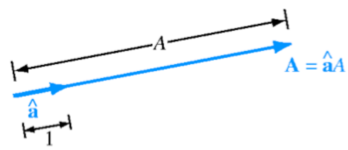
Vector Algebra

Definition

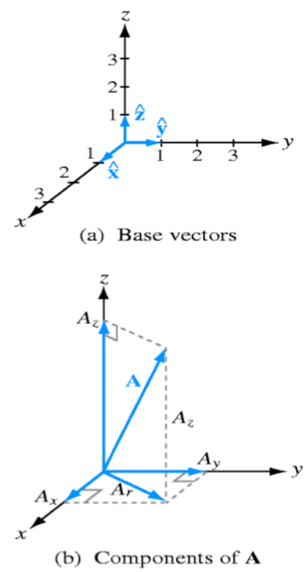
A vector $\mathbf{A}$ has a magnitude $A = |\mathbf{A}|$ and a direction given by the unit vector $\hat{\mathbf{a}} = \mathbf{A} / |\mathbf{A}|$.

In Cartesian coordinates (x, y, z), we find

$$\mathbf{A} = \hat{\mathbf{a}}A = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$$
$$= (A_x, A_y, A_z)$$



Ulaby Figure 3-1



Ulaby Figure 3-2

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6.1

NOTE: This vector notation using boldface type goes back to Josiah Willard Gibbs, a 19-th century American physicist. It is sometimes therefore referred to as GIBBS NOTATION. Maxwell just wrote out all the equations longhand.

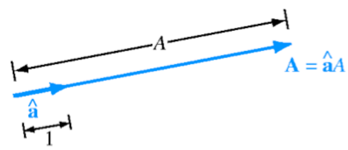
Vector Algebra

Definition

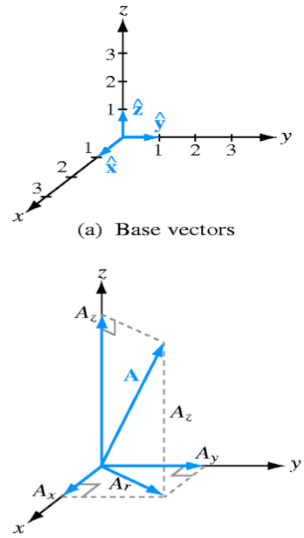
We also have

$$A = \left(A_x^2 + A_y^2 + A_z^2\right)^{1/2}$$

$$\hat{\mathbf{a}} = \frac{\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z}{\left(A_x^2 + A_y^2 + A_z^2\right)^{1/2}}$$



Ulaby Figure 3-1



(a) Base vectors

(b) Components of **A**

Ulaby Figure 3-2

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Vector Algebra

Equality

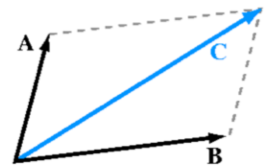
Two vectors **A** and **B** are equal if

$$A_x = B_x,\ A_y = B_y,\ \text{and } A_z = B_z$$

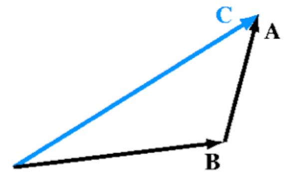
Addition and Subtraction

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \quad \text{(addition is commutative)}$$
$$= \hat{\mathbf{x}}(A_x + B_x) + \hat{\mathbf{y}}(A_y + B_y) + \hat{\mathbf{z}}(A_z + B_z)$$

$$\mathbf{D} = \mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$$
$$= \hat{\mathbf{x}}(A_x - B_x) + \hat{\mathbf{y}}(A_y - B_y) + \hat{\mathbf{z}}(A_z - B_z)$$



(a) Parallelogram rule



(b) Head-to-tail rule

Ulaby Figure 3-3

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6.3

Ulaby et al. makes a distinction between *equal* and *identical* vectors. Identical vectors start at the same point.

Vector Algebra

Position and Distance vectors

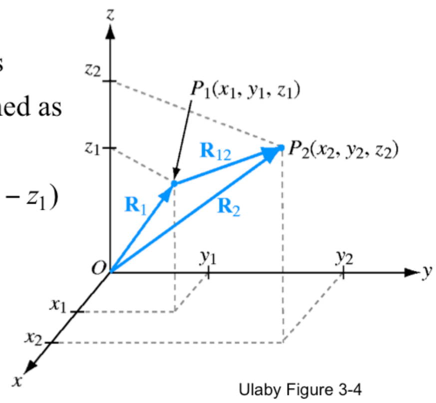
For a point $P(x, y, z)$, the *position vector* **R** goes from the origin to the point

$$\mathbf{R} = \hat{\mathbf{x}}\, x + \hat{\mathbf{y}}\, y + \hat{\mathbf{z}}\, z$$

The distance vector between two points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is defined as

$$\mathbf{R}_{12} = \mathbf{R}_2 - \mathbf{R}_1$$
$$= \hat{\mathbf{x}}(x_2 - x_1) + \hat{\mathbf{y}}(y_2 - y_1) + \hat{\mathbf{z}}(z_2 - z_1)$$

where $\mathbf{R}_i = \hat{\mathbf{x}}\, x_i + \hat{\mathbf{y}}\, y_i + \hat{\mathbf{z}}\, z_i,\ i = 1, 2$



Ulaby Figure 3-4

The distance d between two points equals $|\mathbf{R}_{12}|$

$$d = |\mathbf{R}_{12}|$$
$$= \left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]^{1/2}$$

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6.4

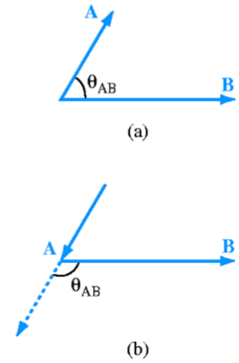
Note that the point 1 corresponds to the tail and the point 2 corresponds to the head of the vector. That way R_2 = R_1 + R_{12}.

Vector Algebra

Multiplication — Simple Product

When a vector **A** is multiplied by a scalar k, the magnitude is multiplied by k and the direction is unchanged

$$\mathbf{B} = k\mathbf{A} = \hat{\mathbf{a}}(kA)$$
$$= \hat{\mathbf{x}}(kA_x) + \hat{\mathbf{y}}(kA_y) + \hat{\mathbf{z}}(kA_z)$$



Multiplication — Scalar or Dot Product

The scalar or dot product of two vectors **A** and **B** is defined by

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta_{AB}$$

where θ_{AB} is the angle between **A** and **B**

Ulaby Figure 3-5

When $\theta_{AB} < 90^o$, $\mathbf{A} \cdot \mathbf{B} > 0$
When $\theta_{AB} > 90^o$, $\mathbf{A} \cdot \mathbf{B} < 0$
When $\theta_{AB} = 90^o$, $\mathbf{A} \cdot \mathbf{B} = 0$, and the vectors are called *orthogonal*

6.5

There are three types of multiplication that we must consider: (1) A scalar (single number) x a vector -> a vector, (2) (a vector) x (a vector) -> scalar [the dot product], and (3) (a vector) x (a vector) -> (a vector) [the cross product].

NOTE: Strictly speaking the cross product does not produce a vector; it produces a pseudo-vector. A pseudo-vector is actually an AREA. The length of the vector corresponds to the magnitude of the area and its orientation is at right angles to the plane of the area. The difference between a pseudo-vector and a real vector is not important in this course.

Vector Algebra

Multiplication — Scalar or Dot Product

We have

$$\hat{\mathbf{x}}\cdot\hat{\mathbf{x}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{z}}\cdot\hat{\mathbf{z}} = 1$$

$$\hat{\mathbf{x}}\cdot\hat{\mathbf{y}} = \hat{\mathbf{y}}\cdot\hat{\mathbf{z}} = \hat{\mathbf{x}}\cdot\hat{\mathbf{z}} = 0$$

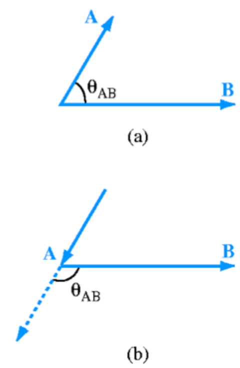
so that

$$\begin{aligned}\mathbf{A}\cdot\mathbf{B} &= \left(\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z\right)\cdot\left(\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z\right)\\ &= A_xB_x + A_yB_y + A_zB_z\end{aligned}$$

Other properties:

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A} \quad \text{(commutative)}$$

$$\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C} \quad \text{(distributive)}$$



Ulaby Figure 3-5

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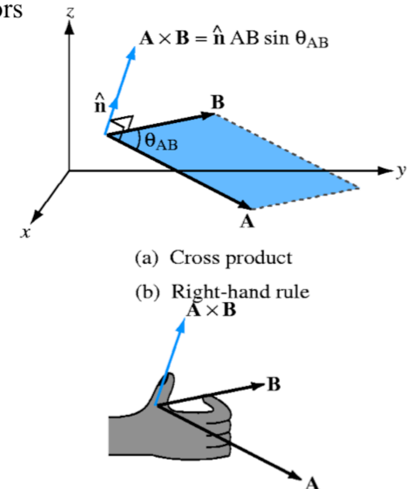
Vector Algebra

Multiplication — Vector or Cross Product

The vector or cross product of two vectors **A** and **B** is defined by

$$\mathbf{A}\times\mathbf{B} = \hat{\mathbf{n}}\, AB\sin\theta_{AB}$$

where θ_{AB} is the angle *from* **A** *to* **B** and $\hat{\mathbf{n}}$ is a unit vector determined by the right-hand rule



Ulaby Figure 3-6

The vector product is *anti-commutative*

$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$$

and distributive

$$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B}+\mathbf{A}\times\mathbf{C}$$

$$\hat{\mathbf{x}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}\times\hat{\mathbf{z}} = 0$$

$$\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}},\quad \hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}},\quad \hat{\mathbf{z}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}$$

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6.7

The direction in which \theta_{AB} is determined matters in this case.

Vector Algebra

Multiplication — Vector or Cross Product

We have

$$\hat{\mathbf{x}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}\times\hat{\mathbf{z}} = 0$$

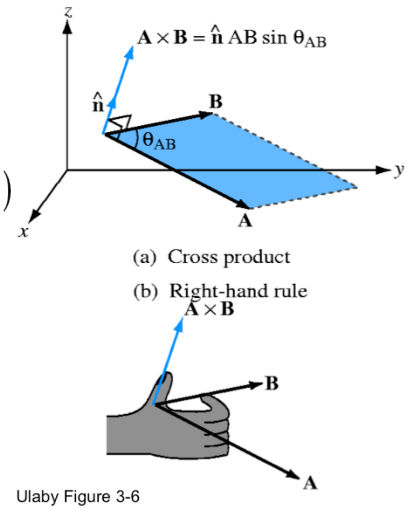
$$\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}, \quad \hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}}, \quad \hat{\mathbf{z}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}$$

so that

$$\begin{aligned}\mathbf{A}\times\mathbf{B} &= \left(\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z\right)\times\left(\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z\right) \\ &= \hat{\mathbf{x}}(A_yB_z - A_zB_y) + \hat{\mathbf{y}}(A_zB_x - A_xB_z) \\ &\quad + \hat{\mathbf{z}}(A_xB_y - A_yB_x)\end{aligned}$$

We may also write

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$



Ulaby Figure 3-6

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This determinant form is very useful!

Vector Algebra

Summary of vector products

Product type	Product elements	Representation
Simple product	(Scalar) x (Vector) → Vector	$\mathbf{C} = k\mathbf{A}$
Scalar product or Dot product	(Vector) x (Vector) → Scalar	$\mathbf{C} = \mathbf{A} \cdot \mathbf{B}$
Vector product or Cross product	(Vector) x (Vector) → Vector	$\mathbf{C} = \mathbf{A} \times \mathbf{B}$

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6.9

Strictly speaker, the (vector)x(vector) product produces a PSEUDO-VECTOR or a 2-form. It can be thought of as an area. The magnitude of the area is given by the length of the pseudo-vector, and the direction of the pseudo-vector is orthogonal to the plane of the area.

Vector Algebra

Vectors and Angles: Ulaby et al. Example 3-1

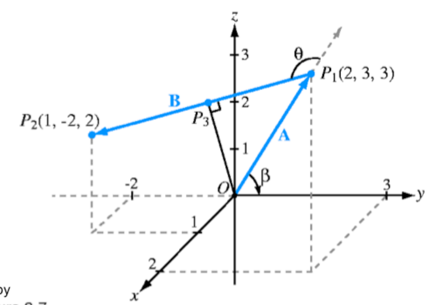
Question: In Cartesian coordinates, vector **A** is directed from the origin to the point $P_1(2, 3, 3)$, and vector **B** is directed from P_1 to $P_2(1, -2, 2)$. Find (a) the vector **A**, its magnitude A, and its unit vector $\hat{\mathbf{a}}$, (b) the angle that **A** makes with the y-axis, (c) vector **B**, (d) the angle between **A** and **B**, and (e) the perpendicular distance from the origin to **B**.

Answer: (a)

$$\mathbf{A} = \hat{\mathbf{x}}\,2 + \hat{\mathbf{y}}3 + \hat{\mathbf{z}}\,3$$

$$A = \sqrt{4+9+9} = \sqrt{22}$$

$$\hat{\mathbf{a}} = \mathbf{A}\,/\,A = \left(\hat{\mathbf{x}}\,2 + \hat{\mathbf{y}}3 + \hat{\mathbf{z}}\,3\right)\Big/\sqrt{22}$$



Ulaby
Figure 3-7

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6.10

Vector Algebra

Vectors and Angles: Ulaby et al. Example 3-1

Answer (continued): (b) The angle β between **A** and the y-axis may be found from the expression $\mathbf{A}\cdot\hat{\mathbf{y}} = A\cos\beta$, which implies

$$\beta = \cos^{-1}\left(\frac{\mathbf{A}\cdot\hat{\mathbf{y}}}{A}\right) = \cos^{-1}\left(\frac{3}{\sqrt{22}}\right) = 0.879 \text{ rads} = 50.2°$$

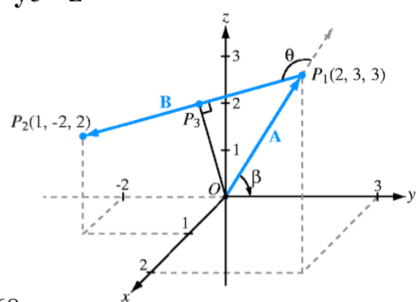
(c) $\mathbf{B} = \hat{\mathbf{x}}\,(1-2) + \hat{\mathbf{y}}(-2-3) + \hat{\mathbf{z}}\,(2-3) = -\hat{\mathbf{x}} - \hat{\mathbf{y}}5 - \hat{\mathbf{z}}$

(d) $\theta = \cos^{-1}\left(\frac{\mathbf{A}\cdot\mathbf{B}}{AB}\right) = \cos^{-1}\left(\frac{-20}{\sqrt{22}\sqrt{27}}\right)$

$= 2.533 \text{ rads} = 145.1°$

(e) The points OP_1P_3 form a right triangle. The magnitude of the line segment OP_3 is given by

$A\sin(\pi-\theta) = \sqrt{22}\sin(0.609) = 2.68$



Ulaby Figure 3-7

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Vector Algebra

Triple Scalar Product

This product can be written in the equivalent forms

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

The equivalence can be demonstrated from the determinant representation

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

The absolute value of this scalar product is the volume of the parallelepiped whose sides are the vectors **A**, **B**, and **C.**

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6.12

Note that it is important to keep the right cyclic order. Otherwise, the sign changes.

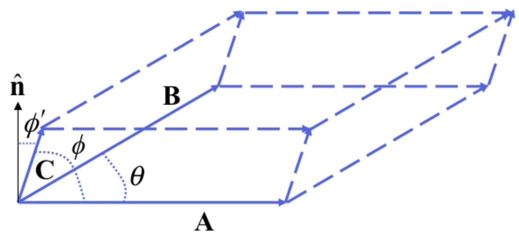
Vector Algebra

Triple Scalar Product

This product can be written in the equivalent forms

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

It corresponds to the volume of a parallelepiped



$$\hat{\mathbf{n}} = \frac{\mathbf{A}\times\mathbf{B}}{|\mathbf{A}\times\mathbf{B}|}$$

$$\phi' = \frac{\pi}{2} - \phi$$

Surface area $= |\mathbf{A}\times\mathbf{B}| = |\mathbf{A}||\mathbf{B}|\sin\theta$

Volume $= |\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})| = |\mathbf{A}||\mathbf{B}||\mathbf{C}|\sin\theta\cos\phi' = |\mathbf{A}||\mathbf{B}||\mathbf{C}|\sin\theta\sin\phi$

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6.13

The triple scalar product can be interpreted "physically" as the volume of a parallelepiped.

Vector Algebra

Triple Vector Product

This product is defined as

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) \quad \text{and we note} \quad \mathbf{A}\times(\mathbf{B}\times\mathbf{C}) \neq (\mathbf{A}\times\mathbf{B})\times\mathbf{C}$$

This relationship can also be written in the form

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - \mathbf{C}(\mathbf{A}\cdot\mathbf{B})$$

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6.14

The first note implies that this relationship is not associative.

Orthogonal Coordinate Systems

Three coordinate systems

Coordinate systems (with three coordinates) specify points in space

Orthogonal systems have coordinates that are mutually perpendicular

— at least locally

There are three coordinate systems that are often used

- Cartesian coordinates (x, y, z): Simplest and orthogonal everywhere
- Cylindrical coordinates (r, θ, z)
  — used with optical fibers, coaxial cables, cylindrical waveguides
- Spherical coordinates (R, θ, ϕ)
  — used with antenna radiation, radar, earth-ionosphere waveguide

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6.15

Three Coordinate Systems

Summary of vector and differential relations

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, θ, z	R, θ, ϕ
Vector, $\mathbf{A} =$	$\hat{\mathbf{x}} A_x + \hat{\mathbf{y}} A_y + \hat{\mathbf{z}} A_z$	$\hat{\mathbf{r}} A_r + \hat{\boldsymbol{\varphi}} A_\phi + \hat{\mathbf{z}} A_z$	$\hat{\mathbf{R}} A_R + \hat{\boldsymbol{\theta}} A_\theta + \hat{\boldsymbol{\varphi}} A_\phi$
Magnitude, $\lvert\mathbf{A}\rvert =$	$\left(A_x^2 + A_y^2 + A_z^2\right)^{1/2}$	$\left(A_r^2 + A_\phi^2 + A_z^2\right)^{1/2}$	$\left(A_R^2 + A_\theta^2 + A_\phi^2\right)^{1/2}$
Position vector	$\hat{\mathbf{x}} x_1 + \hat{\mathbf{y}} y_1 + \hat{\mathbf{z}} z_1$ for $P(x_1, y_1, z_1)$	$\hat{\mathbf{r}} r_1 + \hat{\mathbf{z}} z_1$ for $P(r_1, \phi_1, z_1)$	$\hat{\mathbf{R}} R_1$ for $P(R_1, \theta_1, \phi_1)$
Dot product, $\mathbf{A} \cdot \mathbf{B}$	$A_x B_x + A_y B_y + A_z B_z$	$A_r B_r + A_\phi B_\phi + A_z B_z$	$A_R B_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product, $\mathbf{A} \times \mathbf{B}$	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\varphi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\varphi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length, $d\mathbf{l}$	$\hat{\mathbf{x}}\, dx + \hat{\mathbf{y}}\, dy + \hat{\mathbf{z}}\, dz$	$\hat{\mathbf{r}}\, dr + \hat{\boldsymbol{\varphi}}\, r d\phi + \hat{\mathbf{z}}\, dz$	$\hat{\mathbf{R}}\, dR + \hat{\boldsymbol{\theta}}\, R d\theta + \hat{\boldsymbol{\varphi}}\, R \sin\theta dz$
Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\, dy dz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\, dz dx$ $d\mathbf{s}_z = \hat{\mathbf{z}}\, dx dy$	$d\mathbf{s}_r = \hat{\mathbf{r}}\, r d\phi dz$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\varphi}}\, dz dr$ $d\mathbf{s}_z = \hat{\mathbf{z}}\, r dr d\phi$	$d\mathbf{s}_R = \hat{\mathbf{R}}\, R^2 \sin\theta d\theta d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}\, R \sin\theta d\phi dR$ $d\mathbf{s}_z = \hat{\boldsymbol{\varphi}}\, R dR d\theta$
Differential volume	$dv = dx\, dy\, dz$	$dv = r dr\, d\phi\, dz$	$dv = R^2 \sin\theta\, dR\, d\theta\, d\phi$

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6.16

Cartesian Coordinates

Differential Relations

Length: $d\mathbf{l} = \hat{\mathbf{x}}\,dl_x + \hat{\mathbf{y}}\,dl_y + \hat{\mathbf{z}}\,dl_z = \hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$

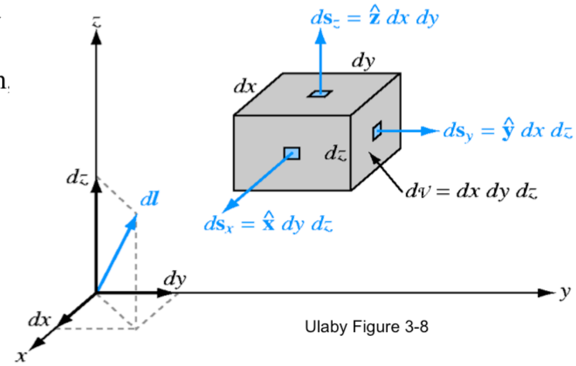
Surface area: $d\mathbf{s}_x = \hat{\mathbf{x}}\,dl_y\,dl_z = \hat{\mathbf{x}}\,dy\,dz$ (y-z plane)

$d\mathbf{s}_y = \hat{\mathbf{y}}\,dz\,dx = \hat{\mathbf{y}}\,dx\,dz$ (x-z plane)

$d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$ (x-y plane)

Volume: $dv = dx\,dy\,dz$

For each coordinate system, there are three differential lengths that also determine the differential areas and volumes



Ulaby Figure 3-8

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6.17

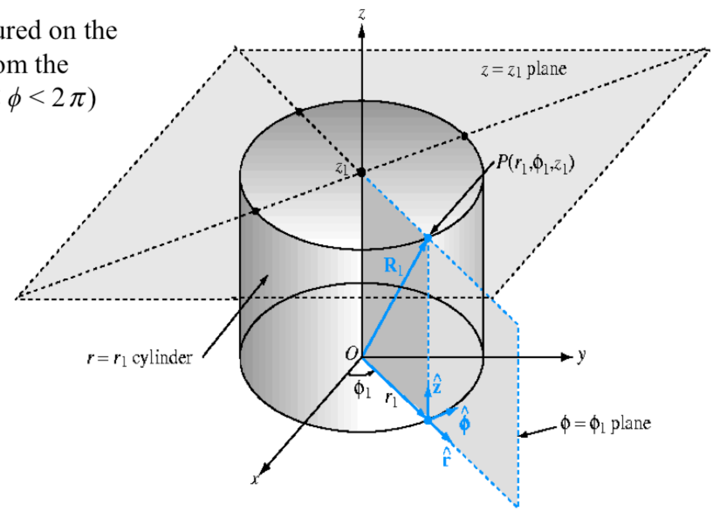
NOTE: It doesn't matter whether you write the differentials as dzdx or dxdz. The usual convention is the way that Ulaby et al. did it --- in the order in which the elements are usually listed. I did it in cyclical order in the table and both ways here.

Cartesian coordinates are the most important because they are globally, not just locally orthogonal AND their differential lengths do not change, depending on position. The orientation of their unit vectors does not change with position. Hence, they are simplest to use. We use other coordinate systems because they conform better to the complexity of real geometries in some cases.

Cylindrical Coordinates

r = distance from the z-axis ($0 < r < \infty$)

ϕ = angle measured on the x-y plane from the $+x$-axis ($0 < \phi < 2\pi$)



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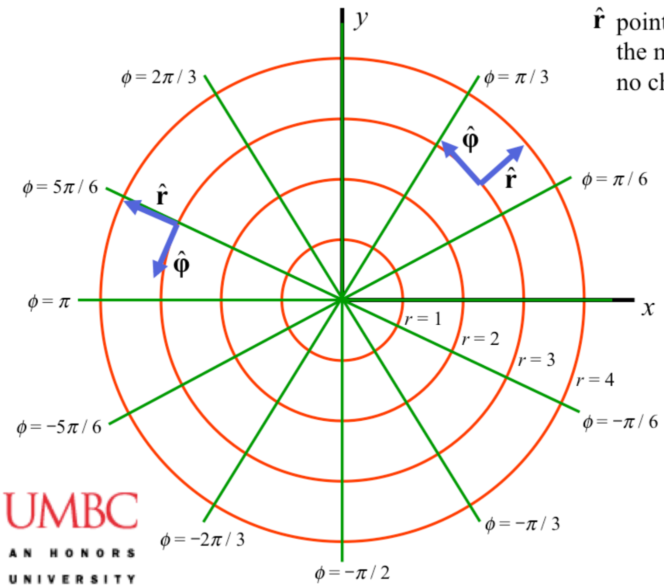
Ulaby Figure 3-9

6.18

We will shortly give explicit relations between cylindrical and Cartesian coordinates

Cylindrical Coordinates



$\hat{\mathbf{r}}$ points in the direction of the maximum change in r; no change in f, z

$\hat{\boldsymbol{\varphi}}$ points in the direction of the maximum change in f; no change in $r; z$

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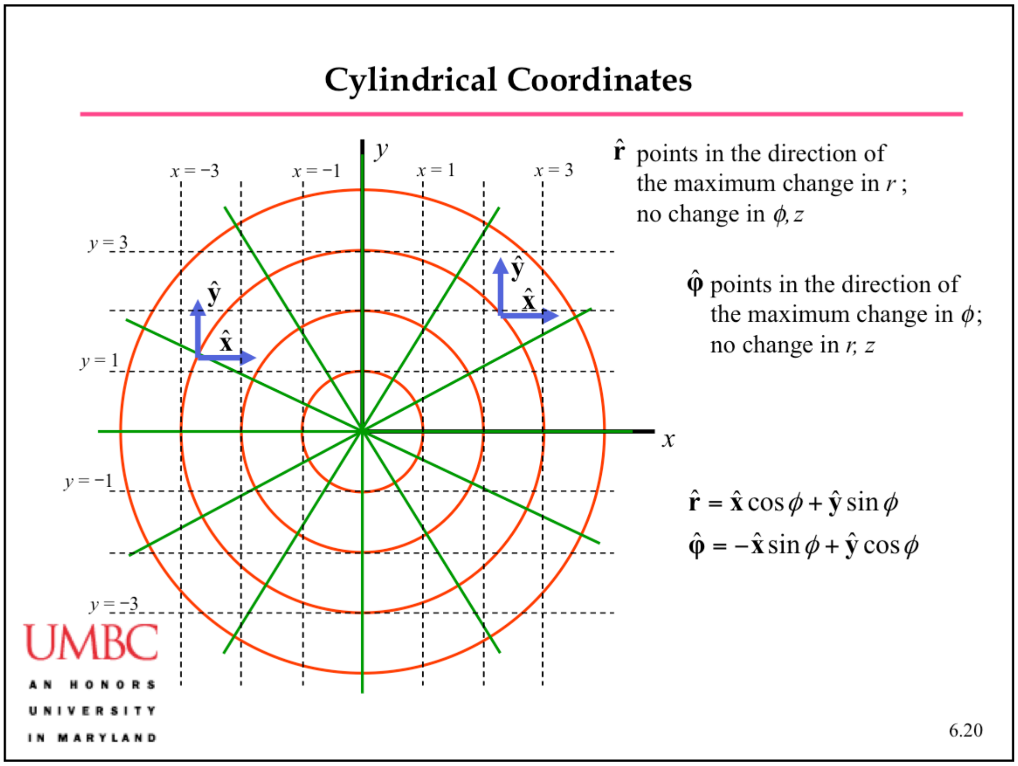
6.19

In polar or cylindrical coordinates, the direction of r-hat and phi-hat are position dependent, since the direction of the maximum change of r and phi are position-dependent

By contrast, there is no change in the direction of maximum change of x and y, and so x-hat and y-hat always point in the same direction. The direction in which r changes maximally is also the direction in which phi and z do not change because we have an ORTHOGONAL coordinate system.



In polar or cylindrical coordinates, the direct of r-hat and phi-hat are position dependent, since the direction of the maximum change of r and phi are position-dependent

By contrast, there is no change in the direction of maximum change of x and y, and so x-hat and y-hat always point in the same direction. The direction in which r changes maximally is also the direction in which phi and z do not change because we have an ORTHOGONAL coordinate system.

Ulaby et al. 2010 Module 3.1. Show that both r-hat and phi-hat depend on phi.

Cylindrical Coordinates

Base vectors

$$\hat{\mathbf{r}}\times\hat{\boldsymbol{\varphi}}=\hat{\mathbf{z}},\quad \hat{\boldsymbol{\varphi}}\times\hat{\mathbf{z}}=\hat{\mathbf{r}},\quad \hat{\mathbf{z}}\times\hat{\mathbf{r}}=\hat{\boldsymbol{\varphi}}$$

$$\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}=\hat{\boldsymbol{\varphi}}\cdot\hat{\boldsymbol{\varphi}}=\hat{\mathbf{z}}\cdot\hat{\mathbf{z}}=1,\quad \hat{\mathbf{r}}\times\hat{\mathbf{r}}=\hat{\boldsymbol{\varphi}}\times\hat{\boldsymbol{\varphi}}=\hat{\mathbf{z}}\times\hat{\mathbf{z}}=0$$

$$\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\varphi}}=\hat{\mathbf{r}}\cdot\hat{\mathbf{z}}=\hat{\boldsymbol{\varphi}}\cdot\hat{\mathbf{z}}=0$$

Vector relations

$$\mathbf{A}=\hat{\mathbf{a}}A=\hat{\mathbf{r}}A_r+\hat{\boldsymbol{\varphi}}A_\phi+\hat{\mathbf{z}}A_z$$

$$A=|\mathbf{A}|=\sqrt{\mathbf{A}\cdot\mathbf{A}}=\left(A_r^2+A_\phi^2+A_z^2\right)^{1/2}$$

Letting $P=P(r_1,\phi_1,z_1)$, the position vector $\mathbf{R}_1=OP=\hat{\mathbf{r}}\,r_1+\hat{\mathbf{z}}\,z_1$

We note that the direction of $\hat{\mathbf{r}}$ depends on ϕ

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6.21

The dependence of the unit vectors on the local coordinates is a property of all orthogonal coordinate systems, EXCEPT CARTESIAN

Cylindrical Coordinates

Differential relations

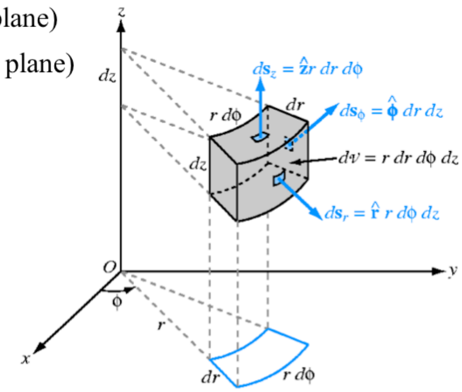
Length: $d\mathbf{l} = \hat{\mathbf{r}}\,dl_r + \hat{\boldsymbol{\phi}}\,dl_\phi + \hat{\mathbf{z}}\,dl_z = \hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}\,r d\phi + \hat{\mathbf{z}}\,dz$

Surface area: $d\mathbf{s}_r = \hat{\mathbf{r}}\,r d\phi\, dz$ (ϕ-z plane)

$d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dr\,dz$ (r-z plane)

$d\mathbf{s}_z = \hat{\mathbf{z}}\,r dr\,d\phi$ (r-ϕ plane)

Volume: $dv = r dr\, d\phi\, dz$



Ulaby Figure 3-10

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Divergence

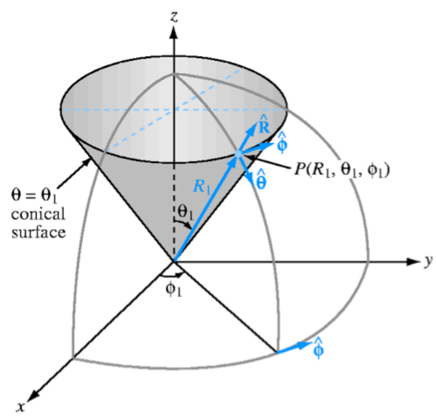
6.22

Spherical Coordinates

R = distance from the origin ($0 < R < \infty$)

θ = angle measured from the $+z$-axis ($0 < \theta < \pi$) = *zenith angle*

ϕ = angle measured from the $+x$-axis on the x-y plane ($0 < \phi < 2\pi$)
= *azimuthal angle*



Ulaby Figure 3-13 6.23

In an analogy to the globe, theta corresponds to latitude and is measured from the north pole, not the equator; phi corresponds to longitude and is measured from an "arbitrarily chosen" x-axis. (Sometimes we use this geometry where the symmetry is broken, and the choice is no longer arbitrary.) On our globe, we measure longitude from the Greenwich Observatory. While not arbitrary, this choice was long disputed. Historically, the French measured longitude for a long time from the Paris Observatory! This issue was not settled until the late 19-th century.

Spherical Coordinates

Base vectors

$$\hat{\mathbf{R}}\times\hat{\boldsymbol{\theta}}=\hat{\boldsymbol{\varphi}},\quad \hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\varphi}}=\hat{\mathbf{R}},\quad \hat{\boldsymbol{\varphi}}\times\hat{\mathbf{R}}=\hat{\boldsymbol{\theta}}$$

$$\hat{\mathbf{R}}\cdot\hat{\mathbf{R}}=\hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\theta}}=\hat{\boldsymbol{\varphi}}\cdot\hat{\boldsymbol{\varphi}}=1,\quad \hat{\mathbf{R}}\times\hat{\mathbf{R}}=\hat{\boldsymbol{\varphi}}\times\hat{\boldsymbol{\varphi}}=\hat{\boldsymbol{\theta}}\times\hat{\boldsymbol{\theta}}=0$$

$$\hat{\mathbf{R}}\cdot\hat{\boldsymbol{\theta}}=\hat{\mathbf{R}}\cdot\hat{\boldsymbol{\varphi}}=\hat{\boldsymbol{\theta}}\cdot\hat{\boldsymbol{\varphi}}=0$$

Vector relations

$$\mathbf{A}=\hat{\mathbf{a}}A=\hat{\mathbf{R}}A_R+\hat{\boldsymbol{\theta}}A_\theta+\hat{\boldsymbol{\varphi}}A_\phi$$

$$A=|\mathbf{A}|=\sqrt{\mathbf{A}\cdot\mathbf{A}}=\left(A_R^2+A_\theta^2+A_\phi^2\right)^{1/2}$$

Letting $P=P(R_1,\theta_1,\phi_1)$, the position vector $\mathbf{R}_1=OP=\hat{\mathbf{R}}R_1$

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6.24

In this case, all three directions of the base vectors depend on the position.

Spherical Coordinates

Differential relations

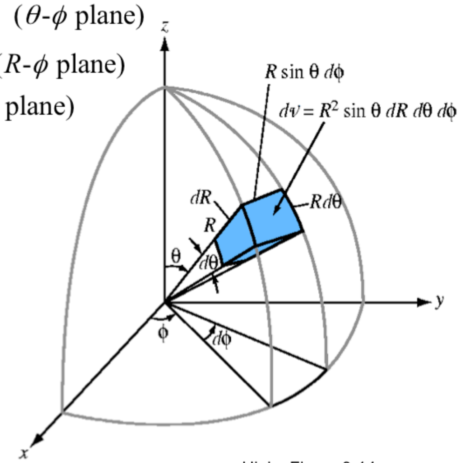
Length: $d\mathbf{l} = \hat{\mathbf{R}}\,dl_R + \hat{\boldsymbol{\theta}}\,dl_\theta + \hat{\boldsymbol{\varphi}}\,dl_\phi = \hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}\,R\,d\theta + \hat{\boldsymbol{\varphi}}\,R\sin\theta\,d\phi$

Surface area: $d\mathbf{s}_R = \hat{\mathbf{R}}\,R^2\sin\theta\,d\theta\,d\phi$ (θ-ϕ plane)

$d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}\,R\sin\theta\,dR\,d\phi$ (R-ϕ plane)

$d\mathbf{s}_\phi = \hat{\boldsymbol{\varphi}}\,R\,dR\,d\theta$ (R-θ plane)

Volume: $dv = R^2\sin\theta\,dR\,d\theta\,d\phi$



Ulaby Figure 3-14

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The dependence of the unit vectors on the local coordinates is a property of all orthogonal coordinate systems, EXCEPT CARTESIAN

Spherical Coordinates

Charge in a Sphere: Ulaby et al. Example 3-6

Question: A sphere of radius 2 cm contains a charge of density ρ_V given by

$$\rho_V = 4\cos^2\theta$$

What is the total charge?

Answer: After converting from cm to m,

$$Q = \int_v \rho_V \, dv$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{R=0}^{2\times10^{-2}} \left(4\cos^2\theta\right) R^2 \sin\theta \, dR \, d\theta \, d\phi$$

$$= 4\int_0^{2\pi}\int_0^{\pi}\left(\frac{R^3}{3}\right)\Bigg|_0^{2\times10^{-2}} \sin\theta\cos^2\theta \, d\theta \, d\phi$$

$$= \frac{64}{9}\times10^{-6}\int_0^{2\pi} d\phi = \frac{128\pi}{9}\times10^{-6} = 44.68\ \mu\text{C}$$

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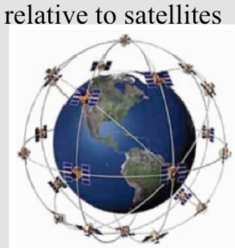
6.26

Note that this example only requires “simple” integral calculus. ALL of Ulaby et al.’s examples on coordinates involve “simple” integral calculus. If you are not completely comfortable with this, you should review it!

Tech Brief 5: GPS

- Originally developed by DOD
- Originally 24 satellites, now 31
- Operation principles
  - Satellites constantly broadcast a message containing
    - Their location
    - Message time
    - System health and rough orbit details
  - GPS receivers use triangulation to determine location relative to satellites
    - Determine distances to each satellite by solving an equation including the sat. position and time, multiplying by the speed of light.



$$d_1^2 = (x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2 = c\,[(t_1 + t_0)]^2\,,$$
$$d_2^2 = (x_2 - x_0)^2 + (y_2 - y_0)^2 + (z_2 - z_0)^2 = c\,[(t_2 + t_0)]^2\,,$$
$$d_3^2 = (x_3 - x_0)^2 + (y_3 - y_0)^2 + (z_3 - z_0)^2 = c\,[(t_3 + t_0)]^2\,,$$
$$d_4^2 = (x_4 - x_0)^2 + (y_4 - y_0)^2 + (z_4 - z_0)^2 = c\,[(t_4 + t_0)]^2\,.$$

Four satellites are needed to correct for the imprecise receiver clock (quartz).

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6.27

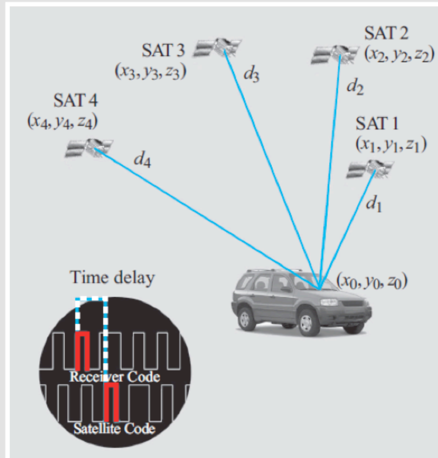
Tech Brief 5: Differential GPS

•GPS accuracy: 20-30m

•Differential GPS (DGPS) uses a static reference of known location in the receiver's area to correct for inaccuracy factors

- •Time-delay errors (speed of light differences
- •Multipath interference
- •Satellite location errors

•Reference receiver calculates correction factors and transmits to DGPS receivers in the area



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