(1) MATH 407

4/30/18

* <u>Def.</u> Monic polynomials: leading coefficient is 1 (so poly non-zero)

* If $\deg(p) = n$

$$p = a_0 + \dots + a_n x^n$$

then is $a_n\left(\frac{a_0}{a_n} + \dots + \frac{a_{n-1}x^{n-1}}{a_n} + x^n\right)$

* $F[x]$ satisfies all of field axioms except multiplicative inverses.

If $\deg(p) \geq 0$ and $p(x)f(x) = p(x)g(x)$, then $f(x) = g(x)$

* No zero divisors: $p(x)g(x) = 0$ iff either $p = 0$ or $g = 0$.

* $F[x]$ is integral domain

* Divisibility: $f(x) \mid g(x)$ iff there is $q(x)$ s.t. $f(x)q(x) = g(x)$

$$\langle f \rangle = fF[x] = \{fq : q \in F[x]\}$$

* <u>Def.</u> $f \in F[x]$, $\deg(f) \geq 1$ is irreducible iff $f(x) = g(x)h(x)$ implies that either g is constant or h is constant

* **Lemma**: Any polynomial $f(x)$ (w/ $\deg(f) \geq 1$) has an irreducible factor

Pf. Let $D_f = \{g \in F[x] : g \mid f, \deg(g) \geq 1\}$

There is a polynomial $d \in D_f$ of minimum degree
Assert d is irreducible, else $d = a(x)b(x)$, a, b lesser $\deg \geq 1$

* **Lemma**: a) Divisibility of polynomials is transitive
b) If $f \mid g$ and $g \mid f$, then $\deg(f) = \deg(g)$

* If $p(x)$ is linear, $\deg(p) = 1$
$\Rightarrow p = mx + b = m(x + b/m)$
Any linear polynomial is irreducible

* **Def.** If $c \in F$ and $f(c) = 0$, then c is a root, or zero of f

* **Lemma** $(x - c) \mid (x^k - c^k)$, any $k \geq 1$, $c \in F$

Pf. $$(x - c)\underbrace{(x^{k-1} + cx^{k-2} + \dots + c^{k-2}x + c^{k-1})}_{q_k}$$

$$= x^k + cx^{k-1} + \dots + c^{k-2}x^2 + c^{k-1}x$$
$$- (cx^{k-1} + \dots + c^{k-1}x)$$
$$= x^k - c^k$$

(3)

If $x=1$, $(1+c+\dots+c^{k-1})(1-c)$
$\Rightarrow 1-c^k$, so $(1+c\cdot 1+\dots+c^{k-1}) = \dfrac{1-c^k}{1-c}$

* **Thm** $(x-c) \mid (f(x)-f(c))$ for any f, $\deg(f)\geq 1$

Pf. $f(x) = \sum_{k=0}^{n} a_k x^k$

$f(c) = \sum_{k=0}^{n} a_k c^k$

$$(f(x)-f(c)) = \sum_{k=0}^{n} a_k (x^k - c^k)$$

$$= \sum_{k=0}^{n} a_k q_k (x-c)$$

$$= \left(\sum_{k=0}^{n} a_k q_k(x)\right)(x-c)$$

* **Remainder thm.** a) $f\in F[x]$, $(x-c)\mid f$ iff $f(c)=0$
b) $f(x) = (x-c)q(x) + f(c)$

Pf. a) $f(x) = f(x) - f(c)$

Conversely, $(x-c) \mid f(x)$
$f(x) = (x-c)q(x)$
$f(c) = (c-c)q(c) = 0$

b) $f(x) - f(c) = (x-c)q(x)$

so $f(x) = (x-c)q(x) + f(c)$

* Cor. Let $c \neq d$, $f(x) = (x-d)g(x)$

then $f(c) = 0 \Rightarrow g(c) = 0$

* Cor. If $f \in F[x]$ and $\deg(f) \geq 1$ then f has at most n distinct zeros.

Pf. Let $\{x_1, \ldots, x_k\}$ distinct zeros f

$f(x) = (x - x_1)g(x)$

$\{x_2, \ldots, x_k\}$ roots of g

$\deg(g) = n-1$

so $k-1 \leq n-1$

so $k \leq n$

* Thm $T: f \to \hat{f}$, $\hat{f}(c) = f(c)$, $\forall c \in F$

is linear from $F[x]$ to F^F.

T is 1-1 iff F is infinite.