CMPE 320: Probability, Statistics, and Random Processes

Lecture 9: Expectation

Spring 2018

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Announcements

- HW#3 will be returned today
- HW#4 solutions will be posted tomorrow (Tues.) around 5pm
- Midterm exam will be on Wednesday 3/7 during the class
  - Will cover everything up to Expectation (Sec. 2.4)
  - Closed book, closed note.
  - Calculators are allowed (No smartphones, tablets, laptop PCs allowed)

Expectation

- Suppose you spin a wheel of fortune many times. At each spin, one of the numbers $m_1, m_2, \ldots, m_n$ comes up with corresponding probability $p_1, p_2, \ldots, p_n$, and this is the money you get. What is the money that you expect to get per spin?

Suppose you spin it k times

Let k_i be the number of times that m_i is the outcome.

Total reward $= m_1 k_1 + m_2 k_2 + \cdots + m_n k_n$

Reward per spin $= \frac{\text{Total}}{k} = \frac{m_1 k_1 + m_2 k_2 + \cdots + m_n k_n}{k}$

If k is large, reasonable $\frac{k_i}{k} \approx p_i$

So expectation per spin is $m_1 p_1 + m_2 p_2 + \cdots + m_n p_n$

Expectation of a RV X

- We define the expected value (also called expectation or mean) of a random variable X with PMF p_X by

$$E[X] = \sum_{x} x \, P(X = x) = \boxed{\sum_{x} x \, P_X(x)}$$

notation for expectation

summate over all possible values that X can take

Example 2.2. Consider two independent coin tosses, each with a 3/4 probability of a head, and let X be the number of heads obtained. This is a binomial random variable with parameters $n = 2$ and $p = 3/4$. Its PMF is

$$p_X(k) = \begin{cases} (1/4)^2, & \text{if } k = 0, \\ 2 \cdot (1/4) \cdot (3/4), & \text{if } k = 1, \\ (3/4)^2, & \text{if } k = 2, \end{cases}$$

$\leftarrow \binom{2}{0}\left(\frac{3}{4}\right)^0\left(1-\frac{3}{4}\right)^2 = \left(\frac{1}{4}\right)^2$

$\leftarrow \binom{2}{1}\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)$

$\leftarrow \binom{2}{2}\left(\frac{3}{4}\right)^2\left(\frac{1}{4}\right)^0$

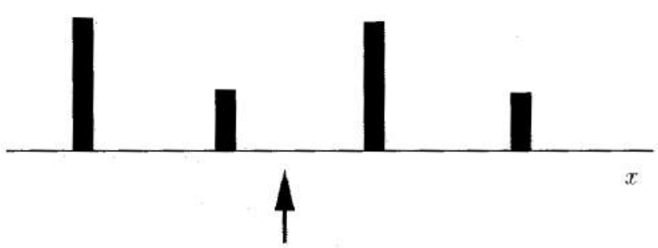
Compute the expectation of X.

$$E[X] = \sum_x x P_X(x) = \sum_{k=0}^{2} k P_X(k) = 0 \cdot \left(\tfrac{1}{4}\right)^2 + 1 \cdot 2 \cdot \left(\tfrac{1}{4}\right)\left(\tfrac{3}{4}\right) + 2 \cdot \left(\tfrac{3}{4}\right)^2 \leftarrow$$

$$= \frac{6}{16} + \frac{18}{16}$$

$$= \frac{3}{2}$$

Expectation is the center of gravity of PMF



Center of gravity
c = mean = E[X]

Moments, variance, standard deviation

- *n*-th moment of X = expectation of X^n $\quad n = 1, 2, 3 \cdots$

  2nd moment of $X = E[X^2]$

- Variance = Expected value of $(X - E[X])^2$

  = 2nd centralized moment

  $$\boxed{Var(X) = E[\underbrace{(X - E[X])^2}]}$$

  Since this is always nonnegative

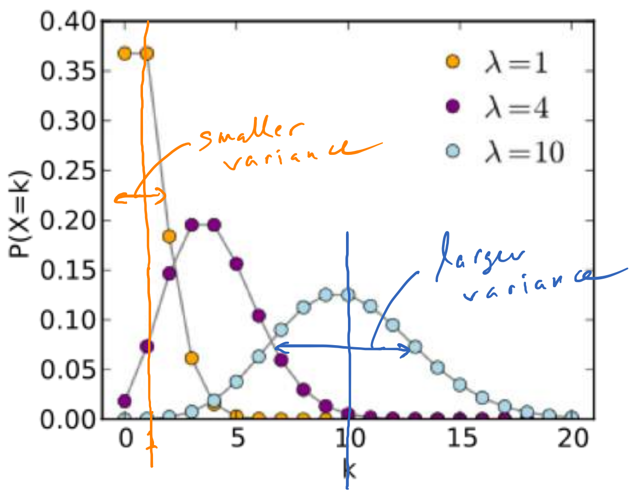
  $Var(X) \geq 0$

- Standard deviation $\quad \sigma_X = \sqrt{var(X)}$ → has the same unit as X

  X inches → var(X) $inch^2$, σ_X inch

Variance (and standard deviation) captures the dispersion around mean of PMF

Poisson PMFs (E [X] = λ) Poisson PMF, $var(X) = \lambda$



Smaller variance

⇒ peaky PMF

⇒ RV is more concentrated around the mean

What if $var(X) = 0$

→ $X = E[X]$

RV X is no longer "random"

Example 2.3. Consider the random variable X of Example 2.1, which has the PMF

$$p_X(x) = \begin{cases} 1/9, & \text{if } x \text{ is an integer in the range } [-4,4], \\ 0, & \text{otherwise.} \end{cases}$$

Compute the variance of X.

$$\mathrm{var}(X) = E[(X - E[X])^2]$$

First compute the mean $E[X]$

$$E[X] = \sum_x x p_X(x) = -4 \cdot \frac{1}{9} - 3 \cdot \left(\frac{1}{9}\right) \cdots + 4\left(\frac{1}{9}\right) = \frac{1}{9}(-4-3-2-1+0+1+2+3+4) = 0$$

$$Z = (X - E[X])^2 = X^2$$ What is the PMF of Z?

$$Z \in \{0, 1^2, 2^2, \ldots, 4^2\}$$

$$P(Z=0) = \frac{1}{9} \quad P(Z=1^2) = P(X=1 \text{ or } X=-1) = \frac{1}{9} + \frac{1}{9} = \frac{2}{9} = P(Z=2^2) = \cdots P(Z=4^2)$$

$$P_Z(z) = \begin{cases} \frac{1}{9} & \text{if } z = 0 \\ \frac{2}{9} & \text{if } z = 1, 2^2, 3^2, 4^2 \\ 0 & \text{otherwise} \end{cases}$$

$$\underset{\mathrm{var}(X)}{\overset{\parallel}{E[Z]}} = \sum_z z P_Z(z) = 0 \cdot \frac{1}{9} + 1 \cdot \left(\frac{2}{9}\right) + 2^2 \cdot \left(\frac{2}{9}\right) + 3^2\left(\frac{2}{9}\right) + 4^2\left(\frac{2}{9}\right)$$

$$= \frac{60}{9}$$

Expected value rule of a function of RV

- Let g(X) be a function of RV X. Then, the expected value of the RV g(X) is given by

$$E[g(X)] = \sum_x g(x) P_X(x)$$

- Variance of X is E[g(X)] with g(X) = (X – E[X])^2

$$\mathrm{var}(X) = E[(X - E[X])^2] = \sum_x (x - E[X])^2 P_X(x)$$

Example 2.3. Consider the random variable X of Example 2.1, which has the PMF

$$p_X(x) = \begin{cases} 1/9, & \text{if } x \text{ is an integer in the range } [-4,4], \\ 0, & \text{otherwise.} \end{cases}$$

Compute the variance of X using the expected value rule for a function of RV.

$$E[X] = 0 \qquad var(X) = E[(X-E[X])^2] = E[X^2]$$

$$g(X) = X^2$$

$$E[X^2] = E[g(X)] = \sum_x g(x) P_X(x) = \sum_x x^2 P_X(x)$$

$$= (-4)^2 \frac{1}{9} + (-3)^2\left(\frac{1}{9}\right) + (-2)^2\left(\frac{1}{9}\right) + (-1)^2\left(\frac{1}{9}\right) + 0^2\left(\frac{1}{9}\right)$$

$$+ 4^2\left(\frac{1}{9}\right) + 3^2\left(\frac{1}{9}\right) + 2^2\left(\frac{1}{9}\right) + 1^2\left(\frac{1}{9}\right)$$

$$= \frac{60}{9}$$

Variance in terms of moment expression

$$\boxed{var(X) = \underbrace{E[X^2]}_{\text{2nd moment of } X} - \underbrace{(E[X])^2}_{(\text{mean of } X)^2}}$$

Why? $$var(X) = E[(X-E[X])^2]$$

$$= \sum_x (x - E[X])^2 P_X(x)$$

$$= \sum_x (x^2 - \underline{2x E[X]} + (E[X])^2) P_X(x)$$

$$= \underbrace{\sum_x x^2 P_X(x)} - \underline{2E[X]} \underbrace{\sum_x x P_X(x)} + (E[X])^2 \underbrace{\sum_x P_X(x)}$$

$$= E[X^2] - 2(E[X])^2 + (E[X])^2 = E[X^2] - (E[X])^2$$

Properties of mean and variance

- Mean of Y = a X + b $E[aX+b] = aE[X] + b$ (Expectation is linear)

$$E[Y] = E[aX+b] = \sum_x (ax+b)\, P_X(x)$$
$$= a\underbrace{\sum_x x P_X}_{E[X]} + b\underbrace{\sum_x P_X(x)}_{1} = aE[X] + b$$

- Variance of Y = a X + b $\mathrm{var}(aX+b) = a^2\,\mathrm{var}(X)$

$$\mathrm{var}(Y) = \mathrm{var}(aX+b) = \sum_x (aX+\cancel{b} - \underbrace{E[aX+b]}_{= aE[X] + \cancel{b}})^2 P_X(x)$$
$$= \sum_x (aX - aE[X])^2 P_X(x) = a^2 \underbrace{\sum_x (X - E[X])^2 P_X(x)}_{} = a^2\,\mathrm{var}(X)$$

Example 2.4. Average Speed Versus Average Time. If the weather is good (which happens with probability 0.6), Alice walks the 2 miles to class at a speed of $V = 5$ miles per hour, and otherwise rides her motorcycle at a speed of $V = 30$ miles per hour. What is the mean of the time T to get to class?

Also what is the variance of T?

① Use the PMF of T

$$P_T(t) = \begin{cases} 0.6 & \text{if } t = \frac{2}{5} \\ 0.4 & \text{if } t = \frac{2}{30} \end{cases}$$

$$E[T] = \left(\frac{2}{5}\right)\cdot 0.6 + \left(\frac{2}{30}\right)\cdot 0.4 = \frac{4}{15}$$

$\rightarrow \mathrm{var}(T) = E[T^2] - (E[T])^2 = \left(\frac{22}{225}\right) - \left(\frac{4}{15}\right)^2 = \frac{2}{25}$

$E[T^2] = \left(\frac{2}{5}\right)^2 \cdot 0.6 + \left(\frac{2}{30}\right)^2 \cdot 0.4 = \frac{22}{225}$

$E[T] = \frac{4}{15}$

② Use the PMF of V

$$P_V(v) = \begin{cases} 0.6 & \text{if } v = 5 \\ 0.4 & \text{if } v = 30 \end{cases}$$

$$T = \frac{2}{V} = g(V)$$

$$E[T] = E\left[\frac{2}{V}\right] = \frac{2}{5}\cdot 0.6 + \frac{2}{30}\cdot 0.4 = \frac{4}{15}$$

What about $\frac{2}{E[V]}$?

$E[V] = 5 \cdot 0.6 + 30 \cdot 0.4 = 15$

$\frac{2}{E[V]} = \frac{2}{15} \neq E\left[\frac{2}{V}\right]$

Mean and variance of Bernoulli RV

Bernoulli RV $X = \begin{cases} 1 & \text{with prob } p \\ 0 & \text{" } 1-p \end{cases}$

$$P_X(x) = \begin{cases} p & \text{if } x=1 \\ 1-p & \text{if } x=0 \end{cases}$$

$$E[X] = 1 \cdot p + 0 \cdot (1-p) = p$$

$$\mathrm{var}(X) = E[X^2] - (E[X])^2 = p - p^2 = p(1-p)$$

$$E[X^2] = 1^2 \cdot p + 0^2 (1-p) = p$$

Mean and variance of uniform RV

– What is the mean and variance of a fair die?

$$P_X(x) = \begin{cases} \frac{1}{6} & \text{if } x = 1, 2, \dots, 6 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \sum_x x P_X(x) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \dots + 6 \cdot \frac{1}{6} = \frac{1}{6}(1+2+\dots+6) = \frac{7}{2}$$

$$\mathrm{var}[X] = E[X^2] - (E[X])^2 = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

$$E[X^2] = \sum_x x^2 P_X(x) = 1^2 \cdot \frac{1}{6} + 2^2 \cdot \frac{1}{6} + \dots + 6^2 \cdot \frac{1}{6} = \frac{1}{6}(1^2 + 2^2 + \dots + 6^2) = \frac{91}{6}$$

How about in general: X is uniform over $[a, b]$? (p. 89)

$$E[X] = \frac{a+b}{2}, \quad \mathrm{var}(X) = (b-a)(b-a+2)/12$$

Mean and variance of Poisson RV

$$P_X(k) = e^{-\lambda}\frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

$$E[X] = \sum_{k=0}^{\infty} k P_X(k) = \sum_{k=1}^{\infty} \cancel{k} e^{-\lambda}\frac{\lambda^k}{\cancel{k!}_{(k-1)!}} = \sum_{k=1}^{\infty} e^{-\lambda}\frac{\lambda^{k-1}\lambda}{(k-1)!}$$

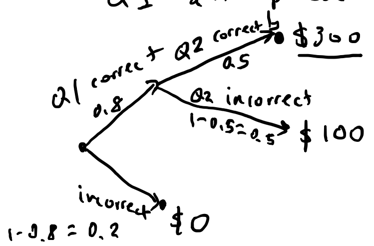
$$= \lambda \underbrace{\sum_{k=1}^{\infty} e^{-\lambda}\frac{\lambda^{k-1}}{(k-1)!}} = \lambda \underbrace{\sum_{k'=0}^{\infty} e^{-\lambda}\frac{\lambda^{k'}}{k'!}}_{1} = \lambda$$

$\mathrm{var}(X) = \lambda$ [Example 2.20]

Decision making using expected values

Consider a quiz game where a person is given two questions and must decide which one to answer first. Question 1 will be answered correctly with probability 0.8, and the person will then receive as prize \$100, while question 2 will be answered correctly with probability 0.5, and the person will then receive as prize \$200. If the first question attempted is answered incorrectly, the quiz terminates, i.e., the person is not allowed to attempt the second question. If the first question is answered correctly, the person is allowed to attempt the second question. Which question should be answered first to maximize the expected value of the total prize money received?

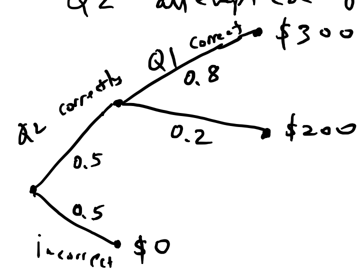
Q1 attempted first



$E[X] = \$300 \times 0.8 \times 0.5 + \$100 \times 0.8 \times 0.5 + \$0 \times 0.2$

$= \$120 + \$40 = \$160$

Q2 attempted first



$E[X] = \$300 \times 0.5 \times 0.8 + \$200 \times 0.5 \times 0.2$

$= \$140$