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MATH 407 - Modern Algebra and Number Theory: HW 06

2.3 **1** Consider the following permutations in S_7

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 5 & 4 & 6 & 1 & 7 \end{pmatrix} \qquad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 5 & 7 & 4 & 6 & 3 \end{pmatrix}$$

Compute the following products:

b $\tau\sigma$

Ans

$$\tau\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 5 & 4 & 6 & 1 & 7 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 1 & 5 & 7 & 4 & 6 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 6 & 7 & 4 & 1 & 5 \end{pmatrix}$$

□

f $\tau^{-1}\sigma\tau$

Ans

$$\tau^{-1} = \begin{pmatrix} 2 & 1 & 5 & 7 & 4 & 6 & 3 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 & 6 & 7 & 4 & 1 & 5 \\ 1 & 7 & 6 & 4 & 5 & 2 & 3 \end{pmatrix}$$

$$\tau^{-1}\sigma = \begin{pmatrix} 2 & 3 & 6 & 7 & 4 & 1 & 5 \\ 1 & 7 & 6 & 4 & 5 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 5 & 4 & 6 & 1 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 & 6 & 7 & 4 & 1 & 5 \\ \\ 3 & 7 & 1 & 4 & 6 & 2 & 5 \end{pmatrix}$$

$$\tau^{-1}\sigma\tau = \begin{pmatrix} 2 & 3 & 6 & 7 & 4 & 1 & 5 \\ \\ 3 & 7 & 1 & 4 & 6 & 2 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \\ 3 & 2 & 5 & 4 & 6 & 1 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & 3 & 6 & 7 & 4 & 1 & 5 \\ \\ 5 & 7 & 3 & 4 & 1 & 2 & 6 \end{pmatrix}$$

□

3 Write $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \\ 3 & 4 & 10 & 5 & 7 & 8 & 2 & 6 & 9 & 1 \end{pmatrix}$ as a product of disjoint cycles and as a product of transpositions. Construct its associated diagram, find its inverse, and find its order.

Ans The product of disjoint cycles:

$$\{(1,3,10)(2,4,5,7)(6,8)\}$$

The product of transpositions:

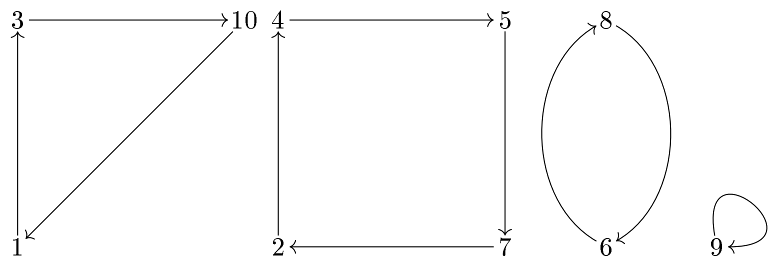
$$\{(1,3,10)(2,4,5,7)(6,8)\} = \{(1,3)(3,10)(2,4)(4,5)(5,7)(6,8)\}$$

Reconstructing the permutation based on the product of transpositions:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \\ 3 & 4 & 10 & 5 & 7 & 8 & 2 & 6 & 9 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 & 10 & 2 & 4 & 5 & 7 & 6 & 8 & 9 \\ \\ 3 & 10 & 1 & 4 & 5 & 7 & 2 & 8 & 6 & 9 \end{pmatrix}$$

Constructing the associated diagrams



The inverse of the permutation:

$$\sigma^{-1} = \begin{pmatrix} 3 & 4 & 10 & 5 & 7 & 8 & 2 & 6 & 9 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 10 & 7 & 1 & 2 & 4 & 8 & 5 & 6 & 9 & 3 \end{pmatrix}$$

Since

$$\begin{aligned} o(\sigma) &= lcm(\text{length(cycles)}) \\ &= lcm(\{2,4,3\}) \\ &= 12 \end{aligned}$$

□

5 Let $3 \leq m \leq n$. Calculate $\sigma\tau^{-1}$ for the cycles $\sigma = (1, 2, \ldots, m-1)$ and $\tau = (1, 2, \ldots, m-1, m)$ in S_n.

Ans Given

$$\tau = \begin{pmatrix} 1 & 2 & 3 & \ldots & \text{m-2} & \text{m-1} & \text{m} \\ 2 & 3 & 4 & \ldots & \text{m-1} & \text{m} & 1 \end{pmatrix}$$

$$\tau^{-1} = \begin{pmatrix} 2 & 3 & 4 & \ldots & \text{m-1} & \text{m} & 1 \\ 1 & 2 & 3 & \ldots & \text{m-2} & \text{m-1} & \text{m} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & \ldots & \text{m-1} & \text{m} \\ \text{m} & 1 & 2 & 3 & \ldots & \text{m-2} & \text{m-1} \end{pmatrix}$$

Therefore the product

$$\sigma\tau^{-1} = \begin{pmatrix} 1 & 2 & 3 & \dots & \text{m-2} & \text{m-1} \\ 2 & 3 & 4 & \dots & \text{m-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & \dots & \text{m-1} & \text{m} \\ \text{m} & 1 & 2 & \dots & \text{m-2} & \text{m-1} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & \dots & \text{m-1} & \text{m} \\ \text{m} & 2 & 3 & \dots & \text{m-1} & 1 \end{pmatrix}$$

$$= (1, m)$$ □

11 Prove that in S_n, with $n \geq 3$, any even permutation is a product of cycles of length three.

Hint: $(a, b)(b, c) = (a, b, c)$ and $(a, b)(c, d) = (a, b, c)(b, c, d)$.

Ans To show any even permutation in S_n, with $n \geq 3$ is a product of cycles of length three, consider the case where the pair of transpositions are disjoint:

$$(a, b)(c, d) = (a, b, c)(b, c, d)$$

The product yields cycles of length three.

The other case would be the transpositions consisting of repeating elements, such as

$$(a, b)(b, c) = (a, b, c)$$

where the product yields cycles of length three as well. □

15 For $\alpha, \beta \in S_n$, let $\alpha \sim \beta$ if there exists $\sigma \in S_n$ such that $\sigma\alpha\sigma^{-1} = \beta$. Show that $\sim$ is an equivalence relation on S_n.

Ans To prove **reflexivity**, let $\alpha = 1_S$

Then,

$$\begin{aligned} \sigma\alpha\sigma^{-1} &= \sigma\sigma^{-1} \cdot 1_S \\ &= 1 \cdot 1_S \\ &= \alpha \end{aligned}$$

for any $\alpha \in S_n$

Therefore, $\alpha \sim \alpha$

To prove **symmetry**, let $\alpha \sim \beta$

Then, there exists $\sigma \in S_n$ such that $\sigma\alpha\sigma^{-1} = \beta$

Thus

$$\begin{aligned} \sigma\alpha\sigma^{-1} &= \beta \\ \Rightarrow \sigma^{-1}\sigma\alpha\sigma^{-1} &= \sigma^{-1}\beta \\ \alpha\sigma^{-1} &= \sigma^{-1}\beta \\ \Rightarrow \alpha\sigma^{-1}\sigma &= \sigma^{-1}\beta\sigma \\ \alpha &= \sigma^{-1}\beta\sigma \\ \Rightarrow \alpha &= \sigma^{-1}\beta(\sigma^{-1})^{-1} \end{aligned}$$

Or $\sigma^{-1}\beta(\sigma^{-1})^{-1} = \alpha$

Implying $\beta \sim \alpha$

The prove **transitivity**, let $\alpha \sim \beta$ and $\beta \sim \gamma$

Then, there exists $\sigma_1, \sigma_2 \in S_n$ such that $\sigma_1\alpha\sigma_1^{-1} = \beta$ and $\sigma_2\beta\sigma_2^{-1} = \gamma$

Then $\sigma_2\sigma_1\alpha\sigma_1^{-1}\sigma_2^{-1} = \gamma$,

Or $(\sigma_2\sigma_1)\alpha(\sigma_1\sigma_2)^{-1} = \gamma$

Thus, $\alpha \sim \gamma$

Therefore, $\sim$ is an equivalence relation on S_n $\square$

16 View S_3 as a subset of S_5, in the obvious way. For $\sigma, \tau \in S_5$, define $\sigma \sim \tau$ if $\sigma\tau^{-1} \in S_3$.

a Show that $\sim$ is an equivalence relation on S_5.

Ans To prove **reflexive**, let $\sigma \in S_3$

Then,

$$\sigma\sigma^{-1} = 1_{S_3} \in S_3$$

Thus, $\sigma \sim \sigma$

To prove **symmetric**, let $\sigma, \tau \in S_3$

Then

$$\sigma \sim \tau \in S_3$$

$$\Rightarrow \sigma\tau^{-1} \in S_3$$
$$\Rightarrow (\sigma\tau^{-1})^{-1} \in S_3$$
$$\Rightarrow \sigma^{-1}\tau \in S_3$$

Thus, $\tau \sim \sigma$

To prove **transitivity**, let $\sigma, \tau, \upsilon \in S_3$.
Then, $\sigma \sim \tau \Rightarrow \sigma\tau^{-1} \in S_3$
and $\tau \sim \upsilon \Rightarrow \tau\upsilon^{-1} \in S_3$
Thus,

$$(\sigma\tau^{-1})(\tau\upsilon^{-1}) \in S_3$$
$$\sigma(\tau^{-1}\tau)\upsilon^{-1} \in S_3$$
$$\sigma\upsilon^{-1} \in S_3$$
$$\Rightarrow \sigma \sim \upsilon \in S_3$$

Therefore, $\sim$ is an equivalence relation on S_5 □

b Find the equivalence class of (4, 5).

Ans Since $(4,5) \in S_3$, $\Rightarrow (4,5)(5,4) \in S_3$ and $(4,5) \sim (4,5)$
Similarly,

$$(4,5)(1,2,3)(5,4), (4,5)(1,3,2)(5,4) \in S_3$$

and

$$(4,5)(1,2)(5,4), (4,5)(1,3)(5,4), (4,5)(2,3)(5,4) \in S_3$$

Therefore,

$$[(4,5)] = \{(4,5), (1,2,3)(4,5), (1,3,2)(4,5), (1,2)(4,5), (1,3)(4,5), (2,3)(4,5)\}$$ □

c Find the equivalence class of (1, 2, 3, 4, 5).

Ans Since

$$(1,2,3,4,5) = (1,2)(1,3)(1,4)(1,5)$$
and $(1,2)(1,3)(1,4)(4,1)(5,1) = (1,2)(1,3)$

$$\in S_3$$

Then $(1,2,3,4,5) \sim (1,4)(1,5)$

Similarly,

$$\begin{aligned}\{(1,2)(1,3)(1,4)(4,1)(5,1)(1,3,2),\\ (1,2)(1,3)(1,4)(4,1)(5,1)(1,2),\\ (1,2)(1,3)(1,4)(4,1)(5,1)(1,3),\\ (1,2)(1,3)(1,4)(4,1)(5,1)(2,3),\\ (1,2)(1,3)(1,4)(4,1)(5,1)(1,2,3)\} \in S_3\end{aligned}$$

Therefore,

$$\begin{aligned}[(1,2,3,4,5)] = \{&(1,4)(1,5)(1,3,2), (1,4)(1,5)(1,2),\\ &(1,4)(1,5)(1,3), (1,4)(1,5)(2,3), (1,4)(1,5)(1,2,3)\}\end{aligned} \quad \square$$

d Determine the total number of equivalence classes.

Ans S_3 contains $3! = 6$ elements, and S_5 contains $5! = 120$ elements

Therefore, the number of equivalence classes are $120/6 = 20$ $\square$