CMPE 320: Probability, Statistics, and Random Processes

Lecture 11: Conditioning of RVs

Spring 2018

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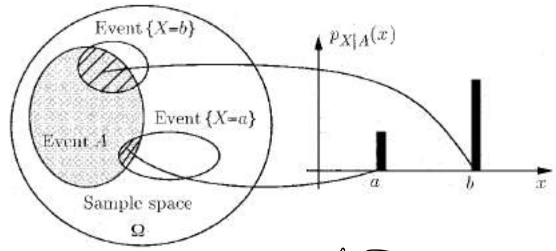
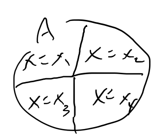
Conditional PMF

Recall conditional probability of an event A given B has happened

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$



The conditional PMF of a RV X conditioned on event A



$$\boxed{P_{X|A}(x) = P(X=x \mid A) = \frac{P(\{X=x\} \cap A)}{P(A)}}$$

$$\sum_x P_{X|A}(x) = \sum_x \frac{P(\{X=x\} \cap A)}{P(A)} = \frac{1}{P(A)} \sum_x P(\underbrace{\{X=x\} \cap A}_{\text{disjoint for different } x}) = \frac{1}{P(A)} P(A) = 1$$

Example

- X is the roll of a fair die. A is the event that the roll is an even number. What is $p_{X|A}(x)$?

$$P_{X|A}(x) = \frac{P(\{X=x\} \cap A)}{P(A)} = \frac{P(X=x \text{ and the roll is even})}{P(\text{roll is even})}$$

$$= \begin{cases} 0, & x = 1, 3, 5, \\ \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}, & x = 2, 4, 6 \end{cases}$$

Example 2.13. A student will take a certain test repeatedly, up to a maximum of n times, each time with a probability p of passing, independent of the number of previous attempts. What is the PMF of the number of attempts, given that the student passes the test?

A : the student passes the test $= X \le n$

X : number of attempts until succeeding (allowing ∞ trials)

$$P_{X|A}(x) = \frac{P(\{X=x\} \cap A)}{P(A)} = \begin{cases} 0 & \text{if } x > n \\ \dfrac{(1-p)^{x-1} p}{\sum_{m=1}^{n} (1-p)^{m-1} p}, & \text{if } x \le n \end{cases}$$

$P(X=1) = p$ (passing the test in the first attempt)

$P(X=2) = (1-p)\,p$ (first fail, then pass)

$\vdots$

$P_X(k) = P(X=k) = (1-p)^{k-1} p$ for $k = 1, 2, \ldots$

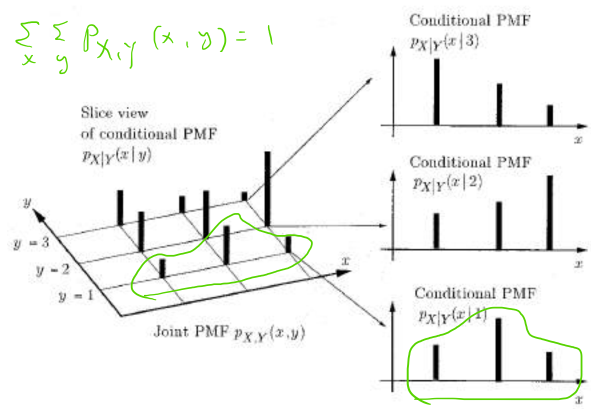
$$P(A) = P(X \le n) = \sum_{m=1}^{n} P_X(m) = \sum_{m=1}^{n} (1-p)^{m-1} p$$

Conditioning of a RV on another

- Let X and Y are RVs associated with the same experiment. Knowing Y takes a particular value y may provide partial knowledge on X.

$$P_{X|Y}(x|y) = P(X=x \mid Y=y)$$
$$= \frac{P(X=x, Y=y)}{P(Y=y)}$$
$$= \boxed{\frac{P_{X,Y}(x,y)}{P_Y(y)}}$$

$$\sum_x \sum_y P_{X,Y}(x,y) = 1$$



$$\sum_x P_{X|Y}(x|y) = 1 \leftarrow$$

Joint PMF from conditional PMF

From the prev. slide,

$$P_{X,Y}(x,y) = P_{X|Y}(x|y)\, P_Y(y)$$

Likewise

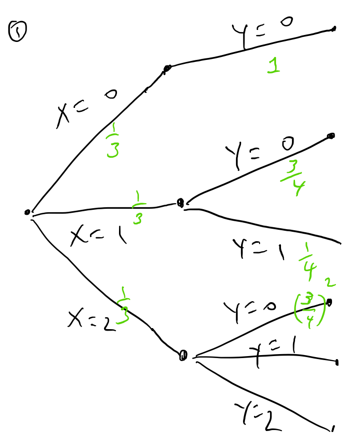
$$P_{X,Y}(x,y) = P_{Y|X}(y|x)\, P_X(x)$$

Example 2.14. Professor May B. Right often has her facts wrong, and answers each of her students' questions incorrectly with probability 1/4, independent of other questions. In each lecture, May is asked 0, 1, or 2 questions with equal probability 1/3. Let X and Y be the number of questions May is asked and the number of questions she answers wrong in a given lecture, respectively.

(1) Compute the joint PMF $P_{X,Y}(x,y)$.

(2) What is the probability that at least one answer is answered wrong?

①



$P_{X,Y}(0,0) = P_X(0)\, P_{Y|X}(0|0) = \frac{1}{3} \cdot 1 = \frac{1}{3}$

$P_{X,Y}(1,0) = P_X(1)\, P_{Y|X}(0|1) = \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$

$P_{X,Y}(1,1) = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$

$P_{X,Y}(2,0) = \frac{1}{3} \cdot \left(\frac{3}{4}\right)^2 = \frac{3}{16}$

$\binom{2}{1}\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)$ $P_{X,Y}(2,1) = \frac{1}{3} \cdot \frac{6}{16} = \frac{1}{8}$

$\left(\frac{1}{4}\right)^2$ $P_{X,Y}(2,2) = \frac{1}{3} \cdot \left(\frac{1}{4}\right)^2 = \frac{1}{48}$

② $P_{X,Y}(1,1) + P_{X,Y}(2,1) + P_{X,Y}(2,2)$

$= \frac{1}{12} + \frac{1}{8} + \frac{1}{48}$

$= \frac{1}{6}$

Marginal PMF from conditional PMFs

$$P_X(x) = \sum_y P_{X,Y}(x,y) = \underline{\sum_y P_Y(y)\, P_{X|Y}(x|y)}$$

- If $A_1, A_2, \cdots, A_n$ are disjoint, and their union is Ω

$$\underline{P_X(x)} = P(X=x \cap A_1) + P(X=x \cap A_2) + \cdots + P(X=x \cap A_n)$$

$$= \sum_{i=1}^n P(X=x \cap A_i) = \sum_{i=1}^n P(A_i)\, P(X=x | A_i)$$

$$= \underline{\sum_{i=1}^n P(A_i)\, P_{X|A_i}(x)}$$

For an event B

$$P_{X|B}(x) = \underline{\sum_{i=1}^n P(A_i | B)\, P_{X|A_i \cap B}(x)}$$

Example 2.15. Consider a transmitter that is sending messages over a computer network. Let us define the following two random variables:

X : the travel time of a given message, Y : the length of the given message. Given

$$p_Y(y) = \begin{cases} 5/6, & \text{if } y = 10^2, \\ 1/6, & \text{if } y = 10^4. \end{cases} \quad p_{X|Y}(x\,|\,10^2) = \begin{cases} 1/2, & \text{if } x = 10^{-2}, \\ 1/3, & \text{if } x = 10^{-1}, \\ 1/6, & \text{if } x = 1, \end{cases} \quad p_{X|Y}(x\,|\,10^4) = \begin{cases} 1/2, & \text{if } x = 1, \\ 1/3, & \text{if } x = 10, \\ 1/6, & \text{if } x = 100 \end{cases}$$

find the PMF of the travel time of a message.

$$P_X(1) = \sum_y P_Y(y) P_{X|Y}(1|y) = P_Y(10^2) P_{X|Y}(1|10^2) + P_Y(10^4) P_{X|Y}(1|10^4)$$
$$= \frac{5}{6} \cdot \frac{1}{6} + \frac{1}{6} \cdot \frac{1}{2} = \frac{2}{9}$$

$$P_X(10^{-2}) = \sum_y P_Y(y) P_{X|Y}(10^{-2}|y) = \frac{5}{6} \cdot \frac{1}{2} + \frac{1}{6} \cdot 0 = \frac{5}{12}$$

$P_X(10^{-1}) =$

$P_X(10) =$ $\vdots$

$P_X(100) =$

Conditional expectation

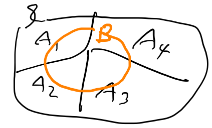
- Instead of the ordinary PMF, use conditional PMF

$$E[X|A] = \sum_x x\, P_{X|A}(x)$$

$$E[g(X)|A] = \sum_x g(x)\, P_{X|A}(x)$$

$$E[X|Y=y] = \sum_x x\, P_{X|Y}(x|y)$$

Total expectation



If $A_1, A_2 \cdots, A_n$ are disjoint and form a partition of Ω

$$E[X] = \sum_x x P_X(x) \quad \text{— 3 slides back}$$

$$= \sum_x x \sum_i P_{X|A_i}(x) P(A_i) = \sum_i P(A_i) \underbrace{\sum_x x P_{X|A_i}(x)}$$

$$= \sum_i P(A_i) E[X|A_i]$$

Likewise $E[X] = \sum_y P_Y(y) E[X|Y=y] \leftarrow A_i = \{Y=y_i\}$

Also, $E[X|B] = \sum_i P(A_i|B) E[X|A_i \cap B]$

Example 2.16. Messages transmitted by a computer in Boston through a data network are destined for New York with probability 0.5, for Chicago with probability 0.3, and for San Francisco with probability 0.2. The transit time X of a message is random. Its mean is 0.05 seconds if it is destined for New York, 0.1 seconds if it is destined for Chicago, and 0.3 seconds if it is destined for San Francisco.

What is E[X]?

$$E[X] = \sum_i P(A_i) E[X|A_i]$$

$$= 0.5 \times 0.05 + 0.3 \times 0.1 + 0.2 \times 0.3$$

$$= 0.115$$

Example 2.17. Mean and Variance of the Geometric. You write a software program over and over, and each time there is probability p that it works correctly, independent of previous attempts. What is the mean and variance of X, the number of tries until the program works correctly?