

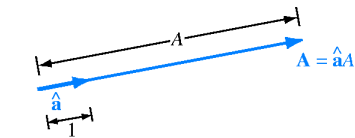
Vector Algebra

Definition

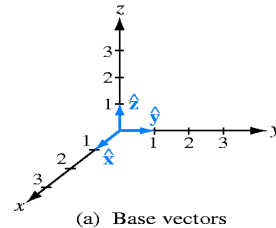
A vector \mathbf{A} has a magnitude $A = |\mathbf{A}|$ and a direction given by the unit vector $\hat{\mathbf{a}} = \mathbf{A} / |\mathbf{A}|$.

In Cartesian coordinates (x, y, z) , we find

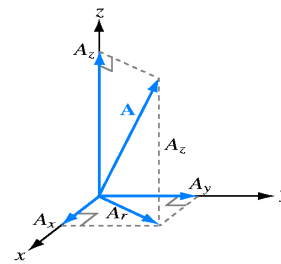
$$\begin{aligned}\mathbf{A} &= \hat{\mathbf{a}}A = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z \\ &= (A_x, A_y, A_z)\end{aligned}$$



Ulaby Figure 3-1



(a) Base vectors



(b) Components of \mathbf{A}

Ulaby Figure 3-2

6.1

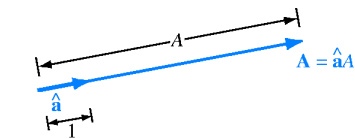
Vector Algebra

Definition

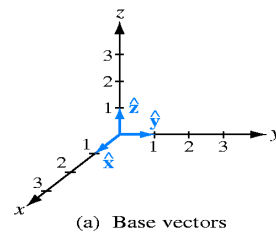
We also have

$$A = (A_x^2 + A_y^2 + A_z^2)^{1/2}$$

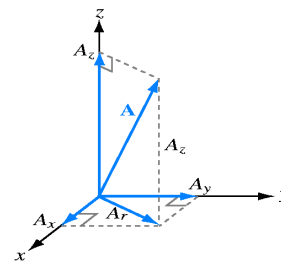
$$\hat{\mathbf{a}} = \frac{\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z}{(A_x^2 + A_y^2 + A_z^2)^{1/2}}$$



Ulaby Figure 3-1



(a) Base vectors



(b) Components of \mathbf{A}

Ulaby Figure 3-2

6.2

Vector Algebra

Equality

Two vectors **A** and **B** are equal if

$$A_x = B_x, A_y = B_y, \text{ and } A_z = B_z$$

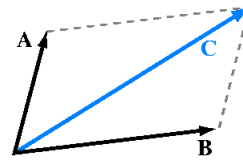
Addition and Subtraction

$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \quad (\text{addition is commutative})$$

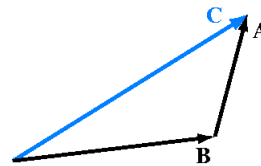
$$= \hat{x}(A_x + B_x) + \hat{y}(A_y + B_y) + \hat{z}(A_z + B_z)$$

$$\mathbf{D} = \mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$$

$$= \hat{x}(A_x - B_x) + \hat{y}(A_y - B_y) + \hat{z}(A_z - B_z)$$



(a) Parallelogram rule



(b) Head-to-tail rule

Ulaby Figure 3-3

Vector Algebra

Position and Distance vectors

For a point $P(x, y, z)$, the *position vector* **R** goes from the origin to the point

$$\mathbf{R} = \hat{x}x + \hat{y}y + \hat{z}z$$

The distance vector between two points

$P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is defined as

$$\mathbf{R}_{12} = \mathbf{R}_2 - \mathbf{R}_1$$

$$= \hat{x}(x_2 - x_1) + \hat{y}(y_2 - y_1) + \hat{z}(z_2 - z_1)$$

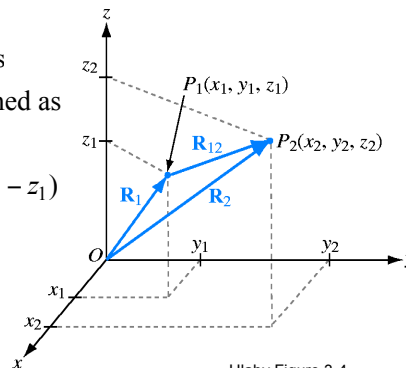
where $\mathbf{R}_i = \hat{x}x_i + \hat{y}y_i + \hat{z}z_i$, $i = 1, 2$

The distance d between two points

equals $|\mathbf{R}_{12}|$

$$d = |\mathbf{R}_{12}|$$

$$= \left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \right]^{1/2}$$



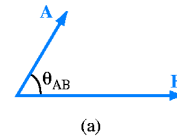
Ulaby Figure 3-4

Vector Algebra

Multiplication — Simple Product

When a vector \mathbf{A} is multiplied by a scalar k , the magnitude is multiplied by k and the direction is unchanged

$$\begin{aligned}\mathbf{B} &= k\mathbf{A} = \hat{\mathbf{a}}(kA) \\ &= \hat{\mathbf{x}}(kA_x) + \hat{\mathbf{y}}(kA_y) + \hat{\mathbf{z}}(kA_z)\end{aligned}$$



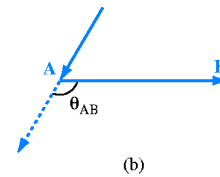
Multiplication — Scalar or Dot Product

The scalar or dot product of two vectors

\mathbf{A} and \mathbf{B} is defined by

$$\mathbf{A} \cdot \mathbf{B} = AB \cos \theta_{AB}$$

where θ_{AB} is the angle between \mathbf{A} and \mathbf{B}



Ulab Figure 3-5

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When $\theta_{AB} < 90^\circ$, $\mathbf{A} \cdot \mathbf{B} > 0$

When $\theta_{AB} > 90^\circ$, $\mathbf{A} \cdot \mathbf{B} < 0$

When $\theta_{AB} = 90^\circ$, $\mathbf{A} \cdot \mathbf{B} = 0$, and the vectors are called *orthogonal*

6.5

Vector Algebra

Multiplication — Scalar or Dot Product

We have

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$$

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{x}} \cdot \hat{\mathbf{z}} = 0$$

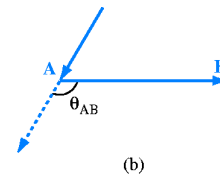
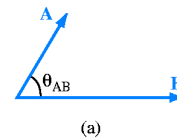
so that

$$\begin{aligned}\mathbf{A} \cdot \mathbf{B} &= (\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z) \cdot (\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z) \\ &= A_xB_x + A_yB_y + A_zB_z\end{aligned}$$

Other properties:

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \quad (\text{commutative})$$

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \quad (\text{distributive})$$



Ulab Figure 3-5

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6.6

Vector Algebra

Multiplication — Vector or Cross Product

The vector or cross product of two vectors

\mathbf{A} and \mathbf{B} is defined by

$$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{n}} AB \sin \theta_{AB}$$

where θ_{AB} is the angle from \mathbf{A} to \mathbf{B}
and $\hat{\mathbf{n}}$ is a unit vector determined by
the right-hand rule

The vector product is *anti-commutative*

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$

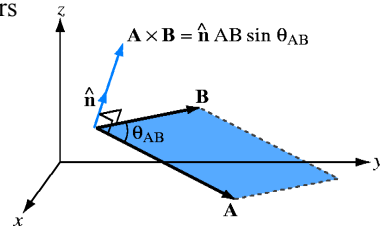
and distributive

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$$



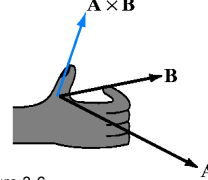
$$\hat{\mathbf{x}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0$$

$$\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}, \quad \hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}, \quad \hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}$$



(a) Cross product

(b) Right-hand rule



Ulab Figure 3-6

6.7

Vector Algebra

Multiplication — Vector or Cross Product

We have

$$\hat{\mathbf{x}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0$$

$$\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}, \quad \hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}, \quad \hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}$$

so that

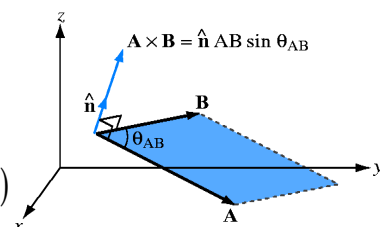
$$\begin{aligned} \mathbf{A} \times \mathbf{B} &= (\hat{\mathbf{x}} A_x + \hat{\mathbf{y}} A_y + \hat{\mathbf{z}} A_z) \times (\hat{\mathbf{x}} B_x + \hat{\mathbf{y}} B_y + \hat{\mathbf{z}} B_z) \\ &= \hat{\mathbf{x}}(A_y B_z - A_z B_y) + \hat{\mathbf{y}}(A_z B_x - A_x B_z) \\ &\quad + \hat{\mathbf{z}}(A_x B_y - A_y B_x) \end{aligned}$$

We may also write

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

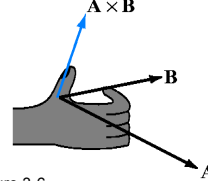


This determinant form is very useful!



(a) Cross product

(b) Right-hand rule



Ulab Figure 3-6

6.8

Vector Algebra

Summary of vector products

Product type	Product elements	Representation
Simple product	(Scalar) \times (Vector) \rightarrow Vector	$\mathbf{C} = k\mathbf{A}$
Scalar product or Dot product	(Vector) \times (Vector) \rightarrow Scalar	$C = \mathbf{A} \cdot \mathbf{B}$
Vector product or Cross product	(Vector) \times (Vector) \rightarrow Vector	$\mathbf{C} = \mathbf{A} \times \mathbf{B}$

Vector Algebra

Vectors and Angles: Ulaby and Ravaioli Example 3-1

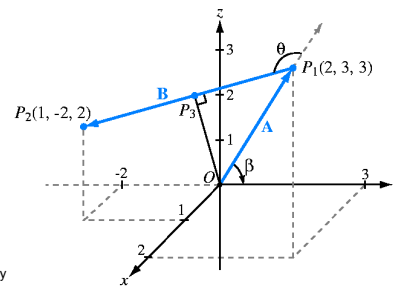
Question: In Cartesian coordinates, vector \mathbf{A} is directed from the origin to the point $P_1(2, 3, 3)$, and vector \mathbf{B} is directed from P_1 to $P_2(1, -2, 2)$. Find (a) the vector \mathbf{A} , its magnitude A , and its unit vector $\hat{\mathbf{a}}$, (b) the angle that \mathbf{A} makes with the y -axis, (c) vector \mathbf{B} , (d) the angle between \mathbf{A} and \mathbf{B} , and (e) the perpendicular distance from the origin to \mathbf{B} .

Answer: (a)

$$\mathbf{A} = \hat{\mathbf{x}} 2 + \hat{\mathbf{y}} 3 + \hat{\mathbf{z}} 3$$

$$A = \sqrt{4 + 9 + 9} = \sqrt{22}$$

$$\hat{\mathbf{a}} = \mathbf{A} / A = (\hat{\mathbf{x}} 2 + \hat{\mathbf{y}} 3 + \hat{\mathbf{z}} 3) / \sqrt{22}$$



Ulaby
Figure 3-7

Vector Algebra

Vectors and Angles: Ulaby and Ravaioli Example 3-1

Answer (continued): (b) The angle β between \mathbf{A} and the y -axis may be found from the expression $\mathbf{A} \cdot \hat{\mathbf{y}} = A \cos \beta$, which implies

$$\beta = \cos^{-1} \left(\frac{\mathbf{A} \cdot \hat{\mathbf{y}}}{A} \right) = \cos^{-1} \left(\frac{3}{\sqrt{22}} \right) = 0.879 \text{ rads} = 50.2^\circ$$

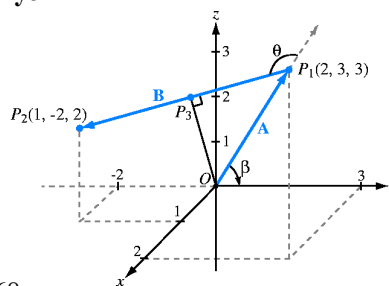
(c) $\mathbf{B} = \hat{\mathbf{x}}(1-2) + \hat{\mathbf{y}}(-2-3) + \hat{\mathbf{z}}(2-3) = -\hat{\mathbf{x}} - \hat{\mathbf{y}}5 - \hat{\mathbf{z}}$

(d) $\theta = \cos^{-1} \left(\frac{\mathbf{A} \cdot \mathbf{B}}{AB} \right) = \cos^{-1} \left(\frac{-20}{\sqrt{22}\sqrt{27}} \right)$
 $= 2.533 \text{ rads} = 145.1^\circ$

(e) The points OP_1P_3 form a right triangle.

The magnitude of the line segment OP_3 is given by

$$A \sin(\pi - \theta) = \sqrt{22} \sin(0.609) = 2.68$$



Ulaby Figure 3-7

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Vector Algebra

Triple Scalar Product

This product can be written in the equivalent forms

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

The equivalence can be demonstrated from the determinant representation

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

The absolute value of this scalar product is the volume of the parallelepiped whose sides are the vectors \mathbf{A} , \mathbf{B} , and \mathbf{C} .

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6.12

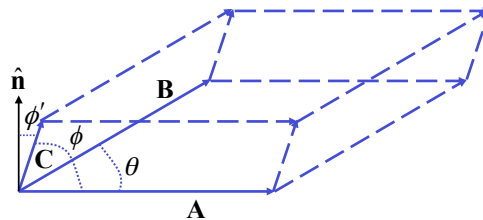
Vector Algebra

Triple Scalar Product

This product can be written in the equivalent forms

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

It corresponds to the volume of a parallelepiped



$$\hat{\mathbf{n}} = \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|}$$

$$\phi' = \frac{\pi}{2} - \phi$$



$$\text{Surface area} = |\mathbf{A} \times \mathbf{B}| = |\mathbf{A}| |\mathbf{B}| \sin \theta$$

$$\text{Volume} = |\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})| = |\mathbf{A}| |\mathbf{B}| |\mathbf{C}| \sin \theta \cos \phi' = |\mathbf{A}| |\mathbf{B}| |\mathbf{C}| \sin \theta \sin \phi$$

6.13

Vector Algebra

Triple Vector Product

This product is defined as

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \quad \text{and we note} \quad \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$$

This relationship can also be written in the form

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$



6.14

Orthogonal Coordinate Systems

Three coordinate systems

Coordinate systems (with three coordinates) specify points in space
Orthogonal systems have coordinates that are mutually perpendicular
— at least locally

There are three coordinate systems that are often used

- Cartesian coordinates (x, y, z) : Simplest and orthogonal everywhere
- Cylindrical coordinates (r, ϕ, z)
— used with optical fibers, coaxial cables, cylindrical waveguides
- Spherical coordinates (R, θ, ϕ)
— used with antenna radiation, radar, earth-ionosphere waveguide

Three Coordinate Systems

Summary of vector and differential relations

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, θ, z	R, θ, ϕ
Vector, $\mathbf{A} =$	$\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$
Magnitude, $ \mathbf{A} =$	$(A_x^2 + A_y^2 + A_z^2)^{1/2}$	$(A_r^2 + A_\phi^2 + A_z^2)^{1/2}$	$(A_R^2 + A_\theta^2 + A_\phi^2)^{1/2}$
Position vector for $P(x_1, y_1, z_1)$	$\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$	$\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1$ for $P(r_1, \phi_1, z_1)$	$\hat{\mathbf{R}}R_1$ for $P(R_1, \theta_1, \phi_1)$
Dot product, $\mathbf{A} \cdot \mathbf{B}$	$A_xB_x + A_yB_y + A_zB_z$	$A_rB_r + A_\phi B_\phi + A_zB_z$	$A_RB_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product, $\mathbf{A} \times \mathbf{B}$	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length, $d\mathbf{l}$	$\hat{\mathbf{x}}dx + \hat{\mathbf{y}}dy + \hat{\mathbf{z}}dz$	$\hat{\mathbf{r}}dr + \hat{\boldsymbol{\phi}}r d\phi + \hat{\mathbf{z}}dz$	$\hat{\mathbf{R}}dR + \hat{\boldsymbol{\theta}}R d\theta + \hat{\boldsymbol{\phi}}R \sin\theta d\phi$
Differential surface areas	$ds_x = \hat{\mathbf{x}}dydz$ $ds_y = \hat{\mathbf{y}}zdx$ $ds_z = \hat{\mathbf{z}}dxdy$	$ds_r = \hat{\mathbf{r}}rd\phi dz$ $ds_\phi = \hat{\boldsymbol{\phi}}dzdr$ $ds_z = \hat{\mathbf{z}}rdrd\phi$	$ds_R = \hat{\mathbf{R}}R^2 \sin\theta d\theta d\phi$ $ds_\theta = \hat{\boldsymbol{\theta}}R \sin\theta dR d\phi$ $ds_\phi = \hat{\boldsymbol{\phi}}R dR d\theta$
Differential volume	$dv = dx dy dz$	$dv = r dr d\phi dz$	$dv = R^2 \sin\theta dR d\theta d\phi$

Cartesian Coordinates

Differential Relations

Length: $d\mathbf{l} = \hat{\mathbf{x}} dl_x + \hat{\mathbf{y}} dl_y + \hat{\mathbf{z}} dl_z = \hat{\mathbf{x}} dx + \hat{\mathbf{y}} dy + \hat{\mathbf{z}} dz$

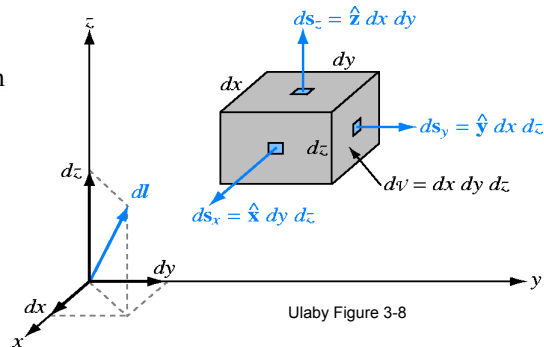
Surface area: $ds_x = \hat{\mathbf{x}} dl_y dl_z = \hat{\mathbf{x}} dy dz$ (y-z plane)

$ds_y = \hat{\mathbf{y}} dz dx = \hat{\mathbf{y}} dx dz$ (x-z plane)

$ds_z = \hat{\mathbf{z}} dx dy$ (x-y plane)

Volume: $dv = dx dy dz$

For each coordinate system there are three differential lengths that also determine the differential areas and volumes



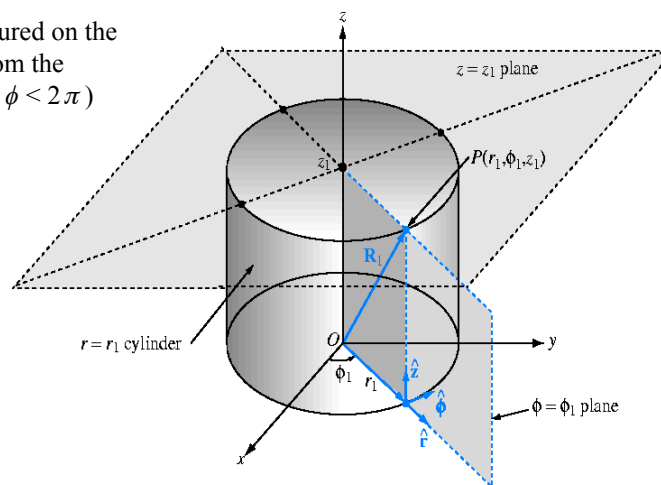
Ulaby Figure 3-8

6.17

Cylindrical Coordinates

r = distance from the z -axis ($0 < r < \infty$)

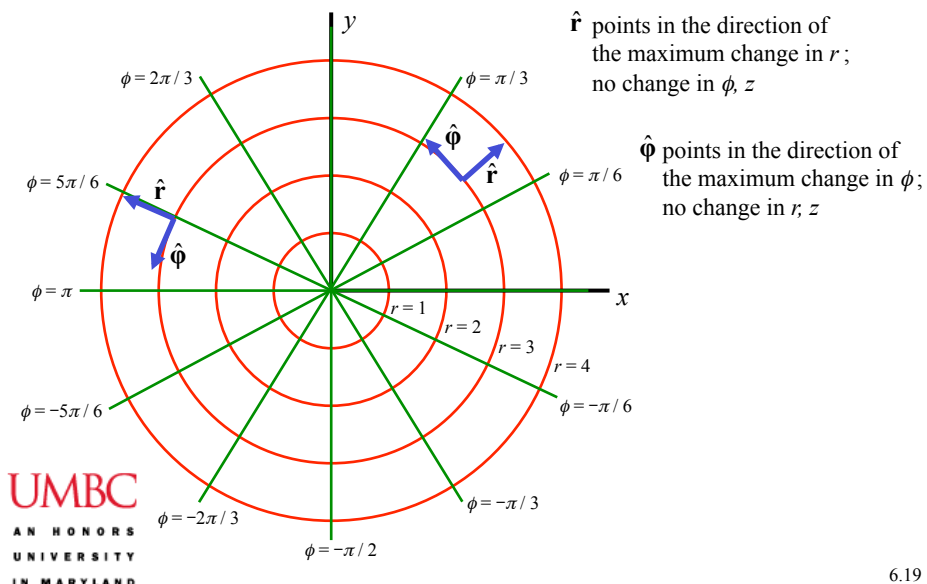
ϕ = angle measured on the x - y plane from the $+x$ -axis ($0 < \phi < 2\pi$)



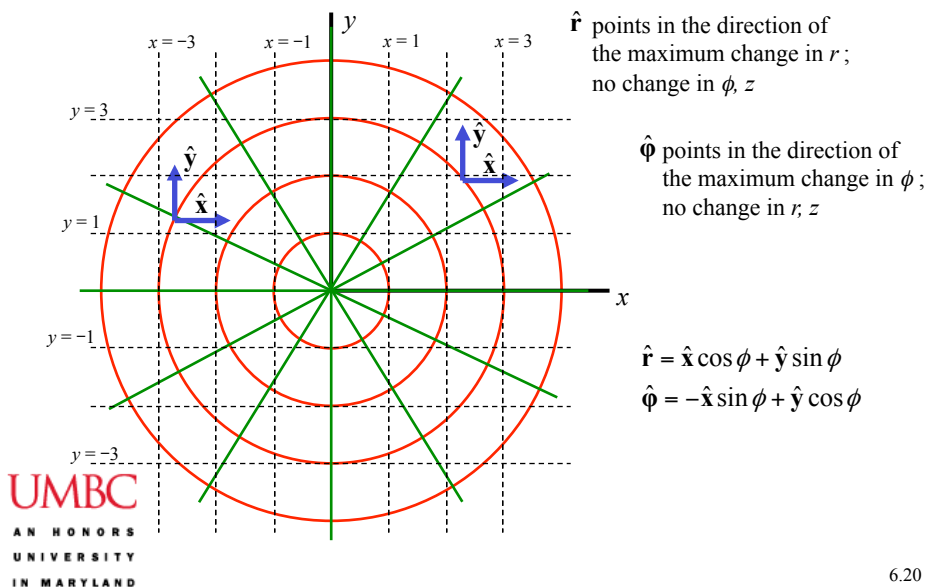
Ulaby Figure 3-9

6.18

Cylindrical Coordinates



Cylindrical Coordinates



Cylindrical Coordinates

Base vectors

$$\begin{aligned}\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}} &= \hat{\mathbf{z}}, & \hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}} &= \hat{\mathbf{r}}, & \hat{\mathbf{z}} \times \hat{\mathbf{r}} &= \hat{\boldsymbol{\phi}} \\ \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} &= \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1, & \hat{\mathbf{r}} \times \hat{\mathbf{r}} &= \hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0 \\ \hat{\mathbf{r}} \cdot \hat{\boldsymbol{\phi}} &= \hat{\mathbf{r}} \cdot \hat{\mathbf{z}} = \hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{z}} = 0\end{aligned}$$

Vector relations

$$\mathbf{A} = \hat{\mathbf{r}} A_r + \hat{\boldsymbol{\phi}} A_\phi + \hat{\mathbf{z}} A_z$$

$$A = |\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}} = \left(A_r^2 + A_\phi^2 + A_z^2 \right)^{1/2}$$

Letting $P = P(r_1, \phi_1, z_1)$, the position vector $\mathbf{R}_1 = OP = \hat{\mathbf{r}} r_1 + \hat{\mathbf{z}} z_1$

We note that the direction of $\hat{\mathbf{r}}$ depends on ϕ

Cylindrical Coordinates

Differential relations

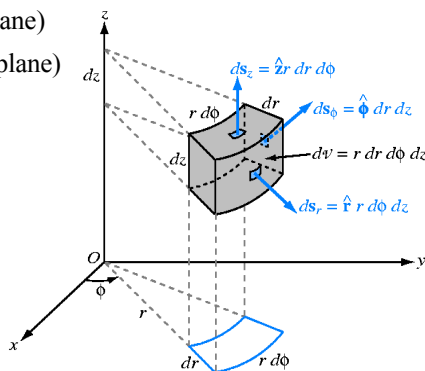
Length: $d\mathbf{l} = \hat{\mathbf{r}} dl_r + \hat{\boldsymbol{\phi}} dl_\phi + \hat{\mathbf{z}} dl_z = \hat{\mathbf{r}} dr + \hat{\boldsymbol{\phi}} r d\phi + \hat{\mathbf{z}} dz$

Surface area: $d\mathbf{s}_r = \hat{\mathbf{r}} r d\phi dz$ (ϕ - z plane)

$$d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}} dr dz \quad (r\text{-}z \text{ plane})$$

$$d\mathbf{s}_z = \hat{\mathbf{z}} r dr d\phi \quad (r\text{-}\phi \text{ plane})$$

Volume: $dv = r dr d\phi dz$



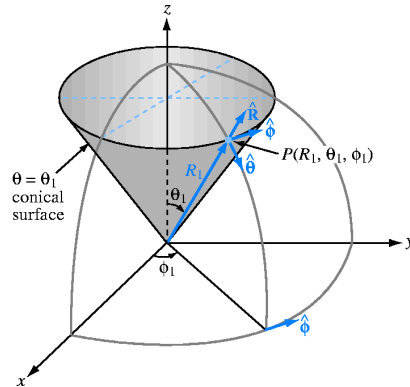
Ulab Figure 3-10

Spherical Coordinates

R = distance from the origin ($0 < R < \infty$)

θ = angle measured from the $+z$ -axis ($0 < \theta < \pi$) = *zenith angle*

ϕ = angle measured from the $+x$ -axis on the x - y plane ($0 < \phi < 2\pi$)
= *azimuthal angle*



Ulaby Figure 3-13 6.23

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Spherical Coordinates

Base vectors

$$\hat{\mathbf{R}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}, \quad \hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}, \quad \hat{\boldsymbol{\phi}} \times \hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$$

$$\hat{\mathbf{R}} \cdot \hat{\mathbf{R}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = 1, \quad \hat{\mathbf{R}} \times \hat{\mathbf{R}} = \hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{\phi}} = 0$$

$$\hat{\mathbf{R}} \cdot \hat{\boldsymbol{\theta}} = \hat{\mathbf{R}} \cdot \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\phi}} = 0$$

Vector relations

$$\mathbf{A} = \hat{\mathbf{a}} A = \hat{\mathbf{R}} A_R + \hat{\boldsymbol{\theta}} A_\theta + \hat{\boldsymbol{\phi}} A_\phi$$

$$A = |\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}} = \left(A_R^2 + A_\theta^2 + A_\phi^2 \right)^{1/2}$$

Letting $P = P(R_1, \theta_1, \phi_1)$, the position vector $\mathbf{R}_1 = OP = \hat{\mathbf{R}} R_1$

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Spherical Coordinates

Differential relations

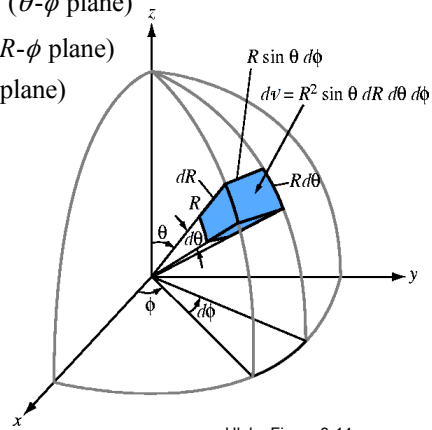
Length: $d\mathbf{l} = \hat{\mathbf{R}} dl_R + \hat{\boldsymbol{\theta}} dl_\theta + \hat{\boldsymbol{\phi}} dl_\phi = \hat{\mathbf{R}} dR + \hat{\boldsymbol{\theta}} R d\theta + \hat{\boldsymbol{\phi}} R \sin \theta d\phi$

Surface area: $d\mathbf{s}_R = \hat{\mathbf{R}} R^2 \sin \theta d\theta d\phi$ (θ - ϕ plane)

$d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}} R \sin \theta dR d\phi$ (R - ϕ plane)

$d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}} R dR d\theta$ (R - θ plane)

Volume: $dv = R^2 \sin \theta dR d\theta d\phi$



Ulaby Figure 3-14

6.25

Spherical Coordinates

Charge in a Sphere: Ulaby and Ravaioli Example 3-6

Question: A sphere of radius 2 cm contains a charge of density ρ_V given by

$$\rho_V = 4 \cos^2 \theta$$

What is the total charge?

Answer: After converting from cm to m,

$$\begin{aligned} Q &= \int_V \rho_V dv \\ &= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \int_{R=0}^{2 \times 10^{-2}} (4 \cos^2 \theta) R^2 \sin \theta dR d\theta d\phi \\ &= 4 \int_0^{2\pi} \int_0^{\pi} \left(\frac{R^3}{3} \right) \bigg|_0^{2 \times 10^{-2}} \sin \theta \cos^2 \theta d\theta d\phi \\ &= \frac{64}{9} \times 10^{-6} \int_0^{2\pi} d\phi = \frac{128\pi}{9} \times 10^{-6} = 44.68 \mu\text{C} \end{aligned}$$

6.26

Tech Brief 5: GPS

- Originally developed by DOD
- Originally 24 satellites, now 31
- Operation principles
 - Satellites constantly broadcast a message containing
 - Their location
 - Message time
 - System health and rough orbit details
 - GPS receivers use triangulation to determine location relative to satellites
 - Determine distances to each satellite by solving an equation including the sat. position and time, multiplying by the speed of light.



$$d_1^2 = (x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2 = c [(t_1 + t_0)]^2,$$

$$d_2^2 = (x_2 - x_0)^2 + (y_2 - y_0)^2 + (z_2 - z_0)^2 = c [(t_2 + t_0)]^2,$$

$$d_3^2 = (x_3 - x_0)^2 + (y_3 - y_0)^2 + (z_3 - z_0)^2 = c [(t_3 + t_0)]^2,$$

$$d_4^2 = (x_4 - x_0)^2 + (y_4 - y_0)^2 + (z_4 - z_0)^2 = c [(t_4 + t_0)]^2.$$



Four satellites are needed to correct for the imprecise receiver clock (quartz).

6.27

Tech Brief 5: Differential GPS

- GPS accuracy: 20-30m
- Differential GPS (DGPS) uses a static reference of known location in the receiver's area to correct for inaccuracy factors
 - Time-delay errors (speed of light differences)
 - Multipath interference
 - Satellite location errors
- Reference receiver calculates correction factors and transmits to DGPS receivers in the area

