Second Midterm Examination Version 2 Solutions

1. The resistance should equal 75 Ω. The impedance of a shorted stub is given by $Z = jZ_0 \tan\beta d$, where Z_0 is the line impedance of the stub, β is the wavenumber and d is the stub length. We must have $\tan\beta d = 1$, which implies $\beta d = 2\pi d/\lambda = \pi/4$, from which we find $d/\lambda = 1/8$. A frequency of 100 MHz corresponds to $\lambda = 3$ m. We conclude that $d = 0.375$ m $= 37.5$ cm.

2. The definition of the gradient is

$$\nabla\psi = \hat{\mathbf{n}} \lim_{\Delta\mathbf{l}\to 0}\left[\frac{\psi(\mathbf{R}+\Delta\mathbf{l}) - \psi(\mathbf{R})}{|\Delta\mathbf{l}|}\right],$$

where $\hat{\mathbf{n}} = \Delta\mathbf{l}/|\Delta\mathbf{l}|$. We now let

$$\mathbf{A} = \hat{\mathbf{x}}\frac{\partial\psi}{\partial x} + \hat{\mathbf{y}}\frac{\partial\psi}{\partial y} + \hat{\mathbf{z}}\frac{\partial\psi}{\partial z}.$$

We have

$$\frac{\psi(\mathbf{R}+\Delta\mathbf{l}) - \psi(\mathbf{R})}{|\Delta\mathbf{l}|} = \frac{\partial\psi}{\partial x}\frac{\Delta l_x}{|\Delta\mathbf{l}|} + \frac{\partial\psi}{\partial y}\frac{\Delta l_y}{|\Delta\mathbf{l}|} + \frac{\partial\psi}{\partial z}\frac{\Delta l_z}{|\Delta\mathbf{l}|} = \mathbf{A}\cdot\hat{\mathbf{n}} = A\cos\theta,$$

where A is the magnitude of $\mathbf{A}$ and θ is the angle between $\mathbf{A}$ and $\hat{\mathbf{n}}$. The maximum value is A and occurs when $\hat{\mathbf{n}} = \mathbf{A}/A$. We thus find $\nabla\psi = (\mathbf{A}/A)A = \mathbf{A}$.

3. Maxwell's equations in differential form are

$$\nabla\cdot\mathbf{D}(\mathbf{R},t) = \rho_V(\mathbf{R},t), \qquad \nabla\times\mathbf{E}(\mathbf{R},t) = -\frac{\partial\mathbf{B}(\mathbf{R},t)}{\partial t},$$
$$\nabla\cdot\mathbf{B}(\mathbf{R},t) = 0, \qquad \nabla\times\mathbf{H}(\mathbf{R},t) = \mathbf{J}(\mathbf{R},t) + \frac{\partial\mathbf{D}(\mathbf{R},t)}{\partial t}.$$

Gauss's theorem states:

$$\int_v \nabla\cdot\mathbf{A}(\mathbf{R})\,dv = \oint_S \mathbf{A}(\mathbf{R})\cdot d\mathbf{S},$$

where the surface S encloses the volume v and $\mathbf{A}$ is any vector. Stokes's theorem states:

$$\int_S [\nabla\times\mathbf{A}(\mathbf{R})]\cdot d\mathbf{S} = \oint_C \mathbf{A}(\mathbf{R})\cdot d\mathbf{l},$$

where S is any surface that is connected to the closed contour C. Using Gauss's theorem on the two divergence equations and Stokes's theorem on the two curl equations yields Maxwell's equations in integral form,

$$\oint_S \mathbf{D}\cdot\mathbf{S} = Q = \int_v \rho_V\,dv, \quad \oint_C \mathbf{E}\cdot d\mathbf{l} = -\int_S \frac{\partial\mathbf{B}}{\partial t}\cdot d\mathbf{S},$$
$$\oint_S \mathbf{B}\cdot\mathbf{S} = 0, \qquad \oint_C \mathbf{H}\cdot d\mathbf{l} = \int_S\left(\mathbf{J} + \frac{\partial\mathbf{D}}{\partial t}\right)\cdot d\mathbf{S}.$$

4. If the inner conductor of the capacitor has a charge $+q$, we find from Gauss's law that

$$\mathbf{E}(r,\phi,z) = \hat{\mathbf{r}}\frac{q}{2\pi\epsilon r L}.$$

   Taking the outer conductor as the ground, we find that the voltage as a function of radius is given by $V(r,\theta,z) = V(r) = (q/2\pi\epsilon L)\ln(b/r)$, and the voltage on the inner cylinder is given by $V \equiv V(a) = (q/2\pi\epsilon L)\ln(b/a)$. The change in energy when we move charge from the ground to the inner conductor is given by $dU = V dq$. Letting Q equal the total charge that is moved from ground to the inner conductor, we find that $U = (Q^2/4\pi\epsilon L)\ln(b/a)$. We next calculate

$$\begin{aligned} I &= \frac{\epsilon}{2}\int_v |\mathbf{E}|^2\,dv = \frac{\epsilon}{2}\left(\frac{Q}{2\pi\epsilon L}\right)^2 \int_0^L dz \int_0^{2\pi} d\phi \int_a^b r dr (1/r^2) \\ &= \frac{Q^2}{4\pi\epsilon L}\ln(b/a) = U. \end{aligned}$$

5. From the Biot-Savart law, we have that the magnetic flux from a current carrying conductor, in which the current is flowing in the z-direction is given by $\mathbf{B} = (\mu_0 I/4\pi r)\hat{\boldsymbol{\phi}}$, where r is the distance from the wire. The force per unit length that one wire exerts on the other is given by $F = \mu_0 I^2/4\pi r$. Since $\mu_0 = 4\pi \times 10^{-7}$ H/m, we find that $F = 10^{-7}$ N/m. The two wires attract. The force per unit length that is exerted by the wire carrying 10 A on the wire that is 1 m from it equals 10^{-6} N/m. The total force per unit length that is exerted on this wire is 9×10^{-7} N/m toward the wire carrying 10 A. The force per unit length that the wire carrying 10 A exerts on the fiber that is 2 m from it equals 5×10^{-7} N/m. The total force per unit length acting on this wire is 6×10^{-7} N/m. The net difference of 3×10^{-7} N/m tends to pull the two original wires apart.