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MATH 407: HW 10

3.6 **5** Show that no proper subgroup of S_4 contains both $(1, 2, 3, 4)$ and $(1, 2)$.

Pf. $\square$

9 A rigid motion of a cube can be thought of either as a permutation of its eight vertices or as a permutation of its six sides. Find a rigid motion of the cube that has order 3, and express the permutation that represents it in both ways, as a permutation on eight elements and as a permutation on six elements.

Pf. $\square$

10 Show that the following matrices form a subgroup of $\mathrm{GL}_2(\mathbb{C})$ isomorphic to D_4:

$$\pm\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm\begin{bmatrix} \mathrm{i} & 0 \\ 0 & \text{-i} \end{bmatrix}, \pm\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \pm\begin{bmatrix} 0 & \mathrm{i} \\ \text{-i} & 0 \end{bmatrix}$$

Pf. $\square$

15 **(a)** Show that $A_4 = \{\sigma \in S_4 \mid \sigma = \tau^2 \text{ for some } \tau \in S_4\}$

Pf. $\square$

(b) Show that $A_5 = \{\sigma \in S_5 \mid \sigma = \tau^2 \text{ for some } \tau \in S_5\}$

Pf. $\square$

(c) Show that $A_6 = \{\sigma \in S_6 \mid \sigma = \tau^2 \text{ for some } \tau \in S_6\}$

Pf. $\square$

(d) What can you say about A_n if $n > 6$?

Pf. $\square$

17 For any elements $\sigma, \tau \in S_n$, show that $\sigma\tau\sigma^{-1}\tau^{-1} \in A_n$.

Pf. $\square$

21 Find the center of the dihedral group D_n.

Hint: Consider two cases, depending on whether n is odd or even.

Pf. $\square$

24 Show that the product of two transpositions is one of (i) the identity; (ii) a 3-cycle; (iii) a product of two (nondisjoint) 3-cycles. Deduce that every element of A_n can be written as a product of 3-cycles.

Pf. $\square$

3.7 **4** Let G be an abelian group, and let n be any positive integer. Show that the function $\phi : G \to G$ defined by $\phi(x) = x^n$ is a homomorphism.

Pf. $\square$

6 Define $\phi : \mathbb{C}^\times \to \mathbb{R}^\times$ by $\phi(a+bi) = a^2+bi$, for all $a+bi \in \mathbb{C}^\times$. Show that ϕ is a homomorphism.

Pf. $\square$

7 **b** $\phi : \mathbb{R} \to \mathrm{GL}_2(\mathbb{R})$ defined by $\phi(a) = \begin{bmatrix} 1 & \text{a} \\ 0 & 1 \end{bmatrix}$

Pf. $\square$

d $\phi : \mathrm{GL}_2(\mathbb{R}) \to \mathbb{R}^\times$ defined by $\phi\left(\begin{bmatrix} \text{a} & \text{b} \\ \text{c} & \text{d} \end{bmatrix}\right) = ab$

Pf. $\square$

10 Let G be the group of affine functions from $\mathbb{R}$ into $\mathbb{R}$, as defined in Exercise 10 of Section 3.1. Define $\phi : G \to \mathbb{R}^\times$ as follows: for any function $f_{m,b} \in G$, let $\phi(f_{m,b}) = m$. Prove that ϕ is a group homomorphism, and find its kernel and image.

Pf. $\square$

14 Recall that the center of a group G is $\{x \in G \mid xh = gx \text{ for all } g \in G\}$. Prove that the center of any group is a normal subgroup.

Pf. $\square$

18 Let the dihedral group D_n be given by elements a of order n and b of order 2, where $ba = a^{-1}b$. Show that any subgroup of $\langle a \rangle$ is normal in D_n.

Pf. □

3.8 **4** For each of the subgroups $\{e, a^2\}$ and $\{e, b\}$ of D_4, list all left and right cosets.

Pf. □

9 Let G be a finite group, and let n be a divisor of $|G|$ Show that if H is the only subgroup of G of order n, then H must be normal in G.

Pf. □

12 Let H and K be normal subgroups of G such that $H \cap K = \langle e \rangle$. Show that $hk = kh$ for all $h \in H$ and $k \in K$.

Pf. □

18 Compute the factor group $(\mathbb{Z}_6 \times \mathbb{Z}_4)/\langle (3, 2) \rangle$.

Pf. □

19 Show that $(\mathbb{Z} \times \mathbb{Z})/\langle (0, 1) \rangle$ is an infinite cyclic group.

Pf. □

23 **a**. Show that G is a subgroup of $\mathrm{GL}_2((\mathbb{Z}_5)$.

Pf. □

b. Show that the subset N of all matrices in G of the form $\begin{bmatrix} 1 & \mathrm{c} \\ 0 & 1 \end{bmatrix}$, with $c \in \mathbb{Z}_5$, is a normal subgroup of G.

Pf. □

c. Show that the factor group G/N is cyclic of order 4.

Pf. □