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CMPE 320: HW 05

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1. A stock market trader buys 100 shares of stock A and 200 shares of stock B. Let X and Y be the price changes of A and B, respectively, over a certain time period. And assume that the joint PMF of X and Y is uniform over the set of integers x and y satisfying

$$-2 \leq x \leq 4 \qquad\qquad -1 \leq y - x \leq 1.$$

(a) Find the marginal PMFs and the means of X and Y.

Given:

$$X \in \{x : -2 \leq x \leq 4\}$$

$$Y \in \{y : x - 1 \leq y \leq x + 1,\ x \in X\}$$

Therefore, the pairs (x, y) consist of:

$$\begin{aligned}(x, y) \in \{&(-2, -3), (-2, -2), (-2, -1),\\ &(-1, -2), (-1, -1), (-1, 0), \ldots,\\ &(4, 3), (4, 4), (4, 5)\}\end{aligned}$$

Totalling in $7 \times 3 = 21$ pairs.

Therefore, the joint PMF is

$$p_{X,Y}(x, y) = \begin{cases} 1/21, & \text{if } -2 \leq x \leq 4, -1 \leq y - x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

The marginal PMF are

$$\begin{aligned} p_X(x) &= \sum_y p_{X,Y}(x, y) \\ &= \begin{cases} 3/21, & \text{if } -2 \leq x \leq 4, \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

$$p_Y(y) = \sum_x p_{X,Y}(x,y)$$

$$= \begin{cases} 1/21, & \text{if } y = -3, 5, \\ 2/21, & \text{if } y = -2, 4, \\ 3/21, & \text{if } -1 \le x \le 3, \\ 0, & \text{otherwise} \end{cases}$$

The means,

$$E[X] = \sum_x x \cdot p_X(x)$$

$$= \frac{3}{21}((-2) + (-1) + 0 + 1 + 2 + 3 + 4)$$

$$= \frac{3}{21}(7)$$

$$= 1$$

$$E[Y] = \sum_y y \cdot p_Y(y)$$

$$= \frac{1}{21}((-3) + 5) + \frac{2}{21}((-2) + 4) + \frac{3}{21}((-1) + 0 + 1 + 2 + 3)$$

$$= 1 \qquad \square$$

(b) Find the mean of the trader's profit.

$$100E[X] + 200E[Y] = 100(1) + 200(1)$$

$$= 300 \qquad \square$$

2. The MIT football team wins any one game with probability p, and loses it with probability

$1-p$. Its performance in each game is independent of its performance in other games. Let L_1 be the number of losses before its first win, and let L_2 be the number of losses after its first win and before its second win. Find the joint PMF of L_1 and L_2.

For $L_1 = 0$, $L_2 = 0$,

$$\begin{aligned} P(L_1 = 0, L_2 = 0) &= p \cdot p \\ &= p^2 \end{aligned}$$

For $L_1 = 0$, $L_2 = 1$,

$$\begin{aligned} P(L_1 = 0, L_2 = 1) &= p \cdot ((1-p) \cdot p) \\ &= p^2(1-p) \end{aligned}$$

Similarly, for $L_1 = 1$, $L_2 = 0$,

$$\begin{aligned} P(L_1 = 1, L_2 = 0) &= ((1-p) \cdot p) \cdot p \\ &= p^2(1-p) \end{aligned}$$

For $L_1 = 0$, $L_2 = 2$,

$$\begin{aligned} P(L_1 = 0, L_2 = 2) &= p \cdot ((1-p) \cdot (1-p) \cdot p) \\ &= p^2(1-p)^2 \end{aligned}$$

For $L_1 = 0$, $L_2 = 3$,

$$\begin{aligned} P(L_1 = 0, L_2 = 3) &= p \cdot ((1-p) \cdot (1-p) \cdot (1-p) \cdot p) \\ &= p^2(1-p)^3 \end{aligned}$$

And so on. Therefore, the general expression is:

$$p^2(1-p)^{L_1+L_2}$$

with the PMF:

$$p_{L_1,L_2}(L_1, L_2) = p^2(1-p)^{L_1+L_2} \qquad \square$$

3. A class of n students take a test in which each student gets an A with probability p, a B with probability q, and a grade below B with probability $1-p-q$, independently of any other student. If X and Y are the numbers of students that get an A and a B, respectively. calculate the joint PMF $p_{x,y}$.

Let $r = 1 - p - q$
Then, the multinomial distribution

$$p_{X,Y}(x, y) = \frac{n!}{x!y!(n-x-y)!} \cdot p^x \cdot q^y \cdot r^{(n-x-j)} \text{ for } y = 0, 1, 2..., \ 0 \le x + y \le n \qquad \square$$

4. Your probability class has 250 undergraduate students and 50 graduate students. The probability of an undergraduate (or graduate) student getting an A is 1/3 (or 1/2, respectively). Let X be the number of students that get an A in your class.

(a) Calculate $E[X]$ by first finding the PMF of X

Let x_i for $i = 1, 2, \ldots, 300$ represent the event
where if $x_i = 1$, student i gets an A, and $x_i = 0$ otherwise
The PMF, p_X:

$$p_X(x_i \mid \text{Undergraduate}) = \begin{cases} 1/3, & \text{if } x_i = 1, \\ 2/3, & \text{if } x_i = 0 \end{cases}$$

$$p_X(x_i \mid \text{Graduate}) = \begin{cases} 1/2, & \text{if } x_i = 1, \\ 1/2, & \text{if } x_i = 0 \end{cases}$$

Therefore,

$$E[X] = E\sum_{i=1}^{300} x_i$$

$$= 300E[x_i]$$

$$= 300\left(\frac{1}{3} \cdot \frac{5}{6} + \frac{1}{2} \cdot \frac{1}{6}\right)$$

$$\approx 109$$

$\square$

(b) Calculate $E[X]$ by viewing X as a sum of random variables, whose mean is easily calculated.

Let Y and Z represent the number of undergraduate and graduate students who receive an A, respectively
Therefore,

$$X = Y + Z$$

Thus, the expectation is

$$\begin{aligned} E[X] &= E[Y] + E[Z] \\ &= 250 \cdot \frac{1}{3} + 50 \cdot \frac{1}{2} \\ &\approx 109 \end{aligned}$$

$\square$

5. A scalper is considering buying tickets for a particular game. The price of the tickets is \$75, and the scalper will sell them at \$150. However, if she can't sell them at \$150, she won't sell them at all. Given that the demand for tickets is a binomial random variable with parameters $n = 10$ and $p = 1/2$, how many tickets should she buy in order to maximize her expected profit?

Let i be the number of tickets,

$$\begin{aligned} i &= (n+1)p \\ &= (10+1)(0.5) \\ &= 5.5 \approx 6 \end{aligned}$$

Therefore, she should buy 6 tickets in order to maximize her expected profit $\square$

6. Suppose that X and Y are independent discrete random variables with the same geometric PMF:

$$p_X(k) = p_Y(k) = p(1-p)^{k-1} \qquad\qquad k = 1, 2, \ldots,$$

where p is a scalar with $0, p < 1$. Show that for any integer $n \geq 2$, the conditional PMF

$$P(X = k \mid X + Y = n)$$

is uniform.

$$\begin{aligned}
P(X = k \mid X + Y = n) &= \frac{P(X = k, X + Y = n)}{P(X + Y = n)} \\
&= \frac{P(X = k, Y = n - k)}{P(Y = n - k)} \\
&= \frac{P(X = k, Y = n - k)}{\sum_{k=0}^{n} P(X = k, Y = n - k)} \\
&= \frac{P(X = k) \cdot P(Y = n - k)}{\sum_{k=0}^{n} P(X = k, Y = n - k)} \\
&= \frac{p \cdot (1-p)^{k-1} \cdot p \cdot (1-p)^{n-k}}{\sum_{k=0}^{n} p \cdot p^{k-1} \cdot p \cdot (1-p)^{n-k}} \\
&= \frac{p^2 \cdot (1-p)^{n-1}}{p^2 \cdot (1-p)^{n-1} \cdot (n+1)} \\
&= \frac{1}{n+1}
\end{aligned}$$

Therefore, $P(X = k \mid X + Y = n)$ is uniform $\square$

7. Consider four independent rolls of a 6-sides die. Let X be the number of 1s and let Y be the number of 2s obtained. What is the joint PMF of X and Y?

Let R represent the number yielded after a roll

For a single trial,

$$P(R = 1) = P(R = 2) = \frac{1}{6}$$

and

$$P(R = 3, 4, 5, 6) = \frac{4}{6}$$

Therefore, the multinomial distribution

$$
\begin{aligned}
p_{X,Y}(x,y) &= \frac{4!}{x!y!(4-(x+y))!} \cdot P(R=1)^x \cdot P(R=2)^y \cdot P(R=3)^{(4-(x+y))} \\
&= \frac{4!}{x!y!(4-(x+y))!} \cdot \left(\frac{1}{6}\right)^x \cdot \left(\frac{1}{6}\right)^y \cdot \left(\frac{4}{6}\right)^{(4-x-y)} \qquad \square
\end{aligned}
$$

8. Alvin shops for probability books for K hours, where K is a random variable that is equally likely to be 1, 2, 3, or 4. The number of books N that he buys is random and depends on how long he shops according to the conditional PMF

$$p_{N|K}(n \mid k) = \frac{1}{k}, \text{ for } n = 1, \dots, k$$

(a) Find the joint PMF of K and N

Since K is equally likely to be 1, 2, 3, or 4

$$p_K(k) = \frac{1}{4}, \text{ for } k = 1, 2, 3, 4$$

Therefore

$$
\begin{aligned}
p_{N,K}(n,k) &= p_{N|K}(n \mid k) \cdot p_K(k) \\
&= \frac{1}{k} \cdot \frac{1}{4}, \text{ for } k = 1,2,3,4,\ n = 1,\dots,k \\
&= \begin{cases} \frac{1}{4k}, & \text{if } k = 1,2,3,4,\ n = 1,\dots,k, \\ 0, & \text{otherwise} \end{cases}
\end{aligned}
$$

$\square$

(b) Find the marginal PMF of N

The marginal PMF is given by

$$p_N(n) = \sum_{n}^{4} p_{N,K}(n,k)$$

Therefore,

$$p_N(n=1) = \frac{1}{4} + \frac{1}{8} + \frac{1}{12} + \frac{1}{16}$$

$$= \frac{25}{48}$$

$$p_N(n=2) = \frac{1}{8} + \frac{1}{12} + \frac{1}{16}$$

$$= \frac{13}{48}$$

$$p_N(n=3) = \frac{1}{12} + \frac{1}{16}$$

$$= \frac{7}{48}$$

$$p_N(n=4) = \frac{1}{16}$$

$$p_N(n>4) = 0$$

□

(c) Find the conditional PMF of K given that $N = 2$

$$p_{K|2}(k \mid 2) = \frac{p_{N,K}(2,k)}{p_N(2)}$$

$$= \begin{cases} \frac{6}{13}, & \text{if } n = 2, \\ \frac{4}{13}, & \text{if } n = 3, \\ \frac{3}{13}, & \text{if } n = 4, \\ 0, & \text{otherwise} \end{cases}$$

□

(d) Find the conditional mean and variance of K, given that he bought at least 2 but no more than 3 books.

$$p_{K|(2\le n\le 3)}(k) = \frac{P(K=k, (2 \le n \le 3))}{p_N(2) + p_N(3)}$$

where $p_N(2) + p_N(3) = \frac{5}{12}$

$$P(K=k,(2\le n\le 3)) = \begin{cases} \frac{1}{8}, & \text{if } k=2, \\ \frac{1}{6}, & \text{if } k=3, \\ \frac{1}{8}, & \text{if } k=4, \\ 0, & \text{otherwise} \end{cases}$$

Therefore,

$$p_{K\mid(2\le n\le 3)}(k) = \begin{cases} \frac{3}{10}, & \text{if } k=2, \\ \frac{4}{10}, & \text{if } k=3, \\ \frac{3}{10}, & \text{if } k=4, \\ 0, & \text{otherwise} \end{cases}$$

Conditional mean

$$E[K \mid (2\le n\le 3)] = 3$$

Conditional variance

$$\begin{aligned} var(K \mid (2\le n\le 3)) &= E[K - E[K \mid (2\le n\le 3)]^2 \mid (2\le n\le 3)] \\ &= \frac{3}{10}(2-3)^2 + \frac{2}{5}(0) + \frac{3}{10}(4-3)^2 \\ &= \frac{3}{5} \end{aligned}$$

□

(e) The cost of each book is a random variable with mean \$30. What is the expected value of his total expenditure? *Hint:* Condition on the events $\{N=1\},\ldots,\{N=4\}$, and use the total expectation theorem.

Let $T = C_1 + \ldots + C_N$, where $E[C_i] = 30$ for $i = 1,\ldots,N$

$$\begin{aligned} E[T] &= E[E[T \mid N]] \\ &= E[N\cdot 30] \\ &= 30E[N] \\ &= 30(1\cdot\frac{25}{48} + 2\cdot\frac{13}{48} + 3\cdot\frac{7}{48} + 4\cdot\frac{3}{48}) \end{aligned}$$

$$= 52.5 \qquad \square$$