## Math-Phys Quiz 2 Questions: Given 02/14/2017

1. To two significant figures, what are: $\pi$, $\pi/2$, $\sqrt{2}/2$, $\sqrt{3}/2$, $e$, $1/e$. To two significant figures, what is 1 radian in degrees.

2. Consider the equation that relates the phasors $\tilde{I}$ and $\tilde{V}$ in an $RC$ circuit,

$$R\,\mathrm{Re}\left[\tilde{I}\exp(j\omega t)\right] + \frac{1}{C}\,\mathrm{Re}\left[\frac{1}{j\omega}\tilde{I}\exp(j\omega t)\right] = \mathrm{Re}\left[V_\mathrm{S}\exp(j\omega t)\right]$$

Show that this equation holds at all points in time if and only if

$$\left(R + \frac{j}{\omega C}\right)\tilde{I} = \tilde{V}_\mathrm{S}$$

3. Demonstrate the equation for a geometric series $\sum_{n=0}^{m} x^n = (1-x^{m+1})/(1-x)$. What is the condition for this result to hold in the limit as $m \to \infty$ and what does it become? Use this result to show that $0.6666\ldots = 2/3$.

## Exam Quiz 2 Questions: Given 02/14/2017

1. **Menyuk Slide no. 3.15 — Modified from Paul:**

What are the per unit length capacitance and inductance of a two-wire line whose wires have a radius of 10 mils and a separation of 50 mils? What is the characteristic impedance and the velocity of propagation? (These dimensions are typical for ribbon cables used to interconnect components.) You will need to use the following formulae from the table on slide 3.7:

$$L' = \frac{\mu}{\pi}\ln\left[(d/2a) + \sqrt{(d/2a)^2 - 1}\right], \qquad C' = \frac{\pi\epsilon}{\ln\left[(d/2a) + \sqrt{(d/2a)^2 - 1}\right]},$$

where $d$ is the separation between the wires, and $a$ is the radius of the wires. You should use $\mu$ and $\epsilon$ for the vacuum: $\mu = 4\pi \times 10^{-7}$ H/m and $\epsilon = 1/\mu c^2$, where $c$ is the speed of light.

2. **Menyuk Slides no. 3.10–3.14 — Voltage-Current Relations:**

In the lossless limit, the telegrapher's equation may be written

$$-\frac{\partial v(z,t)}{\partial z} = L'\frac{\partial i(z,t)}{\partial t}, \qquad -\frac{\partial i(z,t)}{\partial z} = C'\frac{\partial v(z,t)}{\partial t}$$

Show that the solution to this equation may be written

$$v(z,t) = v^+\left(t - \frac{z}{u_\mathrm{p}}\right) + v^-\left(t + \frac{z}{u_\mathrm{p}}\right),$$
$$i(z,t) = \frac{1}{Z_0}v^+\left(t - \frac{z}{u_\mathrm{p}}\right) - \frac{1}{Z_0}v^-\left(t + \frac{z}{u_\mathrm{p}}\right),$$

where $v^+$ and $v_-$ are arbitrary functions whose second derivatives exist, and $u_\mathrm{p} = 1/\sqrt{L'C'}$, $Z_0 = \sqrt{L'/C'}$.

## Math-Physics Quiz 2 Solutions:

1. $\pi = 3.1$, $\pi/2 = 1.6$, $\sqrt{2}/2 = 0.71$, $\sqrt{3}/2 = 0.87$, $e = 2.7$, $1/e = 0.37$, 1 radian $= 57°$.

2. Consider the equation that relates the phasors $\tilde{I}$ and $\tilde{V}$ in an $RC$ circuit,

$$R\,\mathrm{Re}\left[\tilde{I}\exp(j\omega t)\right] + \frac{1}{C}\,\mathrm{Re}\left[\frac{1}{j\omega}\tilde{I}\exp(j\omega t)\right] = \mathrm{Re}\left[V_{\mathrm{S}}\exp(j\omega t)\right]$$

   Show that this equation holds at all points in time if and only if

$$\left(R + \frac{j}{\omega C}\right)\tilde{I} = \tilde{V}_{\mathrm{S}}$$

3. If we let $S_m = \sum_{n=0}^{m} x^m$, we find that

$$xS_m = \sum_{n=1}^{m+1} x^n = S_m - 1 + x^{m+1}.$$

   Solving for $S_m$, we obtain $S_m = (1 - x^{m+1})/(1 - x)$. In order for this series to have a limit as $m \to \infty$, we must have $|x| < 1$. In this case, we find that the limit is $S_\infty = 1/(1 - x)$. To use this result to find $0.6666\ldots$, we write

$$0.6666\ldots = 6\sum_{n=1}^{\infty}(1/10)^n = 6/(1 - 1/10) - 6 = (60 - 54)/9 = 2/3$$

## Exam Quiz 2 Solutions:

1. In this case, we have $d/2a = 50/20 = 2.5$. We have $\mu = 1.2566 \times 10^{-6}$ H/m and $\epsilon = 8.8542 \times 10^{-12}$ F/m. We infer $L' = [1.2566 \times 10^{-6}/3.1426] \times \ln[2.5 + (2.5^2 - 1)] = 6.3 \times 10^{-7}$ H/m or 630 nH/m, where we keep two significant figures. We also infer $C' = 1/(c^2 L') = 1.8 \times 10^{-11}$ F/m or 18 pF/m. To two significant figures, the characteristic impedance is given by $Z_0 = \sqrt{L'/C'} = 190\ \Omega$. The velocity of propagation is given by $u_{\mathrm{p}} = 3.0 \times 10^8$ m/s, which is the speed of light in the vacuum. We only report two significant figures because we only specified the dimensions of the two-wire line to two significant figures.

2. We first eliminate the current from the telegrapher's equation to obtain

$$\frac{\partial^2 v(z,t)}{\partial z^2} = L'C'\frac{\partial^2 v(z,t)}{\partial t^2}$$

   If we let $\theta = t - z/u_{\mathrm{p}}$, we find

$$\frac{\partial}{\partial z^2}v^+\left(t - \frac{z}{u_{\mathrm{p}}}\right) = \frac{1}{u_{\mathrm{p}}^2}\frac{d^2 v^+(\theta)}{d\theta^2} \qquad \text{and} \qquad \frac{\partial}{\partial t^2}v^+\left(t - \frac{z}{u_{\mathrm{p}}}\right) = \frac{d^2 v^+(\theta)}{d\theta^2}$$

After substitution into the wave equation, we obtain an identity. Hence, any function $v^+(\theta)$ will satisfy the wave equation. The proof for $v^-$ is analogous. To obtain the result for the propagation equations, we write

$$i(z,t) = i^+\left(t - \frac{z}{u_\mathrm{p}}\right) + i^-\left(t + \frac{z}{u_\mathrm{p}}\right),$$

and we substitute into either one of the telegrapher's equations. The result follows from equating terms.