Problem Set #6 Solutions

1. We use the expression $\mathbf{T} = \mathbf{m} \times \mathbf{B}$ with $\mathbf{m} = \hat{\mathbf{n}}NIA$. We have $\hat{\mathbf{n}} = -\hat{\mathbf{x}}\cos 30° + \hat{\mathbf{y}}\sin 30° = -\hat{\mathbf{x}}0.866 + \hat{\mathbf{y}}0.500$, $N = 20$, $I = 0.5$ A, and $A = 0.08$ m^2. We thus find $\mathbf{m} = (-\hat{\mathbf{x}}0.866 + \hat{\mathbf{y}}0.500) \times 20 \times 0.5 \times 0.08 = -\hat{\mathbf{x}}0.693 + \hat{\mathbf{y}}0.4$ A·m^2. Using $\mathbf{B} = \hat{\mathbf{y}}1.2$ T, we find $\mathbf{T} = \mathbf{m} \times \mathbf{B} = -\hat{\mathbf{z}}0.83$ N·m (newton-meters), where we report two significant figures. Since, the torque is directed in the $-z$-direction, the rotation will be in the $-\hat{\boldsymbol{\phi}}$-direction, which is clockwise when viewed from above.

2. Since the two radial segments satisfy $d\mathbf{l} \times \hat{\mathbf{R}} = 0$ at every point along them, these two segments contribute nothing. Since the $d\mathbf{l}$ is at right angles to $\hat{\mathbf{R}}$ at every point along the angular segments, we obtain simply, using the right-hand rule,

$$\mathbf{H} = \hat{\mathbf{z}}\frac{I}{4\pi}\int_0^\theta d\theta' \left[b \cdot \left(-\frac{1}{b^2}\right) + a \cdot \left(\frac{1}{a^2}\right)\right] = \hat{\mathbf{z}}\frac{I\theta}{4\pi}\left(\frac{1}{a} - \frac{1}{b}\right) = \hat{\mathbf{z}}\frac{I\theta}{4\pi}\frac{b-a}{ab}.$$

3. Since parallel currents attract and anti-parallel current repel, the force must be in the $+y$-direction. We have

$$\mathbf{F}\,(\mathrm{N}) = 2 \times \hat{\mathbf{y}}\frac{4\pi \times 10^{-7}\,(\mathrm{H/m}) \times 5\,(\mathrm{A}) \times 10\,(\mathrm{A}) \times 2\,(\mathrm{m})}{2\pi \times 1\,(\mathrm{m})} = 4.0 \times 10^{-5}\,\mathrm{N}.$$

4. a. Since we are very far away from the current, and the current is oriented in the $+\hat{\mathbf{z}}$-direction, we find $\mathbf{A} = \hat{\mathbf{z}}\mu IL/R$, where R is the distance from the little current element. Taking the current element at the origin, we may write this result in cylindrical coordinates as $\mathbf{A} = \hat{\mathbf{z}}\mu IL/(r^2 + z^2)^{1/2}$ and in Cartesian coordinates as $\mathbf{A} = \hat{\mathbf{z}}\mu IL/(x^2 + y^2 + z^2)^{1/2}$.

   b. It is easiest to solve for the field in cylindrical coordinates. Using the expression $\nabla \times \mathbf{A}$ at the back of Ulaby's book and noting that only $A_z \neq 0$, we obtain

$$\nabla \times \mathbf{A} = \hat{\mathbf{r}}\frac{1}{r}\frac{\partial A_z}{\partial \phi} - \hat{\boldsymbol{\phi}}\frac{\partial A_z}{\partial r} = \hat{\boldsymbol{\phi}}\frac{\mu IL}{4\pi}\frac{r}{(r^2 + z^2)^{3/2}},$$

   which implies $\mathbf{H} = \mathbf{B}/\mu = \hat{\boldsymbol{\phi}}ILr/(r^2 + z^2)^{3/2}$. In spherical coordinates, this expression becomes $\mathbf{H} = \hat{\boldsymbol{\phi}}IL\sin\theta/R^2$, and in Cartesian coordinates, this expression becomes $\mathbf{H} = (-\hat{\mathbf{x}}y + \hat{\mathbf{y}}x)IL/(x^2 + y^2 + z^2)^{3/2}$.

5. Since the normal component (y-component) of the $\mathbf{H}$-field is zero in medium 1, it must be zero in medium 2, and we have $H_{1y} = H_{2y} = 0$. From the expression for the tangential $\mathbf{H}$-field components, $\hat{\mathbf{n}}_2 \times (\mathbf{H}_1 - \mathbf{H}_2) = \mathbf{J}_\mathrm{S}$, and noting that $\hat{\mathbf{n}}_2 = \hat{\mathbf{y}}$, we find

$H_{1z} - H_{2z} = J_{Sx}$ and $H_{1x} - H_{2x} = -J_{Sz}$. After substitution, we find $H_{1x} = H_{2x} = 0$, and $H_{2z} = H_{1z} - J_{Sx} = 4\,(\mathrm{A/m})$. We conclude $\mathbf{H}_2 = \hat{\mathbf{z}}4\,\mathrm{A/m}$.

6. From Ampere's law and the assumption that the current distribution in each wire is axially symmetric, we find that the field generated by the wire located at $x = 0$ is given by $\mathbf{B} = \hat{\mathbf{y}}(\mu I/2\pi x)$ in the section shown in the figure. Similarly, the field generated by the wire located at $x = d$ is given by $\mathbf{B} = \hat{\mathbf{y}}[\mu I/2\pi(d-x)]$. Hence, the magnetic flux through a section of length l is given by

$$\Phi = \frac{\mu I l}{2\pi}\int_a^{d-a} dx\left(\frac{1}{x} + \frac{1}{d-x}\right) = \frac{\mu I l}{2\pi}\ln\left(\frac{x}{d-x}\right)\Big|_a^{d-a} = \frac{\mu I l}{\pi}\ln\left(\frac{d-a}{a}\right).$$

We conclude that $L = \Phi/I = (\mu l/\pi)\ln[(d-a)/a]$ and $L' = L/l = (\mu/\pi)\ln[(d-a)/a]$.

7. We use the expression

$$H = \frac{IN}{2\,l}\left(\sin\theta_2 - \sin\theta_1\right).$$

In our case, we have $IN/2\,l = 12$ kA/m. We also have

$$\sin\theta_2 = \frac{10-z}{\left[5^2 + (10-z)^2\right]^{1/2}}, \qquad \sin\theta_1 = -\frac{10+z}{\left[5^2 + (10+z)^2\right]^{1/2}},$$

where z is measured in cm. Thus, the expression that we wish to calculate is

$$H = 12\left\{\frac{10-z}{\left[5^2 + (10-z)^2\right]^{1/2}} + \frac{10-z}{\left[5^2 + (10-z)^2\right]^{1/2}}\right\}\ (\mathrm{kA/m}).$$

The MATLAB code to plot the field is on the next page. [It has some extra "bells and whistles" that increases its length. The meat of the code is just six lines.]

MATLAB CODE:

```
% Solenoid
%
% The H-field along the z-axis of a solenoid is plotted in units
% of kA/m.  Distance along the H-field is measured in centimeters.

% Input parameters
I = 12;      % the current in the solenoid (in amperes)
N = 400;     % the number of turns in the solenoid
a = 5;       % the radius of the solenoid (in cm)
l = 20;      % the length of the solenoid (in cm)
range = [-20 20]; % the range of z-values for which the H-field
                  % is calculated
npoints = 401;    % the number of evaluation points

% H-field amplitude (in kA/m)
H0 = (I/1000)*400/(2*(l/100));

% Set up the z-array
Delta = (range(2) - range(1))/(npoints - 1);    % increment of the z-array
z = range(1):Delta:range(2);                    % Set up the z-array

% Calculate the sin functions
SinTheta1 = (0.5*l - z)./sqrt(a^2 + (0.5*l - z).^2);
SinTheta2 = -(0.5*l + z)./sqrt(a^2 + (0.5*l + z).^2);

% Calculate the H-field
H = H0*(SinTheta1 - SinTheta2);

% Plot the H-field
hold off
plot(z,H,'linewidth',2)
grid
xlabel('z (in cm)');  ylabel('H (in kA/m)');
title('Magnetic Field Along the Axis of a Solenoid')

% plot lines for the boundaries of the solenoid and its maximum H-value
hold on
rvals=axis(gca);
xvm = -0.5*l; xvp = 0.5*l; ymin = rvals(3); ymax= rvals(4);
xl = [xvm xvm]; xr = [xvp xvp]; yplot = [ymin ymax];
plot(xl,yplot,'r',xr,yplot,'r')
Hmax=2*H0; xmin  = rvals(1); xmax = rvals(2);
Hplot = [Hmax Hmax]; xplot = [xmin xmax];
plot(xplot,Hplot,'g')
```

The plot that is produced by MATLAB follows. The extra lines indicate the limits of the solenoid (in red) and the maximum value that would be obtained if we let the solenoid extend to infinite length(in green).