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Initial Value Theorem

$$\lim_{s \to \infty} \left(sF(s) \right) = f\left(0^+\right)$$

• Final Value Theorem

$$\lim_{s\to 0} (sF(s)) = \lim_{t\to \infty} f(t)$$

• The "zero s" theorem

$$\int_{-\infty}^{\infty} x(\tau) d\tau = \int_{-\infty}^{\infty} x(\tau) e^{0\tau} d\tau = \left(\int_{-\infty}^{\infty} x(\tau) e^{s\tau} d\tau \right)_{s=0} = X(0)$$

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2

The Convolution Theorem

- One of the primary application of the Laplace (and later Fourier) Domain(s) is the Convolution Theorem
- What is the LT of the convolution?

$$\mathcal{L}(x * h) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau)h(t-\tau) d\tau \right) e^{-st} dt$$

$$= \int_{-\infty}^{\infty} x(\tau) \left(\int_{-\infty}^{\infty} h(t-\tau)e^{-st} dt \right) d\tau = \int_{-\infty}^{\infty} x(\tau) \left(e^{-s\tau} H(s) \right) d\tau$$
time shift property!
$$= H(s) \int_{-\infty}^{\infty} x(\tau)e^{-s\tau} d\tau = H(s)X(s) = X(s)H(s) \quad (!!)$$

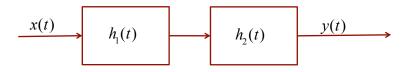
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 $R \supset R_{_{X}} \cap R_{_{H}}$

So what?

Remember our concatenation of systems?



$$y(t) = x * h = x * (h_1 * h_2)$$

$$H(s) = \mathcal{L}(h_1 * h_2) = H_1(s)H_2(s)$$

$$Y(s) = \mathcal{L}(x * h) = X(s)H(s) = X(s)H_1(s)H_2(s)$$

- A convolution in the time domain is equivalent to a multiplication in the frequency domain...
- ...and vice versa

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4

One sided Laplace Transforms

- For *causal* systems, we have h(t) = 0, t < 0
- ...then $\mathcal{L}(h(t)) = \int_{-\infty}^{\infty} h(t)e^{-st} dt = \int_{0}^{\infty} h(t)e^{-st} dt$
- This is so common, that it's called the "one sided Laplace Transform
- What's the RoC?
- ...no strips, no left side!
- This is the form your book (and many other texts) use!

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Properties

- The same properties apply, except for the time derivative property
- Differentiation (1 sided)

$$\mathcal{L}\left(\frac{dx}{dt}\right) = sX(s) - x(0)$$

$$\frac{1}{s+a}, a < 0, \text{Re}[s] < a, -e^{-at}u(-t) \to 0 \text{ as } t \to -\infty$$

$$\mathcal{L}\left(\frac{d^2x}{dt^2}\right) = s^2X(s) - sx(0) - \frac{dx}{dt}\Big|_{t=0}$$

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5



- Stability is generally "bounded input, bounded output" stability
- If $|x(t)| < C_1$ then $|y(t)| < C_2$
- Is our causal integrator $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$ stable?
- To answer the question, we need to either prove that the condition holds or find a function where it doesn't hold
- Let x(t) = u(t), then $y(t) = \int_{0}^{t} 1 d\tau = t$
- The input is bounded, but the output is not, because for any C > 0 I only need wait for t > C for the output to exceed it.
- It is not BIBO stable!

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What about the s-domain

• Where are the poles of the integrator?

$$\mathcal{L}\left(\int_{-\infty}^{t} x(t) dt\right) = \frac{1}{s} \text{ for Re}[s] > 0$$

- What about $\mathcal{L}\left(e^{at}u(t)\right) = H(s) = \frac{1}{s+a}$, a > 0, Re[s] > a
- This also "blows up"

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau)e^{a\tau}u(\tau)d\tau = \int_{0}^{\infty} x(t-\tau)e^{a\tau}u(\tau)d\tau$$

$$|x(t)| \le C$$
, so $y(t) \le \int_{0}^{\infty} C_1 e^{a\tau} d\tau = \infty \Rightarrow$ not stable

A causal system must have its poles in the left half plane!

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Anti causal system



$$\frac{1}{s+a}$$
, $a > 0$, $\text{Re}[s] < -a \Rightarrow -e^{-at}u(-t) \to \infty \text{ as } t \to -\infty$

$$\frac{1}{s+a}$$
, $a < 0$, $\text{Re}[s] < -a \Rightarrow -e^{-at}u(-t) \to 0$ as $t \to -\infty$

- And for two-sided systems, both terms must be stable.
- We will see later that the RoC must include the $j\omega$ axis for the system to be stable.

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Examples

- Find the Laplace Transform of the unit amplitude pulse of duration T by two methods.
- Find the Laplace transform of a causal unit ramp, y(t) = tu(t)
- Find the response of $h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$ to a causal unit ramp
- Find the initial and final values of that response.