① MATH 407

4/6/18

⊛ Isomorphism (cont'd)

Prop. a) $\Phi_1 : G_1 \to G_2$, $\Phi_2 : G_2 \to G_3$
homomorphism implies:
$\Phi_3 = \Phi_2 \circ \Phi_1 : G_1 \to G_3$ homomorphism

b) $\Phi : G_1 \to G_2$ isomorphism implies
$\Phi^{-1} : G_2 \to G_1$ isomorphism

Def. $G_1 \simeq G_2$ iff there is isomorphism $\Phi : G_1 \to G_2$
$\simeq$ Equivalence relation

Prop. If G is cyclic:
a) $[G]_{\simeq} = [\mathbb{Z}]_{\simeq}$ if G is infinite

b) $[G]_{\simeq} = [\mathbb{Z}_n]_{\simeq}$ if G is finite

Prop. 3.4.3: If $\Phi : G_1 \to G_2$ is isomorphic,
a) If $o(a) = n$, then $o(\Phi(a)) = n$
b) If G_1 is abelian, then so is G_2
c) If G_1 is cyclic, then so is G_2

Cor. If Φ is homomorphic
a) $o(\Phi(a)) \mid o(a)$
b) $\Phi(G_1)$ is abelian if G_1 is
c) G_1 is cyclic, then so is $\Phi(G_1)$

Pf. a) $\Phi(a)^k = \Phi(a^k) \; \forall k$

so if $o(a) = n$,

then $\Phi(a^n) = \Phi(e_1) = e_2$

" $(\Phi(a))^n = e_2$

$$o(a) = o(\Phi^{-1}(\Phi(a))) = o(\Phi(a))$$

b) $a, b \in G_1$, $c, d \in G_2$

$c = \Phi(a)$, $d = \Phi(b)$

$cd = \Phi(a)\Phi(b) = \Phi(ab)$

$= \Phi(ba)$

$= \Phi(b)\Phi(a)$

$= dc$

c) $G_1 = \langle a \rangle = \{a^k : k \in \mathbb{Z}\}$

$\Phi(G_1) = \{\Phi(a^k) : k \in \mathbb{Z}\}$

$= \{(\Phi(a))^k : k \in \mathbb{Z}\}$

$= \langle \Phi(a) \rangle$

Prop. Let H_1 be a subgroup of G_1

H_2 be a subgroup of G_2

Let $\Phi : G_1 \to G_2$ be homomorphic

Then,

a) $\Phi(H_1)$ is a subgroup of G_2

b) $\Phi(H_2)$ is a subgroup of G_1

<u>Pf.</u> a) Let $c = \Phi(a)$, $d = \Phi(b)$
$a, b \in H_1$

$$cd^{-1} = \Phi(a)\left(\Phi(b)\right)^{-1}$$
$$= \Phi(a)\Phi(b^{-1})$$
$$= \Phi(ab^{-1}) \subseteq \Phi(H_1)$$

b) Let $\Phi(a) = c \in H_2$
$\Phi(b) = d \in H_2$
Then $\Phi(ab^{-1}) = cd^{-1} \in H_2$

*Homomorphisms preserve subgroups in either images.

<u>Prop 3.4.4</u> Let $\Phi: G_1 \to G_2$ be a homomorphism
Then Φ is 1-1 iff
☆ $\Phi(x) = e_2$ implies $x = e_1$

<u>Pf.</u> If Φ is 1-1 and $\Phi(x) = e_2$ then $x = e_1$,
since $\Phi(e_1) = e_2$

Suppose ☆.
Let $\Phi(a) = \Phi(b)$, some a, b
Then $\Phi(a)\Phi(b)^{-1} = e_2$
$\Rightarrow \Phi(ab^{-1}) = e_2$

$(\longrightarrow)$

Thus, $ab^{-1} = e$, $\Rightarrow a = b$

<u>Thm</u> If $\gcd(n,m) = 1$, then $\mathbb{Z}_n \oplus \mathbb{Z}_m \simeq \mathbb{Z}_{nm}$

Let $k \in \mathbb{Z}_{nm}$ $\quad (k = [k]_{nm})$

$$\Phi(k) = ([k]_n, [k]_m)$$

$$\Phi(hk) = \Phi(h)\Phi(k)$$
$$= ([h]_n [k]_n, [h]_m [k]_m)$$

* Show Φ is 1-1

$$\Phi(k) = ([0]_n, [0]_m)$$

$$k \equiv 0 \pmod n$$
$$k \equiv 0 \pmod m$$
$$\Rightarrow n \mid k,\ m \mid k$$
$$\Rightarrow nm \mid k$$
$$k = 0$$