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CMPE 320: HW 08

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1. Let X have a uniform distribution in the unit interval $[0,1]$, and let Y have an exponential distribution with parameter $\nu = 2$. Assume that X and Y are independent. Let $Z = X + Y$.

(a) Find $P(Y \geq X)$.

Sol. Since X and Y are independent, $f_{X,Y}(x,y) = f_X(x)g_Y(y)$

Therefore, the joint PDF,

$$f_{X,Y}(x,y) = \begin{cases} 2e^{-2y}, & \text{if } 0 \leq x \leq 1,\ y \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$\begin{aligned} P(Y \geq X) &= 1 - P(X \leq Y) \\ &= \iint_{y \geq x} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= 1 - \int_0^1 \int_0^x 2e^{-2y}\,\mathrm{d}x\,\mathrm{d}y \\ &= 1 - \int_0^1 1 - e^{-2x}\,\mathrm{d}x \\ &= \frac{1}{2} - \frac{e^{-2}}{2} \end{aligned}$$

□

(b) Find the conditional PDF of Z given that $Y = y$.

Sol.

$$\begin{aligned} f_{Z|Y=y}(z) &= f_{X+Y|Y=y}(x+y) \\ &= \begin{cases} 1, & \text{if } y \leq z \leq 1+y \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

□

(c) Find the conditional PDF of Y given that $Z = 3$.

Sol. With the laws of conditional probability,

$$f_{Y|3}(y \mid 3) = \frac{f_{Y,Z}(y,3)}{f_Z(3)} = \frac{f_{Z|Y=y}(3 \mid y) f_Y(y)}{f_Z(3)}$$

And,

$$\begin{aligned}
F_Z(3) &= \int_0^1 \int_0^{z-x} f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\
&= \int_0^1 \int_0^{z-x} 2e^{-2y}\,\mathrm{d}x\,\mathrm{d}y \\
&= 1 - \frac{e^{-2(3)+2}}{2} + \frac{e^{-2(3)}}{2} \\
&= e^{-4} - e^{-6}
\end{aligned}$$

Therefore,

$$f_{Y|3}(y \mid 3) = \begin{cases} \dfrac{2e^{-2y}}{e^{-4} - e^{-6}}, & \text{if } 2 \le y \le 3 \\ \\ 0, & \text{otherwise,} \end{cases}$$

□

2. Let P, a random variable which is uniformly distributed between 0 and 1. On any given day, a particular machine is functional with probability P. Furthermore, given the value of P, the status of the machine on different days is independent

(a) Find the probability that the machine is functional on a particular day.

Sol. Let W represent the event that the machine is functional

Then,

$$\begin{aligned}
P(W) &= \int_0^1 P(W \mid X = x) f_X(x)\,\mathrm{d}x \\
&= \int_0^1 x\,\mathrm{d}x \\
&= \frac{1}{2}
\end{aligned}$$

□

(b) We are told that the machine was functional on m out of the last n days. Find the conditional PDF of P. You may use the identity

$$\int_0^1 p^k (1-p)^{n-k} dp = \frac{k!(n-k)!}{(n+1)!}$$

Sol.

$$
\begin{aligned}
P(W_m) &= \int_0^1 P(W_m \mid X = x) f_X(x)\,\mathrm{d}x \\
&= \int_0^1 \binom{n}{m} x^m (1-x)^{n-m} f_X(x)\,\mathrm{d}x \\
&= \binom{n}{m} \frac{m!(n-m)!}{(n+1)!}
\end{aligned}
$$

Therefore, using Bayes rule,

$$
\begin{aligned}
f_{X|W_m}(x) &= \frac{P(W_m \mid X = x) f_X(x)}{P(W_m)} \\
&= \frac{x^m (1-x)^{n-m}}{\frac{m!(n-m)!}{(n+1)!}},\ 0 \le q \le 1,\ n \ge m
\end{aligned}
$$

□

(c) Find the conditional probability that the machine is functional today given that it was functional on m out of the last n days.

Sol.

$$
\begin{aligned}
P(W_m) &= \int_0^1 P(W_m \mid X = x) f_X(x)\,\mathrm{d}x \\
&= \int_0^1 \binom{n}{m} x^m (1-x)^{n-m} f_X(x)\,\mathrm{d}x \\
&= \binom{n}{m} \frac{m!(n-m)!}{(n+1)!}
\end{aligned}
$$

□

3. Let $B \triangleq \{a < X \le b\}$. Derive a general expression for $E[X \mid B]$ if X is a continuous RV. Let $X : N(0,1)$ with $B = \{-1 < X \le 2\}$. Compute $E[X \mid B]$.

Sol. □

4. A particular model of an HDTV is manufactured in three different plants, say, A, B and C, of the same company. Because the workers at A, B and C are not equally experienced, the quality of the units differs from plant to plant. The pdf's of the time-to-failure X, in years, are

$$
\begin{aligned}
f_X(x) &= \frac{1}{5}\exp(-x/5)u(x) \text{ for } A \\
f_X(x) &= \frac{1}{6.5}\exp(-x/6.5)u(x) \text{ for } B
\end{aligned}
$$

$$f_X(x) = \frac{1}{10}\exp(-x/10)u(x) \text{ for } C,$$

where $u(x)$ is the unit step. Plant A produces three times as many units as B, which produces twice as many as C. The TVs are all sent to a central warehouse, intermingled, and shipped to retail stores all around the country. What is the expected lifetime of a unit purchased at random?

Sol. The expectations of the exponential distributions, $1/\lambda$,

$$E[A] = 5$$
$$E[B] = 6.5$$
$$E[C] = 10$$

Given the ratio of the units is $6:2:1$,

$$P(A) = \frac{6}{9}$$
$$P(B) = \frac{2}{9}$$
$$P(C) = \frac{1}{9}$$

Therefore, the expected lifetime of a unit purchased at random,

$$\begin{aligned} E &= \frac{5\times 6}{9} + \frac{6.5\times 2}{9} + \frac{10\times 1}{9} \\ &= \frac{53}{6} \end{aligned}$$

□

5. The coordinate X and Y of a point are independent zero mean normal random variables with common variances σ^2. Given that the point is at a distance of at least c from the origin, find the conditional joint PDF of X and Y.

Sol. Given,

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(\frac{-x^2}{2\sigma^2}\right)$$
$$f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left(\frac{-y^2}{2\sigma^2}\right)$$

Since X and Y are assumed independent

$$f_{(X,Y)}(x,y) = f_X(x)f_Y(y) = \frac{1}{2\pi\sigma^2}\exp\left(\frac{-(x^2+y^2)}{2\sigma^2}\right)$$

Therefore,

$$\begin{aligned} P(x^2+y^2 \geq c^2) &= \int f_{X,Y}(x,y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \frac{1}{2\pi\sigma^2}\int_c^\infty \exp\left(\frac{-r^2}{2\sigma^2}\right)2\pi r\,\mathrm{d}r \end{aligned}$$ □

6. Alexei is vacationing in Monte Carlo. The amount X (in dollars) he takes to the casino each evening is a random variable with a PDF of the form

$$f_X(x) = \begin{cases} ax, & \text{if } 0 \leq x \leq 40, \\ 0, & \text{otherwise} \end{cases}$$

At the end of each night, the amount Y that he has when leaving the casino is uniformly distributed between zero and twice the amount that he came with.

(a) Determine the joint PDF $f_{X,Y}(x,y)$.

Sol. Since

$$\begin{aligned} \int_0^{40} ax\,\mathrm{d}x &= 1 \\ a\frac{40^2}{2} &= 1 \\ \implies a &= \frac{1}{800} \end{aligned}$$

The uniform distribution,

$$f_{Y|X}(y \mid x) = \begin{cases} \frac{1}{2x}, & \text{if } 0 \leq y \leq 2x, \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$\begin{aligned} f_{X,Y}(x,y) &= \frac{f_{Y|X}(y \mid x)}{f_X(x)} \\ &= \begin{cases} \frac{ax}{2x}, & \text{if } 0 \leq x \leq 40,\ 0 \leq y \leq 2x, \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{a}{2}, & \text{if } 0 \leq x \leq 40,\ 0 \leq y \leq 2x, \\ 0, & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{1}{1600}, & \text{if } 0 \leq x \leq 40,\ 0 \leq y \leq 2x, \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$ □

(b) What is the probability that on a given night Alexei makes a positive profit at the casino?

Sol. Since the probability of loss is $0 \le y \le x$, and the probability of profit is $x \le y \le 2x$,

$$P(\text{profit}) = P(y - x > 0) = \frac{1}{2} \qquad \square$$

(c) Find the PDF of Alexei's profit $Y - X$ on a particular night, and also determine its expected value.

Sol.

$$P(x - y < m) = \begin{cases} \displaystyle\int_m^{40} \frac{x+m}{2x} f(x)\,\mathrm{d}x, & \text{if } t > 0, \\ \displaystyle\int_{-m}^{40} \frac{x-m}{2x} f(x)\,\mathrm{d}x, & \text{otherwise} \end{cases}$$

Since $f(x) = ax$,

$$\begin{aligned} P(x - y < m) &= \begin{cases} \displaystyle\int_m^{40} \frac{x+m}{2} a\,\mathrm{d}x, & \text{if } t > 0, \\ \displaystyle\int_{-m}^{40} \frac{x-m}{2} a\,\mathrm{d}x, & \text{otherwise} \end{cases} \\ &= \begin{cases} \displaystyle\int_m^{40} \frac{x+m}{2} a\,\mathrm{d}x, & \text{if } t > 0, \\ \displaystyle\int_{-m}^{40} \frac{x-m}{2} a\,\mathrm{d}x, & \text{otherwise} \end{cases} \\ &= \begin{cases} \displaystyle\frac{1}{1600}\left(\frac{x^2}{2} + mx\right)\Bigg|_m^{40}, & \text{if } t > 0, \\ \displaystyle\frac{1}{1600}\left(\frac{x^2}{2} - mx\right)\Bigg|_{-m}^{40}, & \text{otherwise} \end{cases} \\ &= \begin{cases} \displaystyle\frac{1}{1600}\left(800 - \frac{3m^2}{2} + 40m\right), & \text{if } t > 0, \\ \displaystyle\frac{1}{1600}\left(800 + \frac{m^2}{2} - 40m\right), & \text{otherwise} \end{cases} \end{aligned}$$

Therefore, the PDF

$$= \begin{cases} \displaystyle\frac{1}{1600}(40 - 3m), & \text{if } t > 0, \\ \displaystyle\frac{1}{1600}(m - 40), & \text{otherwise} \end{cases}$$

And the expected value,

$$
\begin{aligned}
&= \frac{1}{1600}\left(\int_0^{40}(40-3m)m\,\mathrm{d}m + \int_{-40}^{0}(m-40)m\,\mathrm{d}m\right) \\
&= \frac{1}{1600}\left((20m^2-m^3)\Big|_0^{40} + (\frac{m^3}{3}-20m^3)\Big|_{-40}^{0}\right) \\
&= \frac{1}{1600}\left(-32000+\frac{160000}{3}\right) \\
&= \frac{40}{3} \qquad \square
\end{aligned}
$$

7. Consider a communication channel corrupted by noise. Let X be the value of the transmitted signal and Y be the value of the received signal. Assume that the conditional density of Y given $\{X = x\}$ is Gaussian, that is,

$$f_{Y|X}(y \mid x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(\frac{-(y-x)^2}{2\sigma^2}\right),$$

and X is uniformly distributed on $[-1,1]$. What is the conditional pdf of X given Y, that is, $f_{X|Y}(x \mid y)$

Sol. Given,

$$f_X(x) = \begin{cases} \frac{1}{2}, & \text{if } -1 \le x \le 1 \\ 0, & \text{otherwise,} \end{cases}$$

And,

$$
\begin{aligned}
f_Y(y) &= \int_{\infty} f_{X,Y}(x,y)\,\mathrm{d}x \\
&= \int_{\infty} f_{Y|X}(y \mid x) f_X(x)\,\mathrm{d}x \\
&= \int_{-1}^{1} \frac{1}{2}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(\frac{-(y-x)^2}{2\sigma^2}\right)\mathrm{d}x \\
&= \frac{1}{2}\left(\phi\left(\frac{y-1}{\sigma}\right) - \phi\left(\frac{y+1}{\sigma}\right)\right)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
f_{X|Y}(x \mid y) &= \frac{f_{Y|X}(y \mid x) f_X(x)}{f_Y(y)} \\
&= \frac{\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(\frac{-(y-x)^2}{2\sigma^2}\right) f_X(x)}{\phi\left(\frac{y-1}{\sigma}\right) - \phi\left(\frac{y+1}{\sigma}\right)} \qquad \square
\end{aligned}
$$

8. A U.S. defense radar scans the skies for unidentified flying objects (UFOs). Let M be the event that a UFO is present and M^C the event that a UFO is absent. Let $f_{X|M}(x \mid M) = \frac{1}{\sqrt{2\pi}} \exp(-0.5[x-r]^2)$ be the conditional pdf of the radar return signal X when a UFO is actually there, and let $f_{X|M^C}(x \mid M) = \frac{1}{\sqrt{2\pi}} \exp(-0.5[x]^2)$ be the conditional pdf of the radar return signal X when there is no UFO. To be specific, let $r = 1$ and let the *alert level* be $x_A = 0.5$. Let A denote the event of an alert, that is, $\{X > x_A\}$. Compute $P[A \mid M]$, $P[A^C \mid M]$, $P[A \mid M^C]$, $P[A^C \mid M^C]$.

Assume that $P[M] = 10^{-3}$. Compute $P[A \mid M]$, $P[A^C \mid M]$, $P[A \mid M^C]$, $P[A^C \mid M^C]$.

Repeat for $P[M] = 10^{-6}$

Sol. $\square$