Phasor-Domain Equations

Transmission line equations — time domain:

$$-\frac{\partial v(z,t)}{\partial z} = R'\, i(z,t) + L' \frac{\partial i(z,t)}{\partial t}$$

$$-\frac{\partial i(z,t)}{\partial z} = G'\, v(z,t) + C' \frac{\partial v(z,t)}{\partial t}$$

Time domain → Phasor domain

$$v(z,t) = \mathrm{Re}\left[\tilde{V}(z)\exp(j\omega t)\right], \quad i(z,t) = \mathrm{Re}\left[\tilde{I}(z)\exp(j\omega t)\right]$$

Transmission line equations — phasor domain:

$$-\frac{d\tilde{V}(z)}{dz} = (R' + j\omega L')\tilde{I}(z)$$

$$-\frac{d\tilde{I}(z)}{dz} = (G' + j\omega C')\tilde{V}(z)$$

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4.1

Phasor-Domain Equations

Advantages of Phasor-Domain Representation

• Reduction from partial differential equation to ordinary differential equation

— so that it is easier to solve

• Allows generalization: $L' \rightarrow L'(\omega), \quad C' \rightarrow C'(\omega),$

$R' \rightarrow R'(\omega), \quad G' \rightarrow G'(\omega)$

Advantages of Time-Domain Representation

• Allows study of transients

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4.2

Laplace transforms make it possible to study both transients and a generalized response. That lies beyond the scope of this course.

Phasor-Domain Equations

Second-Order Equations:

$$\frac{d^2\tilde{V}(z)}{dz^2} - \gamma^2\tilde{V}(z) = 0, \quad \frac{d^2\tilde{I}(z)}{dz^2} - \gamma^2\tilde{I}(z) = 0$$

with

$$\gamma = \alpha + j\beta = \sqrt{(R' + j\omega L')(G' + j\omega C')}$$

$\gamma =$ complex propagation constant
$\alpha = \mathrm{Re}(\gamma) =$ attenuation constant (Np/m)
$\beta = \mathrm{Im}(\gamma) =$ phase constant or wavenumber (rad/m)

NOTES:
- We pick α and β so that both are positive
- In a passive medium, α is always positive; it can be negative in an active medium

[An active medium, like a laser, has an energy source; a passive medium does not and must always lose energy.]

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4.3

Phasor-Domain Equations

The second-order equations have the general solutions

$$\tilde{V}(z) = V_0^+ \exp(-\gamma z) + V_0^- \exp(\gamma z), \quad \tilde{I}(z) = I_0^+ \exp(-\gamma z) + I_0^- \exp(\gamma z)$$

- $\tilde{V}(z)$ and $\tilde{I}(z)$ are not independent
- V_0^+ and V_0^- are arbitrary constants, while I_0^+ and I_0^- are not
- We may also write:

$$\tilde{I}(z) = \frac{V_0^+}{Z_0} \exp(-\gamma z) - \frac{V_0^-}{Z_0} \exp(\gamma z)$$

where

$$Z_0 = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}}$$

is the characteristic impedance

NOTE: Z_0 (Ulaby et al.'s notation) $\rightarrow Z_C$ (Paul's notation)

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4.4

Note analogy of the phasor arbitrary constants V_0^+ and V_0^- to the time domain arbitrary functions V^+(t) and V^-(t)

Note that V_0^+ corresponds to forward propagation and V_0^- corresponds to backward propagation, as we will show

Phasor-Domain Equations

Returning to the time domain:

We first write

$$V_0^+ = |V_0^+|\exp(j\phi^+), \quad V_0^- = |V_0^-|\exp(j\phi^-)$$

from which we obtain

$$\begin{aligned}
v(z,t) &= \mathrm{Re}\left[\tilde{V}(z)\exp(j\omega t)\right] \\
&= \mathrm{Re}\left\{\left[V_0^+\exp(-\gamma z) + V_0^-\exp(\gamma z)\right]\exp(j\omega t)\right\} \\
&= \mathrm{Re}\left\{|V_0^+|\exp(j\phi^+)\exp(j\omega t)\exp[-(\alpha + j\beta)z]\right. \\
&\qquad \left. + |V_0^-|\exp(j\phi^-)\exp(j\omega t)\exp[(\alpha + j\beta)z]\right\} \\
&= \underbrace{|V_0^+|\exp(-\alpha z)\cos(\omega t - \beta z + \phi^+)}_{\text{forward propagation with attenuation}} + \underbrace{|V_0^-|\exp(\alpha z)\cos(\omega t + \beta z + \phi^-)}_{\text{backward propagation with attenuation}}
\end{aligned}$$

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Phase velocity $= u_{\mathrm{p}} = \omega/\beta$

4.5

Phasor-Domain Equations

Example: Ulaby et al. Exercise 2.4

Question: A two-wire air line has the following parameters: $R' = 0.404$ mΩ/m, $L' = 2.00\ \mu$H/m, $G' = 0$, $C' = 5.56$ pF/m. For operation at 5 kHz, determine (a) the attenuation coefficient α, (b) the wavenumber β, (c) the phase velocity u_p, and the characteristic impedance Z_0.

Answer: $\omega = 2 \times 3.14159 \times (5\times10^3\ \mathrm{s}^{-1}) = 3.14159\times10^4\ \mathrm{s}^{-1}$. $R' + j\omega L' = (4.04000\times10^{-4} + j\times3.14159\times10^4 \times 2.00000\times10^{-6})\ \Omega/\mathrm{m} = (4.04000\times10^{-4} + j\times6.28318\times10^{-2})\ \Omega/\mathrm{m} = 6.28319\times10^{-2} \times \exp(j\times1.56436)\ \Omega/\mathrm{m}$. $G' + j\omega C' = j\times1.74673\times10^{-7}\ \Omega^{-1}/\mathrm{m} = 1.74673\times10^{-7} \times \exp(j\times1.57080)\ \Omega^{-1}/\mathrm{m}$. Note the small difference in phases! Six digits of accuracy are needed to keep three digits in the attenuation coefficient. $\gamma^2 = 1.09750\times10^{-8} \times \exp(j\times3.13156)\ \mathrm{m}^{-2}$, so that $\gamma = 1.04762\times10^{-4} \times \exp(j\times1.56758)\ \mathrm{m}^{-1} = 3.37\times10^{-7} + j\times1.04761\times10^{-4}$, so that

$\alpha = 3.37\times10^{-7}$ Np/m and $\beta = 1.05\times10^{-4}$ rad/m.

We have $u_p = \omega / \beta = (3.142/1.048) \times10^8 = 3.00 \times10^8$ m/s

and $Z_0 = \{(6.283\times10^{-2} / 1.747\times10^{-7}) \exp[j\times(1.56436 - 1.57080)]\}^{1/2} = (600 - j\times1.93)\ \Omega$

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4.6

A better approach to calculating the loss coefficient is to use the expansion, sqrt(1+x) = 1 + (1/2)x - (1/8)x^2 +…, so that (e + jA)jB, where e is small becomes (-1)AB[1 – j(e/A)] and sqrt[(e+jA)jB] = jsqrt(AB)*[1 – j(e/2A)]. The x^2 contribution will affect beta in the 6-th digit!

My answer for the imaginary part of Z_0 differs from Ulaby et al.'s in the second digit. There may be rounding problems in his calculation, or he may only be keeping one significant figure in the loss.

Lossless Transmission Line

Specializing to the case of no loss ($R' = 0$, $G' = 0$):

$$\alpha = 0,\ \beta = \omega\sqrt{L'C'},\ Z_0 = \sqrt{L'/C'},\ u_{\mathrm{p}} = 1/\sqrt{L'C'}$$

We will later show that for any TEM transmission line:

$$L'C' = \mu\varepsilon$$

$\mu =$ magnetic permeability
$\varepsilon =$ electrical permittivity

- For any insulating material that would be used in a transmission line, $\mu = \mu_0 = 4\pi\times10^{-7}$ H/m, where μ_0 is the vacuum permeability.
  — The value for μ_0 is exact and defines the relation between **B** and **H**.
- By contrast, the values for ε differ significantly for different materials. We write: $\varepsilon = \varepsilon_{\mathrm{r}}\,\varepsilon_0$, where ε_{r} is referred to as the relative permittivity, and $\varepsilon_0 = 1/\mu_0 c^2 \simeq 8.854\times10^{-12}$ F/m is the vacuum permittivity

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4.7

When we say "any" TEM guide, we are assuming that losses are low.

Lossless Transmission Line

We now have:

$$\beta = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_0\varepsilon_0\varepsilon_\mathrm{r}} = (\omega / c)\sqrt{\varepsilon_\mathrm{r}}, \quad u_\mathrm{p} = 1/\sqrt{\mu\varepsilon} = c/\sqrt{\varepsilon_\mathrm{r}}$$

In optics, we define an *index of refraction*, $n = c / u_\mathrm{p}$, so that $n = \sqrt{\varepsilon_\mathrm{r}}$

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4.8

Lossless Transmission Line

The impedance relation:

$$Z_0 = \sqrt{L'/C'} = \sqrt{\mu_0/\varepsilon_0\varepsilon_r}K = \sqrt{\mu_0/\varepsilon_0}\left(K/\sqrt{\varepsilon_r}\right);\ 377\left(K/\sqrt{\varepsilon_r}\right)\Omega$$

is a bit more complex. It involves a geometric factor K.

Table of Geometric Factor K

	Coaxial	Two wire	Parallel plane
K	$\frac{1}{2\pi}\ln(b/a)$	$\frac{1}{\pi}\ln\left[(d/2a)+\sqrt{(d/2a)^2-1}\right]$	$\frac{d}{w}$

The vacuum impedance of 377 Ω is a very important number

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- Loads with lower impedance have large magnetic near fields
- Loads with higher impedance have large electric near fields

EMI properties are very different in the two cases!

4.9

At 377 ohms, the magnetic and electric fields are exactly balanced. That is why waves propagate with this impedance.

Conductive shields --- the first thing to try --- work well when the load impedance is large, but work poorly when the load impedance is small (currents are large; charge accumulation is small). One can absorb the field if the metal shield is not too good a conductor, but this approach always fails when the frequency gets really low. Instead, you should use high-permeability material like iron. Motors and loop antennas are examples of low impedance sources that require low impedance shielding.

See, R. Schmitt, ELECTROMAGNETICS EXPLAINED, Chapter 9.

Lossless Transmission Line

Voltage Reflection:

When $\alpha = 0$, our phasor relations become

$$\tilde{V}(z) = V_0^+ \exp(-j\beta z) + V_0^- \exp(j\beta z),$$

$$\tilde{I}(z) = \frac{V_0^+}{Z_0}\exp(-j\beta z) - \frac{V_0^-}{Z_0}\exp(j\beta z)$$

At the load, we have

$$Z_L = \frac{\tilde{V}_L}{\tilde{I}_L}, \quad \text{with} \quad \tilde{V}_L = V_0^+ + V_0^- \text{ and } \tilde{I}_L = \frac{V_0^+}{Z_0} - \frac{V_0^-}{Z_0}$$

which implies

$$V_0^- = \frac{Z_L - Z_0}{Z_L + Z_0} V_0^+ \equiv \Gamma V_0^+$$

and defines the voltage reflection coefficient Γ

Load impedances and reflection coefficients are usually complex!

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4.10

This result is like our previous time domain calculation, except that we now consider cases in which Gamma is complex. Note that in a lossless line with a complex load, the impedance is NEVER matched.

Lossless Transmission Line

Example: Ulaby et al. Exercise 2.7

Question: A 50 Ω lossless transmission line is terminated in a load impedance $Z_L = (30 - j200)\ \Omega$. Calculate the voltage reflection coefficient at the load.

Answer: $\Gamma = (Z_L - Z_0) / (Z_L + Z_0) = (30 - j200 - 50) / (30 - j200 + 50) = (-20 - j200) / (80 - j200) = 201\exp(-j1.67) / 215\exp(-j1.19) = 0.93\exp(-j0.48)$. NOTE: -0.48 rads $= -28^{\circ}$.

Writing $\Gamma = |\Gamma| \exp(j\theta_r)$, we find $\theta_r = -0.48$ rads.

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Lossless Transmission Line

Standing Waves

After using the relation $V_0^- = \Gamma V_0^+$ in the phasor equations,

$$\tilde{V}(z) = V_0^+\left[\exp(-j\beta z) + \Gamma\exp(j\beta z)\right] = V_0^+\left[\exp(-j\beta z) + |\Gamma|\exp(j\theta_\mathrm{r} + j\beta z)\right],$$

$$\tilde{I}(z) = \frac{V_0^+}{Z_0}\left[\exp(-j\beta z) - \Gamma\exp(j\beta z)\right] = \frac{V_0^+}{Z_0}\left[\exp(-j\beta z) - |\Gamma|\exp(j\theta_\mathrm{r} + j\beta z)\right]$$

We then find

$$|\tilde{V}(z)| = |V_0^+|\left[1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta z + \theta_\mathrm{r})\right]^{1/2},$$

so that the amplitude of the voltage varies sinusoidally with z.

- This pattern is called a *standing wave*.
- It comes from the interference of forward- and backward-propagating waves

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4.12

Why do we care about standing waves? One reason, as we will see, is that they allow us to transform impedances.

In contrast to the transient time domain variations that we looked at earlier, this variation is steady-state. Ultimately, the variations come from the sinusoidal variation of the source, as well as from the interference of the forward and backward waves.

Lossless Transmission Line

Standing Waves

- The voltage and current maxima and minima are 180° out of phase
- The amplitude multiplies a $\cos(\omega t)$ dependence with a complicated but periodic z-variation.
- The maxima are spaced $\lambda/2$ apart, and so are the minima.

Ulaby Figure 2-11

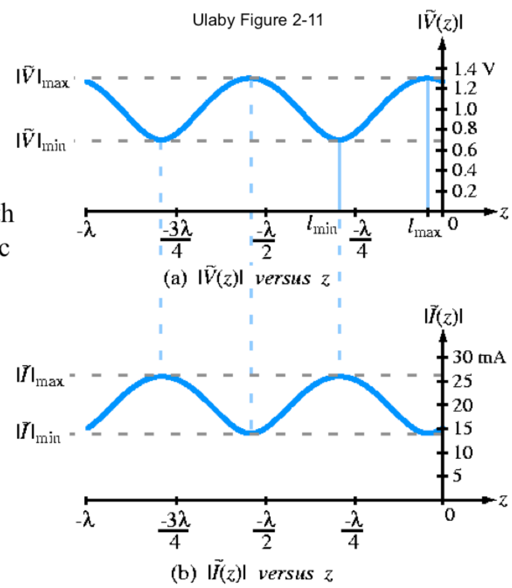

Figure Parameters:
$|\Gamma| = 0.3$, $\theta_r = 30°$
$Z_0 = 50\ \Omega$
$|V_0^+| = 1.3$ V

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4.13

Ulaby et al. take z = 0 AT THE LOAD, not at the generator. So, distance is counted backwards from the load in the figure. This definition is useful when discussing input impedances, which we will do later. Paul essentially does the same thing when discussing phasors by defining d = -(z – L) and then using d when doing most of his calculations.

Lossless Transmission Line

Standing Waves

The total dependence of the standing wave voltage is

$$v(z,t) = |V_0^+| \left[1+|\Gamma|^2 +2|\Gamma|\cos(2\beta z+\theta_r)\right]^{1/2}$$
$$\times\cos\left\{\omega t - \tan^{-1}\left[\frac{\sin(\beta z-\phi^+)-|\Gamma|\sin(\beta z+\theta_r+\phi^+)}{\cos(\beta z-\phi^+)+|\Gamma|\cos(\beta z+\theta_r+\phi^+)}\right]\right\}$$

Ulaby et al. 2010 CD

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4.14

As a consequence, we find that as time increases, the position of the peak appears to slow down and speed up in general, slowing to a stop when Gamma = 1.0 and there are standing waves.

See the animations in Ulaby et al. Module 2.4. Set generator and load impedances to 50 ohms; show impedance matched signal. Increase the impedance to

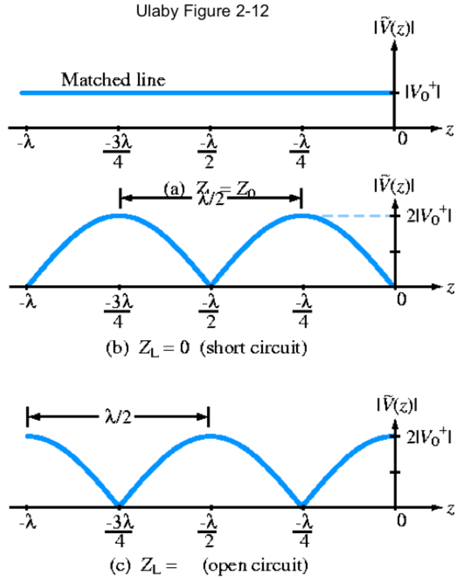
100 ohms, 200 ohms, 500 ohms. Note the increasing “bumpiness” of the voltage and current plots as a function of time.

Lossless Transmission Line

Standing Waves

- For a matched load ($Z_L = Z_0$), there is no standing wave
- For a short-circuited load ($Z_L = 0$) or an open-circuited load ($Z_L = \infty$), there are complete reflections and an oscillation depth of 100%
- With $|\Gamma| = 1$, there are points where the voltage is exactly zero, spaced $\lambda / 2$ apart

Ulaby Figure 2-12



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4.15

For a short circuit, the voltage at the load must be zero. For an open circuit, the reflected voltage equals the incoming voltage, so that the current is zero, and the voltage is twice the forward-going voltage.

When the oscillation depth is 100%, there are points where there is no voltage at all, spaced lambda/2 apart

Lossless Transmission Line

Standing Waves

- We will designate the location of the maxima as $l_{\mathrm{max}} = -z$, so that l_{max} is a positive number (since the load is at $z = 0$).

  $$-z = l_{\mathrm{max}} = \frac{\theta_{\mathrm{r}} + 2n\pi}{2\beta} = \frac{\theta_{\mathrm{r}}\lambda}{4\pi} + \frac{n\lambda}{2}$$

  Only n-values that satisfy $l_{\mathrm{max}} \geq 0$ are allowed
- The voltage standing wave ratio (VSWR or SWR) gives the gives the oscillation depth. It is defined:

  $$S = \frac{|\tilde{V}|_{\mathrm{max}}}{|\tilde{V}|_{\mathrm{min}}} = \frac{1 + |\Gamma|}{1 - |\Gamma|}$$

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Lossless Transmission Line

Example: Ulaby et al. Exercise 2.11

Question: A 140 Ω lossless line is terminated in a load impedance Z_L = (280 + j 182) Ω. If λ = 72 cm, find (a) the reflection coefficient Γ, (b) the VSWR S, (c) the locations of the voltage maxima and minima.

Answer: (a) $\Gamma = (Z_L - Z_0) / (Z_L + Z_0) = (280 + j182 - 140) / (280 + j182 + 140) = 230 \exp(j0.915) / 458 \exp(j0.409) = 0.50 \exp(j0.51)$. NOTE: 0.51 rads = 29°. (b) $S = (1 + |\Gamma|) / (1 - |\Gamma|) = (1 + 0.502) / (1 - 0.502) = 3.0$. (c) $l_{max} = (0.506 \times 72 / 4\pi + n \times 72 / 2)$ cm = 2.9 cm, 39 cm, 75 cm,…; $l_{min} = (0.506 \times 72 / 4\pi + 18 + n \times 72 / 2)$ cm = 21 cm, 57 cm, 93 cm,…

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4.17

Ulaby et al. 2010 module 2.4: lambda = 72 cm → (with epsilon_r = 1.0), frequency = 0.416667E9 Hz. With that, the module gives close to the correct answers.

Ulaby et al. module 2.5 also gives approximate answers, although it is not possible to put in the exact frequency.

Lossless Transmission Line

Input Impedance

With standing waves, the voltage-to-current ratio, which is referred to as the *input impedance*, varies as a function of position

$$Z_{\text{in}}(z) = \frac{\tilde{V}(z)}{\tilde{I}(z)} = \frac{V_0^+\left[\exp(-j\beta z) + \Gamma\exp(j\beta z)\right]}{V_0^+\left[\exp(-j\beta z) - \Gamma\exp(j\beta z)\right]} Z_0 = Z_0 \frac{\left[1+\Gamma\exp(2j\beta z)\right]}{\left[1-\Gamma\exp(2j\beta z)\right]},$$

Of particular interest is the input impedance at the generator, $z = -l$

$$Z_{\text{in}}(-l) = Z_0\left[\frac{1+\Gamma\exp(-2j\beta l)}{1-\Gamma\exp(-2j\beta l)}\right]$$

$$= Z_0\left(\frac{Z_{\text{L}}\cos\beta l + jZ_0\sin\beta l}{Z_0\cos\beta l + jZ_{\text{L}}\sin\beta l}\right) = Z_0\left(\frac{Z_{\text{L}} + jZ_0\tan\beta l}{Z_0 + jZ_{\text{L}}\tan\beta l}\right)$$

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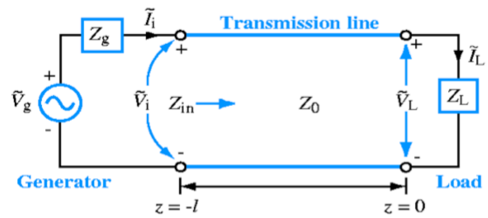
4.18

Lossless Transmission Line

Input Impedance

From the standpoint of the generator circuit, the transmission line appears as an input impedance $Z_{in} \equiv Z_{in}(-l)$, so that

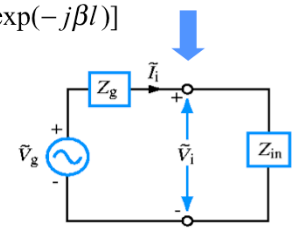
$$\tilde{V}_i = \tilde{I}_i Z_{in} = \frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}}$$



From the standpoint of the transmission line

$$\tilde{V}_i = \tilde{V}(-l) = V_0^+[\exp(j\beta l) + \Gamma \exp(-j\beta l)]$$

Combining these relations…



Ulaby Figure 2-14

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4.19

Lossless Transmission Line

Input Impedance

…we conclude

$$V_0^+ = \left(\frac{\tilde{V}_g Z_{in}}{Z_g + Z_{in}} \right) \left(\frac{1}{\exp(j\beta l) + \Gamma \exp(-j\beta l)} \right)$$

Thus, we can now relate the wave parameters,

$$V_0^+, \quad V_0^- (= \Gamma V_0^+), \quad I_0^+ (= V_0^+ / Z_0), \quad I_0^- (= -V_0^- / Z_0 = -\Gamma V_0^+ / Z_0),$$

to the transmission line parameters

$$Z_g, \quad Z_L, \quad Z_0 = \sqrt{L'/C'}, \quad u_p = 1/\sqrt{L'C'}, \quad l$$

and the input parameters

$$\tilde{V}_g, \quad f = \omega / 2\pi$$

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4.20

Lossless Transmission Line

Paul Example 6.7 (extended)

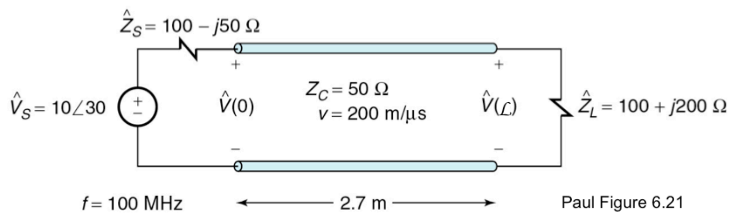
Question: A line 2.7 m in length is excited by a 100 MHz source as shown in the figure. Determine the source and load voltages. Determine the voltages and the current everywhere in the transmission line

Answer: We will work in Ulaby et al.'s notation, and our first task is to translate Paul's problem specification into that notation:

$$\hat{V}_S \to \tilde{V}_g = 10\exp(j30^o) = 10\exp(j\pi/6) = 10\exp(j0.524) = (8.66 + 5.00j)\text{ V}$$

$$\hat{V}(0) \to \tilde{V}_i = \tilde{V}(-l); \quad \hat{V}(\mathcal{L}) \to \tilde{V}_L = \tilde{V}(0); \quad v \to u_p = 200\text{ m}/\mu\text{s} = 2\times10^8\text{ m/s}$$

$$\hat{Z}_S \to Z_g = 100 - j50\,\Omega; \quad \hat{Z}_L \to Z_L = 100 + j200\,\Omega; \quad Z_C \to Z_0 = 50\,\Omega;$$



Paul Figure 6.21

4.21

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Lossless Transmission Line

Paul Example 6.7

Answer (continued):*

(1) *Find the reflection coefficient:*

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{50 + j200}{150 + j200} = \frac{206\exp(j1.326)}{250\exp(j0.927)} = 0.825\exp(j0.399)$$

(2) *Find the propagation factor*

$$\beta l = (2\pi f / u_p) l = (6.28\times 10^8 / 2\times 10^8)\times 2.7 = 8.48;$$

$$\exp(j\beta l) = \exp(j0.7\pi) = \exp(j2.20);$$

$$\exp(-j\beta l) = \exp(-j0.7\pi) = \exp(-j2.20);$$

$$\exp(-j2\beta l) = \exp(-j1.4\pi) = \exp(-j4.40) = \exp(j1.88)$$

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*NOTE: I am using MATLAB to calculate values. So, the calculations are good to 15 places, although I only report three.

4.22

This example was just sufficiently complex that I was better off using MATLAB, rather than my hand calculator.

Lossless Transmission Line

Paul Example 6.7

Answer (continued):

(3) *Find the input impedance:*

$$\Gamma \exp(-j2\beta l) = 0.825 \exp[j(0.399 + 1.885)]$$
$$= 0.825 \exp(j2.283) = -0.539 + j0.624;$$
$$1 + \Gamma \exp(-j2\beta l) = 0.461 + j0.624;$$
$$1 - \Gamma \exp(-j2\beta l) = 1.539 - j0.624;$$

$$Z_{\text{in}} = Z_0 \frac{1 + \Gamma \exp(-j2\beta l)}{1 - \Gamma \exp(-j2\beta l)} = 50 \frac{0.461 + j0.624}{1.539 - j0.624} = 50 \frac{0.776 \exp(j0.935)}{1.661 \exp(-j0.385)}$$
$$= 23.35 \exp(j1.320) = 5.80 + j22.62$$

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Lossless Transmission Line

Paul Example 6.7

Answer (continued):

(4) *Find the input voltage:*

$$Z_{\mathrm{g}} + Z_{\mathrm{in}} = (100 - j50) + (5.80 + j22.6)$$
$$= 106 - j27.4 = 109\exp(-j0.253)$$

$$\tilde{V}_{\mathrm{i}} = \tilde{V}_{\mathrm{g}} \frac{Z_{\mathrm{in}}}{Z_{\mathrm{g}} + Z_{\mathrm{in}}} = 10\exp(j0.524)\frac{23.4\exp(j1.320)}{109\exp(-j0.253)}$$
$$= 2.14\exp(j2.10) = 2.14\angle 120^{\circ}$$

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Lossless Transmission Line

Paul Example 6.7

Answer (continued):

(5) *Find the load voltage:*

$$V_0^+ = \tilde{V}_\mathrm{i}\frac{\exp(-j\beta l)}{1+\Gamma\exp(-j2\beta l)} = 2.14\exp(j2.10)\frac{\exp(-j2.20)}{0.776\exp(j0.935)}$$
$$= 2.75\exp(-j1.037)$$

$$1+\Gamma = 1 + (0.760 + j0.320) = 1.79\exp(j0.180)$$

$$\tilde{V}_\mathrm{L} = V_0^+(1+\Gamma) = [2.75\exp(-j1.037)][1.79\exp(j0.180)]$$
$$= 4.93\exp(-j0.857) = 4.93\angle -49^\mathrm{o}$$

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4.25

Lossless Transmission Line

Paul Example 6.7

Answer (continued):

(6) *Write the time domain voltages:*

$$v_{\mathrm{i}}(t) = \mathrm{Re}\left[\tilde{V}_{\mathrm{i}} \exp(j\omega t)\right] = 2.14\cos(6.28\times 10^{8} t + 120^{\mathrm{o}})$$

$$v_{\mathrm{L}}(t) = \mathrm{Re}\left[\tilde{V}_{\mathrm{L}} \exp(j\omega t)\right] = 4.93\cos(6.28\times 10^{8} t - 49^{\mathrm{o}})$$

(7) *Find the phasor domain voltage and current in the transmission line*

$$\begin{aligned}\tilde{V}(z) &= V_0^{+}\exp(-j\beta z) + \Gamma V_0^{+}\exp(j\beta z)\\ &= 2.75\exp(-j\beta z - j1.037) + 2.27\exp(j\beta z - j0.638)\ \ (\mathrm{V})\end{aligned}$$

$$\begin{aligned}\tilde{I}(z) &= \frac{V_0^{+}}{Z_0}\exp(-j\beta z) - \frac{\Gamma V_0^{+}}{Z_0}\exp(j\beta z)\\ &= 55.1\exp(-j\beta z - j1.037) + 45.4\exp(j\beta z + j2.503)\ \ (\mathrm{mA})\end{aligned}$$

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Ulaby et al. have additional examples. See Ulaby et al. 2010, example 2-7.

Lossless Transmission Line

Paul Example 6.7

Answer (continued):

(8) *Find the time domain voltage and current in the transmission line:*

$$v(z,t) = \mathrm{Re}\left[2.75\exp(j\omega t - j\beta z - j1.037) + 2.27\exp(j\omega t + j\beta z - j0.638)\right]$$

$$= 2.75\cos(\omega t - \beta z - 1.037) + 2.27\cos(\omega t + \beta z - 0.638)\ \text{(V)}$$

$$i(z,t) = \mathrm{Re}\left[55.1\exp(j\omega t - j\beta z - j1.037) + 45.4\exp(j\omega t + j\beta z + j2.503)\right]$$

$$= 55.1\cos(\omega t - \beta z - 1.037) + 45.4\cos(\omega t + \beta z + 2.503)\ \text{(mA)}$$

(9) *Checks:*

$$v(0,t) = 2.75\cos(\omega t - 1.037) + 2.27\cos(\omega t - 0.638)$$

$$= 4.93\cos(\omega t - 0.857)\ \text{(V)}$$

$$v(-l,t) = 2.75\cos(\omega t + 8.48 - 1.037) + 2.27\cos(\omega t - 8.48 - 0.638)$$

$$= 2.14\cos(\omega t + 2.097)\ \text{(V)}$$

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Show this example with Ulaby et al. 2010 Module 2.4

Inputs: Length units = meters, Z_0 = 50, f = 1E8, epsilon_r = 2.25, l = 2.7; Z_L = 100 + j200; V_g = 8.66 + j5.00, Z_g = 100 – j50

Outputs: Z(d) = 223 exp(j 1.107); Gamma(d) = 0.76 + j0.32 = 0.825 exp(j 0.399); V(d) = 4.93 exp(-j 0.857), I(d) = 0.220 exp(-j1.965)