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CMPE 320: HW 07

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1. A radar tends to overestimate the distance of an aircraft, and the error is a normal random variable with a mean of 50 meters and a standard deviation 100 meters. What is the probability that the measured distance will be smaller than the true distance?

$$
\begin{aligned}
P(X < 0) &= P\left(Y < \frac{x-\mu}{\sigma}\right) \\
&= P\left(Y < \frac{0-50}{100}\right) \\
&= P(Y < -0.5) \\
&= \Phi(-0.5) \\
&= 1 - \Phi(0.5) \\
&= 0.3085
\end{aligned}
$$

□

2. Let X be normal with mean 1 and variance 4. Let $Y = 2X + 3$.

(a) Calculate the PDF of Y.

$$
\begin{aligned}
E[Y] &= E[2X+3] \\
&= 2E[X] + 3 \\
&= 2(1) + 3 \\
&= 5
\end{aligned}
$$

$$
\begin{aligned}
var(Y) &= var(2X+3) \\
&= (2)^2 var(X) \\
&= (2)^2 (4)
\end{aligned}
$$

$$= 16$$

The PDF of the normal random variable is

$$f_Y(y) = \begin{cases} \frac{1}{\sqrt{2\pi}} e^{\frac{-(x-5)^2}{32}}, & \text{if } \infty < y \leq \infty, \\ 0, & \text{otherwise,} \end{cases}$$ □

(b) Find $P(Y \geq 0)$.

$$P(Y \geq 0) = P\left(Z \geq \frac{x-\mu}{\sigma}\right)$$

$$= P\left(Z \geq \frac{0-5}{\sqrt{16}}\right)$$

$$= P(Z \geq -1.25)$$

$$= \Phi(-1.25)$$

$$= 1 - \Phi(1.25)$$

$$= 0.1056$$ □

3. A signal of amplitude $s = 2$ is transmitted from a satellite but is corrupted by noise, and the received signal is $X = s + W$, where W is noise. When the weather is good, W is normal with zero mean and variance 1. When the weather is bad, W is normal with zero mean and variance 4. In the absence of any weather information:

(a) Calculate the PDF of X.

Let G represent good weather, and B represent bad weather
where $P(G) = P(B) = \frac{1}{2}$
We are given $W \sim N(0,1)$ given G, and $W \sim N(0,4)$ given B
Therefore, the unconditional PDF pf W,

$$f_W(w) = P(G) \cdot f_{W|G}(w \mid G) + P(B) \cdot f_{W|B}(w \mid B)$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{(1)2\pi}} e^{-\frac{w^2}{(1)2}} + \frac{1}{2} \cdot \frac{1}{\sqrt{(4)2\pi}} e^{-\frac{w^2}{(4)2}}$$

$$= \frac{1}{2\sqrt{2\pi}} \left(e^{-\frac{w^2}{2}} + \frac{1}{2} e^{-\frac{w^2}{(4)2}} \right)$$

$$= \frac{1}{2\sqrt{2\pi}}\left(e^{-\frac{w^2}{2}} + \frac{1}{2}e^{-\frac{w^2}{8}}\right)$$

Since $X = W + 2$,

$$f_X(x) = \frac{1}{2\sqrt{2\pi}}\left(e^{-\frac{(x-2)^2}{2}} + \frac{1}{2}e^{-\frac{(x-2)^2}{8}}\right) \qquad \square$$

(b) Calculate the probability that X is between 1 and 3.

$$P(1 \leq X \leq 3) = P(1 \leq W + 2 \leq 3) = P(-1 \leq W \leq 1)$$

Using the total probability theorem,

$$\begin{aligned}
P(-1 \leq W \leq 1) &= P(G)P(-1 \leq W \leq 1 \mid G) + P(B)P(-1 \leq W \leq 1 \mid B) \\
&= P(G)(N(0,1)) + P(B)(N(0,4)) \\
&= P(G)(\Phi(1) - \Phi(-1)) + P(B)\left(\Phi\left(\frac{1}{2}\right) - \Phi\left(-\frac{1}{2}\right)\right) \\
&= P(G)(\Phi(1) - (1 - \Phi(1))) + P(B)\left(\Phi\left(\frac{1}{2}\right) - \left(1 - \Phi\left(\frac{1}{2}\right)\right)\right) \\
&= P(G)(2\Phi(1) - 1) + P(B)\left(2\Phi\left(\frac{1}{2}\right) - 1\right) \\
&= \frac{1}{2}\left(2\Phi(1) - 1\right) + \frac{1}{2}\left(2\Phi\left(\frac{1}{2}\right) - 1\right) \\
&= \Phi(1) - \frac{1}{2} + \Phi\left(\frac{1}{2}\right) - \frac{1}{2} \\
&= \Phi(1) + \Phi\left(\frac{1}{2}\right) - 1 \\
&= 0.8413 + 0.6915 - 1 \\
&= 0.5328 \qquad \square
\end{aligned}$$

4. Oscar uses his high-speed modem to connect to the Internet. The modem transmits zeros and ones by sending signals -1 and $+1$, respectively. We assume that any given bit has probability p of being a zero. The network cable introduces additive zero-mean Gaussian noise with variance σ^2 (so, the receiver at the other end receives a signal which is the sum of the transmitted signal and the channel noise). The value of the noise is assumed to be independent of the encoded signal value.

(a) Let a be a constant between -1 and 1. The receiver at the other end decides that the signal -1 (respectively, $+1$) was transmitted if the value it receives is less (respectively, more) than a. Find a formula for the probability of making an error.

$$
\begin{aligned}
P(\text{send -1} \mid \text{error}) &= P(\text{error} \mid \text{send -1})P(\text{send -1}) \\
&= P(y-1<a)(1-p) \\
&= P(y<a+1)(1-p) \\
&= (1-P(y \geq a+1))(1-p) \\
&= \left(1-\Phi\left(\frac{a-1}{\sigma}\right)\right)(1-p) \\
&= \Phi\left(\frac{1-a}{\sigma}\right)(1-p)
\end{aligned}
$$

$$
\begin{aligned}
P(\text{send +1} \mid \text{error}) &= P(\text{error} \mid \text{send +1})P(\text{send +1}) \\
&= P(y-1>a)p \\
&= P(y>a+1)p \\
&= \Phi\left(\frac{1+a}{\sigma}\right)p
\end{aligned}
$$

Therefore, by the total probability theorem,

$$
\begin{aligned}
P(\text{error}) &= P(\text{error} \mid \text{send -1})P(\text{send -1}) + P(\text{error} \mid \text{send +1})P(\text{send +1}) \\
&= \Phi\left(\frac{1-a}{\sigma}\right)(1-p) + \Phi\left(\frac{1+a}{\sigma}\right)p
\end{aligned}
$$

□

(b) Find a numerical answer for the question of part (a) assuming that $p = 2/5$, $a = 1/2$ and $\sigma^2 = 1/4$.

$$\Phi\left(\frac{1-a}{\sigma}\right)(1-p)+\Phi\left(\frac{1+a}{\sigma}\right)p = \Phi\left(\frac{1-\frac{1}{2}}{\frac{1}{2}}\right)\left(1-\frac{2}{5}\right)+\Phi\left(\frac{1+\frac{1}{2}}{\frac{1}{2}}\right)\frac{2}{5}$$

$$= \Phi(1)\left(1-\frac{2}{5}\right)+\Phi(3)\left(\frac{2}{5}\right)$$

$$= 0.8413\cdot\frac{3}{5}+0.9987\cdot\frac{2}{5}$$

$$= 0.9043 \qquad \square$$

5. An old modem can take anywhere from 0 to 30 seconds to establish a connection, with all times between 0 and 30 being equally likely.

(a) What is the probability that if you use this modem you will have to wait more than 15 seconds to connect?

For the uniformly distributed normal variable

$$\int_{15}^{30}\frac{1}{30}\,\mathrm{d}x = \frac{1}{30}x\Big|_{15}^{30}$$

$$= 0.5 \qquad \square$$

(b) Given that you have already waited 10 seconds, what is the probability of having to wait at least 10 more seconds?

$$\int_{20}^{30}\frac{1}{20}\,\mathrm{d}x = \frac{1}{20}x\Big|_{10}^{20}$$

$$= 0.5 \qquad \square$$

6. Consider a random variable X with PDF

$$f_X(x) = \begin{cases} 2x/3, & \text{if } 1 < x \leq 2, \\ 0, & \text{otherwise,} \end{cases}$$

and let A be the event $\{X \geq 1.5\}$. Calculate $E[X]$, $P(A)$ and $E[X \mid A]$.

$$
\begin{aligned}
E[x] &= \int xP(x)\,\mathrm{d}x \\
&= \int_1^2 x\frac{2x}{3}\,\mathrm{d}x \\
&= \frac{2}{3}\frac{x^3}{3}\bigg|_1^2 \\
&= \frac{14}{9}
\end{aligned}
$$

$$
\begin{aligned}
P(x) &= \int_{1.5}^2 \frac{2x}{3}\,\mathrm{d}x \\
&= \frac{2}{3}\frac{x^2}{2}\bigg|_{1.5}^2 \\
&= \frac{7}{12}
\end{aligned}
$$

$$
\begin{aligned}
P(X \mid A) &= P(X \mid X \geq 1.5) = 1 \\
E[X \mid A] &= \int_1^2 1x\,\mathrm{d}x \\
&= \frac{x^2}{2}\bigg|_1^2 \\
&= \frac{3}{2}
\end{aligned}
$$

□

7. Dino, the cook, has good days and bad days with equal frequency. On a good day, the time (in hours) it takes Dino to cook a souffle is described by the PDF

$$
f_G(g) = \begin{cases} 2, & \text{if } 1/2 < g \leq 1, \\ 0, & \text{otherwise,} \end{cases}
$$

but on a bad day, the time it takes is described by the PDF

$$f_B(b) = \begin{cases} 1, & \text{if } 1/2 < b \leq 3/2, \\ 0, & \text{otherwise,} \end{cases}$$

Find the conditional probability that today was a bad day, given that it took Dine less than three quarters of an hour to cook a souffle.

Let X represent the condition that it took Dine less than $3/4$ of an hour to cook a souffle

Then

$$P(B \mid X) = \frac{P(X \mid B)P(B)}{P(X \mid B)P(B) + P(X \mid G)P(G)}$$

Since

$$P(X \mid B) = \int_{1/2}^{3/4} 1 \, \mathrm{d}b$$

$$= \frac{1}{4}$$

$$P(X \mid G) = \int_{1/2}^{3/4} 2 \, \mathrm{d}g$$

$$= \frac{1}{2}$$

Therefore,

$$P(B \mid X) = \frac{\frac{1}{4}\frac{1}{2}}{\frac{1}{4}\frac{1}{2} + \frac{1}{2}\frac{1}{2}}$$

$$= \frac{1}{3}$$

□

8. One of the two wheels of fortune, A and B, is selected by the toss of a fair coin, and the wheel chosen is spun once to determine the value of a random variable X. If wheel A is selected, the PDF of X is

$$f_{X|A}(x \mid A) = \begin{cases} 1, & \text{if } 0 < x \leq 1, \\ 0, & \text{otherwise,} \end{cases}$$

If wheel B is selected, the PDF of X is

$$f_{X|B}(x \mid B) = \begin{cases} 3, & \text{if } 0 < w \leq 1/3, \\ 0, & \text{otherwise,} \end{cases}$$

If we are told that the value of X was less than 1/4, what is the conditional probability that wheel A was the one selected.

$$\begin{aligned} P\left(x \leq \frac{1}{4} \mid A\right) &= \int_0^{1/4} f_{X|A}(x \mid A)\, \mathrm{d}x \\ &= \int_0^{1/4} 1\, \mathrm{d}x \\ &= \frac{1}{4} \end{aligned}$$

$$\begin{aligned} P\left(x \leq \frac{1}{4} \mid B\right) &= \int_0^{1/4} f_{X|B}(x \mid B)\, \mathrm{d}x \\ &= \int_0^{1/4} 3\, \mathrm{d}x \\ &= \frac{3}{4} \end{aligned}$$

$$\begin{aligned} P\left(A \mid x \leq \frac{1}{4}\right) &= \frac{P\big(x \leq \frac{1}{4} \mid A\big)P(A)}{P\big(x \leq \frac{1}{4} \mid A\big)P(A) + P\big(x \leq \frac{1}{4} \mid B\big)P(B)} \\ &= \frac{\frac{1}{4} \cdot \frac{1}{2}}{\frac{1}{4} \cdot \frac{1}{2} + \frac{3}{4} \cdot \frac{1}{2}} \\ &= \frac{1}{4} \end{aligned}$$

□