

Difficult Things

- There are four things that happen that complicate the process
 - 1) $M \geq N$
 - 2) Repeated real roots of the form $(s - p_k)^m$
 - 3) Complex roots
 - 4) Exponentials in $H(s)$

Let's Digress just a little

- What is the Laplace Transform of $\delta(t)$?

$$\Delta(s) = \int_{-\infty}^{\infty} \delta(\tau) e^{-s\tau} d\tau = e^{-s\tau} \Big|_{\tau=0} = 1 \quad (!!)$$

- So the Laplace Transform of a delta function is a constant!...
- ...and vice-versa
- So $X_1(s) = 3 \Rightarrow 3\delta(t)$

$$X_2(s) = \frac{-19.5(s-1.487)}{(s+5)(s+2)} = -19.5 \left(\frac{R(-5)}{s+5} + \frac{R(-2)}{s+2} \right) \quad s > -2$$

And a more complicated case

$$H(s) = \frac{A}{s-r} + \frac{C+jD}{s-(\sigma+j\omega)} + \frac{C-jD}{s-(\sigma-j\omega)}, \text{Re}[s] > \max(r, \sigma)$$

Causal!

$$\begin{aligned} h(t) &= Ae^{rt}u(t) + (C+jD)e^{(\sigma+j\omega)t}u(t) + (C-jD)e^{(\sigma-j\omega)t}u(t) \\ &= Ae^{rt}u(t) + Ce^{\sigma t}(e^{j\omega t} + e^{-j\omega t})u(t) + jDe^{\sigma t}(e^{j\omega t} - e^{-j\omega t})u(t) \\ &= Ae^{rt}u(t) + 2Ce^{\sigma t}\cos(\omega t)u(t) + 2De^{\sigma t}\sin(\omega t)u(t) \\ &= (Ae^{rt} + e^{\sigma t}(E\cos\omega t + F\sin\omega t))u(t) \\ &= \text{exponential term} + \text{second order solution from CMPE306!} \end{aligned}$$

Laplace Transforms III: Properties of the Laplace Transform

Summary

- **LCCDE → LT: Put in standard form, find the a, b coefficients, write the rational form, factor & find residuals, convert 1st and 2nd order terms to time domain based on RoC.**

- **$X(s) \rightarrow x(t)$: Use the synthesis equation**

$$x(t) = \frac{1}{2\pi j} \int_{-\infty}^{\infty} X(s) e^{st} ds = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) e^{st} ds$$

- **...which can be complicated because of the integration in the complex plane**
- **...but there are properties that make things simpler**

Key properties

- **Linearity: Of course it is linear!**

$$\mathcal{L}(Ax_1(t) + Bx_2(t)) = AX_1(s) + BX_2(s), \quad R' \supset R_1 \cap R_2$$

- **Time Shift**

$$\mathcal{L}(x(t-t_0)) = e^{-st_0} X(s), \quad R' = R$$

- **Modulation**

$$\mathcal{L}(e^{s_0 t} x(t)) = X(s-s_0), \quad R' = R + \text{Re}[s_0]$$

- **Scaling (remember time scaling!)**

$$\mathcal{L}(x(at)) = \frac{1}{|a|} X\left(\frac{s}{a}\right), \quad R' = aR$$

More Properties

■ Differentiation in the time domain

$$x(t) = \frac{1}{2\pi j} \int_{-\infty}^{\infty} X(s) e^{st} ds = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) e^{st} ds$$

$$\begin{aligned} \frac{dx}{dt} &= \frac{d}{dt} \left[\frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) e^{st} ds \right] = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} \frac{d}{dt} [X(s) e^{st}] ds \\ &= \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} s X(s) e^{st} ds \end{aligned}$$

$$\text{Therefore } \mathcal{L}\left(\frac{dx}{dt}\right) = sX(s) \quad R' \supset R$$

■ Differentiation in the transform domain

$$X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

$$\frac{dX}{ds} = \frac{d}{ds} \left[\int_{-\infty}^{\infty} x(t) e^{-st} dt \right] = \int_{-\infty}^{\infty} x(t) \frac{d}{ds} [e^{-st}] dt = \int_{-\infty}^{\infty} (-t) x(t) e^{-st} dt$$

$$\text{Therefore } \mathcal{L}((-t)x(t)) = \frac{dX}{ds}, \quad R' = R$$

■ Integration in the time domain

$$\begin{aligned} \int_{-\infty}^t x(\tau) d\tau &= \int_{-\infty}^t \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) e^{s\tau} ds d\tau = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s) \int_{-\infty}^t e^{s\tau} d\tau ds \\ &= \int_{\sigma-j\infty}^{\sigma+j\infty} \frac{1}{s} X(s) e^{st} ds \end{aligned}$$

$$\text{Therefore } \mathcal{L}\left(\int_{-\infty}^t x(\tau) d\tau\right) = \frac{1}{s} X(s), \quad R' \supset R \cap \text{Re}[s] > 0$$

- **Initial Value Theorem**

$$\lim_{s \rightarrow \infty} (sF(s)) = f(0^+)$$

- **Final Value Theorem**

$$\lim_{s \rightarrow 0} (sF(s)) = \lim_{t \rightarrow \infty} f(t)$$

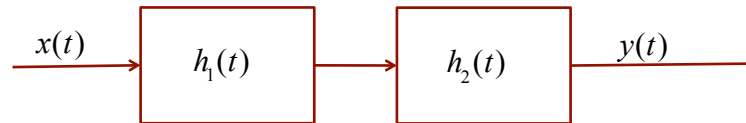
The Convolution Theorem

- **One of the primary application of the Laplace (and later Fourier) Domain(s) is the Convolution Theorem**
- **What is the LT of the convolution?**

$$\begin{aligned} \mathcal{L}(x * h) &= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau \right) e^{-st} dt \\ &= \int_{-\infty}^{\infty} x(\tau) \underbrace{\left(\int_{-\infty}^{\infty} h(t - \tau) e^{-st} dt \right)}_{\text{time shift property!}} d\tau = \int_{-\infty}^{\infty} x(\tau) (e^{-s\tau} H(s)) d\tau \\ &= H(s) \int_{-\infty}^{\infty} x(\tau) e^{-s\tau} d\tau = H(s) X(s) = X(s) H(s) \quad (!!)\end{aligned}$$

So what?

- Remember our concatenation of systems?



$$y(t) = x * h = x * (h_1 * h_2)$$

$$H(s) = \mathcal{L}(h_1 * h_2) = H_1(s)H_2(s)$$

$$Y(s) = \mathcal{L}(x * h) = X(s)H(s) = X(s)H_1(s)H_2(s)$$

- A convolution in the time domain is equivalent to a multiplication in the frequency domain...
- ...and *vice versa*