MATH 407

4/2/18

* Order 6 (contd.)

Must have a $o(a)=3$

$\langle a \rangle \subseteq G$

Must have $b \notin \langle a \rangle$

$\langle a \rangle = \{e, a, a^2\}$

$\langle a \rangle b = \{b, ab, a^2b\}$

$\langle b \rangle = \{e, b\}$

If $ab=ba$, then G was commutative.

or $a^2b=ba$

S_6: $a=(1,2,3)$, $b=(1,2)$

$\langle a \rangle \langle b \rangle$

$= \{a^ib^j : 0 \leq i < 3, 0 \leq j < 2\}$

*Let G be a group, H, K subgroups.

$HK = \{hk : h \in H, k \in K\}$ is a group if ☆

☆ for any $(h,k) \in H \times K$, $h^{-1}kh \in K$

a) $h^{-1}Kh \subseteq K \quad \forall h \in H$

$= \{h^{-1}kh : k \in K\}$

b) $Kh \subseteq hK = \{hk : k \in K\}$

c) For every $(h,k) \in H \times K$, $\exists k' \in K$, $kh = hk'$

b') $Kh = hK$

<u>Pf.</u> $Kh^{-1} \subseteq h^{-1}K \quad \forall h \in H$ ($\because$ condition b))

$hKh^{-1}h \subseteq hh^{-1}Kh$

$hK \subseteq Kh$

a') $h^{-1}Kh = K$

c') For any $(h,k) \in HK$, $\exists k'' \in K$
s.t. $hk = k''h$

<u>Pf.</u> $a, b \in HK$. Is $ab^{-1} \in HK$?

$a = h_1 k_1, \; b = h_2 k_2$

$$
\begin{aligned}
&= (h_1 k_1)(h_2^{-1} k_2^{-1}) \\
&= h_1 (k_1 k_2^{-1}) h_2^{-1} \\
&= h_1 (k_3 h_2^{-1}) \\
&= h_1 (h_2^{-1} k_4) \\
&= (h_1 h_2^{-1}) k_4 \\
&= h_3 k_4 \in HK
\end{aligned}
$$

<u>Cor.</u> $KH = HK$

<u>Pf.</u> $(HK) = (HK)^{-1} = \{(h,k)^{-1} : h \in H, k \in K\}$

$= K^{-1}H^{-1} = KH, \because K^{-1} = K, H^{-1} = H$

Ex: $a\langle b\rangle = a\{e, b\}$
$= \{a, ab\}$

$b\langle a\rangle = \{a, ba\}$
$= \{a, a^2b\}$ (condition ☆ not met)

* G_1, G_2 groups
$G_1 \times G_2$ cartesian product
$= \{(a_1, a_2);\ a_1 \in G_1,\ a_2 \in G_2\}$

↳ $(a_1, a_2) \cdot (b_1, b_2)$
$= (a_1 b_1, a_2 b_2)$

↳ $(a_1, a_2)(b_1, b_2)(c_1, c_2)$
$\Rightarrow (a_1 b_1, a_2 b_2)(c_1, c_2)$
$= ((a_1 b_1)c_1, (a_2 b_2)c_2)$
$= (a_1(b_1 c_1), a_2(b_2 c_2))$
$\vdots$
$(a_1 a_2)((b_1, b_2)(c_1, c_2))$ (associativity)

* $e = \{e_1, e_2\}$
$(a_1, a_2)^{-1} = (a_1^{-1}, a_2^{-1})$

* Direct Product of G_1, G_2

(→)

Thm. $o(a_1, a_2) = \mathrm{lcm}(o(a_1), o(a_2))$

* proof is same as orders in permutation

Pf. $(a_1, a_2)^l = (a_1^l, a_2^l)$ (by induction)

where $l = \mathrm{lcm}(o(a_1), o(a_2))$

$= (e_1, e_2)$

$\Rightarrow o(a_1) \mid l$ and $o(a_2) \mid l$

So,

$\mathrm{lcm}(o(a_1), o(a_2)) \mid l$

* If G is cyclic, then $G = \langle a \rangle$

If $G = S_6$, $G = \langle a, b \rangle$

* If $G_1 \times G_2$, $\langle a_1, a_2 \rangle$

then

$o(G) = o(a_1, a_2)$

$= \mathrm{lcm}(o(a_1), o(a_2))$

but

$|G| = o(a_1) \times o(a_2)$

$= |G_1| \times |G_2|$