CMPE 320: Probability, Statistics, and Random Processes

Lecture 20: Derived distributions

Spring 2018

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Derived distributions

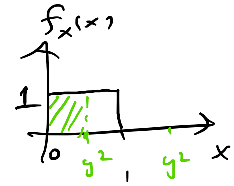
- Given a RV X and its PDF $f_X(x)$, derive the PDF $f_Y(y)$ of Y = g(X)
  - First calculate the CDF F_Y of Y

$$F_Y(y) = P(Y \le y) = P(g(X) \le y) = \int_{\{x : g(x) \le y\}} f_X(x)\, dx$$

  - Differentiate F_Y to obtain the PDF f_Y

$$f_Y(y) = \frac{dF_Y(y)}{dy}$$

Example 4.1. Let X be uniform on $[0, 1]$, and let $Y = \sqrt{X}$.
Compute the PDF of Y.



Note $Y \geq 0$

$$F_Y(y) = P(Y \leq y) = P(\sqrt{X} \leq y) = P(X \leq y^2)$$

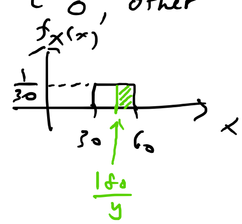
$$= \int_{-\infty}^{y^2} f_X(x)\,dx = \begin{cases} y^2, & 0 \leq y^2 \leq 1 \Rightarrow 0 \leq y \leq 1 \\ 1, & y^2 > 1 \Rightarrow y > 1 \\ 0, & y^2 < 0 \Rightarrow y < 0 \end{cases}$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} 2y, & 0 \leq y \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Example 4.2. John Slow is driving from Boston to the New York area, a distance of 180 miles at a constant speed, whose value is uniformly distributed between 30 and 60 miles per hour. What is the PDF of the duration of the trip?

X : John's speed. Uniform over [30, 60] mph $\Rightarrow f_X(x) = \begin{cases} \frac{1}{30}, & 30 \leq x \leq 60 \\ 0, & \text{other} \end{cases}$

Y : duration of the trip, $Y = \frac{180}{X}$

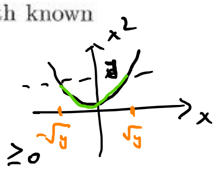


$$F_Y(y) = P(Y \leq y) = P\left(\frac{180}{X} \leq y\right) = P\left(X \geq \frac{180}{y}\right)$$

$$= \begin{cases} \frac{1}{30}\cdot\left(60 - \frac{180}{y}\right) = 2 - \frac{6}{y}, & 30 \leq \frac{180}{y} \leq 60 \Rightarrow 6 \geq y \geq 3 \\ 0, & \frac{180}{y} > 60 \Rightarrow y < 3 \\ 1, & \frac{180}{y} < 30 \Rightarrow y > 6 \end{cases}$$

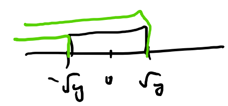
$$f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} \frac{6}{y^2}, & 3 \leq y \leq 6 \\ 0, & \text{otherwise} \end{cases}$$

Example 4.3. Let $Y = g(X) = X^2$, where X is a random variable with known PDF. Compute the PDF of Y.



$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}), \quad y \ge 0$$

$$= \int_{-\sqrt{y}}^{\sqrt{y}} f_X(x)\,dx = F_X(\sqrt{y}) - F_X(-\sqrt{y})$$



$$f_Y(y) = \frac{dF_Y(y)}{dy} = f_X(\sqrt{y})\frac{1}{2\sqrt{y}} - f_X(-\sqrt{y})\left(-\frac{1}{2\sqrt{y}}\right)$$

$$\frac{dF_X(y)}{dy} = f_X(y)$$

$$= \frac{f_X(\sqrt{y}) + f_X(-\sqrt{y})}{2\sqrt{y}}, \quad y \ge 0$$

$$\frac{dF_X(\sqrt{y})}{dy} = f_X(\sqrt{y})\cdot\frac{1}{2\sqrt{y}}$$

$$\boxed{\text{chain rule: } \frac{df(g(x))}{dx} = f'(g(x))\cdot g'(x)}$$

Linear case

$$Y = aX + b \quad (a \neq 0)$$

$$F_Y(y) = P(Y \le y) = P(aX + b \le y) = \begin{cases} P\left(X \le \frac{y-b}{a}\right), & a > 0 \\ P\left(X \ge \frac{y-b}{a}\right), & a < 0 \end{cases}$$

$$= \begin{cases} F_X\left(\frac{y-b}{a}\right), & a > 0 \\ 1 - F_X\left(\frac{y-b}{a}\right), & a < 0 \end{cases}$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} f_X\left(\frac{y-b}{a}\right)\cdot\frac{1}{a}, & a > 0 \\ -f_X\left(\frac{y-b}{a}\right)\cdot\frac{1}{a}, & a < 0 \end{cases} = \boxed{\frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)}$$

Example 4.5. A Linear Function of a Normal Random Variable is Normal. Suppose that X is a normal random variable with mean μ and variance σ^2, and let $Y = aX + b$, where a and b are scalars, with $a \neq 0$. What is the PDF of Y?

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

$$f_Y(y) = \frac{1}{|a|}\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\left(\frac{y-b}{a}-\mu\right)^2}{2\sigma^2}} = \frac{1}{\sqrt{2\pi a^2\sigma^2}} \exp\left(-\frac{(y-(b+a\mu))^2}{2a^2\sigma^2}\right)$$

$\therefore$ Y is also normal with mean $a\mu+b$, var. $a^2\sigma^2$

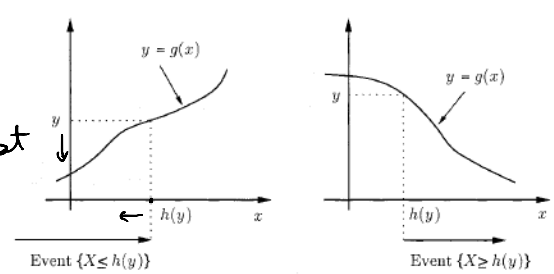
$$\left(\begin{array}{l} E[Y] = E[aX+b] = aE[X]+b = a\mu+b \\ var(Y) = var(aX+b) = a^2 var(X) = a^2\sigma^2 \end{array}\right.$$

Strictly monotonic case

- Y = g(X), but now g(x) is strictly monotonic

$\Rightarrow$ There is an inverse of g such that

$$y = g(x) \Leftrightarrow x = h(y)$$



$$F_Y(y) = P(Y \le y)$$

$$= \begin{cases} P(X \le h(y)), & \text{if increasing} \\ P(X \ge h(y)), & \text{if decreasing} \end{cases} = \begin{cases} F_X(h(y)), & \text{if increasing} \\ 1 - F_X(h(y)), & \text{if decreasing} \end{cases}$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} f_X(h(y)) \cdot \frac{dh(y)}{dy} \\ -f_X(h(y)) \cdot \frac{dh(y)}{dy} \end{cases} = \boxed{f_X(h(y)) \left|\frac{dh(y)}{dy}\right|}$$

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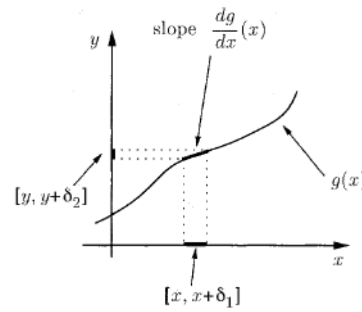
Alternative derivation



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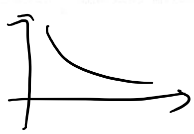
(Revisited)

Example 4.2. John Slow is driving from Boston to the New York area, a distance of 180 miles at a constant speed, whose value is uniformly distributed between 30 and 60 miles per hour. What is the PDF of the duration of the trip?

$Y = \frac{180}{X} = g(X),$



$h(y) = x = \frac{180}{y}$

$$f_Y(y) = f_X(h(y)) \left| \frac{dh(y)}{dy} \right|$$

$$= \begin{cases} \frac{1}{30} \cdot \left| -\frac{180}{y^2} \right| = \frac{6}{y^2}, & 30 \le \frac{180}{y} \le 60 \\ 0, & \text{otherwise} \end{cases}$$

Functions of 2 RVs

Example 4.8. Let X and Y be independent random variables that are uniformly distributed on the interval $[0, 1]$. What is the PDF of the random variable $Z = Y/X$?

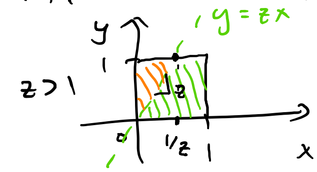
Find the CDF $F_Z(z)$ and differentiate.

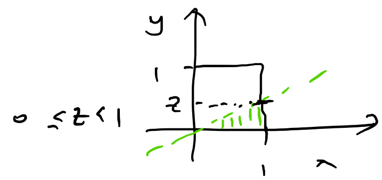
$f_{X,Y}(x,y) = f_X(x) f_Y(y)$

$$F_Z(z) = P(Z \le z) = P\left(\frac{Y}{X} \le z\right) = P(Y \le zX)$$

$$= \begin{cases} \frac{1}{2} \cdot 1 \cdot z, & 0 \le z \le 1 \\ 1 - \frac{1}{2} \cdot 1 \cdot \frac{1}{z}, & z > 1 \\ 0, & \text{o.w} \end{cases}$$

$$f_Z(z) = \begin{cases} \frac{1}{2} & 0 \le z \le 1 \\ \frac{1}{2z^2}, & z > 1 \\ 0, & \text{o.w.} \end{cases}$$





Example 4.9. Romeo and Juliet have a date at a given time, and each, independently, will be late by an amount of time that is exponentially distributed with parameter λ. What is the PDF of the difference between their times of arrival?

X: amount by which Romeo is late, Y: Juliet, $Z = X - Y$

Since X and Y are independent

$$f_{X,Y}(x,y) = f_X(x) f_Y(y) = \begin{cases} \lambda e^{-\lambda x} \cdot \lambda e^{-\lambda y} = \lambda^2 e^{-\lambda(x+y)}, & x \ge 0,\ y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

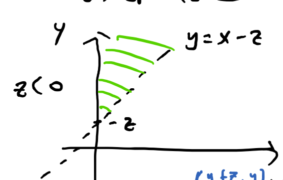
$$F_Z(z) = P(Z \le z) = P(X - Y \le z) = P(Y \ge X - z)$$

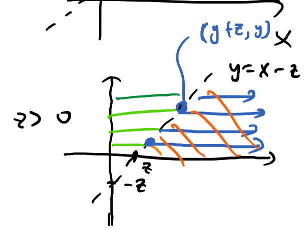
If $-z < 0$ (or $z > 0$), $F_Z(z) = 1 - P(Y < X - z)$

$$= 1 - \int_0^\infty \int_{y+z}^\infty f_{X,Y}(x,y)\,dx\,dy = 1 - \int_0^\infty \int_{y+z}^\infty \lambda^2 e^{-\lambda(x+y)}\,dx\,dy$$

$$= 1 - \int_0^\infty \left[-\lambda e^{-\lambda(x+y)}\right]_{y+z}^\infty dy = 1 - \int_0^\infty \lambda e^{-\lambda(2y+z)}\,dy$$

$$= 1 - \left[-\frac{1}{2} e^{-\lambda(2y+z)}\right]_0^\infty = 1 - \left(\frac{1}{2} e^{-\lambda z}\right)$$





For $z<0$, can do similarly, or use the symmetry
"$Z = X - Y$ and $-Z = Y - X$ must have the same distribution"

$$F_Z(z) = P(Z \le z) = P(-Z \ge -z) = P(Z \ge -z) = 1 - F_Z(-z)$$

$$= 1 - \left(1 - \tfrac{1}{2}e^{+\lambda z}\right) = \tfrac{1}{2}e^{+\lambda z} \quad \text{for } z<0$$

$$\Rightarrow F_Z(z) = \begin{cases} 1 - \frac{1}{2}e^{-\lambda z} & z>0 \\ \frac{1}{2}e^{\lambda z} & z<0 \end{cases}$$

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \begin{cases} +\frac{\lambda}{2}e^{-\lambda z} & z>0 \\ \frac{\lambda}{2}e^{\lambda z} & z<0 \end{cases} = \frac{\lambda}{2}e^{-\lambda|z|}$$

Sum of independent RVs: Convolution

X, Y : independent RVs, What is PDF of $Z = X+Y$

let's compute $f_{X,Z}(x,z)$ and marginalize

$$f_{X,Z}(x,z) = f_{Z|X}(z|x)\, f_X(x)$$

To compute $f_{Z|X}(z|x)$, consider $F_{Z|X}(z|x)$

$$F_{Z|X}(z|x) = P(Z \le z \mid X=x) = P(X+Y \le z \mid X=x)$$

$$= P(x+Y \le z \mid X=x) = P(Y \le z-x) = F_Y(z-x)$$

$$f_{Z|X}(z|x) = \frac{\partial F_{Z|X}}{\partial z} = f_Y(z-x) \Rightarrow f_{X,Z}(x,z) = f_X(x) f_Y(z-x)$$

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Z}(x,z)\,dx = \boxed{\int_{-\infty}^{\infty} f_X(x) f_Y(z-x)\,dx}$$

Convolution of f_X and f_Y

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Example 4.10. The random variables X and Y are independent and uniformly distributed in the interval $[0,1]$. The PDF of $Z = X + Y$ is ?

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)\, f_Y(z-x)\, dx$$

$$= \begin{cases} 0, & \text{if } z < 0 \\ z, & \text{if } 0 \le z \le 1 \\ 1-(z-1) = 2-z, & \text{if } 1 \le z \le 2 \\ 0, & \text{if } z > 2 \end{cases}$$