CMPE 320: Probability, Statistics, and Random Processes

Lecture 5: Independence

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Announcement

- Instructor's O/H on Tuesday (2/13) moved to Friday (2/16) (just for this one time). The time is noon – 1pm. ITE 312
- The TA will still have the O/H on Thursday during noon – 1pm at ITE 353.

Independence

- P(A|B) captures the partial information that B provides about A
- Sometimes observing event B does not provide any information on A

$$\boxed{P(A|B) = P(A)}$$

$\Leftrightarrow$ Events A and B are independent

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = P(A)$$

$$\Rightarrow \boxed{P(A\cap B) = P(A)P(B)} \qquad \text{Also: } P(B|A) = P(B)$$

Disjoint events are not independent

Say $A \cap B = \emptyset$

$$P(A\cap B) = P(\emptyset) = 0 \neq P(A)P(B)$$

unless A or $B = \emptyset$

A and A^c are never independent (again unless A or $A^c = \emptyset$)

In fact, knowing A has happened provides a LOT of information on A^c happening.

Example 1.19. Consider an experiment involving two successive rolls of a 4-sided die in which all 16 possible outcomes are equally likely and have probability 1/16.

(a) Are the events

$$A_i = \{\text{1st roll results in } i\}, \qquad B_j = \{\text{2nd roll results in } j\},$$

independent?

$P(A_i \cap B_j) = P(\text{the outcome is } (i,j)) = \frac{1}{16}$

$P(A_i) = \frac{1}{4}$

$P(B_j) = \frac{1}{4}$

$P(A_i \cap B_j) = P(A_i)\, P(B_j)$

$\Rightarrow A_i$ and B_j are independent

$$A = \{\text{1st roll is a 1}\}, \qquad B = \{\text{sum of the two rolls is a 5}\},$$

$P(A \cap B) = P(\{(1,4)\}) = \frac{1}{16}$

$P(A) = \frac{1}{4}$

$P(B) = P(\{(1,4), (2,3), (3,2), (4,1)\})$

$= P(1,4) + P(2,3) + P(3,2) + P(4,1)$

$= \frac{4}{16} = \frac{1}{4}$

$P(A \cap B) = \frac{1}{16} \qquad P(A)\,P(B) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$

$\therefore$ A and B are independent

$A = \{\text{maximum of the two rolls is 2}\}$, $B = \{\text{minimum of the two rolls is 2}\}$,

$P(A \cap B) = P(\{(2,2)\}) = \frac{1}{16}$

$P(A) = P(\{(1,2),(2,1),(2,2)\}) = \frac{3}{16}$

$P(B) = P(\{(2,2),(2,3),(2,4),(3,2),(4,2)\}) = \frac{5}{16}$

$P(A \cap B) \neq P(A)\,P(B)$

$\therefore$ A and B are dependent

Conditional independence

- Given that event C has happened, additionally knowing that event B happened does not add any information regarding event A

$$\boxed{P(A \mid B \cap C) = P(A \mid C)}$$

$\Leftrightarrow$ Events A and B are conditionally independent (conditioned on C)

$$P(A \mid B \cap C) = \frac{P(A \cap B \mid C)}{P(B \mid C)} = P(A \mid C)$$

cond. prob. definition to $P(\cdot \mid C)$

$\Leftrightarrow$ $\boxed{P(A \cap B \mid C) = P(B \mid C)\,P(A \mid C)}$

Example 1.20. Consider two independent fair coin tosses, in which all four possible outcomes are equally likely. Let

$$H_1 = \{\text{1st toss is a head}\},$$
$$H_2 = \{\text{2nd toss is a head}\},$$
$$D = \{\text{the two tosses have different results}\}.$$

Are H_1 and H_2 independent conditioned on D?

$$P(H_1|D) = \frac{\#(H_1 \cap D)}{\#(D)} = \frac{\#\{HT\}}{\#\{HT, TH\}} = \frac{1}{2}$$

$$P(H_2|D) = \frac{\#\{TH\}}{\#\{HT, TH\}} = \frac{1}{2}$$

$$P(H_1 \cap H_2 | D) = \frac{\# H_1 \cap H_2 \cap D}{\# D} = \frac{0}{2} = 0$$

$\therefore P(H_1 \cap H_2 | D) \neq P(H_1 | D)\, P(H_2 | D) \Rightarrow$ H_1 & H_2 are not conditionally indep.

Example 1.21. There are two coins, a blue and a red one. We choose one of the two at random, each being chosen with probability 1/2, and proceed with two independent tosses. The coins are biased: with the blue coin, the probability of heads in any given toss is 0.99, whereas for the red coin it is 0.01. $P(H_1|B^c)$ $P(H_1|B)$

H_i = {i-th toss resulted in a head}, B = {the blue coin was selected}

1) Are H_1 and H_2 conditionally independent given B? 2) Are H_1 and H_2 (unconditionally) independent?

$$P(H_1) = P((H_1 \cap B) \cup (H_1 \cap B^c)) = P(H_1 \cap B) + P(H_1 \cap B^c)$$
$$= P(B)P(H_1|B) + P(B^c)P(H_1|B^c)$$
$$= \frac{1}{2} \times 0.99 + \frac{1}{2} \times 0.01 = \frac{1}{2}$$

$$P(H_2) = \frac{1}{2}$$

$$P(H_1 \cap H_2) = P(H_1 \cap H_2 | B)\, P(B) + P(H_1 \cap H_2 | B^c)\, P(B^c)$$
$$= 0.99 \times 0.99 \times \frac{1}{2} + 0.01 \times 0.01 \times \frac{1}{2}$$

$\therefore P(H_1 \cap H_2) \neq P(H_1)\, P(H_2)$ $\therefore$ H_1 and H_2 are dependent

Independence of collection of events

- Events A_1, A_2 and A_3 are independent if

$$P(A_1 \cap A_2) = P(A_1)\,P(A_2)$$
$$P(A_2 \cap A_3) = P(A_2)\,P(A_3)$$
$$P(A_3 \cap A_1) = P(A_3)\,P(A_1)$$

pairwise independence

$$P(A_1 \cap A_2 \cap A_3) = P(A_1)\,P(A_2)\,P(A_3)$$

Important to check this one, in addition to the pairwise independence.

Pairwise independence does not imply independence

- 2 independent fair coin tosses

H_1 = {1st coin is a head}, H_2 = {2nd toss is a head}
D = {the two tosses have different results}

H_1 and H_2 are independent by definition

$$P(D|H_1) = \frac{P(D \cap H_1)}{P(H_1)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

HT

$$P(D) = P(\{HT, TH\}) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$P(D|H_1) = P(D)$ $\Rightarrow$ H_1 & D are indep.

Likewise, D and H_2 are also independent.

$$P(H_1 \cap H_2 \cap D) = 0 \overset{?}{\neq} P(H_1)\,P(H_2)\,P(D) = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{8}$$

Extension to more events

- Events $A_1, A_2, \dots, A_n$ are independent if

$$P\left(\bigcap_{i\in S} A_i\right) = \prod_{i\in S} P(A_i) \quad \text{for every subset } S \subset \{1,2,\dots,n\}$$

- It means that the occurrence of any subset of events carries ZERO information on the occurrence (or not occurrence) of the rest

Ex: If A_1, A_2, A_3 and A_4 independent,

$$P(A_1 \cup A_2 \mid A_3 \cap A_4) = P(A_1 \cup A_2)$$

$$P(A_1 \cup A_2^c \mid A_3^c \cap A_4) = P(A_1 \cup A_2^c)$$

Problem 30. A hunter has two hunting dogs. One day, on the trail of some animal, the hunter comes to a place where the road diverges into two paths. He knows that each dog, independent of the other, will choose the correct path with probability p. The hunter decides to let each dog choose a path, and if they agree, take that one, and if they disagree, to randomly pick a path. Is his strategy better than just letting one of the two dogs decide on a path?

$A = \{\text{two dogs agree}\}$ $\quad B = \{\text{pick the correct path}\}$

$$P(B) = P(A \cap B) + P(A^c \cap B)$$

$$P(A \cap B) = P(\{\text{both dogs pick correct path}\})$$

$$= p \times p = p^2$$ ← The dogs are choosing path independently.

$$P(A^c \cap B) = P(A^c)\,P(B \mid A^c) = 2p(1-p) \times \tfrac{1}{2} = p(1-p)$$

$$P(B) = p^2 + p(1-p) = p$$