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MATH 407: HW 08

3.2 **1** In $GL_2(R)$, find the order of each of the following elements.

b

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Ans

$$\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)^3 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)^4 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore,

$$o\left(\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right) = 4$$

□

d

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$

Ans

$$\left(\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}\right)^2 = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore,

$$o\left(\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}\right) = 2 \qquad \square$$

11 Let S be a set, and let a be a fixed element of S. Show that $\{\sigma \in \text{Sym}(S) \mid \sigma(a) = a\}$ is a subgroup of Sym(S).

Ans Let $H = \{\sigma \in \text{Sym}(S) \mid \sigma(a) = a\}$

Since $\sigma(a)$ is the identity function, H is a non-empty subset

For H to be subgroup, $\sigma\tau^{-1} \in H,\ \forall \sigma, \tau \in H$

Let $\sigma, \tau \in H$

Then,

$$\begin{aligned} \sigma(a) &= a \\ \tau(a) &= a \\ \tau^{-1}(a) &= a \end{aligned}$$

Therefore,

$$\begin{aligned} (\sigma \circ \tau^{-1})(a) &= \sigma(\tau^{-1}(a)) \\ &= \sigma(a) \\ &= a \end{aligned}$$

Therefore, $\sigma \circ \tau^{-1} \in H$, and H is a subgroup of Sym(S) $\square$

12 For each of the following groups, find all elements of finite order.

a $\mathbb{R}^\times$

Ans We want elements $r \in \mathbb{R}^\times$ such that $r^n = e = 1$ for $n \in \mathbb{Z}^+$

$$\begin{aligned} r^n &= 1 \\ r &= \pm 1, \text{with } n = 2 \end{aligned}$$

Therefore, $\{-1, 1\} \in \mathbb{R}^\times$ $\square$

b $\mathbb{C}^\times$

Ans We want elements $c \in \mathbb{C}^\times$ such that $c^n = e = 1$ for $n \in \mathbb{Z}^+$

$$c^n = 1$$
$$c = \pm 1, \text{with } n = 2$$
$$c = \pm i, \text{with } n = 4$$

Therefore, $\{-1, 1, -i, i\} \in \mathbb{C}^\times$ $\square$

19 Let G be a group, and let $a \in G$. The set $C(a) = \{x \in G \mid xa = ax\}$ of all elements of G that commute with a is called the **centralizer** of a.

a Show that $C(a)$ is a subgroup of G.

Ans Since $C(a) = \{x \in G \mid xa = ax\}$,

$ea = ab, e \in C(a)$

Therefore, $C(a)$ is a non-empty set

For $C(a)$ to be a subgroup, $xy^{-1} \in C(a),\ \forall x, y \in C(a)$

Let $x, y, \in C(a)$,

Then,

$$xa = ax$$
$$ya = ay$$

And,

$$ya = ay$$
$$(y^{-1}y)ay^{-1} = y^{-1}a(yy^{-1})$$
$$ay^{-1} = y^{-1}a$$

Therefore,

$$\begin{aligned} xa &= ax \\ \Rightarrow (xy^{-1})a &= x(y^{-1}a) \\ &= x(ay^{-1}) \\ &= (xa)y^{-1} \\ &= (ax)y^{-1} \\ &= a(xy^{-1}) \end{aligned}$$

Therefore, $xy^{-1} \in C(a)$ and $C(a)$ is a subgroup of G $\square$

b Show that $\langle a \rangle \subseteq C(a)$.

Ans For $\langle a \rangle$ to be a generator of $C(a)$,

$$\langle a \rangle = \{x \in G \mid x = a^n \text{ for some } n \in \mathbb{Z}\}$$

Let $x = a^n \in \langle a \rangle$

Then,

$$\begin{aligned} xa &= a^n a \\ &= a^{n+1} \\ &= a^{1+n} \\ &= aa^n \\ &= ax \end{aligned}$$

Therefore, $x \in \langle a \rangle \subseteq C(a)$ $\square$

c Compute $C(a)$ if $G = S_3$ and $a = (1, 2, 3)$.

Ans

$$\begin{aligned} (1, 2, 3)(1) &= (1, 2, 3) \\ &= (1, 2, 3) \\ (1, 2, 3)(1, 3, 2) &= (1) \\ (1, 3, 2)(1, 2, 3) &= (1) \end{aligned}$$

Therefore, $C(1, 2, 3) = \{(1), (1, 2, 3), (1, 3, 2)\}$ $\square$

d Compute $C(a)$ if $G = S_3$ and $a = (1, 2)$.

Ans

$$\begin{aligned} (1, 2)(1) &= (1, 2) \\ (1)(1, 2) &= (1, 2) \\ (1, 2)(1, 2) &= (1) \end{aligned}$$

Therefore, $C(1, 2) = \{(1), (1, 2)\}$ $\square$

20 Compute the centralizer in $\mathrm{GL}_2(\mathbb{R})$ of the matrix $\begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$

Ans Let $a = \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$ and the centralizer $C(a) = \{x \in G \mid xa = ax\}$

Let $x = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, so,

$$C(a) \Rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix} = \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix}$$

Therefore,

$$a + c = a$$
$$\Rightarrow c = 0$$
$$b + d = a + b$$
$$\Rightarrow d = a$$

And

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$$

Therefore,

$$C\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = \left\{\begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \middle| \, a, b \in \mathbb{R}\right\}$$ $\square$

25 Let G be a finite group, let $n > 2$ be an integer, and let S be the set of elements of G that have order n. Show that S has an even number of elements.

Ans Let $a \in S$, so $o(a) = n > 2$

Then, $a^n = e$

Consider the element,

$$\begin{aligned} a^{-1} &\in S \\ (a^{-1})^n &= (a^n)^{-1} \\ &= e \end{aligned}$$

Therefore, $o(a^{-1}) = n$ and if $a \in S$ then $a^{-1} \in S$

Since the elements in S can be paired with its inverse,

S has an even number of elements $\square$

3.3 **4** Find the cyclic subgroup generated by $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ in $\mathrm{GL}_2(\mathbb{Z}_3)$.

Ans

$$\left(\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\right)^2 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\left(\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\right)^3 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\left(\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\right)^4 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$\left(\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\right)^5 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}$$

$$\left(\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\right)^6 = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore, the cyclic groups generated are

$$\left\{ \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\} \qquad \square$$

5 Prove that if G_1 and G_2 are abelian groups, then the direct product $G_1 \times G_2$ is abelian.

Ans Let $x_1, x_2 \in G_1$, so $x_1x_2 = x_2x_1$
and $y_1, y_2 \in G_2$ so $y_1y_2 = y_2y_1$
Let $(x_1, y_1), (x_2, y_2) \in G_1G_2$
Consider

$$\begin{aligned} (x_1, y_1) \times (x_2, y_2) &= (x_1x_2, y_1y_2) \\ &= (x_2x_1, y_2y_1) \\ &= (x_2, y_2) \times (x_1, y_1) \\ \Rightarrow G_1 \times G_2 &= G_2 \times G_1 \end{aligned}$$

Therefore, $G_1 \times G_2$ is commutative and therefore abeilian $\square$

8 Let G_1 and G_2 be groups, with subgroups H_1 and H_2, respectively. Show that $\{(x_1, x_2) \mid x_1 \in H_1, x_2 \in H_2\}$ is a subgroup of the direct product $G_1 \times G_2$.

Ans Since H_1 and H_2 are subgroups of G_1 and G_2 respectively,

$$\begin{aligned} e_1 &\in H_1 \\ e_2 &\in H_2 \\ (e_1, e_2) &\in \{(x_1, x_2) \mid x_1 \in H_1, x_2 \in H_2\} \end{aligned}$$

Let K be a non-empty subset
For K to be a subgroup, $xy^{-1} \in K,\ \forall x, y \in K$
Let $(x_1, x_2), (y_1, y_2) \in \{(x_1, x_2) \mid x_1 \in H_1, x_2 \in H_2\}$
Then $x_1, y_1 \in H_1$, $x_2, y_2 \in H_2$
Therefore,

$$\begin{aligned} (x_1, x_2)(y_1, y_2)^{-1} &= (x_1, x_2)(y_1^{-1}, y_2^{-1}) \\ &= (x_1y_1^{-1}, x_2y_2^{-1}) \end{aligned}$$

And since H_1 and H_2 are subgroups,

if $x_1, y_1 \in H_1$, $x_2, y_2 \in H_2$,

then $x_1 y_1^{-1} \in H_1$, $x_2 y_2^{-1} \in H_2$

Therefore, $(x_1, x_2)(y_1, y_2)^{-1} \in \{(x_1, x_2) \mid x_1 \in H_1, x_2 \in H_2\}$ $\square$

11 Let G_1 and G_2 be groups, and let G be the direct product $G_1 \times G_2$. Let $H = \{(x_1, x_2) \in G_1 \times G_2 \mid x_2 = e\}$ and let $K = \{(x_1, x_2) \in G_1 \times G_2 \mid x_1 = e\}$.

a Show that H and K are subgroups of G.

Ans Consider $H = \{(x_1, x_2) \in G_1 \times G_2 \mid x_2 = e\}$

Since $e \in G_1$ and $e \in G_2$

$(e, e) \in \{(x_1, x_2) \in G_1 \times G_2 \mid x_2 = e\}$

A non-empty set L is a subgroup if $xy^{-1} \in L,\ \forall x, y \in L$

Let $(x_1, x_2), (y_1, y_2) \in H$ so,

$x_1, y_1 \in G_1$ and $x_2, y_2 \in G_2$, where $x_2 = y_2 = e$

Therefore,

$$\begin{aligned}(x_1, x_2)(y_1, y_2)^{-1} &= (x_1, x_2)(y_1^{-1}, y_2^{-1}) \\ &= (x_1 y_1^{-1}, x_2 y_2^{-1})\end{aligned}$$

And since G_1 and G_2 are groups,

then $x_1 y_1^{-1} \in G_1$, $x_2 y_2^{-1} \in G_2$

Since $x_2 = y_2 = e$

$$\begin{aligned}(x_1, x_2)(y_1, y_2)^{-1} &= (x_1, x_2)(y_1^{-1}, y_2^{-1}) \\ &= (x_1 y_1^{-1}, e)\end{aligned}$$

Therefore, $H = \{(x_1, x_2) \in G_1 \times G_2 \mid x_2 = e\}$ is a subgroup of $G_1 \times G_2$,

and similarly for $K = \{(x_1, x_2) \in G_1 \times G_2 \mid x_1 = e\}$ $\square$

b Show that $HK = KH = G$.

Ans Given

$$\begin{aligned}H &= \{(x_1, x_2) \in G_1 \times G_2 \mid x_2 = e\} \\ K &= \{(x_1, x_2) \in G_1 \times G_2 \mid x_1 = e\} \\ HK &= \{(x_1, y_1)(x_2, y_2) \mid (x_1, y_1) \in H, (x_2, y_2) \in K\}\end{aligned}$$

Since $(x_1, y_1) \in H, (x_2, y_2) \in K$, $y_1 = x_2 = e$

Therefore

$$\begin{aligned}
HK &= \{(x_1, y_1)(x_2, y_2) \mid (x_1, y_1) \in H, (x_2, y_2) \in K\} \\
&= \{(x_1, e)(e, y_2) \mid (x_1, e) \in G_1, (e, y_2) \in G_2\} \\
&= \{(x_1, y_2) \mid x_1 \in G_1, y_2 \in G_2\} \\
&= G
\end{aligned}$$

Therefore, $HK = G$, and similarly for $KH = G$

$HK = KH = G$ $\square$

c Show that $H \cap K = \{e, e\}$.

Ans Let $(x_1, x_2) \in H \cap K$

Then, $(x_1, x_2) \in H$, $(x_1, x_2) \in K$

If $(x_1, x_2) \in H$, then $x_2 = e$

And if $(x_1, x_2) \in K$, then $x_1 = e$

Therefore, $(x_1, x_2) = (e, e)$ $\square$

13 Let p, q be distinct prime numbers, and let $n = pq$. Show that $HK = \mathbb{Z}_n^\times$, for the subgroups $H = \{[x] \in \mathbb{Z}_n^\times \mid x \equiv 1 \pmod{p}\}$ and $K = \{[y] \in \mathbb{Z}_n^\times \mid y \equiv 1 \pmod{q}\}$ of $\mathbb{Z}_n^\times$.
Hint: You can either use a counting argument to show that HK has $\varphi(n)$ elements, or use the Chinese Remainder Theorem to show that the sets are the same.

Ans Given, HK is a subgroup of $\mathbb{Z}_n^\times$,

If $\forall a, b \in HK$

$$\begin{aligned}
a &\equiv 1 \pmod{pq} \\
b &\equiv 1 \pmod{pq} \\
ab &\equiv 1 \pmod{pq}
\end{aligned}$$

If $x \in H \cap K$

$$\begin{aligned}
x &\equiv 1 \pmod{p} \\
x &\equiv 1 \pmod{q}
\end{aligned}$$

And by the Chinese Remainder Theorem, $x = 1$

Therefore,

$$|HK| = \varphi(n)$$

$$
\begin{aligned}
&= |\mathbb{Z}_n^\times| \\
&\Rightarrow HK = \mathbb{Z}_n^\times
\end{aligned}
$$

$\square$

16 Let G be a group of order 6, and suppose that $a, b \in G$ with a of order 3 and b of order 2. Show that either G is cyclic or $ab \neq ba$.

Ans If $ab \neq ba$, then G is non-abelian

If $ab = ba$, then

$$
\begin{aligned}
(ab)^6 &= a^6 b^6 \\
&= (a^3)^2 (b^2)^3 \\
&\Rightarrow e \ (\because o(a) = 3, o(b) = 2)
\end{aligned}
$$

Since $o(ab) = 6$, G is cyclic generated by ab

Therefore, G is cyclic or $ab \neq ba$ $\square$