MATH 407

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$(1,2)(1,2,3) \neq (1,2,3)(1,2)$ [recap]

<u>Def:</u> $\pi_1, \pi_2 \in S_n$ are disjoint

iff $\pi_1(i) \neq i$ implies $\pi_2(i) = i$

and $\pi_2(i) \neq i$ implies $\pi_1(i) = i$

<u>Example:</u>

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\ 8 & 2 & 10 & 11 & 5 & 9 & 4 & 6 & 1 & 3 & 12 & 7 \end{pmatrix}$$

cycles $= \{(1,8,6,9), (2), (3,10), (4,11,12,7), (5)\}$

$(1,8,6,9)$ $(4,11,12,7)$ are disjoint since they leave each other as fixed points

<u>Thm:</u> If π_1, π_2 are disjoint then $\pi_1 \circ \pi_2 = \pi_2 \circ \pi_1$

<u>Pf.</u> Choose $i \in \{1, \dots, n\}$

a) π_1, π_2 have i as fixed

$\pi_1(\pi_2(i)) = \pi_1(i) = i$

$\pi_2(\pi_1(i)) = \pi_2(i) = i$

b) $\pi_1(i) \neq i \Rightarrow \pi_2(i) = i$

$\pi_1(\pi_1(i)) \neq \pi_1(i)$

$\pi_2(\pi_1(i)) = \pi_1(i)$

$(\longrightarrow)$

$\pi_1(\pi_2(i)) = \pi_1(i)$

c) Same vice versa for $\pi_2(i) \neq i \Rightarrow \pi_1(i) = i$

* π_1^k is disjoint from π_2^l, for any k, l

If $\pi_1^k(i) \neq i$, then $\pi_1(i) \neq i$.
Thus $\pi_2(i) = i$
$\Rightarrow \pi_2^l(i) = i$

* i	$(1,8,6,9)(2)(3,10)(4,11,12,7)(5)(i)$
1	$8 \leftarrow 1 \leftarrow 1 \leftarrow 1 \leftarrow 1$
2	$2 \leftarrow 2 \leftarrow 2 \leftarrow 2 \leftarrow 2$
3	$10 \leftarrow 10 \leftarrow 10 \leftarrow 3 \leftarrow 3$
$\vdots$	

<u>Def.</u> Order of a permutation π

$\pi^{-1}, \pi = (1), \pi^1 = \pi, \ldots \pi^k, \ldots$ (bi-infinite)

If $\pi \in Sym(S)$ and S infinite possibly no repeats.

$S_n = Sym(\mathbb{N}_n)$, there are $i, j = i+k$

$\pi^i = \pi^{i+k} = \pi^j$

$(\rightarrow)$

$\pi^0 = \pi^k$, $k > 0$ minimum

$$k = \text{order}(\pi) = o(\pi)$$

<u>Def.</u> Order of cyclic permutation is length of permutation.

<u>Thm.</u> If π_1, π_2 disjoint permutations,

$k_1 = o(\pi_1)$, $k_2 = o(\pi_2)$

then $k = o(\pi_1, \pi_2) = \text{lcm}(k_1, k_2)$

<u>Pf.</u> $(\pi_1, \pi_2)^{\text{lcm}(k_1,k_2)}$

Let $L = \text{lcm}(k_1, k_2)$

Then $(\pi_1, \pi_2)^L = \pi_1^L \pi_2^L = (1)(1) = (1)$

Thus, $k | L$.

Conversely, to show $L | k$

Look at $(\pi_1, \pi_2)^k = (1)$

$$\pi_1^k \pi_2^k = (1)$$

Let $\pi_1(i) \neq i$ s.t. $\pi_2(i) = i$

$$\pi_1^k(i) = \pi_1^k(\pi_2^k(i)) = \pi_1^k(i) = i$$

$\Rightarrow \pi_1^k = (1) = \pi_1^0$

$k_1 | k$ and $k_2 | k$

$\Rightarrow \text{lcm}(k_1, k_2) | k = L | k$

<u>Thm.</u> Suppose $\sigma \in S_n$ has cycles $\sigma_1, \sigma_2, \ldots, \sigma_\ell$

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Then $o(\sigma) = \mathrm{lcm}(o(\sigma_1), \ldots, o(\sigma_\ell))$