HW#6 Solutions

Problem 1.

(a) Let A be the event that he got a total of three pens. The event A corresponds to getting 1 pen in one of the trips and 2 in the other, or 3 pens in the first trip. So

$$\mathbf{P}(A) = \frac{1}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3} + \frac{1}{3} = \frac{5}{9}.$$

(b) Let B be the event that he visited the supply room twice on the given day. Then,

$$\mathbf{P}(B \,|\, A) = \frac{\mathbf{P}(B \cap A)}{\mathbf{P}(A)} = \frac{(1/3)(1/3) + (1/3)(1/3)}{(5/9)} = \frac{2}{5}.$$

(c) We have

$$\mathbf{E}[N] = \sum_{n=2}^{5} n\mathbf{P}_N(n) = 2\cdot\frac{1}{9} + 3\cdot\frac{5}{9} + 4\cdot\frac{2}{9} + 5\cdot\frac{1}{9} = \frac{10}{3},$$

and

$$\begin{aligned}\mathbf{E}[N \,|\, C] &= \sum_{n=4}^{5} n\mathbf{P}_{N\,|\,C}(n)\\ &= 4\cdot\mathbf{P}(N = 4 \,|\, N > 3) + 5\cdot\mathbf{P}(N = 5 \,|\, N > 3)\\ &= 4\cdot\frac{2/9}{(2/9)+(1/9)} + 5\cdot\frac{1/9}{(2/9)+(1/9)}\\ &= \frac{13}{3}.\end{aligned}$$

(d) We have

$$\begin{aligned}\mathbf{E}[N^2 \,|\, C] &= \sum_{n=4}^{5} n^2\mathbf{P}_{N\,|\,C}(n)\\ &= 4^2\cdot\mathbf{P}(N = 4 \,|\, N > 3) + 5^2\cdot\mathbf{P}(N = 5 \,|\, N > 3)\\ &= 4^2\cdot\frac{2/9}{(2/9)+(1/9)} + 5^2\cdot\frac{1/9}{(2/9)+(1/9)}\\ &= 19.\end{aligned}$$

Uisng also the result from part (c), $\mathbf{E}[N \mid C] = 13/3$, we obtain

$$\sigma^2_{N \mid C} = 19 - \left(\frac{13}{3}\right)^2 = 0.2222.$$

(e) Let C_i be the event that he gets more that three pens on the ith day. Noting that the C_i's are independent, we obtain

$$\mathbf{P}\left(\bigcap_{i=1}^{16} C_i\right) = \prod_{i=1}^{16} \mathbf{P}(C_i) = \left(\frac{2}{9} + \frac{1}{9}\right)^{16} = 3^{-16}.$$

(f) Let N_i be the total number of pens he gets in the ith day, let $X = \sum_{i=1}^{16} N_i$, and let $D = \bigcap_{i=1}^{16} C_i$. Noting that conditional on D, the N_i's are still independent and that $p_{N_i \mid D} = p_{N_i \mid C_i}$, we obtain

$$\sigma^2_{X \mid D} = \sum_{i=1}^{16} \sigma^2_{N_i \mid D} = \sum_{i=1}^{16} \sigma^2_{N_i \mid C_i} = 16 \cdot 0.2222 = 3.5552.$$

Problem 2.

Let A be the event that your detection programs lead you to the correct conclusion about your computer. Let V be the event that your computer has a virus, and let V^c be the event that your computer does not have a virus. We have

$$\mathbf{P}(A) = \mathbf{P}(V)\mathbf{P}(A \mid V) + \mathbf{P}(V^c)\mathbf{P}(A \mid V^c),$$

and $\mathbf{P}(A \mid V)$ and $\mathbf{P}(A \mid V^c)$ can be found using the binomial PMF. Thus we have

$$\begin{aligned}
\mathbf{P}(A \mid V) &= \binom{12}{9} \cdot (0.8)^9 \cdot (0.2)^3 + \binom{12}{10} \cdot (0.8)^{10} \cdot (0.2)^2 \\
&\quad + \binom{12}{11} \cdot (0.8)^{11} \cdot (0.2)^1 + \binom{12}{12} \cdot (0.8)^{12} \cdot (0.2)^0 \\
&= 0.7899.
\end{aligned}$$

using a similar calculation, we find that $\mathbf{P}(A \mid V^c) = 0.9742$, so that

$$\mathbf{P}(A) = 0.65 \cdot 0.7899 + 0.35 \cdot 0.9742 = 0.8544.$$

Problem 3.

(a) Let L_i be the event that Joe played the lottery on week i, and let W_i be the event that he won on week i. The desired probability is

$$\mathbf{P}(L_i \,|\, W_i^c) = \frac{\mathbf{P}(W_i^c \,|\, L_i)\mathbf{P}(L_i)}{\mathbf{P}(W_i^c \,|\, L_i)\mathbf{P}(L_i) + \mathbf{P}(W_i^c \,|\, L_i^c)\mathbf{P}(L_i^c)} = \frac{(1-q)p}{(1-q)p + 1\cdot(1-p)} = \frac{p-pq}{1-pq}.$$

(b) Conditioned on X, the random variable Y is binomial

$$p_{Y\,|\,X}(y\,|\,x) = \begin{cases} \binom{x}{y} q^y (1-q)^{(x-y)} & \text{if } 0 \le y \le x, \\ 0 & \text{otherwise.} \end{cases}$$

(c) Since X has a binomial PMF, we have

$$\begin{aligned} p_{X,Y}(x,y) &= p_{Y\,|\,X}(y\,|\,x) p_X(x) \\ &= \begin{cases} \binom{x}{y} q^y (1-q)^{(x-y)} \binom{n}{x} p^x (1-p)^{(n-x)} & \text{if } 0 \le y \le x \le n, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(d) Using the result from part (c), we could compute the marginal p_Y using the formula

$$p_Y(y) = \sum_{x=y}^{n} p_{X,Y}(x,y),$$

but the algebra is messy. An easier method is based on the fact that Y is just the sum of n independent Bernoulli random variables, each having a probability pq of being 1. Therefore Y has a binomial PMF:

$$p_Y(y) = \begin{cases} \binom{n}{y} (pq)^y (1-pq)^{(n-y)} & \text{if } 0 \le y \le n, \\ 0 & \text{otherwise.} \end{cases}$$

(e) We have

$$\begin{aligned} p_{X\,|\,Y}(x\,|\,y) &= \frac{p_{X,Y}(x,y)}{p_Y(y)} \\ &= \begin{cases} \dfrac{\binom{x}{y} q^y (1-q)^{(x-y)} \binom{n}{x} p^x (1-p)^{(n-x)}}{\binom{n}{y} (pq)^y (1-pq)^{(n-y)}} & 0 \le y \le x \le n, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

(f) From part (a), we know the probability $\mathbf{P}(L_i \mid W_i^c)$ that Joe played the lottery on week i given that he did not win in that week. For each of the $n-y$ weeks when Joe did not win, there are $x-y$ weeks when he played. Thus, X conditioned on $Y=y$ is binomial with parameters $n-y$ and $\mathbf{P}(L_i \mid W_i^c) = (p-pq)/(1-pq)$:

$$p_{X\,|\,Y}(x\,|\,y) = \begin{cases} \binom{n-y}{x-y}\left(\frac{p-pq}{1-pq}\right)^{x-y}\left(1-\frac{p-pq}{1-pq}\right)^{(x-y)}\binom{n}{x} & 0 \le y \le x \le n, \\ 0 & \text{otherwise.} \end{cases}$$

After some algebraic manipulation, the answer to (e) can be shown equal to the above.

Problem 4.

Let X be the reward. The probability that $X=10$ is $\frac{1}{12}$, the probability that $X=6$ is $\frac{1}{6}$, and the probability that $X=2$ is $\frac{1}{4}$. Therefore

$$\mathbf{E}[X] = 0.083 \cdot 10 + 0.167 \cdot 6 + 0.25 \cdot 2 = 2.33,$$

and

$$\mathrm{var}(X) = \mathbf{E}[X^2] - \big(\mathbf{E}[X]\big)^2 = 0.083 \cdot 10^2 + 0.167 \cdot 6^2 + 0.25 \cdot 2^2 - (2.33)^2 = 9.8831.$$

Problem 5.

We have

$$\mathbf{E}[Y] = \mathbf{E}[X^2] = \int_1^2 x^2 f_X(x)\,dx = \int_1^2 \frac{2x^3}{3}\,dx = \frac{2}{3}\frac{x^4}{4}\bigg|_1^2 = \frac{2}{3}\left(\frac{16}{4}-\frac{1}{4}\right) = \frac{5}{2}.$$

To obtain the variance of Y, we first calculate $\mathbf{E}[Y^2]$. We have, after straightforward calculation,

$$\mathbf{E}[Y^2] = \mathbf{E}[X^4] = \int_1^2 x^4 f_X(x)dx = \int_1^2 \frac{2x^5}{3}dx = 7.$$

Thus,

$$\mathrm{var}(Y) = \mathbf{E}[Y^2] - \big(\mathbf{E}[Y]\big)^2 = 7 - \left(\frac{5}{2}\right)^2 = \frac{3}{4}.$$

Problem 6.

This problem does some calculations with a mixed random variable. We can represent the pdf of X as

$$f_X(x) = Ae^{-x}[u(x-1) - u(x-4)] + \frac{1}{4}\delta(x-2) + \frac{1}{4}\delta(x-3).$$

To find the constant A, we must integrate the pdf over all x to get 1.

$$\begin{aligned} A\int_1^4 e^{-x}dx + \frac{1}{4}\int_{-\infty}^{+\infty} \delta(x-2)dx + \frac{1}{4}\int_{-\infty}^{+\infty} \delta(x-3)dx &= 1, \\ A(e^{-1} - e^{-4}) + \frac{1}{4} + \frac{1}{4} &= 1, \end{aligned}$$

which has solution $A = \frac{1}{2}\frac{1}{e^{-1}-e^{-4}} \doteq 1.43$.

Problem 7.

We have

$$f_X(x) = \frac{dF_X}{dx}(x) = \begin{cases} 3a^3x^{-4}, & \text{if } x \geq a, \\ 0, & \text{if } x < a. \end{cases}$$

Also

$$\mathbf{E}[X] = \int_{-\infty}^{\infty} xf_X(x)dx = \int_a^{\infty} x{\cdot}3a^3x^{-4}dx = 3a^3\int_a^{\infty} x^{-3}dx = 3a^3\left(-\frac{1}{2}x^{-2}\right)\Bigg|_a^{\infty} = \frac{3a}{2}.$$

Finally, we have

$$\mathbf{E}[X^2] = \int_{-\infty}^{\infty} x^2f_X(x)dx = \int_a^{\infty} x^2{\cdot}3a^3x^{-4}dx = 3a^3\int_a^{\infty} x^{-2}dx = 3a^3\left(-x^{-1}\right)\Bigg|_a^{\infty} = 3a^2,$$

so the variance is

$$\text{var}(X) = \mathbf{E}[X^2] - \left(\mathbf{E}[X]\right)^2 = 3a^2 - \left(\frac{3a}{2}\right)^2 = \frac{3a^2}{4}.$$

Problem 8.

Let X_1, X_2, X_3 be the results of the three tests, and let

$$X = \max\{X_1, X_2, X_3\}$$

be the final score. We first compute the CDF F_X and then obtain the PMF as

$$p_X(k) = \begin{cases} F_X(k) - F_X(k-1), & \text{if } k = 3, \ldots, 10, \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\begin{aligned} F_X(k) &= \mathbf{P}(X \le k) \\ &= \mathbf{P}(X_1 \le k,\ X_2 \le k,\ X_3 \le k) \\ &= \mathbf{P}(X_1 \le k)\mathbf{P}(X_2 \le k)\mathbf{P}(X_3 \le k) \\ &= \begin{cases} 0 & \text{if } k < 3, \\ \frac{k}{10}\frac{k-1}{9}\frac{k-2}{8}, & \text{if } 3 \le k \le 10, \\ 1 & \text{if } 10 \le k. \end{cases} \end{aligned}$$

Problem 9.

We are given that $F_X(\mu) = 1/2$, or

$$\frac{1}{2} = \int_0^\mu \lambda e^{-\lambda x}\, dx = -e^{-\lambda x}\Big|_0^\mu = 1 - e^{-\lambda\mu},$$

or

$$-\lambda\mu = \ln\frac{1}{2},$$

from which we obtain

$$\mu = \frac{\ln 2}{\lambda}.$$