Vector Algebra
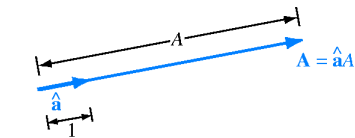
Definition
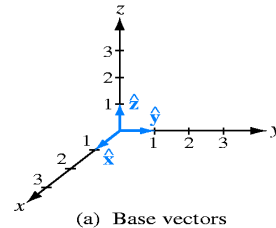
A vector **A** has a magnitude $A = |\mathbf{A}|$ and a direction given by the unit vector $\hat{\mathbf{a}} = \mathbf{A} / |\mathbf{A}|$.

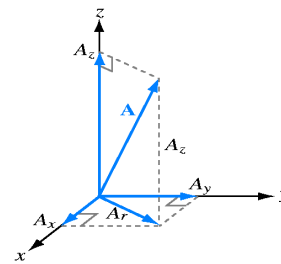In Cartesian coordinates (x, y, z), we find

$$\mathbf{A} = \hat{\mathbf{a}} A = \hat{\mathbf{x}} A_x + \hat{\mathbf{y}} A_y + \hat{\mathbf{z}} A_z$$

$$= (A_x, A_y, A_z)$$

(a) Base vectors

(b) Components of **A**

Ulaby Figure 3-1

Ulaby Figure 3-2

6.1

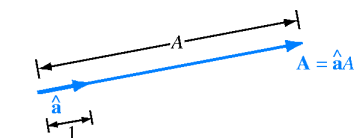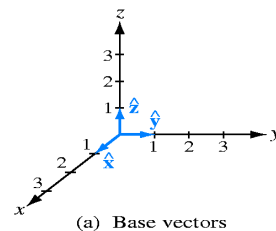---

Vector Algebra

Definition

We also have
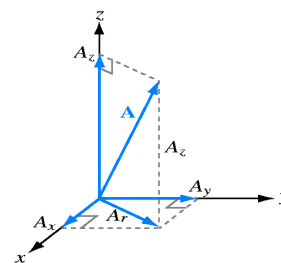
$$A = \left(A_x^2 + A_y^2 + A_z^2 \right)^{1/2}$$

$$\hat{\mathbf{a}} = \frac{\hat{\mathbf{x}} A_x + \hat{\mathbf{y}} A_y + \hat{\mathbf{z}} A_z}{\left(A_x^2 + A_y^2 + A_z^2 \right)^{1/2}}$$

(a) Base vectors

(b) Components of **A**

Ulaby Figure 3-1

Ulaby Figure 3-2

6.2

1

Vector Algebra

Equality

Two vectors **A** and **B** are equal if
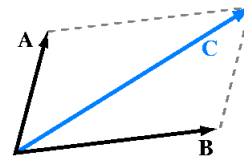
$$A_x = B_x,\ A_y = B_y,\ \text{and}\ A_z = B_z$$

Addition and Subtraction

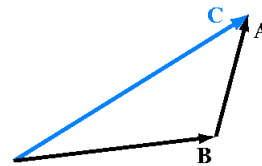$$\mathbf{C} = \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \quad \text{(addition is commutative)}$$
$$= \hat{\mathbf{x}}(A_x + B_x) + \hat{\mathbf{y}}(A_y + B_y) + \hat{\mathbf{z}}(A_z + B_z)$$

$$\mathbf{D} = \mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B})$$
$$= \hat{\mathbf{x}}(A_x - B_x) + \hat{\mathbf{y}}(A_y - B_y) + \hat{\mathbf{z}}(A_z - B_z)$$

(a) Parallelogram rule

(b) Head-to-tail rule

Ulaby Figure 3-3

6.3

Vector Algebra

Position and Distance vectors

For a point $P(x, y, z)$, the *position vector* **R** goes from the origin to the point

$$\mathbf{R} = \hat{\mathbf{x}}\, x + \hat{\mathbf{y}}\, y + \hat{\mathbf{z}}\, z$$

The distance vector between two points
$P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is defined as

$$\mathbf{R}_{12} = \mathbf{R}_2 - \mathbf{R}_1$$
$$= \hat{\mathbf{x}}(x_2 - x_1) + \hat{\mathbf{y}}(y_2 - y_1) + \hat{\mathbf{z}}(z_2 - z_1)$$

where $\mathbf{R}_i = \hat{\mathbf{x}}\, x_i + \hat{\mathbf{y}}\, y_i + \hat{\mathbf{z}}\, z_i,\ i = 1, 2$

The distance d between two points
equals $|\mathbf{R}_{12}|$

$$d = |\mathbf{R}_{12}|$$
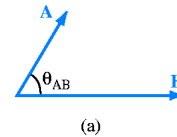$$= \left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \right]^{1/2}$$

Ulaby Figure 3-4

6.4

2
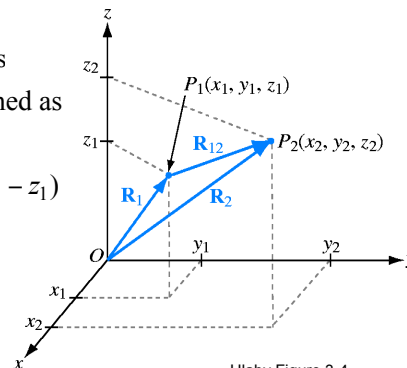
Vector Algebra

Multiplication — Simple Product

When a vector **A** is multiplied by a scalar k, the magnitude is multiplied by k and the direction is unchanged

$$\mathbf{B} = k\mathbf{A} = \hat{\mathbf{a}}(kA)$$

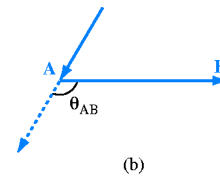$$= \hat{\mathbf{x}}(kA_x) + \hat{\mathbf{y}}(kA_y) + \hat{\mathbf{z}}(kA_z)$$

(a)

Multiplication — Scalar or Dot Product

The scalar or dot product of two vectors

A and **B** is defined by

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta_{AB}$$

where θ_{AB} is the angle between **A** and **B**

(b)

Ulaby Figure 3-5

When $\theta_{AB} < 90^{\circ}$, $\mathbf{A} \cdot \mathbf{B} > 0$
When $\theta_{AB} > 90^{\circ}$, $\mathbf{A} \cdot \mathbf{B} < 0$
When $\theta_{AB} = 90^{\circ}$, $\mathbf{A} \cdot \mathbf{B} = 0$, and the vectors are called *orthogonal*

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6.5

Vector Algebra

Multiplication — Scalar or Dot Product

We have

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{x}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1$$

$$\hat{\mathbf{x}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{x}} \cdot \hat{\mathbf{z}} = 0$$

so that

(a)

$$\mathbf{A} \cdot \mathbf{B} = \left(\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z \right) \cdot \left(\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z \right)$$

$$= A_x B_x + A_y B_y + A_z B_z$$

Other properties:

(b)

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \quad \text{(commutative)}$$

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C} \quad \text{(distributive)}$$
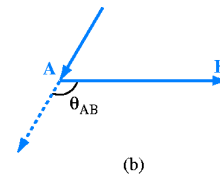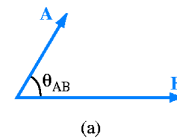
Ulaby Figure 3-5

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6.6

3

Vector Algebra

Multiplication — Vector or Cross Product

The vector or cross product of two vectors **A** and **B** is defined by

$$\mathbf{A}\times\mathbf{B} = \hat{\mathbf{n}}\,AB\sin\theta_{AB}$$

where θ_{AB} is the angle *from* **A** *to* **B** and $\hat{\mathbf{n}}$ is a unit vector determined by the right-hand rule

The vector product is *anti-commutative*

$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}$$

and distributive

$$\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C}$$

$$\hat{\mathbf{x}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}\times\hat{\mathbf{z}} = 0$$

$$\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}, \quad \hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}}, \quad \hat{\mathbf{z}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}$$

$\mathbf{A}\times\mathbf{B} = \hat{\mathbf{n}}\,AB\sin\theta_{AB}$

(a) Cross product

(b) Right-hand rule
$\mathbf{A}\times\mathbf{B}$

Ulaby Figure 3-6

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6.7

Vector Algebra

Multiplication — Vector or Cross Product

We have

$$\hat{\mathbf{x}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}\times\hat{\mathbf{z}} = 0$$

$$\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}, \quad \hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}}, \quad \hat{\mathbf{z}}\times\hat{\mathbf{x}} = \hat{\mathbf{y}}$$
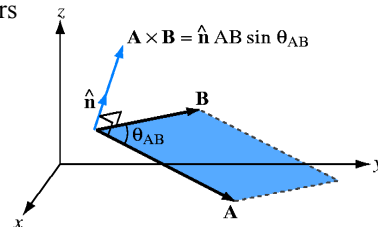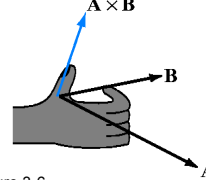
so that

$$\mathbf{A}\times\mathbf{B} = \left(\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z\right)\times\left(\hat{\mathbf{x}}B_x + \hat{\mathbf{y}}B_y + \hat{\mathbf{z}}B_z\right)$$

$$= \hat{\mathbf{x}}(A_yB_z - A_zB_y) + \hat{\mathbf{y}}(A_zB_x - A_xB_z)$$

$$+ \hat{\mathbf{z}}(A_xB_y - A_yB_x)$$

We may also write

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$\mathbf{A}\times\mathbf{B} = \hat{\mathbf{n}}\,AB\sin\theta_{AB}$

(a) Cross product

(b) Right-hand rule
$\mathbf{A}\times\mathbf{B}$

Ulaby Figure 3-6

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This determinant form is very useful!
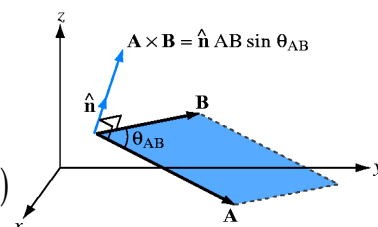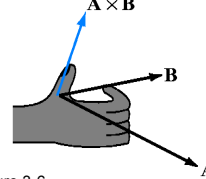
6.8

Vector Algebra

Summary of vector products

Product type	Product elements	Representation
Simple product	(Scalar) x (Vector) → Vector	$\mathbf{C} = k\mathbf{A}$
Scalar product or Dot product	(Vector) x (Vector) → Scalar	$\mathbf{C} = \mathbf{A} \cdot \mathbf{B}$
Vector product or Cross product	(Vector) x (Vector) → Vector	$\mathbf{C} = \mathbf{A} \times \mathbf{B}$

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6.9

Vector Algebra

Vectors and Angles: Ulaby and Ravaioli Example 3-1

Question: In Cartesian coordinates, vector **A** is directed from the origin to the point $P_1(2, 3, 3)$, and vector **B** is directed from P_1 to $P_2(1, -2, 2)$.
Find (a) the vector **A**, its magnitude A, and its unit vector $\hat{\mathbf{a}}$, (b) the angle that **A** makes with the y-axis, (c) vector **B**, (d) the angle between **A** and **B**, and (e) the perpendicular distance from the origin to **B**.

Answer: (a)

$$\mathbf{A} = \hat{\mathbf{x}}\,2 + \hat{\mathbf{y}}\,3 + \hat{\mathbf{z}}\,3$$

$$A = \sqrt{4 + 9 + 9} = \sqrt{22}$$

$$\hat{\mathbf{a}} = \mathbf{A}/A = \left(\hat{\mathbf{x}}\,2 + \hat{\mathbf{y}}\,3 + \hat{\mathbf{z}}\,3\right)\big/\sqrt{22}$$

Ulaby
Figure 3-7

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6.10

5
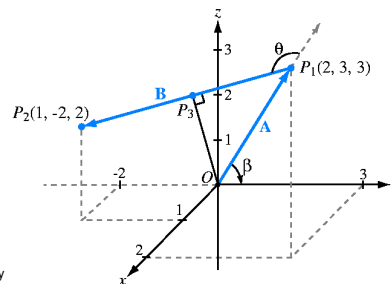
Vector Algebra

Vectors and Angles: Ulaby and Ravaioli Example 3-1

Answer (continued): (b) The angle β between **A** and the *y*-axis may be found from the expression $\mathbf{A} \cdot \hat{\mathbf{y}} = A \cos \beta$, which implies

$$\beta = \cos^{-1} \left(\frac{\mathbf{A} \cdot \hat{\mathbf{y}}}{A} \right) = \cos^{-1} \left(\frac{3}{\sqrt{22}} \right) = 0.879 \text{ rads} = 50.2°$$

(c) $\mathbf{B} = \hat{\mathbf{x}}(1-2) + \hat{\mathbf{y}}(-2-3) + \hat{\mathbf{z}}(2-3) = -\hat{\mathbf{x}} - \hat{\mathbf{y}}5 - \hat{\mathbf{z}}$

(d) $\theta = \cos^{-1} \left(\frac{\mathbf{A} \cdot \mathbf{B}}{A B} \right) = \cos^{-1} \left(\frac{-20}{\sqrt{22}\sqrt{27}} \right)$

$\quad\quad = 2.533 \text{ rads} = 145.1°$

(e) The points $OP_1 P_3$ form a right triangle.
The magnitude of the line segment
OP_3 is given by

$A \sin(\pi - \theta) = \sqrt{22} \sin(0.609) = 2.68$
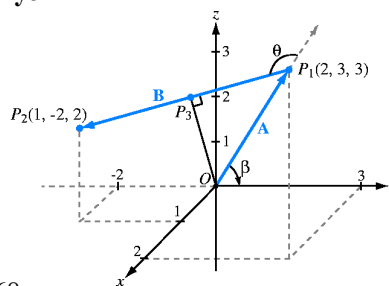
Ulaby Figure 3-7

6.11

Vector Algebra

Triple Scalar Product

This product can be written in the equivalent forms

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

The equivalence can be demonstrated from the determinant representation

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

The absolute value of this scalar product is the volume of the parallelepiped whose sides are the vectors **A**, **B**, and **C.**

6.12

6

Vector Algebra

Triple Scalar Product

This product can be written in the equivalent forms

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$$

It corresponds to the volume of a parallelepiped
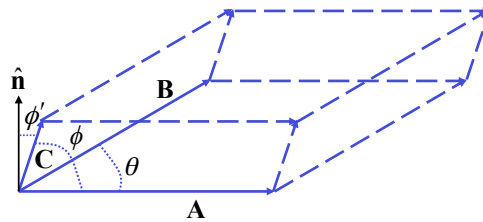
$$\hat{\mathbf{n}} = \frac{\mathbf{A} \times \mathbf{B}}{|\mathbf{A} \times \mathbf{B}|}$$

$$\phi' = \frac{\pi}{2} - \phi$$

Surface area $= |\mathbf{A} \times \mathbf{B}| = |\mathbf{A}||\mathbf{B}| \sin \theta$

Volume $= |\mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})| = |\mathbf{A}||\mathbf{B}||\mathbf{C}| \sin \theta \cos \phi' = |\mathbf{A}||\mathbf{B}||\mathbf{C}| \sin \theta \sin \phi$

6.13

Vector Algebra

Triple Vector Product

This product is defined as

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \quad \text{and we note} \quad \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$$

This relationship can also be written in the form

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$

6.14

Orthogonal Coordinate Systems

Three coordinate systems

Coordinate systems (with three coordinates) specify points in space

Orthogonal systems have coordinates that are mutually perpendicular
— at least locally

There are three coordinate systems that are often used

- Cartesian coordinates (x, y, z): Simplest and orthogonal everywhere
- Cylindrical coordinates (r, θ, z)
  — used with optical fibers, coaxial cables, cylindrical waveguides
- Spherical coordinates (R, θ, ϕ)
  — used with antenna radiation, radar, earth-ionosphere waveguide

6.15

Three Coordinate Systems

Summary of vector and differential relations

	Cartesian Coordinates	Cylindrical Coordinates	Spherical Coordinates
Coordinate variables	x, y, z	r, θ, z	R, θ, ϕ
Vector, $\mathbf{A} =$	$\hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{r}}A_r + \hat{\boldsymbol{\phi}}A_\phi + \hat{\mathbf{z}}A_z$	$\hat{\mathbf{R}}A_R + \hat{\boldsymbol{\theta}}A_\theta + \hat{\boldsymbol{\phi}}A_\phi$
Magnitude, $\|\mathbf{A}\| =$	$\left(A_x^2 + A_y^2 + A_z^2\right)^{1/2}$	$\left(A_r^2 + A_\phi^2 + A_z^2\right)^{1/2}$	$\left(A_R^2 + A_\theta^2 + A_\phi^2\right)^{1/2}$
Position vector	$\hat{\mathbf{x}}x_1 + \hat{\mathbf{y}}y_1 + \hat{\mathbf{z}}z_1$ for $P(x_1, y_1, z_1)$	$\hat{\mathbf{r}}r_1 + \hat{\mathbf{z}}z_1$ for $P(r_1, \phi_1, z_1)$	$\hat{\mathbf{R}}R_1$ for $P(R_1, \theta_1, \phi_1)$
Dot product, $\mathbf{A}\cdot\mathbf{B}$	$A_xB_x + A_yB_y + A_zB_z$	$A_rB_r + A_\phi B_\phi + A_zB_z$	$A_RB_R + A_\theta B_\theta + A_\phi B_\phi$
Cross product, $\mathbf{A}\times\mathbf{B}$	$\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{r}} & \hat{\boldsymbol{\phi}} & \hat{\mathbf{z}} \\ A_r & A_\phi & A_z \\ B_r & B_\phi & B_z \end{vmatrix}$	$\begin{vmatrix} \hat{\mathbf{R}} & \hat{\boldsymbol{\theta}} & \hat{\boldsymbol{\phi}} \\ A_R & A_\theta & A_\phi \\ B_R & B_\theta & B_\phi \end{vmatrix}$
Differential length, d**l**	$\hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{r}}\,dr + \hat{\boldsymbol{\phi}}\,rd\phi + \hat{\mathbf{z}}\,dz$	$\hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}\,Rd\theta + \hat{\boldsymbol{\phi}}\,R\sin\theta dz$
Differential surface areas	$d\mathbf{s}_x = \hat{\mathbf{x}}\,dydz$ $d\mathbf{s}_y = \hat{\mathbf{y}}\,dzdx$ $d\mathbf{s}_z = \hat{\mathbf{z}}\,dxdy$	$d\mathbf{s}_r = \hat{\mathbf{r}}\,rd\phi dz$ $d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}}\,dzdr$ $d\mathbf{s}_z = \hat{\mathbf{z}}\,rdrd\phi$	$d\mathbf{s}_R = \hat{\mathbf{R}}\,R^2\sin\theta d\theta d\phi$ $d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}\,R\sin\theta d\phi dR$ $d\mathbf{s}_z = \hat{\boldsymbol{\phi}}\,RdRd\theta$
Differential volume	$dv = dx\,dy\,dz$	$dv = rdr\,d\phi dz$	$dv = R^2\sin\theta\,dR\,d\theta\,d\phi$

6.16

8

Cartesian Coordinates

Differential Relations

Length: $d\,\mathbf{l} = \hat{\mathbf{x}}\,dl_x + \hat{\mathbf{y}}\,dl_y + \hat{\mathbf{z}}\,dl_z = \hat{\mathbf{x}}\,dx + \hat{\mathbf{y}}\,dy + \hat{\mathbf{z}}\,dz$

Surface area: $d\mathbf{s}_x = \hat{\mathbf{x}}\,dl_y\,dl_z = \hat{\mathbf{x}}\,dy\,dz$ (y-z plane)

$\qquad\qquad\quad d\mathbf{s}_y = \hat{\mathbf{y}}\,dz\,dx = \hat{\mathbf{y}}\,dx\,dz$ (x-z plane)

$\qquad\qquad\quad d\mathbf{s}_z = \hat{\mathbf{z}}\,dx\,dy$ (x-y plane)

Volume: $dv = dx\,dy\,dz$

For each coordinate system there are three differential lengths that also determine the differential areas and volumes

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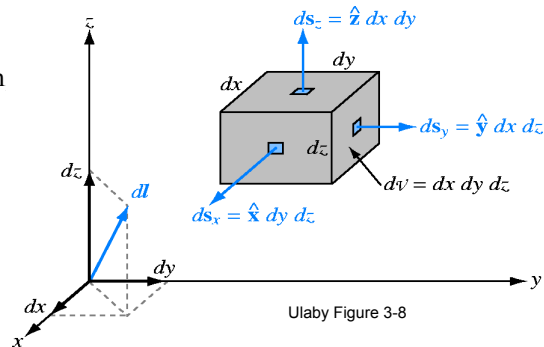
Ulaby Figure 3-8

6.17

Cylindrical Coordinates

r = distance from the z-axis $(0 < r < \infty)$

ϕ = angle measured on the x-y plane from the $+x$-axis $(0 < \phi < 2\pi)$

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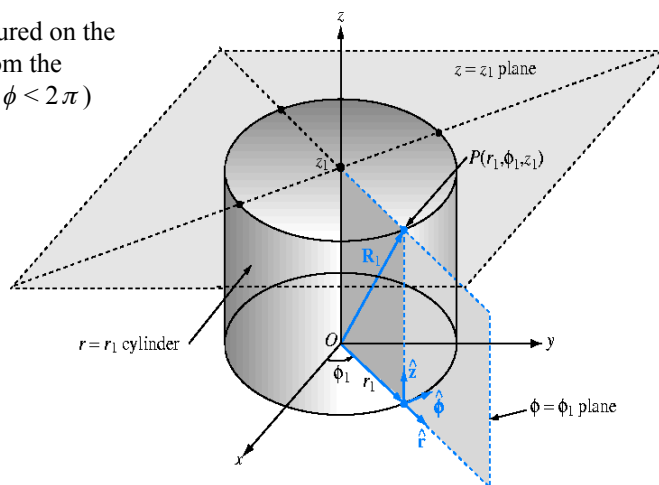
Ulaby Figure 3-9   6.18

Cylindrical Coordinates

$\hat{\mathbf{r}}$ points in the direction of the maximum change in r ; no change in ϕ, z

$\hat{\boldsymbol{\varphi}}$ points in the direction of the maximum change in ϕ ; no change in r, z

$\phi = 2\pi/3$ $\phi = \pi/3$

$\phi = 5\pi/6$ $\phi = \pi/6$

$\phi = \pi$ x

$r = 1$
$r = 2$
$r = 3$
$r = 4$

$\phi = -5\pi/6$ $\phi = -\pi/6$

$\phi = -2\pi/3$ $\phi = -\pi/3$

$\phi = -\pi/2$

$\hat{\boldsymbol{\varphi}}$ $\hat{\mathbf{r}}$

$\hat{\mathbf{r}}$ $\hat{\boldsymbol{\varphi}}$

y

6.19

Cylindrical Coordinates

$\hat{\mathbf{r}}$ points in the direction of the maximum change in r ; no change in ϕ, z

$\hat{\boldsymbol{\varphi}}$ points in the direction of the maximum change in ϕ ; no change in r, z

$x = -3$ $x = -1$ $x = 1$ $x = 3$

$y = 3$
$y = 1$
$y = -1$
$y = -3$

y

x

$\hat{\mathbf{y}}$ $\hat{\mathbf{x}}$

$\hat{\mathbf{y}}$ $\hat{\mathbf{x}}$

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$$
$$\hat{\boldsymbol{\varphi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$$

6.20

Cylindrical Coordinates

Base vectors

$$\hat{\mathbf{r}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}}, \quad \hat{\boldsymbol{\phi}} \times \hat{\mathbf{z}} = \hat{\mathbf{r}}, \quad \hat{\mathbf{z}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}}$$

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1, \quad \hat{\mathbf{r}} \times \hat{\mathbf{r}} = \hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0$$

$$\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\phi}} = \hat{\mathbf{r}} \cdot \hat{\mathbf{z}} = \hat{\boldsymbol{\phi}} \cdot \hat{\mathbf{z}} = 0$$

Vector relations

$$\mathbf{A} = \hat{\mathbf{a}} A = \hat{\mathbf{r}} A_r + \hat{\boldsymbol{\phi}} A_\phi + \hat{\mathbf{z}} A_z$$

$$A = |\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}} = \left(A_r^2 + A_\phi^2 + A_z^2 \right)^{1/2}$$

Letting $P = P(r_1, \phi_1, z_1)$, the position vector $\mathbf{R}_1 = OP = \hat{\mathbf{r}} r_1 + \hat{\mathbf{z}} z_1$

We note that the direction of $\hat{\mathbf{r}}$ depends on ϕ

6.21

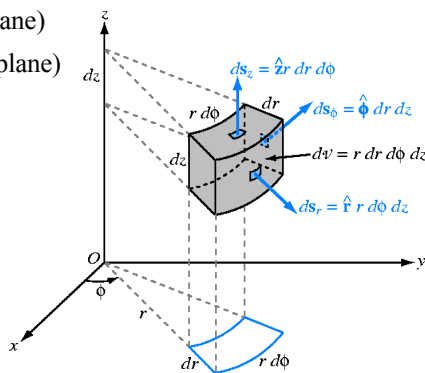## Cylindrical Coordinates

Differential relations

Length: $d\mathbf{l} = \hat{\mathbf{r}} \, dl_r + \hat{\boldsymbol{\phi}} \, dl_\phi + \hat{\mathbf{z}} \, dl_z = \hat{\mathbf{r}} \, dr + \hat{\boldsymbol{\phi}} \, rd\phi + \hat{\mathbf{z}} \, dz$

Surface area: $d\mathbf{s}_r = \hat{\mathbf{r}} \, rd\phi \, dz \quad (\phi\text{-}z \text{ plane})$

$d\mathbf{s}_\phi = \hat{\boldsymbol{\phi}} \, dr \, dz \quad (r\text{-}z \text{ plane})$

$d\mathbf{s}_z = \hat{\mathbf{z}} \, rdr \, d\phi \quad (r\text{-}\phi \text{ plane})$

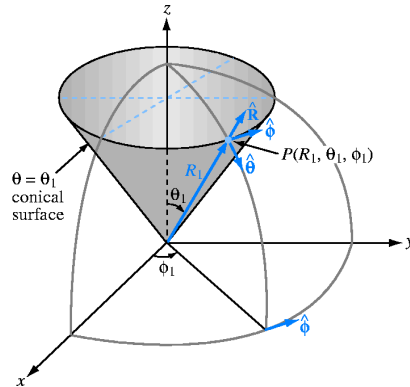Volume: $dv = rdr \, d\phi \, dz$

Ulaby Figure 3-10

Divergence

6.22

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Spherical Coordinates

R = distance from the origin ($0 < R < \infty$)

θ = angle measured from the $+z$-axis ($0 < \theta < \pi$) = *zenith angle*

ϕ = angle measured from the $+x$-axis on the *x-y* plane ($0 < \phi < 2\pi$)
  = *azimuthal angle*

Ulaby Figure 3-13   6.23

Spherical Coordinates

Base vectors

$$\hat{\mathbf{R}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}, \quad \hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\phi}} = \hat{\mathbf{R}}, \quad \hat{\boldsymbol{\phi}} \times \hat{\mathbf{R}} = \hat{\boldsymbol{\theta}}$$

$$\hat{\mathbf{R}} \cdot \hat{\mathbf{R}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = 1, \quad \hat{\mathbf{R}} \times \hat{\mathbf{R}} = \hat{\boldsymbol{\phi}} \times \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{\theta}} = 0$$

$$\hat{\mathbf{R}} \cdot \hat{\boldsymbol{\theta}} = \hat{\mathbf{R}} \cdot \hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\phi}} = 0$$

Vector relations

$$\mathbf{A} = \hat{\mathbf{a}} A = \hat{\mathbf{R}} A_R + \hat{\boldsymbol{\theta}} A_\theta + \hat{\boldsymbol{\phi}} A_\phi$$

$$A = |\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}} = \left(A_R^2 + A_\theta^2 + A_\phi^2 \right)^{1/2}$$

Letting $P = P(R_1, \theta_1, \phi_1)$, the position vector $\mathbf{R}_1 = OP = \hat{\mathbf{R}} R_1$

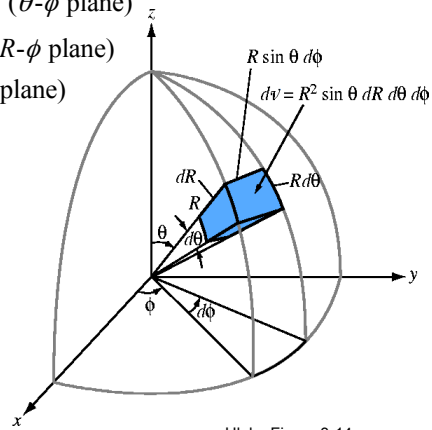6.24

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Spherical Coordinates

Differential relations

Length: $d\mathbf{l} = \hat{\mathbf{R}}\,dl_R + \hat{\boldsymbol{\theta}}\,dl_\theta + \hat{\boldsymbol{\varphi}}\,dl_\phi = \hat{\mathbf{R}}\,dR + \hat{\boldsymbol{\theta}}\,R\,d\theta + \hat{\boldsymbol{\varphi}}\,R\sin\theta\,d\phi$

Surface area: $d\mathbf{s}_R = \hat{\mathbf{R}}\,R^2 \sin\theta\,d\theta\,d\phi$ (θ-ϕ plane)

$d\mathbf{s}_\theta = \hat{\boldsymbol{\theta}}\,R\sin\theta\,dR\,d\phi$ (R-ϕ plane)

$d\mathbf{s}_\phi = \hat{\boldsymbol{\varphi}}\,R\,dR\,d\theta$ (R-θ plane)

Volume: $dv = R^2 \sin\theta\,dR\,d\theta\,d\phi$

Ulaby Figure 3-14

6.25

Spherical Coordinates

Charge in a Sphere: Ulaby and Ravaioli Example 3-6

Question: A sphere of radius 2 cm contains a charge of density ρ_V given by

$$\rho_V = 4\cos^2\theta$$

What is the total charge?

Answer: After converting from cm to m,

$$Q = \int_v \rho_V\,dv$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{R=0}^{2\times10^{-2}} \left(4\cos^2\theta\right) R^2 \sin\theta\,dR\,d\theta\,d\phi$$

$$= 4\int_0^{2\pi}\int_0^{\pi} \left(\frac{R^3}{3}\right)\Bigg|_0^{2\times10^{-2}} \sin\theta\cos^2\theta\,d\theta\,d\phi$$

$$= \frac{64}{9}\times10^{-6}\int_0^{2\pi} d\phi = \frac{128\pi}{9}\times10^{-6} = 44.68\ \mu C$$

6.26

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Tech Brief 5: GPS

- Originally developed by DOD
- Originally 24 satellites, now 31
- Operation principles
  - Satellites constantly broadcast a message containing
    - Their location
    - Message time
    - System health and rough orbit details
  - GPS receivers use triangulation to determine location relative to satellites
    - Determine distances to each satellite by solving an equation including the sat. position and time, multiplying by the speed of light.

$$d_1^2 = (x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2 = c\,[(t_1 + t_0)]^2 ,$$
$$d_2^2 = (x_2 - x_0)^2 + (y_2 - y_0)^2 + (z_2 - z_0)^2 = c\,[(t_2 + t_0)]^2 ,$$
$$d_3^2 = (x_3 - x_0)^2 + (y_3 - y_0)^2 + (z_3 - z_0)^2 = c\,[(t_3 + t_0)]^2 ,$$
$$d_4^2 = (x_4 - x_0)^2 + (y_4 - y_0)^2 + (z_4 - z_0)^2 = c\,[(t_4 + t_0)]^2 .$$

Four satellites are needed to correct for the imprecise receiver clock (quartz).

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6.27

Tech Brief 5: Differential GPS

- GPS accuracy: 20-30m

- Differential GPS (DGPS) uses a static reference of known location in the receiver's area to correct for inaccuracy factors
  - Time-delay errors (speed of light differences
  - Multipath interference
  - Satellite location errors

- Reference receiver calculates correction factors and transmits to DGPS receivers in the area

SAT 3 (x_3, y_3, z_3) d_3
SAT 2 (x_2, y_2, z_2) d_2
SAT 4 (x_4, y_4, z_4) d_4
SAT 1 (x_1, y_1, z_1) d_1
Time delay
Receiver Code
Satellite Code
(x_0, y_0, z_0)

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