Second Midterm Examination Solutions

1. We have $\Gamma = (200 - 100)/(200 + 100) = 1/3$, and since Γ is purely real, we have

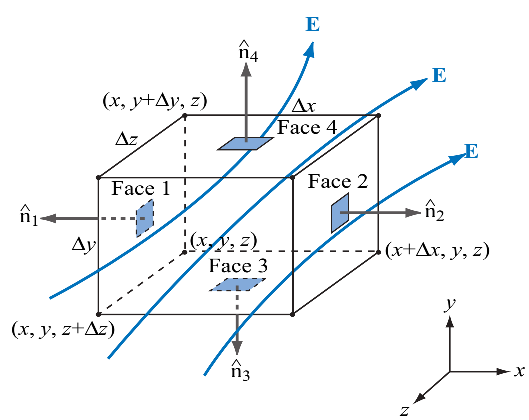
$$S = \frac{1 + |\Gamma|}{1 - |\Gamma|} = \frac{1 + \Gamma}{1 - \Gamma} = 2.$$

   Because Γ is purely real and positive, the voltage has a maximum at the load and every distance $-z = l = n\lambda/2$ away from the load, where $n = 1, 2, 3, \ldots$ So, the first maximum away from the load is at $l = \lambda/2$. The minima occur in between the maxima when $-z = l = (\lambda/4) + n\lambda/2$, where $n = 0, 1, 2, \ldots$ Hence, the first minimum occurs at $l = \lambda/4$

2. Since the load impedance and the characteristic impedance are matched, there is only a forward-propagating wave, and the input impedance must therefore equal the load impedance, which is 50 Ω. Since the source impedance is also 50 Ω, half the power must be dissipated in the source impedance, and the other half must be dissipated in the load impedance. The power dissipated in the generator is $P_{\mathrm{g}} = (1/2)\,\mathrm{Re}(VI^*)$, and we have $I^* = V^*/(Z^*_{\mathrm{in}} + Z^*_{\mathrm{source}}) = (100/100)$ A $= 1$ A. Hence, we have $P_{\mathrm{g}} = 25$ W and $P_{\mathrm{L}} = 25$ W.

3. a. The divergence is defined as the net flux of a vector field into a small volume in the limit as that volume tends to zero. The corresponding mathematical expression is

$$\nabla \cdot \mathbf{E} \equiv \lim_{\Delta v \to 0} \frac{\oint \mathbf{E} \cdot d\mathbf{S}}{\Delta v}.$$



   b. A picture of the flux fields passing through a small volume in Cartesian coordinates is given in the figure above, which is Ulaby et al.'s Fig. 3-21. If we consider the flux out of faces 1 and 2 in the figure to the right, we find that the flux out of face 1 is given by $-E_x(x, y, z)\Delta y \Delta z$, while the flux out of face 2 is given by $E_x(x + \Delta x, y, z)\Delta y\Delta z \simeq E_x(x, y, z)\Delta y\Delta z + (\partial E_x/\partial x)\Delta x\Delta y\Delta z$ Adding the contributions from faces 1 and 2, we obtain $(\partial E_x/\partial x)\Delta x\Delta y\Delta z$. Considering faces 3 and 4 yields analogously $(\partial E_y/\partial y)\ \Delta x\Delta y\Delta z$, and considering faces 5 and 6 yields analogously $(\partial E_z/\partial z)\Delta x\Delta y\Delta z$. Noting that $\Delta v = \Delta x\Delta y\Delta z$ and applying the definition of the divergence, we conclude

$$\nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}.$$

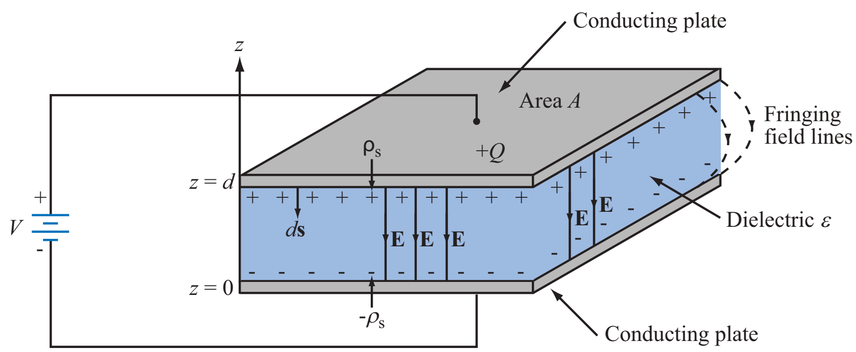
4. The geometry that we are considering is shown below, and we neglect fringing fields. We are going to suppose that we ramp up the voltage v from 0 to V. As that happens the charge q on the positive voltage end of the voltage source ramps up from 0 to Q coulombs. In order to move an increment of charge dq from the plate at the negative end of the voltage source to the positive end, we must do work, since the existing charges on the positive end will exert a force on the charge increment. This work is given by $dW_\mathrm{e} = v\,dq = (q/C)dq$. Adding up all the work that is performed in charging the capacitor, we find

$$W_\mathrm{e} = \int_0^Q \frac{q}{C}\,dq = \frac{1}{2}\frac{Q^2}{C}.$$

Using $C = Q/V$, we also have $W_\mathrm{e} = (1/2)CV^2$. For the parallel plate capacitor, we have $C = \epsilon A/d$, where A is the area of the plate and d is the separation between the plates. We also have $V = Ed$. Substituting for C and V in the expression for W_e, we obtain

$$W_e = \frac{1}{2}\frac{\epsilon A}{d}(Ed)^2 = \frac{1}{2}\epsilon E^2(Ad).$$

Noting that Ad equals the volume between the plates, we conclude that $(1/2)\epsilon E^2$ equals the energy density.



5. a. The equations of motion are: $d\mathbf{x}/dt = \mathbf{u}$ and $d\mathbf{u}/dt = (q/m)(\mathbf{E} + \mathbf{u}\times\mathbf{B})$, where x and u are the position and velocity of the charge.

   b. If $B = 0$, we may choose the direction in which the electric field points for convenience, and we will pick the z-direction. We may also pick the initial starting point of the particle to be $(0,0,0)$. The equations of motion become $dx/dt = u_x$, $dy/dt = u_y$, $dz/dt = u_z$, $du_x/dt = 0$, $du_y/dt = 0$, and $du_z/dt = qE/m$. The motion in the x, y, and z directions is uncoupled. Since there is no change in the x- or y-velocities and we are starting from rest, we find $u_x = 0$, $u_y = 0$, $x = 0$, and $y = 0$. So, there is no motion at all in the x- and y-directions. Since E is constant, we find $u_z = (qE/m)t$ and $z = (qE/m)(t^2/2)$, which is what we want to show. This result corresponds to a constant acceleration in the z-direction.