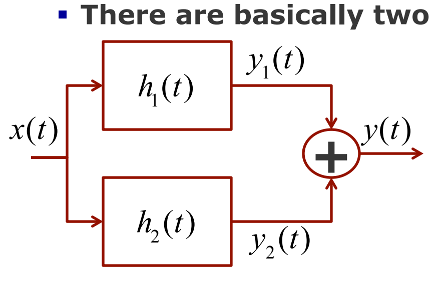
Lecture 6: LTI systems and convolution

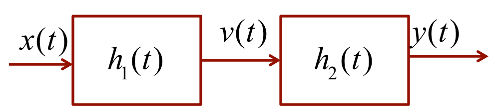
Combinations of LTI systems

- We can (and do) combine multiple LTI systems into a larger (possibly) LTI system…
- …so we need some rules.
- There are basically two combination strategies



Parallel (addition)

$$h(t) = h_1(t) + h_2(t)$$



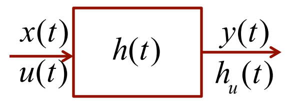
Series (concatenation)

$$h(t) = h_1(t) * h_2(t)$$

- …and the combination equations are the result of the properties of the convolution operator

The step response

- We have seen that the LTI system $\mathcal{H}$ is characterized by its impulse response $h(t)$
- The **step response** is also useful



$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

$$h(t) = \int_{-\infty}^{\infty} \delta(\tau)h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} h(\tau)\delta(t-\tau)\,d\tau$$

$$h_u(t) = \int_{-\infty}^{\infty} u(\tau)h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)\,d\tau = \int_{-\infty}^{t} h(\tau)\,d\tau$$

And we can obtain one from the other

$$\delta(t) = \frac{du(t)}{dt} \Rightarrow h(t) = \frac{dh_u(t)}{dt}$$

- By Liebnitz' Rule for differentiation of integrals

$$h_u(t) = \int_{-\infty}^{t} h(\tau)\,d\tau$$

$$\frac{dh_u(t)}{dt} = \frac{dt}{dt} \times h(t) - \frac{d(-\infty)}{dt} h(-\infty) + \int_{-\infty}^{t} \frac{dh(\tau)}{dt}\,d\tau$$

$$= 1 \times h(t) - 0 + 0 = h(t)$$

LTI Systems and Differential Equations

- LTI systems are often used to describe systems whose behavior is characterized **by linear, constant coefficient differential (or difference) equations**
- …which appear in a wide range of communications, navigation, control, signal processing, coding, and decoding applications
- …(which is why we're studying this)
- The general form is

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}, \quad \frac{d^0 y(t)}{dx^0} \triangleq y(t)$$

- … which is a function of ***N*** derivatives of the output and ***M*** derivatives of the input.

- From MATH225 we know (or should know) that the solution to this general form consists of
  - A ***particular solution*** $y_p(t)$ which satisfies the general form, and,
  - A ***homogeneous solution*** $y_h(t)$ which satisfies

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0$$

  - …with the exact form determined by ***N auxiliary conditions***, which are usually in the form of ***initial conditions***.
- The solution is then $y(t) = y_p(t) + y_h(t)$
- We should also know that $y(t) = Ae^{st}, \ s = \sigma + j\omega$ are ***eigenfunctions*** for the LCCDEs under consideration

- **If** $x(t)$ **is an eigenfunction of an operator** $\mathcal{G}$**, then**

$$\mathcal{G}\left(x(t)\right) = Bx(t)$$

- **… where** B **is a (possibly complex) constant**
- **For the LCCDE given earlier, it is clear that** $x(t) = Ae^{st}$ **is an eigenfunction of**

  k-th order derivative operator $\frac{d^k}{dt^k}$

$$\frac{d^k}{dx^k}\left(Ae^{st}\right) = As^k e^{st}$$

- **And that's why we choose trial solutions of the form**

$$y(t) = Ce^{\alpha t}$$

Working along these lines

- **Consider** $\frac{dy(t)}{dt} + ay(t) = x(t)$**, with** $x(t) = e^{-bt}u(t)$
- **It has the right form!**
- **Our trial particular solution for** $t > 0$ **is** $y_p(t) = Ax(t) = Ae^{-bt}$
- **Plugging and chugging**

$$-bAe^{-bt} + ae^{-bt} = e^{-bt};\ t > 0$$

$$A = \frac{1}{a-b}$$

$$y_p(t) = \frac{1}{a-b}e^{-bt},\ t > 0$$

- **For the homogeneous solution to** $\frac{dy(t)}{dt} + ay(t) = 0$
- **We again guess an eigenfunction** $y_h(t) = Ke^{st}$
- **Giving** $\frac{d}{dt}(Ke^{st}) + aKe^{st} = 0$

$$Kse^{st} + aKe^{st} = 0 \Rightarrow s = -a$$

- **The total solution is then**

$$y(t) = y_p(t) + y_h(t)$$
$$= Ke^{-at} + \left(\frac{1}{a-b}\right)e^{-bt}, t > 0$$

$$y(t) = y_p(t) + y_h(t)$$
$$= Ke^{-at} + 0, t < 0$$

- **With the value at** $t = 0$ being the initial condition Y_I

$$Ke^{-at}\Big|_{t=0} = Y_I = K + \left(\frac{1}{a-b}\right) \Rightarrow K = Y_I - \left(\frac{1}{a-b}\right)$$

System applications

- **In systems application we're much less worried about the particular and homogeneous solutions than we are about the solution with and without an input**
- **The zero state solution**

$$y_{ZS}(t) = y(t)\big|_{Y_I=0} = \left(\frac{1}{a-b}\right)\left(e^{-bt} - e^{-at}\right)u(t)$$

- **The zero input solution** $y_{ZI}(t) = y(t)\big|_{x(t)=0} = y_h(t) = Y_I e^{-at}$
- **And** $y(t) = y_{ZS}(t) + y_{ZI}(t)$

Is the system described linear?

- The linear combinations have to work for any valid inputs, so choose $x_1(t) = \alpha e^{-bt}u(t),\ x_2(t) = -x_1(t)$
- Then, from our solution

$$y_1(t) = Y_I e^{-at} + \frac{1}{\alpha}\left(\frac{1}{a-b}\right)(e^{-bt} - e^{-at})u(t)$$

$$y_2(t) = Y_I e^{-at} - \frac{1}{\alpha}\left(\frac{1}{a-b}\right)(e^{-bt} - e^{-at})u(t)$$

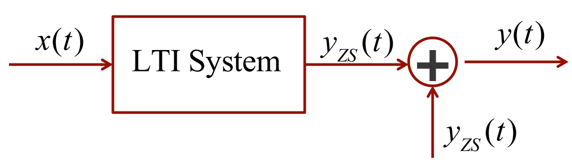
- If we let $x(t) = x_1(t) + x_2(t) = 0$, a linear system will give

$$y(t) = Y_I e^{-at} + \frac{1}{\alpha}\left(\frac{1}{a-b}\right)(e^{-bt} - e^{-at})u(t) + Y_I e^{-at} - \frac{1}{\alpha}\left(\frac{1}{a-b}\right)(e^{-bt} - e^{-at})u(t)$$

$$y(t) = Y_I e^{-at} + Y_I e^{-at} = 2Y_I e^{-at} = 0 \Leftrightarrow Y_I = 0$$

- So this system is linear iff it **is initially at rest!**

In general



- A system described by a LCCDE is linear if and only if it is initially at rest.
- A causal system that is initially at rest is also Time Invariant

Block Diagrams of LCCDE

- How would we implement or synthesize a system if we had a LCCDE?
- Implementing differentiators is problematic because
  - High frequencies in the input are multiplied by a frequency ramp

$$\frac{d\sin(\omega_o t)}{dt} = \omega_o \cos(\omega_0 t)$$

  - Impulses can result from differentiation
- So we transform the equation into an integral equation
- Here are the steps

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}, \quad \frac{d^0 y(t)}{dx^0} \triangleq y(t)$$

$$\text{If } M < N, \text{ let } b_k = 0, \; k = M+1, M+2 \ldots N$$

$$\text{If } N < M, \text{ let } a_k = 0, \; k = N+1, N+2 \ldots M$$

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{N} b_k \frac{d^k x(t)}{dt^k}, \quad \frac{d^0 y(t)}{dx^0} \triangleq y(t)$$

Now form a set of auxiliary variables or , $y_k(t)$ and $x_k(t)$

$$y_0(t) = y(t), \;\; x_0(t) = x(t)$$

$$y_1(t) = \int_{-\infty}^{t} y_o(\tau)\,d\tau, \; x_1(t) = \int_{-\infty}^{t} x_o(\tau)\,d\tau$$

$$y_2(t) = \int_{-\infty}^{t} y_1(\tau)\,d\tau, \; x_2(t) = \int_{-\infty}^{t} x_1(\tau)\,d\tau$$

etc...

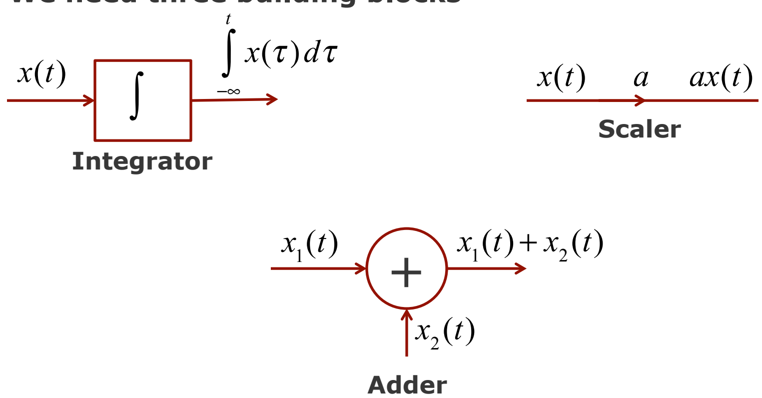
$$y_N(t) = \int_{-\infty}^{t} y_{N-1}(\tau)\,d\tau, \; x_N(t) = \int_{-\infty}^{t} x_{N-1}(\tau)\,d\tau$$

$$\sum_{k=0}^{N} a_k y_{N-k}(t) = \sum_{k=0}^{N} b_k x_{N-k}(t) = \sum_{k=0}^{M} b_k x_{N-k}(t)$$

Where we have integrated both sides N times...
requiring N initial conditions (one for each integral)

Basic Block Diagram Elements

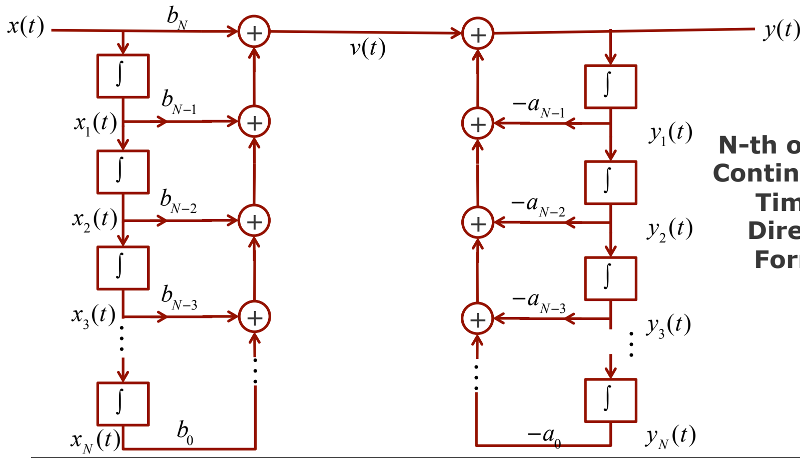
- **We'll now use this "integrated" form to *synthesize* an implementation of the system described by the LCCDE**
- **We need three building blocks**



Synthesis

- **Assume** $a_N = 1$ **(or divide through to make it so)**
- **Rewrite**

$$y(t) = y_0(t) = \sum_{k=0}^{N} b_k x_{N-k}(t) - \sum_{k=0}^{N-1} a_k y_{N-k}(t) = v(t) - \sum_{k=0}^{N-1} a_k y_{N-k}(t)$$



N-th order Continuous Time Direct Form