HW#1 Solutions

Problem 1.

The set $A \cap (A \cup B)^c$ is always empty, because it can be written as $A \cap A^c \cap B^c$ or $\emptyset \cap B^c$, since $A \cap A^c = \emptyset$.

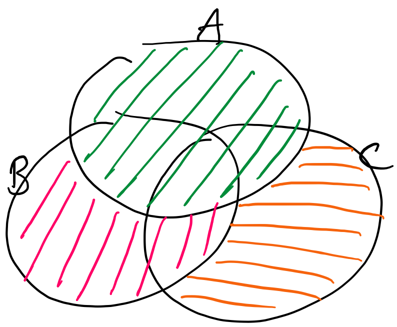
Problem 2. Express $A \cup B \cup C$, B, C and $B \cap C$ as unions of <u>disjoint</u> sets.

$$A \cup B \cup C = A \cup (A^c \cap B) \cap (A^c \cap B^c \cap C)$$

$$(A^c \cap B) \cup (A \cap B) = B$$

$$(A^c \cap B^c \cap C) \cup (A \cap C) \cup (A^c \cap B \cap C) = C$$

$$B \cap C = (A^c \cap B \cap C) \cup (A \cap B \cap C)$$



Using additivity axiom of probability:

$$P(A \cup B \cup C) + P(A \cap B) + P(A \cap C) + P(B \cap C) = P(A) + P(B) + P(C) + P(A \cap B \cap C)$$

Problem 3.

We have $\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B) = 1 - \mathbf{P}(A^c) + \mathbf{P}(B) - \mathbf{P}(A \cap B) = 1 - 0.6 + 0.3 - 0.2 = 0.5$.

Problem 4.

There are 12 possible outcomes,

$$(1,2), (1,3), (1,4), (2,1), (2,3), (2,4), (3,1), (3,2), (3,4), (4,1), (4,2), (4,3),$$

and each has probability $1/12$. The required probabilities are obtained by counting the number of outcomes that satisfy the corresponding criterion and dividing by 12. The answers are: (a) $1/2$. (b) $1/6$. (c) $1/3$.

Problem 5、

We claim that the optimal order is to play the weakest player second (the order in which the other two opponents are played makes no difference). To see this, let p_i be the probability of winning against the opponent played in the ith turn. Then you will win the tournament if you win against the 2nd player (prob. p_2) and also you win against at least one of the two other players [prob. $p_1 + (1 - p_1)p_3 = p_1 + p_3 - p_1p_3$]. Thus the probability of winning the tournament is

$$p_2(p_1 + p_3 - p_1p_3).$$

The order $(1, 2, 3)$ is optimal if and only if the above probability is no less than the probabilities corresponding to the two alternative orders:

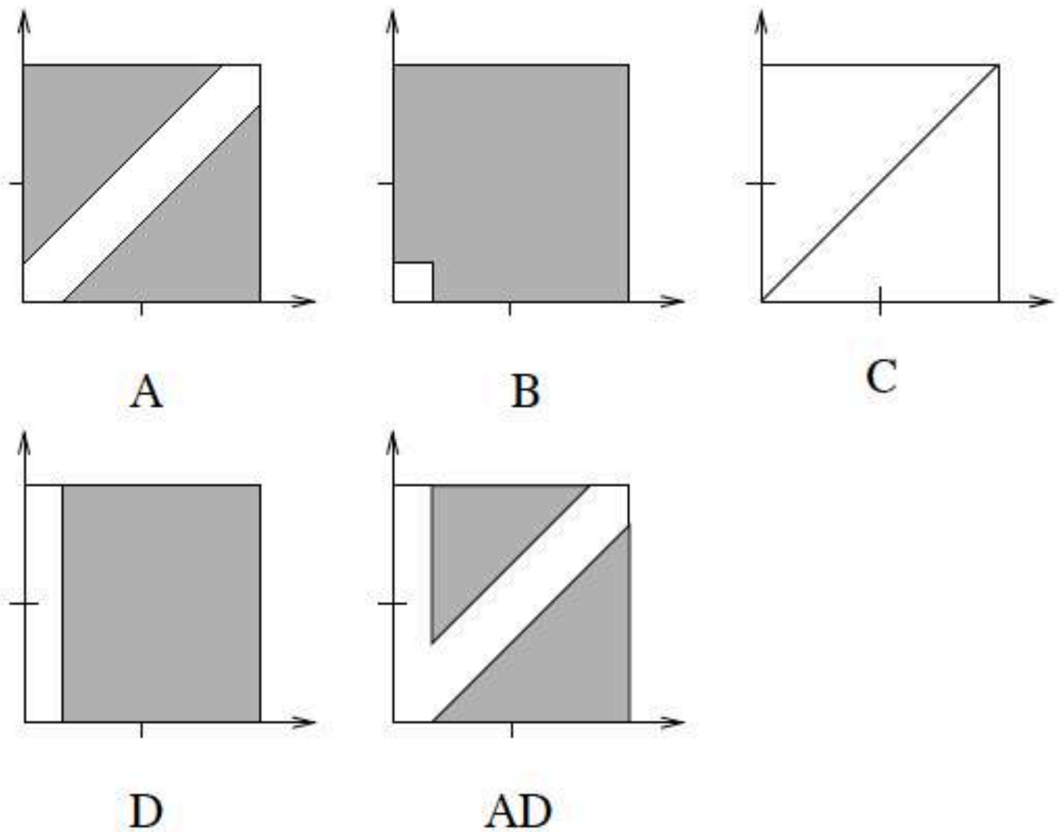
$$p_2(p_1 + p_3 - p_1p_3) \geq p_1(p_2 + p_3 - p_2p_3),$$

$$p_2(p_1 + p_3 - p_1p_3) \geq p_3(p_2 + p_1 - p_2p_1).$$

It can be seen that the first inequality above is equivalent to $p_2 \geq p_1$, while the second inequality above is equivalent to $p_2 \geq p_3$.

Problem 6.

Each of the events of interest can be described by a region in two-dimensional space, where the horizontal axis is Alice's number and the vertical axis is Bob's number; see the accompanying figure.



Note that the events A and $A \cap B$ are identical because A is a subset of B. We are given that the probability of any event is proportional to its area. Since the sample space has an area of 4, the probability of each event must be equal to its area divided by 4 to satisfy the normalization axiom. We then obtain

$$\mathbf{P}(A) = \frac{25}{36}, \quad \mathbf{P}(B) = \frac{35}{36}, \quad \mathbf{P}(A \cap B) = \frac{25}{36}, \quad \mathbf{P}(C) = 0, \quad \mathbf{P}(D) = \frac{5}{6}, \quad \mathbf{P}(A \cap D) = \frac{41}{72}.$$

Problem 7.

Let A be the event that your friend searches disc 1 and finds nothing, and let B_i be the event that your thesis is on disc i. Now note that for $i \neq 1$, we have $\mathbf{P}(A \cap B_i) = \mathbf{P}(B_i)$, so that

$$\mathbf{P}(B_i \mid A) = \frac{\mathbf{P}(A \cap B_i)}{\mathbf{P}(A)} = \frac{\mathbf{P}(B_i)}{\mathbf{P}(A)} = \frac{0.25}{1 - 0.25 \cdot 0.4} = 0.2777.$$

For $i = 1$, we have

$$\mathbf{P}(B_i \mid A) = \frac{\mathbf{P}(A \cap B_i)}{\mathbf{P}(A)}.$$

Since $\mathbf{P}(A \mid B_i) = \mathbf{P}(A \cap B_i)/\mathbf{P}(B_i)$, we have

$$\mathbf{P}(A \cap B_i) = \mathbf{P}(B_i)\mathbf{P}(A \mid B_i) = 0.25(1 - 0.4).$$

Therefore,

$$\mathbf{P}(B_i \mid A) = \frac{0.25(1 - 0.4)}{1 - 0.25 \cdot 0.4} = 0.1666.$$

Problem 8.

The probability of dialing correctly with less or equal to k tries is 1 minus the probability of dialing incorrectly with k successive tries. The latter probability is

$$\frac{9}{10}\frac{8}{9}\cdots\frac{10-k}{11-k},$$

since $\frac{10-i}{11-i}$ is the conditional probability that the ith try is unsuccessful, given that the preceding $i-1$ tries are unsuccessful. Therefore, the desired number of tries is the smallest number $k \geq 1$ such that

$$\frac{9}{10}\frac{8}{9}\cdots\frac{10-k}{11-k} < 0.5.$$

Problem 9.

Let A be the event that the student is not overstressed, and let A^c be the event that the student is in fact overstressed. Now let B be the event that the test results indicate that the student is not overstressed. The desired probability, $\mathbf{P}(A \mid B)$, is found by Bayes' rule:

$$\mathbf{P}(A \mid B) = \frac{\mathbf{P}(A)\mathbf{P}(B \mid A)}{\mathbf{P}(A)\mathbf{P}(B \mid A) + \mathbf{P}(A^c)\mathbf{P}(B \mid A^c)} = \frac{0.005 \cdot 0.95}{0.005 \cdot 0.95 + 0.995 \cdot 0.15} \approx 0.03.$$

Problem 10,

Let A_i be the event corresponding to starting with trail i. We have

$$\mathbf{P}(A_i) = \frac{1}{n}, \qquad i = 1, 2, \ldots, n.$$

Let also B be the event of reaching the destination. We have

$$\mathbf{P}(B \mid A_i) = \frac{1}{1+i}, \qquad i = 1, 2, \ldots, n.$$

Thus, by the total probability theorem, the probability of reaching the destination is

$$\mathbf{P}(B) = \sum_{i=1}^{n} \mathbf{P}(A_i)\mathbf{P}(B \mid A_i) = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{1+i}.$$