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MATH 407: HW 10

3.6 **5** Show that no proper subgroup of S_4 contains both $(1, 2, 3, 4)$ and $(1, 2)$.

Pf. Suppose $H \leq S_4$ with the permutations (1 2 3 4) and (1 2)

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$$

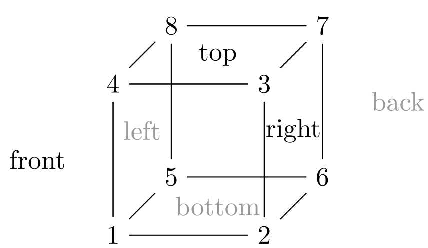
Their product would yield

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix} = S_4$$

Therefore, H is not a proper subgroup of S_4 by contradiction $\square$

9 A rigid motion of a cube can be thought of either as a permutation of its eight vertices or as a permutation of its six sides. Find a rigid motion of the cube that has order 3, and express the permutation that represents it in both ways, as a permutation on eight elements and as a permutation on six elements.

Figure 1: Cube of order 3



Pf. Rigid motion of order 3 will be rotating the cube about a vector passing through $(1, 7)$ $120°$

The permutation of the eight vertices are $(2, 4, 5)(3, 8, 6)$

and the sides are (front, left, back)(top, back, right) $\square$

10 Show that the following matrices form a subgroup of $\mathrm{GL}_2(\mathbb{C})$ isomorphic to D_4:

$$\pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \pm \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

Pf. Let $a = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ and $b = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

Then,

$$a^2 = \left(\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\right)^2 = \begin{bmatrix} \text{-1} & 0 \\ 0 & \text{-1} \end{bmatrix}$$

$$a^3 = \left(\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\right)^3 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$$

$$a^4 = \left(\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\right)^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$b^2 = \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$ab = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

$$a^2b = \begin{bmatrix} \text{-1} & 0 \\ 0 & \text{-1} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \text{-1} \\ \text{-1} & 0 \end{bmatrix}$$

$$a^3b = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Which forms the set

$$\begin{aligned} \mathrm{GL}_2(\mathbb{C}) &= \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}, \text{ with } a^4 = b^2 = e,\ ba = a^{-1}b \\ &\cong D_4 \end{aligned}$$

□

15 (a) Show that $A_4 = \{\sigma \in S_4 \mid \sigma = \tau^2 \text{ for some } \tau \in S_4\}$

Pf.

□

(b) Show that $A_5 = \{\sigma \in S_5 \mid \sigma = \tau^2 \text{ for some } \tau \in S_5\}$

Pf.

□

(c) Show that $A_6 = \{\sigma \in S_6 \mid \sigma = \tau^2 \text{ for some } \tau \in S_6\}$

Pf. $\square$

(d) What can you say about A_n if $n > 6$?

Pf. $\square$

17 For any elements $\sigma, \tau \in S_n$, show that $\sigma\tau\sigma^{-1}\tau^{-1} \in A_n$.

Pf. Let $\sigma, \tau \in S_n$ be the products of m and n transpositions respectively

Thus, $\sigma^{-1}, \tau^{-1} \in S_n$ are also the products of m and n transpositions

Therefore, $\sigma\tau\sigma^{-1}\tau^{-1}$ is a product of $m + n + m + n = 2(m + n)$ transpositions

Since $2 \mid 2(m + n)$, $\sigma\tau\sigma^{-1}\tau^{-1} \in A_n$ $\square$

21 Find the center of the dihedral group D_n.

Hint: Consider two cases, depending on whether n is odd or even.

Pf. The center of a group is the subgroup consisting of all the elements that commute with every other element in the group

The group D_1 is $\mathbb{Z}/2\mathbb{Z}$, and D_2 is $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

Let $n > 2$, and assume yx^k is in the center,

That is, yx^k must commute with x

Then,

$$xyx^kx^{-1} = yx^{k-2} = yx^k$$

Which implies, $2 \equiv 0 \pmod{n}$, (contradiction)

Therefore, x^k is in the center iff $2k \equiv 0 \pmod{n}$

If n is odd, then the center of D_{2m+1} is $\{e\}$

If n is even, then the center of D_{2m} is $\{e, x^m\}$ $\square$

24 Show that the product of two transpositions is one of (i) the identity; (ii) a 3-cycle; (iii) a product of two (non-disjoint) 3-cycles. Deduce that every element of A_n can be written as a product of 3-cycles.

Pf. Consider $\sigma = (1, 2, 3, 4)$

(i) For identity,

$$(1, 2)(1, 2) = (1)$$

(ii) For a 3-cycle,

$$(1, 2)(2, 3) = (1, 3, 2)$$

(iii) For a product of two (non-disjoint) 3-cycles,

$$(1,2)(3,4) = (1,2,3)(1,4,3)$$

Since A_n is a set of even permutations, is it can be expressed as a product of even number of transpositions $\square$

3.7 **4** Let G be an abelian group, and let n be any positive integer. Show that the function $\phi : G \to G$ defined by $\phi(x) = x^n$ is a homomorphism.

Pf. Let $x, y \in G$
Then $\phi(xy) = (xy)^n$
Since G is abelian,

$$\begin{aligned}(xy)^n &= x^n y^n \\ &= \phi(x)\phi(y)\end{aligned}$$

Therefore, $\phi(xy) = \phi(x)\phi(y)\ \forall x, y \in G$ $\square$

6 Define $\phi : \mathbb{C}^\times \to \mathbb{R}^\times$ by $\phi(a+bi) = a^2+b^2$, for all $a+bi \in \mathbb{C}^\times$. Show that ϕ is a homomorphism.

Pf. Let $a + bi, c + di \in \mathbb{C}^\times$
Then $\phi((a + bi)(c + di)) = (ac - bd) + i(ad + bc)$
Therefore,

$$\begin{aligned}\phi((a + bi)(c + di)) &= (ac - bd) + i(ad + bc) \\ &= (ac - bd)^2 + (ad + bc)^2 \\ &= (ac)^2 - 2acbd + (bd)^2 + (ad)^2 + 2adbc + (bc)^2 \\ &= (ac)^2 + (bd)^2 + (ad)^2 + (bc)^2\end{aligned}$$

Also,

$$\begin{aligned}\phi(a + bi)\phi(c + di) &= (a^2 + b^2)(c^2 + d^2) \\ &= a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 \\ &= (ac)^2 + (bd)^2 + (ad)^2 + (bc)^2\end{aligned}$$

Therefore $\phi((a + bi)(c + di)) = \phi(a + bi)\phi(c + di)$ $\square$

7 Which of the following functions are homomorphisms?

b $\phi : \mathbb{R} \to \mathrm{GL}_2(\mathbb{R})$ defined by $\phi(a) = \begin{bmatrix} 1 & \mathrm{a} \\ 0 & 1 \end{bmatrix}$

Pf. Consider,

$$\begin{aligned} \phi(a+b) &= \begin{bmatrix} 1 & \mathrm{a+b} \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & \mathrm{a} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \mathrm{b} \\ 0 & 1 \end{bmatrix} \\ &= \phi(a)\phi(b) \end{aligned}$$

Since $\phi(a+b) = \phi(a)\phi(b)$, ϕ is a homomorphism □

d $\phi : \mathrm{GL}_2(\mathbb{R}) \to \mathbb{R}^\times$ defined by $\phi\left(\begin{bmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{c} & \mathrm{d} \end{bmatrix}\right) = ab$

Pf. For ϕ to be homomorphic, the identity element of $\mathrm{GL}_2(\mathbb{R}) = e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ must be mapped to the identity element of $\mathbb{R}^\times = e_2 = 1$

But

$$\begin{aligned} \phi\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) &= 1 \times 0 \\ &= 0 \\ &\neq e_2 = 1 \end{aligned}$$

Therefore, ϕ is not a homomorphism □

10 Let G be the group of affine functions from $\mathbb{R}$ into $\mathbb{R}$, as defined in Exercise 10 of Section 3.1. Define $\phi : G \to \mathbb{R}^\times$ as follows: for any function $f_{m,b} \in G$, let $\phi(f_{m,b}) = m$. Prove that ϕ is a group homomorphism, and find its kernel and image.

Pf. Given $G = \{f_{m,b} : \mathbb{R} \to \mathbb{R} \mid m \neq 0 \text{ and } f_{m,b}(x) = mx + b\}$
Consider,

$$\begin{aligned} (f_{n,a} \circ f_{m,b}) &= f_{n,a}(f_{m,b}) \\ &= f_{n,a}(mx+b) \\ &= n(mx+b) + a \end{aligned}$$

$$= nmx + bn + a$$
$$= f_{nm,bn+a}$$

Therefore,

$$\begin{aligned}
\phi(f_{n,a} \circ f_{m,b}) &= \phi(f_{nm,bn+a}) \\
&= nm \\
&= \phi(f_{n,a})\phi(f_{m,b})
\end{aligned}$$

Therefore, ϕ is a homomorphism

By definition,

$$\begin{aligned}
\ker(\phi) &= \{f_{m,b} \in G \mid \phi(f_{m,b}) = 1\} \\
&= \{f_{m,b} \in G \mid m = 1\}
\end{aligned}$$

By definition,

$$\begin{aligned}
\mathrm{img}(\phi) &= \{m \in \mathbb{R}^{\times} \mid \phi(f_{m,b}) = m\} \\
&= \mathbb{R}^{\times}
\end{aligned}$$ $\square$

14 Recall that the center of a group G is $\{x \in G \mid xh = gx \text{ for all } g \in G\}$. Prove that the center of any group is a normal subgroup.

Pf. Let H be a subgroup of G, and $x \in H$

Then $xg = gx, \ \forall g \in G$

Therefore,

$$\begin{aligned}
gx &= xg \\
\implies gxg^{-1} &= xgg^{-1} \\
\implies gxg^{-1} &= x \in H
\end{aligned}$$

Thus, $\forall x \in H$, $\forall g \in G$, $gxg^{-1} \in H$

That is, $\{x \in G \mid xh = gx \text{ for all } g \in G\}$ is a normal subgroup of G $\square$

18 Let the dihedral group D_n be given by elements a of order n and b of order 2, where $ba = a^{-1}b$. Show that any subgroup of $\langle a \rangle$ is normal in D_n.

Pf. Let H be a subgroup of $\langle a \rangle$, such that any elements of the form $(a^m)^d \in H$

Also, since $ba = a^{-1}b \in D_n$,

$$\begin{aligned} ba^2 &= a^{-1}ba \\ &= a^{-1}a^{-1}b \\ &= a^{-2}b \end{aligned}$$

Thus, $ba^i = a^{-i}b$ for any i

□

3.8 **4** For each of the subgroups $\{e, a^2\}$ and $\{e, b\}$ of D_4, list all left and right cosets.

Pf. □

9 Let G be a finite group, and let n be a divisor of $|G|$ Show that if H is the only subgroup of G of order n, then H must be normal in G.

Pf. aHa^{-1} is the subgroup of G that is isomorphic to H

Therefore, $aHa^{-1} = H$, $\forall a \in G$

Which is true iff H is a normal subgroup in G □

12 Let H and K be normal subgroups of G such that $H \cap K = \langle e \rangle$. Show that $hk = kh$ for all $h \in H$ and $k \in K$.

Pf. Consider $hkh^{-1}k^{-1}$

Since K is a normal subgroup,

$$\begin{aligned} hkh^{-1}k^{-1} &= (hkh^{-1})k^{-1} \\ &= hkh^{-1}, k^{-1} \in K \\ &= hkh^{-1}k^{-1} \in K \end{aligned}$$

Similarly, since H is a normal subgroup,

$$\begin{aligned} hkh^{-1}k^{-1} &= h(kh^{-1}k^{-1}) \\ &= h, kh^{-1}k^{-1} \in H \\ &= hkh^{-1}k^{-1} \in H \end{aligned}$$

Therefore, $hkh^{-1}k^{-1} \in K \cap H$

But, since $hkh^{-1}k^{-1} \in K \cap H = \{e\}$,

$$\begin{aligned} hkh^{-1}k^{-1} &= e \\ \implies hk(h^{-1}k^{-1}) &= e \\ \implies hk &= kh \end{aligned}$$ $\square$

18 Compute the factor group $(\mathbb{Z}_6 \times \mathbb{Z}_4)/\langle(3,2)\rangle$.

Pf. Given $\langle(3,2)\rangle = \{(3,2),(0,0)\}$

$\mathbb{Z}_6 \times \mathbb{Z}_4$ has 24 elements, with 24/2 left cosets

The factor groups are,

$$\begin{aligned}
(0,1) + \langle(3,2)\rangle &= \{(3,3),(0,1)\} \\
(0,2) + \langle(3,2)\rangle &= \{(3,2),(0,2)\} \\
(0,3) + \langle(3,2)\rangle &= \{(3,1),(0,3)\} \\
(1,0) + \langle(3,2)\rangle &= \{(4,2),(1,0)\} \\
(1,1) + \langle(3,2)\rangle &= \{(4,3),(1,1)\} \\
(1,2) + \langle(3,2)\rangle &= \{(4,0),(1,2)\} \\
(1,3) + \langle(3,2)\rangle &= \{(4,1),(1,3)\} \\
(2,1) + \langle(3,2)\rangle &= \{(5,3),(2,1)\} \\
(2,2) + \langle(3,2)\rangle &= \{(5,0),(2,2)\} \\
(2,3) + \langle(3,2)\rangle &= \{(5,1),(2,3)\}
\end{aligned}$$ $\square$

19 Show that $(\mathbb{Z} \times \mathbb{Z})/\langle(0,1)\rangle$ is an infinite cyclic group.

Pf. $\square$

23 Let G be the set of all matrices in $\mathrm{GL}_2((Z)_5)$ of the form $\begin{bmatrix} \mathrm{m} & \mathrm{b} \\ 0 & 1 \end{bmatrix}$

a. Show that G is a subgroup of $\mathrm{GL}_2((\mathbb{Z}_5)$.

Pf. Given G is non-empty and finite with $\mathbb{Z}_5 = \{[0],[1],[2],[3],[4]\}$

For G to be a subgroup, it is enough to show $xy \in G$, $\forall x, y \in G$

Let $x = \begin{bmatrix} \mathrm{n} & \mathrm{a} \\ 0 & 1 \end{bmatrix}, y = \begin{bmatrix} \mathrm{m} & \mathrm{b} \\ 0 & 1 \end{bmatrix} \in G$

Then,

$$
\begin{aligned}
xy &= \begin{bmatrix} \text{n} & \text{a} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \text{m} & \text{b} \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \text{nm} & \text{nb+a} \\ 0 & 1 \end{bmatrix} \\
&\in G
\end{aligned}
$$

□

b. Show that the subset N of all matrices in G of the form $\begin{bmatrix} 1 & \text{c} \\ 0 & 1 \end{bmatrix}$, with $c \in \mathbb{Z}_5$, is a normal subgroup of G.

Pf. □

c. Show that the factor group G/N is cyclic of order 4.

Pf. □