



## Output of an LTI system: Convolution



## LTI Output

- Starting over again (sorry)
- Assume we know the unit pulse output

$$h^{(p)}(t) = H(p(t, T))$$

the text uses  $\zeta(t)$  for what I have called  $p(t, T)$

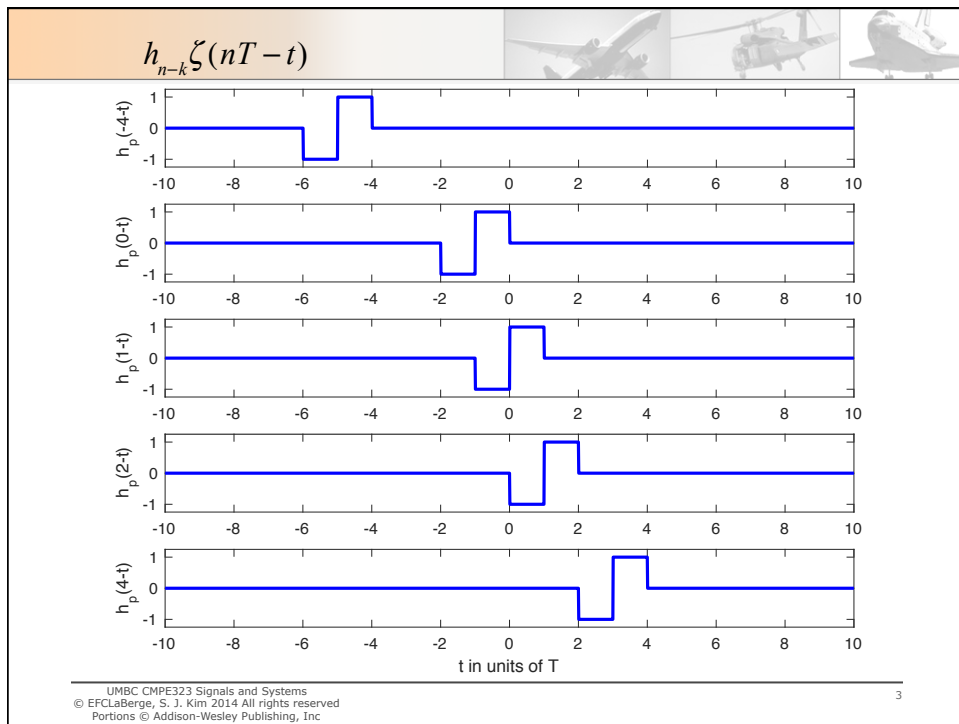
the text uses  $\psi(t)$  for what I have called  $h^{(p)}(t)$

- ...and we can decompose the input

$$x(t) = \sum_{n=-\infty}^{\infty} x_n p(t - nT, T)$$

- Then the output of an LTI system is just

$$y(t) = \sum_{n=-\infty}^{\infty} x_n h(t - nT)$$



**Example (from last time)**

$$h^{(p)}(t) = \zeta(t) = p(t,1) - p(t-1,1)$$

$$x(t) = \sum_{n=0}^3 x_n p(t-nT,1)$$

$$= p(t,1) + 3p(t-1,1) - 2p(t-2,1) - 1p(t-3,1)$$

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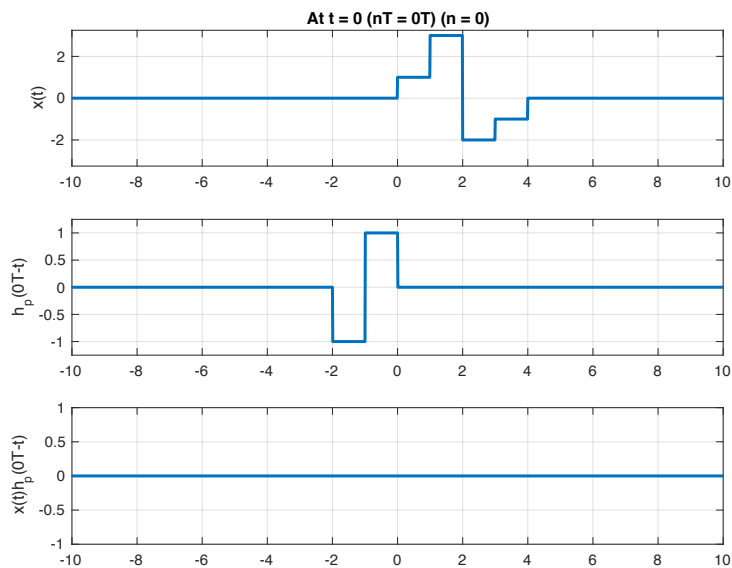
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$$\begin{aligned}
 y(t) &= \sum_{n=0}^3 x_n h^{(p)}(t-n) \\
 &= 1(\zeta(t) - \zeta(t-1)) + 3(\zeta(t-1) - \zeta(t-1-1)) \\
 &\quad - 2(\zeta(t-2) - \zeta(t-2-1)) - (\zeta(t-3) - \zeta(t-3-1)) \\
 &= 1(\zeta(t) - \zeta(t-1)) + 3(\zeta(t-1) - \zeta(t-2)) \\
 &\quad - 2(\zeta(t-2) - \zeta(t-3)) - (\zeta(t-3) - \zeta(t-4)) \\
 &= 1\zeta(t) + (-1+3)\zeta(t-1) + (-3-2)\zeta(t-2) + (+2-1)\zeta(t-3) + (1)\zeta(t-4) \\
 &= (1 \times 1)\zeta(t) + (1 \times (-1) + 3 \times 1)\zeta(t-1) + (3 \times (-1) + -2 \times 1)\zeta(t-2) \\
 &\quad + (-2 \times (-1) + -1 \times 1)\zeta(t-3) + (-1 \times -1)\zeta(t-4) \\
 &= (x_0 h_0)\zeta(t-0) + (x_0 h_1 + x_1 h_0)\zeta(t-1) + (x_1 h_1 + x_2 h_0)\zeta(t-2) \\
 &\quad + (x_2 h_1 + x_3 h_0)\zeta(t-3) + (x_3 h_1)\zeta(t-4) = \sum_{k=0}^4 x_k h_{n-k} \zeta(t-n)
 \end{aligned}$$

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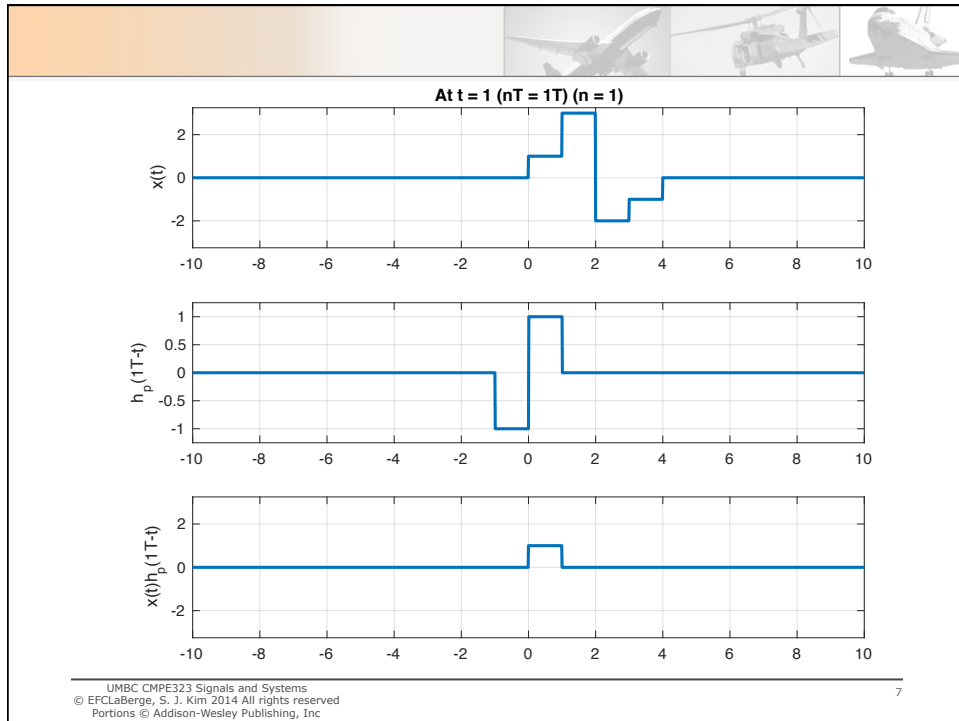
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## And the convolution term

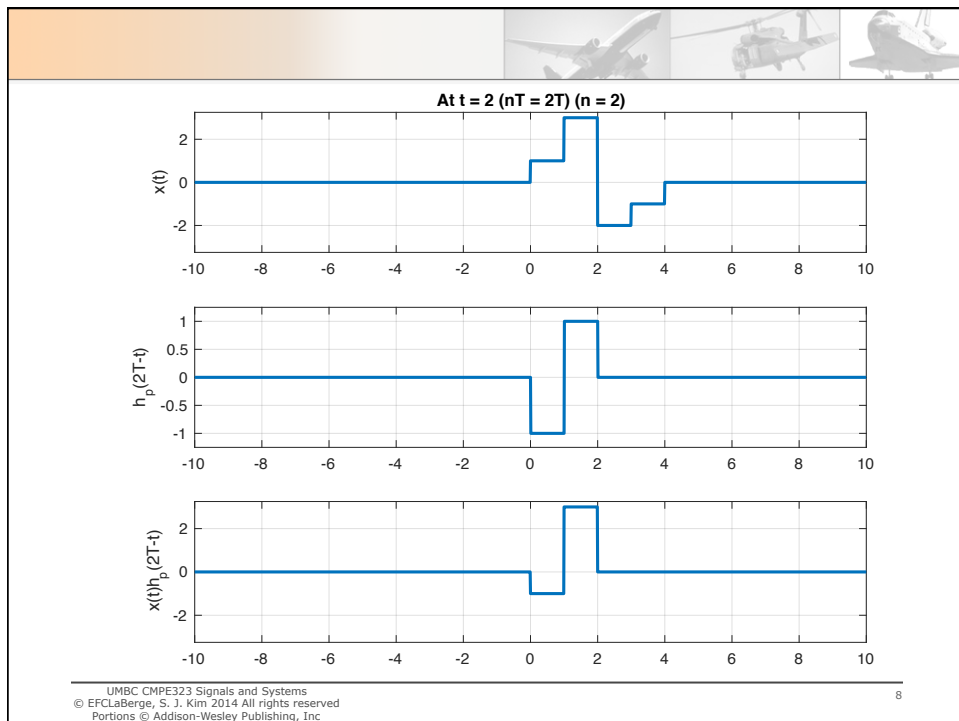


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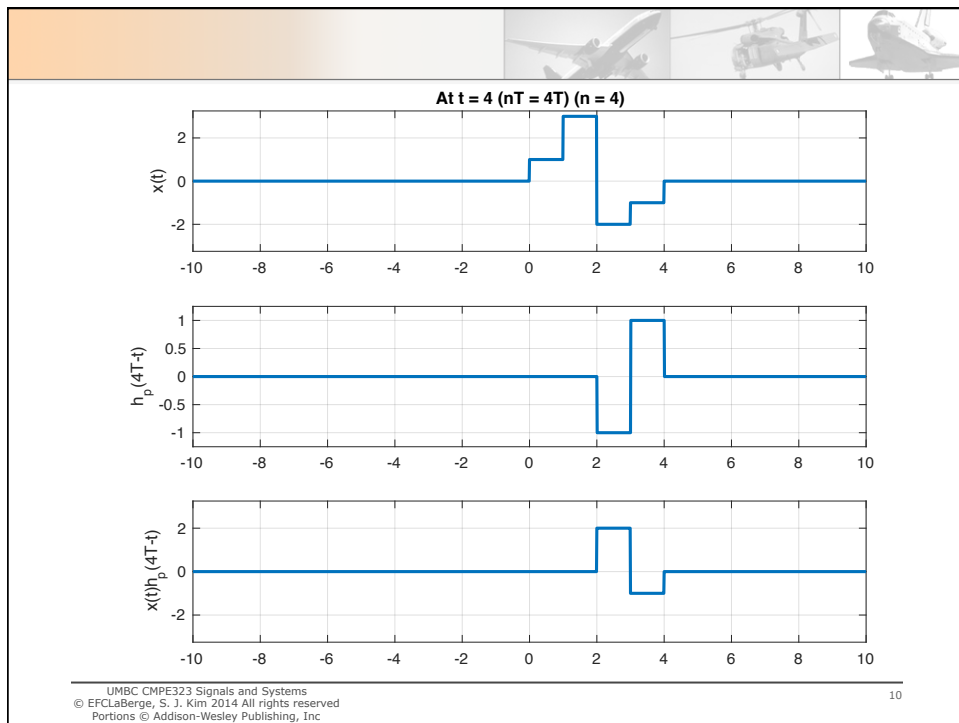
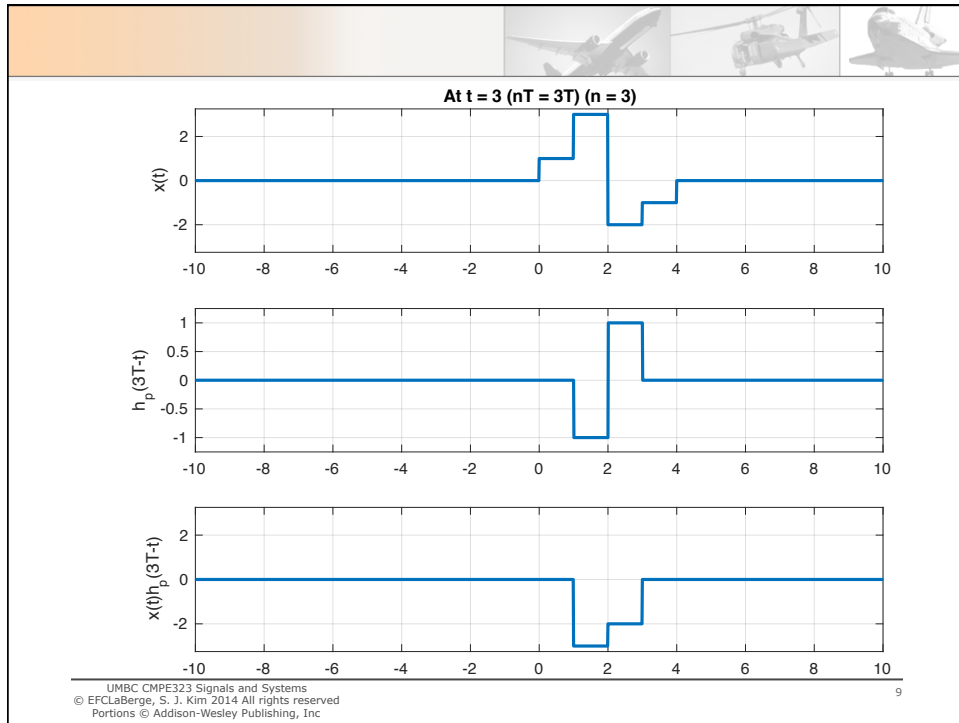
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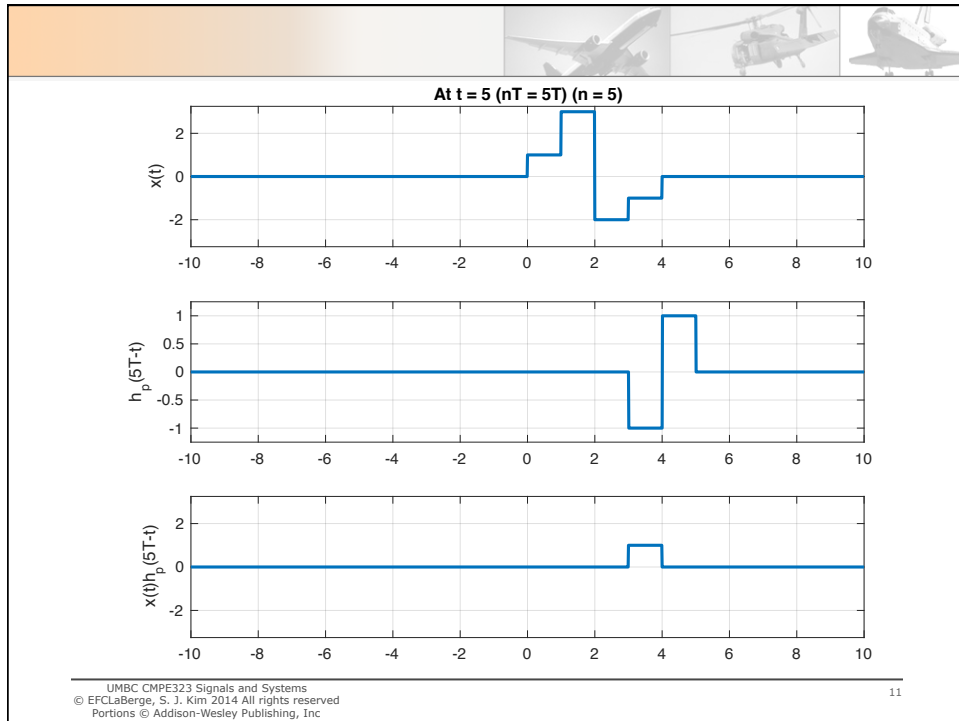


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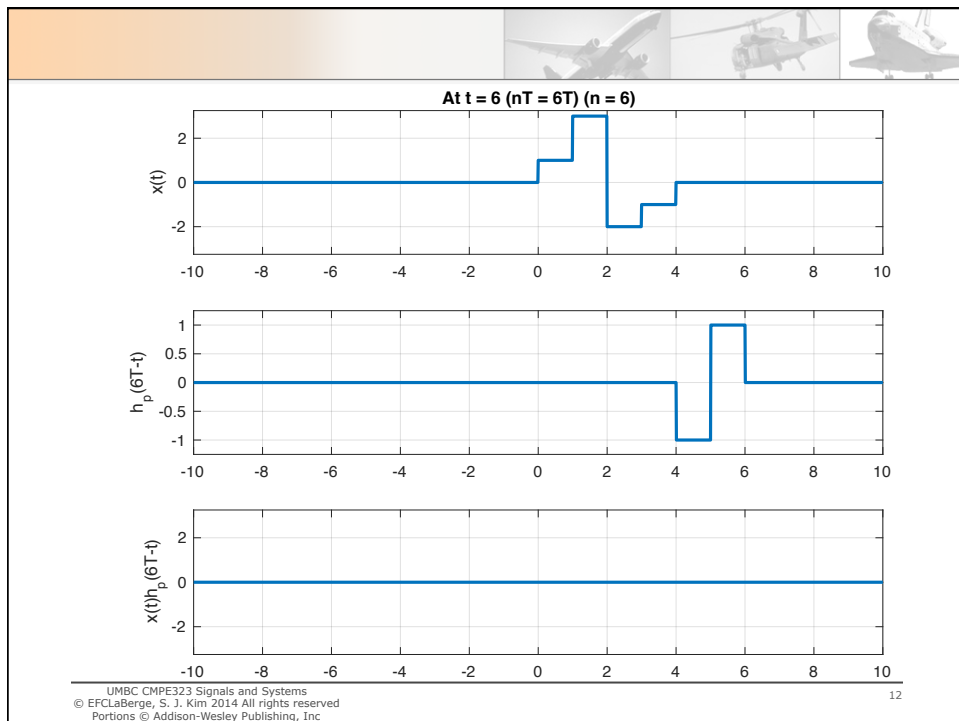


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## Lets make the input pulses shorter

- ...while keeping the energy the same...
- Define  $\xi(t) = \frac{1}{T} p(t, T)$  and let  $H(\xi(t)) = \eta(t)$
- Let the input be

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} T x(nT) \xi(t - nT)$$

- ... a series of delayed pulses of amplitude  $T x(nT)$
- What's the output?
- The system is LTI, so  $\hat{y}(t) = \sum_{n=-\infty}^{\infty} T x(nT) \eta(t - nT)$
- ...no problem.
- What's the limit as  $T \rightarrow 0$  ?

## As T approaches 0

Let  $nT \rightarrow \tau$  a continuous variable

$$T \rightarrow d\tau$$

$$\xi(t) \rightarrow \delta(t)$$

$$\eta(t) \rightarrow h(t)$$

$$\hat{x}(t) \rightarrow x(t)$$

$$\sum_{n=-\infty}^{\infty} \rightarrow \int_{-\infty}^{\infty}$$

$$\hat{y}(t) \rightarrow \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau$$

- Which we call the convolution integral!

## Examples

$$h(t) = 2e^{-2t}u(t)$$

$$x(t) = p(t,1)$$

Find  $y(t)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} p(\tau,1)2e^{-2(t-\tau)}u(t-\tau)d\tau$$

For  $0 < t \leq 1$

$$\int_0^t 2e^{-2(t-\tau)} d\tau = \frac{2e^{-2t}e^{2\tau}}{2} \Big|_0^t = e^{-2t}(e^{2t} - 1) = 1 - e^{-2t}$$

For  $t > 1$

$$\int_0^1 2e^{-2(t-\tau)} d\tau = \frac{2e^{-2t}e^{2\tau}}{2} \Big|_0^1 = e^{-2t}(e^2 - 1)$$