**CMPE 330 Spring 2017 Second Midterm Examination**

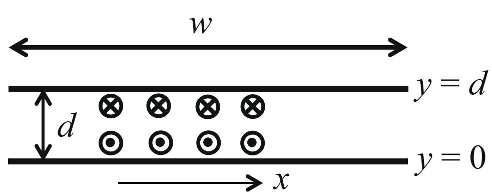
*(Each problem is worth 10 points)*

1. In two dimensions, we relate the Cartesian coordinates $(x, y)$ to the polar coordinates $(r, \phi)$ by the relations $r = (x^2 + y^2)^{1/2}$, $\phi = \tan^{-1}(y/x)$.
   a. What are $x(r, \phi)$ and $y(r, \phi)$?
   b. Write $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ as functions of $r$, $\phi$, $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\phi}}$.

2. Give the fundamental definition of the gradient operator and use the definition to give the form of the gradient operator in Cartesian, cylindrical, and spherical coordinates.

3. Consider a coaxial cable with inner radius $a$ and outer radius $b$. Between the two conductors, the material is characterized by permittivity $\epsilon$, permeability $\mu$, and conductivity $\sigma$. Determine the capacitance per unit length $C'$. What are the inductance per unit length $L'$ and the conductance per unit length $G'$?

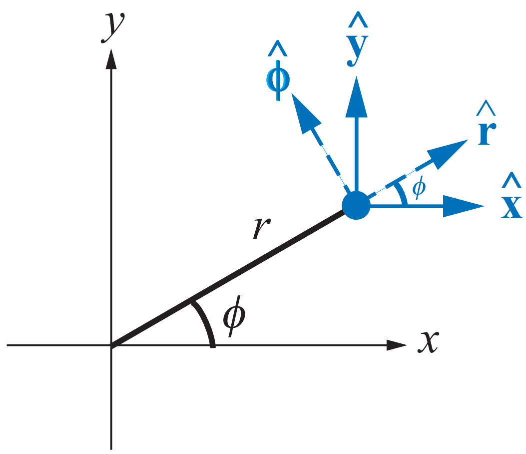
4. Consider a parallel-plate microstrip line of width $w$ and a plate separation $d$ with $d \ll w$. Suppose that the width is in the $x$-direction, the first plate is at $y = 0$, and the second plate is at $y = d$. A current $I$ flows in the $+z$-direction on the plate at $y = 0$ and in the $-z$-direction in the plate at $y = d$. The geometry is shown below. Use Ampere's law to show that the magnetic flux between the plates is given by $\mathbf{B} = -\hat{\mathbf{x}}(\mu I/w)$. Show that the energy per unit length that is contained between the plates is given by $W'_{\mathrm{m}} = (\mu/2)(d/w)I^2$.




5. Consider an infinite planar interface at $z = 0$ that separates a medium with permittivity and permeability $\epsilon_1$ and $\mu_1$ when $z > 0$ and with permittivity and permeability $\epsilon_2$ and $\mu_2$ when $z < 0$. Both media are non-conducting. Derive the relations between $\mathbf{E}$ and $\mathbf{D}$ and between $\mathbf{H}$ and $\mathbf{B}$ in the upper and lower layers.

## Second Midterm Examination Solutions

1. We show the geometry in the figure below:




   a. We have $x = r\cos\phi$ and $y = r\sin\phi$.

   b. From the figure, we infer

$$\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi,$$
$$\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi.$$

2. The fundamental definition of the gradient operator is that

$$\nabla V(\mathbf{R}) = \hat{\mathbf{n}} \lim_{\Delta\mathbf{l}\to 0}\left[\frac{V(\mathbf{R}+\Delta\mathbf{l}) - V(\mathbf{R})}{\Delta\mathbf{l}}\right]_{\max}, \qquad \hat{\mathbf{n}} = \Delta\mathbf{l}/|\Delta\mathbf{l}|.$$

From this general definition, we infer that

$$\nabla V(\mathbf{R}) = \hat{\mathbf{x}}_1\frac{\partial V}{\partial l_1} + \hat{\mathbf{x}}_2\frac{\partial V}{\partial l_2} + \hat{\mathbf{x}}_3\frac{\partial V}{\partial l_3} = \hat{\mathbf{x}}_1\frac{1}{h_1}\frac{\partial V}{\partial x_1} + \hat{\mathbf{x}}_2\frac{1}{h_2}\frac{\partial V}{\partial x_2} + \hat{\mathbf{x}}_3\frac{1}{h_3}\frac{\partial V}{\partial x_3},$$

where $x_1$, $x_2$, and $x_3$ are three coordinates in a general orthogonal coordinate system, $\hat{\mathbf{x}}_1$, $\hat{\mathbf{x}}_2$, and $\hat{\mathbf{x}}_3$ are the unit vectors, and $dl_1 = h_1 dx_1$, $dl_2 = h_2 dx_2$, and $dl_3 = h_3 dx_3$ are the differential lengths. In the Cartesian coordinate system, we have $x_1 = x$, $x_2 = y$, $x_3 = z$ and $h_1 = h_2 = h_3 = 1$, so that

$$\nabla V = \hat{\mathbf{x}}\frac{\partial V}{\partial x} + \hat{\mathbf{y}}\frac{\partial V}{\partial y} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}.$$

In the cylindrical coordinate system, we have $x_1 = r$, $x_2 = \phi$, $x_3 = z$, and $h_1 = 1$, $h_2 = r$, $h_3 = 1$, so that

$$\nabla V = \hat{\mathbf{r}}\frac{\partial V}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial V}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}.$$

In the spherical coordinate system, we have $x_1 = R$, $x_2 = \theta$, $x_3 = \phi$, $h_1 = 1$, $h_2 = R$, $h_3 = R\sin\theta$, so that

$$\nabla V = \hat{\mathbf{R}}\frac{\partial V}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial V}{\partial \theta} + \hat{\boldsymbol{\phi}}\frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}.$$

3. From Gauss's law, we have that the field due a surface charge $\rho_{\mathrm{S}}$ on a long cylinder of radius $a$ and length $L$ is given by $\mathbf{E} = \hat{\mathbf{r}}(\rho_S/\epsilon)(a/r)$, so that the voltage difference between the inner and outer radius is $V = (\rho_{\mathrm{S}}/\epsilon)a\ln(b/a)$. The charge Q is given by $Q = 2\pi\rho_{\mathrm{S}}aL$. We thus have that the capacitance per unit length is given by $C' = C/L = Q/VL = (2\pi\epsilon)/\ln(b/a)$. Recalling that $C'L' = \mu\epsilon$, we find that $L' = (\mu/2\pi)\ln(b/a)$. Recalling that $G' = (\sigma/\epsilon)C'$, we find $G' = (2\pi\sigma)/\ln(b/a)$.

4. Ampere's law states that $\oint \mathbf{B}\cdot d\mathbf{l} = \mu I$, where $I$ is the current flowing through any closed loop. Taking a loop that encloses a portion $\Delta x = l$ of the lower plate, recalling that the field below the lower plate is zero, and noting that the current flowing through that portion is given by $Il/w$, we have $Bl = \mu Il/w$. From the right-hand rule, the field is oriented in the $-x$ direction, so that the field is given by $\mathbf{B} = -\hat{\mathbf{x}}\mu(I/w)$. It follows that $\mathbf{H} = -\hat{\mathbf{x}}(I/w)$, so that the energy density is given by $w_{\mathrm{m}} = (1/2)\mathbf{B}\cdot\mathbf{H} = (\mu/2)(I^2/w^2)$. Since the area between the plates is given by $wd$, we conclude that the energy per unit length is given by $W'_{\mathrm{m}} = (\mu/2)(d/w)I^2$.

5. We first take a small rectangular loop in the $x$-$z$ plane of length $\Delta l$ in the $x$-direction and length $\Delta h$ in the $z$-direction, where $\Delta h \ll \Delta l$. Tracing the field around the loop, we find $\oint \mathbf{E}\cdot d\mathbf{l} = (E_{2x} - E_{1x})\Delta l = 0$, which implies that $E_{2x} = E_{1x}$. We find in a similar way that $E_{2y} = E_{1y}$. Next we consider a cylindrical volume of height $\Delta h$ in the $z$-direction and radius $\Delta r$. Since there is no charge, we find $\int \mathbf{D}\cdot d\mathbf{s} = \hat{\mathbf{z}}\cdot(\mathbf{D}_1 - \mathbf{D}_2) = 0$, which implies $D_{2z} = D_{1z}$. These relations in turn imply $D_{1x}/\epsilon_1 = D_{2x}/\epsilon_2$, $D_{1y}/\epsilon_1 = D_{2y}/\epsilon_2$, and $\epsilon_1 E_{1z} = \epsilon_2 E_{2z}$.

   The relations between $\mathbf{B}$ and $\mathbf{H}$ are similarly obtained. Using a small rectangular loop and the relation $\oint \mathbf{H}\cdot d\mathbf{l} = 0$, we find $H_{2x} = H_{1x}$ and $H_{2y} = H_{1y}$. Using a small cylinder and the relation $\int \mathbf{B}\cdot\mathbf{s} = 0$, we find $B_{2z} = B_{1z}$. From these relationships, we find in turn, $B_{2x}/\mu_2 = B_{1x}/\mu_1$, $B_{2y}/\mu_2 = B_{1y}/\mu_2$, and $\mu_2 H_{2z} = \mu_1 H_{1z}$.