Calculating Small Differences

We often have two quantities x and y that have the relation $y = ax^r$, where a is a positive constant and r is a constant power:

Examples: angular frequency, $\omega = 2\pi f$; $a = 2\pi$, $r = 1$
wavenumber, $\beta = 2\pi/\lambda$; $a = 2\pi$, $r = -1$
frequency, $f = c/\lambda$; $a = c$, $r = -1$
index of refraction, $n = \beta c/\omega = c/u_p$; $a = c$, $r = -1$
index of refraction, $n = \varepsilon_r^{1/2}$; $a = 1$, $r = 1/2$

and two values that differ slightly

$$y_1 = ax_1^r,\ y_2 = ax_2^r;\quad \Delta x = |x_2 - x_1| \ll x_1, x_2$$

We then have

$$\frac{\Delta y}{y_1} = \frac{\Delta y}{y_2} \equiv \frac{\Delta y}{y} = |r|\frac{\Delta x}{x_1} = |r|\frac{\Delta x}{x_2} \equiv |r|\frac{\Delta x}{x}$$

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We then have: $\dfrac{\Delta y}{y} = |r|\dfrac{\Delta x}{x}$

Example 1: $\Delta\lambda/\lambda = \Delta f/f$

Problem: In an optical fiber, a communication signal has a bandwidth of 10 nm at a wavelength of 1.5 μm and an index of refraction of 1.5; what is the bandwidth?

Solution: $\Delta f/f = \Delta\lambda/\lambda = 10/1500 = 6.7\times10^{-3}$, $f = c/\lambda = 2\times10^{15}$ Hz = 200 THz, $\Delta f = (6.7\times10^{-3}) \times 200 = 1.3$ THz

Example 2: $\Delta u_p/u_p = \Delta n/n$

Problem: In an optical fiber, the index of refraction is 1.5 and the birefringence is given by $\Delta n/n = 10^{-6}$; what is the difference in velocity between the two modes? How far do we propagate before a 200 THz signal slips by one period?

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Example 2: $\Delta z / z = \Delta u_p / u_p = \Delta n / n$

Problem: In an optical fiber, the index of refraction is 1.5 and the birefringence is given by $\Delta n / n = 10^{-6}$; what is the difference in velocity between the two modes? How far do we propagate before an optical signal at 200 THz slips by one period?

Solution: $u_p = c / n = 2.0\times10^8$ m/s; $\Delta u_p = (2.0\times10^8) \times 10^{-6} = 200$ m/s;
$\Delta z = u_p T = u_p / f = (2.0\times10^8) / (2.0\times10^{14}) = 10^{-6}$ m; $z = \Delta z\times(n/\Delta n) = 1$ m.
Alternatively: $\Delta z = \lambda / n$, so that $z = (\lambda / n) \times(n/\Delta n) = 1$ m.

Example 3: $\Delta n / n = \Delta\varepsilon_r / 2\varepsilon_r$

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Calculating Small Differences

Statement:

$$y = ax^r \quad \Rightarrow \quad \frac{\Delta y}{y} = |r| \frac{\Delta x}{x}$$

Proof:

From a first-order Taylor expansion, we have

$$y_2 = ax_1^r + rax_1^{r-1}(x_2 - x_1) + \text{ higher order terms} \simeq y_1 + r\frac{y_1}{x_1}(x_2 - x_1)$$

so that

$$\frac{|y_2 - y_1|}{y_1} = |r| \frac{|x_2 - x_1|}{x_1}$$

If we do the Taylor expansion, exchanging 1 and 2, we obtain the same result with 2 replacing 1.

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