Exercise sheet 3
Bifurcation and visualization

Due date:	2019–11–28 (2 weeks)
Tasks:	5

In this exercise, you will study qualitative changes of dynamical systems over changes of their parameters. These changes in the qualitative behavior of the system are called *bifurcations*, "to divide into two, like a fork". The notion of a qualitative change has been made precise, the definitions are given below. The goals for this exercise are

- to familiarize yourself with the mathematical notation of bifurcation theory,
- to understand topological equivalence between systems,
- to know several basic bifurcations present in almost all dynamical systems in the world,
- to be able to visualize qualitative changes of a dynamical system in a bifurcation diagram, and
- to apply these ideas to crowd dynamics.

Why is this related to Machine Learning? Even though it is possible to analyze a given mathematical model formally regardings its bifurcations, there are many systems where such an analysis is not possible. The most difficult examples are systems in the real world, where you cannot study the behavior on pen and paper, only through observations. This is where Machine Learning can assist you, to produce data-driven bifurcation diagrams for real systems (or large, complex simulated systems such as the ones you studied with Vadere) from data. You have to do exactly that in the last task of this exercise.

Dynamical systems with bifurcations

A dynamical system is a set X of states $x \in X$ together with a combination of rules (the evolution operator) $\phi : I \times X \to X$ that change this state over the change of the parameter t, typically considered as "time" $t \in I$ in an index set I. Typically, for discrete dynamical systems, we choose $I \subseteq \mathbb{N}$, for continuous dynamical systems, $I \subseteq \mathbb{R}$. The combination of time, state space, and evolution operator defines the dynamical system and is often stated as a triple (I, X, ϕ). For a discrete system, the evolution by ϕ starting at an initial point $x_0 \in X$ is usually written

$$x_n = \phi(n, x_0),\ x_n \in X,\ n \in I. \tag{1}$$

Many descriptions of continuous dynamical systems do not directly specify the map ϕ, but its derivative with respect to time as a function $v : X \to TX$,

$$\left.\frac{d\phi(t, x)}{dt}\right|_{t=0} = v(x), \tag{2}$$

where TX denotes the tangent bundle of X, and $v(x) \in T_xX$ for all $x \in X$. The symbol T_xX denotes the local tangent space at x, with $TX = \cup_{x \in X} T_xX$. The map v is called *vector field*, as it associates a vector to every point $x \in X$. For more details on tangent bundles, vector fields, and manifolds, I recommend the book of Lee [3]. You do not need these concepts for this exercise. Equivalent (but more informal) notations for the time derivative of the flow at $t = 0$ are $\frac{d}{dt}\phi^t(x)$, $\frac{d}{dt}x$, and $\dot{x}$. Parameters of a dynamical system change its behavior (i.e., the map ϕ) in rather arbitrary, mostly smooth ways. Such parameters can be indicated as a subscript to the evolution operator: the symbol ϕ_α indicates that the operator ϕ depends on a certain number of parameters $\alpha \in \mathbb{R}^k$. A bifurcation analysis of a dynamical system is concerned with qualitative changes of the system when the values of the parameters change. Qualitative change is formalized through the notion of topological equivalence, i.e. a system is *qualitatively the same* as another system, if it is *topologically equivalent*:

Definition 1. ***Topological equivalence.*** *A dynamical system (I, X, ϕ) is topologically equivalent to another dynamical system (I, Y, ψ) if there is a homeomorphism $h : X \to Y$ mapping orbits of the first system onto orbits of the second system, preserving the direction of time.*

Note that this definition does not take into account parameters. There is a separate definition for parameterized systems that includes a second map between the parameter spaces.

Definition 2. ***Steady states / equilibrium points / fixed points.*** *A point $x_0 \in X$ is called an equilibrium (fixed point, steady state) if $\phi(t, x_0) = x_0$ for all $t \in I$.*

To visualize a given vector field v, phase portraits are very powerful. Essentially, they show suitably chosen sample vectors from v and orbits around qualitatively interesting parts of state space. In python, the `matplotlib`

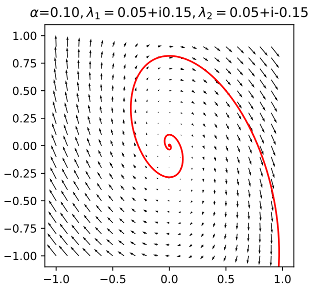


Figure 1: Phase portrait of a dynamical system with state $x \in \mathbb{R}^2$ and parametrized vector field $v_\alpha(x) = A_\alpha x = (\alpha x_1 + \alpha x_2, -0.25x_1)$, with $\alpha = 0.1$. A trajectory is shown in red, and the eigenvalues of the matrix A_α are shown in the title.

package offers a convenient way to visualize vector fields as phase portraits, through the method `streamplot`[1].

Definition 3. ***Bifurcation, see [2, p.57].*** *The appearance of a topologically nonequivalent phase portrait under variation of parameters is called a bifurcation.*

An accessible introduction to dynamical systems and their bifurcations with more details can be found in the book by Kuznetsov [2]. You can download it through your TUM account at Springer[2].

Numerical solution to ordinary differential equations

Equation (2) defines the local derivative of the flow map at every point in the state space. There are thousands of books and hundreds of years of work in mathematics on the solution to the following problem: Which flow map (which function ϕ) satisfies equation (2), i.e. has local derivatives $v(x)$ at every point x? A sub-problem asks for individual trajectories: Given a point x, how does the set $\{\phi(t, x) | t \in I\} \subset X$ look like? This set is called *orbit*. The basic idea to find an orbit numerically is encoded in Euler's algorithm:

1. Start with a point $x_0 \in X$, and set the current iteration number to $n = 1$.
2. To generate a new point x_n on the orbit through x_0, define a small time step $\Delta t \in \mathbb{R}$, and compute
$$x_n = x_{n-1} + \Delta t \cdot v(x_{n-1}). \tag{3}$$
3. Increase n by one, and iterate. If Δt is small enough and v well-behaved, this will create an approximation to a part of the orbit through x_0.

For some systems, it is possible to use a negative value for Δt, and "integrate backwards in time", thus creating the orbit in the other direction. Note that you should avoid using Euler's method as much as possible, as it is not very accurate and you do not have any control over the error to the true orbit. Many robust and accurate solvers exist, in python you should look at `scipy.integrate.solve_ivp` (solves initial value problems).

Note: the number of points per exercise is a rough estimate of how much time you should spend on each task.

[1] `https://matplotlib.org/3.1.1/gallery/images_contours_and_fields/plot_streamplot.html`
[2] `https://link.springer.com/book/10.1007%2F978-1-4757-3978-7`

Task 1/5: Vector fields, orbits, and visualization **Points: 10/100**

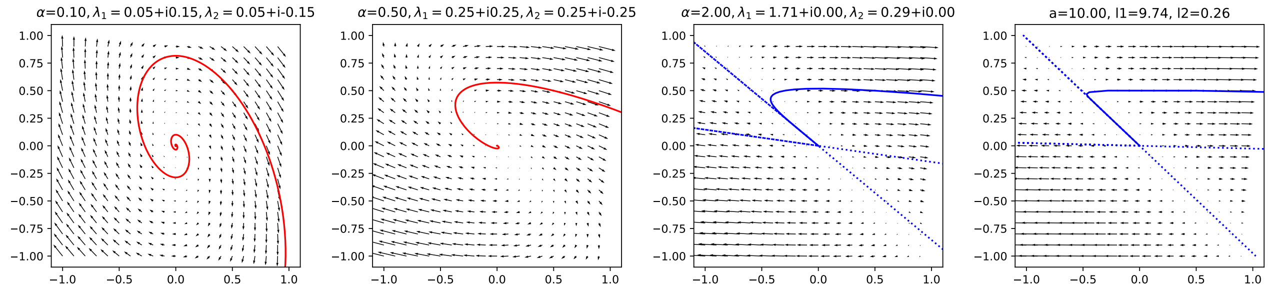
A good way to visualize a dynamical system with a one- or two-dimensional state space is through its *phase portrait*. Consider the following linear dynamical system, with state space $X = \mathbb{R}^2$, $I = \mathbb{R}$, parameter $\alpha \in \mathbb{R}$, and flow ϕ_α defined by

$$\left.\frac{d\phi_\alpha(t,x)}{dt}\right|_{t=0} = A_\alpha x, \tag{4}$$

where $A_\alpha \in \mathbb{R}^{2\times 2}$ is a parametrized matrix

$$A_\alpha = \begin{bmatrix} \alpha & \alpha \\ -\frac{1}{4} & 0 \end{bmatrix}. \tag{5}$$

With the linear system defined in equation (4), construct a figure similar to Fig. 2.5 in [2, p.49] for hyperbolic



(a) (b) (c) (d)

Figure 2: Phase portraits of system (4) with different values of α. The eigenvalues $\lambda_{1,2}$ of A_α are shown in the title of each portrait. Trajectories in red color indicate complex eigenvalues.

equilibria on the plane. The phase portraits may look slightly different, and you may need to use a different, parametrized matrix. Specify the value of the parameter for each of your phase portraits. Are these systems topologically equivalent? Why, or why not? Have a look at [2, p.43ff].

Task 2/5: Common bifurcations in nonlinear systems **Points: 20/100**

Consider a dynamical system on the real line $X = \mathbb{R}$, time $I = \mathbb{R}$, with the evolution described by

$$\dot{x} = \alpha - x^2. \tag{6}$$

For $\alpha > 0$, this system has two steady states at $x_0 = \pm\sqrt{\alpha}$, and for $\alpha < 0$ there are no steady states. What type of bifurcation happens at $\alpha = 0$? Plot the bifurcation diagram of the system for values of α in $(-1, 1)$, visually indicating the stability of the steady states. Then, do the same for the following system:

$$\dot{x} = \alpha - 2x^2 - 2. \tag{7}$$

Are the systems (6) and (7) at $\alpha = 1$ topologically equivalent? Why, or why not? What about the systems at $\alpha = -1$? Argue why the systems have the same normal form.

Task 3/5: Bifurcations in higher dimensions **Points: 10/100**

Bifurcations can happen for dynamical systems with state spaces of arbitrary dimension, and also in more than one parameter. Some bifurcations do not occur if the state space is one-dimensional (and the system is continuous). An important bifurcation for systems with one parameter exists for two-dimensional state spaces: the Andronov-Hopf bifurcation [2, p.57], with the vector field in normal form

$$\begin{aligned} \dot{x}_1 &= \alpha x_1 - x_2 - x_1(x_1^2 + x_2)^2, \\ \dot{x}_2 &= x_1 + \alpha x_2 - x_2(x_1^2 + x_2)^2. \end{aligned} \tag{8}$$

1. Visualize the bifurcation of the system by plotting three phase diagrams at representative values of α.

2. For $\alpha = 1$, numerically compute and visualize two orbits of the system forward in time, starting at the point $(2, 0)$ and at $(0.5, 0)$. You can use Euler's method with a very small time step, or any numerical solver, but you have to descibe how you obtain the results.

Another important bifurcation occurs already in one state space dimension $X = \mathbb{R}$, but with two parameters $\alpha \in \mathbb{R}^2$: the cusp bifurcation, with normal form

$$\dot{x} = \alpha_1 + \alpha_2 x - x^3. \tag{9}$$

Visualize the bifurcation surface (all points (x, α_1, α_2) where $\dot{x} = 0$) of the cusp bifurcation in a 3D plot, with α_1, α_2 on the bottom plane and x in the third direction. Why is it called *cusp bifurcation*?

Task 4/5: Chaotic dynamics **Points: 20/100**

Dynamical systems can behave in very irregular ways, and changes in their parameters can lead to very drastic changes in their behavior. Consider the discrete map

$$x_{n+1} = r x_n (1 - x_n),\ n \in \mathbb{N}, \tag{10}$$

with the parameter $r \in (0, 4]$. Perform the following bifurcation analyses separately:

1. Vary r from 0 to 2. Which bifurcations occur? At which numerical values do you find steady states and limit cycles of the system?

2. Now vary r from 2 to 4. What happens?

3. Plot a bifurcation diagram for r between 0 and 4 (horizontal axis), x between 0 and 1 (vertical axis), indicating the positions of steady states and limit cycles.

The system described by equation (10) is well studied. It is called the "logistic map", and is a good example of chaos in discrete maps on one-dimensional spaces.

Dynamical systems in continuous time cannot have smooth evolution operators that produce chaotic dynamics if the dimension of the state space is smaller than three. The Lorenz attractor [4] is a famous example for a system in three-dimensional space that forms a *strange attractor*, a fractal set on which the dynamics are chaotic.[3] Visualize a single trajectory of the Lorenz system starting at $x_0 = (10, 10, 10)$, with a length of $T_{\text{end}} = 1000$, at the parameter values $\sigma = 10$, $\beta = 8/3$, and $\rho = 28$. What does the attractor look like? The chaotic nature of the system implies that small perturbations in the initial condition will grow larger at an exponential rate, until the error is as large as the diameter of the attractor. Test this by plotting another trajectory from $x_0 = (10 + 10^{-8}, 10, 10)$. At what time is the difference between the points on the trajectory larger than 1?

Now, change the parameter ρ to the value 0.5 and again compute and plot the two trajectories. Is there a bifurcation (or multiple ones) between the value 0.5 and 28? Why, or why not?

Task 5/5: Bifurcations in crowd dynamics **Points: 40/100**

First part: In this task, you have to apply your knowledge about bifurcation theory to analyze a given Vadere scenario. If you want to know more about research in this direction, I recommend taking a look at the research papers by Disselnkötter, Marschler, and Starke [7, 6, 5, 1]. Download the *.scenario file from the Moodle page and simulate it with Vadere. Figure 3 shows the setup. The y-position of the obstacle is the parameter of the scenario, you have to adjust it between $2.0m$ and $4.5m$ to generate the observation data. At $y = 2.0m$, the path is almost completely blocked and the crowd gets stuck after a short time. At $y = 4.5m$, the obstacle is not influencing the movement anymore.

1. Generate the x, y positions for every pedestrian over the complete simulation time (1500 seconds), for all fixed obstacle positions between $y = 2.0m$ and $y = 4.5m$ (you have to decide a reasonable increment of the coordinate between simulations). Note that the random seed of the scenario is fixed, so re-simulating the scenario at the same y position of the obstacle will not change the result. Also note that an extension of the simulation time beyond 1500 seconds might pose problems, because the `targetIds` list in the source only contains a finite number of ids. In this task, we consider this list to be practically infinite, and do not simulate beyond 1500 seconds.

[3] You can find a vector field with parameters for the chaotic regime on Wikipedia, `https://en.wikipedia.org/wiki/Lorenz_system`.

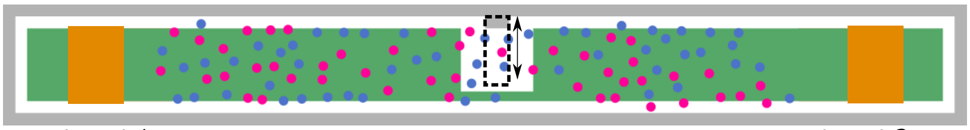


Figure 3: Setup of the bifurcation scenario in Vadere. 100 pedestrians move back and forth between two targets, while an obstacle in the center blocks movement. The red pedestrians are currently moving to the left, the blue ones move to the right, their color changes once they reach the next target and they switch to the other one.

2. Once you have all the positions, visualize the x and y coordinate of one of the 100 pedestrians over time, and for two separate positions of the obstacle (two separate simulations) that you think best represent different behaviors of the system. Figure 4 illustrates the idea for a certain value of the parameter.

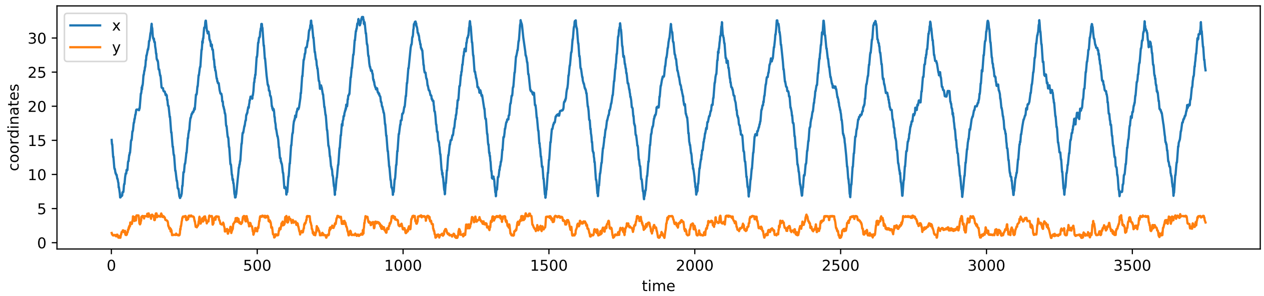


Figure 4: Coordinates of the position of a single pedestrian in the scenario over time, for a fixed value of the parameter (height of the obstacle).

3. What type of bifurcation is occurring in this scenario? At what value of the parameter?

4. Visualize your answer, by showing the similarity to the phase portrait of the normal form of the bifurcation. A hint: plot the x position of a single pedestrian (horizontal axis) again against the x position of the same pedestrian, but a certain number of time steps after / before the current time step (vertical axis). How many time steps have to be in between the two x values for best visualization results? Why?

Second part: Note that this second part of the task will give a maximum of 20 points (plus bonus): construct a simple scenario in Vadere that has a different bifurcation than the one above. You do not have to create a scenario with many pedestrians, even just one may be enough. You can choose one of the bifurcations discussed in this exercise, the lecture, or other bifurcations from the book [2]. Clearly argue why the bifurcation is present in your scenario, how you can detect it numerically, and construct a bifurcation diagram. Plot the vector field of the normal form of the chosen bifurcation in phase portraits. Choosing a bifurcation with a normal form that needs more than one parameter or more than one space dimension will give 10 bonus points.

References

[1] Felix Dietrich, Stefan Disselnkötter, and Gerta Köster. How to get a model in pedestrian dynamics to produce stop and go waves? In Victor L. Knoop and Winnie Daamen, editors, *Traffic and Granular Flow '15*, pages 161–168. Springer International Publishing, 2015. 27–30 October 2015.

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[5] Christian Marschler, Jan Sieber, Rainer Berkemer, Atsushi Kawamoto, and Jens Starke. Implicit methods for equation-free analysis: Convergence results and analysis of emergent waves in microscopic traffic models. *SIAM Journal on Applied Dynamical Systems*, 13(3):1202–1238, 1 2014.

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[7] Jens Starke, Kristian Berg Thomsen, Asger Soerensen, Christian Marschler, Frank Schilder, Anne Dederichs, and Poul Hjorth. Nonlinear effects in examples of crowd evacuation scenarios. In *IEEE 17th International Conference on Intelligent Transportation Systems (ITSC), October 8-11, 2014,Qingdao, China*, Qingdao, China, October 2014.