8x,\ 0 \le x \le \frac{1}{2} \qquad \square$$

(d) Calculate $E[XY]$.

Sol.

$$E[XY] = \int_0^1 \int_0^{1-y} 2xy \,\mathrm{d}y \,\mathrm{d}x$$

$$= \frac{1}{12} \qquad \square$$

(e) Calculate the PDF of Y/X.

Sol.

$$f_X(x) = \int_0^{1-x} 2\,\mathrm{d}y$$
$$= 2(1-x),\ 0 < x < 1$$

Then,

$$f_{Y/X}(x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$
$$= \frac{1}{1-x},\ 0 < y < 1-x$$

□

5. The random variables $X_1, \ldots, X_n$ have common mean μ, common variance σ^2 and, furthermore, $E[X_iX_j = c]$ for every pair of distinct i and j. Derive a formula for the variance $X_1 + \ldots + X_n$, in terms of μ, σ^2, c, and n.

Sol. Given,

$$E[x_i] = \mu,\ var(x_i) = \sigma^2, \text{for } 1 \le i \le n$$

Also, given for every pair i and j, $E[X_iX_j] = c$

Then,

$$cov(X_iX_j) = E[X_iX_j] - E[X_i]E[X_j]$$
$$= c - \mu \cdot \mu$$
$$= c - \mu^2$$

$$var(X_1 + \ldots + X_n) = var(X_1) + var(X_2) + \ldots + var(X_n) + 2cov(X_1, X_2) + 2cov(X_2, X_3) + \ldots + 2cov(X_{n-1}, X_n)$$

$$= \sum_{i=1}^{n} var(X_i) + \sum_{i=1}^{n-1}\sum_{j=2}^{n} 2cov(X_i, X_j)$$

$$= \sum_{i=1}^{n} \sigma^2 + \sum_{i=1}^{n-1}\sum_{j=2}^{n} 2(c - \mu^2)$$

$$= n\sigma^2 + (n-1)2(c - \mu^2)$$

$$= n\sigma^2 + 2(n-1)(c - \mu^2)$$

□

6. Consider n independent tosses of a die. Each toss has probability p_i of resulting in i. Let X_i be the number of tosses that result in i. Show that X_1 and X_2 are negatively correlated (i.e., a large number of ones suggests a smaller number of twos).

Sol. Given $P(x_k = i) = p_i$ for $i = 1,2,3,4,5,6$ and $k = 1,2,\ldots,n$

Then,

$$X_i = \sum_{k=1}^{n} 1_{\{x_k=i\}}$$

Thus,

$$\begin{aligned}
\operatorname{cov}(X_1,X_2) &= E[X_1X_2] - E[X_1]E[X_2] \\
&= E\left[\left(\sum_{k=1}^{n} 1_{\{x_k=1\}}\right)\left(\sum_{k=1}^{n} 1_{\{x_k=2\}}\right)\right] - (np_1)(np_2).
\end{aligned}$$

Since, $1_{\{x_i=1\}}1_{\{x_i=2\}} = 0$,

then $E\left[1_{\{x_i=1\}}1_{\{x_i=2\}}\right] = 0$

$$\begin{aligned}
E\left[1_{\{x_i=1\}}1_{\{x_j=2\}}\right] &= P(x_i = 1, x_j = 2) \\
&= P(x_i = 1)P(x_j = 2) \\
&= p_1p_2, \text{for } i \neq j
\end{aligned}$$

Therefore,

$$\begin{aligned}
\operatorname{cov}(X_1,X_2) &= (n^2 - n)p_1p_2 - n^2p_1p_2 \\
&= -np_1p_2 < 0
\end{aligned}$$

□

7. Let $X = Y - Z$ where Y and Z are nonnegative random variables such that $YZ = 0$.

(a) Show that $\operatorname{cov}(Y,Z) \leq 0$.

Sol.

$$\begin{aligned}
\operatorname{cov}(Y,Z) &= E[(Y - E(Y))(Z - E(Z))] \\
&= E[YZ - 2Y \cdot E[Z] - E[Z]^2] \\
&= E[YZ] - 2 \cdot E[Y] \cdot E[Z] - E[Z]^2 \\
&= -2 \cdot E[Y] \cdot E[Z] - E[Z]^2 < 0
\end{aligned}$$

□

(b) Show that $var(X) \geq var(Y) + var(Z)$.

Sol. Since $X = Y - Z$,

$$\begin{aligned} \text{var}(X) &= \text{var}(Y - Z) \\ &= \text{var}(Y) + \text{var}(Z) + 2 \cdot \text{cov}(Y, Z) \end{aligned}$$

But since $\text{cov}(Y, Z) < 0$,

Then, $\text{var}(X) >= \text{var}(Y) + \text{var}(Z)$ $\square$

(c) Use the result of part (b) to show that

$$var(X) \geq var(\max\{0, X\}) + var(\max\{0, -X\})$$

Sol. If $Y > Z$, then $\max 0, X = X$ and $\max 0, -X = 0$

Therefore,

$$\begin{aligned} \text{var}(\max 0, X) + \text{var}(\max 0, -X) &= \text{var}(X) + \text{var}(0) \\ &= \text{var}(X) \end{aligned}$$

If $Y < Z$, then $\max 0, X = 0$ and $\max 0, -X = -X$

Therefore

$$\begin{aligned} \text{var}(\max 0, X) + \text{var}(\max 0, -X) &= \text{var}(0) + \text{var}(-X) \\ &= \text{var}(X) \end{aligned}$$

If $Y = 0$, $Z = 0$,

then $\max 0, X = X$ and $\max 0, -X = 0$

Therefore,

$$\begin{aligned} \text{var}(\max 0, X) + \text{var}(\max 0, -X) &= \text{var}(0) + \text{var}(0) \\ &= 0 \end{aligned}$$

Therefore, $\text{var}(X) \geq \text{var}(\max\{0, X\}) + \text{var}(\max\{0, -X\})$ $\square$

8. Consider two random variables X and Y. Assume for simplicity that they both have zero mean.

(a) Show that X and $E[X \mid Y]$ are positively correlated.

Sol. Given $E[X] = 0$, $E[Y] = 0$

Then,

$$\begin{aligned} E[X \cdot E[X \mid Y]] &= E[E[X \cdot E[X \mid Y] \mid Y]] \\ &= E[E[X \mid Y] \cdot E[X \mid Y]] \\ &= E[E^2[X \mid Y]] > 0 \end{aligned}$$

Since $E[E[X \mid Y]] = E[X] = 0$,

$\text{cov}(X, E[X \mid Y]) = E[E^2[X \mid Y]] > 0$ □

(b) Show that the correlation coefficient of Y and $E[X \mid Y]$ has the same sign as the correlation coefficient of X and Y.

Sol.

$$\begin{aligned} &\text{cov}(Y, E[X \mid Y]) = E[Y \cdot E[X \mid Y]] \\ &\implies \text{cov}(X,Y) = E[XY] \end{aligned}$$

Since,

$$\begin{aligned} E[XY] &= E[E[XY \mid Y]] \\ &= E[Y \cdot E[X \mid Y]] \\ \implies \text{cov}(X,Y) &= \text{cov}(Y, E[X \mid Y]) \end{aligned}$$ □