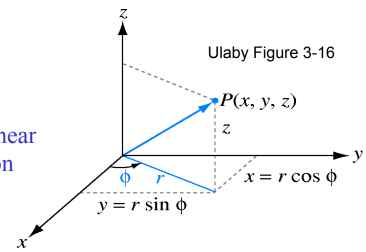
Coordinate Transformations

Cartesian ↔ Cylindrical

Coordinates

$$r = \sqrt{x^2 + y^2}, \quad \phi = \tan^{-1}(y/x), \quad x = r\cos\phi, \quad y = r\sin\phi$$

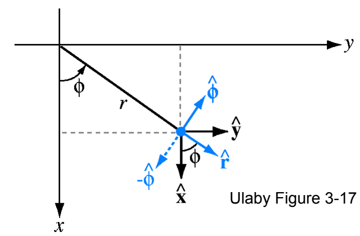
Nonlinear relation



Basis vectors

$$\hat{\mathbf{r}}\cdot\hat{\mathbf{x}} = \cos\phi, \quad \hat{\mathbf{r}}\cdot\hat{\mathbf{y}} = \sin\phi, \quad \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{x}} = -\sin\phi, \quad \hat{\boldsymbol{\phi}}\cdot\hat{\mathbf{y}} = \cos\phi$$

Linear rotation



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Coordinate Transformations

Cartesian ↔ Cylindrical

Basis vectors

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi, \quad \hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi \quad \text{[forward rotation]}$$

$$\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi, \quad \hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi \quad \text{[backward rotation]}$$

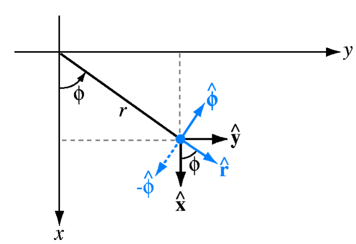
Arbitrary vector components transform similarly

$$A_r = A_x\cos\phi + A_y\sin\phi,$$

$$A_\phi = -A_x\sin\phi + A_y\cos\phi$$

$$A_x = A_r\cos\phi - A_\phi\sin\phi,$$

$$A_y = A_r\sin\phi + A_\phi\cos\phi$$



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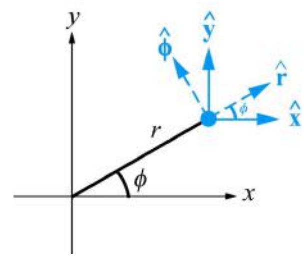
Coordinate Transformations

Example: Polar Coordinates

Question: In two dimensions (where all scalar and vector fields do not depend on z), we relate the Cartesian coordinates (x,y) to the polar coordinates (r,ϕ) using the relations $r=(x^2+y^2)^{1/2}$, $\phi=\tan^{-1}(y/x)$. What are $x(r,\phi)$ and $y(r,\phi)$? Write $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ as functions of r, ϕ, $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\phi}}$.

Answer: We have $x=r\cos\phi$, $y=r\sin\phi$. From the figure, we infer

$$\hat{\mathbf{x}}=\hat{\mathbf{r}}\cos\phi-\hat{\boldsymbol{\phi}}\sin\phi,$$
$$\hat{\mathbf{y}}=\hat{\mathbf{r}}\sin\phi+\hat{\boldsymbol{\phi}}\cos\phi.$$



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7.3

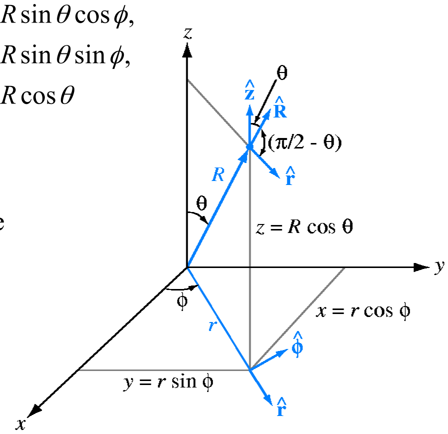
Coordinate Transformations

Cartesian ⟷ Spherical

Coordinates

$$R=\sqrt{x^2+y^2+z^2},\qquad x=R\sin\theta\cos\phi,$$
$$\theta=\tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right),\qquad y=R\sin\theta\sin\phi,$$
$$\phi=\tan^{-1}(y/x)\qquad z=R\cos\theta$$

From here, the basis vectors and the vector representations can be found



Ulaby Figure 3-18

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7.4

Coordinate Transformations

Cartesian ⟷ Spherical

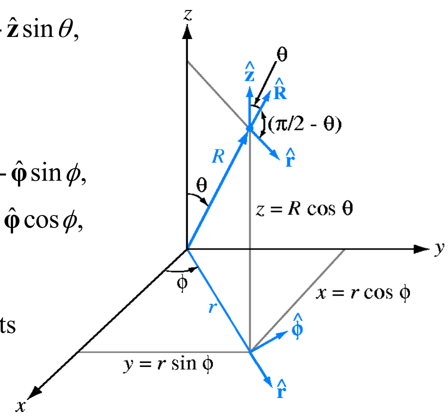
Using the geometry we find

$$\hat{\mathbf{R}} = \hat{\mathbf{x}}\sin\theta\cos\phi + \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta,$$
$$\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}}\cos\theta\cos\phi + \hat{\mathbf{y}}\cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta,$$
$$\hat{\boldsymbol{\varphi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$$

Inverting these relations we obtain

$$\hat{\mathbf{x}} = \hat{\mathbf{R}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\varphi}}\sin\phi,$$
$$\hat{\mathbf{y}} = \hat{\mathbf{R}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\varphi}}\cos\phi,$$
$$\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$$

Arbitrary vector components again transform similarly



Ulaby Figure 3-18

Coordinate Transformation Relations

Transformations	Coordinate Variables	Unit Vectors	Vector Components
Cartesian to Cylindrical	$r = \sqrt{x^2+y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$ $\hat{\boldsymbol{\varphi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_r = A_x\cos\phi + A_y\sin\phi$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$ $A_z = A_z$
Cylindrical to Cartesian	$x = r\cos\phi$ $y = r\sin\phi$ $z = z$	$\hat{\mathbf{x}} = \hat{\mathbf{r}}\cos\phi - \hat{\boldsymbol{\varphi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\phi + \hat{\boldsymbol{\varphi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_x = A_r\cos\phi - A_\phi\sin\phi$ $A_y = A_r\sin\phi + A_\phi\cos\phi$ $A_z = A_z$
Cylindrical to Spherical	$R = \sqrt{r^2+z^2}$ $\theta = \tan^{-1}(r/z)$ $\phi = \phi$	$\hat{\mathbf{R}} = \hat{\mathbf{r}}\sin\theta + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{r}}\cos\theta - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}}$	$A_R = A_r\sin\theta + A_z\cos\theta$ $A_\theta = A_r\cos\theta - A_z\sin\theta$ $A_\phi = A_\phi$
Spherical to Cylindrical	$r = R\sin\theta$ $\phi = \phi$ $z = R\cos\theta$	$\hat{\mathbf{r}} = \hat{\mathbf{R}}\sin\theta + \hat{\boldsymbol{\theta}}\cos\theta$ $\hat{\boldsymbol{\varphi}} = \hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_r = A_R\sin\theta + A_\theta\cos\theta$ $A_\phi = A_\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$

Coordinate Transformation Relations

Transformations	Coordinate Variables	Unit Vectors	Vector Components
Cartesian to Spherical	$R = \sqrt{x^2+y^2+z^2}$ $\theta = \tan^{-1}\left[\sqrt{x^2+y^2}/z\right]$ $\phi = \tan^{-1}(y/x)$	$\hat{\mathbf{R}} = \hat{\mathbf{x}}\sin\theta\cos\phi + \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}}\cos\theta\cos\phi + \hat{\mathbf{y}}\cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$	$A_R = A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta$ $A_\theta = A_x\cos\theta\cos\phi + A_y\cos\theta\sin\phi - A_z\sin\theta$ $A_\phi = -A_x\sin\phi + A_y\cos\phi$
Spherical to Cartesian	$x = R\sin\theta\cos\phi$ $y = R\sin\theta\sin\phi$ $z = R\cos\theta$	$\hat{\mathbf{x}} = \hat{\mathbf{R}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{R}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{R}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$	$A_x = A_R\sin\theta\cos\phi + A_\theta\cos\theta\cos\phi - A_\phi\sin\phi$ $A_y = A_R\sin\theta\sin\phi + A_\theta\cos\theta\sin\phi + A_\phi\sin\phi$ $A_z = A_R\cos\theta - A_\theta\sin\theta$

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7.7

Coordinate Transformations

Example: Ulaby and Ravaioli 3-8

Question: Express the vector $\mathbf{A} = \hat{\mathbf{x}}(x+y) + \hat{\mathbf{y}}(y-x) + \hat{\mathbf{z}}\,z$ in spherical coordinates.

Answer: We have

$$\begin{aligned} A_R &= A_x\sin\theta\cos\phi + A_y\sin\theta\sin\phi + A_z\cos\theta \\ &= (x+y)\sin\theta\cos\phi + (y-x)\sin\theta\sin\phi + z\cos\theta \\ &= (R\sin\theta\cos\phi + R\sin\theta\sin\phi)\sin\theta\cos\phi \\ &\qquad +(R\sin\theta\sin\phi - R\sin\theta\cos\phi)\sin\theta\sin\phi + R\cos^2\theta \\ &= R\sin^2\theta + R\cos^2\theta = R \end{aligned}$$

Similarly, we have $A_\theta = 0$, $A_\phi = -R\sin\theta$

Collecting terms, we conclude

$$\mathbf{A} = \hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi = \hat{\mathbf{R}}R - \hat{\boldsymbol{\phi}}R\sin\theta$$

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7.8

Distance Transformations

From the basic Cartesian relation

$$d = |\mathbf{R}_{12}| = \left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]^{1/2}$$

we obtain in cylindrical coordinates

$$d = \left[(r_2 \cos\phi_2 - r_1 \cos\phi_1)^2 + (r_2 \sin\phi_2 - r_1 \sin\phi_1)^2 + (z_2 - z_1)^2\right]^{1/2}$$

$$= \left[r_2^2 + r_1^2 - 2r_1 r_2 \cos(\phi_2 - \phi_1) + (z_2 - z_1)^2\right]^{1/2}$$

and in spherical coordinates

$$d = \left\{R_2^2 + R_1^2 - 2R_1 R_2 \left[\cos\theta_2 \cos\theta_1 + \sin\theta_2 \sin\theta_1 \cos(\phi_2 - \phi_1)\right]\right\}^{1/2}$$

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7.9

Vector Differential Operators

There are three vector differential operators that are important

Gradient:	Operates on a scalar and produces a vector
Divergence:	Operates on a vector and produces a scalar
Curl:	Operates on a vector and produces a vector

These operators operate in the three-dimensional space that we inhabit!

These operators generalize in different ways the elementary notion of a derivative acting in one dimension

$$\frac{dV(x)}{dx} = \underbrace{\lim}_{\Delta x \to 0} \frac{V(x + \Delta x) - V(x)}{\Delta x}$$

or the partial derivative of one variable in three dimensions

$$\frac{\partial V(x,y,z)}{\partial x} = \underbrace{\lim}_{\Delta x \to 0} \frac{V(x + \Delta x, y, z) - V(x,y,z)}{\Delta x}$$

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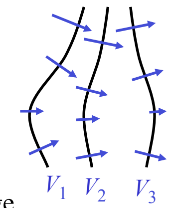
7.10

Gradient

Definition: For a scalar $V(x, y, z)$

$$\nabla V(x,y,z) = \hat{\mathbf{x}}\frac{\partial V}{\partial x} + \hat{\mathbf{y}}\frac{\partial V}{\partial y} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}$$



Physical meaning:

The direction is orthogonal to the "contour lines" of constant V. The magnitude gives the rate of change along the vector (is inversely proportional to the separation of the contour lines).

A more general definition

$$\nabla V(\mathbf{R}) = \hat{\mathbf{n}} \underbrace{\lim}_{\Delta\mathbf{l}\to 0}\left[\frac{V(\mathbf{R}+\Delta\mathbf{l}) - V(\mathbf{R})}{|\Delta\mathbf{l}|}\right]_{\max}, \text{ where } \hat{\mathbf{n}} = \Delta\mathbf{l}/|\Delta\mathbf{l}|$$

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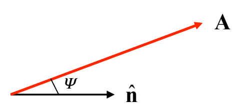
Gradient

From the more general definition

$$\nabla V(\mathbf{R}) = \hat{\mathbf{n}} \underbrace{\lim}_{\Delta\mathbf{l}\to 0}\left[\frac{V(\mathbf{R}+\Delta\mathbf{l}) - V(\mathbf{R})}{|\Delta\mathbf{l}|}\right]_{\max}, \text{ where } \hat{\mathbf{n}} = \Delta\mathbf{l}/|\Delta\mathbf{l}|$$

Prove

$$\nabla V(x,y,z) = \hat{\mathbf{x}}\frac{\partial V}{\partial x} + \hat{\mathbf{y}}\frac{\partial V}{\partial y} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}$$



Proof: Let

$$\mathbf{A} = \hat{\mathbf{x}}\frac{\partial V}{\partial x} + \hat{\mathbf{y}}\frac{\partial V}{\partial y} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}$$

We have

$$\frac{V(\mathbf{R}+\Delta\mathbf{l}) - V(\mathbf{R})}{|\Delta\mathbf{l}|} = \frac{\partial V(\mathbf{R})}{\partial x}\frac{\Delta l_x}{|\Delta\mathbf{l}|} + \frac{\partial V(\mathbf{R})}{\partial y}\frac{\Delta l_y}{|\Delta\mathbf{l}|} + \frac{\partial V(\mathbf{R})}{\partial z}\frac{\Delta l_z}{|\Delta\mathbf{l}|} = \mathbf{A}\cdot\hat{\mathbf{n}} = A\cos\psi$$

The maximum value is A and occurs when $\hat{\mathbf{n}} = \mathbf{A}/A$

We thus find $\nabla V(\mathbf{R}) = (\mathbf{A}/A)A = \mathbf{A}$ Q.E.D.

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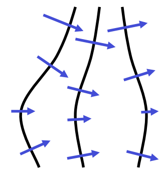
Gradient

Why is this important?

(1) Charges create a voltage field (electrostatic potential) $V(x, y, z)$

(2) The gradient of the voltage equals the negative of the electric field $-\mathbf{E}(x, y, z)$*

(3) The electric field exerts a force on charges



Line Integral Theorem

$$\int_{\mathbf{R}_1}^{\mathbf{R}_2} \left[\nabla V(\mathbf{R})\right] \cdot d\mathbf{l} = V(\mathbf{R}_2) - V(\mathbf{R}_1)$$

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*More precisely, the electrostatic potential generates the static portion of the electric field; there can also be a portion that is generated by the time variations of the magnetic field.

7.13

Gradient

Expression in cylindrical coordinates

$$\nabla V(r,\phi,z) = \hat{\mathbf{r}}\frac{\partial V}{\partial l_r} + \hat{\boldsymbol{\varphi}}\frac{\partial V}{\partial l_\phi} + \hat{\mathbf{z}}\frac{\partial V}{\partial l_z} = \hat{\mathbf{r}}\frac{\partial V}{\partial r} + \hat{\boldsymbol{\varphi}}\frac{1}{r}\frac{\partial V}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}$$

Expression in spherical coordinates

$$\nabla V(R,\theta,\phi) = \hat{\mathbf{R}}\frac{\partial V}{\partial l_R} + \hat{\boldsymbol{\theta}}\frac{\partial V}{\partial l_\theta} + \hat{\boldsymbol{\varphi}}\frac{\partial V}{\partial l_\phi} = \hat{\mathbf{R}}\frac{\partial V}{\partial R} + \hat{\boldsymbol{\theta}}\frac{1}{R}\frac{\partial V}{\partial \theta} + \hat{\boldsymbol{\varphi}}\frac{1}{R\sin\theta}\frac{\partial V}{\partial \phi}$$

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7.14

Divergence

Definition: For a vector $\mathbf{E}(x, y, z)$

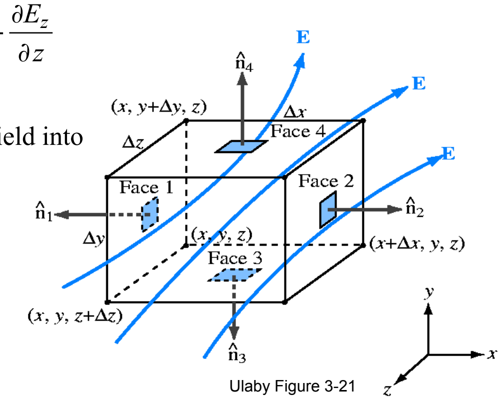
$$\nabla \cdot \mathbf{E}(x,y,z) = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

Physical meaning:

The net flow of the vector field into a small volume.

A more general definition

$$\nabla \cdot \mathbf{E} = \underbrace{\lim}_{\Delta v \to 0} \frac{\oint_S \mathbf{E} \cdot d\mathbf{s}}{\Delta v}$$



Ulaby Figure 3-21

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7.15

Divergence

Why is this important?

The charge density at any point in space equals the divergence of the electric field

Divergence Theorem (Gauss's Theorem):

Inverting the general definition of the divergence

$$\int_V \nabla \cdot \mathbf{E}\, dv = \oint_S \mathbf{E} \cdot d\mathbf{s}$$

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7.16

Divergence

Expression in cylindrical coordinates:

From the general definition, we find

$$
\begin{aligned}
\nabla\cdot\mathbf{E} = \underbrace{\lim}_{\Delta v\to 0}\frac{1}{\Delta v}\Big\{&\big[E_r(r+\Delta l_r,\phi,z)\Delta s_r(r+\Delta l_r,\phi,z)-E_r(r,\phi,z)\Delta s_r(r,\phi,z)\big]\\
&+\big[E_\phi(r,\phi+\Delta l_\phi,z)\Delta s_\phi(r,\phi+\Delta l_\phi,z)-E_\phi(r,\phi,z)\Delta s_\phi(r,\phi,z)\big]\\
&+\big[E_z(r,\phi,z+\Delta l_z)\Delta s_z(r,\phi,z+\Delta l_z)-E_z(r,\phi,z)\Delta s_z(r,\phi,z)\big]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
= \underbrace{\lim}_{\Delta r,\Delta\phi,\Delta z\to 0}\frac{1}{r\,\Delta r\,\Delta\phi\,\Delta z}\Big\{&\big[E_r(r+\Delta r,\phi,z)(r+\Delta r)\Delta\phi\,\Delta z-E_r(r,\phi,z)r\,\Delta\phi\,\Delta z\big]\\
&+\big[E_\phi(r,\phi+r\,\Delta\phi,z)\Delta r\,\Delta z-E_\phi(r,\phi,z)\Delta r\,\Delta z\big]\\
&+\big[E_z(r,\phi,z+\Delta z)r\,\Delta r\,\Delta z-E_z(r,\phi,z)r\,\Delta r\,\Delta z\big]\Big\}
\end{aligned}
$$

$$
=\frac{1}{r}\frac{\partial rE_r}{\partial r}+\frac{1}{r}\frac{\partial E_\phi}{\partial\phi}+\frac{\partial E_z}{\partial z}
$$

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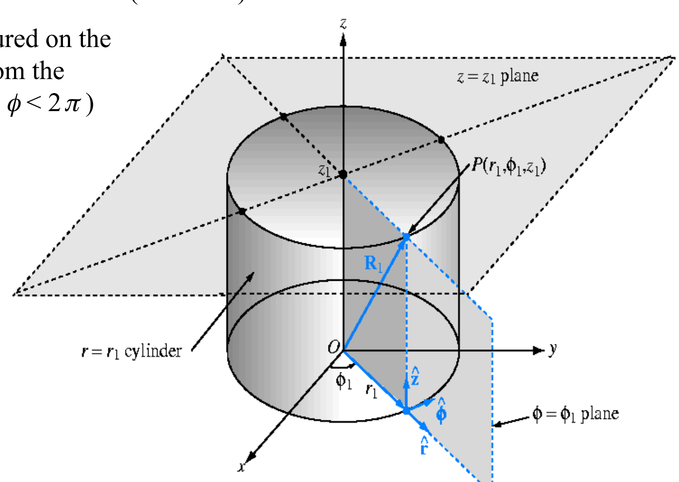
7.17

Cylindrical Coordinates

r = distance from the z-axis ($0 < r < \infty$)

ϕ = angle measured on the x-y plane from the $+x$-axis ($0 < \phi < 2\pi$)



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Ulaby Figure 3-9

7.18

Curl

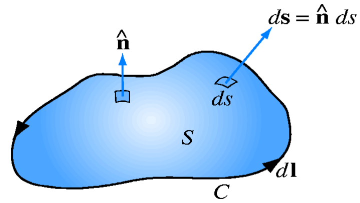
Definition: For a vector $\mathbf{B}(x, y, z)$

$$\nabla \times \mathbf{B}(x,y,z) = \hat{\mathbf{x}}\left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right) + \hat{\mathbf{y}}\left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right) + \hat{\mathbf{z}}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)$$

$$= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

Physical meaning:

The circulation of a vector around the perimeter of a small area



Ulaby Figure 3-23

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7.19

Curl

A more general definition

$$\nabla \times \mathbf{B} = \underbrace{\lim}_{\Delta s \to 0} \frac{\left[\hat{\mathbf{n}} \oint_C \mathbf{B} \cdot d\mathbf{l}\right]_{\max}}{\Delta s},$$

where Δs is oriented to maximize the result and $\hat{\mathbf{n}}$ is normal to Δs

Circulation Theorem (Stokes Theorem)

$$\int_S (\nabla \times \mathbf{B}) \cdot d\mathbf{s} = \oint_C \mathbf{B} \cdot d\mathbf{l}$$

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7.20

Curl

Two important relations

(1) $\nabla \cdot (\nabla \times \mathbf{A}) = 0,$ for any vector $\mathbf{A}$

(2) $\nabla \times (\nabla V) = 0,$ for any scalar V

Two definitions

A vector field $\mathbf{B}$ is *solenoidal* if $\nabla \cdot \mathbf{B} = 0$

— Magnetic fields (or more precisely fluxes) are always solenoidal

A vector field $\mathbf{E}$ is *irrotational* if $\nabla \times \mathbf{E} = 0$

— Electric fields generated by charges are irrotational*

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*But electric fields generated by time-varying magnetic fields are not irrotational

Laplacian

Definition (Scalar)

$$\nabla^2 V = \nabla \cdot (\nabla V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Definition (Vector)

$$\nabla^2 \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla \times (\nabla \times \mathbf{E}) = \frac{\partial^2 \mathbf{E}}{\partial x^2} + \frac{\partial^2 \mathbf{E}}{\partial y^2} + \frac{\partial^2 \mathbf{E}}{\partial z^2}$$

In Cartesian coordinates, the scalar and vector forms look the same. Hence, we call them both the Laplacian

In other coordinate systems, that is not true!

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This quantity appears in the relationship of voltage to charge and the propagation wave equations.

General Forms for Differential Operators

Ulaby has expressions for differential operators in the back of the book
— Where do they come from?

The expressions look messy for cylindrical and spherical systems
… In fact, they are messy!

It is easier to derive general expressions, where the symmetry is apparent, and then apply them to specific cases:

An overview:

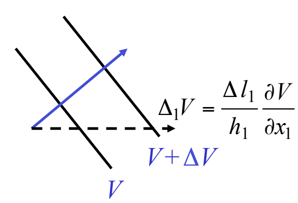
(x_1, x_2, x_3) is an orthogonal coordinate system with differential lengths

$$dl_1 = h_1 dx_1, \quad dl_2 = h_2 dx_2, \quad dl_3 = h_3 dx_3$$

General Forms for Differential Operators

Gradient:

$$\nabla V = \frac{\hat{\mathbf{x}}_1}{h_1}\frac{\partial V}{\partial x_1} + \frac{\hat{\mathbf{x}}_2}{h_2}\frac{\partial V}{\partial x_2} + \frac{\hat{\mathbf{x}}_3}{h_3}\frac{\partial V}{\partial x_3}$$



We sum vectorially the contributions

$$\lim_{\Delta l_j \to 0} \frac{\Delta_j V}{\Delta l_j} \text{ where } \Delta_j V = V(\mathbf{R} + \Delta \mathbf{l}_j) - V(\mathbf{R})$$

Divergence:

$$\nabla \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial x_1} h_2 h_3 A_1 + \frac{\partial}{\partial x_2} h_1 h_3 A_2 + \frac{\partial}{\partial x_3} h_1 h_2 A_3\right)$$

In all three directions, we add the net flux Φ out of a small volume.

General Forms for Differential Operators

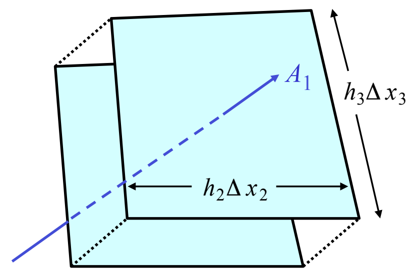
Divergence (continued):

For the 1-direction, we have

$$\Delta\Phi_1 = \Delta_1\left[A_1(h_2\Delta x_2)(h_3\Delta x_3)\right]$$
$$\simeq \Delta x_1 \Delta x_2 \Delta x_3 \frac{\partial}{\partial x_1} h_2 h_3 A_1$$

so that

$$\lim_{\Delta v \to 0} \frac{\Delta\Phi_1}{\Delta v} = \frac{1}{h_1 h_2 h_3} \frac{\partial}{\partial x_1} h_2 h_3 A_1$$



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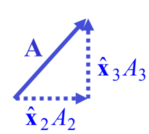
General Forms for Differential Operators

Curl:

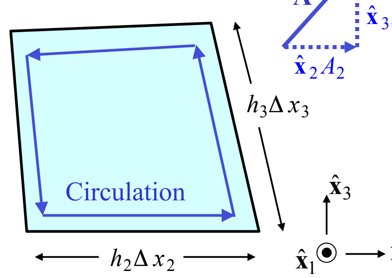
$$\nabla\times\mathbf{A} = \hat{\mathbf{x}}_1\left[\frac{1}{h_2h_3}\left(\frac{\partial}{\partial x_2}h_3A_3 - \frac{\partial}{\partial x_3}h_2A_2\right)\right] + \hat{\mathbf{x}}_2\left[\frac{1}{h_1h_3}\left(\frac{\partial}{\partial x_3}h_1A_1 - \frac{\partial}{\partial x_1}h_3A_3\right)\right]$$
$$+\,\hat{\mathbf{x}}_3\left[\frac{1}{h_1h_2}\left(\frac{\partial}{\partial x_1}h_2A_2 - \frac{\partial}{\partial x_2}h_1A_1\right)\right]$$

We sum vectorially the circulations in all three directions



Considering the 1-direction:

The circulation in the 1-direction may be written (through second order, using the trapezoidal rule)



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General Forms for Differential Operators

Curl (continued):

$$\Delta C_1 = \frac{1}{2}\big[(h_2 A_2)(x_2+\Delta x_2, x_3) + (h_2 A_2)(x_2, x_3)\big]\Delta x_2$$
$$+\frac{1}{2}\big[(h_3 A_3)(x_2+\Delta x_2, x_3+\Delta x_3) + (h_3 A_3)(x_2+\Delta x_2, x_3)\big]\Delta x_3$$
$$-\frac{1}{2}\big[(h_2 A_2)(x_2, x_3+\Delta x_3) + (h_2 A_2)(x_2+\Delta x_2, x_3+\Delta x_3)\big]\Delta x_2$$
$$-\frac{1}{2}\big[(h_3 A_3)(x_2, x_3) + (h_3 A_3)(x_2, x_3+h_3\Delta x_3)\big]\Delta x_3$$
$$+\text{cubic and higher order terms}$$

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General Forms for Differential Operators

Curl (continued):

$$\Delta C_1 = \frac{1}{2}\left[2h_2 A_2 + \frac{\partial h_2 A_2}{\partial x_2}\Delta x_2\right]_{x_2,x_3}\Delta x_2$$
$$+\frac{1}{2}\left[2h_3 A_3 + 2\frac{\partial h_3 A_3}{\partial x_2}\Delta x_2 + \frac{\partial h_3 A_3}{\partial x_3}\Delta x_3\right]_{x_2,x_3}\Delta x_3$$
$$-\frac{1}{2}\left[2h_2 A_2 + 2\frac{\partial h_2 A_2}{\partial x_3}\Delta x_3 + \frac{\partial h_2 A_2}{\partial x_2}\Delta x_2\right]_{x_2,x_3}\Delta x_2$$
$$-\frac{1}{2}\left[2h_3 A_3 + \frac{\partial h_3 A_3}{\partial x_3}\Delta x_3\right]_{x_2,x_3}\Delta x_3 + \text{cubic and higher order terms}$$

$$\Rightarrow \Delta \mathrm{C}_1 = \left(\frac{\partial h_3 A_3}{\partial x_2} - \frac{\partial h_2 A_2}{\partial x_3}\right)\Delta x_2 \Delta x_3 + \text{higher order terms}$$

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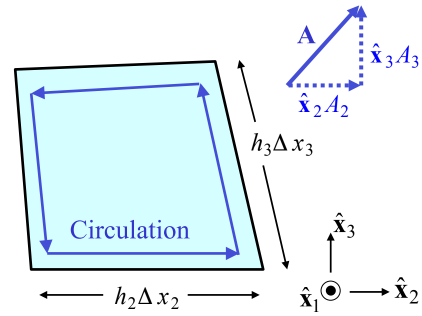
General Forms for Differential Operators

Curl (continued):

so that we conclude

$$\lim_{\Delta s_1 \to 0} \frac{\Delta C_1}{\Delta s_1} = \frac{1}{h_2 h_3}\left(\frac{\partial}{\partial x_2} h_3 A_3 - \frac{\partial}{\partial x_3} h_2 A_2\right)$$



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General Forms for Differential Operators

Applications

Cartesian coordinates: $x_1 = x, \quad x_2 = y, \quad x_3 = z; \quad h_1 = 1, \quad h_2 = 1, \quad h_3 = 1$

Cylindrical coordinates: $x_1 = r, \quad x_2 = \phi, \quad x_3 = z; \quad h_1 = 1, \quad h_2 = r, \quad h_3 = 1$

Spherical coordinates:

$$x_1 = R, \quad x_2 = \theta, \quad x_3 = \phi; \quad h_1 = 1, \quad h_2 = R, \quad h_3 = R\sin\theta$$

Substitution of these special cases into our general formulae will yield the results at the end of Ulaby and Ravaioli's book.

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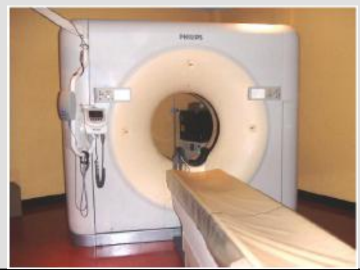
Tech Brief 6: X-Ray Computed Tomography

- Invented in 1972; 1979 Nobel Prize for Physiology or Medicine
- Provides 3D images of objects
- Operation principles
  - X-Ray source with fan-beam profile
  - 700 X-ray detectors instead of film
  - Rotates incrementally 1000 times per circle (0.36º)
  - One circle is one “slice” (a few millimeters thick)
  - Patient moves through the device for different slices
  - 3D image is reconstructed from the collected data, which represents the integrated absorption profile of the object in a lower dimensional form



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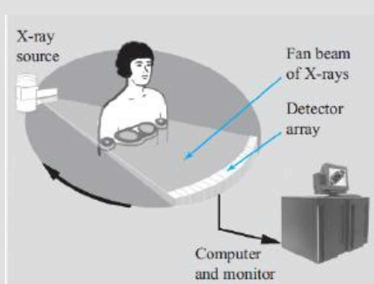
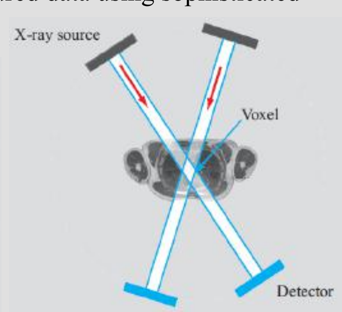
Tech Brief 6: X-Ray Computed Tomography

Image Reconstruction
- Each slice contains 700k measurements (700 detectors, 1000 angles)
- Each measurement is the integrated x-ray absorption along the path from source to detector
- Each small volume element (voxel) is contained in all 1000 angle measurements
- Computer reconstructs 3D data from the measured data using sophisticated matrix computation





(b) Detector measures integrated attenuation along anatomical path

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Assignment

Reading: Ulaby, Chapter 4

Problem Set 4: Some notes.

- There are 10 problems. As always, YOU MUST SHOW YOUR WORK TO GET FULL CREDIT!
- You will have to use the tables in the back of Ulaby's book for for the differential operators for several of the problems.
- These problems are not hard (with one exception), but they do take time. Get started early!

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