Name: ____________________

1. (10 points) Complete the OPT(i, j) table for the sequence $\langle W, G, H, T, H, T, W, T, W, H, H, W, T, H \rangle$. Values of i are on the vertical axis, and j is on the horizontal axis. *Show all of your work!*

	6	7	8	9	10	11	12	13	14
1	1	1	1	1	2	2	2	*3*	*3*
2	0	0	0	1	2	2	*2*	2	2
3	0	0	0	1	1	1	1	*1*	1
4	0	0	0	1	1	1	1	1	1
5	0	0	0	0	0	*0*	1	1	1
6	0	0	0	0	0	0	1	1	1
7	0	0	0	0	0	0	0	1	1
8	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0

Solution

$OPT(5, 11)$: $a_{11} = H$ can not pair with a_5 or a_6, so $OPT(5, 11) = OPT(5, 10) = 0$.

$OPT(3, 13)$: $a_{13} = T$ could pair with $a_7 = W$. Therefore, $OPT(3, 13)$ is the maximum of $OPT(3, 12) = 1$ and $OPT(3, 6) + OPT(8, 12) + 1 = 1$, so $OPT(3, 13) = 1$.

$OPT(2, 12)$: $a_{12} = W$ could pair with a_4 or a_6. Therefore, $OPT(2, 12)$ is the maximum of $OPT(2, 11) = 2$, $OPT(2, 3) + OPT(5, 11) + 1 = 1$, and $OPT(2, 5) + OPT(7, 11) + 1 = 1$, so $OPT(2, 12) = 2$.

$OPT(1, 13)$: a_{13} could pair with a_1 or a_7. Therefore, $OPT(1, 13)$ is the maximum of $OPT(1, 12) = 2$, $OPT(1, 0) + OPT(2, 12) + 1 = 3$, and $OPT(1, 6) + OPT(8, 12) + 1 = 2$, so $OPT(1, 13) = 3$.

$OPT(1, 14)$: a_{14} could pair with a_2. Therefore, $OPT(1, 14)$ is the maximum of $OPT(1, 13) = 3$ and $OPT(1, 1) + OPT(3, 13) + 1 = 2$, so $OPT(1, 14) = 3$.

2. (10 points) A trucking company ships boxes between Boston and New York. Each truck can carry at most W pounds of cargo. Packages arrive at the Boston station one by one, and each package i has weight w_i. The trucking station is small, so at most one truck can be at the station at any time. The packages must be shipped in the order they arrived; otherwise, a customer might be upset to learn that a package that arrived after hers made it to New York faster. The company is using a simple greedy algorithm for packing: load packages in the order they arrive, and whenever the next package does not fit, send the truck on its way. The company wants to be sure it is using as few trucks as possible. Show that for a specific set of packages with given weights, the company's algorithm has the greedy choice property.

Solution

Let $x_1, x_2, \ldots, x_n$ denote the packages in order and let $T = \{T_1, T_2, \ldots, T_k\}$ be an optimal solution, meaning that T_1 contains the first k_1 packages, T_2 contains the next k_2 packages, etc., the total weight of the packages in T_i is less than or equal to W, and k is minimal. Let $T_1 = \langle x_1, x_2, \ldots x_{k_1} \rangle$ and $T_2 = \langle x_{k_1+1}, x_{k_1+2}, \ldots, x_{k_1+k_2} \rangle$. Suppose T_1 does not use the greedy choice; then some number of initial packages from T_2, say $\langle x_{k_1+1}, \ldots, x_{k_1+m} \rangle$, could be loaded into T_1 without exceeding the capacity of T_1. Let

$$T_1' = \langle x_1, x_2, \ldots, x_{k_1}, x_{k_1+1}, \ldots, x_{k_1+m} \rangle.$$

and let T_2' be the packages left in T_2 after shifting the first m to T_1,

$$T_2' = \langle x_{k_1+m+1}, x_{k_1+m+2}, \ldots, x_{k_1+k_2} \rangle.$$

Then $T' = \{T_1', T_2', T_3, \ldots, T_k\}$ is an optimal solution that uses the greedy choice: all the packages are in some truck since we have only shifted m packages from T_2 to T_1'; each truck is within its capacity, since we have increased the load in T_1' to the maximum it can carry, decreased the load in T_2', and left all others unchanged; and the number of trucks is still k, which is assumed to be minimal. Therefore, the company's algorithm has the greedy choice property.

Note: It is sufficient to prove the greedy choice for the first truck. Once we have done that, we could apply the same argument to $\{T_2', T_3, \ldots, T_k\}$ and so on until we have shown that making the greedy choice for each truck still produces an optimal solution.