# CMPE323 DTFT and DFT

Chapter 8




## Properties of Fourier Transform

- **Duality:** $X(f)=\mathcal{F}\big(x(t)\big)\Rightarrow x(f)=\mathcal{F}\big(X(-t)\big)$ and $x(-f)=\mathcal{F}\big(X(t)\big)$
- **Linearity:** $z(t)=ax(t)+by(t)\Rightarrow Y(f)=aX(f)+bY(f)$
- **Time Shift:** $\mathcal{F}\big(x(t-t_0)\big)=e^{-j2\pi f t_0}\mathcal{F}\big(x(t)\big)$
- **Scaling:** For $a\neq 0\in\mathbb{R},\ \mathcal{F}\big(x(at)\big)=\frac{1}{|a|}X\left(\frac{f}{a}\right)$
- **Modulation:** $\mathcal{F}\left(x(t)e^{j2\pi f_0 t}\right)=X(f-f_0)$
- **Conjugation:** $\mathcal{F}\big(x^*(t)\big)=X^*(-f)$
- **Parseval:** $\int_{-\infty}^{\infty}x(t)y^*(t)\,dt=\int_{-\infty}^{\infty}X(f)Y^*(f)\,df=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)Y^*(\omega)\,d\omega$
- **Rayleigh** $\int_{-\infty}^{\infty}|x(t)|^2\,dt=\int_{-\infty}^{\infty}|X(f)|^2\,df=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)Y^*(\omega)\,d\omega$

## Advanced Properties of Fourier Transform

- **Integration:** $\mathcal{F}\left(\int_{-\infty}^{t} x(t)\,dt\right) = \frac{X(f)}{j2\pi f} + \frac{1}{2}X(0)\delta(f)$
- **Differentiation:** $\mathcal{F}\left(\frac{d}{dt}x(t)\right) = j2\pi fX(f)$
- **Moments:** $\int_{-\infty}^{\infty} t^n x(t)\,dt = \left(\frac{j}{2\pi}\right)^n \frac{d^n}{df^n}X(f)\Big|_{f=0}$
- **Real Signals: The Fourier transform of a real signal is EVEN in magnitude and ODD in phase**



## Sampling as multiplication

- **We can look at the act of sampling as a multiplication in the time domain**

$$x_S(t) = x(t) \times \left(\sum_{k=-\infty}^{\infty} \delta(t - n\Delta t)\right)$$

- **So that (assuming we will be doing an integration at some time)**

$$x_S(t) = \begin{cases} x(k\Delta t) & t = k\Delta t \\ 0 & elsewhere! \end{cases}$$

- **This series of impulse must have infinite frequency content (like an impulse), despite the frequency content of** $x(t)$

## Fourier Transforms with MATLAB

$$x_S(t) = x(t)\left(\sum_{k=-\infty}^{\infty} \delta(t-k\Delta t)\right) = \sum_{k=-\infty}^{\infty} x(t)\delta(t-k\Delta t)$$

$$X_S(j\omega) = \int_{-\infty}^{\infty} x_S(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x(k\Delta t)\delta(t-k\Delta t)e^{-j\omega t}\,dt$$

$$\Omega \triangleq \omega\Delta t$$

$$= \sum_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t)\delta(t-k\Delta t)e^{-j\omega t}\,dt \qquad \text{with units } \frac{\text{radians}}{\text{sec}} \times \frac{\text{seconds}}{\text{sample}}$$

$$= \sum_{-\infty}^{\infty} x(k\Delta t)e^{-j\omega k\Delta t} = \sum_{-\infty}^{\infty} x(k\Delta t)e^{-j\Omega k} \qquad = \frac{\text{rad}}{\text{sample}}$$

$\triangleq$ Discrete Time Fourier Transform (DTFT)



## Alternatively, we can recognize

$$x_S(t) = x(t)\times s(t), \quad s(t) = \sum_{k=-\infty}^{\infty} \delta(t-n\Delta t)$$

$$X_S(j\omega) = X(j\omega) * S(j\omega)$$

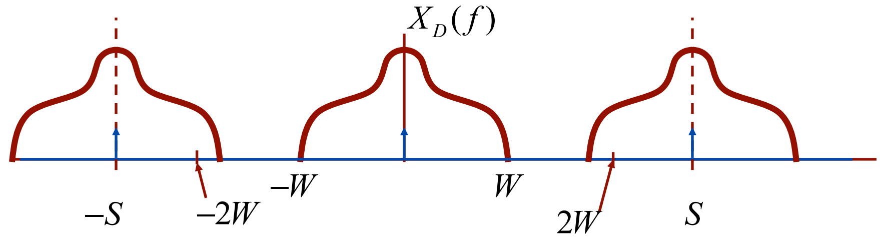
$$= \int_{-\infty}^{\infty} X\big(j(\omega-\alpha)\big)S(j\alpha)\,d\alpha \qquad S(j\omega) = \sum_{n=-\infty}^{\infty}\delta\left(\omega - \frac{2\pi n}{\Delta t}\right)$$

$$X_S(j\omega) = \int_{-\infty}^{\infty} X\big(j\omega-\alpha\big)\sum_{n=-\infty}^{\infty}\delta\left(\alpha - \frac{2\pi n}{\Delta t}\right)d\alpha$$

$$= \sum_{n=-\infty}^{\infty} X\left(\omega - \frac{2\pi n}{\Delta t}\right)$$

## The frequency domain view!

- **Start with a band limited signal** $x(nt_s) \leftrightarrow X_D(f)$




- **Sampling is multiplication in the time domain**

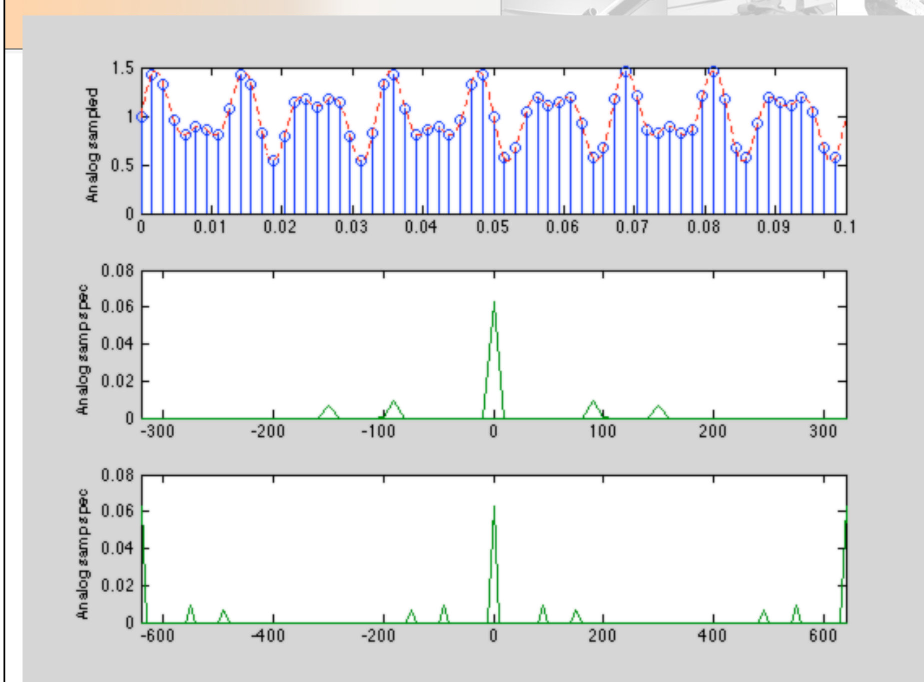
$$x(nt_s) = x(t) \times \sum_{n=-\infty}^{\infty} \delta(t - nt_s) = x(t) \times \sum_{n=-\infty}^{\infty} \delta\left(t - \frac{n}{S}\right)$$

- **So it is *convolution* in the frequency domain**

$$X_D(f) = X(f) * \mathcal{F}\left(\sum_{n=-\infty}^{\infty} \delta\left(t - \frac{n}{S}\right)\right) = X(f) * \sum_{n=-\infty}^{\infty} \delta\left(f - nS\right)$$

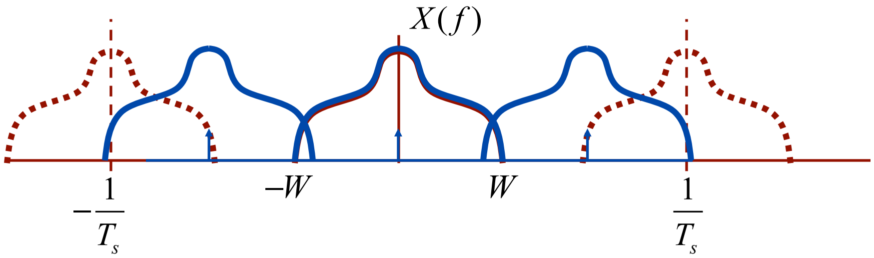
$$= \frac{1}{S} X\left(\frac{f}{S}\right) * \sum_{n=-\infty}^{\infty} \delta\left(\frac{f}{S} - n\right) = t_S \sum_{n=-\infty}^{\infty} X\left(\frac{f}{S} - n\right)$$



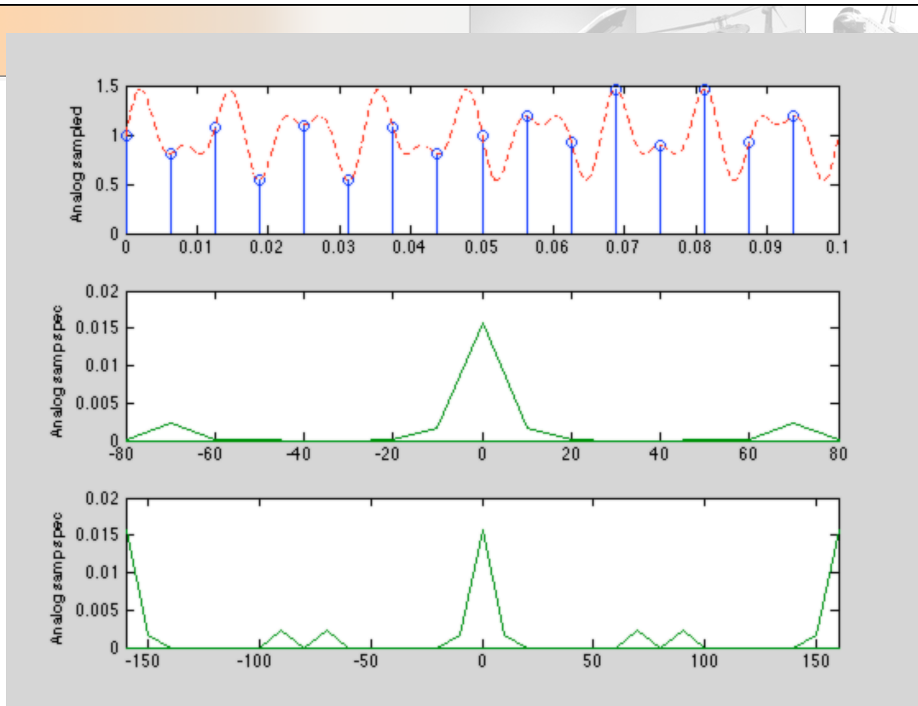
## The aliasing problem

- What if we don't satisfy the Nyquist criteria?




- The sampled spectra overlap…
- …creating distortion…
- …that can not be eliminated!
- This is called ***aliasing*** (acting by another name)





## The DTFT is periodic in frequency!

$$= \sum_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(t)\delta(t-k\Delta t)e^{-j\omega t}\,dt$$

$$X_S(j\omega) = \sum_{-\infty}^{\infty} x(k\Delta t)e^{-j\omega k\Delta t}$$

$$X_S\left(j\left(\omega+\omega_S\right)\right) = \sum_{-\infty}^{\infty} x(k\Delta t)e^{-j\left(\omega+2\pi f_s\right)k\Delta t}$$

$$= \sum_{-\infty}^{\infty} x(k\Delta t)e^{-j\left(\omega+\frac{2\pi}{\Delta t}\right)k\Delta t} = \sum_{-\infty}^{\infty} x(k\Delta t)e^{-j\omega k\Delta t}e^{-j2\pi k}$$

$$= \sum_{-\infty}^{\infty} x(k\Delta t)e^{-j\omega k\Delta t} = X_S(j\omega)$$

- **The periodicity is an artifact of sampling in the time domain…it's actually our aliasing phenomenon!**



## Sampled periodic signals

- **We know that a sampled signal has a periodic spectrum…**
- **…and a periodic signal has a sampled (discrete spectrum due to the Fourier Series.**
- **…so we should get both impulses ($\delta\left[n-\frac{k}{N}\right]$) and periodicity…**
- **Thus *both* the time sequence and the frequency domain sequence will be periodic**
- **Let's see how this works**

  Assume $x(t)$ is periodic, with period $T = Nt_s$

$$c_n = \frac{1}{T}\int_{0-t_s/2}^{T-t_s/2} x(t)e^{-\frac{j2\pi nt}{T}}\,dt = \frac{1}{T}\int_{0-t_s/2}^{T-t_s/2} x_1(t)e^{-\frac{j2\pi nt}{T}}\,dt$$

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi nt/T} \qquad X_P(f) = \sum_{n=-\infty}^{\infty} c_n\delta\left(f-\frac{f}{T}\right)$$

Now sample $x(t)$ to get $x(nt_s) = \sum_{k=-\infty}^{\infty} x[k]\delta(t-kt_s)$

$$\tilde{c}_n = \frac{1}{T}\int_{0-t_s/2}^{T-t_s/2} \sum_{n=-\infty}^{\infty} x[k]\delta(t-kt_s)e^{-\frac{j2\pi nt}{T}}\,dt$$

Swap the order $\tilde{c}_n = \frac{1}{T}\sum_{n=-\infty}^{\infty}\int_{0-t_s/2}^{T-t_s/2} x[k]\delta(t-kt_s)e^{-\frac{j2\pi nt}{T}}\,dt$

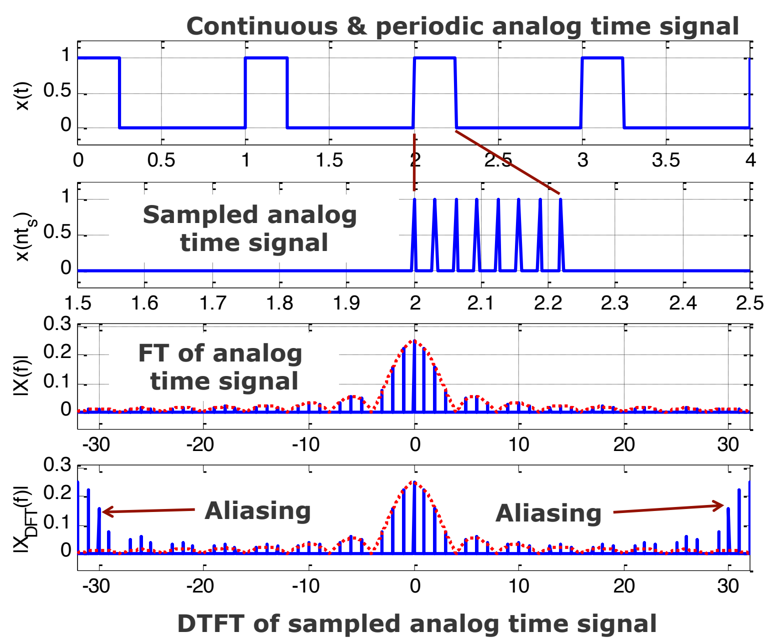
The argument of the $\delta$ function is only zero for $k = 0,1,2,\ldots N-1$

$$\tilde{c}_n = \frac{1}{T}\sum_{k=0}^{N-1} x[k]e^{-\frac{j2\pi nkt_s}{T}} = \frac{1}{T}X_1(F)\bigg|_{F=\frac{n}{Nt_s}} = \frac{1}{T}\sum_{k=0}^{N-1} x_1[k]e^{-\frac{j2\pi nk}{N}}$$

This is still the "continuous time" Fourier series (series of impulses)

To get to discrete time, we have to take the sample time into account and multiply the coefficients by $t_s$

$\tilde{\tilde{c}}_n = \frac{1}{N}\sum_{k=0}^{N-1} x_1[k]e^{-\frac{j2\pi nk}{N}}$ as the coef. of the Discrete Fourier Series (DFS)




Co

# The Discrete Fourier Transform (DFT)

- The DFT is the "unscaled" version

$$X_{DFT}[k] = X_1(F)\Big|_{F=\frac{n}{Nt_s}} = \sum_{k=0}^{N-1} x_1[k] e^{-\frac{j2\pi nk}{N}}, \; k = 0,1,2,\ldots N-1$$

- ...and is defined only over one period of $F$, usually $0 \le F < 1$ or $-\frac{1}{2} \le F < \frac{1}{2}$

- The Discrete Fourier Series

$$X_{DFS}[k] = \frac{1}{N} X_1(F)\Bigg|_{F=\frac{n}{Nt_s}} = \frac{1}{N}\sum_{k=0}^{N-1} x_1[k] e^{-\frac{j2\pi nk}{N}} = \tilde{\tilde{c}}_k$$

- Related to the "discrete approximation" of the analog

$$c_k = X[k] = \frac{1}{T}\int_T x_1(t) e^{-2\pi kt/T} dt$$

$$\to \frac{1}{Nt_s}\sum_{n=0}^{N-1} x_1[n] e^{-j2\pi nkt_s/Nt_s} \times t_s = \frac{1}{N}\sum_{n=0}^{N-1} x_1[n] e^{-j2\pi nk/N}$$