CMPE 320: Probability, Statistics, and Random Processes

Lecture 18: Conditioning

Spring 2018

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Conditioning with continuous RVs

- Similar to discrete RVs, one can condition a RV on an event or another RV
- Can define conditional PDFs, conditional expectations, and independence
- Mostly similar to the discrete counterpart; some subtleties do arise

When conditioning on $\{Y = y\}$ as this event has probability 0

Conditioning a RV on an event

- Conditional PDF of a RV X given an event A with P(A) > 0 is defined as the nonnegative function $f_{X|A}$ satisfying

$$P(X \in B \mid A) = \int_B f_{X|A}(x)\, dx$$

for any subset B of real line

- If B = ℝ (entire real line): $P(X \in \mathbb{R} \mid A) = 1 = \int_{-\infty}^{\infty} f_{X|A}(x)\, dx$
- If event A is in the form of $\{X \in A\}$:

$$P(X \in B \mid X \in A) = \frac{P(X \in B,\, X \in A)}{P(X \in A)} = \frac{\int_{A \cap B} f_X(x)\, dx}{P(X \in A)}$$

$$= \int_B \underbrace{\begin{cases} \dfrac{f_X(x)}{P(X \in A)} & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}}_{f_{X|\{X \in A\}}(x)} dx$$

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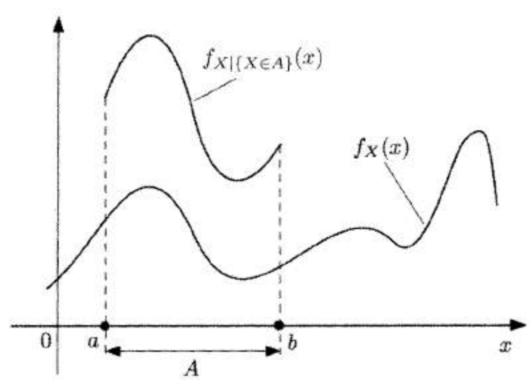


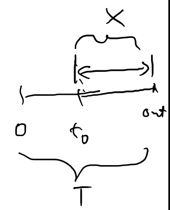
$$f_{X|\{X \in A\}}(x) = \begin{cases} \dfrac{f_X(x)}{P(X \in A)} & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

- The conditional PDF is 0 outside the conditioning set $\{X \in A\}$
- Inside the conditioning set it is just a scaled version of $f_X(x)$ (The scaling makes sure that it integrates to 1)

Exponential RV T

$$f_T(t)=\begin{cases}\lambda e^{-\lambda t}, & t\ge 0\\ 0, & \text{otherwise}\end{cases}$$

Example 3.13. The Exponential Random Variable is Memoryless. The time T until a new light bulb burns out is an exponential random variable with parameter λ. Ariadne turns the light on, leaves the room, and when she returns, t_0 time units later, finds that the light bulb is still on, which corresponds to the event $A = \{T > t_0\}$. Let X be the additional time until the light bulb burns out. What is the conditional CDF of X, given the event A?



First, what is the conditional PDF of T given A?

$$f_{T|A}(t)=\begin{cases}\dfrac{f_T(t)}{P(T>t_0)} & \text{if } t>t_0\\ 0 & \text{otherwise}\end{cases} = \begin{cases}\dfrac{\lambda e^{-\lambda t}}{e^{-\lambda t_0}} = \lambda e^{-\lambda(t-t_0)}, & t\ge t_0\\ 0, & \text{otherwise}\end{cases}$$

$$P(A)=P(T>t_0)=\int_{t_0}^{\infty} f_T(t)\,dt=\int_{t_0}^{\infty}\lambda e^{-\lambda t}\,dt = -e^{-\lambda t}\Big|_{t_0}^{\infty} = e^{-\lambda t_0}$$

Note $X = T - t_0$

$$P(X\le x_0 \mid T>t_0) = P(T\le x_0+t_0 \mid T>t_0) = \int_{-\infty}^{x_0+t_0} f_{T|A}(t)\,dt = \int_{t_0}^{x_0+t_0}\lambda e^{-\lambda(\overbrace{t-t_0}^{=x})}\,dt$$

$$= \int_0^{x_0}\lambda e^{-\lambda x}\,dx = \begin{cases}1-e^{-\lambda x_0}, & x_0\ge 0\\ 0, & \text{otherwise}\end{cases}$$

⇒ Same as not having the conditioning ⇒ Memoryless

Conditioning multiple RVs on an event

Let $f_{X,Y}(x,y)$ be the joint PDF of X and Y.

Event C of the form $C=\{(X,Y)\in A\}$

$$\boxed{f_{X,Y|C}(x,y)=\begin{cases}\dfrac{f_{X,Y}(x,y)}{P(C)} & \text{if } (x,y)\in C\\ 0 & \text{otherwise}\end{cases}}$$

$f_{X|C}(x) = ?$

$$f_{X|C}(x)=\int_{-\infty}^{\infty} f_{X,Y|C}(x,y)\,dy$$

: Useful when conditioning event is defined through another RV

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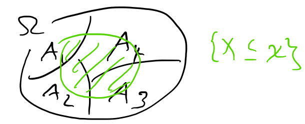
Total probability theorem for PDF



Let $A_1, A_2, \cdots, A_n$ form a partition of Ω

$$\underbrace{P(X \le x)}_{F_X(x)} = \sum_{i=1}^{n} P(X \le x \cap A_i) = \sum_{i=1}^{n} \underbrace{P(X \le x \mid A_i)}_{\int_{-\infty}^{x} f_{X|A_i}(t)\,dt} P(A_i)$$

Take derivatives on both sides

$$\boxed{f_X(x) = \sum_{i=1}^{n} f_{X|A_i}(x)\, P(A_i)}$$

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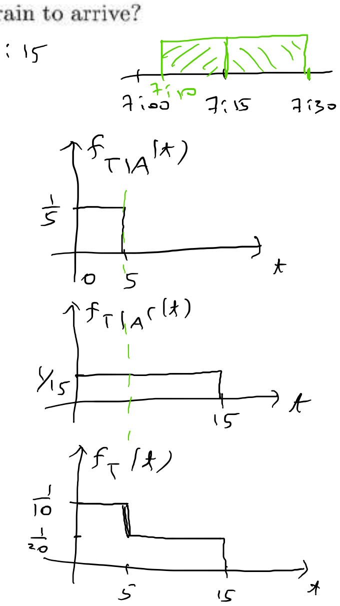
Example 3.14. The metro train arrives at the station near your home every quarter hour starting at 6:00 a.m. You walk into the station every morning between 7:10 and 7:30 a.m., and your arrival time is a uniform random variable over this interval. What is the PDF of the time you have to wait for the first train to arrive?

T : time of your wait A : arrive before 7:15



$$f_T(t) = \underbrace{f_{T|A}(t)}_{\substack{\text{uniform bet'n} \\ [0,\,5]\text{ mins}}} \cdot P(A) + \underbrace{f_{T|A^c}(t)}_{\substack{\text{uniform bet'n} \\ [0,\,15]\text{ mins}}} P(A^c)$$

$$P(A) = \frac{5}{20} = \frac{1}{4} \qquad P(A^c) = 1 - \frac{1}{4} = \frac{3}{4}$$

$$f_T(t) = \begin{cases} \frac{1}{5} \cdot \frac{1}{4} + \frac{1}{15} \cdot \frac{3}{4} = \frac{1}{10}, & 0 \le t \le 5 \\ 0 \cdot \frac{1}{4} + \frac{1}{15} \cdot \frac{3}{4} = \frac{1}{20}, & 5 < t \le 15 \end{cases}$$

Conditioning on event {Y = y}

- For an event of the form {Y = y}, how do we obtain the conditional PDF?

Can we use $f_{X,Y|\{Y=y\}} = \begin{cases} \dfrac{f_{X,Y}(x,y)}{P(Y=y)}, & \text{if } Y=y \\ 0, & \text{otherwise} \end{cases}$?

$= 0$ for a continuous RV Y: $\int_y^y f_Y(t)\,dt = 0$

It does not work since $P(Y=y)=0$

Instead, the conditional PDF for X given $Y=y$ is defined

$$\boxed{f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}}$$

$$\int_{-\infty}^{\infty} f_{X|Y}(x|y)\,dx = \int_{-\infty}^{\infty} \frac{f_{X,Y}(x,y)}{f_Y(y)}\,dx = \frac{f_Y(y)}{f_Y(y)} = 1$$

(It is a valid PDF.)

Interpretation of conditional PDF

For small positive numbers δ_1 and δ_2

$$P(x \le X \le x+\delta_1 \mid y \le Y \le y+\delta_2) = \frac{P(x \le X \le x+\delta_1,\ y \le Y \le y+\delta_2)}{P(y \le Y \le y+\delta_2)}$$

$$= \frac{\int_x^{x+\delta_1}\int_y^{y+\delta_2} f_{X,Y}(t,u)\,du\,dt}{\int_y^{y+\delta_2} f_Y(u)\,du} \simeq \frac{f_{X,Y}(x,y)\,\delta_1\delta_2}{f_Y(y)\,\delta_2}$$

$$= f_{X|Y}(x|y)\,\delta_1$$

By $\delta_1 \to 0$, $f_{X|Y}(x|y)$ is the probability that $x \in [x, x+\delta_1]$ given $Y=y$

In general, $P(X \in A \mid Y=y) = \int_A f_{X|Y}(x|y)\,dx$

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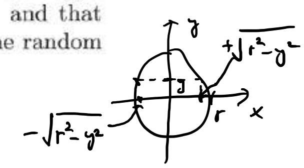
Example 3.15. Circular Uniform PDF. Ben throws a dart at a circular target of radius r. We assume that he always hits the target, and that all points of impact (x, y) are equally likely, so that the joint PDF of the random variables X and Y is uniform. Compute $f_{X|Y}(x|y)$.



$$f_{X,Y}(x,y) = \begin{cases} c & \text{if } (x,y) \in C = \{(x,y) \mid x^2+y^2 \le r^2\} \\ 0 & \text{otherwise} \end{cases}$$

$$\iint f_{X,Y}(x,y)\,dx\,dy = 1 \Rightarrow c \cdot \pi r^2 = 1 \Rightarrow c = \frac{1}{\pi r^2}$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \begin{cases} \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{1}{\pi r^2}\,dx = \frac{2\sqrt{r^2-y^2}}{\pi r^2}, & -r \le y \le r \\ 0, & \text{otherwise} \end{cases}$$

$$f_{X|Y}(x|y) = \begin{cases} \dfrac{\frac{1}{\pi r^2}}{\frac{2\sqrt{r^2-y^2}}{\pi r^2}} = \dfrac{1}{2\sqrt{r^2-y^2}}, & (x,y) \in C \\ 0, & \text{otherwise} \end{cases}$$

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Example 3.16. The speed of a typical vehicle that drives past a police radar is modeled as an exponentially distributed random variable X with mean 50 miles per hour. The police radar's measurement Y of the vehicle's speed has an error which is modeled as a normal random variable with zero mean and standard deviation equal to one tenth of the vehicle's speed. What is the joint PDF of X and Y?

$$\boxed{f_{X,Y}(x,y) = f_{X|Y}(x|y)\, f_Y(y)}$$

$$\boxed{f_{X,Y}(x,y) = f_{Y|X}(y|x)\, \underline{f_X(x)}}$$

$$Y = X + \varepsilon$$

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \ge 0 \\ 0, & \text{otherwise} \end{cases} = \begin{cases} \frac{1}{50} e^{-x/50}, & x \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi\left(\frac{x}{10}\right)^2}} e^{-\frac{(y-x)^2}{2\left(\frac{x}{10}\right)^2}}$$

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\sqrt{2\pi\left(\frac{x}{10}\right)^2}} e^{-\frac{(y-x)^2}{2\left(\frac{x}{10}\right)^2}} \cdot \frac{1}{50} e^{-x/50}, & x \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

More than 2 RVs

- Extensions are natural. For example, for 3 RVs X, Y, and Z:

$$f_{X,Y|Z}(x,y|z) = \frac{f_{X,Y,Z}(x,y,z)}{f_Z(z)}$$

$$f_{X|Y,Z}(x|y,z) = \frac{f_{X,Y,Z}(x,y,z)}{f_{Y,Z}(y,z)}$$

- Multiplication rule

$$f_{X,Y,Z}(x,y,z) = f_{X|Y,Z}(x|y,z)\, f_{Y,Z}(y,z)$$
$$= f_{X|Y,Z}(x|y,z)\, f_{Y|Z}(y|z)\, f_Z(z)$$