CMPE 320: Probability, Statistics, and Random Processes

Lecture 2: Probabilistic Models

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Probabilistic model

- A mathematical description of an uncertain situation
- Elements of probabilistic model
  - **Sample space Ω**: set of all possible outcomes

2 coin tosses $\{HH, HT, TH, TT\}$

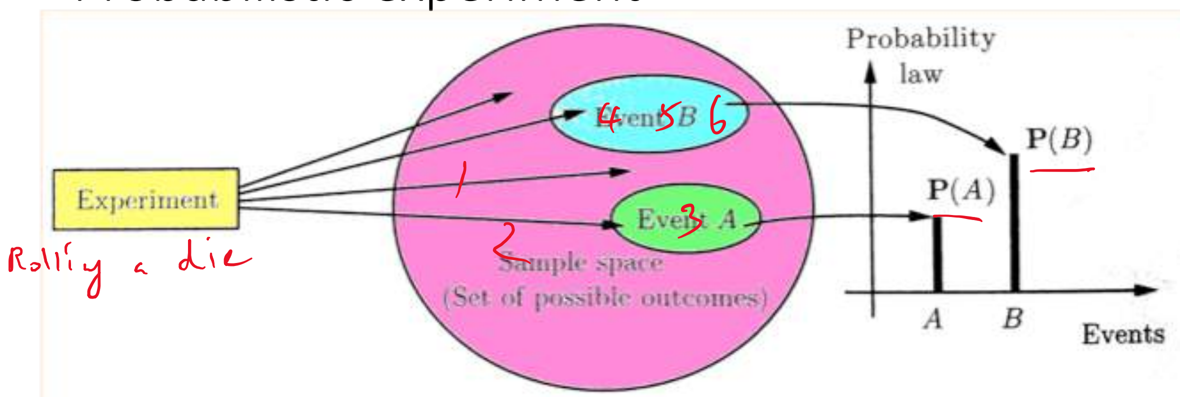
  - **Probability law P(A)**: assigns to a set of possible outcomes ("event") a nonnegative number ("probability")

Event that both coins yield heads $= \{HH\}$ $P(\{HH\}) = \frac{1}{4}$

Event that two coins have different faces $= \{HT, TH\}$

$P(\{HT, TH\}) = \frac{1}{2}$

Probabilistic experiment

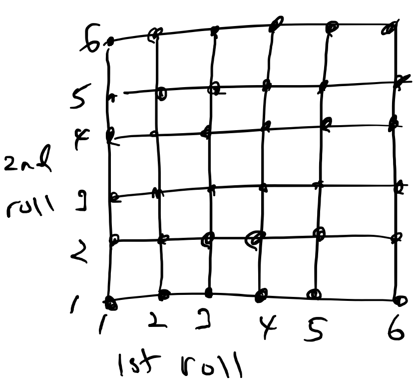


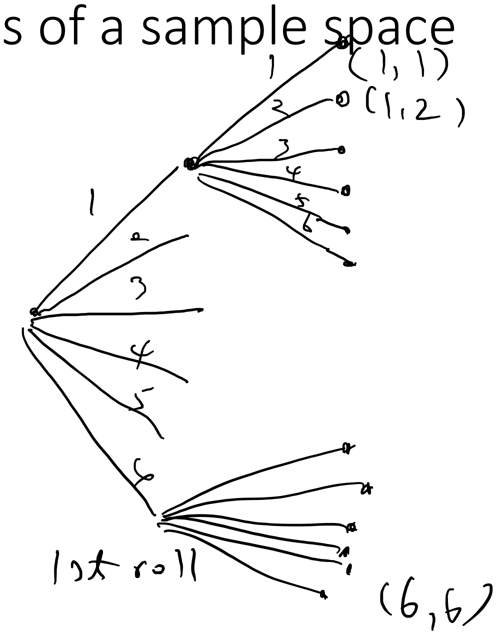
- An **experiment** produces one of possible outcomes
- An **event** is a collection of outcomes (including Ω and ∅)

Event A = roll of the die is 3
Event B = roll of the die ≥ 4

Equivalent descriptions of a sample space

- Two rolls of dice





Probability axioms

- Probability law P(A) specifies the likelihood of event A
- P must satisfy a certain set of rules called probability axioms

1) Nonnegativity $P(A) \geq 0$ for any event A
2) Additivity If A and B are disjoint $(A \cap B = \emptyset)$
   $P(A \cup B) = P(A) + P(B)$
3) Normalization
   $P(\Omega) = 1$ [probability of entire sample space must equal to 1]

All properties of probability are derived from the axioms

- P(Ø) = 0 $\quad \emptyset \cap \Omega = \emptyset \quad P(\underbrace{\emptyset \cup \Omega}_{\Omega}) = P(\emptyset) + P(\Omega)$
  $P(\Omega) = 1 = P(\emptyset) + \underbrace{P(\Omega)}_{1} \Rightarrow P(\emptyset) = 0$
- $P(A^c) = 1 - P(A)$: $A^c \cap A = \emptyset \Rightarrow P(\underbrace{A^c \cup A}) = P(A^c) + P(A) = 1$
- For disjoint events A_1, A_2, A_3: $P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3)$

$P(A_1 \cup A_2 \cup A_3) = P(A_1 \cup (A_2 \cup A_3)) = P(A_1) + P(A_2 \cup A_3)$

disjoint disjoint

$= P(A_1) + P(A_2) + P(A_3)$

Coin toss example

- Consider an experiment of a single coin toss
  - Sample space: $\Omega = \{H, T\}$
  - Set of events: $\{H\}, \{T\}, \{H, T\}, \emptyset$
- How should we define the probability law?
  - Let's assume that the coin toss is fair. $P(\{H\}) = P(\{T\})$ ✓

$P(\Omega) = P(\{H, T\}) = P(\{H\}) + P(\{T\}) = 1$ ✓

$P(\{H\}) = P(\{T\}) = 0.5$, $P(\Omega) = 1$, $P(\emptyset) = 0$

3 coin tosses

- Sample space : $\{HHH, HHT, HTH, THH, THT, TTH, HTT, TTT\}$
- Assume the coin tosses are fair ➔ Each outcome has probability: $\frac{1}{8}$
- What is the probability that exactly 2 heads occur?

$A = \{HHT, HTH, THH\}$

$P(A) = P(HHT) \overset{\text{disjoint}}{+} P(HTH) + P(THH) = \frac{3}{8}$

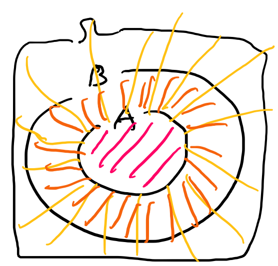
Discrete versus continuous

- A single spin of a wheel of fortune yields a number in interval [0,1]
- Assuming a fair wheel, the probability of single outcome must be 0 (why?)

If not, probability will blow up due to additivity axiom

- In this example, it makes sense to assign probability (b-a) to any interval [a,b], that is, probability is essentially measuring the length

Properties of probability laws



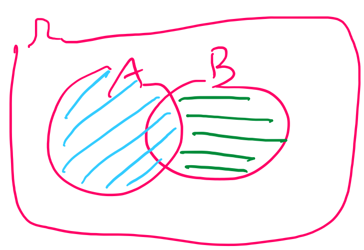
- If $A \subset B$, then $P(A) \leq P(B)$

$B = A \cup (A^c \cap B) \Rightarrow P(B) = P(A) + \overbrace{P(A^c \cap B)}^{\geq 0}$

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ ←

$A \cup B = A \cup (B \cap A^c)$

$(B \cap A^c) \cup (A \cap B) = B$



$P(A \cup B) = P(A) + P(B \cap A^c)$

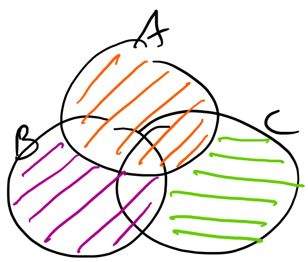
$P(B \cap A^c) + P(A \cap B) = P(B)$ (+

$P(A \cup B) + P(A \cap B) = P(A) + P(B)$ ←

- $P(A \cup B) \leq P(A) + P(B)$

$$P(A \cup B) = P(A) + P(B) - \underbrace{P(A \cap B)}_{\geq 0}$$

- $P(A \cup B \cup C) = P(A) + P(A^c \cap B) + P(A^c \cap B^c \cap C)$



- Express $P(A \cup B \cup C)$ in terms of P(A), P(B), P(C) and the probabilities of various intersections of A, B, and C

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$
$$- P(A \cap B) - P(B \cap C) - P(C \cap A)$$
$$+ P(A \cap B \cap C)$$