

- 4.2** **1** Use the division algorithm to find the quotient and remainder when $f(x)$ is divided by $g(x)$ over the field of rational numbers \mathbb{Q} .

c $f(x) = x^5 + 1, \quad g(x) = x + 1$

Pf.

$$\begin{array}{r}
 x^4 - x^3 + x^2 - x + 1 \\
 \hline
 x + 1 \quad x^5 \qquad \qquad \qquad + 1 \\
 \quad - x^5 - x^4 \\
 \quad \hline
 \qquad - x^4 \\
 \qquad \quad x^4 + x^3 \\
 \qquad \quad \hline
 \qquad \qquad x^3 \\
 \qquad \quad - x^3 - x^2 \\
 \qquad \quad \hline
 \qquad \qquad - x^2 \\
 \qquad \quad \quad x^2 + x \\
 \qquad \quad \quad \hline
 \qquad \qquad \qquad x + 1 \\
 \qquad \quad \quad - x - 1 \\
 \qquad \quad \quad \hline
 \qquad \qquad \qquad \qquad 0
 \end{array}$$

Therefore,

$$\begin{aligned}
 f(x) &= g(x)(x^4 - x^3 + x^2 - x + 1) + (0) \\
 &= (x + 1)(x^4 - x^3 + x^2 - x + 1) + (0) \pmod{\mathbb{Q}}
 \end{aligned}$$

□

- 2** Use the division algorithm to find the quotient and remainder when $f(x)$ is divided by $g(x)$ over the indicated field.

c $f(x) = x^5 + 2x^3 + 3x^2 + x - 1, \quad g(x) = x^2 + 5$ over \mathbb{Z}_7

Pf.

$$\begin{aligned}
 f(x) &= x^5 + 2x^3 + 3x^2 + x - 1 \\
 &\equiv x^5 + 2x^3 + 3x^2 + x + 6 \pmod{\mathbb{Z}_7} \\
 g(x) &= x^2 + 5 \\
 &\equiv x^2 + 5 \pmod{\mathbb{Z}_7}
 \end{aligned}$$

$$\begin{array}{r}
\begin{array}{cccccc}
& & & +x^3 & & +4x & +3 \\
x^2 + 5 &) & +x^5 & +0x^4 & +2x^3 & +3x^2 & +x & +6 \\
- & (+x^5 & & +5x^3) & & & & \\
\hline
& & +4x^3 & & +3x^2 & & +x & \\
- & (+4x^3 & & & +6x) & & & \\
\hline
& & & +3x^2 & & +2x & & \\
- & (+3x^2 & & +2x) & & & & \\
\hline
& & & & & & & +6
\end{array}
\end{array}$$

Therefore,

$$\begin{aligned}
f(x) &= g(x)(x^3 + 4x + 3) + 6 \\
&= (x^2 + 5)(x^3 + 4x + 3) + 6 \pmod{\mathbb{Z}_7}
\end{aligned}$$

□

3 Find the greatest common divisor of $f(x)$ and f' , over \mathbb{Q} .

d $f(x) = x^4 + 2x^3 + 3x^2 + 2x + 1$

Pf. Given $f(x) = x^4 + 2x^3 + 3x^2 + 2x + 1$

and $f' = 4x^3 + 6x^2 + 6x + 2$

And, $\frac{f(x)}{f'}$,

$$\begin{array}{r}
\begin{array}{cc}
& \frac{1}{4}x + \frac{1}{8} \\
4x^3 + 6x^2 + 6x + 2 &) \quad x^4 + 2x^3 + 3x^2 + 2x + 1 \\
- & x^4 - \frac{3}{2}x^3 - \frac{3}{2}x^2 - \frac{1}{2}x \\
\hline
& \frac{1}{2}x^3 + \frac{3}{2}x^2 + \frac{3}{2}x + 1 \\
- & \frac{1}{2}x^3 - \frac{3}{4}x^2 - \frac{3}{4}x - \frac{1}{4} \\
\hline
& \frac{3}{4}x^2 + \frac{3}{4}x + \frac{3}{4}
\end{array}
\end{array}$$

Multiplying the remainder with a non-zero constant keeps it unchanged, and therefore,

$$\begin{aligned}
\text{remainder} &= \left(\frac{3}{4}x^2 + \frac{3}{4}x + \frac{3}{4} \right) \frac{4}{3} \\
&= x^2 + x + 1
\end{aligned}$$

Thus,

$$\begin{aligned}
&\gcd(x^4 + 2x^3 + 3x^2 + 2x + 1, 4x^3 + 6x^2 + 6x + 2) \\
&= \gcd(4x^3 + 6x^2 + 6x + 2, x^2 + x + 1)
\end{aligned}$$

Dividing as before,

$$\begin{array}{r} \phantom{x^2 + x + 1) } 4x + 2 \\ \hline x^2 + x + 1) 4x^3 + 6x^2 + 6x + 2 \\ \phantom{x^2 + x + 1) } - 4x^3 - 4x^2 - 4x \\ \hline \phantom{x^2 + x + 1) } 2x^2 + 2x + 2 \\ \phantom{x^2 + x + 1) } - 2x^2 - 2x - 2 \\ \hline \phantom{x^2 + x + 1) } 0 \end{array}$$

Therefore,

$$\gcd(x^4 + 2x^3 + 3x^2 + 2x + 1, 4x^3 + 6x^2 + 6x + 2) = x^2 + x + 1 \quad \square$$

5 Find the greatest common divisor of the given polynomials, over the given field.

c $x^5 + 4x^4 + 6x^3 + 6x^2 + 5x + 2$ and $x^4 + 3x^2 + 3x + 6$ over \mathbb{Z}_7

Pf. Doing long division until remainder is 0,

$$\begin{array}{r}
 \begin{array}{r}
 x \qquad +4 \\
 \hline
 \end{array} \\
 x^4 + 3x^2 + 3x + 6 \quad) \quad \begin{array}{r}
 +x^5 \quad +4x^4 \quad +6x^3 \quad +6x^2 \quad +5x \quad +2 \\
 -(x^5 \qquad \qquad +3x^3 \quad +3x^2 \quad +6x) \\
 \hline
 \qquad \qquad +4x^4 \quad +3x^3 \quad +3x^2 \quad +x \\
 \qquad \qquad -(4x^4 \qquad \qquad +5x^2 \quad +5x \quad +3) \\
 \hline
 \qquad \qquad \qquad +3x^3 \quad +5x^2 \quad +3x \quad +6
 \end{array}
 \end{array}$$

$$\begin{array}{r}
 \begin{array}{ccccc}
 & 5x & & +1 & \\
 \hline
 3x^3 + 5x^2 + 3x + 6 & \Big) & +x^4 & & +3x^2 & +3x & +6 \\
 & & -(x^4 & +4x^3 & +x^2 & +2x) & \\
 \hline
 & & & +3x^3 & +2x^2 & +x & +6 \\
 & & & -(3x^3 & +5x^2 & +3x & +6) \\
 \hline
 & & & & +4x^2 & +5x &
 \end{array}
 \end{array}$$

$$\begin{array}{r}
 \begin{array}{cc}
 6x & +6 \\
 \hline
 4x^2 + 5x &) \quad +3x^3 & +5x^2 & +3x & +6 \\
 \quad \quad \quad & -(3x^3 & +2x^2) \\
 \hline
 & & +3x^2 & +3x & +6 \\
 & & -(3x^2 & +2x) \\
 \hline
 & & & +x & +6
 \end{array}
 \end{array}$$

$$\begin{array}{r} 4x +2 \\ \hline x+6 +4x^2 +5x \\ -(4x^2 +3x) \\ \hline +2x \\ -(2x +5) \\ \hline +2 \end{array}$$

$$\begin{array}{r} 4x3 \\ \hline 2\)+x6 \\ -(x) \\ \hline 6 \\ -(6) \\ \hline 0 \end{array}$$

Therefore, $\gcd(x^5 + 4x^4 + 6x^3 + 6x^2 + 5x + 2, x^4 + 3x^2 + 3x + 6) = 2 \in \mathbb{Z}_7$

☐

- 9** Let $a \in \mathbb{R}$, and let $f(x) \in \mathbb{R}[x]$, with derivative $f'(x)$. Show that the remainder when $f(x)$ is divided by $(x - a)^2$ is $f'(a)(x - a) + f(a)$.

Pf. By the division algorithm, there exists unique polynomials $q(x), r(x) \in F[x]$, such that $f(x) = q(x)(x - a)^2 + r(x)$, where $\deg(x) < 2$

Let $r(x) = bx + c$

Then, $f(a) = 0 + r(a) = ba + c$

Deriving,

$$f'(x) = (q'(x)(x - a) + 2q(x))(x - a) + b$$

So, $f'(a) = b$

Also,

$$\begin{aligned} c &= f(a) - ba \\ &= f(a) - f'(a)a \end{aligned}$$

Therefore,

$$\begin{aligned} r(x) &= f'(a)x + f(a) - f(a)a \\ &= f'(a)(x - a) + f(a) \end{aligned}$$

☐

11 Find the irreducible factors of $x^6 - 1$ over \mathbb{R} .

Pf. Factoring $x^6 - 1$,

$$\begin{aligned}x^6 - 1 &= (x^3)^2 - 1^2 \\&= (x^3 - 1)(x^3 + 1) \\&= (x^3 - 1^3)(x^3 + 1^3) \\&= (x^3 - 1^3)(x + 1)(x^2 - x + 1) \\&= (x^3 - 1^3)(x + 1)(x^2 - x + 1) \\&= (x - 1)(x^2 + x + 1)(x + 1)(x^2 - x + 1)\end{aligned}$$

Factors of degree 1, $(x - 1)$ and $(x + 1)$ are irreducible

Also, both factors $(x^2 + x + 1)$ and $(x^2 - x + 1)$ have no roots in \mathbb{R} since their discriminant $(b^2 - 4ac)$ are less than zero.

Therefore, all factors of $x^6 - 1$; $(x - 1)$, $(x^2 + x + 1)$, $(x + 1)$, and $(x^2 - x + 1)$ are irreducible over \mathbb{R} □

18 Compute the following products.

b $(a + bx)(c + dx) \equiv ??? \pmod{x^2 - 2}$ over \mathbb{Q} .

Pf. Since $x^2 \equiv 2 \pmod{x^2 - 2}$

$$\begin{aligned}(a + bx)(c + dx) &= ac + adx + cbx + bdx^2 \\&= ac + adx + cbx + 2bd \pmod{x^2 - 2} \\&= (ac + 2bd) + (ad + cb)x \pmod{x^2 - 2}\end{aligned} \quad \square$$