HW#2 Solutions

Problem 1.

We are given that $P[D] = 0.001$, where D is the event 'disease is present.'. Let T denote the event 'test is positive,' so that T^c is the event 'test is negative.' We are additionally given $P[T|D] = 1$ and $P[T|D^c] = 0.005$. We are asked to compute $P[D|T]$, i.e. the probability that 'disease is present given the test is positive.' We use Bayes' rule and Theorem as follows

$$\begin{aligned}
P[D|T] &= \frac{P[DT]}{P[T]} \\
&= \frac{P[T|D]P[D]}{P[T|D]P[D] + P[T|D^c]P[D^c]} \\
&= \frac{1 \times 0.001}{1 \times 0.001 + 0.005 \times 0.999} \\
&= \frac{1}{1 + 4.995} \approx 0.167.
\end{aligned}$$

Thus in only about 17% of the cases will a positive test result actually confirm that you suffer from the disease. The other 83% of the time you will be needlessly worried!

Problem 2.

Directly from the problem statement

$$\begin{aligned}
P[X = 3] &= 3 \cdot P[X = 1], \\
P[X = 2] &= 2 \cdot P[X = 1].
\end{aligned}$$

But we also know $P[X = 3] + P[X = 2] + P[X = 1] = 1$ which is always true by axiom 2 $P[\Omega] = 1$. Therefore $P[X = 1] = 1/6$, $P[X = 2] = 1/3$, and $P[X = 3] = 1/2$. Using Bayes' Theorem, we then compute

$$\begin{aligned}
P[X = 1|Y = 1] &= \frac{P[Y = 1|X = 1]P[X = 1]}{\sum_{i=1}^{3} P[Y = 1|X = i]P[X = i]} \\
&= \frac{(1-\alpha)1/6}{(1-\alpha)\frac{1}{6} + \frac{\beta}{2}\frac{1}{3} + \frac{\gamma}{2}\frac{1}{2}} \\
&= \frac{1-\alpha}{1 - \alpha + \beta + \frac{3}{2}\gamma}.
\end{aligned}$$

Problem 3.

. Let

$$\begin{aligned}
A &\triangleq \{\text{examinee knows}\}, \\
B &\triangleq \{\text{examinee guesses}\}, \text{ and} \\
C &\triangleq \{\text{getting right answer}\}.
\end{aligned}$$

Then $P[A] = p$, $P[B] = 1 - p$, $P[C|A] = 1$, and $P[C|B] = 1/m$. So

$$P[A|C] = \frac{P[C|A]P[A]}{P[C]}$$

$$
\begin{aligned}
&= \frac{1 \cdot p}{P[C|A]P[A] + P[C|B]P[B]} \\
&= \frac{p}{p + \frac{1}{m}(1-p)} \\
&= \frac{mp}{mp + (1-p)}.
\end{aligned}
$$

Problem 4.

Let

$$
\begin{aligned}
\widetilde{A} &\triangleq \{\text{random drawn chip} \in A\}, \\
\widetilde{B} &\triangleq \{\text{random drawn chip} \in B\}, \text{ and} \\
\widetilde{C} &\triangleq \{\text{random drawn chip} \in C\}.
\end{aligned}
$$

Also, let $D \triangleq \{\text{random drawn chip is defective}\}$. Then

$$
\begin{aligned}
P[D] &= P[D|\widetilde{A}]P[\widetilde{A}] + P[D|\widetilde{B}]P[\widetilde{B}] + P[D|\widetilde{C}]P[\widetilde{C}] \\
&= 0.05 \times 0.25 + 0.04 \times 0.35 + 0.02 \times 0.40 \\
&= 0.0345.
\end{aligned}
$$

Hence

$$
\begin{aligned}
P[\widetilde{A}|D] &= \frac{P[D|\widetilde{A}]P[\widetilde{A}]}{P[D]} = \frac{0.05 \times 0.25}{0.0345} \doteq 0.363 \\
P[\widetilde{B}|D] &= \frac{P[D|\widetilde{B}]P[\widetilde{B}]}{P[D]} = \frac{0.04 \times 0.35}{0.0345} \doteq 0.406 \\
P[\widetilde{C}|D] &= \frac{P[D|\widetilde{C}]P[\widetilde{C}]}{P[D]} = \frac{0.02 \times 0.40}{0.0345} \doteq 0.232
\end{aligned}
$$

Problem 5.

Clearly

$$
P[A] = \frac{4}{52} \quad \text{and} \quad P[B] = \frac{26}{52} = \frac{1}{2}.
$$

Then $P[AB] = P[\{\text{pick one of two red aces in 52 cards}\}] = \frac{2}{52}$. Is $P[AB] = P[A]P[B]$? Now

$$
\begin{aligned}
P[AB] = \frac{2}{52} &= \frac{4}{52}\frac{1}{2} \\
&= P[A]P[B],
\end{aligned}
$$

so, yes A and B are independent events.

Problem 6.

Since it is a fair die, the successive tosses are independent with probability $p = 1/6$ for each face. From the provided information, we equivalently want the probability of getting a total of 5 on the two remaining tosses. This can happen in just 4 equally likely outcomes, i.e. (4,1), (3,2), (2,3), and (1,4). The desired probability this then $4/36 = 1/9$.

Problem 7.

The probability that the customer receives service 3 out of 4 times is given by the binomial formula:

$$\binom{4}{3} \cdot \left(\frac{3}{4}\right)^3 \cdot \left(\frac{1}{4}\right) = \frac{27}{64}.$$

Problem 8.

Let p_i and q_i be respectively the probability and the area of face i. We have

$$p_i = \frac{q_i}{\sum_{j=1}^{6} q_j}, \qquad i = 1, \ldots, 6.$$

Thus the probability that we get doubles is

$$\sum_{i=1}^{6} p_i^2 = \frac{\sum_{i=1}^{6} q_i^2}{\left(\sum_{j=1}^{6} q_j\right)^2} = \frac{2(1.5)^2 + 2(0.4)^2 + 2(0.4 \cdot 1.5)^2}{\big(2(1.5) + 2(0.4) + 2(0.4 \cdot 1.5)\big)^2} = 0.2216.$$