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CMPE 320: HW 08

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1. Let  $X$  have a uniform distribution in the unit interval  $[0, 1]$ , and let  $Y$  have an exponential distribution with parameter  $\nu = 2$ . Assume that  $X$  and  $Y$  are independent. Let  $Z = X + Y$ .

(a) Find  $P(Y \geq X)$ .

**Sol.** Since  $X$  and  $Y$  are independent,  $f_{X,Y}(x,y) = f_X(x)g_Y(y)$

Therefore, the joint PDF,

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-2y}, & \text{if } 0 \leq x \leq 1, y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$\begin{aligned} P(Y \geq X) &= 1 - P(X \leq Y) \\ &= \iint_{y \geq x} f_{X,Y}(x,y) \, dx \, dy \\ &= 1 - \int_0^1 \int_0^x 2e^{-2y} \, dx \, dy \\ &= 1 - \int_0^1 1 - e^{-2x} \, dx \\ &= \frac{1}{2} - \frac{e^{-2}}{2} \end{aligned} \quad \square$$

(b) Find the conditional PDF of  $Z$  given that  $Y = y$ .

**Sol.**

$$\begin{aligned} f_{Z|Y=y}(z) &= f_{X+Y|Y=y}(x+y) \\ &= \begin{cases} 1, & \text{if } y \leq z \leq 1+y \\ 0, & \text{otherwise,} \end{cases} \end{aligned} \quad \square$$

(c) Find the conditional PDF of  $Y$  given that  $Z = 3$ .

**Sol.** With the laws of conditional probability,

$$f_{Y|3}(y | 3) = \frac{f_{Y,Z}(y, 3)}{f_Z(3)} = \frac{f_{Z|Y=y}(3 | y)f_Y(y)}{f_Z(3)}$$

And,

$$\begin{aligned} F_Z(3) &= \int_0^1 \int_0^{z-x} f_{X,Y}(x, y) dx dy \\ &= \int_0^1 \int_0^{z-x} 2e^{-2y} dx dy \\ &= 1 - \frac{e^{-2(3)+2}}{2} + \frac{e^{-2(3)}}{2} \\ &= e^{-4} - e^{-6} \end{aligned}$$

Therefore,

$$f_{Y|3}(y | 3) = \begin{cases} \frac{2e^{-2y}}{e^{-4}-e^{-6}}, & \text{if } 2 \leq y \leq 3 \\ 0, & \text{otherwise,} \end{cases} \quad \square$$

2. Let  $P$ , a random variable which is uniformly distributed between 0 and 1. On any given day, a particular machine is functional with probability  $P$ . Furthermore, given the value of  $P$ , the status of the machine on different days is independent

- (a) Find the probability that the machine is functional on a particular day.

**Sol.** Let  $W$  represent the event that the machine is functional

Then,

$$\begin{aligned} P(W) &= \int_0^1 P(W | X = x)f_X(x) dx \\ &= \int_0^1 x dx \\ &= \frac{1}{2} \end{aligned} \quad \square$$

- (b) We are told that the machine was functional on  $m$  out of the last  $n$  days. Find the conditional PDF of  $P$ . You may use the identity

$$\int_0^1 p^k(1-p)^{n-k} dp = \frac{k!(n-k)!}{(n+1)!}$$

**Sol.**

$$\begin{aligned}
 P(W_m) &= \int_0^1 P(W_m | X = x) f_X(x) dx \\
 &= \int_0^1 \binom{n}{m} x^m (1-x)^{n-m} f_X(x) dx \\
 &= \binom{n}{m} \frac{m!(n-m)!}{(n+1)!}
 \end{aligned}$$

Therefore, using Bayes rule,

$$\begin{aligned}
 f_{X|W_m}(x) &= \frac{P(W_m | X = x) f_X(x)}{P(W_m)} \\
 &= \frac{x^m (1-x)^{n-m}}{\frac{m!(n-m)!}{(n+1)!}}, \quad 0 \leq x \leq 1, \quad n \geq m
 \end{aligned}
 \quad \square$$

- (c) Find the conditional probability that the machine is functional today given that it was functional on  $m$  out of the last  $n$  days.

**Sol.**

$$\begin{aligned}
 P(W_m) &= \int_0^1 P(W_m | X = x) f_X(x) dx \\
 &= \int_0^1 \binom{n}{m} x^m (1-x)^{n-m} f_X(x) dx \\
 &= \binom{n}{m} \frac{m!(n-m)!}{(n+1)!}
 \end{aligned}
 \quad \square$$

3. Let  $B \triangleq \{a < X \leq b\}$ . Derive a general expression for  $E[X | B]$  if  $X$  is a continuous RV. Let  $X : N(0, 1)$  with  $B = \{-1 < X \leq 2\}$ . Compute  $E[X | B]$ .

**Sol.**  $\square$

4. A particular model of an HDTV is manufactured in three different plants, say,  $A$ ,  $B$  and  $C$ , of the same company. Because the workers at  $A$ ,  $B$  and  $C$  are not equally experienced, the quality of the units differs from plant to plant. The pdf's of the time-to-failure  $X$ , in years, are

$$\begin{aligned}
 f_X(x) &= \frac{1}{5} \exp(-x/5) u(x) \text{ for } A \\
 f_X(x) &= \frac{1}{6.5} \exp(-x/6.5) u(x) \text{ for } B
 \end{aligned}$$

$$f_X(x) = \frac{1}{10} \exp(-x/10)u(x) \text{ for } C,$$

where  $u(x)$  is the unit step. Plant A produces three times as many units as B, which produces twice as many as C. The TVs are all sent to a central warehouse, intermingled, and shipped to retail stores all around the country. What is the expected lifetime of a unit purchased at random?

**Sol.** The expectations of the exponential distributions,  $1/\lambda$ ,

$$E[A] = 5$$

$$E[B] = 6.5$$

$$E[C] = 10$$

Given the ratio of the units is 6 : 2 : 1,

$$P(A) = \frac{6}{9}$$

$$P(B) = \frac{2}{9}$$

$$P(C) = \frac{1}{9}$$

Therefore, the expected lifetime of a unit purchased at random,

$$\begin{aligned} E &= \frac{5 \times 6}{9} + \frac{6.5 \times 2}{9} + \frac{10 \times 1}{9} \\ &= \frac{53}{6} \end{aligned}$$

□

5. The coordinate  $X$  and  $Y$  of a point are independent zero mean normal random variables with common variances  $\sigma^2$ . Given that the point is at a distance of at least  $c$  from the origin, find the conditional joint PDF of  $X$  and  $Y$ .

**Sol.**

□

6. Alexei is vacationing in Monte Carol. The amount  $X$  (in dollars) he takes to the casino each evening is a random variable with a PDF of the form

$$f_X(x) = \begin{cases} ax, & \text{if } 0 \leq x \leq 40, \\ 0, & \text{otherwise} \end{cases}$$

At the end of each night, the amount  $Y$  that he has when leaving the casino is uniformly distributed between

zero and twice the amount that he came with.

(a) Determine the joint PDF  $f_{X,Y}(x,y)$ .

**Sol.** Since

$$\begin{aligned}\int_0^{40} ax \, dx &= 1 \\ a \frac{40^2}{2} &= 1 \\ \Rightarrow a &= \frac{1}{800}\end{aligned}$$

□

(b) What is the probability that on a given night Alexei makes a positive profit at the casino?

**Sol.**

□

(c) Find the PDF of Alexei's profit  $Y - X$  on a particular night, and also determine its expected value.

**Sol.**

□

7. Consider a communication channel corrupted by noise. Let  $X$  be the value of the transmitted signal and  $Y$  be the value of the received signal. Assume that the conditional density of  $Y$  given  $\{X = x\}$  is Gaussian, that is,

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-x)^2}{2\sigma^2}\right),$$

and  $X$  is uniformly distributed on  $[-1, 1]$ . What is the conditional pdf of  $X$  given  $Y$ , that is,  $f_{X|Y}(x|y)$

**Sol.** Given,

$$f_X(x) = \begin{cases} \frac{1}{2}, & \text{if } -1 \leq x \leq 1 \\ 0, & \text{otherwise,} \end{cases}$$

And,

$$\begin{aligned}f_Y(y) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y) \, dx \\ &= \int_{-\infty}^{\infty} f_{Y|X}(y|x) f_X(x) \, dx \\ &= \int_{-1}^1 \frac{1}{2} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(y-x)^2}{2\sigma^2}\right) \, dx\end{aligned}$$

$$= \frac{1}{2} \left( \phi \left( \frac{y-1}{\sigma} \right) - \phi \left( \frac{y+1}{\sigma} \right) \right)$$

Therefore,

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{f_{Y|X}(y|x)f_X(x)}{f_Y(y)} \\ &= \frac{\frac{1}{\sqrt{2\pi}\sigma^2} \exp\left(-\frac{(y-x)^2}{2\sigma^2}\right) f_X(x)}{\phi\left(\frac{y-1}{\sigma}\right) - \phi\left(\frac{y+1}{\sigma}\right)} \end{aligned} \quad \square$$

8. A U.S. defense radar scans the skies for unidentified flying objects (UFOs). Let  $M$  be the event that a UFO is present and  $M^C$  the event that a UFO is absent. Let  $f_{X|M}(x|M) = \frac{1}{\sqrt{2\pi}} \exp(-0.5[x-r]^2)$  be the conditional pdf of the radar return signal  $X$  when a UFO is actually there, and let  $f_{X|M^C}(x|M^C) = \frac{1}{\sqrt{2\pi}} \exp(-0.5[x]^2)$  be the conditional pdf of the radar return signal  $X$  when there is no UFO. To be specific, let  $r = 1$  and let the *alert level* be  $x_A = 0.5$ . Let  $A$  denote the event of an alert, that is,  $\{X > x_A\}$ . Compute  $P[A|M]$ ,  $P[A^C|M]$ ,  $P[A|M^C]$ ,  $P[A^C|M^C]$ .

Assume that  $P[M] = 10^{-3}$ . Compute  $P[A|M]$ ,  $P[A^C|M]$ ,  $P[A|M^C]$ ,  $P[A^C|M^C]$ .

Repeat for  $P[M] = 10^{-6}$

**Sol.**

□