ORI (2025) Linear Programming TD

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October 2025

LP problems, feasible region

Exercise 1 : Put the following LP problem in normal form, represent its feasible region and conjecture the optimal solution by using the graphical method.

$$\begin{array}{rl} \max & 2x_1 + x_2 \\ \text{s.t.} & x_1 + x_2 \leq 4 \\ & -x_1 + x_2 \leq 2 \\ & x_1,\ x_2 \geq 0 \end{array}$$

Exercise 2 : Put the following LP problem in normal form, represent its feasible region and conjecture the optimal solution by using the graphical method.

$$\begin{array}{rl} \max & x_1 + 2x_2 \\ \text{s.t.} & -3x_1 + 2x_2 \leq 2 \\ & x_1 - 2x_2 \geq -4 \\ & x_1 + x_2 \leq 5 \\ & x_1,\ x_2 \geq 0 \end{array}$$

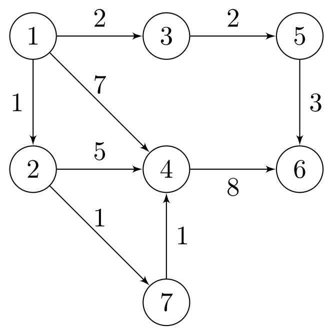
Modelization

Exercise 3 : linearization Consider a WiFi network whose relay stations can be placed at designated locations $i \in C = \{1, \ldots, n\}$. We want to minimize the number of such relay stations while still providing a high-quality service. To assess the quality of the service, we have a grid of test points $j \in T = \{1, \ldots, m\}$ in which we **must** ensure that there is at least one relay station providing a signal of strength greater or equal to a quantity q.

1) Model the problem by highlighting the objective, defining the decision variables and specifying the constraints.

2) If your previous model is not already an LP problem, rephrase it as an LP problem.

Exercise 4 : maxflow as an LP problem Consider the following graph, the goal is to maximize the flow from 1 to 6 while respecting the maximal capacity of every edge. This is a famous problem called *maxflow* and we will show that it can be rephrased by an LP problem.



1) What should be the decision variables ? And the objective ?

2) Write linear constraints to ensure that the capacity of each edge of the graph is not exceeded.

3) Write linear constraints to ensure that the solution is physically feasible, i.e. that the flow is conserved at every node of the graph.

Exercise 5 : A bakery plans its monthly production of cakes for one month. We assume that it is able to foresee the maximal amount of cakes that can be sold on day i, denoted by v_i. On day i, the bakery can produce up to m_i cakes that will be sold for c euros each and whose production cost p euros each. We will try to maximize the profit.

1) Describe this situation by an LP problem.

2) For the rest of the exercise we suppose that the bakery can store cakes, with a storage capacity of at most m_s cakes. Storage comes with a price of c_s euros per cake and per day of storage. Write the new LP problem. **Hint :** do not forget to ensure (with a constraint) that at the end of each day a cake is either stored or sold !

3) Let us now assume that when we produce at least one cake, there is a fixed cost (called activation cost) to pay. This corresponds for instance to the cost of turning on a machine or paying an extra employee. We will see different way of integrating this reality in our model.

a) Let us assume that we produce cakes on a daily basis. Write the new LP model. Do you think it is a good approach ? Why ?

b) We now decide to not produce cakes when the storage is sufficient to answer the demand. Write what changes compared to the previous model. Do you think we can be even smarter ? Why ?

c) Let us now assume that we **decide** on which days we produce cakes. Write what changes in the LP model and explain why it is a better approach compared to the two previous ones.

d) What difference do you make between a variable and a parameter ? In questions b) and c), the decision to produce or not plays one role and then the other. Explain which is which and why.

Simplex algorithm

Exercise 6 : Put the following LP problem in normal form, and solve it by using the simplex algorithm.

$$\begin{aligned} \max \quad & 2x_1 + x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 4 \\ & -x_1 + x_2 \leq 2 \\ & x_1,\ x_2 \geq 0 \end{aligned}$$

Exercise 7 : Put the following LP problem in normal form, represent its feasible region, conjecture the optimal solution by using the graphical method and solve it by using the simplex algorithm.

$$\begin{aligned} \max \quad & x_1 + 2x_2 \\ \text{s.t.} \quad & -3x_1 + 2x_2 \leq 2 \\ & -x_1 + 2x_2 \leq 4 \\ & x_1 + x_2 \leq 5 \\ & x_1,\ x_2 \geq 0 \end{aligned}$$

Duality and complementary slackness

Exercise 8 : Compute the dual of the problem from Exercise 7. Using complementary slackness, check that the optimal solution obtained in Exercise 7 is indeed optimal. Justify why the dual problem also has an optimal solution and compute it.

Exercise 9 : Consider the following LP problem :

$$\begin{aligned} \min \quad & 4x_1 + 5x_2 + 7x_3 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 && = 1 \\ & 5x_2 + 3x_3 && \geq 2 \\ & x_2 + 2x_3 && \geq 1 \\ & 4x_2 + x_3 && \leq 2 \\ & x_i \geq 0 \quad \forall i \in 1, 2, 3 \end{aligned}$$

The optimal solution is $\left(\frac{2}{7}, \frac{3}{7}, \frac{2}{7}\right)$.

1) Write the dual problem.
2) Justify that the dual problem has an optimal solution and compute it.

3) Check your answer as follows : assuming that your answer to 2) is correct, it will then give a solution the primal problem (the dual of the dual). Verify that the solution you obtained this way is indeed $\left(\frac{2}{7}, \frac{3}{7}, \frac{2}{7}\right)$.