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MATH 407: HW 13

4.2 **1** Use the division algorithm to find the quotient and remainder when $f(x)$ is divided by $g(x)$ over the field of rational numbers $\mathbb{Q}$.

c $f(x) = x^5 + 1, \quad g(x) = x + 1$

Pf.

$$
\begin{array}{r|r}
 & x^4 - x^3 + x^2 - x + 1 \\ \hline
x+1 & x^5 \qquad\qquad\qquad\qquad + 1 \\
 & -x^5 - x^4 \qquad\qquad\qquad\quad \\ \cline{2-2}
 & -x^4 \qquad\qquad\qquad\quad \\
 & x^4 + x^3 \qquad\qquad\quad \\ \cline{2-2}
 & x^3 \qquad\qquad\quad \\
 & -x^3 - x^2 \qquad\quad \\ \cline{2-2}
 & -x^2 \qquad\quad \\
 & x^2 + x \qquad \\ \cline{2-2}
 & x + 1 \\
 & -x - 1 \\ \cline{2-2}
 & 0
\end{array}
$$

Therefore,

$$
\begin{aligned}
f(x) &= g(x)(x^4 - x^3 + x^2 - x + 1) + (0) \\
&= (x + 1)(x^4 - x^3 + x^2 - x + 1) + (0) \pmod{\mathbb{Q}}
\end{aligned}
$$

□

2 Use the division algorithm to find the quotient and remainder when $f(x)$ is divided by $g(x)$ over the indicated field.

c $f(x) = x^5 + 2x^3 + 3x^2 + x - 1, \quad g(x) = x^2 + 5$ over $\mathbb{Z}_7$

Pf.

$$
\begin{aligned}
f(x) &= x^5 + 2x^3 + 3x^2 + x - 1 \\
&\equiv x^5 + 2x^3 + 3x^2 + x + 6 \pmod{\mathbb{Z}_7} \\
g(x) &= x^2 + 5 \\
&\equiv x^2 + 5 \pmod{\mathbb{Z}_7}
\end{aligned}
$$

$$
\begin{array}{r|rrrrrr}
 & & & +x^3 & & +4x & +3 \\
\cline{2-7}
x^2+5 & +x^5 & +0x^4 & +2x^3 & +3x^2 & +x & +6 \\
 & -(+x^5 & & +5x^3) & & & \\
\cline{2-6}
 & & & +4x^3 & +3x^2 & +x & \\
 & & & -(+4x^3 & & +6x) & \\
\cline{4-6}
 & & & & +3x^2 & +2x & \\
 & & & & -(+3x^2 & +2x) & \\
\cline{5-7}
 & & & & & & +6
\end{array}
$$

Therefore,

$$
\begin{aligned}
f(x) &= g(x)(x^3+4x+3)+6 \\
&= (x^2+5)(x^3+4x+3)+6 \pmod{\mathbb{Z}_7}
\end{aligned}
$$

□

3 Find the greatest common divisor of $f(x)$ and f', over $\mathbb{Q}$.

d $f(x) = x^4+2x^3+3x^2+2x+1$

Pf. Given $f(x) = x^4+2x^3+3x^2+2x+1$

and $f' = 4x^3+6x^2+6x+2$

And, $\frac{f(x)}{f'}$,

$$
\begin{array}{r|rrrrr}
 & & & & \frac{1}{4}x & +\frac{1}{8} \\
\cline{2-6}
4x^3+6x^2+6x+2 & x^4 & +2x^3 & +3x^2 & +2x & +1 \\
 & -x^4 & -\frac{3}{2}x^3 & -\frac{3}{2}x^2 & -\frac{1}{2}x & \\
\cline{2-5}
 & & \frac{1}{2}x^3 & +\frac{3}{2}x^2 & +\frac{3}{2}x & +1 \\
 & & -\frac{1}{2}x^3 & -\frac{3}{4}x^2 & -\frac{3}{4}x & -\frac{1}{4} \\
\cline{3-6}
 & & & \frac{3}{4}x^2 & +\frac{3}{4}x & +\frac{3}{4}
\end{array}
$$

Multiplying the remainder with a non-zero constant keeps it unchanged, and therefore,

$$
\begin{aligned}
\text{remainder} &= \left(\frac{3}{4}x^2+\frac{3}{4}x+\frac{3}{4}\right)\frac{4}{3} \\
&= x^2+x+1
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\gcd(x^4+2x^3+3x^2+2x+1, 4x^3+6x^2+6x+2) \\
&= \gcd(4x^3+6x^2+6x+2, x^2+x+1)
\end{aligned}
$$

Dividing as before,

$$
\begin{array}{r|l}
 & 4x+2 \\ \cline{2-2}
x^2+x+1 & 4x^3+6x^2+6x+2 \\
 & -4x^3-4x^2-4x \\ \cline{2-2}
 & 2x^2+2x+2 \\
 & -2x^2-2x-2 \\ \cline{2-2}
 & 0
\end{array}
$$

Therefore,

$$\gcd(x^4+2x^3+3x^2+2x+1, 4x^3+6x^2+6x+2) = x^2+x+1 \qquad \square$$

5 Find the greatest common divisor of the given polynomials, over the given field.

c $x^5+4x^4+6x^3+6x^2+5x+2$ and x^4+3x^2+3x+6 over $\mathbb{Z}_7$

Pf. Doing long division until remainder is 0,

$$
\begin{array}{r|cccccc}
 & x & +4 & & & & \\ \cline{2-7}
x^4+3x^2+3x+6 & +x^5 & +4x^4 & +6x^3 & +6x^2 & +5x & +2 \\
 & -(x^5 & & +3x^3 & +3x^2 & +6x) & \\ \cline{2-6}
 & & +4x^4 & +3x^3 & +3x^2 & +x & \\
 & & -(4x^4 & & +5x^2 & +5x & +3) \\ \cline{3-7}
 & & & +3x^3 & +5x^2 & +3x & +6
\end{array}
$$

$$
\begin{array}{r|ccccc}
 & 5x & +1 & & & \\ \cline{2-6}
3x^3+5x^2+3x+6 & +x^4 & & +3x^2 & +3x & +6 \\
 & -(x^4 & +4x^3 & +x^2 & +2x) & \\ \cline{2-6}
 & & +3x^3 & +2x^2 & +x & +6 \\
 & & -(3x^3 & +5x^2 & +3x & +6) \\ \cline{3-6}
 & & & +4x^2 & +5x &
\end{array}
$$

$$
\begin{array}{r|cccc}
 & 6x & +6 & & \\ \cline{2-5}
4x^2+5x & +3x^3 & +5x^2 & +3x & +6 \\
 & -(3x^3 & +2x^2) & & \\ \cline{2-5}
 & & +3x^2 & +3x & +6 \\
 & & -(3x^2 & +2x) & \\ \cline{3-5}
 & & & +x & +6
\end{array}
$$

$$
\begin{array}{r|rrr}
 & 4x & +2 & \\
\hline
x+6 & +4x^2 & +5x & \\
 & -(4x^2 & +3x) & \\
\hline
 & & +2x & \\
 & & -(2x & +5) \\
\hline
 & & & +2
\end{array}
$$

$$
\begin{array}{r|rr}
 & 4x & +3 \\
\hline
2 & +x & +6 \\
 & -(x) & \\
\hline
 & & +6 \\
 & & -(6) \\
\hline
 & & 0
\end{array}
$$

Therefore, $\gcd(x^5 + 4x^4 + 6x^3 + 6x^2 + 5x + 2, x^4 + 3x^2 + 3x + 6) = 2 \in \mathbb{Z}_7$ □

9 Let $a \in \mathbb{R}$, and let $f(x) \in \mathbb{R}[x]$, with derivative $f'(x)$. Show that the remainder when $f(x)$ is divided by $(x-a)^2$ is $f'(a)(x-a) + f(a)$.

Pf. By the division algorithm, there exists unique polynomials $q(x), r(x) \in F[x]$, such that $f(x) = q(x)(x-a)^2 + r(x)$, where $\deg(x) < 2$

Let $r(x) = bx + c$

Then, $f(a) = 0 + r(a) = ba + c$

Deriving,

$$f'(x) = (q'(x)(x-a) + 2q(x))(x-a) + b$$

So, $f'(a) = b$

Also,

$$
\begin{aligned}
c &= f(a) - ba \\
&= f(a) - f'(a)a
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
r(x) &= f'(a)x + f(a) - f(a)a \\
&= f'(a)(x-a) + f(a)
\end{aligned}
$$

□

11 Find the irreducible factors of $x^6 - 1$ over $\mathbb{R}$.

Pf. Factoring $x^6 - 1$,

$$\begin{aligned}
x^6 - 1 &= (x^3)^2 - 1^2 \\
&= (x^3 - 1)(x^3 + 1) \\
&= (x^3 - 1^3)(x^3 + 1^3) \\
&= (x^3 - 1^3)(x + 1)(x^2 - x + 1) \\
&= (x^3 - 1^3)(x + 1)(x^2 - x + 1) \\
&= (x - 1)(x^2 + x + 1)(x + 1)(x^2 - x + 1)
\end{aligned}$$

Factors of degree 1, $(x - 1)$ and $(x + 1)$ are irreducible

Also, both factors $(x^2 + x + 1)$ and $(x^2 - x + 1)$ have no roots in $\mathbb{R}$ since their discriminant $(b^2 - 4ac)$ are less than zero.

Therefore, all factors of $x^6 - 1$; $(x - 1)$,$(x^2 + x + 1)$,$(x + 1)$, and $(x^2 - x + 1)$ are irreducible over $\mathbb{R}$ $\square$

18 Compute the following products.

b $(a + bx)(c + dx) \equiv\, ???\ (\text{mod } x^2 - 2)$ over $\mathbb{Q}$.

Pf. Since $x^2 \equiv 2\ (\text{mod } x^2 - 2)$

$$\begin{aligned}
(a + bx)(c + dx) &= ac + adx + cbx + bdx^2 \\
&= ac + adx + cbx + 2bd\ (\text{mod } x^2 - 2) \\
&= (ac + 2bd) + (ad + cb)x\ (\text{mod } x^2 - 2)
\end{aligned}$$

$\square$