Math-Phys Quiz 8 Question:

1. The field from a circular disk of charge along the axis of the disk may be written (Ulaby et al., Example 4-5)

$$\mathbf{E} = \hat{\mathbf{z}} \frac{\rho_{\mathrm{S}} h}{2\epsilon_0} \int_0^a \frac{r\, dr}{(r^2 + h^2)^{3/2}}.$$

   Do the integration to obtain an explicit expression.

2. Show using Taylor expansion that $(1+x)^\alpha \simeq 1 + \alpha x$ when $x \ll 1$. Use this result to show that

$$\frac{y^2}{(y^2 - a^2)^{3/2}} - \frac{y^2}{(y^2 + a^2)^{3/2}} \simeq \frac{3a^2}{y^3}$$

   when $a \ll y$.

3. Derive Coulomb's law from Gauss's law.

Exam Quiz 8 Questions:

1. **Modified from slides 8.16, 8.17:**
   Two point charges with $q_1 = 4 \times 10^{-5}$ C and $q_2 = -8 \times 10^{-5}$ C are located in free space at points with Cartesian coordinates (2, 6, −2) and (−6, 2, −4), respectively. Find (a) the electric field $\mathbf{E}$ at (6, 2, −4) and (b) the force on a 16×10^{-5} C charge located at that point. All distances are in meters.

Math-Physics Quiz 8 Solutions:

1. We have

$$\mathbf{E} = \hat{\mathbf{z}}\frac{\rho_{\mathrm{S}} h}{2\epsilon_0}\left[\frac{-2}{(r^2+h^2)^{1/2}}\bigg|_0^a\right] = \pm\hat{\mathbf{z}}\frac{\rho_{\mathrm{S}}}{2\epsilon_0}\left[1 - \frac{|h|}{(a^2+h^2)^{1/2}}\right],$$

   where the positive sign applies when $h > 0$ and the negative sign applies when $h < 0$.

2. To first order, we have that the first two terms in a Taylor expansion at the point $x = 0$ are:

$$(1+x)^\alpha = (1+x)^\alpha\big|_{x=0} + \left[\frac{d(1+x)^\alpha}{dx}\bigg|_{x=0}\right]x = 1 + \alpha x.$$

   We may rewrite the terms in the next expression as

$$\frac{y^2}{(y^2 \mp a^2)^{3/2}} = \frac{1}{y}\left(1 \mp \frac{a^2}{y^2}\right)^{-3/2},$$

   so that $x = \mp(a^2/y^2)$, $\alpha = -3/2$, and our expression becomes

$$\frac{1}{y}\left[\left(1 + \frac{3}{2}\frac{a^2}{y^2}\right) - \left(1 - \frac{3}{2}\frac{a^2}{y^2}\right)\right] = \frac{3a^2}{y^3}.$$

3. Gauss's law is $\nabla \cdot \mathbf{D} = \rho_{\mathrm{V}}$, which, using Gauss's theorem becomes,

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{D}\, d\mathcal{V} = \int_S \mathbf{D} \cdot \hat{\mathbf{n}}\, ds = \int_{\mathcal{V}} \rho_{\mathrm{V}}\, d\mathcal{V} = Q,$$

   where Q is the total charge in the volume. The charge in Coulomb's law is a point charge, and we take our volume to be the sphere that is centered at the charge and whose surface is a distance R from the center. We then find from symmetry that $\mathbf{D} = \hat{\mathbf{R}}D$ and $\hat{\mathbf{n}} = \hat{\mathbf{R}}$, so that we have

$$Q = \int_S D\, ds = 4\pi R^2 D,$$

   where we recall that the surface of sphere has area $4\pi R^2$. We conclude

$$\mathbf{D} = \frac{Q\hat{\mathbf{R}}}{4\pi R^2} \qquad \text{and} \qquad \mathbf{E} = \frac{Q\hat{\mathbf{R}}}{4\pi\epsilon R^2},$$

   where we recall that $\mathbf{E} = \mathbf{D}/\epsilon$.

Exam Quiz 8 Solution:

1. a. Since $\epsilon = \epsilon_0$ and there are two charges, we have

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\left[\frac{q_1(\mathbf{R}-\mathbf{R}_1)}{|\mathbf{R}-\mathbf{R}_1|^3} + \frac{q_2(\mathbf{R}-\mathbf{R}_2)}{|\mathbf{R}-\mathbf{R}_2|^3}\right],$$

where $\mathbf{R}_1 = \hat{\mathbf{x}}2 + \hat{\mathbf{y}}6 - \hat{\mathbf{z}}2$, $\mathbf{R}_2 = -\hat{\mathbf{x}}3 + \hat{\mathbf{y}}2 - \hat{\mathbf{z}}4$, $\mathbf{R}_3 = \hat{\mathbf{x}}3 + \hat{\mathbf{y}}2 - \hat{\mathbf{z}}4$. After substitution, we find

$$\begin{aligned}\mathbf{E} &= \frac{1}{4\pi\epsilon_0}\left[\frac{4\times 2(\hat{\mathbf{x}}2 - \hat{\mathbf{y}}2 - \hat{\mathbf{z}}1)}{8\times 27} - \frac{8\times 2(\hat{\mathbf{x}}6)}{8\times 216}\right]\times 10^{-5}\\ &= \frac{\hat{\mathbf{x}} - \hat{\mathbf{y}}4 - \hat{\mathbf{z}}2}{216\pi\epsilon_0}\times 10^{-5} = (\hat{\mathbf{x}} - \hat{\mathbf{y}}4 - \hat{\mathbf{z}}2)\times(1.664\times 10^3)\ \text{(V/m)}.\end{aligned}$$

b. Using the force equation, we have

$$\mathbf{F} = q_3\mathbf{E} = (16\times 10^{-5})\times\frac{\hat{\mathbf{x}} - \hat{\mathbf{y}}4 - \hat{\mathbf{z}}2}{216\pi\epsilon_0}\times 10^{-5} = \frac{\hat{\mathbf{x}}2 - \hat{\mathbf{y}}8 - \hat{\mathbf{z}}4}{27\pi\epsilon_0}\times 10^{-10} = 0.266\ \text{(N)}.$$