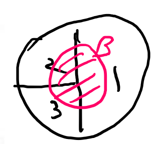
CMPE 320: Probability, Statistics, and Random Processes

Lecture 4: Total probability; Bayes' rule

Spring 2018

Seung-Jun Kim

Example 1.13. You enter a chess tournament where your probability of winning a game is 0.3 against half the players (call them type 1), 0.4 against a quarter of the players (call them type 2), and 0.5 against the remaining quarter of the players (call them type 3). You play a game against a randomly chosen opponent. What is the probability of winning?



A_i : play against player type i $\quad$ B : win the game

$P(B|A_1) = 0.3 \qquad P(B|A_2) = 0.4 \qquad P(B|A_3) = 0.5$

$P(A_1) = 0.5 \qquad P(A_2) = 0.25 \qquad P(A_3) = 0.25$

$P(B) = ?$

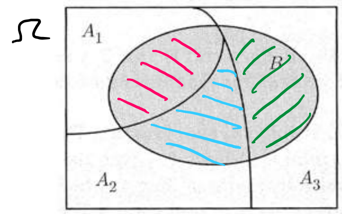
$$B = \underbrace{(A_1 \cap B)} \cup \underbrace{(A_2 \cap B)} \cup \underbrace{(A_3 \cap B)}$$

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B)$$

total prob.

$$= \boxed{P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + P(B|A_3)P(A_3)}$$

The total probability theorem



- Let $A_1, \ldots, A_n$ be disjoint events that partition the sample space, and assume $P(A_i) > 0$ for all i. Then, for any event B,

$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + \cdots + P(A_n \cap B)$$

$(\because$ Additivity axiom$)$

cond. probability

$$= \underbrace{P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \cdots + P(A_n)P(B|A_n)}_{\text{Total probability}}$$

Example 1.14. You roll a fair four-sided die. If the result is 1 or 2, you roll once more but otherwise, you stop. What is the probability that the sum total of your rolls is at least 4?

A_i: outcome of die is i, B: sum of rolls ≥ 4

$$P(B) = P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3) + P(A_4)P(B|A_4)$$

$$= \frac{1}{4}\cdot\frac{1}{2} + \frac{1}{4}\cdot\frac{3}{4} + \frac{1}{4}\cdot 0 + \frac{1}{4}\cdot 1$$

$$= \frac{9}{16}$$

Example 1.15. Alice is taking a probability class and at the end of each week she can be either up-to-date or she may have fallen behind. If she is up-to-date in a given week, the probability that she will be up-to-date (or behind) in the next week is 0.8 (or 0.2, respectively). If she is behind in a given week, the probability that she will be up-to-date (or behind) in the next week is 0.4 (or 0.6, respectively). Alice is (by default) up-to-date when she starts the class. What is the probability that she is up-to-date after three weeks?

Bayes' Rule

- Relates conditional probability P(A|B) to P(B|A)

What for? Think A: cause B: effect

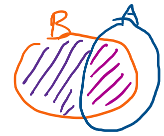
P(A|B) corresponds to inferring the cause to an effect

But often we can model P(B|A) much better

- Often used in conjunction with the total probability theorem

$P(B|A) = \frac{P(A\cap B)}{P(A)}$

Example 1.18. The False-Positive Puzzle. A test for a certain rare disease is assumed to be correct 95% of the time: if a person has the disease, the test results are positive with probability 0.95, and if the person does not have the disease, the test results are negative with probability 0.95. A random person drawn from a certain population has 0.001 of having the disease.



A: has disease (cause) B: test is positive (effect)

$P(B|A) = 0.95$ $P(B^c|A^c) = 0.95$ $P(A) = 0.001$

$P(A|B) = ?$ Bayes' rule

$$P(A|B) = \underbrace{\frac{P(A\cap B)}{P(B)}}_{\text{def. cond. prob.}} \overset{\text{cond. prob}}{=} \boxed{\frac{P(A)P(B|A)}{P(B)}} = \frac{P(A)P(B|A)}{\underbrace{P(B|A)P(A)+P(B|A^c)P(A^c)}_{\text{total prob.}}}$$

$$= \frac{0.001\times 0.95}{0.95\times 0.001 + (1-0.95)\times(1-0.001)} = 0.0187$$

Bayes' rule

- Let $A_1, \ldots, A_n$ be disjoint events that partition the sample space, and assume $P(A_i) > 0$ for all i. Then, for any event B,

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{P(B)}$$

$$= \frac{P(A_i)P(B|A_i)}{P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + \cdots + P(A_n)P(B|A_n)}$$

Example 1.16. Let us return to the radar detection problem of Example 1.9 and Fig. 1.9. Let

$$A = \{\text{an aircraft is present}\},$$

$$B = \{\text{the radar generates an alarm}\}.$$

We are given that

$$\mathbf{P}(A) = 0.05, \qquad \mathbf{P}(B \mid A) = 0.99, \qquad \mathbf{P}(B \mid A^c) = 0.1.$$

What is Pr(aircraft present | alarm)?

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B \mid A)}{P(B)} = \frac{P(A)P(B \mid A)}{P(B \mid A)P(A) + P(B \mid A^c)P(A^c)}$$

$$= \frac{0.05 \times 0.99}{0.99 \times 0.05 + 0.1 \times (1 - 0.05)}$$