CMPE 212

Principles of Digital Design

Lecture 4

Boolean Algebra

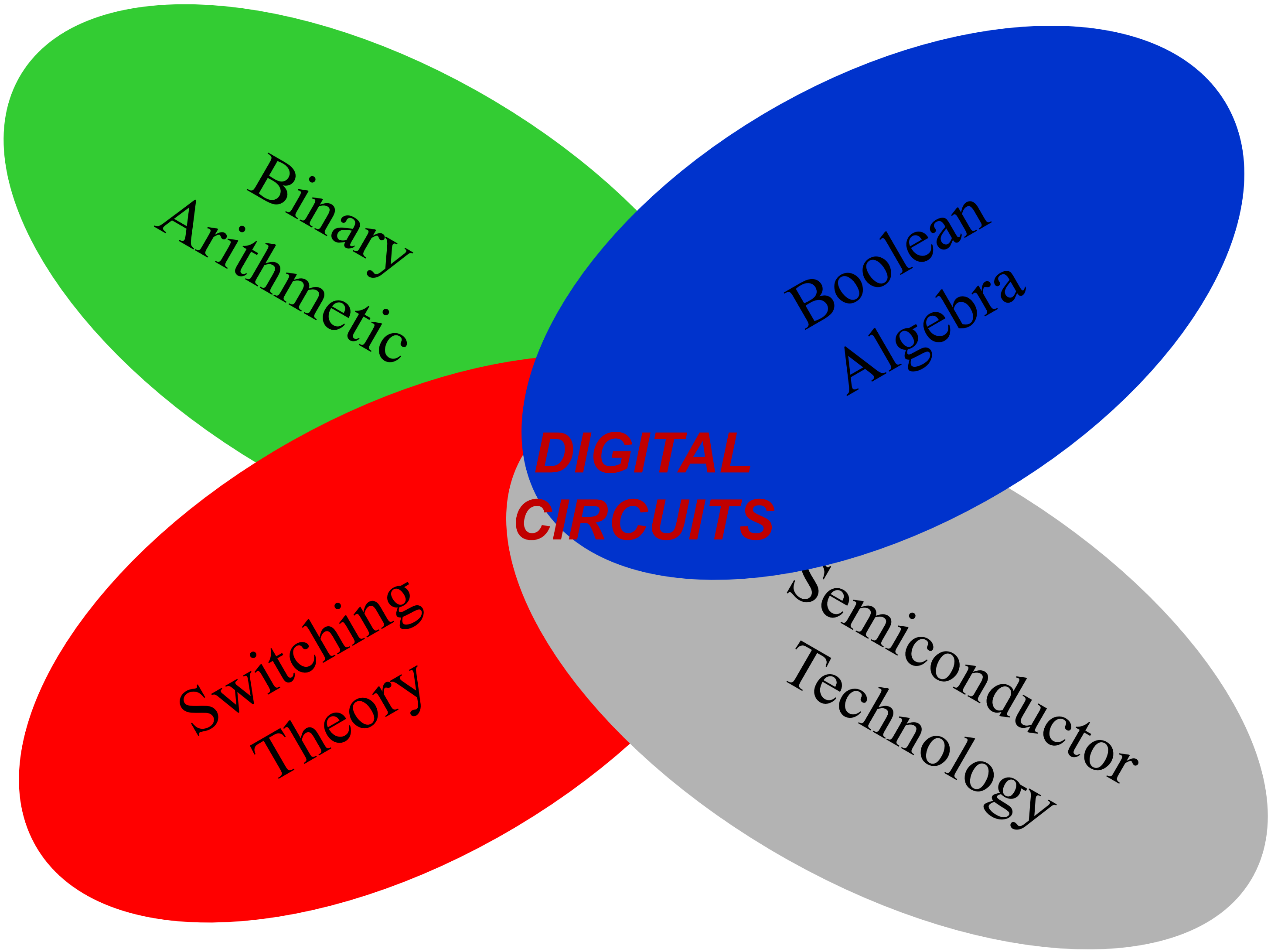
February 3, 2016

www.csee.umbc.edu/~younis/CMPE212/CMPE212.htm

Lecture's Overview

- ***Previous Lecture:***
  - ➔ Representation of signed fixed numbers
    (signed magnitude, 1's complement, 2's complement, excess)
  - ➔ Addition and subtraction of signed binary fixed numbers
    (sign handling, overflow and underflow, effect of the representation)
  - ➔ Fixed number representation
    (arithmetic overflow)
- ***This Lecture:***
  - ➔ Boolean Algebra
  - ➔ Simplification of Boolean expressions
  - ➔ Boolean algebra and switching circuits

Digital Systems



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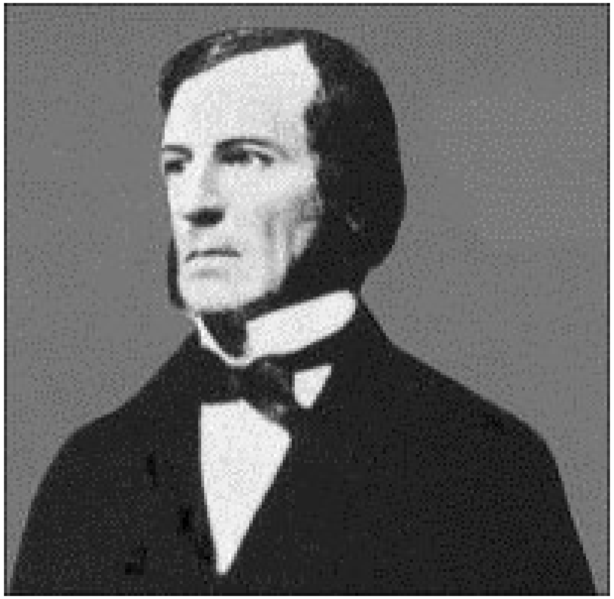
Boolean Algebra

- A set of Axioms developed by George Boole
- Formulation based on set theory

An Axiom or Postulate

- A self-evident or universally recognized truth
- An established rule, principle, or law
- A self-evident principle or one that is accepted as true without proof as the basis for argument
- A postulate – Understood as the truth



George Boole

1815-1864

- Born, Lincoln, England
- Professor of Math., Queen's College, Cork, Ireland
- Book, *The Laws of Thought*, 1853

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Basic Postulates

Postulate 1: A Boolean algebra is a closed algebraic system

- A set K containing two or more elements and two binary operators:
  - "+", also called "OR"
  - "·", also called "AND"
  - Such that for any pair of elements, a and b in K, the results of $a + b$ and $a \cdot b$ also belong to K

Postulate 2: Existence of 1 and 0 (Identity Elements)

- There exist unique *0* and *1* elements in K, such that for every element a in K
  - $a + 0 = a$
  - $a \cdot 1 = a$

Where: *0* and *1* are the identity elements for "+" and "·" operation, respectively

Postulate 3: Commutativity of the + and · operators

- for any elements *a* and *b* in K:
  - $a + b = b + a$
  - $a \cdot b = b \cdot a$

Basic Postulates (Cont.)

Postulate 4: Associativity of the + and · operators

- $a + (b + c) = (a + b) + c$
- $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

Postulate 5: Distributivity of + and + over ·

- $a + (b \cdot c) = (a + b) \cdot (a + c)$
- $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$

➢ Important:

dot (·) operation has precedence over + operation:

$$a + b \cdot c = a + (b \cdot c) \neq (a + b) \cdot c$$

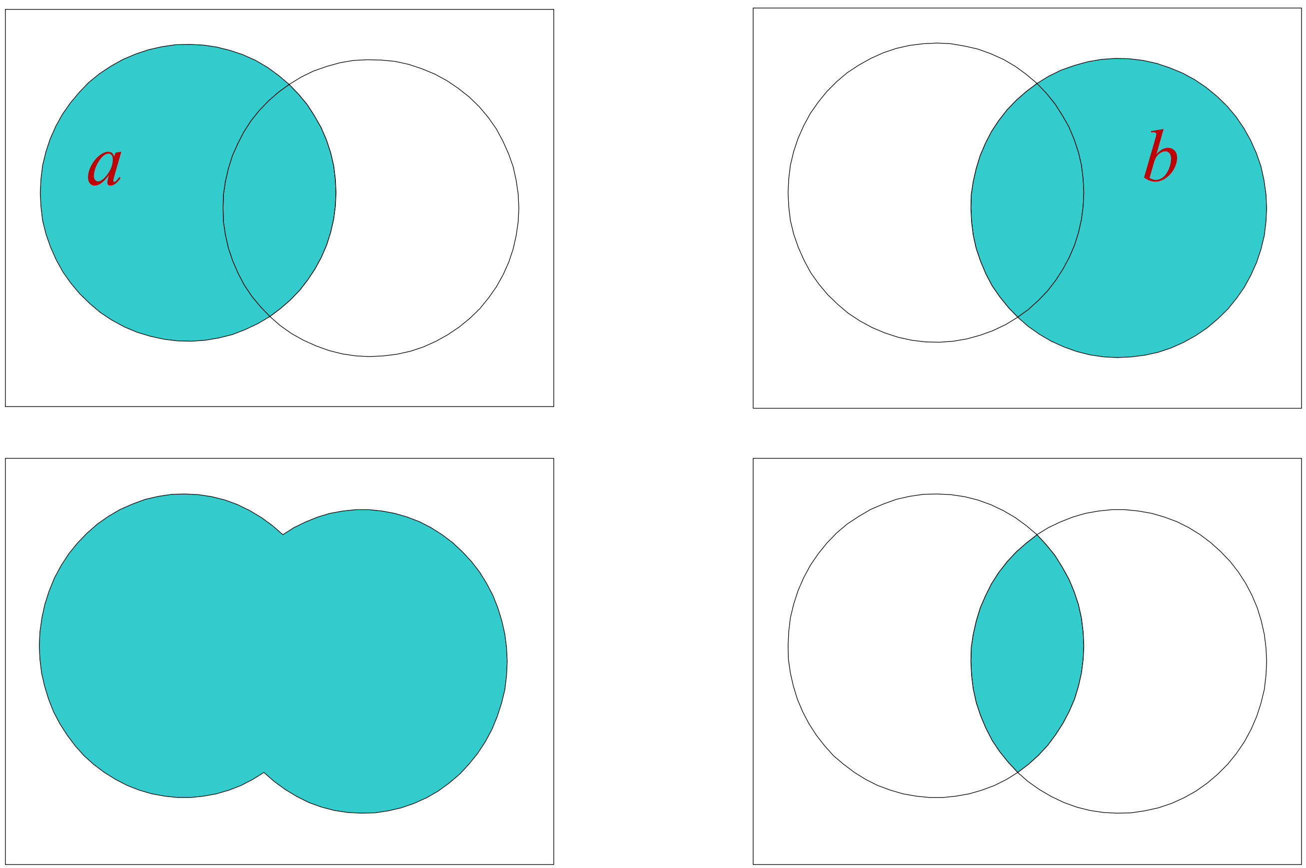
Postulate 5: Existence of the complement

➢ for every element a in K, there exist a unique element $\bar{a}$ (called complement of a) in K such that:

- $a + \bar{a} = 1$
- $a \cdot \bar{a} = 0$

Venn Diagram Representation

- The "." and "+" operations are equivalent to the intersection and union operations on sets
- The Boolean axioms can be illustrated using Venn diagrams

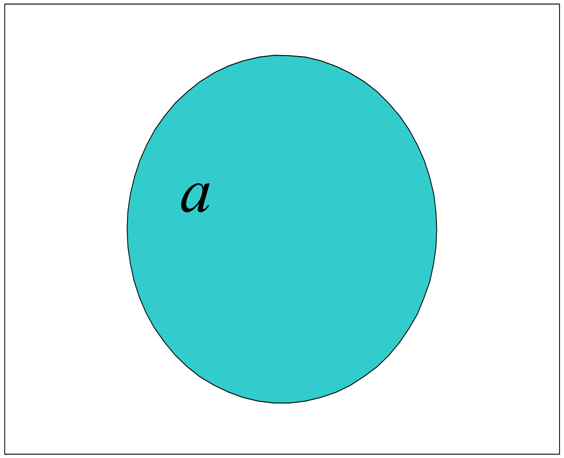


$$a + b = b + a$$

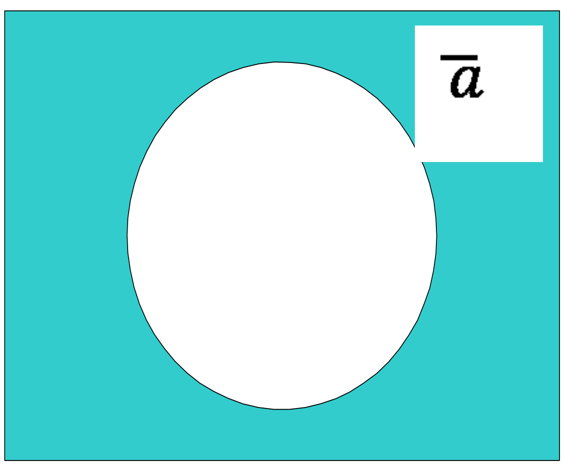
$$a \cdot b = b \cdot a$$

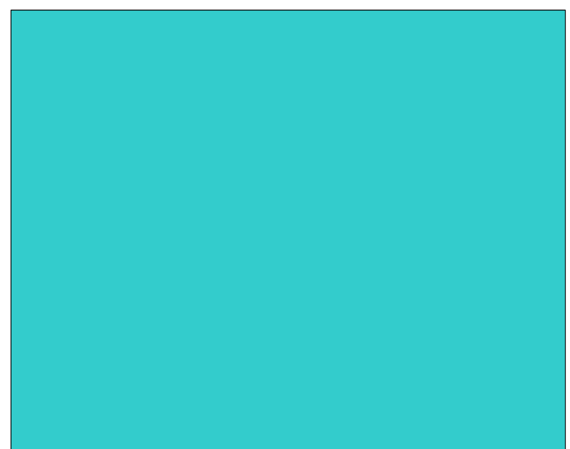
Venn Diagram (Cont.)

Postulate 6: existence of complement



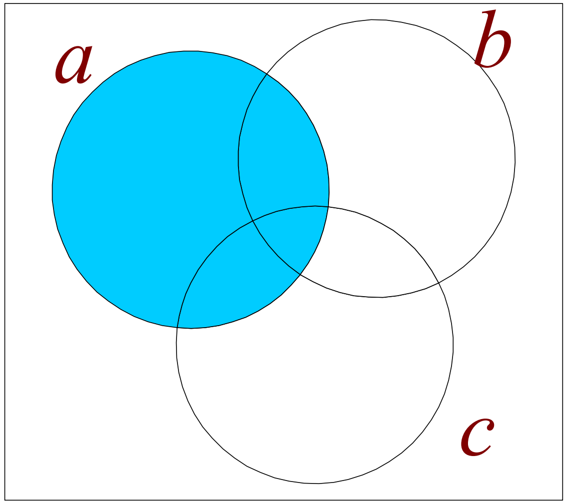
$$a \cdot \overline{a} = 0$$



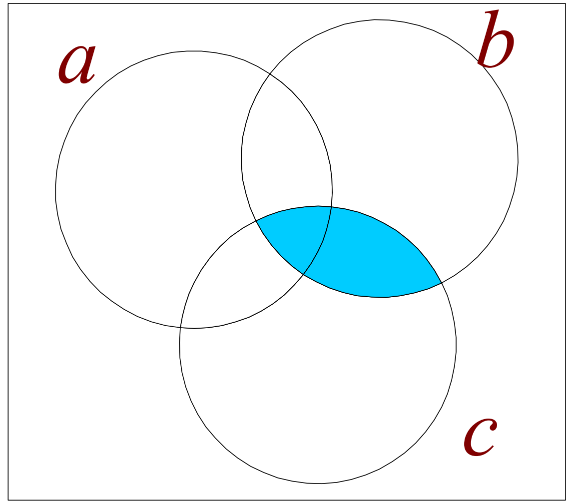


$$a + \overline{a} = 1$$

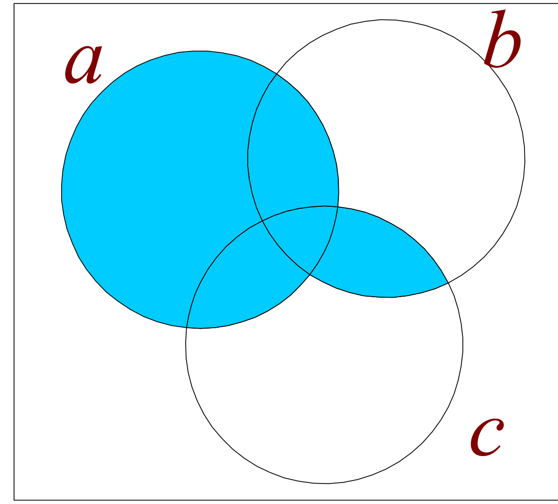
Postulate 3



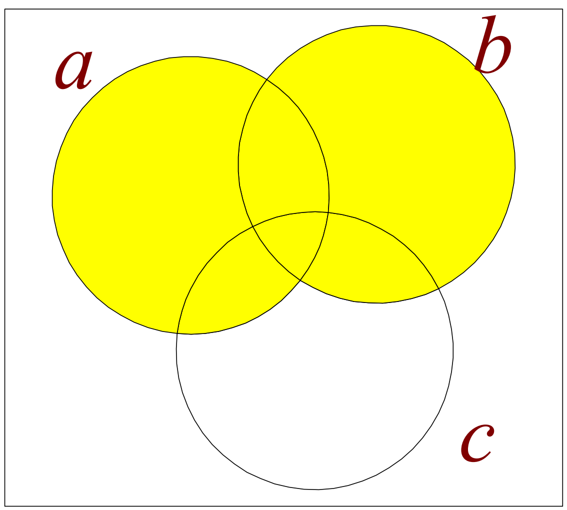
Set a is shaded



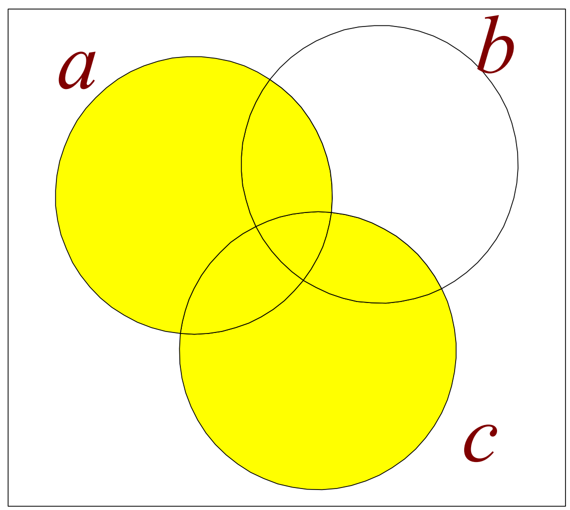
Set $b \cdot c$ is shaded



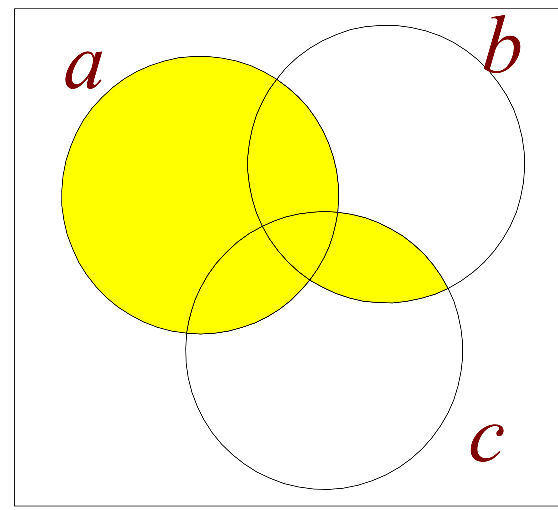
Set $a + b \cdot c$



Set $a + b$ is shaded



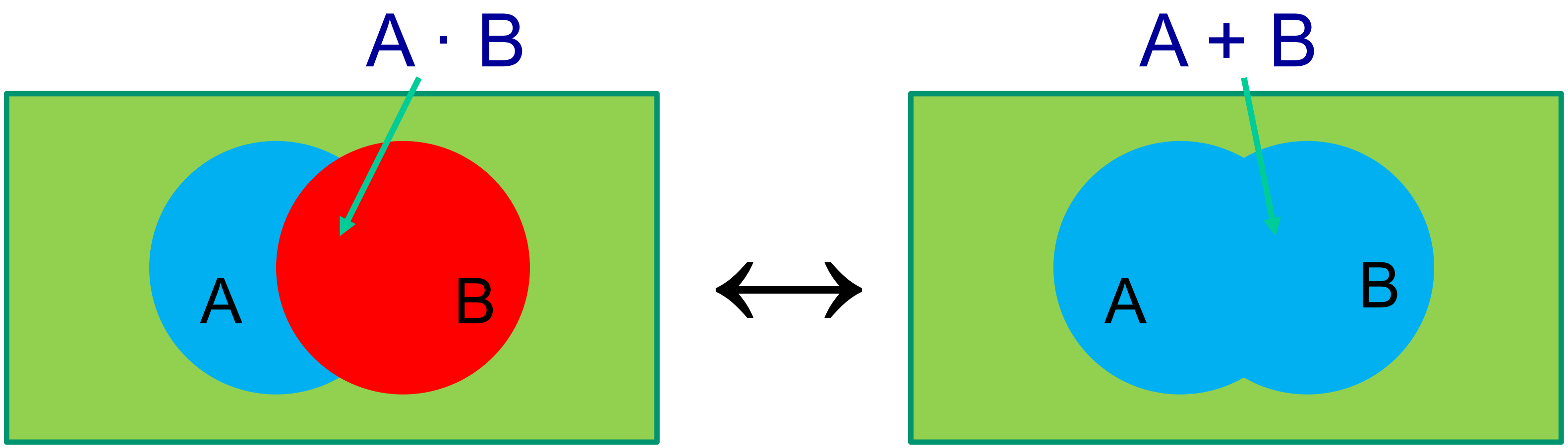
Set $a + c$ is shaded



Set $(a + b)(a + c)$

The Duality Principle

- Each postulate of Boolean algebra contains a pair of expressions or equations such that one is transformed into the other and vice-versa by interchanging the operators, $+ \leftrightarrow \cdot$, and identity elements, $0 \leftrightarrow 1$
- The two expressions are called the duals of each other



- Example: $a \cdot (b+c) = a \cdot b + a \cdot c$

  is dual for: $a + (b \cdot c) = (a + b) \cdot (a + c)$
- Should not alter the position of parentheses if they are present
- The principle of duality is used extensively in proving Boolean algebra theorems

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More Examples of Duals

Postulate	Duals	
	Expression 1	Expression 2
1	$a, b, a + b \in K$	$a, b, a \cdot b \in K$
2	$a + 0 = a$	$a \cdot 1 = a$
3	$a + b = b + a$	$a \cdot b = b \cdot a$
4	$a + (b + c) = (a + b) + c$	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$
5	$a + (b \cdot c) = (a + b) \cdot (a + c)$	$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$
6	$a + \overline{a} = 1$	$a \cdot \overline{a} = 0$

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Properties of Boolean Algebra

	Relationship	Dual	Property
Postulates	$A\,B = B\,A$	$A + B = B + A$	Commutative
	$A\,(B + C) = A\,B + A\,C$	$A + B\,C = (A + B)\,(A + C)$	Distributive
	$1\,A = A$	$0 + A = A$	Identity
	$A\,\overline{A} = 0$	$A + \overline{A} = 1$	Complement
Theorems	$0\,A = 0$	$1 + A = 1$	Zero and one theorems
	$A\,A = A$	$A + A = A$	Idempotence
	$A\,(B\,C) = (A\,B)\,C$	$A + (B + C) = (A + B) + C$	Associative
	$\overline{\overline{A}} = A$		Involution
	$\overline{A\,B} = \overline{A} + \overline{B}$	$\overline{A + B} = \overline{A}\,\overline{B}$	DeMorgan's Theorem
	$AB + \overline{A}C + BC = AB + \overline{A}C$	$(A + B)(\overline{A} + C)(B + C) = (A + B)(\overline{A} + C)$	Consensus Theorem
	$A\,(A + B) = A$	$A + A\,B = A$	Absorption Theorem

Principle of duality: The dual of a Boolean function is obtained by replacing AND with OR and OR with AND, 1s ith 0s, and 0s with 1s.

Theorem: Idempotency (Invariance)

- For all elements "a" in K: $a + a = a$; $a \cdot a = a$.

Proof:

$a + a$	$=$	$(a + a)1$,	(identity element)
	$=$	$(a + a)(a + \bar{a})$,	(complement)
	$=$	$a + a\bar{a}$,	(distributivity)
	$=$	$a + 0$,	(complement)
	$=$	a,	(identity element)
$a \cdot a$	$=$	$(a \cdot a) + 0$,	(identity element)
	$=$	$(a \cdot a) + (a \cdot \bar{a})$,	(complement)
	$=$	$a \cdot (a + \bar{a})$,	(distributivity)
	$=$	$a \cdot 1$,	(complement)
	$=$	a,	(identity element)

More Theorems

Theorem: Null Elements

- $a + 1 = 1$, for + operator, and $a \cdot 0 = 0$, for $\cdot$ operator

<u>Proof:</u>

$$
\begin{aligned}
a + 1 &= (a + 1) \cdot 1, && \text{(identity element)} \\
&= 1 \cdot (a + 1), && \text{(commutativity)} \\
&= (a + \bar{a}) \cdot (a + 1), && \text{(complement)} \\
&= a + \bar{a} \cdot 1, && \text{(distributivity)} \\
&= a + \bar{a}, && \text{(identity element)} \\
&= 1, && \text{(complement)}
\end{aligned}
$$

Similar proof can be made for: $a \cdot 0 = 0$

Theorem: Involution ($\bar{\bar{a}} = a$)

Proof: $a + \bar{a} = 1$ and $a \cdot \bar{a} = 0$, (complements)

or $\bar{a} + a = 1$ and $\bar{a} \cdot a = 0$, (commutativity)

i.e., a is complement of $\bar{a}$, since the complement of $\bar{a}$ is unique. Therefore, $\bar{\bar{a}} = a$

More Theorems

Theorem: Absorption $a + a\,b = a$, and $a\,(a + b) = a$

<u>Proof:</u>

$$a + a\,b = a\,1 + a\,b, \quad \text{(identity element)}$$
$$= a\,(1 + b), \quad \text{(distributivity)}$$
$$= a\,1, \quad \text{(identity element)}$$
$$= a, \quad \text{(identity element)}$$

Similar proof can be made for: $a\,(a + b) = a$

Theorem: $a + \bar{a}\,b = a + b$ and $a\,(\bar{a} + b) = a\,b$

<u>Proof:</u>

$$a + \bar{a}\,b = (a + \bar{a})(a + b), \quad \text{(distributivity)}$$
$$= 1\,(a + b), \quad \text{(complement)}$$
$$= (a + b), \quad \text{(identity element)}$$

Similar proof can be made for: $a\,(\bar{a} + b) = a\,b$

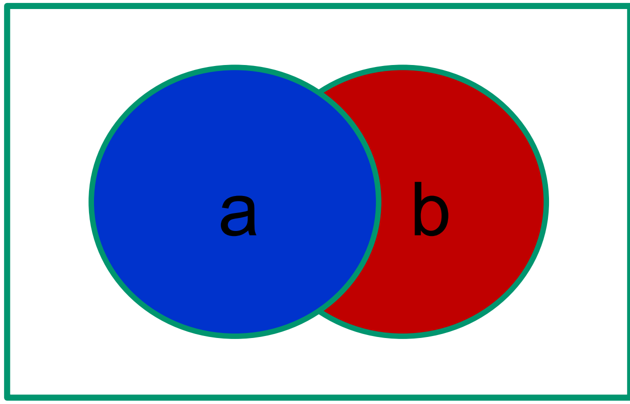


Figure is courtesy of Vishwani D. Agrawal

DeMorgan's Theorem

DeMorgan's Theorem:

$$\overline{a+b} = \bar{a} \,.\, \bar{b},$$

and

$$\overline{a\,.\,b} = \bar{a} + \bar{b}$$

Proof: From the complement postulate $\bar{x} \,.\, x = 0$. Thus is suffice to show that $(a\,.\,b)\,.\,\overline{(a.b)} = (a\,.\,b)\,.\,(\bar{a}+\bar{b}) = 0$

$$
\begin{aligned}
(a\,.\,b)\,.(\bar{a}+\bar{b}) &= (a\,.\,b)\,.\,\bar{a} + (a\,.\,b)\,.\,\bar{b} && \text{(Distributivity)}\\
&= \bar{a}\,.\,(a\,.\,b) + (a\,.\,b)\,.\,\bar{b} && \text{(Commutativity)}\\
&= (\bar{a}\,.\,a)\,.\,b + a\,.\,(b\,.\,\bar{b}) && \text{(Associativity)}\\
&= 0\,.\,b + a\,.\,0 && \text{(Complement)}\\
&= 0 + 0 && \text{(Null element)}\\
&= 0
\end{aligned}
$$

The other part of the theorem can be proven in a similar manner

Generalization:

$$\overline{a+b+c+\cdots+z} = \bar{a}\,.\,\bar{b}\,.\,\bar{c}\,.\cdots\,\bar{z},$$

and

$$\overline{a\,.\,b.\,c.\cdots.\,z} = \bar{a}\,.\,\bar{b}\,.\,\bar{c}\,.\cdots\,\bar{z},$$

Consensus Theorem

Consensus Theorem:

$$a\,.\,b + \bar{a}\,.\,c + b\,.\,c = a\,.\,b + \bar{a}\,.\,c$$

and

$$(a + b)\,.\,(\bar{a} + c)\,.\,(b + c) = (a + b)\,.\,(\bar{a} + c)$$

Proof:

$$\begin{aligned}
a\,.\,b + \bar{a}\,.\,c + b\,.\,c &= a\,.\,b + \bar{a}\,.\,c + 1.\,b\,.\,c && \text{(Identity element)}\\
&= a\,.\,b + \bar{a}\,.\,c + (a + \bar{a})\,.\,b\,.\,c && \text{(Complement)}\\
&= a\,.\,b + \bar{a}\,.\,c + a\,.\,b\,.\,c + \bar{a}\,.\,b\,.\,c && \text{(Distributivity)}\\
&= (a\,.\,b + a\,.\,b\,.\,c) + (\bar{a}\,.\,c + \bar{a}\,.\,b\,.\,c) && \text{(Commutativity)}\\
&= a\,.\,b + \bar{a}\,.\,c && \text{(Absorption)}
\end{aligned}$$

The 2nd part of the theorem can be proven in a similar manner

Example:

$AB + \bar{A}CD + BCD = AB + \bar{A}CD$ *(replace c in the theorem with CD)*

Simplification of Boolean Expressions

- Boolean expressions correspond to switching circuit
- Postulates and theorems are used in simplifying Boolean expressions to minimize the corresponding switching circuit
- DeMorgan's and consensus theorems are particularly very useful

<u>Example:</u>

$$
\begin{aligned}
ABC + \bar{A}D + \bar{B}D + CD &= ABC + (\bar{A} + \bar{B})D + CD && \text{(Distibutivity)} \\
&= ABC + \overline{AB}D + CD && \text{(DeMorgan)} \\
&= ABC + \overline{AB}D && \text{(Consensus)} \\
&= ABC + (\bar{A} + \bar{B})D && \text{(DeMorgan)} \\
&= ABC + \bar{A}D + \bar{B}D && \text{(Distibutivity)}
\end{aligned}
$$

<u>Example:</u>

$$
\begin{aligned}
\overline{a(b+c) + \bar{a}b} &= \overline{ab + ac + \bar{a}b} && \text{(Distibutivity)} \\
&= \overline{ab + \bar{a}b + ac} = \overline{(a + \bar{a})b + ac} && \text{(Commutativity, Distibutivity)} \\
&= \overline{b + ac} && \text{(Complement)} \\
&= \bar{b}(\overline{ac}) = \bar{b}(\bar{a} + \bar{c}) && \text{(DeMorgan)}
\end{aligned}
$$

Next, Switching Algebra

- Set K contains two elements, {0, 1}, also called {false, true}, or {off, on}, etc.
- Two operations are defined as, + ≡ *OR*,   · ≡ *AND*.

+	**0**	**1**
0	0	1
1	1	1

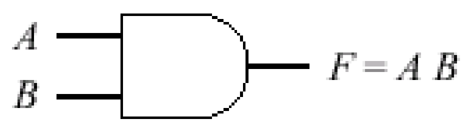
·	**0**	**1**
0	0	0
1	0	1

- More operations will be define as using *AND* and *OR*
- Realization of these basic logical functions is referred to as gates.

Logic Gates and Their Symbols

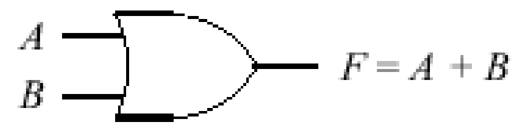
- Logic symbols for AND, OR, buffer, and NOT Boolean functions
- Note the use of the "inversion bubble."
- Be careful about the "nose" of the gate when drawing AND vs. OR.

A	B	F
0	0	0
0	1	0
1	0	0
1	1	1



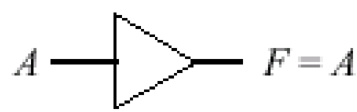
AND

A	B	F
0	0	0
0	1	1
1	0	1
1	1	1



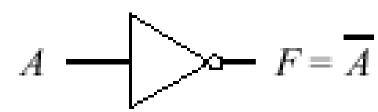
OR

A	F
0	0
1	1



Buffer

A	F
0	1
1	0



NOT (Inverter)

* Slide is courtesy of M. Murdocca and V. Heuring

Conclusion

❑ Summary

➔ Boolean Algebra
(History, postulates, theorems)

➔ Simplification of Boolean expressions
(how to put postulates and theorem to work)

➔ Boolean algebra and switching circuits
(Logic gates and algebraic method for minimization)

❑ Next Lecture

➔ Multiplication and division of binary numbers

➔ Binary codes (BCD, Character representation)

➔ Representations for floating point numbers

Reading assignment: section 2.1 in the textbook