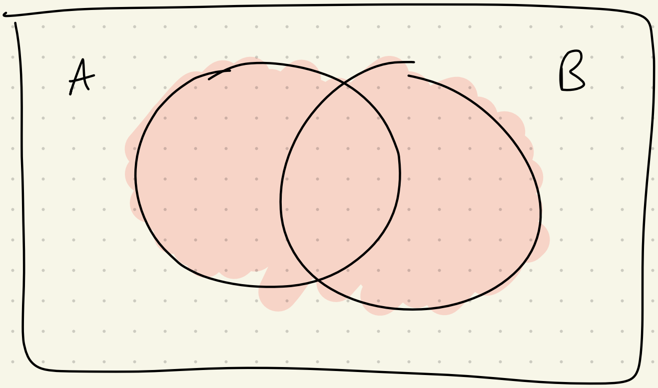
1a)



$A \setminus B \Rightarrow x \in A \wedge x \notin B$

$(A \setminus B)^c \Rightarrow x \in U \wedge (x \notin A \vee x \in B)$

$(A \setminus B)^c \setminus B \Rightarrow x \in U \wedge (x \notin A \vee x \notin B)$

$((A \setminus B)^c \setminus B)^c \Rightarrow x \in A \vee x \in B$

$\Rightarrow A \cup B$

b) just proved that it's only true for

$A \cup B$.

counter example:

$A = \{1, 2, 3\}$

$B = \{3, 5, 6\}$

$((A \setminus B)^c \setminus B)^c = \{1, 2, 3, 5, 6\}$

$A \cap B = \{3\}$ $\rightarrow$ not equal.

2a)

a function is injective iff: $f: X \mapsto Y$

$\forall x_1, x_2 \in X : x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2)$

$\Leftrightarrow \forall x_1, x_2 \in X : f(x_1) = f(x_2) \rightarrow x_1 = x_2$

each distinct element $x \in X$ maps to a different element in Y $\Rightarrow$ one-to-one

surjective: $f: X \mapsto Y$

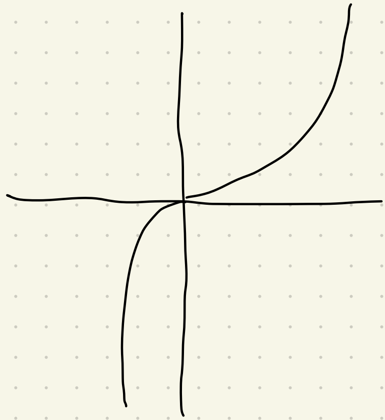
$f(X) = Y \Leftrightarrow \forall y \in Y : \exists x \in X : f(x) = y$

each $y \in Y$ has a corresponding $x \in X$

bijective: $f: X \mapsto Y$

both injective and surjective

$f^{-1}: Y \mapsto X$ does exist

b) 

we need to prove

$(x_1 \neq x_2) \rightarrow (f(x_1) \neq f(x_2))$

case $x_1 < x_2 < 0$:

$f(x_1) = x_1^3$, $f(x_2) = x_2^3$

$\Rightarrow f(x_1) < f(x_2)$

case $x_1 < 0 \leq x_2$:

$f(x_1) = x_1^3$, $f(x_2) = x_2^2$

$\Rightarrow f(x_1) < f(x_2)$

case $0 \leq x_1 < x_2$

$f(x_1) = x_1^2$, $f(x_2) = x_2^2$

$\Rightarrow f(x_1) < f(x_2)$

$\Rightarrow (x_1 < x_2) \rightarrow f(x_1) < f(x_2)$ $\therefore$ injective

$\Rightarrow f(x_1) \neq f(x_2)$

c) we need to prove $\forall y \in Y : \exists x \in X :$

$f(x) = y$

case $y < 0$:

take $x = \sqrt[3]{y}$

then $x < 0$ and $f(x) = x^3 = (\sqrt[3]{y})^3 = y$

case $y \geq 0$:

take $x = \sqrt[2]{y}$

then $x \geq 0$ and $f(x) = x^2 = (\sqrt[2]{y})^2 = y$

$\Rightarrow$ for any real y, we find x such that $f(x) = y$ $\therefore$ surjective

$\Rightarrow$ both injective and surjective so bijective.

3a)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 5 & 1 & 4 & 6 \end{pmatrix} = \alpha$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 2 & 6 & 5 & 3 \end{pmatrix} = \beta$$

b)

$$\left.\begin{array}{lll} 1<2 & \text{but} & 2>1 \\ 3<4 & \text{but} & 4>3 \\ 5<6 & \text{but} & 6>5 \end{array}\right\} \sigma_1$$

$\Rightarrow \sigma_1$ is odd since there are 3 disorders

$$\left.\begin{array}{lll} 1<3 & \text{but} & 2>1 \\ 2<3 & \text{but} & 3>1 \\ 4<6 & \text{but} & 5>4 \\ 5<6 & \text{but} & 6>4 \end{array}\right\} \sigma_2$$

$\Rightarrow \sigma_2$ is even since 4 disorders

$\mathrm{sgn}(\sigma_1) \cdot \mathrm{sgn}(\sigma_2) =$ odd $\cdot$ even $=$ odd

$1<2$ but $3>2$

$1<4$ but $3>1$

$2<4$ but $2>1$

$3<4$ but $5>1$

$3<5$ but $5>4$

there are 5 disodes so odd so $\mathrm{sgn}(\sigma_1) \cdot \mathrm{sgn}(\sigma_2) = \mathrm{sgn}(\alpha)$

c)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{pmatrix}$$

$$\sigma_1^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

$\Rightarrow$ order of σ_1 is 2

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix}$$

$$\sigma_2^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 6 & 4 & 5 \end{pmatrix}$$

$$\sigma_2^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 6 & 4 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

$\Rightarrow$ order of σ_2 is 3

4a)

$$21 = 13 \cdot 1 + 8 \qquad a = b + r_1$$

$$13 = 8 \cdot 1 + 5 \qquad b = r_1 + r_2$$

$$8 = 5 \cdot 1 + 3 \qquad r_1 = r_2 + r_3$$

$$5 = 3 \cdot 1 + 2 \qquad r_2 = r_3 + r_4$$

$$3 = 2 \cdot 1 + 1 \qquad r_3 = r_4 + r_5$$

$$2 = 1 \cdot 1 + \boxed{1} \qquad r_4 = r_5 + \boxed{r_6}$$

$$1 = 1 \cdot 1 + 0 \qquad r_5 = r_6$$

$$\boxed{\Rightarrow 21 \cdot -8 + 13 \cdot 13 = 1}$$

$$\begin{aligned}
r_6 &= r_4 - r_5 \\
&= (r_2 - r_3) - (r_3 - r_4) \\
&= r_2 - 2r_3 + r_4 \\
&= (b - r_1) - 2(r_1 - r_2) + (r_2 - r_3) \\
&= b - 3r_1 + 3r_2 - r_3 \\
&= b - 3(a-b) + 3(b - r_1) - (r_1 - r_2) \\
&= b - 3a + 3b + 3b - 3r_1 - r_1 + r_2 \\
&= 7b - 3a - 4(a-b) + (b - r_1) \\
&= 12b - 7a - r_1 = 12b - 7a - (a-b) \\
&= \boxed{-8a + 13b}
\end{aligned}$$

b)

$\varphi(77)$

$= \varphi(11 \cdot 7)$

$= \varphi(11) \cdot \varphi(7)$

$= (11-1) \cdot (7-1)$

$= 10 \cdot 6$

$= 60$

c) $3^{62} \bmod 77$

due to Euler Totient Function Theorem,

$3^{\varphi(77)} = 1 \bmod 77$

$= 3^{60} = 1 \bmod 77$

$\Rightarrow 3^{62} = 3^{60+2} \bmod 77$

$= 1 \cdot 3^{2} \bmod 77$

$= 9 \bmod 77$

5b) $(a+1)x = 2$

$\therefore$ the system has no solution when

det $= 0$ $\therefore$ $a = -1$

$-x - y = 0$

$\Rightarrow x = -y$

$\Rightarrow -y + y = 0 \neq 2$

c) infinite solutions when determinant $= 0$, but no way to make equations consistent with $a = -1$, therefore there aren't infinite solutions.

a) $\begin{pmatrix} 1 & 1 \\ a & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-a-1} \begin{pmatrix} -1 & -1 \\ -a & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$= \frac{1}{-a-1} \begin{pmatrix} -2 \\ -2a \end{pmatrix}$$

$$\Rightarrow \quad x = \frac{-2}{-a-1} = \frac{2}{a+1}$$

$$y = \frac{-2a}{-a-1} = \frac{2a}{a+1}$$

and $a \neq -1$ is the unique solution

unique solution when determinant $\neq 0$

d) for $n \times n$ matrix, eigenvalue λ and eigenvector $\vec{v}$ satisfy $A\vec{v} = \lambda\vec{v}$, $\vec{v} \neq \vec{0}$

$$A\vec{v} - \lambda\vec{v} = (A - \lambda I) \cdot \vec{v} = \vec{0} \Rightarrow |A - \lambda I| = 0$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 4 & 5 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 3-\lambda \end{vmatrix}$$

$$= 1-\lambda(2-\lambda)(3-\lambda)$$

$$= 0$$

$$\Rightarrow \lambda_1 = 1 , \quad \lambda_2 = 2 , \quad \lambda_3 = 3$$

$$A - \lambda_1 I = \begin{pmatrix} 0 & 4 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & 2 \end{pmatrix}$$

$$A - \lambda_2 I = \begin{pmatrix} -1 & 4 & 5 \\ 0 & 0 & 6 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A - \lambda_3 I = \begin{pmatrix} -2 & 4 & 5 \\ 0 & -1 & 6 \\ 0 & 0 & 0 \end{pmatrix}$$

case λ_1:

$$\begin{pmatrix} 0 & 4 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$v_3 = 0, \quad v_2 = 0, \quad v_1 = 0 \quad \Rightarrow \vec{v_1} = \vec{0}$

case λ_2:

$-v_1 + 4v_2 + 5v_3 = 0$

$v_3 = 0 \quad \therefore \quad v_1 = 4v_2 \quad \therefore \quad \vec{v_2} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

case λ_3:

$$-v_2 + 6v_3 = 0$$

$$-2v_1 + 4v_2 + 5v_3 = 0$$

but $v_3 = 0 \Rightarrow v_2 = 0, \ v_1 = 0$

$$\vec{v_3} = \vec{0}$$