Problem Set #6 Solutions

1. a. We have $\mathbf{B} = 0$, so that $\mathbf{F} = 0$ N.

   b. Since we have $\mathbf{F}_\mathrm{m} = \hat{\boldsymbol{\phi}} 2\cos\phi$ N and the change in position of the wire as it rotates in ϕ is given by $d\mathbf{l} = \hat{\boldsymbol{\phi}} r\, d\phi$ m, we have that the total work is given by

$$W_\mathrm{m} = \oint \mathbf{F}_\mathrm{m} \cdot d\mathbf{l} = 2.4 \int_{\phi=0}^{2\pi} \cos\phi \, d\phi = 0 \text{ J}$$

   c. The force is a maximum when $\cos\phi$ is a maximum, which occurs when $\cos\phi = 1$ or $\phi = 0$.

2. As Ulaby, et al. show in their example 5-2, the field in this geometry is given by

$$\mathbf{H} = \hat{\boldsymbol{\phi}} \frac{I}{4\pi r} (\cos\theta_1 - \cos\theta_2).$$

In this problem, in contrast to Ulaby and Ravaioli's example 5-2, we have that both θ_1 and θ_2 are greater than $\pi/2$, which implies that both $\cos\theta_1$ and $\cos\theta_2$ will be negative. Specifically, we have

$$\begin{aligned} \cos\theta_1 &= -\cos(\pi - \theta_1) = -\frac{z_1}{\sqrt{z_1^2 + r^2}} = -\frac{1}{\sqrt{1^2 + 2^2}} = -0.4472, \\ cos\theta_2 &= -\cos(\pi - \theta_2) = -\frac{z_2}{\sqrt{z_2^2 + r^2}} = -\frac{6}{\sqrt{6^2 + 2^2}} = -0.9487, \end{aligned}$$

so that

$$\mathbf{H} = \hat{\boldsymbol{\phi}} \frac{10}{(4\pi)(2)} (-0.4472 + 0.9487) = \hat{\boldsymbol{\phi}} 0.1995 \to \hat{\boldsymbol{\phi}} 200 \text{ mA/m}.$$

3. a. Using the expression

$$\mathbf{H} = \frac{1}{4\pi} \int_S \frac{\mathbf{J}_\mathrm{S} \times \hat{\mathbf{R}}}{R^2} \, dS,$$

   along with the relations, $\mathbf{J} = -\hat{\mathbf{y}}(I/w)$ and $\hat{\mathbf{R}}(x, y) = -\hat{\mathbf{x}}x - \hat{\mathbf{y}}y + \hat{\mathbf{z}}h$ at a point on the current surface (x, y), we find

$$\begin{aligned} \mathbf{H}(0, 0, h) &= -\frac{I}{4\pi w} \int_{-\infty}^{infty} dy \int_{-w/2}^{w/2} dx \, \frac{\hat{\mathbf{z}}x + \hat{\mathbf{x}}h}{(x^2 + y^2 + h^2)^{3/2}} \\ &= -\frac{I}{2\pi w} \int_{-w/2}^{w/2} dx \, \frac{\hat{\mathbf{z}}x + \hat{\mathbf{x}}h}{x^2 + h^2} = -\hat{\mathbf{x}} \frac{I}{\pi w} \tan^{-1}\left(\frac{w}{2h}\right) \text{ A/m}. \end{aligned}$$

From symmetry, we have have $\mathbf{H}(0, y, h) = \mathbf{H}(0, 0, h)$ at any y.

b. Writing the force per unit length as $\mathbf{F}'_{\mathrm{m}}$, we have

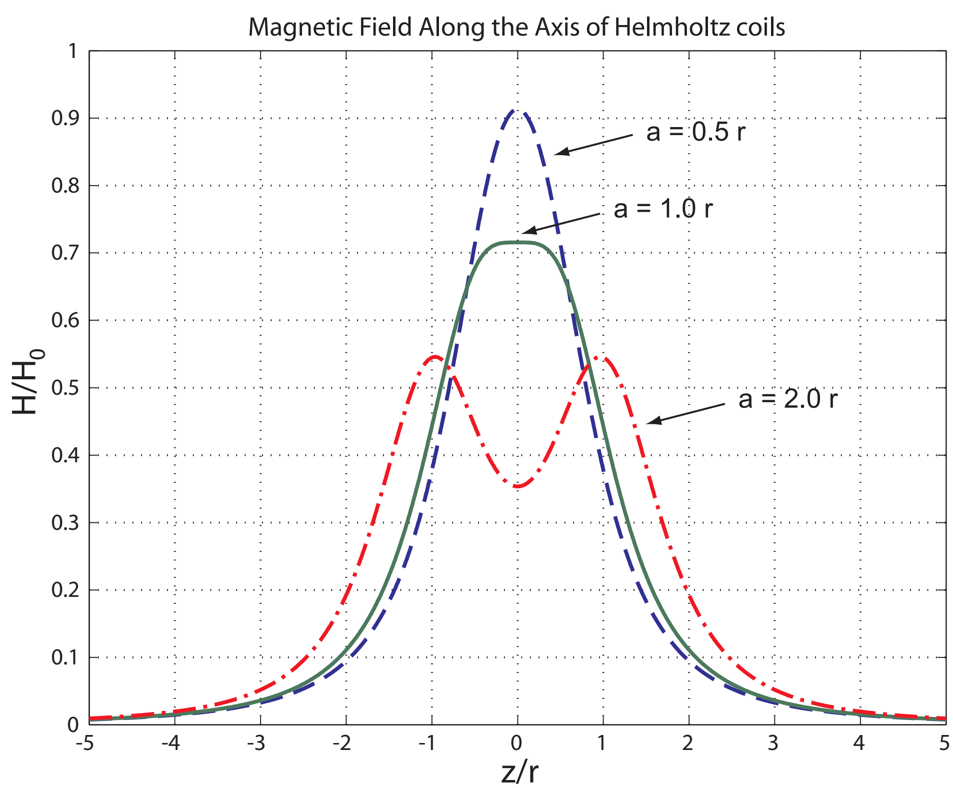
$$\mathbf{F}'_{\mathrm{m}} = \frac{1}{2Y} \int_{-Y}^{Y} dy \, (\hat{\mathbf{y}} I \times \mu_0 \mathbf{H}) = \hat{\mathbf{z}} \frac{\mu_0 I^2}{\pi w} \tan^{-1}\left(\frac{w}{2h}\right) \text{ N/m}$$

Since the force is in the $+z$-direction, it is repulsive.

4. Using the formula for a single loop, given by Ulaby's Eq. (5.34) and adding the result for two current loops displaced along the z-axis by $\pm a/2$, we find that along the z-axis, $\mathbf{H}(z) = \hat{\mathbf{z}} H(z)$, where

$$H(z) = \frac{Ir^2}{2} \left\{ \frac{1}{[r^2 + (z - a/2)^2]^{3/2}} + \frac{1}{[r^2 + (z + a/2)^2]^{3/2}} \right\}.$$

a. In order to plot the magnetic field vs. z, we must first normalize both the field and the z-axis in a sensible way. It makes sense in this problem to normalize the magnetic field with respect to $H_0 = I/r$ and the z-axis with respect to r. Doing that, we obtain the plot below



The listing of the MATLAB code that produced this figure is given by

```
% Helmholtz
%
% The H-field along the z-axis of Helmholtz rings is plotted
% in normalized units.  The H-field is normalized to
% I/r = (current/ring radius).  The distance along the z-axis
% is normalized to the ring radius

% Input parameters
Zmax = 5;     % the range of z-values is (-Zmax,Zmax) in
              % normalized units
ratio = [0.5 1.0 2.0]'    % the ratio of the separation to
                              % the radius in normalized units
npoints = 1001;    % the number of evaluation points

% Set up fields
   %Fix z-values and corresponding ONES array
   Delta = 2*Zmax/(npoints-1);   Z = -Zmax:Delta:Zmax;
   IZ = ones(size(Z));

   %Set up a ONES array for the ratio
   IR = ones(size(ratio));

   %Set up the + and - Helmholtz coil distances
   Zplus = IR*Z + 0.5*ratio*IZ;    Zminus = IR*Z - 0.5*ratio*IZ;

% Calculate the H-field
H = 0.5*((1 + Zplus.^2).^(-3/2) + (1 + Zminus.^2).^(-3/2));

% Plot the H-field
hold off
plot(Z,H,'linewidth',2)
grid
xlabel('z/r');  ylabel('H/H_0');
title('Magnetic Field Along the Axis of Two Helmholtz Rings')
```

b. We first note that when we exchange z with $-z$, the function $H(z)$ is unchanged. Hence, the magnetic field $H(z)$ is an even function, which means that it is symmetric around $z = 0$, which in turn implies that all odd derivatives of $H(z)$ must be zero at $z = 0$. We don't have to calculate anything to obtain this result! We next calculate the second derivative of $H(z)$ with respect to z. We obtain

$$\frac{d^2H(z)}{dz^2} = \frac{I}{2r^3}\left\{\left[1+\left(\frac{z}{r}-\frac{a}{2r}\right)^2\right]^{-7/2}\left[12\left(\frac{z}{r}-\frac{a}{2r}\right)^2-3\right]\right. \\ \left. +\left[1+\left(\frac{z}{r}+\frac{a}{2r}\right)^2\right]^{-7/2}\left[12\left(\frac{z}{r}+\frac{a}{2r}\right)^2-3\right]\right\}.$$

When $z = 0$, we find that $d^2H(z)/dz^2 = 0$ if $a = r$. Referring to the figure that we just plotted, we can see that when $a < r$, the field is curved upward at $z = 0$, while when $a > r$, it is curved downward. When $a = r$, it is flat at the origin. This effect can be used to create highly intense, constant magnetic fields over a short region.

5. The point of this problem is to give you the opportunity to verify the vector potential formula in a particular instance and compare the calculational convenience of using it rather than calculating the magnetic field directly.

a. In this instance only A_z and J_z are non-zero, and the vector Poisson equation becomes,

$$\frac{\partial A_z^2}{\partial x^2} + \frac{\partial A_z^2}{\partial y^2} + \frac{\partial A_z^2}{\partial z^2} = -\mu_0 J_z. \tag{5.1}$$

In this instance, we have

$$J_z = 4J_0, \quad \frac{\partial A_z^2}{\partial x^2} = -2\mu_0 J_0, \quad \frac{\partial A_z^2}{\partial y^2} = -2\mu_0 J_0, \quad \frac{\partial A_z^2}{\partial z^2} = 0. \tag{5.2}$$

Substituting the values from (5.2) into (5.1), we confirm the expression for $\mathbf{A}$.

b. We have

$$\mathbf{H} = \frac{1}{\mu_0}\nabla \times \mathbf{A} = \frac{1}{\mu_0} = \hat{\mathbf{x}}\frac{\partial A_z}{\partial y} - \hat{\mathbf{y}}\frac{\partial A_z}{\partial x} = \frac{J_0}{2}(\hat{\mathbf{y}}x - \hat{\mathbf{x}}y).$$

c. Taking a closed circle in the x-y plane surrounding the origin, we have $\oint \mathbf{H}\cdot d\mathbf{l} = 2\pi a H = \int_S \mathbf{J}\cdot d\mathbf{S} = 4\pi a^2 J_0$. The field must point in the $\hat{\boldsymbol{\phi}}$-direction. We conclude that $\mathbf{H} = \hat{\boldsymbol{\phi}} 2aJ_0$. We have that $\hat{\boldsymbol{\phi}} = (\hat{\mathbf{y}}x - \hat{\mathbf{x}}y)/(x^2+y^2)^{1/2}$. We also have that $a = (x^2+y^2)^{1/2}$. After substitution of these values, we obtain the same result as before. I will point out that we have "cheated" slightly. We have assumed that the current flow is the same everywhere in space. However, the magnetic field that we have calculated is not uniform. There is an origin where it goes to zero. How can that be? Answer: We can add a constant vector to the magnetic field without in any way violating Ampere's law. Ampere's law does not uniquely define the magnetic field. We need to use appropriate boundary conditions as well in order to determine the field uniquely.

6. We have $\hat{\mathbf{n}}_1 = -\hat{\mathbf{n}}_2 = (\hat{\mathbf{x}} - \hat{\mathbf{y}})/\sqrt{2}$. It follows that $\mathbf{B}_{1\mathrm{n}} = \hat{\mathbf{n}}_1(\hat{\mathbf{n}}_1\cdot\mathbf{B}) = -\hat{\mathbf{n}}_1(\sqrt{2}/2) = -(1/2)(\hat{\mathbf{x}} - \hat{\mathbf{y}})$ and that $\mathbf{B}_{1\mathrm{t}} = \mathbf{B}_1 - \mathbf{B}_{1\mathrm{n}} = (5/2)(\hat{\mathbf{x}} + \hat{\mathbf{y}})$. We have $\mathbf{B}_{2\mathrm{n}} = \mathbf{B}_{1\mathrm{n}} = -(1/2)(\hat{\mathbf{x}} - \hat{\mathbf{y}})$ and $\mathbf{B}_{2\mathrm{t}} = (\mu_2/\mu_1)\mathbf{B}_{1\mathrm{t}} = (5\mu_2/2\mu_1)(\hat{\mathbf{x}} + \hat{\mathbf{y}})$, so that $\mathbf{B}_2 = \hat{\mathbf{x}}[(5\mu_2 - \mu_1)/2\mu_1] + \hat{\mathbf{y}}[(5\mu_2 + \mu_1)/2\mu_1]$. When $\mu_1 = 3\mu_2$, we obtain $\mathbf{B}_2 = \hat{\mathbf{x}}(1/3) + \hat{\mathbf{y}}(4/3)$ Tesla.

7. We have for the magnetic field, $H = I/2\pi r$. The magnetic energy is given by

$$W_\mathrm{m} = \frac{\mu}{2}L\int_0^{2\pi} d\phi \int_{R_1}^{R_2}\left(\frac{I}{2\pi r}\right)^2 r\,dr = \frac{\mu_0 I^2}{4\pi}L\int_{R_1}^{R_2}\frac{1}{r}dr = \mu_0\frac{I^2}{4\pi}L\ln(R_2/R_1).$$

Substituting $\mu = \mu_0 = 1.257 \times 10^{-6}$, $L = 5$ m, $R_1 = 0.02$ m, $R_2 = 0.10$ m, and $I = 10$ A, we find $W_\mathrm{m} = (\mu_0/4\pi)LI^2\ln(R_2/R_1) = (1.0\times10^{-7})(5)(10^2)\ln(5) = 8.047\times 10^{-5}$ J. We conclude that $W_\mathrm{m} = 80.4\ \mu$J.