

2.3 1 Consider the following permutations in S_7

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 5 & 4 & 6 & 1 & 7 \end{pmatrix} \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 5 & 7 & 4 & 6 & 3 \end{pmatrix}$$

Compute the following products:

b $\tau\sigma$

Ans

$$\begin{aligned} \tau\sigma &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 5 & 4 & 6 & 1 & 7 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 5 & 7 & 4 & 6 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 6 & 7 & 4 & 1 & 5 \end{pmatrix} \end{aligned}$$

□

f $\tau^{-1}\sigma\tau$

Ans

$$\begin{aligned} \tau^{-1} &= \begin{pmatrix} 2 & 1 & 5 & 7 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 3 & 6 & 7 & 4 & 1 & 5 \\ 1 & 7 & 6 & 4 & 5 & 2 & 3 \end{pmatrix} \\ \tau^{-1}\sigma &= \begin{pmatrix} 2 & 3 & 6 & 7 & 4 & 1 & 5 \\ 1 & 7 & 6 & 4 & 5 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 5 & 4 & 6 & 1 & 7 \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
&= \begin{pmatrix} 2 & 3 & 6 & 7 & 4 & 1 & 5 \\ 3 & 7 & 1 & 4 & 6 & 2 & 5 \end{pmatrix} \\
\tau^{-1}\sigma\tau &= \begin{pmatrix} 2 & 3 & 6 & 7 & 4 & 1 & 5 \\ 3 & 7 & 1 & 4 & 6 & 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 5 & 4 & 6 & 1 & 7 \end{pmatrix} \\
&= \begin{pmatrix} 2 & 3 & 6 & 7 & 4 & 1 & 5 \\ 5 & 7 & 3 & 4 & 1 & 2 & 6 \end{pmatrix} \quad \square
\end{aligned}$$

3 Write $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 4 & 10 & 5 & 7 & 8 & 2 & 6 & 9 & 1 \end{pmatrix}$ as a product of disjoint cycles and as a product of transpositions. Construct its associated diagram, find its inverse, and find its order.

Ans The product of disjoint cycles:

$$\{(1, 3, 10)(2, 4, 5, 7)(6, 8)\}$$

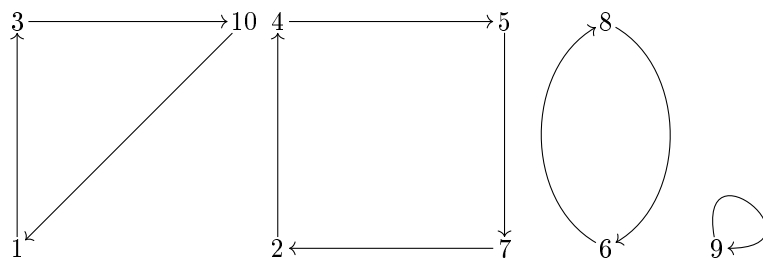
The product of transpositions:

$$\{(1, 3, 10)(2, 4, 5, 7)(6, 8)\} = \{(1, 3)(3, 10)(2, 4)(4, 5)(5, 7)(6, 8)\}$$

Reconstructing the permutation based on the product of transpositions:

$$\begin{aligned}
\sigma &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 4 & 10 & 5 & 7 & 8 & 2 & 6 & 9 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 3 & 10 & 2 & 4 & 5 & 7 & 6 & 8 & 9 \\ 3 & 10 & 1 & 4 & 5 & 7 & 2 & 8 & 6 & 9 \end{pmatrix}
\end{aligned}$$

Constructing the associated diagrams



The inverse of the permutation:

$$\begin{aligned}\sigma^{-1} &= \begin{pmatrix} 3 & 4 & 10 & 5 & 7 & 8 & 2 & 6 & 9 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 10 & 7 & 1 & 2 & 4 & 8 & 5 & 6 & 9 & 3 \end{pmatrix}\end{aligned}$$

Since

$$\begin{aligned}o(\sigma) &= lcm(\text{length}(\text{cycles})) \\ &= lcm(\{2, 4, 3\}) \\ &= 12\end{aligned}$$

□

5 Let $3 \leq m \leq n$. Calculate $\sigma\tau^{-1}$ for the cycles $\sigma = (1, 2, \dots, m-1)$ and $\tau = (1, 2, \dots, m-1, m)$ in S_n .

Ans Given

$$\begin{aligned}\tau &= \begin{pmatrix} 1 & 2 & 3 & \dots & m-2 & m-1 & m \\ 2 & 3 & 4 & \dots & m-1 & m & 1 \end{pmatrix} \\ \tau^{-1} &= \begin{pmatrix} 2 & 3 & 4 & \dots & m-1 & m & 1 \\ 1 & 2 & 3 & \dots & m-2 & m-1 & m \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & m-1 & m \\ m & 1 & 2 & 3 & \dots & m-2 & m-1 \end{pmatrix}\end{aligned}$$

Therefore the product

$$\begin{aligned}
 \sigma\tau^{-1} &= \begin{pmatrix} 1 & 2 & 3 & \dots & m-2 & m-1 \\ 2 & 3 & 4 & \dots & m-1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & \dots & m-1 & m \\ m & 1 & 2 & \dots & m-2 & m-1 \end{pmatrix} \\
 &= \begin{pmatrix} 1 & 2 & 3 & \dots & m-1 & m \\ m & 2 & 3 & \dots & m-1 & 1 \end{pmatrix} \\
 &= (1, m)
 \end{aligned}$$

□

11 Prove that in S_n , with $n \geq 3$, any even permutation is a product of cycles of length three.

Hint: $(a, b)(b, c) = (a, b, c)$ and $(a, b)(c, d) = (a, b, c)(b, c, d)$.

Ans To show any even permutation in S_n , with $n \geq 3$ is a product of cycles of length three, consider the case where the pair of transpositions are disjoint:

$$(a, b)(c, d) = (a, b, c)(b, c, d)$$

The product yields cycles of length three.

The other case would be the transpositions consisting of repeating elements, such as

$$(a, b)(b, c) = (a, b, c)$$

where the product yields cycles of length three as well.

□

15 For $\alpha, \beta \in S_n$, let $\alpha \sim \beta$ if there exists $\sigma \in S_n$ such that $\sigma\alpha\sigma^{-1} = \beta$. Show that \sim is an equivalence relation on S_n .

Ans To prove **reflexivity**, let $\alpha = 1_S$

Then,

$$\begin{aligned}
 \sigma\alpha\sigma^{-1} &= \sigma\sigma^{-1} \cdot 1_S \\
 &= 1 \cdot 1_S \\
 &= \alpha
 \end{aligned}$$

for any $\alpha \in S_n$

Therefore, $\alpha \sim \alpha$

To prove **symmetry**, let $\alpha \sim \beta$

Then, there exists $\sigma \in S_n$ such that $\sigma\alpha\sigma^{-1} = \beta$

Thus

$$\begin{aligned}\sigma\alpha\sigma^{-1} &= \beta \\ \Rightarrow \sigma^{-1}\sigma\alpha\sigma^{-1} &= \sigma^{-1}\beta \\ \alpha\sigma^{-1} &= \sigma^{-1}\beta \\ \Rightarrow \alpha\sigma^{-1}\sigma &= \sigma^{-1}\beta\sigma \\ \alpha &= \sigma^{-1}\beta\sigma \\ \Rightarrow \alpha &= \sigma^{-1}\beta(\sigma^{-1})^{-1}\end{aligned}$$

Or $\sigma^{-1}\beta(\sigma^{-1})^{-1} = \alpha$

Implying $\beta \sim \alpha$

To prove **transitivity**, let $\alpha \sim \beta$ and $\beta \sim \gamma$

Then, there exists $\sigma_1, \sigma_2 \in S_n$ such that $\sigma_1\alpha\sigma_1^{-1} = \beta$ and $\sigma_2\beta\sigma_2^{-1} = \gamma$

Then $\sigma_2\sigma_1\alpha\sigma_1^{-1}\sigma_2^{-1} = \gamma$,

Or $(\sigma_2\sigma_1)\alpha(\sigma_1\sigma_2)^{-1} = \gamma$

Thus, $\alpha \sim \gamma$

Therefore, \sim is an equivalence relation on S_n

□

16 View S_3 as a subset of S_5 , in the obvious way. For $\sigma, \tau \in S_5$, define $\sigma \sim \tau$ if $\sigma\tau^{-1} \in S_3$.

a Show that \sim is an equivalence relation on S_5 .

Ans To prove **reflexive**, let $\sigma \in S_3$

Then,

$$\sigma\sigma^{-1} = 1_{S_3} \in S_3$$

Thus, $\sigma \sim \sigma$

To prove **symmetric**, let $\sigma, \tau \in S_3$

Then

$$\sigma \sim \tau \Rightarrow \sigma\tau^{-1} \in S_3$$

$$\begin{aligned}\Rightarrow \sigma\tau^{-1} &\in S_3 \\ \Rightarrow (\sigma\tau^{-1})^{-1} &\in S_3 \\ \Rightarrow \sigma^{-1}\tau &\in S_3\end{aligned}$$

Thus, $\tau \sim \sigma$

To prove **transitivity**, let $\sigma, \tau, v \in S_3$.

Then, $\sigma \sim \tau \Rightarrow \sigma\tau^{-1} \in S_3$

and $\tau \sim v \Rightarrow \tau v^{-1} \in S_3$

Thus,

$$\begin{aligned}(\sigma\tau^{-1})(\tau v^{-1}) &\in S_3 \\ \sigma(\tau^{-1}\tau)v^{-1} &\in S_3 \\ \sigma v^{-1} &\in S_3 \\ \Rightarrow \sigma \sim v &\in S_3\end{aligned}$$

Therefore, \sim is an equivalence relation on S_5

□

b Find the equivalence class of $(4, 5)$.

Ans Since $(4, 5) \in S_3, \Rightarrow (4, 5)(5, 4) \in S_3$ and $(4, 5) \sim (4, 5)$

Similarly,

$$(4, 5)(1, 2, 3)(5, 4), (4, 5)(1, 3, 2)(5, 4) \in S_3$$

and

$$(4, 5)(1, 2)(5, 4), (4, 5)(1, 3)(5, 4), (4, 5)(2, 3)(5, 4) \in S_3$$

Therefore,

$$[(4, 5)] = \{(4, 5), (1, 2, 3)(4, 5), (1, 3, 2)(4, 5), (1, 2)(4, 5), (1, 3)(4, 5), (2, 3)(4, 5)\} \quad \square$$

c Find the equivalence class of $(1, 2, 3, 4, 5)$.

Ans

□

d Determine the total number of equivalence classes.

Ans S_3 contains $3! = 6$ elements, and S_5 contains $5! = 120$ elements

Therefore, the number of equivalence classes are $120/6 = 20$

□