HW#7 Solutions

Problem 1.

Let X denote the error of the radar. The desired probability is

$$\mathbf{P}(X < 0) = \mathbf{P}\left(\frac{X-50}{100} \leq \frac{-50}{100}\right) = \Phi(-0.5) = 1 - \Phi(0.5) = 1 - 0.6915 = 0.3085,$$

where the value of $\Phi(0.5)$ is obtained from the normal tables.

Problem 2.

(a) The random variable Y is normal with mean 5 and variance 16. Therefore,

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\,4} e^{-(y-5)^2/32}.$$

(b) We have

$$\mathbf{P}(Y \geq 0) = \mathbf{P}\left(\frac{Y-5}{4} \geq -\frac{5}{4}\right) = \mathbf{P}\left(\frac{Y-5}{4} \leq \frac{5}{4}\right) = \Phi\left(\frac{5}{4}\right) = 0.8944.$$

Problem 3.

(a) Let G and B be the events of good and bad weather, respectively. We are given that

$$\mathbf{P}(G) = \mathbf{P}(G) = \frac{1}{2}.$$

We first calculate the PDF of W by using the density version of the total probability theorem:

$$f_W(w) = \mathbf{P}(G) \cdot f_{W|G}(w) + \mathbf{P}(B) \cdot f_{W|B}(w).$$

The conditional densities $f_{W|G}$ and $f_{W|B}$ are zero mean normal with variance 1 and 4, respectively, so

$$f_W(w) = \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} e^{-w^2/2} + \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} e^{-w^2/8}.$$

We find the PDF of X by using the change of variables $X = 2 + W$:

$$f_X(x) = f_W(x-2) = \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} e^{-(x-2)^2/2} + \frac{1}{2} \cdot \frac{1}{\sqrt{2\pi}} e^{-(x-2)^2/8}.$$

(b) We have

$$\mathbf{P}(1 \le X \le 3) = \mathbf{P}(1 \le 2 + W \le 3) = \mathbf{P}(-1 \le W \le 1).$$

We now use the total probability theorem and the normal tables to calculate the above probability. We have

$$\begin{aligned}
\mathbf{P}(-1 \le W \le 1) &= \mathbf{P}(G) \cdot \mathbf{P}(-1 \le W \le 1 \,|\, G) + \mathbf{P}(B) \cdot \mathbf{P}(-1 \le W \le 1 \,|\, B) \\
&= \mathbf{P}(G) \cdot \mathbf{P}(-1 \le W \le 1 \,|\, G) + \mathbf{P}(B) \cdot \mathbf{P}\left(-\frac{1}{2} \le W \le \frac{1}{2} \,|\, B\right) \\
&= \mathbf{P}(G)\left\{\mathbf{P}(W \le 1 \,|\, G) - \mathbf{P}(W \le -1 \,|\, G)\right\} \\
&\quad + \mathbf{P}(B)\left\{\mathbf{P}\left(W \le \frac{1}{2} \,|\, B\right) - \mathbf{P}\left(W \le -\frac{1}{2} \,|\, B\right)\right\} \\
&= \frac{1}{2}\left\{\Phi(1) - \Phi(-1)\right\} + \frac{1}{2}\left\{\Phi\left(\frac{1}{2}\right) - \left(1 - \Phi\left(\frac{1}{2}\right)\right)\right\} \\
&= \frac{1}{2}\left\{2\Phi(1) - 1\right\} + \frac{1}{2}\left\{2\Phi\left(\frac{1}{2}\right) - 1\right\} \\
&\approx \frac{1}{2}\left\{2 \cdot 0.8413 - 1\right\} + \frac{1}{2}\left\{2 \cdot 0.6915 - 1\right\} \\
&= 0.5328.
\end{aligned}$$

Problem 4.

(a) Let Z be the random variable representing the additive zero-mean Gaussian noise, and note that Z is zero-mean normal with variance σ^2. There are two ways for errors to occur. (1) The true encoded signal could be -1 (received signal is $Z - 1$) but we would conclude that the encoded signal of $+1$ was sent if $Z - 1 > a$. (2) The true encoded signal could be $+1$ (received signal is $Z + 1$) but we could conclude that the encoded signal of -1 was sent if $Z + 1 < a$. Therefore

$$\begin{aligned}
\mathbf{P}(\text{error}) &= p \cdot \mathbf{P}(Z - 1 > a) + (1 - p) \cdot \mathbf{P}(Z + 1 < a) \\
&= p \cdot \left(1 - \Phi\left(\frac{a - (-1)}{\sigma}\right)\right) + (1 - p) \cdot \Phi\left(\frac{a - 1}{\sigma}\right) \\
&= p - p \cdot \Phi\left(\frac{1 + a}{\sigma}\right) + (1 - p) \cdot \left(1 - \Phi\left(\frac{1 - a}{\sigma}\right)\right) \\
&= 1 - p \cdot \Phi\left(\frac{1 + a}{\sigma}\right) - (1 - p) \cdot \Phi\left(\frac{1 - a}{\sigma}\right).
\end{aligned}$$

(b) We have

$$\mathbf{P}(\text{error}) = 1 - 0.4 \cdot \Phi\left(\frac{3/2}{1/2}\right) - 0.6 \cdot \Phi\left(\frac{1/2}{1/2}\right).$$

Problem 5.

(a) Since the time for connection is uniformly distributed, the probability that you will wait more than 15 seconds is

$$\int_{15}^{30} \frac{1}{30} = \frac{1}{2}.$$

(b) Given that you have waited 10 seconds without connection, the time for the connection is uniformly distributed between 10 and 30. Thus the probability that you will wait another 10 seconds is

$$\int_{20}^{30} \frac{1}{20} = \frac{1}{2}.$$

Problem 6.

We have

$$\mathbf{E}[X] = \int_{1}^{2} \frac{2x^2}{3} dx = \frac{2}{3}\frac{x^3}{3}\Big|_1^2 = \frac{2}{3}\left(\frac{8}{3} - \frac{1}{3}\right) = \frac{14}{9}.$$

Also

$$\mathbf{P}(A) = \int_{1.5}^{2} \frac{2x}{3} dx = \frac{2}{3}\frac{x^2}{2}\Big|_{1.5}^2 = \frac{2}{3}\left(\frac{4}{2} - \frac{9/4}{2}\right) = \frac{7}{12}.$$

The conditional PDF $f_{X|A}$ is given by

$$f_{X|A}(x) = \begin{cases} \dfrac{f_X(x)}{\mathbf{P}(A)} & \text{if } 1.5 \le x \le 2, \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} \dfrac{2x}{3\cdot(7/12)} & \text{if } 1.5 \le x \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$\mathbf{E}[X \mid A] = \int_{1.5}^{2} \frac{2x^2}{3\cdot(7/12)} dx = \frac{8}{7}\frac{x^3}{3}\Big|_{1.5}^2 = \frac{8}{21}\left(8 - \frac{27}{8}\right) = \frac{37}{21}.$$

Problem 7.

Let G denote the event that Dino has a good day and let B denote the event that Dino has a bad day. We are given that $\mathbf{P}(G) = \mathbf{P}(B) = .5$. Let T be the time it takes Dino to cook a souffle, so that $f_{T|G}$ is uniform between $1/2$ and 1, and $f_{T|B}$ is uniform between $1/2$ and $3/2$. We need to find $\mathbf{P}(B \,|\, T \le 3/4)$.

Using Bayes' rule, we have

$$\mathbf{P}(B \,|\, T \le 3/4) = \frac{\mathbf{P}(T \le 3/4 \,|\, B)\mathbf{P}(B)}{\mathbf{P}(T \le 3/4)} = \frac{\mathbf{P}(T \le 3/4 \,|\, B)\mathbf{P}(B)}{\mathbf{P}(T \le 3/4 \,|\, B)\mathbf{P}(B) + \mathbf{P}(T \le 3/4 \,|\, G)\mathbf{P}(G)}$$

Evaluating this expression, we find that

$$\mathbf{P}(B \,|\, T \le 3/4) = \frac{(\frac{1}{4})(\frac{1}{2})}{(\frac{1}{4})(\frac{1}{2}) + (\frac{1}{2})(\frac{1}{2})} = \frac{1}{3}.$$

Problem 8.

We have

$$\begin{aligned}
\mathbf{P}(A \,|\, X < 1/4) &= \frac{\mathbf{P}(A)\mathbf{P}(X \le 1/4 \,|\, A)}{\mathbf{P}(A)\mathbf{P}(X \le 1/4 \,|\, A) + \mathbf{P}(B)\mathbf{P}(X \le 1/4 \,|\, B)} \\
&= \frac{\mathbf{P}(A)\int_0^{1/4} f_{X|A}(x|A)\,dx}{\mathbf{P}(A)\int_0^{1/4} f_{X|A}(x|A)\,dx + \mathbf{P}(B)\int_0^{1/4} f_{X|B}(x|B)\,dx} \\
&= \frac{0.5\int_0^{1/4} 1\,dx}{0.5\int_0^{1/4} 1\,dx + 0.5\int_0^{1/4} 3\,dx} \\
&= \frac{1}{4}.
\end{aligned}$$