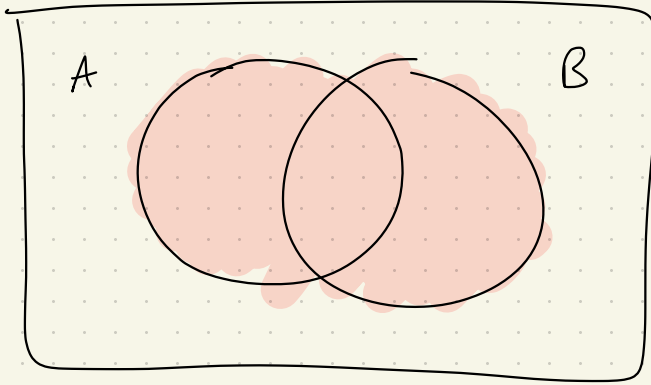


1a)



$$A \setminus B \Rightarrow x \in A \wedge x \notin B$$

$$(A \setminus B)^c \Rightarrow x \in U \wedge (x \notin A \vee x \in B)$$

$$(A \setminus B)^c \setminus B \Rightarrow x \in U \wedge (x \notin A \vee x \notin B)$$

$$\begin{aligned} ((A \setminus B)^c \setminus B)^c &\Rightarrow x \in A \vee x \in B \\ &\Rightarrow A \cup B \end{aligned}$$

b) just proved that it's only true for $A \cup B$.

counter example:

$$A = \{1, 2, 3\}$$

$$B = \{3, 5, 6\}$$

$$((A \setminus B)^c \setminus B)^c = \{1, 2, 3, 5, 6\}$$

$$A \cap B = \{3\} \Rightarrow \text{not equal.}$$

2a)

a function is injective if: $f: X \mapsto Y$

$$\forall x_1, x_2 \in X : x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2)$$

$$\Leftrightarrow \forall x_1, x_2 \in X : f(x_1) = f(x_2) \rightarrow x_1 = x_2$$

each distinct element $x \in X$ maps to a different element in Y \Rightarrow one-to-one

surjective: $f: X \mapsto Y$

$$f(X) = Y \Leftrightarrow \forall y \in Y : \exists x \in X : f(x) = y$$

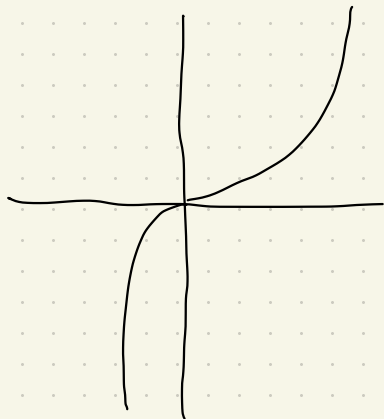
each $y \in Y$ has a corresponding $x \in X$

bijective: $f: X \mapsto Y$

both injective and surjective

$f^{-1}: Y \mapsto X$ does exist

b)



we need to prove
 $(x_1 \neq x_2) \rightarrow (f(x_1) \neq f(x_2))$

case $x_1 < x_2 < 0$:

$$f(x_1) = x_1^3, \quad f(x_2) = x_2^3$$

$$\Rightarrow f(x_1) < f(x_2)$$

case $x_1 < 0 \leq x_2$:

$$f(x_1) = x_1^3, \quad f(x_2) = x_2^2$$

$$\Rightarrow f(x_1) < f(x_2)$$

case $0 \leq x_1 < x_2$

$$f(x_1) = x_1^2, \quad f(x_2) = x_2^2$$

$$\Rightarrow f(x_1) < f(x_2)$$

$$\Rightarrow (x_1 < x_2) \rightarrow f(x_1) < f(x_2) \therefore \text{injective}$$

$$\Rightarrow f(x_1) \neq f(x_2)$$

c) we need to prove $\forall y \in Y : \exists x \in X :$

$$f(x) = y$$

case $y < 0 :$

$$\text{take } x = \sqrt[3]{y}$$

$$\text{then } x < 0 \text{ and } f(x) = x^3 = (\sqrt[3]{y})^3 = y$$

case $y \geq 0 :$

$$\text{take } x = \sqrt{y}$$

$$\text{then } x \geq 0 \text{ and } f(x) = x^2 = (\sqrt{y})^2 = y$$

\Rightarrow for any real y , we find x such that $f(x) = y \quad \therefore$ surjective

\Rightarrow both injective and surjective so bijective.

3a)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 2 & 5 & 1 & 4 & 6 \end{pmatrix} = \alpha$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 2 & 6 & 5 & 3 \end{pmatrix} = \beta$$

b)

$$\left. \begin{array}{l} 1 < 2 \quad \text{but} \quad 2 > 1 \\ 3 < 4 \quad \text{but} \quad 4 > 3 \\ 5 < 6 \quad \text{but} \quad 6 > 5 \end{array} \right\} \sigma_1$$

$\Rightarrow \sigma_1$ is odd since there are 3 disorders

$$\left. \begin{array}{l} 1 < 3 \quad \text{but} \quad 2 > 1 \\ 2 < 3 \quad \text{but} \quad 3 > 1 \\ 4 < 6 \quad \text{but} \quad 5 > 4 \\ 5 < 6 \quad \text{but} \quad 6 > 4 \end{array} \right\} \sigma_2$$

$\Rightarrow \sigma_2$ is even since 4 disorders

$$\text{sgn}(\sigma_1) \cdot \text{sgn}(\sigma_2) = \text{odd} \cdot \text{even} = \text{odd}$$

$$1 < 2 \quad \text{but} \quad 3 > 2$$

$$1 < 4 \quad \text{but} \quad 3 > 1$$

$$2 < 4 \quad \text{but} \quad 2 > 1$$

$$3 < 4 \quad \text{but} \quad 5 > 1$$

$$3 < 5 \quad \text{but} \quad 5 > 4$$

there are 5 disorders, so odd so

$$\text{sgn}(\sigma_1) =$$

$$\text{sgn}(\sigma_2) =$$

$$\text{sgn}(\alpha)$$

c)

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{pmatrix}$$

$$\sigma_1^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

\Rightarrow order of σ_1 is 2

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix}$$

$$\sigma_2^2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 6 & 4 & 5 \end{pmatrix}$$

$$\sigma_2^3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 6 & 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}$$

\Rightarrow order of σ_2 is 3

4a)

$$21 = 13 \cdot 1 + 8$$

$$13 = 8 \cdot 1 + 5$$

$$8 = 5 \cdot 1 + 3$$

$$5 = 3 \cdot 1 + 2$$

$$3 = 2 \cdot 1 + 1$$

$$2 = 1 \cdot 1 + \boxed{1}$$

$$1 = 1 \cdot 1 + 0$$

$$a = b + r_1$$

$$b = r_1 + r_2$$

$$r_1 = r_2 + r_3$$

$$r_2 = r_3 + r_4$$

$$r_3 = r_4 + r_5$$

$$r_4 = r_5 + \boxed{r_6}$$

$$r_5 = r_6$$

$$r_6 = r_4 - r_5$$

$$= (r_2 - r_3) - (r_3 - r_4)$$

$$= r_2 - 2r_3 + r_4$$

$$= (b - r_1) - 2(r_1 - r_2) + (r_2 - r_3)$$

$$= b - 3r_1 + 3r_2 - r_3$$

$$= b - 3(a - b) + 3(b - r_1) - (r_1 - r_2)$$

$$= b - 3a + 3b + 3b - 3r_1 - r_1 + r_2$$

$$= 7b - 3a - 4(a - b) + (b - r_1)$$

$$= 12b - 7a - r_1 = 12b - 7a - (a - b) \\ = \boxed{-8a + 13b}$$

\Rightarrow

$$21 \cdot -8 + 13 \cdot 13 \\ = 1$$

b)

$$\varphi(77)$$

$$= \varphi(11 \cdot 7)$$

$$= \varphi(11) \cdot \varphi(7)$$

$$= (11-1) \cdot (7-1)$$

$$= 10 \cdot 6$$

$$= 60$$

c) $3^{62} \bmod 77$

due to Euler Totient Function Theorem,

$$3^{\varphi(77)} = 1 \bmod 77$$

$$= 3^{60} = 1 \bmod 77$$

$$\Rightarrow 3^{62} = 3^{60+2} \bmod 77$$

$$= 1 \cdot 3^2 \bmod 77$$

$$= 9 \bmod 77$$

$$5b) (a+1)x = 2$$

\therefore the system has no solution when

$$\det = 0 \quad \therefore a = -1$$

$$-x - y = 0$$

$$\Rightarrow x = -y$$

$$\Rightarrow -y + y = 0 \neq 2$$

c) infinite solutions when determinant $\neq 0$

$$\therefore a \neq -1$$

$$a) \begin{pmatrix} 1 & 1 \\ a & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{-a-1} \begin{pmatrix} -1 & -1 \\ -a & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

$$= \frac{1}{-a-1} \begin{pmatrix} -2 \\ -2a \end{pmatrix}$$

$$\Rightarrow x = \frac{-2}{-a-1} = \frac{2}{a+1}$$

$$y = \frac{-2a}{-a-1} = \frac{2a}{a+1}$$

and $a \neq -1$ is the unique solution
 unique solution when determinant $\neq 0$

d) for $n \times n$ matrix, eigenvalue λ and
 eigenvector \vec{v} satisfy $A\vec{v} = \lambda\vec{v}$, $\vec{v} \neq \vec{0}$

$$A\vec{v} - \lambda\vec{v} = (A - \lambda I) \cdot \vec{v} = \vec{0} \Rightarrow |A - \lambda I| = 0$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 4 & 5 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 3-\lambda \end{vmatrix}$$

$$= (1-\lambda)(2-\lambda)(3-\lambda)$$

$$= 0$$

$$\Rightarrow \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$$

$$A - \lambda_1 I = \begin{pmatrix} 0 & 4 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & 2 \end{pmatrix}$$

$$A - \lambda_2 I = \begin{pmatrix} -1 & 4 & 5 \\ 0 & 0 & 6 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A - \lambda_3 I = \begin{pmatrix} -2 & 4 & 5 \\ 0 & -1 & 6 \\ 0 & 0 & 0 \end{pmatrix}$$

case λ_1 :

$$\begin{pmatrix} 0 & 4 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$v_3 = 0, \quad v_2 = 0, \quad v_1 = 0 \quad \Rightarrow \vec{v}_1 = \vec{0}$$

case λ_2 :

$$-v_1 + 4v_2 + 5v_3 = 0$$

$$v_3 = 0 \quad \therefore \quad v_1 = 4v_2 \quad \therefore \quad \vec{v}_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$$

case λ_3 :

$$-v_2 + 6v_3 = 0$$

$$-2v_1 + 4v_2 + 5v_3 = 0$$

$$\text{but } v_3 = 0 \Rightarrow v_2 = 0, v_1 = 0$$

$$\vec{v_3} = \vec{0}$$