ORII (2025) Optimization Exercises

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Convexity

Exercise 1 : Show that if $(f_i)_i$ is a family of convex functions defined on an interval I, then $\max_i f_i$ is also a convex function on I.

Consider the function taking as input a symmetric matrix of size $n \times n$ and returning its largest eigenvalue. Show that it is a convex function.

Exercise 2 : Consider the family of functions $f_\alpha : (x, y) \mapsto x^4 + y^4 + 2\alpha xy$.

1) For which values of α is f_α convex ? Concave ?

2) For $\alpha = 1$, show that f_1 has a global minimum over $\mathbb{R}^2$ and compute the value of f_1 at this minimum.

Unconstrained Optimization

Exercise 3 : For the following functions, find the critical points and, when you can, say whether they are minima, maxima or saddle points.

1) $f(x, y) = x^2 + y^2 - xy$

2) $f(x, y) = x^2 - y^2 - xy$

3) $f(x, y, z) = x^4 + y^2 + z^2 - 4x - 2y - 2z + 4$

Exercise 4 : Gradient descent Let A be a symmetric positive semi-definite matrix whose eigenvalues are ordered $0 \leq \lambda_1 \leq \cdots \leq \lambda_n$, we want to solve $Ax = b$ iteratively using the gradient algorithm with constant step. We start from an arbitrary x_0 and define $x_{k+1} = x_k + \eta r_k$, with $r_k = b - Ax_k$.

1) Let x^* be the solution to the problem, we introduce the error $e_k = x_k - x^*$. Compute $e_k - e_0$. To make the computations easier, you may first want to compute $r_{k+1} - r_k$.

2) Justify that the algorithm converges if and only if η is smaller than $\frac{2}{\lambda_n}$.

3) Show that the best choice of η is $\eta = \frac{2}{\lambda_1 + \lambda_n}$.

Exercise 4b : Conjugate gradient We want to do something similar but with a dynamic choice for η, so we reuse the above definitions but we additionally define : $\eta_k = \frac{r_k \cdot r_k}{Ar_k \cdot r_k}$, where $\cdot$ is the usual scalar product.

We also define $J(x) = \frac{1}{2}Ax \cdot x - b \cdot x$ and assume that A is positive definitive, i.e. $\lambda_1 > 0$.

1) Show that J is differentiable, what is ∇J ?

2) Show that η_k is the unique global minimizer of $\eta \mapsto J(x_k + \eta r_k)$.

3) From the Taylor series expansion of J at x_k, show that

$$J(x_{k+1}) - J(x_k) = -\frac{1}{2}\frac{||r_k||^4}{Ar_k \cdot r_k}$$

.

4) Show that for any $k \geq 0$, we have $J(x_k) - J(x^*) = \frac{1}{2}A^{-1}r_k \cdot r_k$ and deduce that

$$\frac{J(x_{k+1}) - J(x^*)}{J(x_k) - J(x^*)} = 1 - \frac{||r_k||^4}{(A^{-1}r_k \cdot r_k)(Ar_k \cdot r_k)}.$$

5) We recall that the **condition number** of A for the Euclidean norm is $\kappa(A) = \frac{\lambda_n}{\lambda_1}$. Show that

$$J(x_{k+1}) - J(x^*) \leq (J(x_0) - J(x^*))\left(\frac{\kappa(A) - 1}{\kappa(A) + 1}\right)^{2k}.$$

Hint : you may use Kantorovich's inequality

$$||x||^4 \leq (Ax \cdot x)(A^{-1}x \cdot x) \leq \frac{(\lambda_1 + \lambda_n)^2}{4\lambda_1\lambda_n}||x||^4.$$

6) Justify that $||x_k - x^*||^2 \leq \frac{2}{\lambda_1}||J(x_k) - J(x^*)||$ and conclude on the convergence of the conjugate gradient method.

Constrained Optimization

Exercise 5 : You are given a fence of maximal length L and must use it to delimit the biggest possible rectangular field. Assuming that the land is perfectly flat, justify why the optimal rectangle is actually a square.

Exercise 6 : Consider the following benefit function $B = -4q_1^2 - 6q_2^2 - 3q_1q_2 + 256q_1 + 222q_2$ that we want to maximize under the following constraints : $3q_1 + q_2 \leq 246$ and $q_1 + 5q_2 \leq 212$.

1) Justify why this problem has a unique optimal solution.

2) Find this optimal solution.

Exercise 7 : Solve the following LP problem (without using the simplex algorithm) :

$$\begin{aligned} \max \quad & 4x_1 - 3x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 7 \\ & 2x_1 + 5x_2 \geq 8 \\ & x_1,\ x_2 \geq 0 \end{aligned}$$

Exercise 8 : Solve the following non-linear parametric problem :

$$\begin{aligned} \max \quad & (x_1 - 1)(6 - x_2) \\ \text{s.t.} \quad & x_1 \geq 2 \\ & 0 \leq x_2 \leq 5 \\ & x_1 - mx_2 \leq 0 \end{aligned}$$