MATH 407
3/12/18

* Transpositions: (a,b)
$$(a,b)(a,b) = (1)$$
$$(a,b) = (a,b)^{-1}$$

Thm: Every $\pi \in S_n$ is the product of transpositions (for $n \geq 2$)

Pf. Assume true for $n=k$. Let $\pi \in S_{k+1}$
(Regard $\sigma \in S_n$ as an element of S_{n+1} w/ $\sigma(n+1) = (n+1)$)

i) $\pi(k+1) = (k+1)$

ii) $\pi(k+1) = i \leq k$

Let $\tau = (i, k+1)$

$\tau \circ \pi (k+1) = (k+1)$

$\tau \circ \pi = \tau_1 \circ \dots \circ \tau_l$ $\quad$ τ_j transpositions

$$\pi = \tau \circ (\tau \circ \pi) = \tau \circ \tau_1 \circ \dots \circ \tau_l$$
(since τ is its own inverse)

Ex. $(1, 2, \dots, n) = (1, n) \circ (1, n-1) \circ \dots \circ (1,3) \circ (1,2)$

Pf. $(a_1, a_2, \dots, a_n)$
$= (a_1, a_n) \dots (a_1, a_3)(a_1, a_2)$

②

<u>Thm.</u> If $\pi \in S_n$ is $\tau_1 \ldots \tau_k$
and $\sigma_1 \ldots \sigma_\ell$
then $k \equiv \ell \pmod 2$

* A_n := even permutations (k is even)

<u>Pf.</u> Suppose k is even, ℓ is odd.

$$(\tau_1 \ldots \tau_k)(\sigma_\ell \ldots \sigma_1) = (1)$$

$$(\sigma_1 \ldots \sigma_\ell)^{-1} = \sigma_\ell^{-1} \sigma_{\ell-1}^{-1} \ldots \sigma_2^{-1} \sigma_1^{-1}$$

$$= \sigma_\ell \sigma_{\ell-1} \ldots \sigma_2 \sigma_1 \quad \text{transpositions}$$

$\therefore k + \ell$ is odd

* Let $(1) = \tau_1 \ldots \tau_k \ldots \tau_{2m+1}$.
Assume $2m+1$ is as small as possible.
Let $\tau_1 = (a, b)$
$(1) = (a,b)(c,d) \ldots \ldots \quad 2m+1 \geq 3$

Assume fewest # occurrences of a.

If $(c,d) = (a,b)$ then
$(1) = \tau_3 \ldots \tau_{2m+1}$
$(c,d) = (a,c) = \tau_2$

$(a,b)(a,c) = (a,c)(b,c)$ (contradiction)

$\tau_2 = (c,d)$, $a \notin \{c,d\}$

* Let $\tau_1 \dots \tau_{2m+1}$ have earliest subsequent occurrence of a

$\tau_i = (a,r)$, $i \geq 3$

α) $\tau_{i-1} = (u,v)$, $i-1 \geq 2$

$\underset{\tau_{i-1}}{(u,v)}\underset{\tau_i}{(a,r)} = \underset{\tau_{i-1}}{(a,r)}\underset{\tau_i}{(u,v)}$ (contradiction)

β) $\underset{\tau_{i-1}}{(u,v)}\underset{\tau_i}{(a,v)} = (a,u)(u,v)$ (contradiction)

* $S_n = A_n \cup A_n^c$ if τ is transposition:

$A_n^c = \tau A_n = \{\tau\pi : \pi \in A_n\}$

$\underset{\underset{n!}{\parallel}}{|S_n|} = \underset{\underset{n!/2}{\parallel}}{|A_n|} \cup A_n^c$