Maxwell's Equations

The complete equations in differential form:

$$\nabla \cdot \mathbf{D}(\mathbf{R},t) = \rho_V(\mathbf{R},t), \qquad \nabla \times \mathbf{E}(\mathbf{R},t) = -\frac{\partial \mathbf{B}(\mathbf{R},t)}{\partial t},$$

$$\nabla \cdot \mathbf{B}(\mathbf{R},t) = 0, \qquad \nabla \times \mathbf{H}(\mathbf{R},t) = \mathbf{J}(\mathbf{R},t) + \frac{\partial \mathbf{D}(\mathbf{R},t)}{\partial t}$$

with the constitutive relations:

$$\mathbf{D}(\mathbf{R},t) = \varepsilon(\mathbf{R})\mathbf{E}(\mathbf{R},t), \qquad \mathbf{B}(\mathbf{R},t) = \mu(\mathbf{R})\mathbf{H}(\mathbf{R},t)$$

Other important forms that we will see:

(1) Integral form (using Stokes and Gauss's theorems)

(2) Phasor/Frequency domain form

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8.1

Note that we are assuming that epsilon and mu are time-independent. When they are not, the generalization in the time domain involves convolutions. They multiply in the frequency domain, not the time domain!

The form that we have here is the CLASSIC form that is most often seen. Arguably, the integral form is more important. It was the original form of the equations, since it is this form that can be measured experimentally, and it is also the form that is the basis for most numerical approaches. This form is most useful when finding analytical solutions or obtaining analytical approximations before numerical calculations.

Maxwell's Equations

Time-independent (Static) Forms:

$$\nabla \cdot \mathbf{D}(\mathbf{R},t) = \rho_V(\mathbf{R},t), \qquad \nabla \times \mathbf{E}(\mathbf{R},t) = 0,$$

$$\nabla \cdot \mathbf{B}(\mathbf{R},t) = 0, \qquad \nabla \times \mathbf{H}(\mathbf{R},t) = \mathbf{J}(\mathbf{R},t)$$

The electric and magnetic fields decouple; they can be treated independently!

This observation is the starting point for electrostatics and magnetostatics

When can we neglect the time variations?

In the same limit that circuit theory holds

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8.2

Maxwell's Equations

Time-independent (Static) Forms:

$$\nabla \cdot \mathbf{D}(\mathbf{R},t) = \rho_V(\mathbf{R},t), \qquad \nabla \times \mathbf{E}(\mathbf{R},t) = 0,$$

$$\nabla \cdot \mathbf{B}(\mathbf{R},t) = 0, \qquad \nabla \times \mathbf{H}(\mathbf{R},t) = \mathbf{J}(\mathbf{R},t)$$

The electric and magnetic fields decouple; they can be treated independently!

This observation is the starting point for electrostatics and magnetostatics

So, why bother with statics?

(1) **Important applications:** near fields of radiating systems; inductors and capacitors; electrostatic discharge

(2) **Visualizing fields:** The full system is complex; it contains radiative contributions; charge contributions; current contributions. It is important to learn about each of them separately.

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8.3

NOTE: When we get to radiation; we will be treating it separately from current and charge sources. We will not get to the point where we put all three pieces together.

Constitutive Relations

$$\mathbf{D}(\mathbf{R},t) = \varepsilon(\mathbf{R})\mathbf{E}(\mathbf{R},t), \qquad \mathbf{B}(\mathbf{R},t) = \mu(\mathbf{R})\mathbf{H}(\mathbf{R},t)$$

Where do they come from?

There are two kinds of charge:

free (these flow in conductors) [This charge is included in ρ_V]

bound (these are dipole charges in dielectrics)

[This charge is what determines ε!]

In statics, the bound charges always tend to cancel the free charges.

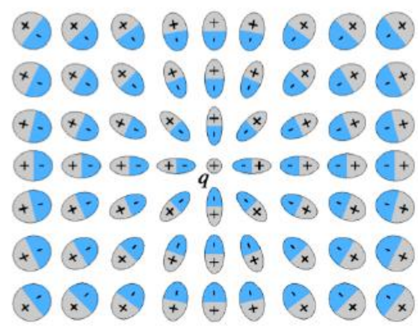
Thus, we have:

$$\varepsilon > \varepsilon_0, \quad |\mathbf{D}| > |\mathbf{E}|$$

NOTE:

D = field response to free charge

E = field response to total charge



Ulaby Figure 1-6

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8.4

Note that in dynamics, epsilon -> epsilon(omega) and can have any magnitude. In particular, there can be resonant frequencies of the molecule where it becomes very large.

In plasmas (ionized gases), the charges are often included in epsilon even though they are not bound for computational convenience. The distinction between free and bound charges becomes fuzzy in semiconductors and the separation is made in a way that is computationally or analytically convenient.

Constitutive Relations

$$\mathbf{D}(\mathbf{R},t) = \varepsilon(\mathbf{R})\mathbf{E}(\mathbf{R},t), \qquad \mathbf{B}(\mathbf{R},t) = \mu(\mathbf{R})\mathbf{H}(\mathbf{R},t)$$

Where do they come from?

There are two kinds of charge:

free (these flow in conductors) [This charge is included in ρ_V]

bound (these are dipole charges in dielectrics)

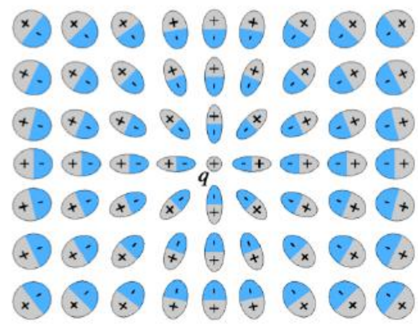
[This charge is what determines ε!]

- Approximate calculations of ε are possible semi-classically

BUT

- Exact calculations usually require quantum mechanics

An important exception is plasmas, which can be treated classically



Ulaby Figure 1-6

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8.5

Calculations of epsilon can be very difficult and are definitely part of physics. Engineers typically use known values of epsilon that have come from measurements and/or physics calculations.

Constitutive Relations

$$\mathbf{D}(\mathbf{R},t) = \varepsilon(\mathbf{R})\mathbf{E}(\mathbf{R},t), \qquad \mathbf{B}(\mathbf{R},t) = \mu(\mathbf{R})\mathbf{H}(\mathbf{R},t)$$

Where do they come from?

There are also two kinds of current:

free (these flow in conductors) [This current is included in **J**]

bound (in magnetic materials)

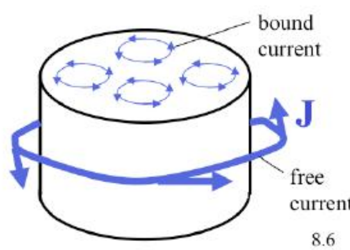
Important note: Static current always flows in loops!

That is a consequence of $\nabla \cdot \mathbf{B} = 0$

NOTE:

H = field response to free current

B = field response to total current



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8.6

If we include “displacement current,” the current due to a changing electric field --- then current always flows in loops, period!

Physicists have searched for years for magnetic monopoles --- the analog of an electric charge, but they have never found it, although there have been some false alarms…

Constitutive Relations

$$\mathbf{D}(\mathbf{R},t) = \varepsilon(\mathbf{R})\mathbf{E}(\mathbf{R},t), \qquad \mathbf{B}(\mathbf{R},t) = \mu(\mathbf{R})\mathbf{H}(\mathbf{R},t)$$

Where do they come from?

There are also two kinds of current:

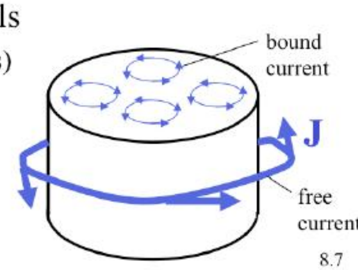
free (these flow in conductors) [This current is included in **J**]

bound (in magnetic materials)

BUT: the behavior of magnetic materials is complicated even in the static limit!

There are three kinds of magnetic materials

- Ferromagnetic (permanent magnets)
- Paramagnetic
- Diamagnetic



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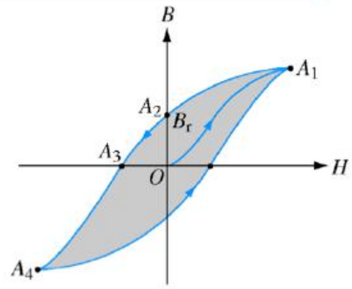
8.7

Constitutive Relations

Ferromagnetism: $\mathbf{B} \neq \mu \mathbf{H}$

These materials are highly nonlinear and have hysteresis

Quantum mechanics must be used to explain this phenomenon; no semi-classical explanation is possible



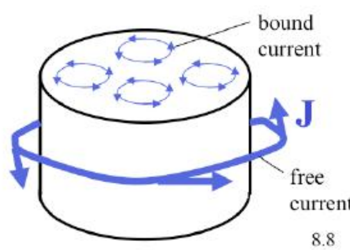
Ulaby Figure 5-22

Paramagnetism: $\mathbf{B} \approx \mu \mathbf{H}$

A small amount of hysteresis may be present

The bound flow is in the same direction as the free flow and enhances it ($\mu > \mu_0$)

Semi-classical explanation is not possible



8.8

Permanent magnets are usually treated as a constant source of B, with a known spatial profile that we will discuss. Physically, the magnetic moments of the electrons all line up.

Paramagnetism is due to the intrinsic magnetic moment of the electron or “spin”. Of course, the electron is really a point charge and is not really spinning. The magnetic moment is just there.

Constitutive Relations

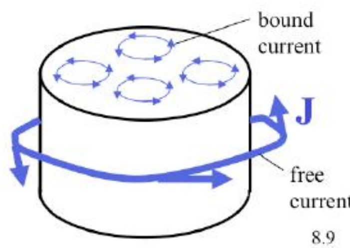
Diamagnetism: $\mathbf{B} = \mu \mathbf{H}$

No hysteresis is present

The bound flow is in the opposite direction from the free flow and decreases it ($\mu_r < 1$)

Semi-classical explanation is possible

Fortunately, in almost all dielectric materials: $\mathbf{B} = \mu_0 \mathbf{H}$



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8.9

Diamagnetism is due to the orbital motion of the electrons in their molecules, which creates a current.

Charge and Current Distributions

Volume charge density

$$\rho_V = \lim_{\Delta v \to 0} \frac{\Delta q}{\Delta v} = \frac{dq}{dv}, \quad \text{where } \Delta q \text{ is the charge is a small volume } \Delta v$$

Conversely, we have in a finite volume v:

$$Q = \int_v \rho_V \, dv$$

It is useful to define analogous surface and line charge densities

Surface and line charge densities

$$\rho_S = \lim_{\Delta s \to 0} \frac{\Delta q}{\Delta s} = \frac{dq}{ds}, \quad \rho_l = \lim_{\Delta l \to 0} \frac{\Delta q}{\Delta l} = \frac{dq}{dl}$$

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The converses are:

$$Q = \int_s \rho_S \, ds, \quad Q = \int_l \rho_l \, dl$$

8.10

Charge and Current Distributions

Surface Charge Distribution: Ulaby et al. Example 4-2

Question: A circular disk of electric charge is azimuthally symmetric and increases linearly with r from 0 to 6 $\mathrm{C/m^2}$ at $r = 3$ cm. What is the total charge on the surface?

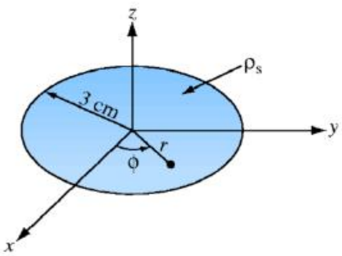
Answer: We have

$$\rho_S = \frac{6r}{3\times10^{-2}} = 2\times10^2 r$$

so that

$$Q = \int_s \rho_S \, ds = \int_{\phi=0}^{2\pi}\int_{r=0}^{3\times10^{-2}} \left(2\times10^2 r\right) r\,dr\,d\phi$$

$$= 2\pi\times2\times10^2 \left.\frac{r^3}{3}\right|_0^{3\times10^{-2}} = 11.3 \text{ mC}$$



Surface charge distribution

Ulaby Figure 4-1(b)

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Charge and Current Distributions

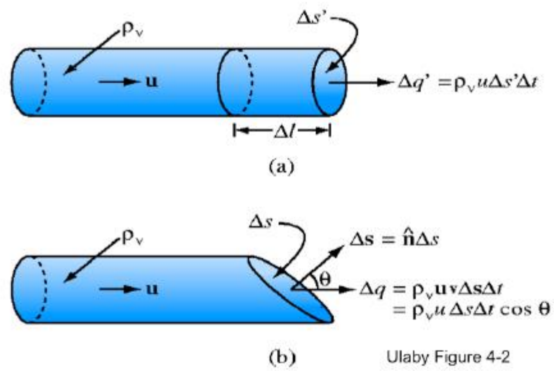
Current density

The increment of current that flows through a surface $\Delta \mathbf{s}$ in a time Δt is given by:

$$\Delta I = \frac{\Delta q}{\Delta t} = \rho_V \mathbf{u} \cdot \Delta \mathbf{s} = \mathbf{J} \cdot \Delta \mathbf{s}, \quad \text{where} \quad \mathbf{J} = \rho_V \mathbf{u}$$

The converse is:

$$I = \int_s \mathbf{J} \cdot d\mathbf{s}$$



Ulaby Figure 4-2

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8.12

Charge and Current Distributions

Coulomb's Law:

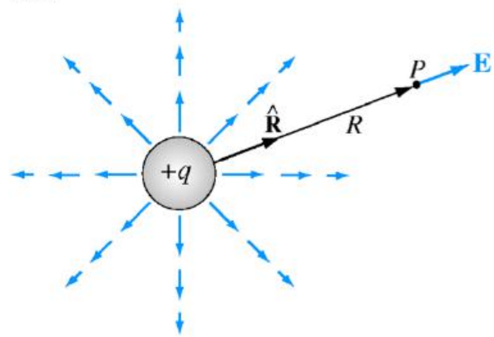
1) An isolated charge q induces an electric field $\mathbf{E}$ at every point in space and at the point P, the field $\mathbf{E}$ is given by

$$\mathbf{E} = \hat{\mathbf{R}} \frac{q}{4\pi\varepsilon R^2}, \quad \text{where}$$

$\hat{\mathbf{R}}$ = unit vector pointing from q to P.

R = distance between q and P.

ε = dielectric constant



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Ulaby Figure 4-2

Charge and Current Distributions

Coulomb's Law:

2) An electric field **E** at a point *P* in space, which may be due to one charge or many charges, induces a force on a charge q' that is given by

$$\mathbf{F} = q'\mathbf{E}$$

NOTE: The only way to detect the presence of a field (electric or magnetic) is by the force that it exerts on charges.

— In this sense, the fields are an abstraction, albeit a very useful one

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8.14

This point is of great importance in physics because in quantum mechanics these fields become probability density functions for photons that can "collapse" instantaneously when a photon is detected. Ascribing an independent reality to the waves then becomes uncomfortable.

On the other hand, if one creates a theory without using fields, which is possible to do, relativity demands that the action is delayed. That is what makes the concept of a field so useful. Without it, you would say that when you make an electron oscillate, it gives up energy that may reappear later when another electron oscillates. Energy is not conserved. Using the field picture, you say that the electron radiates energy into the field that travels through space until it encounters another electron and then transfers its energy to that electron. So, energy is conserved. That is much nicer, of course, but all that we can observe is that one electron loses its energy and --- after some time --- another electron gains energy.

One can avoid both issues --- and be more correct quantum mechanically --- by saying that the energy is given up to photons, not to the field. There is no collapse of photons, they just give up their energy (and very existence) to something else and disappear. Of course, photons can only be observed --- like fields --- by the effect that they have on charged particles.

At the same time, photons are not bullets. They go where the fields tell them to go "on average" and one must still calculate the fields.

Charge and Current Distributions

Multiple Point Charges:

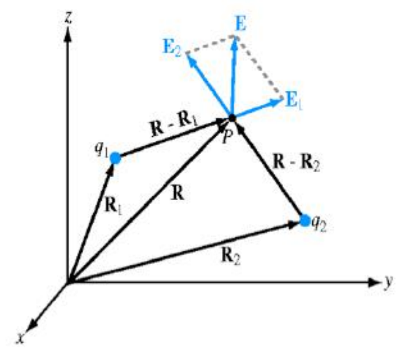
When we have multiple point charges, we add the field contributions from each of them vectorially

$$\mathbf{E} = \frac{1}{4\pi\varepsilon}\sum_{i=1}^{N}\frac{q_i\left(\mathbf{R}-\mathbf{R}_i\right)}{\left|\mathbf{R}-\mathbf{R}_i\right|^3}, \quad \text{where}$$

N = the total number of charges

q_i = amount of the i-th charge

$\mathbf{R}_i$ = position vector of the i-th charge



IMPORTANT NOTE: A charge does not contribute to the electric field at its own location!

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Ulaby Figure 4-4 8.15

If a charge did contribute to the field at its own location, that would lead to infinite results. This is the so-called self-infinity problem. For a finite number of point charges, it is not a problem. But suppose an electron has several parts (or covers some volume continuously), do the differt parts interact and exert forces on each other?

Charge and Current Distributions

Electric Field Due to Two Point Charges: Ulaby et al. Example 4-3

Question: Two point charges with $q_1 = 2 \times 10^{-5}$ C and $q_2 = -4 \times 10^{-5}$ C are located at (1, 3, –1) and (–3, 1, –2), respectively, in a Cartesian coordinate system. Find (a) the electric field **E** at (3, 1, –2) and (b) the force on a charge $q_3 = 8 \times 10^{-5}$ C located at that point.

Answer: (a) Since $\varepsilon = \varepsilon_0$ and there are two charges, we have

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1(\mathbf{R}-\mathbf{R}_1)}{|\mathbf{R}-\mathbf{R}_1|^3} + \frac{q_2(\mathbf{R}-\mathbf{R}_2)}{|\mathbf{R}-\mathbf{R}_2|^3}\right]$$

with

$$\mathbf{R}_1 = \hat{\mathbf{x}} + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}$$
$$\mathbf{R}_2 = -\hat{\mathbf{x}}3 + \hat{\mathbf{y}} - \hat{\mathbf{z}}2$$
$$\mathbf{R} = \hat{\mathbf{x}}3 + \hat{\mathbf{y}} - \hat{\mathbf{z}}2$$

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8.16

Charge and Current Distributions

Electric Field Due to Two Point Charges: Ulaby et al. Example 4-3

Answer: (a) [continued] After substitution, we find

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\left[\frac{2\left(\hat{\mathbf{x}}2-\hat{\mathbf{y}}2-\hat{\mathbf{z}}\right)}{27} - \frac{4\left(\hat{\mathbf{x}}6\right)}{216}\right]\times 10^{-5} = \frac{\hat{\mathbf{x}}-\hat{\mathbf{y}}4-\hat{\mathbf{z}}2}{108\pi\varepsilon_0}\times 10^{-5}\ \text{V/m}$$

(b) Using the force equation, we have

$$\mathbf{F} = q_3\mathbf{E} = 8\times 10^{-5}\times\frac{\hat{\mathbf{x}}-\hat{\mathbf{y}}4-\hat{\mathbf{z}}2}{108\pi\varepsilon_0}\times 10^{-5} = \frac{\hat{\mathbf{x}}2-\hat{\mathbf{y}}8-\hat{\mathbf{z}}4}{27\pi\varepsilon_0}\times 10^{-10}\ \text{N}$$

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Ulaby 2007/2010 Demos/Modules

8.17

Do Ulaby 2007 Demos 4.1 – 4.5. Do Ulaby 2007 Modules 4.1 – 4.5, except voltage. Show how to set up charges in Ulaby et al. 2010 modules

Charge and Current Distributions

Continuous Charge Distributions:

When we have a continuous charge density, each increment of charge $dq = \rho_V \, dv'$ contributes

$$d\mathbf{E} = \hat{\mathbf{R}}' \frac{dq}{4\pi\varepsilon R'^2} = \hat{\mathbf{R}}' \frac{\rho_V \, dv'}{4\pi\varepsilon R'^2}, \quad \text{where}$$

$\mathbf{R}'$ = vector from differential volume dv' to point P

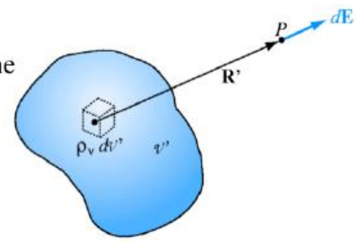
Integrating over a complete volume, we obtain:

$$\mathbf{E} = \int_{v'} d\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{v'} \hat{\mathbf{R}}' \frac{\rho_V \, dv'}{R'^2}$$

For surface and line distributions, these become

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{s'} \hat{\mathbf{R}}' \frac{\rho_S \, ds'}{R'^2} \quad \text{(surface)}$$

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{l'} \hat{\mathbf{R}}' \frac{\rho_l \, dl'}{R'^2} \quad \text{(line)}$$



Ulaby Figure 4-5

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8.18

Charge and Current Distributions

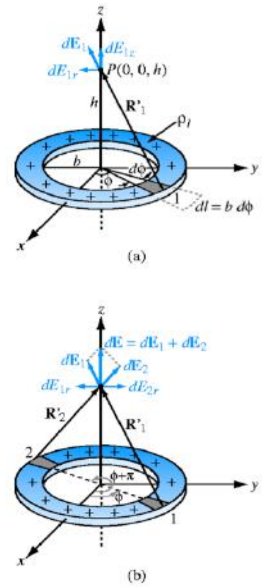
Example: An Infinite Charge Sheet

Question: An infinite plane of charge is located on the x-y plane. What is the electric field at the point $P(0, 0, h)$?

Answer: We will build up the answer in two parts. The first part is to integrate over a ring of charge (Ulaby et al., Example 4-4) and then integrate over many rings.

Integration over the ring:

$$\mathbf{R}_1' = -\hat{\mathbf{r}}b + \hat{\mathbf{z}}h = -\hat{\mathbf{x}}b\cos\phi - \hat{\mathbf{y}}b\sin\phi + \hat{\mathbf{z}}h$$



Ulaby Figure 4-6

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8.19

Our procedure is a little different from Ulaby et al.'s. So, we just use R1-prime.

Charge and Current Distributions

Example: An Infinite Charge Sheet

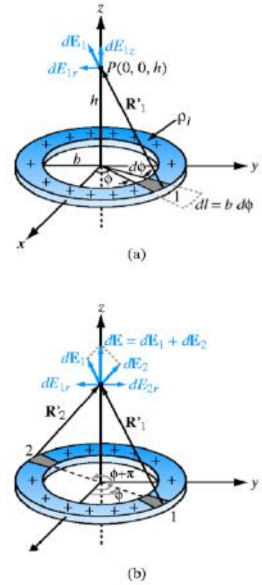
Answer (continued): *Integration over the ring:*

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int_{l'} \hat{\mathbf{R}}' \frac{\rho_l \, dl'}{R'^2}$$

$$= \frac{\rho_l b}{4\pi\varepsilon_0}\int_{\phi=0}^{2\pi} \frac{-\hat{\mathbf{x}} b\cos\phi - \hat{\mathbf{y}} b\sin\phi + \hat{\mathbf{z}} h}{\left(b^2 + h^2\right)^{3/2}} \, d\phi$$

$$= \hat{\mathbf{z}} \frac{\rho_l b h}{2\varepsilon_0 \left(b^2 + h^2\right)^{3/2}}$$

Due to the symmetry, only the z-component appears!



Ulaby Figure 4-6

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8.20

Symmetry considerations like the one here play a very important role in reducing the work of calculations. One should always look for them!

Charge and Current Distributions

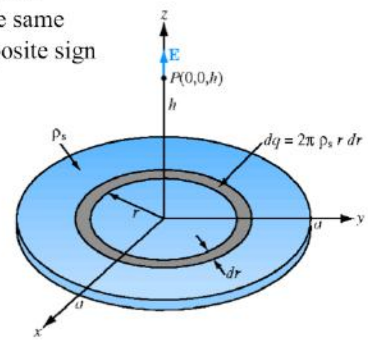
Example: An Infinite Charge Sheet

Answer (continued): *Integration over a circle:*
Making the replacements, $b \to r$ and $\rho_l \to \rho_S\, dr$
and integrating from $r = 0$ to $r = a$, we find

$$\mathbf{E} = \hat{\mathbf{z}}\frac{\rho_S h}{2\varepsilon_0}\int_{r=0}^{a}\frac{r}{\left(r^2+h^2\right)^{3/2}}dr$$

$$= \hat{\mathbf{z}}\frac{\rho_S}{2\varepsilon_0}\left[1-\frac{h}{\sqrt{a^2+h^2}}\right] \quad (h > 0)$$

When $h < 0$, the answer is the same with the opposite sign



Integration over the plane: When $a \to \infty$, we find

$$\mathbf{E} = \hat{\mathbf{z}}\frac{\rho_S}{2\varepsilon_0}$$

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NOTE: This simple, yet important result can also be obtained directly from Gauss's law

Ulaby Figure 4-7 8.21

This simple, yet important result can be obtained in many different ways, including directly from the integral form of Gauss's law. We will shortly discuss how to do this sort of calculation.