MATH 407
3/26/18

Subgroups $H \neq \emptyset$ of group G.
i) a, b in H implies $a, b \in H$
ii) a in H implies $a^{-1} \in H$

Thm. $H \neq \emptyset \in G$ is subgroup iff $a, b \in H \Rightarrow ab^{-1} \in H$

Examples $\{e\}, G$
If $a \in G$ then $\langle a \rangle = \{a^n : n \in G\}$ cyclic subgroup generated by a $n \to a^n$ injection $\mathbb{Z} \to G$

Finite $\langle a \rangle = \{e, a, a^2, \ldots, a^{k-1}\}$
$a^k = e$
$k = o(a)$, order of a

Thm. If $\{H_1, \ldots, H_i, \ldots\}$ is a collection of subgroups of G then $H = \bigcap_i H_i$ is a subgroup.

Pf Let a, b be in H. Thus, $ab \in H_i$, $\forall i$
Consequently, $ab^{-1} \in H_i$, $\forall i$
Thus, $ab^{-1} \in H$

If H subgroup of G, $a \in H$ implies $\langle a \rangle \subseteq H$

Thm. Let $S \subseteq G$. There is a smallest subgroup $\langle S \rangle$ of G containing S.

②

Df. $\langle S \rangle = \bigcap \{H : H \text{ subgroup of } G,\ S \subseteq H\}$

Example. $\langle \emptyset \rangle = \{e\} = \langle e \rangle$

$\langle G \rangle = G$

$a \in G,\ \langle a \rangle$ (cyclic)

Example. $G = GL_n(F)$

$$H = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$$

$A \in H \Rightarrow A^2 = I \in H$

$B \in H$

$AB \in H,\ BA \in H$

$\langle A \rangle = \{I, A\}$

$$H = \left\langle \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \right\rangle \text{ (non cyclic)}$$

* Let $H \subseteq G$ be a subgroup. If $a, b \in G$, define $a \sim_H b$ iff $ab^{-1} \in H$

Reflexivity: $a \sim_H a$

$$aa^{-1} = e \in H$$

Symmetry: $a \sim_H b$. Is $b \sim_H a$?

$$(ab^{-1})^{-1} = ba^{-1} \in H$$

Transitivity: Let $a \sim_H b$, $b \sim_H c$

Then,

$$ab^{-1} \in H, \ bc^{-1} \in H$$

$$(ab^{-1})(bc^{-1}) \in H$$

So $ac^{-1} \in H$

$$\Rightarrow a \sim_H c$$

↳ * Example: $G = S_5$, $H = S_3$, $\sigma\tau^{-1} \in S_3$

Then $\sigma \sim_H \tau$

* $[b]_{\sim H}$ or $[b]_H = \{a : a \sim_H b\}$

$= \{a : ab^{-1} \in H\}$

$= \{a : a \in Hb\}$, $Hb = \{hb : h \in H\}$

$$m_b^r(H) = \{m_b^r(h) : h \in H\} = \{hb : h \in H\}$$

④

$[b]_H = Hb$ (right coset of H)

$[e]_h = H$

$\{[b]_H : b \in G\} = \{[e]_H, [b]_H, \dots [b_\ell]_H \dots\}$

$= \{H, Hb_1, \dots Hb_\ell \dots\}$

$\Rightarrow |H| = |Hb_i| \quad \forall i$

Suppose $|G| < \infty$

Then $|H| < \infty$ and $\{H, Hb_1, \dots, Hb_{k-1}\}$

$|G| = |H| + |Hb_1| + \dots + \dots |Hb_{k-1}| = k|H|$

(LaGrange's Thm)