ORI (2025) Linear Programming

Simon Abelard
(Inspired by a course of Mauro Passacantando)

November 2025

Course agenda

Two courses:

- OR I: Linear programming
- OR II: Optimization

Evaluation:

- OR I Exam (2hrs) on December 11^{th} or 18^{th} TBC
- OR II Exam (2hrs, 10-12am) on January 23^{rd}

Authorized material:

- No calculator, no course/exercise sheets
- One sheet that you fill as you want (choose carefully!)
- This sheet is personal and has your name on it

Introductory example

- A farmer has 12 hectares of land to cultivate tomatoes and/or potatoes.
- He has 70kg of tomato seeds, 18t of tubers and 160t of manure.
- Assuming that the market is able to absorb all of the production and that prices are stable, the estimated profit is 3000 euros per hectare for tomatoes and 5000 euros per hectare for potatoes.
- Tomatoes needs 7kg of seeds and 10t of manure per hectare, while potatoes require 3t tubers and 20t manure.

Goal: The farmer needs to determine how to allocate land to tomatoes and potatoes so as to maximize the income.

Decomposition of the problem

Data available: The farmer has 12 hectares and 160t of manure.

veg	qty	profits/ha	needs / ha	manure /ha
tomatoes	70kg	3000	7kg	10t
potatoes	18t	5000	3t	20t

Decision: how many hectares of potatoes/tomatoes to plant?

Objective: maximize the income.

Constraints: limited resources (land, manure, seeds, tubers)

Mathematical modeling

Decision variables: x_T and x_P the number of hectares dedicated to cultivate tomatoes and potatoes

Objective: maximize income $= 3x_T + 5x_P$ (in kilo-euros)

Constraints: limits on resources
Available land: $x_T + x_P \leq 12$
Available seeds: $7x_T \leq 70$
Available tubers: $3x_P \leq 18$
Available manure: $10x_T + 20x_P \leq 160$

Non negative variables: both x_T and x_P are non negative.

Matrix modeling

$$
\begin{aligned}
\max 3x_T + 5x_P & \\
x_T + x_P & \leq 12 \\
7x_T & \leq 70 \\
3x_P & \leq 18 \\
10x_T + 20x_P & \leq 160 \\
-x_T & \leq 0 \\
-x_P & \leq 0
\end{aligned}
$$

Matrix modeling

$$
\begin{aligned}
\max 3x_T + 5x_P & \\
x_T + x_P &\leq 12 \\
x_T &\leq 10 \\
x_P &\leq 6 \\
x_T + 2x_P &\leq 16 \\
-x_T &\leq 0 \\
-x_P &\leq 0
\end{aligned}
$$

$\begin{cases} \max & c^T x \\ Ax \leq b \end{cases}$ where $x = \begin{pmatrix} x_T \\ x_P \end{pmatrix}$, $c =$,

$A =$ and $b =$

Matrix modeling

$$\begin{aligned} \max 3x_T + 5x_P & \\ x_T + x_P &\leq 12 \\ x_T &\leq 10 \\ x_P &\leq 6 \\ x_T + 2x_P &\leq 16 \\ -x_T &\leq 0 \\ -x_P &\leq 0 \end{aligned}$$

$$\begin{cases} \max \quad c^T x \\ Ax \leq b \end{cases} \text{ where } x = \begin{pmatrix} x_T \\ x_P \end{pmatrix}, \ c = \begin{pmatrix} 3 \\ 5 \end{pmatrix},$$

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$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 2 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \text{ and } b =$$

Matrix modeling

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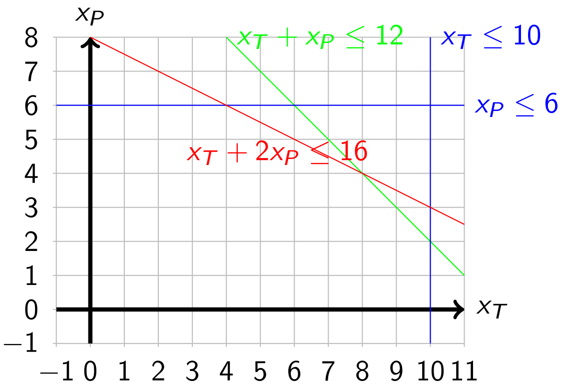
$$

Graphical representation: constraints

$$
\begin{aligned}
\max\ & 3x_T + 5x_P \\
& x_T + x_P && \leq 12 \\
& x_T && \leq 10 \\
& \qquad\quad x_P && \leq 6 \\
& x_T + 2x_P && \leq 16 \\
& -x_T && \leq 0 \\
& \qquad - x_P && \leq 0
\end{aligned}
$$



Graphical representation: feasible solutions

$$
\begin{aligned}
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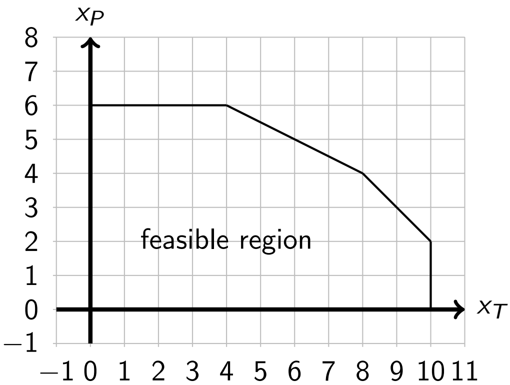

Level sets of the objective

For a **fixed** value v the level set associated to v is

$$L(v) = \{x = (x_T, x_P) \mid c^T x = v\}.$$

In other words the sets of x such that the objective equals v.
Remark: it is always a line!

$$\begin{aligned}
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x_T + x_P &\le 12 \\
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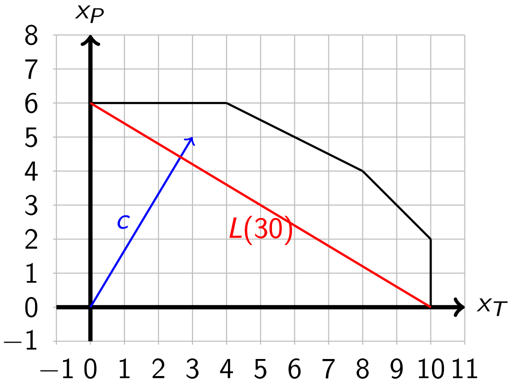

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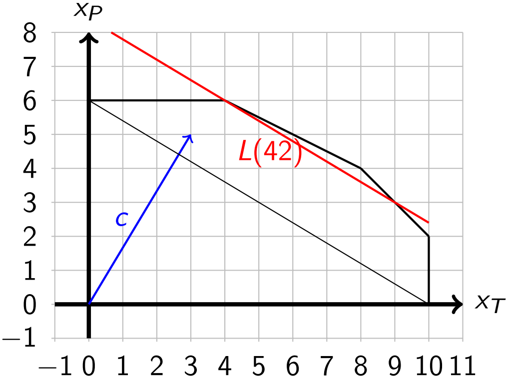

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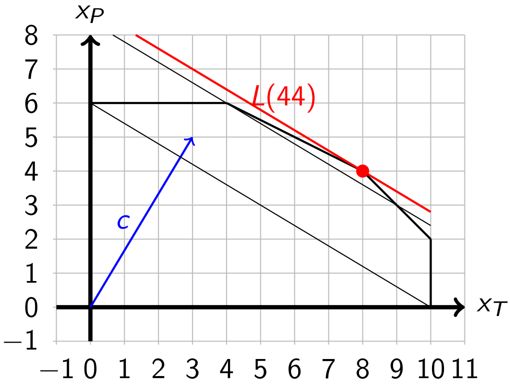

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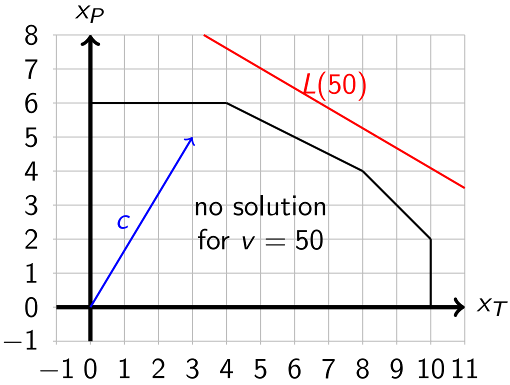

Optimal solution

The optimal solution is $(8, 4)$ and yields an income of 44k.
It is a **vertex** of the feasible region, we will see that it is always true (when optimum is not unique a whole edge can be optimal).

$$
\begin{aligned}
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x_T & \leq 10 \\
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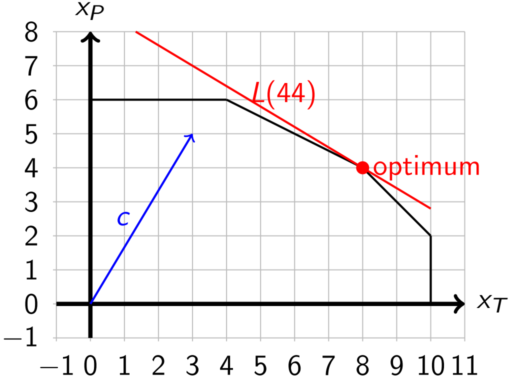
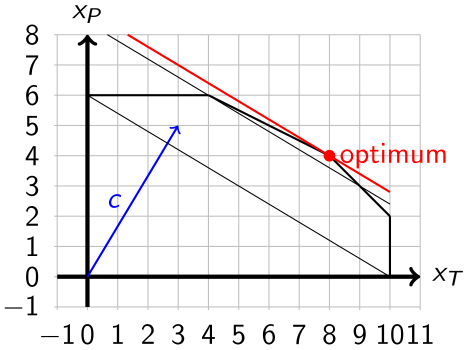


Finding an optimal solution on the graph

The following method is **not** a proof but can be useful to quickly find the optimal solution or check that your answer is correct.

- Draw the constraints
- Draw feasible region
- Draw the vector c
- Draw level lines $L(v)$ (orthogonal to c)
- Follow direction of c



The optimal solution is the last summit (or edge) reached before the level line exits the feasible region.

A bit of formalism

Our introductory example showed:

- How to represent a linear programming problem with matrices
- How to represent the feasible region as a polygon
- How to identify the optimal solution on the graph

Now we will:

- Formally define a linear programming (LP) problem
- Introduce the mathematics behind the graphical method
- Show how things generalize for more than 2 variables

Canonical form of an LP problem

Definition: Linear Programming Problem

An LP problem is the maximization (resp minimization) of a linear function in n variables subject to linear constraints (inequalities)

General form

$$\begin{cases} \max(\min) \;\; c^T x \\ A_1 x \leq b_1 \\ A_2 x \geq b_2 \\ A_3 x = b_3 \end{cases}$$

Canonical form

$$\begin{cases} \max \;\; c^T x \\ Ax \leq b \end{cases}$$

Every LP problem has a canonical form.

Putting a problem in canonical form

Theorem

Any linear programming problem can be written in canonical form.

Proof: we transform a General form into a Canonical form.

- Objective: $\min c^T x = -\max(-c^T x)$
- Inequalities: $A_2 x \geq b_2$ is equivalent to $-A_2 x \leq -b_2$
- Equalities: $A_3 x = b_3$ is equivalent to $\begin{cases} A_3 x \leq b_3 \\ -A_3 x \leq -b_3 \end{cases}$

The geometry of linear programming: convex hull

Definition

A vector $x \in \mathbb{R}^n$ is a convex combination of the vectors $x_1, \ldots x_k$ if there exists $\alpha_1, \ldots \alpha_k$ such that

$$x = \sum_{i=1}^{k} \alpha_k x_k \quad \text{and} \quad \sum_{i=1}^{k} \alpha_k = 1.$$

Example: $(2,2) = \frac{1}{3}(4,0) + \frac{2}{3}(1,3)$
$x = (2,2)$ is a convex combination of $x_1 = (4,0)$ and $x_2 = (1,3)$.

Definition

The **Convex Hull** of a set K, denoted by $\mathrm{Conv(K)}$ is the set of all convex combinations of elements of K.

Examples: the convex hull of points A and B is the segment $[AB]$.
In $\mathbb{R}^2$, the convex hull of n points is the polygon they define.

The geometry of linear programming: conical hull

Definition

A vector $x \in \mathbb{R}^n$ is a conical combination of the vectors $x_1, \ldots x_k$ if there exists nonnegative $\alpha_1, \ldots \alpha_k$ such that

$$x = \sum_{i=1}^{k} \alpha_k x_k.$$

Example: $(4,4) = \frac{8}{7}(2,3) + \frac{4}{7}(3,1)$
$x = (4,4)$ is a conical combination of $x_1 = (2,3)$ and $x_2 = (3,1)$.

Definition

The **Conical Hull** of a set K, denoted by $\mathrm{Cone(K)}$ is the set of all conical combinations of elements of K.

Example: in $\mathbb{R}^2$ the conical hull of two points A and B is the half cone defined by $(0,0)$, A and B.

Polyhedra (aka feasible regions)

Definitions

A **closed half-space** is a part of the Euclidean space defined by a hyperplane, including the hyperplane itself.
A **polyhedron** is the intersection of a finite number of closed hyperplanes of $\mathbb{R}^n$.

Link with linear programming

In $\mathbb{R}^n$, a linear constraint on x defines a closed hyperplane.
In $\mathbb{R}^n$, $Ax \leq b$ defines a polyhedron, hence **the feasible region of any LP problem is a polyhedron**.

Warning: polyhedra can be unbounded!
For instance, $P = \{x_1 \geq 1, x_2 \geq 1, x_1 + x_2 \geq 3\}$ is unbounded.
When it happens, the optimum can be unbounded too.

Recession directions

Definition

A **recession direction** of a polyhedron P is a vector d such that $x + \alpha d \in P$ for any $x \in P$ and $\alpha > 0$.

Example: with $P = \{x_1 \geq 1, x_2 \geq 1, x_1 + x_2 \geq 3\}$, $d = (1, 1)$ is a recession direction of P.

Consequence: if my objective is $\max x_1 + x_2$ and my constraints are the one defining P then the problem is unbounded, I can make the objective as large as I want by increasing x_1 or x_2.

Remark: in real life, unbounded problems are rare (in the case of the farmer, no way of getting infinite income from a finite land).

Vertices of a polyhedron

Definition

A **vertex** of a polyhedron P is a point $x \in P$ such that x cannot be expressed as a convex combination of other points in P.

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Example: with $P = \{x_1 \geq 1, x_2 \geq 1, x_1 + x_2 \geq 3\}$, $(1, 2)$ and $(2, 1)$ are the only **vertices** of P.

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Example: with $P = \{x_1 \geq 1, x_2 \geq 1, x_1 + x_2 \geq 3\}$, $(1, 2)$ and $(2, 1)$ are the only **vertices** of P.

Remarks: we have seen in the first example that the optimal solution was found at a vertex of the feasible region.
When two distinct vertices are optimal solutions, the whole edge between them is also optimal (the objective is a linear function).

Decomposition of polyhedra and optimal solutions

Theorem

If P is a polyhedron, there is a finite set $P_1, ..., P_k$ of points in P and a (possibly empty) set of recession directions $d_1, \ldots, d_\ell$ s.t.:

$$P = \mathrm{Conv}(\{P_1, \ldots, P_k\}) + \mathrm{Cone}(\{d_1, \ldots, d_\ell\}).$$

Corollary: if P is bounded, then $P = \mathrm{Conv}(\mathrm{Vertices}(V))$.

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Corollary: if P is bounded, then $P = \mathrm{Conv}(\mathrm{Vertices}(V))$.

In this course we will see the following options:

- Feasible region is empty: no solution
- Feasible region is a bounded polyhedron: at least one optimal solution on a vertex
- Feasible region is unbounded but objective does not grow indefinitely: at least one optimal solution on a vertex
- Feasible region is unbounded and objective can grow indefinitely: unbounded problem

The fundamental theorem of linear programming

Consider an LP problem in canonical form:

$$\begin{cases} \max \quad c^T x \\ Ax \leq b \quad (i.e. \ \ x \in P) \end{cases}$$

Fundamental Theorem of LP

If $P = \mathrm{Conv}(\{v_1, \dots, v_k\}) + \mathrm{Cone}(\{d_1, \dots, d_\ell\})$.

- If there is a finite solution, then there is an i such that v_i is an optimal solution (a vertex is optimal).
- There is a finite solution if and only if for every $0 \leq i \leq \ell$ we have $c^T d_i \leq 0$ (the objective does not grow indefinitely).

The fundamental theorem of LP – Examples

Example 1

$$\begin{cases} \max \quad 2x_1 - 3x_2 \\ x_1 \geq 1 \\ x_2 \geq 1 \\ x_1 + x_2 \geq 3 \end{cases}$$

Vertices: $(1,2)$ and $(2,1)$ Directions: $(1,0)$ and $(0,1)$.
Unbounded problem because $c^T(1,0) = 2 > 0$.

The fundamental theorem of LP – Examples

Example 1

$$\begin{cases} \max \quad 2x_1 - 3x_2 \\ x_1 \geq 1 \\ x_2 \geq 1 \\ x_1 + x_2 \geq 3 \end{cases}$$

Vertices: $(1, 2)$ and $(2, 1)$ Directions: $(1, 0)$ and $(0, 1)$.
Unbounded problem because $c^T(1, 0) = 2 > 0$.

Example 2

$$\begin{cases} \max \quad -2x_1 - 3x_2 \\ x_1 \geq 1 \\ x_2 \geq 1 \\ x_1 + x_2 \geq 3 \end{cases}$$

Now there is no regression direction.
Optimum is either at $(1, 2)$ (value -8) or $(2, 1)$ (value -7) hence $(2, 1)$ is optimal.

Duality

Consider an LP problem called the *primal problem*:

$$\begin{cases} \max \quad c^T x \\ Ax \leq b \end{cases}$$

Definition

The following LP problem is called the *dual* of the above problem:

$$\begin{cases} \min \quad y^T b \\ y^T A = c, \quad y \geq 0 \end{cases}$$

Dictionary of the correspondence dual-primal:

	Primal	*Dual*
Objective	max	min
Variables	n	m
Constraints	m	n

Dual of an LP problem – Example

Primal LP problem:

$$\begin{cases} \max 4x_1 + 5x_2 \\ \qquad x_1 \leq 1 \\ \qquad x_2 \leq 2 \\ x_1 + x_2 \leq 3 \end{cases}$$

Dual of an LP problem – Example

Primal LP problem:

$$\begin{cases} \max 4x_1 + 5x_2 \\ \qquad x_1 \leq 1 \\ \qquad x_2 \leq 2 \\ x_1 + x_2 \leq 3 \end{cases}$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix},\ b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \text{ and } c = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

Dual of an LP problem – Example

Primal LP problem:

$$\begin{cases} \max 4x_1 + 5x_2 \\ \qquad x_1 \leq 1 \\ \qquad x_2 \leq 2 \\ x_1 + x_2 \leq 3 \end{cases}$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}, \; b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \text{ and } c = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

Dual LP problem:

$$\begin{cases} \min y_1 + 2y_2 + 3y_3 \\ y_1 + 0 * y_2 + y_3 = 4 & \text{(first column of A)} \\ 0 * y_1 + y_2 + y_3 = 5 & \text{(second column of A)} \\ y_1, y_2, y_3 \geq 0 \end{cases}$$

Why the name dual?

Consider an LP problem $\mathcal{P}$ in canonical form:

$$\begin{cases} \max \quad c^T x \\ Ax \leq b \quad (i.e. \;\; x \in P) \end{cases}$$

Such that P is not empty. Let $\mathcal{D}$ be its dual.

Strong duality theorem

- If $\mathcal{D}$ is infeasible ($P_{\mathcal{D}} = \varnothing$) then $\mathcal{P}$ is unbounded.
- If $\mathcal{D}$ has an optimal solution reaching value v, then so does $\mathcal{P}$.

Why bother? **Because the dual may be simpler to solve.**
If $\mathcal{P}$ has many variables but only 2 constraints then $\mathcal{D}$ will have only 2 variables and so we can represent it graphically!

Example of using duality

What is the optimal solution of $\begin{cases} \min y_1 + y_2 + y_3 + y_4 \\ y_1 - y_2 + y_3 - y_4 = 2 \\ y_1 + 2y_2 - y_3 - y_4 = 1 \\ y_1, y_2, y_3, y_4 \geq 0 \end{cases}$

Example of using duality

What is the optimal solution of $\begin{cases} \min y_1 + y_2 + y_3 + y_4 \\ y_1 - y_2 + y_3 - y_4 = 2 \\ y_1 + 2y_2 - y_3 - y_4 = 1 \\ y_1, y_2, y_3, y_4 \geq 0 \end{cases}$

The dual LP problem is

$$\begin{cases} \max 2x_1 + x_2 \\ x_1 + x_2 \leq 1 \\ -x_1 + 2x_2 \leq 1 \\ x_1 - x_2 \leq 1 \\ -x_1 - x_2 \leq 1 \end{cases}$$

It is easy to solve (maximize x_1) and we can see that the optimal solution is $(1, 0)$, at which the objective is 2.

By strong duality we know that the minimum of the primal problem is also 2, without knowing where it is reached. We will see a way to also find that out.

Complementary slackness

Question: given a feasible solution, is it optimal?

Theorem: Complementary slackness

Let x be a feasible solution of $\mathcal{P}$. It is optimal if and only if there exists y such that

$$\begin{cases} y^T A = c, & \text{(dual feasibility)} \\ y \geq 0, & \text{(dual feasibility)} \\ y^T(b - Ax) = 0. & \text{(complementary slackness)} \end{cases}$$

In that case, y is an optimal solution of the dual $\mathcal{D}$.

Remark: complementary slackness is an optimality test but also allows to compute an optimal solution of $\mathcal{P}$ from an optimal solution of $\mathcal{D}$ and vice versa.

Example

Let us consider the following primal LP problem:

$$\begin{cases} \max 3x_1 + 4x_2 \\ 2x_1 + x_2 \leq 3 \\ x_1 + 2x_2 \leq 3 \\ -x_1 \leq 0 \\ -x_2 \leq 0 \end{cases}$$

Example

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Complementary slackness:

$$\begin{cases} 2y_1 + y_2 - y_3 = 3 \\ y_1 + 2y_2 - y_4 = 4 \\ y_1(2x_1 + x_2 - 3) = 0 \\ y_2(x_1 + 2x_2 - 3) = 0 \\ y_3 x_1 = 0 \\ y_4 x_2 = 0 \\ \forall 1 \leq i \leq 4, y_i \geq 0 \end{cases}$$

Example

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Complementary slackness:

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The feasible solution $x = (0, 0)$ is not optimal (no solution in y). The feasible $x = (1, 1)$ is optimal and the corresponding solution of the dual is $(2/3, 5/3, 0, 0)$.

Algebraic characterization of vertices

How to transform the graphical method into an algorithm?

Consider an LP problem $\mathcal{P}$ in canonical form with A an $m \times n$ matrix of rank n:

$$\begin{cases} \max \quad c^T x \\ Ax \le b \quad (i.e. \;\; x \in P) \end{cases}$$

Definition

A basis B is a set of n indices of rows of A such that the corresponding sub-matrix A_B is invertible.

We write $A = \begin{pmatrix} A_B \\ A_N \end{pmatrix}$ and $b = \begin{pmatrix} b_B \\ b_N \end{pmatrix}$.

Given a basis B we say that the vector $\bar{x} = A_B^{-1} b_B$ is a primal basis solution. It is feasible if $A_N \bar{x} \le b_N$.

Example

$$\begin{cases} \max 2x_1 + x_2 \\ \qquad x_1 \le 2 \\ x_1 + x_2 \le 3 \\ \quad -x_1 \le 0 \\ \quad -x_2 \le 0 \end{cases}$$

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix},\ b = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 0 \end{pmatrix} \text{ and } c = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Example

$$\begin{cases} \max 2x_1 + x_2 \\ \qquad x_1 \leq 2 \\ x_1 + x_2 \leq 3 \\ \quad -x_1 \leq 0 \\ \quad -x_2 \leq 0 \end{cases}$$

$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \; b = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 0 \end{pmatrix} \text{ and } c = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

For $B = (1, 2)$, $A_B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ is an invertible matrix.

The related solution is $\bar{x} = A_B^{-1} b_B = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

$\bar{x}$ is feasible since $A_N \bar{x} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix} \leq b_N = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Example

$$\begin{cases} \max 2x_1 + x_2 \\ x_1 \leq 2 \\ x_1 + x_2 \leq 3 \\ -x_1 \leq 0 \\ -x_2 \leq 0 \end{cases} \qquad A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \; b = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 0 \end{pmatrix} \text{ and } c = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

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$B = (1, 3)$ is not a basis because $A_B = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$ is not invertible.

$B = (2, 4)$ is a basis since $A_B = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$ but it is not feasible.

Link between bases and vertices

Theorem: correspondence between vertices and bases

The point of coordinates given by $\bar{x}$ is a vertex of the polyhedron P if and only if $\bar{x}$ is a feasible basis solution.

Definition: dual basis

Given a basis B, let $y_N = 0$ and $y_B^T = c^T A_B^{-1}$. The vector $y = \begin{pmatrix} y_B \\ y_N \end{pmatrix}$ is called a dual basis solution of B.
We say that it is feasible if $y_B \geq 0$.

Theorem: sufficient optimality condition

Let $\bar{x}$ be a feasible basis solution corresponding to basis B. If the dual basis solution $\bar{y}$ is feasible, then $\bar{x}$ is an optimal solution.

Example

Consider the previous problem:

$$\begin{cases} \max 2x_1 + x_2 \\ \quad x_1 \leq 2 \\ x_1 + x_2 \leq 3 \\ \quad -x_1 \leq 0 \\ \quad -x_2 \leq 0 \end{cases} \qquad A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix},\ b = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 0 \end{pmatrix} \text{ and } c = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Let $\bar{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ be the feasible solution corresponding to $B = (1, 2)$.

Is it optimal?

We compute $y_B = c^T A_B^{-1} = (2, 1) \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = (1, 1)$.

Since y_B is feasible (both coordinates are non negative), $\bar{x}$ is an optimal solution.

The simplex algorithm (overview)

The simplex algorithm is a special case of gradient descent, it seems quite complex but follows simple rules:

- Start from a vertex (not always easy to find, in this course it is given to you and most often will be the origin $(0, 0)$)
- If $y_B \geq 0$ we have found the optimal solution (yay!).
- The vertex is a base B, find an index exiting B. Which one? The smallest $i \in B$ such that $y_i < 0$. (i makes B non optimal)
- Find an index entering B. Which one?
  The one that increases the gain the most.
- Repeat steps 2, 3 and 4.

Remarks: steps 3 & 4 can be tuned but must avoid cycling (i.e. endlessly switching between 2 non-optimal vertices).
Step 4 also includes a check to detect unbounded problems.

The simplex algorithm, detail

- **Step 1:** find a basis B such that the related basis solution $\bar{x} = A_B^{-1} b_B$ is feasible.
- **Step 2:** Compute the dual basis solution $\bar{y}$ defined by $\bar{y}_B^T = c^T A_B^{-1}$ and $\bar{y}_N = 0$. If $\bar{y}_B \geq 0$, return $\bar{x}$ as it is optimal.
- **Step 3:** Let h be the first index in B for which $\bar{y}_i < 0$, h is the index that will exit the basis B.
- **Step 4:** Set $W = -A_B^{-1}$ and W^h its h-th column. If for all $i \in N$ $A_i W^h \leq 0$ then return $+\infty$ (unbounded problem). Else, among all $i \in N$ such that $A_i W^h > 0$, find the smallest i that minimizes $\frac{b_i - A_i \bar{x}}{A_i W^h}$. This is the entering index.
- **Step 5:** Update the basis B: index $h \in B$ leaves B and is replaced by the index found in step 4. Restart the process from Step 2 with the updated B.

The simplex algorithm, remarks

- Step 1 is not always easy, sometimes need to solve another simplex to find a B. In our exercises, $\bar{x} = 0$ is often a vertex.
- In step 3, any index corresponding to a negative y_j would fit, the choice is important to avoid cycling. Another option could be to take $j = \arg\min_i \bar{y}_i$ (remove the "worst" index).
- Likewise in step 4, any index in N such that $A_i W^h > 0$ would fit, taking the one we chose helps the algorithm to converge faster (in a way we try to pick the "best" index).

Theorem

The simplex algorithms converges to an optimal solution (if it exists) after a finite number of steps. Idea: there is a finite number of vertices and we avoid cycling between them.

Remark: we could try all vertices to find an optimal one, but there is an exponential number of them. The simplex algorithm remains exponential but still much faster than naive enumeration.

Simplex algorithm, example

Consider the previous problem:

$$\begin{cases} \max 2x_1 + x_2 \\ \quad x_1 \leq 2 \\ x_1 + x_2 \leq 3 \\ \quad -x_1 \leq 0 \\ \quad -x_2 \leq 0 \end{cases} \qquad A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \; b = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 0 \end{pmatrix} \text{ and } c = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Let us start from $(0, 0)$, i.e. $B = (3, 4)$.

Iteration 1: $A_B = -I_2 = A_B^{-1}$, $b_B = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ hence $\bar{x} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

We have $y_B^T = c^T A_B^{-1} = (-2, -1)$ so index 3 exits B.

The first column of $-A_B^{-1}$ is $W = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ hence $A_1 W = A_2 W = 1$.

We have $\theta_1 = 2$, $\theta_2 = 3$ so index 1 replaces index 3 in the base B.

Simplex algorithm, example (continued)

Consider the previous problem:

$$\begin{cases} \max 2x_1 + x_2 \\ \quad x_1 \leq 2 \\ x_1 + x_2 \leq 3 \\ \quad -x_1 \leq 0 \\ \quad -x_2 \leq 0 \end{cases} \qquad A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \; b = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 0 \end{pmatrix} \text{ and } c = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Iteration 2: $B = (1, 4)$ so $A_B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = A_B^{-1}$ and $\bar{x} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$.

We have $y_B^T = c^T A_B^{-1} = (2, -1)$ so index 4 exits B.

The second column of $-A_B^{-1}$ is $W = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ so $A_3 W = 0$, $A_2 W = 1$.

This means that index 2 replaces index 4 in B

Simplex algorithm, example (the end)

Consider the previous problem:

$$\begin{cases} \max 2x_1 + x_2 \\ \quad x_1 \le 2 \\ x_1 + x_2 \le 3 \\ \quad -x_1 \le 0 \\ \quad -x_2 \le 0 \end{cases} \qquad A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \; b = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 0 \end{pmatrix} \text{ and } c = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Iteration 3: $B = (1, 2)$ so $A_B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $A_B^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$.

We have $\bar{x} = A_B^{-1} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

We have $y_B^T = c^T A_B^{-1} = (1, 1) \ge 0$ so $\bar{x}$ is optimal.

Conclusion: $(x_1, x_2) = (2, 1)$ is an optimal solution of the LP problem, for which the objective reaches the maximal value 5.

Further topics

This course on Linear Programming opens a range of questions:

- What if x is an integer or a Boolean? $\rightsquigarrow$ ILP (Integer Linear Programming), e.g. branch-and-bound algorithms
- What if the data of the problem changes over time (e.g. fluctuating prices and availability) $\rightsquigarrow$ Sensitivity analysis
- What if the problems are non-linear? $\rightsquigarrow$ next part of the class

Spoiler: in the end everything becomes linear algebra!