Laplace Transforms IV: Other Properties of the Laplace Transform

- **Initial Value Theorem**

$$\lim_{s\to\infty}\left(sF(s)\right) = f\left(0^{+}\right)$$

- **Final Value Theorem**

$$\lim_{s\to 0}\left(sF(s)\right) = \lim_{t\to\infty} f(t)$$

- **The "zero s" theorem**

$$\int_{-\infty}^{\infty} x(\tau)\,d\tau = \int_{-\infty}^{\infty} x(\tau)e^{0\tau}\,d\tau = \left(\int_{-\infty}^{\infty} x(\tau)e^{s\tau}\,d\tau\right)_{s=0} = X(0)$$

The Convolution Theorem

- **One of the primary application of the Laplace (and later Fourier) Domain(s) is the Convolution Theorem**
- **What is the LT of the convolution?**

$$\mathcal{L}\left(x*h\right)=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}x(\tau)h(t-\tau)\,d\tau\right)e^{-st}\,dt$$

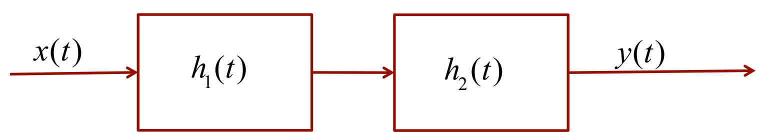
$$=\int_{-\infty}^{\infty}x(\tau)\underbrace{\left(\int_{-\infty}^{\infty}h(t-\tau)e^{-st}\,dt\right)}_{\text{time shift property!}}d\tau=\int_{-\infty}^{\infty}x(\tau)\left(e^{-s\tau}H(s)\right)d\tau$$

$$=H(s)\int_{-\infty}^{\infty}x(\tau)e^{-s\tau}\,d\tau=H(s)X(s)=X(s)H(s)\quad(!!)$$

$$R\supset R_X\cap R_H$$

So what?

- **Remember our concatenation of systems?**



$$y(t)=x*h=x*(h_1*h_2)$$

$$H(s)=\mathcal{L}\left(h_1*h_2\right)=H_1(s)H_2(s)$$

$$Y(s)=\mathcal{L}\left(x*h\right)=X(s)H(s)=X(s)H_1(s)H_2(s)$$

- **A convolution in the time domain is equivalent to a multiplication in the frequency domain…**
- **…and *vice versa***

One sided Laplace Transforms

- **For *causal* systems, we have** $h(t) = 0, t < 0$
- **...then** $\mathcal{L}\left(h(t)\right) = \int_{-\infty}^{\infty} h(t)e^{-st}\,dt = \int_{0}^{\infty} h(t)e^{-st}\,dt$
- **This is so common, that it's called the "one sided Laplace Transform**
- **What's the RoC?**
- **...no strips, no left side!**
- **This is the form your book (and many other texts) use!**

Properties

- **The same properties apply, except for the time derivative property**
- **Differentiation (1 sided)**

$$\mathcal{L}\left(\frac{dx}{dt}\right) = sX(s) - x(0)$$

$$\frac{1}{s+a}, a < 0, \mathrm{Re}\left[s\right] < a, -e^{-at}u(-t) \to 0 \text{ as } t \to -\infty$$

$$\mathcal{L}\left(\frac{d^2x}{dt^2}\right) = s^2X(s) - sx(0) - \left.\frac{dx}{dt}\right|_{t=0}$$

Stability

- Stability is generally "bounded input, bounded output" stability
- If $|x(t)| < C_1$ then $|y(t)| < C_2$
- Is our causal integrator $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$ stable?
- To answer the question, we need to either prove that the condition holds or find a function where it doesn't hold
- Let $x(t) = u(t)$, then $y(t) = \int_{0}^{t} 1 \; d\tau = t$
- The input is bounded, but the output is not, because for any $C > 0$ I only need wait for $t > C$ for the output to exceed it.
- It is ***not*** BIBO stable!

What about the s-domain

- Where are the poles of the integrator?

$$\mathcal{L}\left(\int_{-\infty}^{t} x(t)\,dt \right) = \frac{1}{s} \text{ for } \mathrm{Re}\left[s\right] > 0$$

- What about $\mathcal{L}\left(e^{at}u(t)\right) = H(s) = \dfrac{1}{s+a},\ a > 0,\ \mathrm{Re}\left[s\right] > a$
- This also "blows up"

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)e^{a\tau}u(\tau)\,d\tau = \int_{0}^{\infty} x(t-\tau)e^{a\tau}u(\tau)\,d\tau$$

$$|x(t)| \le C, \text{ so } y(t) \le \int_{0}^{\infty} C_1 e^{a\tau}\,d\tau = \infty \Rightarrow \text{not stable}$$

- A causal system must have its poles in the left half plane!

Anti causal system

- **The reverse is true for anti causal systems**

$$\frac{1}{s+a}, a>0, \mathrm{Re}\left[s\right] < -a \Rightarrow -e^{-at}u(-t) \to \infty \text{ as } t \to -\infty$$

$$\frac{1}{s+a}, a<0, \mathrm{Re}\left[s\right] < -a \Rightarrow -e^{-at}u(-t) \to 0 \text{ as } t \to -\infty$$

- **And for two-sided systems, both terms must be stable.**
- **We will see later that the RoC must include the** $j\omega$ **axis for the system to be stable.**

Examples

- **Find the Laplace Transform of the unit amplitude pulse of duration T by two methods.**
- **Find the Laplace transform of a causal unit ramp,**
  $y(t) = tu(t)$
- **Find the response of** $h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$ **to a causal unit ramp**
- **Find the initial and final values of that response.**