CMPE 320: Probability, Statistics, and Random Processes

Lecture 3: Conditional probability

Spring 2018

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Conditional probability

- Conditional probability characterizes the likelihood of an outcome based on related partial information
- For example, in an experiment with two rolls of a die, given that the sum of two rolls is 9, how likely is it that the first roll is 6?

partial information outcome of interest

- More generally, given that the outcome is within event B, what is the likelihood that the outcome also belongs to event A?

P(A|B) outcome of interest

partial info

Motivating example

- For an experiment of rolling a die
- Given that the outcome is even $B = \{2, 4, 6\}$
- What is the likelihood that the outcome is 2? $A = \{2\}$

Equally likely outcome $\Rightarrow \frac{1}{3}$

$$P(A|B) = \frac{\text{number of elements in } A \cap B}{\text{numbers of elements in } B}$$

Definition of conditional probability

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}$$

- Is it a valid probability law? (That is, does it satisfy the axioms?)

1) Nonnegativity : $P(A|B) = \frac{P(A \cap B)}{P(B)} \geq 0$

2) Normalization : $P(\Omega|B) = \frac{P(\Omega \cap B)}{P(B)} = \frac{P(B)}{P(B)} = 1$

3) Additivity : For disjoint A_1 and A_2

$$P(\underline{A_1 \cup A_2} | B) = \frac{P((A_1 \cup A_2) \cap B)}{P(B)} = \frac{P(\overbrace{(A_1 \cap B)} \cup \overbrace{(A_2 \cap B)})}{P(B)}$$

disjoint

$$= \frac{P(A_1 \cap B) + P(A_2 \cap B)}{P(B)} \longrightarrow = \underbrace{P(A_1|B)} + \underbrace{P(A_2|B)}$$

Another interpretation

- $P(B|B) = \frac{P(B)}{P(B)} = 1 \quad \leftrightarrow \quad P(\Omega) = 1$

$\Rightarrow$ B is taking the role of universal set Ω

Conditional probability $P(A|B)$ is the probability law defined on this new universe B

Three coin tosses

$\Omega = \{HHH \ \ HHT \ \ HTH \ \ THH$
$HTT \ \ THT \ \ TTH \ \ TTT\}$

$B = \{HHH, HHT, HTH, HTT\}$

- A = {more heads than tails come up}, B = {1st toss is a head} $P(A|B) = ?$

$P(A|B) = P(A \cap B) / P(B)$

$P(B) = P(1^{st} \text{ toss} = H) = \frac{1}{2}$

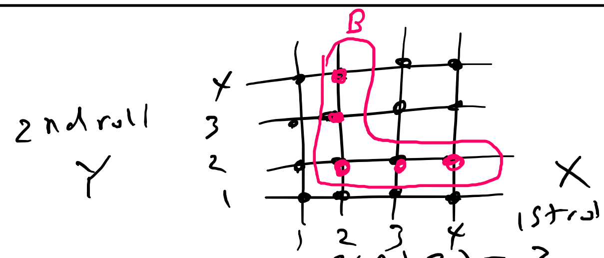
$P(A \cap B) = \frac{3}{8}$ $\quad A = \{HHH, HHT, HTH, THH\}$

$A \cap B = \{HHH, HHT, HTH\}$

$P(A|B) = \frac{3/8}{1/2} = \frac{3}{4}$

Alternative: $\frac{3}{4}$

4-sided die



- A 4-sided die was rolled twice
- A = {max(X,Y) = m}, B = {min(X,Y) = 2} (m=1,2,3,4)

$P(A|B) = ?$

Equally likely outcomes $\Rightarrow P(A|B) = \dfrac{\#(A \cap B)}{\underbrace{\#(B)}_{5}}$

$m=1: \quad A \cap B = \emptyset \quad \Rightarrow P(A|B) = 0$

$m=2: \quad A \cap B = \{(2,2)\} \quad \Rightarrow P(A|B) = \frac{1}{5}$

$m=3: \quad A \cap B = \{(2,3),(3,2)\} \quad \Rightarrow P(A|B) = \frac{2}{5}$

$m=4: \quad A \cap B = \{(2,4),(4,2)\} \quad \Rightarrow P(A|B) = \frac{2}{5}$

Using conditional probability for modeling

- $\boxed{P(A \cap B) = P(B)\, P(A|B)}$ $\Leftarrow$ $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$

Example 1.9. Radar Detection. If an aircraft is present in a certain area, a radar detects it and generates an alarm signal with probability 0.99. If an aircraft is not present, the radar generates a (false) alarm, with probability 0.10. We assume that an aircraft is present with probability 0.05. What is the probability of no aircraft presence and a false alarm? What is the probability of aircraft presence and no detection?

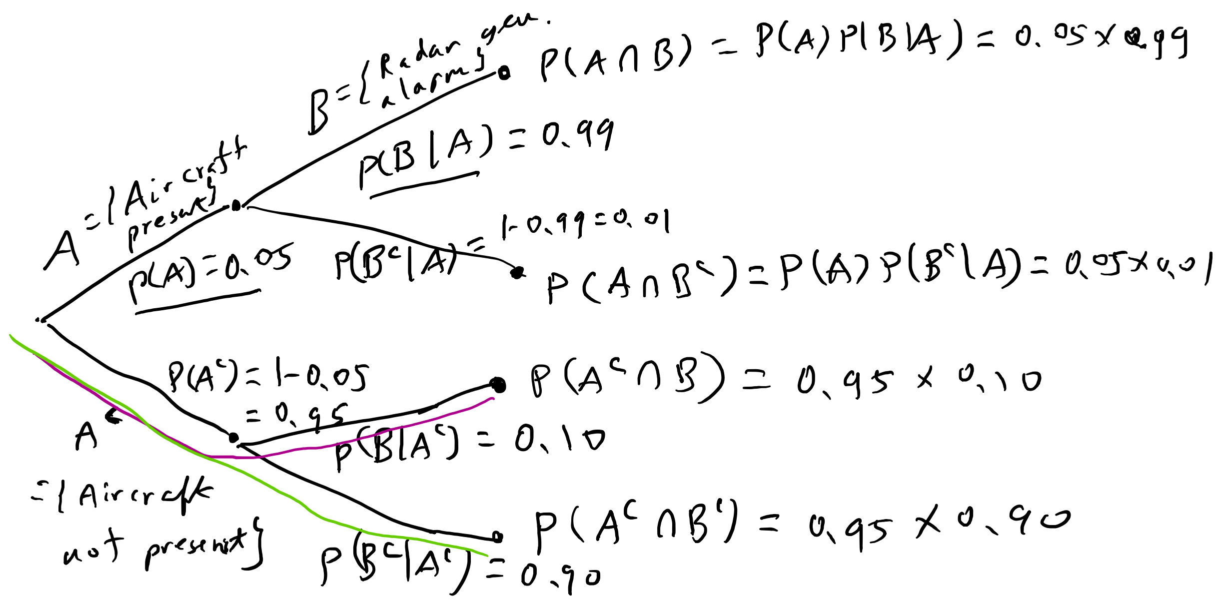
A = {aircraft is present} B = {radar generates an alarm}

$P(B|A) = 0.99$

$P(B|A^c) = 0.10$

$P(A) = 0.05$

$P(A^c \cap B) = ?$ 0.95×0.10 $\qquad P(A \cap B^c) = ?$ 0.05×0.01



Generalization

$$P(A_1 \cap A_2) = P(A_1)\, P(A_2 | A_1) = P(A_2) P(A_1 | A_2)$$

$$P(A_1 \cap \underbrace{A_2 \cap A_3}) = P(A_1)\, \underline{P(A_2 \cap A_3 | A_1)}$$

$$= P(A_1)\, P(A_2 | A_1)\, P(A_3 | A_2 \cap A_1)$$

$$P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1)\, P(A_2 | A_1)\, P(A_3 | A_1 \cap A_2)$$

$$\cdot P(A_4 | A_1 \cap A_2 \cap A_3)$$

$$\cdots P(A_n | A_{n-1} \cap A_{n-2} \cap \cdots \cap A_1)$$

Example 1.10. Three cards are drawn from an ordinary 52-card deck without replacement (drawn cards are not placed back in the deck). We wish to find the probability that none of the three cards is a heart. We assume that at each step,

Example 1.11. A class consisting of 4 graduate and 12 undergraduate students is randomly divided into 4 groups of 4. What is the probability that each group includes a graduate student?