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MATH 407 HW 03

1.2 **7** Let m and n be positive integers such that $m + n = 57$ and $[m, n] = 680$. Find m and n.

Ans $m + n = 57$ and $[m, n] = 680$

Let n be represented by m, such that,

$$m \cdot (57 - m) = 680$$

$$\Rightarrow 0 = m^2 - 57m + 680$$

$$= \frac{57 \pm \sqrt{57^2 - 4 \cdot 680}}{2}$$

$$= \frac{57 \pm 23}{2}$$

$$m = 40, 17$$

$\therefore m = 40$, $n = 17$ □

10 Show that $a\mathbb{Z} \cap b\mathbb{Z} = [a, b]\mathbb{Z}$.

Ans Let $x \in [a, b]\mathbb{Z}$

Since, $a \; a, b|$ and $b \; a, b|$,

$a \mid x$ and $b \mid x$

Therefore, $x \in a\mathbb{Z} \cap b\mathbb{Z}$

Conversely,

since $a \; a, b|$, then $[a, b] \in a\mathbb{Z}$

and $b \; a, b|$, then $[a, b] \in b\mathbb{Z}$

Then, $[a, b] \in (a \cap b)\mathbb{Z}$

$\therefore a\mathbb{Z} \cap b\mathbb{Z} = [a, b]\mathbb{Z}$ □

16 A positive integer a is called a **square** if $a = n^2$ for some $n \in \mathbb{Z}$. Show that the integer $a > 1$ is an integer if and only if every exponent in its prime factorization is even.

Ans Suppose $a = p_1^{r_1} \cdot p_2^{r_2} \cdot \ldots \cdot p_k^{r_k}$, where r_i is even.

Also, let $n = p_1^{r_1/2} \cdot p_2^{r_2/2} \cdot \ldots \cdot p_k^{r_k/2}$

Then, $a = n^2$

Conversely, suppose $a = n^2$.
Let $n = p_1^{s_1} \cdot p_2^{s_2} \cdot \ldots \cdot p_l^{s_l}$
Then, $a = p_1^{2s_1} \cdot p_2^{2s_2} \cdot \ldots \cdot p_l^{2s_l}$
Therefore, all the primes of a have even powers. $\square$

20 A positive integer is called **square-free** if it is a product of distinct primes. Prove that every positive integer can be written uniquely as a product of a square and a square-free integer.

Ans `NOTE: I needed help on this question`
Let $x \in \mathbb{Z}^+$ not be a square or a square-free integer.
Let

$$x = p_1^{r_1} \cdot p_2^{r_2} \cdot \ldots \cdot p_k^{r_k}$$

where r_i for $1 \leq i \leq k$ can be either odd or even.
However, they all cannot be even since x is not a square integer.

All the odd r_i's can be represented as the sum of r_i with all the even r_i's.
For example,
if $r_i = 5$ (a prime integer),
then $r_i = 1 + 4$

If all the even r_i's and all the exponents that are 1 are grouped together,
x can be represented as:

$$x = p_1^{s_1} \cdot p_2^{s_2} \cdot \ldots \cdot p_k^{s_k} \cdot p_{k+1} \cdot p_{k+2} \cdot \ldots \cdot p_{k+r}$$

where all the s_i's are even.
This implies that x is the product of a square and square free integer. $\square$