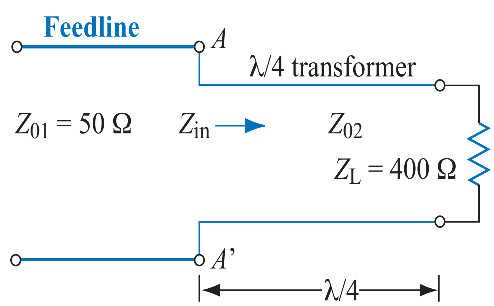
Math-Phys Quiz 5 Questions:

1. **modified from Slide 5.4:**
A $Z_{01} = 50\ \Omega$ transmission line is to be matched to a resistive load with $Z_L = 400\ \Omega$ as shown to the right. What should be the characteristic impedance Z_{02} of a quarter-wave transformer? Give the result to three significant figures.

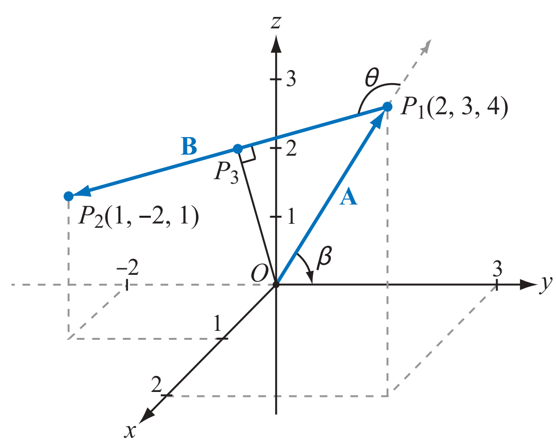


2. Show in two dimensions that the definition of the dot product $\mathbf{A} \cdot \mathbf{B} = AB \cos\theta_{AB}$, where $\cos\theta_{AB}$ is the angle between the two vectors is equivalent to the definition $\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y$, where $\mathbf{A} = (A_x, A_y)$ and $\mathbf{B} = (B_x, B_y)$.

3. (a) Show that $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A})$. (b) Show that $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C}$.

Exam Quiz 5 Questions:

1. **modified from Slide 6.10:**
In Cartesian coordinates, vector $\mathbf{A}$ is directed from the origin to the point $P_1(2, 3, 4)$ and vector $\mathbf{B}$ is directed from P_1 to $P_2(1, -2, 1)$, as shown on the right. Find (a) the vector $\mathbf{A}$, its magnitude A, and its unit vector $\hat{\mathbf{a}}$, (b) the angle that $\mathbf{A}$ makes with the y-axis, (c) vector $\mathbf{B}$, (d) the angle between $\mathbf{A}$ and $\mathbf{B}$, and (e) the perpendicular distance from the origin to $\mathbf{B}$. Give numbers to three significant figures.



2. **Slides 6.17, 6.22, and 6.25:**
Obtain the differential relations for the lengths, surface areas, and volumes in Cartesian, cylindrical, and spherical coordinates, using pictures that illustrate the geometry.

Math-Phys Quiz 5 Solutions:

1. We have $Z_{02} = \sqrt{50 \times 400} = \sqrt{8000} = 200\sqrt{2} = 282\Omega$ to three significant figures.

2. In two dimensions, we have $A_x = A\cos\theta_A$, $A_y = A\sin\theta_B$, $B_x = B\cos\theta_B$, and $B_y = B\sin\theta_B$, where θ_A and θ_B are the angles that the vectors $\mathbf{A}$ and $\mathbf{A}$ make with respect to the x- and y-coordinates. We now find $A_xB_x + A_yB_y = AB[cos\theta_A \cos\theta_B + \sin\theta_A \sin_t heta_B] = AB\cos(\theta_B - \theta_A) = AB\cos\theta_{AB}$, where we used the cosine addition formula.

3. For (a), we have by definition

$$\begin{aligned}\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) &= \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \\ &= A_xB_yC_z + C_xA_yB_z + B_xC_yA_z - A_xC_yB_z - C_xB_yA_z - B_xA_yC_z,\end{aligned}$$

which is manifestly invariant under permutations, $\mathbf{A} \to \mathbf{C}$, $\mathbf{C} \to \mathbf{B}$, $\mathbf{B} \to \mathbf{A}$. For (b), we may write

$$\mathbf{B}\times\mathbf{C} = \hat{\mathbf{x}}(B_yC_z - B_zC_y) + \hat{\mathbf{y}}(B_zC_x - B_xC_z) + \hat{\mathbf{z}}(B_xC_y - B_yC_x),$$

so that

$$\begin{aligned}\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) &= \hat{\mathbf{x}}[A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z)] \\ &\quad + \hat{\mathbf{y}}[A_z(B_yC_z - B_zC_y) - A_x(B_xC_y - B_yC_x)] \\ &\qquad + \hat{\mathbf{z}}[A_x(B_zC_x - B_xC_z) - A_y(B_yC_z - B_zC_y)] \\ &= \hat{\mathbf{x}}[B_x(A_yC_y + A_zC_z) - C_x(A_yB_y + A_zB_z)] \\ &\quad + \hat{\mathbf{y}}[B_y(A_zC_z + A_xC_x) - C_y(A_zB_z + A_xB_x)] \\ &\qquad + \hat{\mathbf{z}}[B_z(A_xC_x + A_yC_y) - C_z(A_xB_x + A_yB_y)] \\ &= \hat{\mathbf{x}}[B_x(A_xC_x + A_yC_y + A_zC_z) - C_x(A_xB_x + A_yB_y + A_zB_z)] \\ &\quad + \hat{\mathbf{y}}[B_y(A_xC_x + A_yC_y + A_zC_z) - C_y(A_xB_x + A_yB_y + A_zB_z)] \\ &\qquad + \hat{\mathbf{z}}[B_z(A_xC_x + A_yC_y + A_zC_z) - C_z(A_xB_x + A_yB_y + A_zB_z)] \\ &= (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}.\end{aligned}$$

The use of permutations is necessary to make this calculation efficient. Another approach, which is much faster, is to take advantage of our freedom to choose the x, y, and z-axes. We may first choose the x-axis so that it is in the direction of the vector $\mathbf{B}$ and $\mathbf{B} = \hat{\mathbf{x}}B_x$. We next choose the y-axis so that $\mathbf{C}$ is in the x-y plane and $\mathbf{C} = \hat{\mathbf{x}}C_x + \hat{\mathbf{y}}C_y$. The z-axis is now fixed, and in general $\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$. We now have $\mathbf{B}\times\mathbf{C} = \hat{\mathbf{z}}B_xC_y$. It follows that

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \hat{\mathbf{x}}(A_yC_y)B_x - \hat{\mathbf{y}}(A_xB_x)C_y.$$

We also have

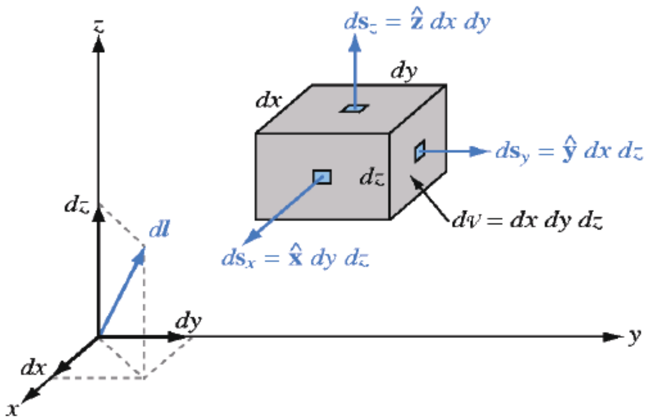
$$(\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C} = (A_xC_x + A_yC_y)\hat{\mathbf{x}}B_x - A_xB_x(\hat{\mathbf{x}}C_x + \hat{\mathbf{y}}C_y)$$
$$= \hat{\mathbf{x}}(A_yC_y)B_x - \hat{\mathbf{y}}(A_xB_x)C_y,$$

which proves the identity.

Exam Quiz 5 Solutions:

1. We have
   (a) $\mathbf{A} = \hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 + \hat{\mathbf{z}}4$, $A = \sqrt{4+9+16} = \sqrt{29} = 5.39$, $\hat{\mathbf{a}} = (1/\sqrt{29})(\hat{\mathbf{x}}2 + \hat{\mathbf{y}}3 + \hat{\mathbf{z}}) = \hat{\mathbf{x}}0.371 + \hat{\mathbf{y}}0.557 + \hat{\mathbf{z}}0.743$.
   (b) $\cos\beta = \hat{\mathbf{a}}\cdot\hat{\mathbf{y}} = 0.557$, from which we find $\beta = \cos^{-1}(0.557) = 0.980$ rad $= 56.1°$.
   (c) $\mathbf{B} = \hat{\mathbf{x}}(2-1) + \hat{\mathbf{y}}(3+2) + \hat{\mathbf{y}}(4-1) = \hat{\mathbf{x}}1 + \hat{\mathbf{y}}5 + \hat{\mathbf{z}}3$.
   (d) $B = \sqrt{35} = 5.92$, from which we obtain $\cos\theta_{AB} = \cos\theta = \mathbf{A}\cdot\mathbf{B}/AB = 0.910$, from which we conclude $\theta_{AB} = \cos^{-1}\theta_{AB} = 0.427$ rad $= 24.5°$.
   (e) The distance is given by $A\sin\theta_{AB} = 2.23$

2. **Solution: Cartesian Coordinates:**
   Length: $d\mathbf{l} = \hat{\mathbf{x}}\,dl_x + \hat{\mathbf{y}}\,dl_y + \hat{\mathbf{z}}\,dl_z = \hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy + \hat{\mathbf{z}}dz$
   Surface area (y-z plane): $d\mathbf{s}_x = \hat{\mathbf{x}}\,dl_y\,dl_z = \hat{\mathbf{x}}\,dy\,dz$
   (x-z plane): $d\mathbf{s}_y = \hat{\mathbf{y}}\,dl_x\,dl_z = \hat{\mathbf{x}}\,dx\,dz$
   (x-y plane): $d\mathbf{s}_z = \hat{\mathbf{z}}\,dl_x\,dl_y = \hat{\mathbf{x}}\,dx\,dy$
   Volume: $dv = dx\,dy\,dz$



2. **Solution: Cylindrical Coordinates:**

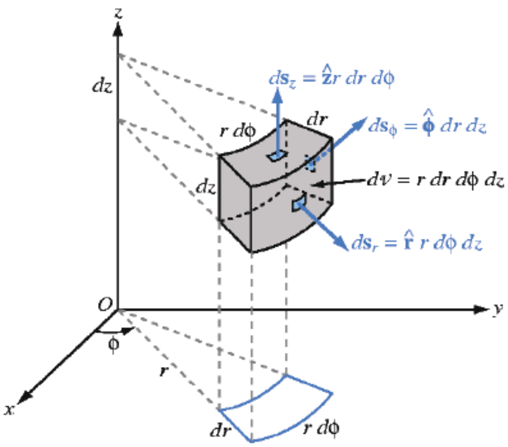
Length: $d\mathbf{l} = \hat{\mathbf{r}}\, dl_r + \hat{\boldsymbol{\phi}}\, dl_\phi + \hat{\mathbf{z}}\, dl_z = \hat{\mathbf{r}}\, dr + \hat{\boldsymbol{\phi}} r\, d\phi + \hat{\mathbf{z}}\, dz$

Surface area (ϕ-z plane): $d\mathbf{s}_r = \hat{\mathbf{r}} r\, d\phi dz$

(r-z plane): $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\, dr\, dz$

(r-ϕ plane): $d\mathbf{s}_z = \hat{\mathbf{z}} r\, dr\, d\phi$

Volume: $dv = r\, dr\, d\phi\, dz$



Solution: Spherical Coordinates:

Length: $d\mathbf{l} = \hat{\mathbf{R}}\, dl_R + \hat{\boldsymbol{\theta}}\, dl_\theta + \hat{\boldsymbol{\phi}}\, dl_\phi = \hat{\mathbf{R}}\, dR + \hat{\boldsymbol{\theta}}\, R\, d\theta + \hat{\boldsymbol{\phi}}\, R \sin\theta\, d\phi$

Surface area (θ-ϕ plane): $d\mathbf{s}_R = \hat{\mathbf{R}} R^2 \sin\theta\, d\theta\, d\phi$

(R-ϕ plane): $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}\, R \sin\theta\, dR\, d\phi$

(R-θ plane): $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\, R\, dR\, d\theta$

Volume: $dv = R^2\, dR\, d\theta\, d\phi$