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MATH 407: HW 08

3.4 **4** Show that $\mathbb{Z}_5^\times$ is not isomorphic to $\mathbb{Z}_8^\times$ by showing that the first group has an element of order 4 but the second group does not

The elements in each of the groups

$$\{[1],[2],[3],[4]\} \in \mathbb{Z}_5^\times$$
$$\{[1],[3],[5],[7]\} \in \mathbb{Z}_8^\times$$

For $\mathbb{Z}_5^\times$

$$[5]^2 = [1]$$

Therefore, $o()$

$\square$

7 Let G be a group. Show that the group $(G, *)$ defined in Exercise 3 of Section 3. 1 is isomorphic to G.

$\square$

11 Let G be the set of all matrices in $GL_2(\mathbb{Z}_3)$ of the form $\begin{bmatrix} m & b \\ 0 & 1 \end{bmatrix}$. That is, $m, b \in \mathbb{Z}_3$ and $m \neq [0]_3$. Show that G is a subgroup of $GL_2(\mathbb{Z}_3)$ that is isomorphic to S_3.

$\square$

14 Let $G = \{x \in \mathbb{R} \mid x > 0$ and $x \neq 1\}$, and define $*$ on G by $a * b = a^{\ln b}$. Show that G is isomorphic to the multiplicative group $\mathbb{R}^\times$. (See Exercise 9 of Section 3.1.)

$\square$

17 Let $\phi : G_1 \to G_2$ be a group isomorphism. Prove that if H is asubgroup of G_1, then $\phi(H) = \{y \in G_2 \mid y = \phi(h)$ for some $h \in H\}$ is a subgroup of G_2.

Since $\phi : G_1 \to G_2$ is a group isomorphism, $\phi(e_1) = e_2$

Since H is a subgroup,

$$\begin{aligned} e_1 &\in H \\ \Rightarrow e_2 &\in \phi(H) \end{aligned}$$

A non-empty set G is a subgroup if $xy^{-1} \in G$, $\forall\, x, y \in G$

Let $x, y \in \phi(H)$

Then, there exists $h_1, h_2 \in H$, such that

$$\begin{aligned} \phi(h_1) &= x \\ \phi(h_2) &= y \end{aligned}$$

Also, since ϕ is homomorphic,

$$\begin{aligned} \phi(h_2^{-1}) &= (\phi(h_2))^{-1} \\ &= y^{-1} \\ \phi(h_1 h_2^{-1}) &= \phi(h_1)\phi(h_2^{-1}) \\ &= xy^{-1} \end{aligned}$$

Since H is a subgroup, $h_1 h_2^{-1} \in H$, $\forall\, h_1, h_2 \in H$

Therefore,

$$\begin{aligned} \phi(h_1 h_2^{-1}) &= xy^{-1} \\ &\in \phi(H) \end{aligned}$$

That is, $\phi(h_1 h_2^{-1}) \in \phi(H)$, $\forall\, x, y \in \phi(H)$ □

24 Let $G = \mathbb{R} - -1$. Define $*$ on G by $a * b = a + b + ab$. Show that G is isomorphic to the multiplicative group $\mathbb{R}^\times$. (See Exercise 13 of Section 3.1.)

Hint: Remember that an isomorphism maps identity to identity. Use this fact to help find the necessary mapping.

□

26 Let G_1 and G_2 be groups. A function from G into G_2 that preserves products but is not necessarily a one-to-one correspondence will be called a group homomorphism, from the Greek word *homos* meaning same. Show that $\phi : \mathrm{GL}_2(\mathbb{R}) \to \mathbb{R}^{\times}$ defined by $\phi(A) = \det(A)$ for all matrices $A \in \mathrm{GL}_2(\mathbb{R})$ is a group homomorphism.

□

3.5 **2** Let G be a group and let $a \in G$ be an element of order 30. List the powers of a that have order 2, order 3 or order 5.

□

3 Give the subgroup diagrams of the following groups.

a $\mathbb{Z}_{24}$

□

b $\mathbb{Z}_{36}$

□

10 Find all cyclic subgroups of $\mathbb{Z}_6 \times \mathbb{Z}_3$

□

12 Let a, b be positive integers, and let $d = \gcd(a, b)$ and $m = \mathrm{lcm}(a, b)$. Use Proposition 3.5.5 to prove that $\mathbb{Z}_a \times \mathbb{Z}_b \cong \mathbb{Z}_d \times \mathbb{Z}_m$

□

13 Show that in a finite cyclic group of order n, the equation $x^m = e$ has exactly m solutions, for each positive integer m that is a divisor of n.

□

17 Let G be the set of all 3×3 matrices of the form $\begin{bmatrix} 1 & b & c \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$.

a Show that if $a, b, c \in \mathbb{Z}_3$, the G is a group with exponent 3.

□

b Show that if $a, b, c \in \mathbb{Z}_2$, the G is a group with exponent 4.

$\square$

19 Let $n = 2^k$ for $k > 2$. Prove that $\mathbb{Z}_n^\times$ is not cyclic.

Hint: Show that ± 1 satisfy the equation $x^2 = 1$, and that this is impossible in any cyclic group.

$\square$