Output of an LTI system: Convolution

LTI Output

- **Starting over again (sorry)**
- **Assume we know the unit pulse output**

$$h^{(p)}(t) = H\left(p(t,T)\right)$$

the text uses $\zeta(t)$ for what I have called $p(t,T)$

the text uses $\psi(t)$ for what I have called $h^{(p)}(t)$

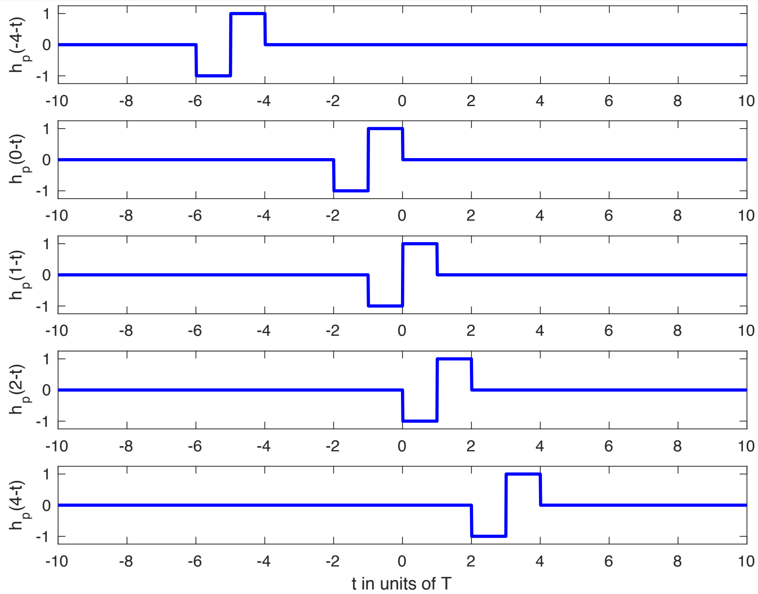
- **…and we can decompose the input**

$$x(t) = \sum_{n=-\infty}^{\infty} x_n p(t-nT,T)$$

- **Then the output of an LTI system is just**

$$y(t) = \sum_{n=-\infty}^{\infty} x_n h(t-nT)$$

$h_{n-k}\zeta(nT-t)$



Example (from last time

$$h^{(p)}(t) = \zeta(t) = p(t,1) - p(t-1,1)$$

$$x(t) = \sum_{n=0}^{3} x_n p(t-nT,1)$$

$$= p(t,1) + 3p(t-1,1) - 2p(t-2,1) - 1p(t-3,1)$$

$$y(t)=\sum_{n=0}^{3} x_n h^{(p)}(t-n)$$

$$=1\big(\zeta(t)-\zeta(t-1)\big)+3\big(\zeta(t-1)-\zeta(t-1-1)\big)$$

$$-2\big(\zeta(t-2)-\zeta(t-2-1)\big)-\big(\zeta(t-3)-\zeta(t-3-1)\big)$$

$$=1\big(\zeta(t)-\zeta(t-1)\big)+3\big(\zeta(t-1)-\zeta(t-2)\big)$$

$$-2\big(\zeta(t-2)-\zeta(t-3)\big)-\big(\zeta(t-3)-\zeta(t-4)\big)$$

$$=1\zeta(t)+\big(-1+3\big)\zeta(t-1)+\big(-3-2\big)\zeta(t-2)+\big(+2-1\big)\zeta(t-3)+\big(1\big)\zeta(t-4)$$

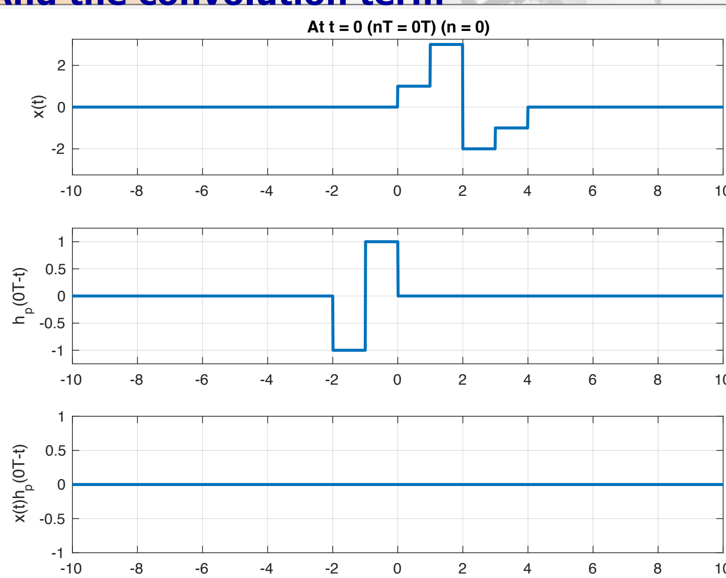
$$=(1\times 1)\zeta(t)+\big(1\times(-1)+3\times 1\big)\zeta(t-1)+\big(3\times(-1)+-2\times 1\big)\zeta(t-2)$$

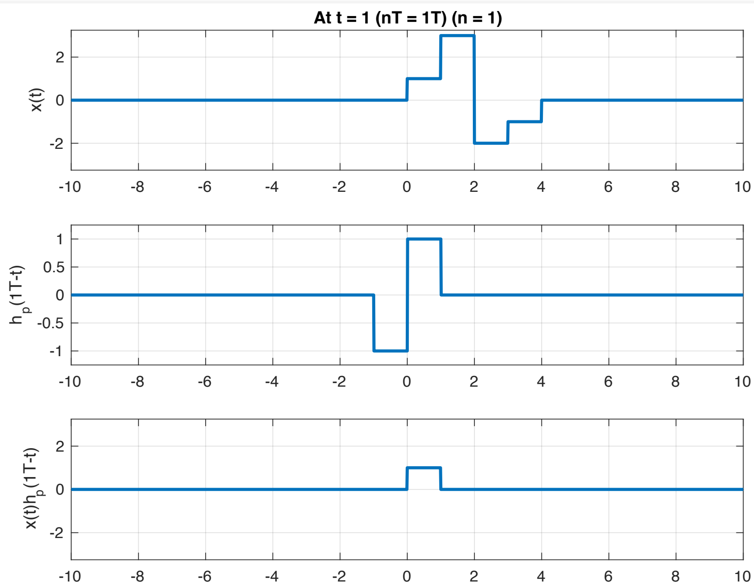
$$+\big(-2\times(-1)+-1\times 1\big)\zeta(t-3)+\big(-1\times -1\big)\zeta(t-4)$$

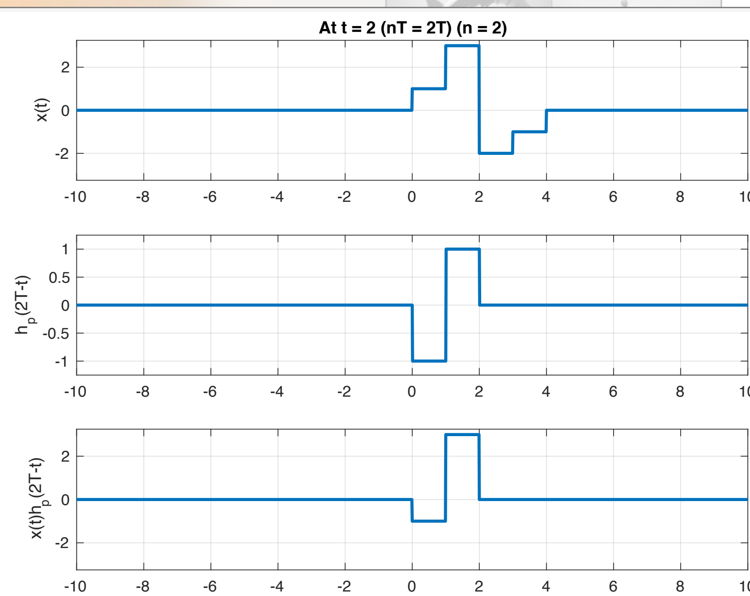
$$=(x_0h_0)\zeta(t-0)+\big(x_0h_1+x_1h_0\big)\zeta(t-1)+\big(x_1h_1+x_2h_0\big)\zeta(t-2)$$

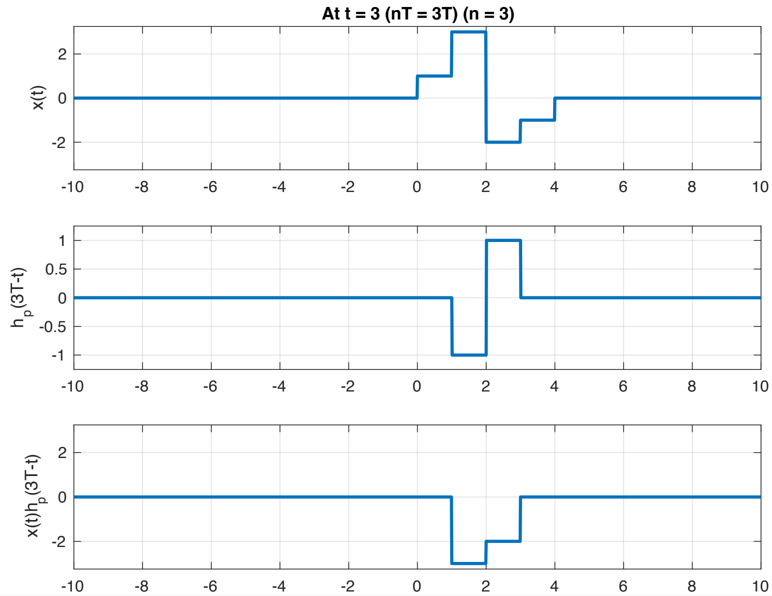
$$+\big(x_2h_1+x_3h_0\big)\zeta(t-3)+\big(x_3h_1\big)\zeta(t-4)\ =\sum_{k=0}^{4} x_k h_{n-k}\zeta(t-n)$$

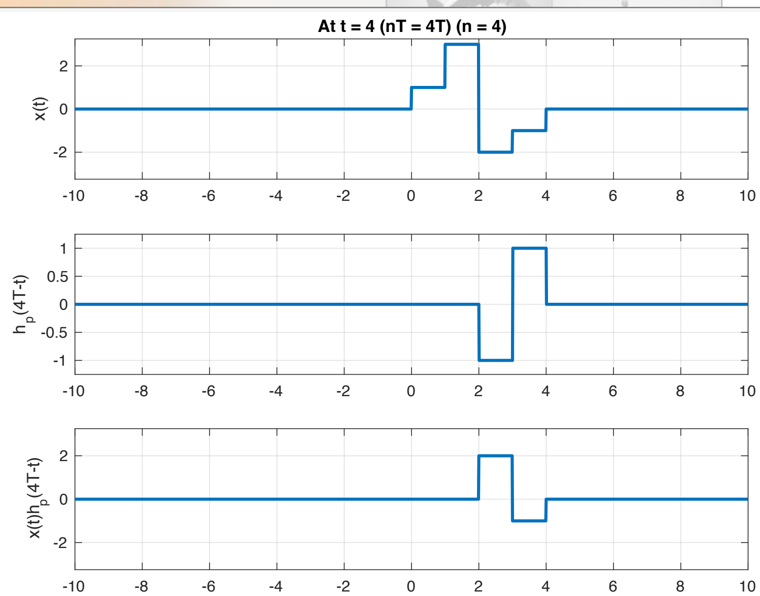
And the convolution term



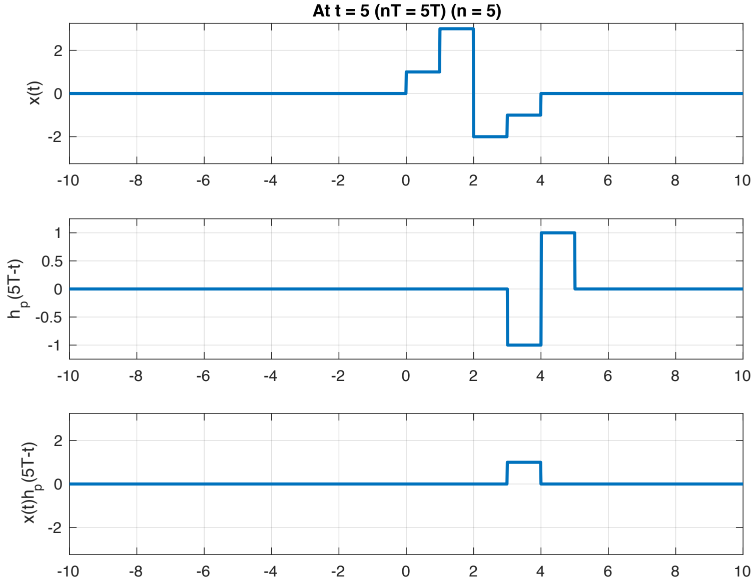

At t = 1 (nT = 1T) (n = 1)
x(t)
$h_p(1T\text{-}t)$
$x(t)h_p(1T\text{-}t)$
2
0
-2
1
0.5
0
-0.5
-1
-10 -8 -6 -4 -2 0 2 4 6 8 10


At t = 2 (nT = 2T) (n = 2)
x(t)
$h_p(2T\text{-}t)$
$x(t)h_p(2T\text{-}t)$
2
0
-2
1
0.5
0
-0.5
-1
-10 -8 -6 -4 -2 0 2 4 6 8 10

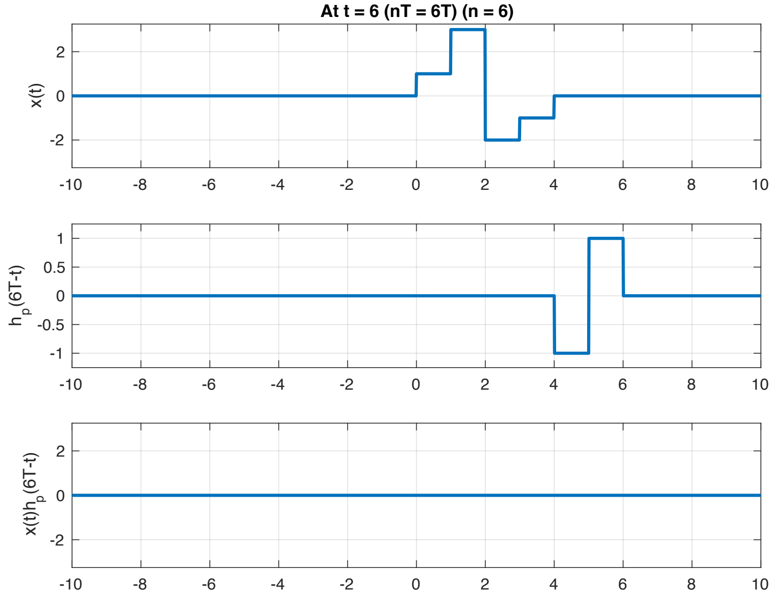

At t = 3 (nT = 3T) (n = 3)
x(t)
$h_p(3T-t)$
$x(t)h_p(3T-t)$


At t = 4 (nT = 4T) (n = 4)
x(t)
$h_p(4T-t)$
$x(t)h_p(4T-t)$

At t = 5 (nT = 5T) (n = 5)



At t = 6 (nT = 6T) (n = 6)



Lets make the input pulses shorter

- …while keeping the energy the same…
- Define $\xi(t) = \frac{1}{T} p(t,T)$ and let $H(\xi(t)) = \eta(t)$
- Let the input be

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} Tx(nT)\xi(t-nT)$$

- … a series of delayed pulses of amplitude $Tx(nT)$
- What's the output?
- The system is LTI, so $\hat{y}(t) = \sum_{n=-\infty}^{\infty} Tx(nT)\eta(t-nT)$
- …no problem.
- What's the limit as $T \to 0$?

As T approaches 0

Let $nT \to \tau$ a continuous variable

$$T \to d\tau$$

$$\xi(t) \to \delta(t)$$

$$\eta(t) \to h(t)$$

$$\hat{x}(t) \to x(t)$$

$$\sum_{n=-\infty}^{\infty} \to \int_{-\infty}^{\infty}$$

$$\hat{y}(t) \to \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

- Which we call the convolution integral!

Examples

$$h(t) = 2e^{-2t}u(t)$$

$$x(t) = p(t,1)$$

Find $y(t)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} p(\tau,1)2e^{-2(t-\tau)}u(t-\tau)\,d\tau$$

For $0 < t \le 1$

$$\int_{0}^{t} 2e^{-2(t-\tau)}\,d\tau = \left.\frac{2e^{-2t}e^{2\tau}}{2}\right|_{0}^{t} = e^{-2t}\left(e^{2t}-1\right) = 1-e^{-2t}$$

For $t > 1$

$$\int_{0}^{1} 2e^{-2(t-\tau)}\,d\tau = \left.\frac{2e^{-2t}e^{2\tau}}{2}\right|_{0}^{1} = e^{-2t}\left(e^{2}-1\right)$$