Problem Set #7 Solutions

1. There are no forces in the z-direction. As a consequence, there is no motion in the z-direction, and the charged particle only moves in x and y. The easiest way to verify that the equation is correct is by substitution. We first substitute $t = 0$, and we see that $u_x(t = 0) = u_{x0}$ and $u_y(t = 0) = u_{y0}$, so that u_{x0} and u_{y0} correspond to the initial conditions. Substituting this solution into the original equation, we have

$$\begin{aligned}\frac{du_x}{dt} &= -\omega_{\mathrm{L}} u_{x0} \sin\omega_{\mathrm{L}} t + \omega_{\mathrm{L}} u_{y0} \cos\omega_{\mathrm{L}} t + \frac{E}{B}\omega_{\mathrm{L}} \cos(\omega_{\mathrm{L}} t) \\ &= \frac{qB}{m}\left[-u_{x0} \sin\omega_{\mathrm{L}} t + u_{y0} \cos\omega_{\mathrm{L}} t + \frac{E}{B}(\cos\omega_{\mathrm{L}} t - 1)\right] + \frac{qE}{m} \\ &= \frac{qB}{m} u_y + \frac{qE}{m}.\end{aligned}$$

The equation for du_y/dt can be verified in a similar fashion. To obtain the motion of the charged particle, we integrate the equations for u_x and u_y to obtain

$$\begin{aligned}x(t) &= x_0 + \frac{u_{x0}}{\omega_{\mathrm{L}}} \sin\omega_{\mathrm{L}} t + \frac{u_{y0} + E/B}{\omega_{\mathrm{L}}}(1 - \cos\omega_{\mathrm{L}} t), \\ y(t) &= y_0 - \frac{u_{x0}}{\omega_{\mathrm{L}}}(1 - \cos\omega_{\mathrm{L}} t) + \frac{u_{y0} + E/B}{\omega_{\mathrm{L}}} \sin\omega_{\mathrm{L}} t - \frac{E}{B} t.\end{aligned}$$

We see that the electric field in the x-direction produces a drift in the $-y$ direction. More generally, this drift is in the direction $\mathbf{E} \times \mathbf{B}$ and is called the $\mathbf{E} \times \mathbf{B}$ drift.

2. At $t = 0$, the derivative of the flux is zero, and there there is no current flow. At times $\omega t = \pi/2$ or $\omega t = 3\pi/4$, the derivative of the flux is negative and the flux is in the $+z$-direction. The current flows in a direction to maintain this flux and hence it flows in the $+\hat{\boldsymbol{\phi}}$-direction.

3. We have that 6,000 revolutions per minute equals 100 revolutions per second, so that $\omega = 628\ \mathrm{s}^{-1}$ to three significant figures. We also have for the area of the loop $A = 6\ \mathrm{cm}^2 = 6.00 \times 10^{-4}\ \mathrm{m}^2$. Hence, we have $\Phi = (50.0 \times 10^{-3})(6.00 \times 10^{-4}) \cos(200\pi t) = (3.00 \times 10^{-5}) \cos(200\pi t)$ V-s. We have for the current flow

$$I_{\mathrm{EMF}} = \frac{V_{\mathrm{EMF}}}{\Omega} = -\frac{1}{\Omega}\frac{d\Phi}{dt} = \frac{(3.00 \times 10^{-5})(6.28 \times 10^2)}{1.0} \sin(200\pi t) \to 19 \sin(200\pi t)\ \mathrm{mA}$$

To obtain an accuracy of 2 significant figures, we must calculate $\sin(\omega t)$ accurately to two significant figures. That means that the error in ωt must be less than approximately 0.01, which in turn means that as t increases, we must calculate ωt with increasing accuracy. After 100 seconds, when the loop has undergone 10^4 revolutions, the allowed error is less than 10^{-6}!

4. At a radial distance r from the center of the disk, the velocity is given by $\mathbf{u} = \hat{\boldsymbol{\phi}}\omega r$. It follows that the motional EMF is given by

$$\int_0^a \hat{\mathbf{r}} \cdot (\mathbf{u} \times \mathbf{B})\, dr = \int_0^a \omega B_0 r\, dr = \frac{1}{2}\omega B_0 a^2$$

5. Since $\mathbf{J} = \sigma\mathbf{E}$, where $\mathbf{J}$ is the conduction current, and $\mathbf{E}$ is the electric field, we have that the magnitude of the conduction current is given by σE_0, where E_0 is the magnitude of the electric field. Since $\mathbf{J}_\mathrm{d} = \epsilon\partial\mathbf{E}/\partial t$, where $\mathbf{J}_\mathrm{d}$ is the displacement current, we have that the magnitude of the displacement current is given by $\omega\epsilon E_0 = 2\pi f E_0$, where f is the frequency of the field. It follows that the ratio R of the two currents is given by

$$R = \frac{\sigma}{2\pi \times 81\epsilon_0 f} = \frac{8.877 \times 10^8 \text{ Hz}}{f},$$

where we have substituted $\sigma = 4$ S/m. Substituting the frequencies, we find: (a) $R(1 \text{ kHz}) = 1.8 \times 10^5$, (b) $R(1 \text{ MHz}) = 8.9 \times 10^2$, (c) $R(1 \text{ GHz}) = 0.18$, (d) $R(100 \text{ GHz}) = 8.9 \times 10^{-3}$. We go from a limit where we are dominated by conduction to a limit where we are dominated by radiation. In the former limit, waves cannot penetrate a single wavelength into the medium. In the latter limit, waves can penetrate and propagate some distance before damping.

6. In this case, we have $\nabla \cdot \mathbf{J} = -6y\cos(\omega t)$. It follows from the conservation of charge that $\partial\rho/\partial t = -\nabla \cdot \mathbf{J} = 9y^2 \cos\omega t$. Integrating this relationship, we obtain $\rho = (9y^2/\omega)\sin\omega t$.

7. From the expression for the electric field, we have

$$\nabla \times \mathbf{E} = \hat{\mathbf{y}} k E_0 \sin ay \sin(\omega t - kz) - \hat{\mathbf{z}} a E_0 \cos ay \cos(\omega t - kz)$$

Using Faraday's law,

$$\nabla \times \mathbf{E} = -\mu\frac{\partial \mathbf{H}}{\partial t},$$

we conclude

$$\mathbf{H} = \hat{\mathbf{y}}\frac{k}{\mu\omega}E_0 \sin ay \cos(\omega t - kz) + \hat{\mathbf{z}}\frac{a}{\mu\omega}E_0 \cos ay \sin(\omega t - kz).$$