CMPE 320: Probability, Statistics, and Random Processes

Lecture 23: Review

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Problem 5. Consider a triangle and a point chosen within the triangle according to the uniform probability law. Let X be the distance from the point to the base of the triangle. Given the height of the triangle, find the CDF and the PDF of X.

CDF of X $\quad F_X(x) = \underline{P(X \le x)} = \frac{\text{Area}(\ )}{\text{Area}(\ )}$

$=$

$b : a = h : x$

$f_Y(y) = c \cdot y \qquad \int f_X(x)\,dx = \int_0^h c\,x\,dx = c \cdot \frac{1}{2}x^2\Big|_0^h = 1 = \frac{ch^2}{2} \quad c = \frac{2}{h^2}$

$f_Y(y) = \frac{2}{h^2}y \quad 0 \le y \le h$

$Y = aX + b$

$X = h - Y = -Y + h$

$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$

PDF of X? $\quad f_X(x) = \frac{1}{|-1|} f_Y(h-x) = \frac{2}{h^2}(h-x) \quad 0 \le x \le h$

Problem 18. Let X be a random variable with PDF

$$f_X(x) = \begin{cases} x/4, & \text{if } 1 < x \leq 3, \\ 0, & \text{otherwise,} \end{cases}$$

and let A be the event $\{X \geq 2\}$.

(a) Find $\mathbf{E}[X]$, $\mathbf{P}(A)$, $f_{X|A}(x)$, and $\mathbf{E}[X \mid A]$.

(b) Let $Y = X^2$. Find $\mathbf{E}[Y]$ and $\text{var}(Y)$.

$$(a)\ E[X] = \int x\, f_x(x)\,dx = \int_1^3 x \cdot \frac{x}{4}\,dx = \left.\frac{x^3}{12}\right|_1^3 = \frac{3^3 - 1^3}{12} = \frac{26}{12} = \frac{13}{6}$$

$$P(A) = \int_A f_x(x)\,dx = \int_2^\infty f_x(x)\,dx = \int_2^3 \frac{x}{4}\,dx = \left.\frac{x^2}{8}\right|_2^3 = \frac{3^2 - 2^2}{8} = \frac{5}{8}$$

$$f_{x|A}(x) = \begin{cases} \dfrac{f_x(x)}{P(A)} & \text{if } x \in A \\ 0 & \text{o.w} \end{cases} = \begin{cases} \dfrac{x/4}{5/8} & \text{if } 2 \leq x \leq 3 \\ 0 & \text{, otherwise} \end{cases}$$

$$E[X|A] = \int_2^3 x \cdot f_{x|A}(x)\,dx = \int_2^3 x \cdot \frac{x}{4} \cdot \frac{8}{5}\,dx = \ldots$$

$$(b)\ E[Y] = E[X^2] = \int x^2 \frac{x}{4}\,dx$$

Homework #9

Problem 1. Let X be a random variable with PDF f_X. Find the PDF of the random variable $|X|$ in the following three cases.

(a) X is exponentially distributed with parameter λ.

(b) X is uniformly distributed in the interval $[-1, 2]$.

(c) f_X is a general PDF.

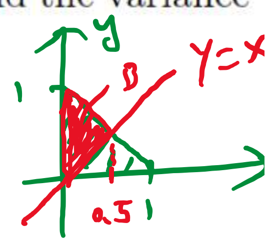
Problem 2. Let X and Y be independent random variables, uniformly distributed in the interval $[0, 1]$. Find the CDF and the PDF of $|X - Y|$.

Problem 3. Your driving time to work is between 30 and 45 minutes if the day is sunny, and between 40 and 60 minutes if the day is rainy, with all times being equally likely in each case. Assume that a day is sunny with probability 2/3 and rainy with probability 1/3.

(a) Find the PDF, the mean, and the variance of your driving time.

(b) On a given day your driving time was 45 minutes. What is the probability that this particular day was rainy?

(c) Your distance to work is 20 miles. What is the PDF, the mean, and the variance of your average speed (driving distance over driving time)?



Problem 4. The random variables X and Y have the joint PDF

$$f_{X,Y}(x,y) = \begin{cases} 2, & \text{if } x > 0 \text{ and } y > 0 \text{ and } x + y \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

$P(B) = \int_B f_{X,Y}(x,y$

Let A be the event $\{Y \le 0.5\}$ and let B be the event $\{Y > X\}$.

$= \frac{1}{2}$

(a) Calculate $\mathbf{P}(B \mid A)$.

(b) Calculate $f_{X|Y}(x \mid 0.5)$. Calculate also the conditional expectation and the conditional variance of X, given that $Y = 0.5$.

$f_{X,Y|B}(x,y) = \begin{cases} 2 \cdot 2 = 4 & \text{if } x \\ 0 & \text{, o.w} \end{cases}$

(c) Calculate $f_{X|B}(x)$ $= \begin{cases} \frac{f_X(x)}{P(B)} & \text{if } x \in B \\ 0 & \text{other} \end{cases}$

(d) Calculate $\mathbf{E}[XY]$.

(e) Calculate the PDF of Y/X.

$f_{X|B}(x) = \int f_{X,Y|B}(x,y)\,dy =$

Problem 5. The random variables $X_1, \ldots, X_n$ have common mean μ, common variance σ^2 and, furthermore, $\mathbf{E}[X_iX_j] = c$ for every pair of distinct i and j. Derive a formula for the variance of $X_1 + \cdots + X_n$, in terms of μ, σ^2, c, and n.

$$E[g(X,Y)] = \iint g(x,y)\, f_{X,Y}(x,y)\, dx\, dy$$

Problem 6. Consider n independent tosses of a die. Each toss has probability p_i of resulting in i. Let X_i be the number of tosses that result in i. Show that X_1 and X_2 are negatively correlated (i.e., a large number of ones suggests a smaller number of twos).

$$E[(X_1 - E(X_1))(X_2 - E(X_2))] \qquad E[X_1] = n \cdot \tfrac{1}{6} = E[X_2]$$

$$= \boxed{E[X_1X_2]} - E[X_1]E[X_2] \qquad E[E[X_1X_2 \mid X_1]] = E[X_1 \underbrace{E[X_2 \mid X_1]}]$$

$$= \frac{5(n^2-n)}{6^3} - \frac{n^2}{36} = \frac{5n^2-5n-6n^2}{6^3} \qquad = E[X_1(n-X_1)\tfrac{1}{6}]$$

Problem 7. Let $X = Y - Z$ where Y and Z are nonnegative random variables such that $YZ = 0$.

$$= \frac{-n^2-5n}{6^3} < 0 \qquad = \frac{1}{6}E[nX_1 - X_1^2]$$

(a) Show that $\operatorname{cov}(Y, Z) \leq 0$.

(b) Show that $\operatorname{var}(X) \geq \operatorname{var}(Y) + \operatorname{var}(Z)$.

$$= \frac{1}{6}\left(nE[X_1] - E[X_1^2]\right)$$

(c) Use the result of part (b) to show that

$$= \frac{1}{6}\left(n \cdot \frac{1}{6}n - \frac{n^2+5n}{36}\right)$$

$$E[X] = E[E[X \mid Y]]$$

$$\operatorname{var}(X) \geq \operatorname{var}\big(\max\{0, X\}\big) + \operatorname{var}\big(\max\{0, -X\}\big). \quad = \frac{1}{6} \cdot \frac{5n^2-5n}{36} = \boxed{\frac{5n^2-5n}{6^3}}$$

$$E(X_1^2) = \operatorname{var}(X_1) + (E(X_1))^2 = n \cdot \frac{1}{6} \cdot \frac{5}{6} + \left(\frac{1}{6}n\right)^2 = \frac{n^2+5n}{36}$$

Problem 8. Consider two random variables X and Y. Assume for simplicity that they both have zero mean.

(a) Show that X and $\mathbf{E}[X \mid Y]$ are positively correlated.

(b) Show that the correlation coefficient of Y and $\mathbf{E}[X \mid Y]$ has the same sign as the correlation coefficient of X and Y.

(a) $\operatorname{cov}(X, E[X \mid Y]) = E[(X - \underbrace{E[X]}_{0})(E[X \mid Y] - E[E[X \mid Y]])]$

$$E[E[X \mid Y]] = E[X] = 0$$

$$= E[X E[X \mid Y]] =$$

$$= E[E[X E[X \mid Y] \mid Y]]$$

$$= E[E[X \mid Y] E[X \mid Y]]$$

Show X and $E[X|Y]$ are positively corr'ed

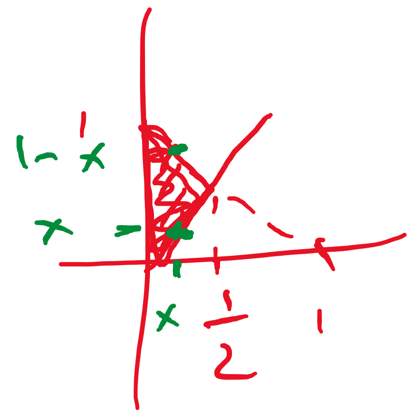
$\text{cov} = E[X\, E[X|Y]] > 0 \qquad E[X] = E[E[X|Y]]$

$= E\big[E[\cancel{X}\, \underbrace{E[X|Y]}_{g(Y)} \,|\, Y]\big]$

$= E\big[E[X|Y]\, E[X|Y]\big]$

$= E\big[E[X|Y]^2\big] \geq 0$

cor coeff.



$f_{X|B}(x) = \int f_{X,Y|B}(x,y)\, dy$

$$= \begin{cases} \int_x^{1-x} 4\, dy = 4(1-2x) & 0 \leq x \leq \frac{1}{2} \\ 0 & \text{o.w.} \end{cases}$$

$X = Y - Z$ $\quad$ $YZ = 0$

$0 = E[YZ] \neq E[Y]E[Z]$

$$\mathrm{var}(X) = \mathrm{var}(Y - Z)$$

$$= \mathrm{var}(Y) + \mathrm{var}(-Z) + 2\,\mathrm{cov}(Y, -Z)$$

$$= \mathrm{var}(Y) + \mathrm{var}(Z) + 2\left(\underbrace{E[Y(-Z)]}_{=0} - E[Y]E[-Z]\right)$$

$$= \mathrm{var}(Y) + \mathrm{var}(Z) + 2\underbrace{E[Y]}_{\geq 0}\underbrace{E[Z]}_{\geq 0}$$

$$\mathrm{var}(X) \geq \mathrm{var}(Y) + \mathrm{var}(Z)$$