CMPE323 FT, FS, DTFT, DFT: Making Sense of it all

I'm confused!

- We have these things that sound similar, but aren't the same.
- How do I keep it straight?
- Let's start with a piecewise continuous in time waveform that is not periodic $x(t)$
- To analyze this is the frequency domain, we use the Fourier Transform, which essentially gives us the "voltage spectral density"

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}\,dt = X(f)$$

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}\,df = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

- Piecewise continuous in time but not periodic → Fourier Transform

I'm confused (part 2)

- **Now let our function be piecewise continuous in time, but periodic with period** T
- **In this case, the analysis is the Fourier Series**
- **In the frequency domain, we talk about the Fourier Coefficients,** c_k **, which are related to *harmonics* of the period of the time waveform**

$$c_k = \frac{1}{T}\int_{\alpha}^{\alpha+T} x(t)e^{-j\omega_0 kt}\,dt = \frac{1}{T}\int_{\alpha}^{\alpha+T} x(t)e^{-j\frac{2\pi}{T}kt}\,dt$$

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j\omega_o t} = \sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi}{T}kt}$$

- **Periodic in time results in discrete in frequencies, with spacing of** $\Delta f = 1/T$

I'm confused, part 3

- **Now let's assume that we have a waveform that we will assume is not periodic in time, but is sampled in time at constant intervals** Δt
- **In the frequency domain, this gives us the Discrete Time Fourier Transform (DTFT), which is a (piecewise) continuous, periodic waveform as a function of frequency**

$$X_S(\omega) = \int_{-\infty}^{\infty}\left(\sum_{n=-\infty}^{\infty} x(t)\delta(t-n\Delta t)\right)e^{-j\omega t}\,dt$$

$$= \sum_{n=-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\left(x(t)e^{-j\omega t}\right)\delta(t-n\Delta t)\,dt\right] = \sum_{n=-\infty}^{\infty} x(n\Delta t)e^{-j\omega n\Delta t} = DTFT$$

$$x_S(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_S(\omega)e^{j\omega t}\,d\omega$$

- **Not periodic in time, sampled in time → periodic and continuous in frequency**

I'm confused (part 4)

- **Now let's consider a periodic, piecewise continuous time waveform that is sampled at Δt**
- **We start by computing the coefficients of the Fourier Series, using $\hat{c}_k$ to remind ourselves that these are the coefficients from the sampled waveform**
- **Furthermore, assume that Δt is chosen such that**

$$N = \frac{T}{\Delta t} > 2 \in \mathbb{N}$$

$$\hat{c}_k = \frac{1}{T}\int_{\alpha}^{\alpha+T} x(t)\sum_{n=-\infty}^{\infty}\delta(t-n\Delta t)e^{-j\frac{2\pi}{T}kt}\,dt = \frac{1}{T}\sum_{n=-\infty}^{\infty}\int_{\alpha}^{\alpha+T}\left(x(t)e^{-j\frac{2\pi}{T}kt}\right)\delta(t-n\Delta t)\,dt$$

Let $\alpha = 0^-$, then $\int_{0^-}^{T^-}$ picks out $n = 0,1,2,\ldots,N-1,$ all other terms are 0

$$\hat{c}_k == \frac{1}{N\Delta t}\sum_{n=0}^{N-1}x(n\Delta t)e^{-j\frac{2\pi}{N\Delta t}kn\Delta t} = \frac{1}{N\Delta t}\sum_{n=0}^{N-1}x(n\Delta t)e^{-j\frac{2\pi kn}{N}}$$

- **This expression is the Discrete Fourier Series (DFS)**

continued

- **By analogy with**

$$c_k = \frac{1}{T}X\left(\frac{2\pi k}{T}\right)\text{(frequency in rad/s) or }\frac{1}{T}X\left(\frac{k}{T}\right)\text{(frequency in Hz)}$$

$$\hat{c}_k = \frac{1}{T}X_S\left(\frac{k}{T}\right) \Rightarrow X_S\left(\frac{k}{T}\right) = \sum_{n=0}^{N-1}x(n\Delta t)e^{-j\frac{2\pi kn}{N}}$$

- **Where $X_S\left(\frac{k}{T}\right)$ is the Discrete Fourier Transform (DFT)**