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MATH 407

2/21/18

<u>Thm</u> If $A (\neq \emptyset) \subseteq \mathbb{N}$ then A is range of finite or infinite strictly increasing sequence.

<u>Pf.</u> Let $A_1 = A$ and $a_1 = \min(A_1)$

Induction step:

Suppose

$$a_1 < a_2 < \dots < a_k$$

have been found so that if

$a \in A_{k+1} = A_1 \setminus \{a_1, \dots, a_k\}$

then $a > a_k$.

Define $a_{k+1} = \min(A_{k+1})$ if $A_{k+1} \neq \emptyset$.

(*) <u>Eratosthenes Sieve</u>

Let $\mathbb{N}' = \mathbb{N} \setminus \{1\}$

Let $p_1 = 2 = \min(\mathbb{N}')$

Let $\mathbb{N}^2 = \mathbb{N}' \setminus p_1 \mathbb{N}$

Let $p_2 = 3 = \min(\mathbb{N}^2)$

Induction step:

Suppose:

$p_1 < p_2 < \dots < p_k$ are primes and any remaining prime $p > p_k$. Let $\mathbb{N}^{k+1} = \mathbb{N}^k \setminus p_k \mathbb{N}$

$= \mathbb{N}' \setminus (p_1 \mathbb{N} \cup \dots \cup p_k \mathbb{N})$

i) If $\mathbb{N}^{k+1} = \emptyset$, stop

$\{p_1, \dots, p_k\}$ are all primes

Else $\mathbb{N}^{k+1} \neq \emptyset$, $\exists n \in \mathbb{N}^{k+1}$ $p_j \nmid n$

any $1 \leq j \leq k$

$p_{k+1} = \min\left(\mathbb{N}^{k+1}\right)$ is prime

* <u>Thm</u> (Euclid) There are infinitely many primes.

$n = p_1 p_2 \cdots p_{k-1}$

We have $1 = \left(\frac{p_1 \cdots p_k}{p_i}\right) p_i + (-1) n$

$\Rightarrow (n, p_1) = 1$

* <u>Thm</u> (Fundamental Theorem of Arithmetic) Every $n \in \mathbb{N}'$ may be uniquely expressed as $q_1^{\alpha_1}, \ldots, q_k^{\alpha_k}$

where q_i are distinct primes

$\alpha_i \in \mathbb{N}$

$n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \cdots p^0 = 1$

where $r_i \in \mathbb{Z}^+$ all i

Ultimately, $r_i = 0$ all i

$p_i^0 = 1$ such i

<u>Pf</u>. Induction on $n \in \mathbb{N}'$

$$2 = p_1^1 \left(\prod_j p_j^0\right)$$

Suppose the proposition is true $\forall m < n$

n has a prime factor p_i some i

Let $m = \frac{n}{p_i} < n$. If $m = 1$, done. $n = p_i^1 \prod_{i \neq j} p_j^0$

$\Rightarrow m = p_1^{s_1} p_2^{s_2} \cdots p_i^{s_i} \cdots$