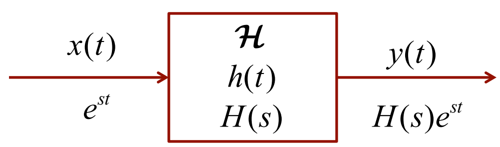
CMPE323: The Laplace Transform

Part I

Back to eigenfunctions

- **Remember that the eigenfunctions for LTI systems are complex, not merely imaginary exponentials**



- **We find the result by convolution**

$$y(t) = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau = e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau}\, d\tau = H(s) e^{st}$$

$$H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau}\, d\tau, \text{ where integral exists}$$

- **Two-sided Laplace Transform**

What does all that mean?

- **Consider our causal real exponential** $h(t) = e^{-at}u(t)$, **with an exponential input**

$$H(s) = \int_{-\infty}^{\infty} e^{-a\tau}u(t)e^{-s\tau}\,d\tau = \int_{0}^{\infty} e^{-(s+a)\tau}\,d\tau = \frac{1}{s+a}$$

- **…which will exist EXCEPT when** $s + a < 0$**, in which case the integral becomes infinite!**
- **So to properly express** $H(s)$ **, we need the form and the condition**

$$H(s) = \frac{1}{s+a}, s > -a$$

- **The condition is called the Region of Convergence**
- **A Laplace Transform is not complete without the RoC**

The Region of Convergence

- **The range of values for which the Laplace Transform converges is called the Region of Convergence (RoC)**
- **…and must be specified with each LT we do!**
- **Why? Doesn't this just make things more complicated?**
- **Consider the anticausal signal** $x(t) = -e^{-at}u(-t)$ **for real values of** a

$$X(s) = \int_{-\infty}^{\infty} -e^{-at}u(-t)e^{-st}\,dt = -\int_{-\infty}^{0} e^{-at}e^{-st}\,dt$$

Converges when $\sigma + a < 0 \Rightarrow \sigma < -a$

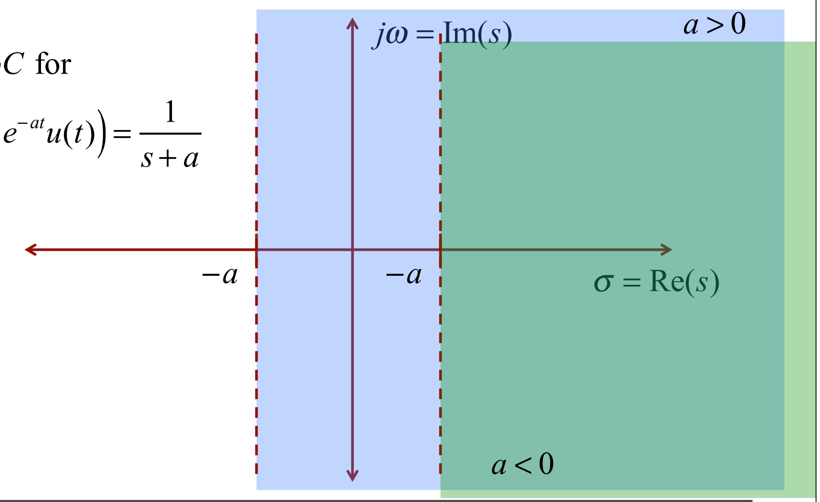
$$X(s) = \frac{1}{s+a}, \sigma < -a$$

The s-plane

- **It is common to display these conditions graphically, using a Cartesian system known as the *s-plane***

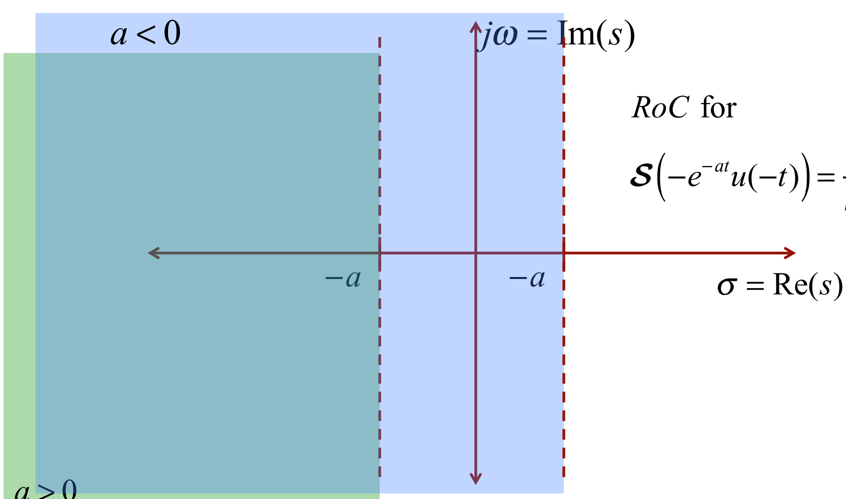
RoC for

$$\mathcal{S}\left(e^{-at}u(t)\right) = \frac{1}{s+a}$$



The s-plane

- **The ROC *must* be specified**



RoC for

$$\mathcal{S}\left(-e^{-at}u(-t)\right) = \frac{1}{s+a}$$

LCCDE

Assume we have a LCCDE

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}, \quad \frac{d^0 y(t)}{dx^0} \triangleq y(t)$$

With $x(t) = e^{st}$ **and, therefore,** $y(t) = H(s)e^{st}$

$$\sum_{k=0}^{N} a_k H(s) s^k e^{st} = \sum_{k=0}^{M} b_k s^k e^{st}, \quad \frac{d^0 y(t)}{dx^0} \triangleq y(t)$$

$$H(s)e^{st} \sum_{k=0}^{N} a_k s^k = e^{st} \sum_{k=0}^{M} b_k s^k,$$

$$H(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

Rational Functions

- **The typical case is that the Laplace Transform can be expressed as a rational polynomials**

$$X(s) = \frac{B(s)}{A(s)}, \text{ for } s \text{ in RoC}$$

$$= \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

- $X(s)$ **will be of this form whenever** $x(t)$ **satisfies a LCCDE** $$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}, \quad \frac{d^0 y(t)}{dx^0} \triangleq y(t)$$
- **…or is a sum of complex exponentials**

Poles and zeros

- **There are M roots of the numerator polynomial**

$$B(s)=\sum_{k=0}^{M} b_k s^k = b_M s^M + b_{M-1}s^{M-1}+\cdots+b_1 s+b_0=0$$

$$=\underbrace{K}_{\substack{\text{a gain}\\ \text{or scale}\\ \text{factor}\\ \textit{not}\text{ a}\\ \text{function}\\ \text{of } s}}\prod_{k=1}^{M}(s-z_k)=0\Rightarrow s\in\left\{z_1,z_2,\ldots,z_M\right\}$$

- **These are called the zeros of the Laplace function** $X(s)$
- **…because** $\left|X(s)\right|=0$ at $s=z_k$
- **(most of the time, anyway)**

Poles and zeros

- **Similarly, There are N roots of the denominator polynomial**

$$A(s)=\sum_{k=0}^{N} a_k s^k = a_N s^N + a_{N-1}s^{N-1}+\cdots+a_1 s+a_0=0$$

$$=\underbrace{G}_{\substack{\text{a gain}\\ \text{or scale}\\ \text{factor}\\ \textit{not}\text{ a}\\ \text{function}\\ \text{of } s}}\prod_{k=1}^{N}(s-p_k)=0\Rightarrow s\in\left\{p_1,p_2,\ldots,p_N\right\}$$

- **These are called the poles of the Laplace function** $X(s)$
- **…because** $\left|X(s)\right|\to\infty$ as $s\to p_k$ **…**
- **…except when**

$p_k=z_k=s^*$ for some k, in which case, as we approach s^*,

$$\lim_{s\to s^*}\frac{(s-s^*)}{(s-s^*)}=\frac{0}{0};\quad \text{L'Hopital's Rule:}\quad \lim_{s\to 0}\frac{1}{1}=1 \text{ and the zero and pole cancel!}$$

Let's see:

$$x(t) = \frac{1}{2}e^{-3t}u(t) + \frac{1}{2}e^{-t}u(t)$$

Find X(s)

Find the poles and zeros

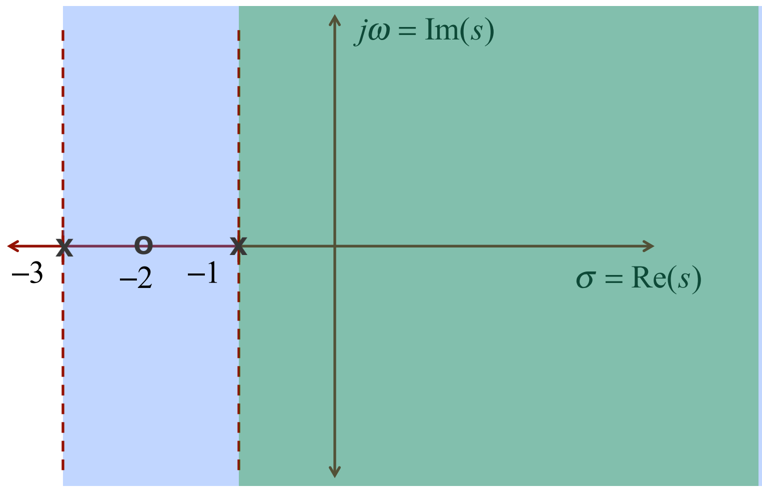
$$X(s) = \frac{s+2}{s^2+4s+3} \Rightarrow B(s) = s+2, A(s) = s^2+4s+3$$

$$\Rightarrow M = 1, b_1 = 1, b_0 = 2, N = 2, a_2 = 1, a_1 = 4, a_0 = 3$$

zeros are the roots of $B(s)$, $s+2=0 \Rightarrow s=-2$

poles are the roots of $A(s)$, $(s+3)(s+1)=0 \Rightarrow s=-3,-1$

- **Poles must be outside the RoC**
- **Zeros may be inside or outside**

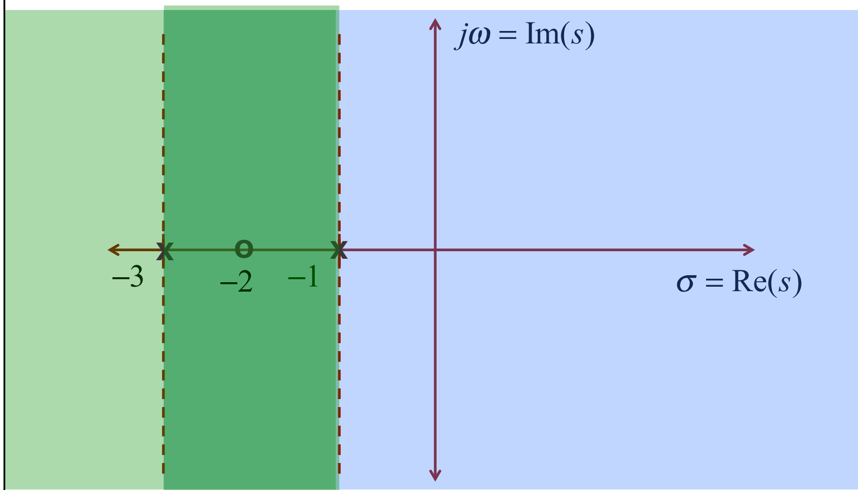


A two-sided signal

$$x(t) = \frac{1}{2}e^{-3t}u(t) - \frac{1}{2}e^{-t}u(-t)$$

Find X(s)

- **Same poles and zeros, but different time function!**
- **Poles outside, Zeros be inside**

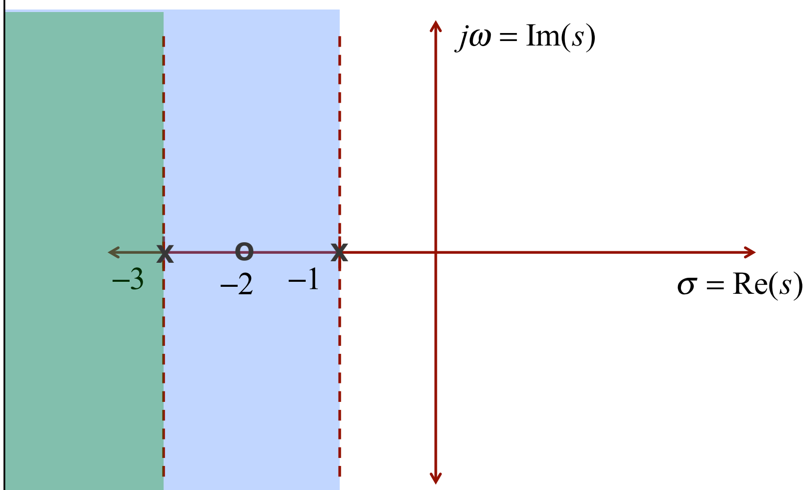


A left-sided signal

$$x(t) = -\frac{1}{2}e^{-3t}u(-t) - \frac{1}{2}e^{-t}u(-t)$$

Find X(s)

- **Same poles and zeros, but different time function!**
- **Poles outside, Zeros be inside**



Inverse Laplace Transforms

- **The equivalent synthesis equation is** $x(t) = \int_{-\infty}^{\infty} X(s)e^{st}\,ds$
- **…and this is often performed via a *Cauchy* integration…**
- **…which I'm not going into right now.**
- **In most cases, the Cauchy integration reduces to the Cauchy Residue Theorem…**
- **…and residues reduce to Partial Fraction Expansion (PFE)**
- **We generally use PFE when** $X(s)$ **is a rational polynomial, as is usually the case.**

PFE Process

- **Factor numerator and denominator polynomials. For the moment, we'll assume that all the roots are distinct and real.**

$$X(s) = \frac{K\prod_{k=1}^{M}(s-z_k)}{\prod_{k=1}^{N}(s-p_k)}$$

- **We want to expand**

$$X(s) = \sum_{k=1}^{N}\frac{R(p_k)}{(s-p_k)} = \frac{R(p_1)}{(s-p_1} + \frac{R(p_2)}{(s-p_2)} + \cdots + \frac{R(p_3)}{(s-p_3)}$$

- **… where $R(p_k)$ are called the *residuals***

See http://lpsa.swarthmore.edu/BackGround/PartialFraction/PartialFraction.html

- **For each distinct p_m, form**

$$(s-p_m)X(s) = \frac{K\prod_{k=1}^{M}(s-z_k)}{\prod_{\substack{k=1\\k\neq m}}^{N}(s-p_k)} = R(p_m) + \sum_{\substack{k=1\\k\neq m}}^{N}(s-p_m)\frac{R(p_k)}{(s-p_k)}$$

- **…and evaluate at $s = p_m$**

$$\left.\frac{K\prod_{k=1}^{M}(s-z_k)}{\prod_{\substack{k=1\\k\neq m}}^{N}(s-p_k)}\right|_{s=p_m} = R(p_m) + \sum_{\substack{k=1\\k\neq m}}^{N}0\times\frac{R(p_k)}{(s-p_k)} = R(p_m)$$

PFE Example #1

- $X(s) = \dfrac{s+2}{(s+1)(s+3)}, \ \mathrm{ROC} = \mathrm{Re}(s) > -1$

$$= \frac{R(-1)}{s-(-1)} + \frac{R(-3)}{s-(-3)}$$

$$R(-1) = \left.\frac{(s+1)(s+2)}{(s+1)(s+3)}\right|_{s=-1} = \frac{(-1+2)}{(-1+3)} = \frac{1}{2}$$

$$R(-3) = \left.\frac{(s+3)(s+2)}{(s+1)(s+3)}\right|_{s=-1} = \frac{(-3+2)}{(-3+1)} = \frac{-1}{-2} = \frac{1}{2}$$

$$X(s) = \frac{\frac{1}{2}}{s+1} + \frac{\frac{1}{2}}{s+3} \Rightarrow x(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

Difficult Things

- **There are four things that happen that complicate the process**
- **1)** $M \geq N$
- **2) Repeated real roots of the form** $(s - p_k)^m$
- **3) Complex roots**
- **4) Exponentials in** $H(s)$

M >= N

$$X(s)=\frac{6s^2+3s+2}{2s^2+14s+20}$$

Use synthetic division to divide the denominator into the numerator until $M < N$

					3
2	14	20	6	3	2
			6	42	60
			0	−39	−58

$$X(s)=3+\frac{-39s-58}{2s^2+14s+20}$$

$$=3+\frac{-39(s-1.487)}{2(s+5)(s+2)}=3+\frac{-19.5(s-1.487)}{(s+5)(s+2)}$$

Multiple real roots

$$X(s)=\frac{s+3}{s(s+2)^2(s+5)}$$

$$=\frac{R(0)}{s}+\frac{R_1(-2)}{s+2}+\frac{R_2(-2)}{(s+2)^2}+\frac{R(-5)}{s+5}$$

$R(0), R_2(-2), R(-5)$ found using standard techniques

$$R_1(-2)=\left[\frac{d}{ds}(s+2)^2X(s)\right]_{s=-2}$$

$$=\frac{d}{ds}\left(\frac{s+3}{s(s+5)}\right)_{s=-2}\qquad=\frac{-7}{36}$$