CMPE 320: Probability, Statistics, and Random Processes

Lecture 7: Counting

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Counting in probability

- Calculating probability often involves counting
  - Probability of getting an even number from rolling a die event A

Individual events are equally likely

$$P(A) = \frac{\#\text{ elements in } A}{\#\text{ elements in } \Omega} = \frac{3}{6} = \frac{1}{2}$$

  - Probability of getting 1 head in 3 coin tosses $\to B$

HTT, THT, TTH : equally likely

$P(B) = P(\text{individual outcomes}) \times \text{their number}$

$= \left(\frac{1}{2}\right)^3 \times 3$

Counting principle

- Divide and conquer

First stage: n_1 choices

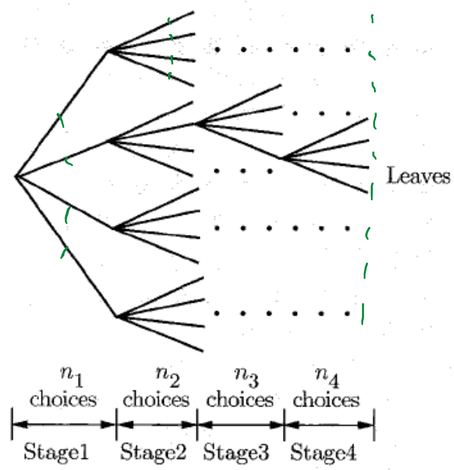
For each of first stage choices

2nd stage: n_2 choices

⋮

r-th stage: n_r choices

Total possibilities: $\boxed{n_1 \times n_2 \times \cdots \times n_r}$



Count the number of all subsets of an n-element set

$n = 4$

$\{1, 2, 3, 4\}$

First element → either include in the subset or not : 2 choices

2nd element → 2 choices

⋮

n-th element → 2 choices

$$\underbrace{2 \times 2 \times \cdots \times 2}_{n \text{ times}} = \boxed{2^n}$$

Permutations

$_nP_k$ $\binom{n}{k} = {}_nC_k$

- Count the number of different ways to pick *k* out of *n* objects and arrange them in a sequence (the order matters!)

2-permutation of letters A, B, C, and D → $4 \times 3 = 12$

AB BA AC CA AD DA BC CB BD DB CD DC

First stage : n choices
2nd stage : $(n-1)$ choices
⋮
k-th stage : $(n-k+1)$ choices

$\Rightarrow n(n-1)(n-2) \cdots (n-k+1)$

$$= \frac{n(n-1)\cdots 1}{(n-k)(n-k-1)\cdots 1} = \boxed{\frac{n!}{(n-k)!}}$$

If $k = n \Rightarrow n!$

Example 1.29. You have n_1 classical music CDs, n_2 rock music CDs, and n_3 country music CDs. In how many different ways can you arrange them so that the CDs of the same type are contiguous?

Combinations

- Count the number of different ways to pick *k* out of *n* objects without ordering them
  - Forming a committee of k people out of n people
  - Counting the number of k-element subsets of an n-element set

Example 1.31. We have a group of n persons. Consider clubs that consist of a special person from the group (the club leader) and a number (possibly zero) of additional club members. Let us count the number of possible clubs of this type.

Partitions

- ``n choose k” can be viewed as partitioning n elements into two parts: the part with k elements and the other with (n-k)
- Given n elements, count the number of ways to partition these into r parts with n_1, n_2, … n_r elements. ($\sum_{i=1}^{r} n_i = n$)

Example 1.32. Anagrams. How many different words (letter sequences) can be obtained by rearranging the letters in the word TATTOO?