SABELLA :)

HW 10

1) Let $M \in M_n$ be the 2×2 block matrix (3.3.5). If D is invertible, then the Schur complement of D in M is $M/D = A - BD^{-1}C$. Show that $\det(M) = (\det D)\det(M/D)$.

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in M_n, \text{ in which } A \in M_r \text{ and } D \in M_{n-r}$$

Suppose we multiply M by $\begin{bmatrix} I_r & -BD^{-1} \\ 0 & I_r \end{bmatrix}$:

$$\begin{bmatrix} I_r & -BD^{-1} \\ 0 & I_r \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A-BD^{-1}C & 0 \\ C & D \end{bmatrix}.$$

Now, by Theorem 3.3.4,

$$\det\begin{bmatrix} A-BD^{-1}C & 0 \\ C & D \end{bmatrix} = \det\begin{bmatrix} I_r & 0 \\ 0 & D \end{bmatrix}\det\begin{bmatrix} I_r & 0 \\ C & I_{n-r} \end{bmatrix}\det\begin{bmatrix} A-BD^{-1}C & 0 \\ 0 & I_{n-r} \end{bmatrix}$$

$$= \det D \cdot (1) \cdot \det(A-BD^{-1}C)$$

$$= (\det D)\det(A-BD^{-1}C)$$

$(M/D = A - BD^{-1}C)$

$$= (\det D)\det(M/D)$$

4) Compute

$$\det \begin{bmatrix} 0 & 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 3 & 5 & 6 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

We have

$$M = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ \hline 3 & 5 & 6 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \to \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}, \text{ want } \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

$$\left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ \hline 3 & 5 & 6 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c|c} 0 & I_3 \\ \hline I_3 & 0 \end{array}\right] = \begin{bmatrix} 0 & 0 & 0 & 1 & 4 & 6 \\ 0 & 0 & 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & 3 \\ 3 & 5 & 6 & 0 & 0 & 0 \\ 0 & 2 & 4 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 & 6 & 0 & 0 & 0 \\ 0 & 2 & 5 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 5 & 6 \\ 0 & 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

$$\det M = \det \begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix} \det \begin{bmatrix} 0 & I_3 \\ I_3 & 0 \end{bmatrix} = \det \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \det A \det B$$

$$\det A = 1 \cdot 2 \cdot 3 = 6 \qquad \det B = 3 \cdot 2 \cdot 1 = 6$$

$$\Rightarrow \det M = 36$$

5) Let $A, B \in \mathbb{M}_{m \times n}$ and suppose that $r = \operatorname{rank} A \geq 1$. Show that the following are equivalent:

(a) $\operatorname{col} A = \operatorname{col} B$

(b) There are full-rank matrices $X \in \mathbb{M}_{m \times r}$ and $Y, Z \in \mathbb{M}_{r \times n}$ such that $A = XY$ and $B = XZ$.

(c) There is an invertible $S \in \mathbb{M}_n$ such that $B = AS$.

Hint: If $A = XY$ is a full-rank factorization, then $A = [X \ 0_{n \times (n-r)}] W$, in which $W = \begin{bmatrix} Y \\ Y_2 \end{bmatrix} \in \mathbb{M}_n$ is invertible.

(a) $\Rightarrow$ (b) Suppose that $\operatorname{col} A = \operatorname{col} B$. Then every column of A can be written as a linear combination of the columns of B, and vice versa, hence, $B = AC$ for some $C \in \mathbb{M}_{n \times n}$. Now, since $r = \operatorname{rank} A \geq 1$, we can factorize A as

$$A = XY, \quad X \in \mathbb{M}_{m \times r}, \quad Y \in \mathbb{M}_{r \times n},$$

where X and Y have rank r. Now,

$$B = AC$$

$$\Rightarrow B = XYC.$$

Let $YC = Z$, $Z \in \mathbb{M}_{r \times n}$. Then

$$B = XZ, \quad X \in \mathbb{M}_{m \times r}, \quad Z \in \mathbb{M}_{r \times n}.$$

Hence, there are full-rank matrices $X \in \mathbb{M}_{m \times r}$ and $Y, Z \in \mathbb{M}_{r \times n}$ such that $A = XY$ and $B = XZ$.

(b) $\Rightarrow$ (c) Suppose there are full-rank matrices $X \in M_{m\times r}$ and $Y, Z \in M_{r\times n}$ such that $A = XY$ and $B = XZ$. Let $A = [X \; 0_{n\times(n-r)}]W$, where $W = \begin{bmatrix} Y \\ Y_2 \end{bmatrix} \in M_n$ is invertible, and $B = [X \; 0_{n\times(n-r)}]V$, where $V = \begin{bmatrix} Z \\ Z_2 \end{bmatrix} \in M_n$ is invertible. Now,

$$A = XY = [X \; 0_{n\times(n-r)}]W, \quad B = XZ = [X \; 0_{n\times(n-r)}]V.$$

We can solve for $[X \; 0_{n\times(n-r)}]$ in both equations,

$$A = [X \; 0_{n\times(n-r)}]W \qquad B = [X \; 0_{n\times(n-r)}]V$$

$$\Rightarrow AW^{-1} = [X \; 0_{n\times(n-r)}] \qquad BV^{-1} = [X \; 0_{n\times n-r}]$$

$$\Rightarrow AW^{-1} = BV^{-1}$$

$$\Rightarrow AW^{-1}V = B.$$

Let $W^{-1}V = S$. Then there is an invertible $S \in M_n$ such that $B = AS$.

(c) $\Rightarrow$ (a) Suppose there is an invertible $S \in M_n$ such that $B = AS$. Then each column of B is a linear combination of the columns of A. Hence, $\operatorname{col} B \subseteq \operatorname{col} A$. Now, S is invertible, so we can write $A = BS^{-1}$. Here, each column of A is a linear combination of the columns of B. Thus, $\operatorname{col} A \subseteq \operatorname{col} B$. Now, since $\operatorname{col} B \subseteq \operatorname{col} A$ and $\operatorname{col} A \subseteq \operatorname{col} B$, we can conclude that $\operatorname{col} A = \operatorname{col} B$.

HW 9

1) Let $A \in M_{m\times n}(\mathbb{F})$ and see Figure 4.1. Prove that $\operatorname{col} A = \mathbb{F}^m$ if and only if the rows of A are linearly independent.

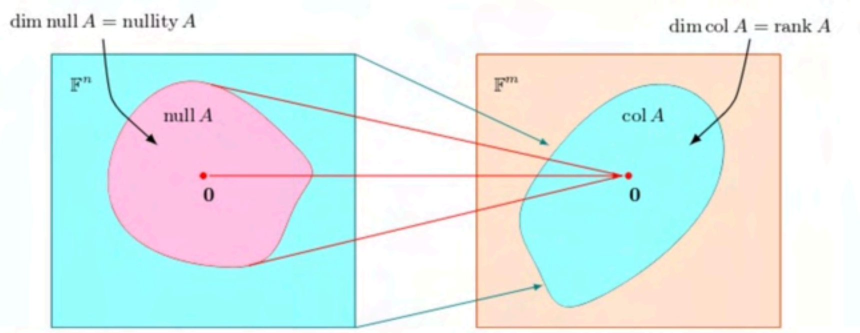


Figure 4.1 For $A \in M_{m\times n}(\mathbb{F})$, the column space of A need not equal $\mathbb{F}^m$; this occurs if and only if A has linearly independent rows. The null space of A need not be $\{\mathbf{0}\}$; this occurs if and only if A has linearly independent columns. The rank-nullity theorem says that $\operatorname{rank} A + \operatorname{nullity} A = n$.

Proof. ($\Rightarrow$) Suppose that $\operatorname{col} A = \mathbb{F}^m$. Then, for any vector $v \in \mathbb{F}^m$, we can write v as a linear combination of the columns of A. hence, $\operatorname{rank} A \geq m$, but since the maximum row rank is m, we have $\operatorname{rank} A = \text{row rank } A = m$. Hence, there is a pivot in every row of A, which implies that the rows of A are linearly independent.

($\Leftarrow$) Suppose that the rows of A are linearly independent. Then there are m pivot rows, hence $\text{row rank } A = \operatorname{rank} A = m$. Now, by the Fundamental Theorem of Linear Algebra,

$$\operatorname{rank} A = \dim \operatorname{col} A$$

Hence, $\dim \operatorname{col} A = m$, which means we have m linearly independent columns in A. By the Dimension theorem, the columns of A form a basis for $\mathbb{F}^m$. Therefore, $\operatorname{col} A = \mathbb{F}^m$. ∎

2) Let V be a nonzero n-dimensional $\mathbb{F}$-vector space and let $T \in \mathcal{L}(V, \mathbb{F})$ be a nonzero linear transformation. Why is $\dim \ker T = n-1$?

Note that $\ker T = \{v \in V \mid Tv = 0\}$. By the Rank-Nullity Theorem,

$$\dim V = \dim \ker T + \dim \operatorname{range} T.$$

We know that V is n-dimensional, so $\dim V = n$. However, T is a nonzero linear transformation, which implies that $\operatorname{range} T = \mathbb{F}$, where $\dim \operatorname{range} T = \dim \mathbb{F} = 1$. Therefore,

$$n = \dim \ker T + 1$$

$$\Rightarrow \dim \ker T = n-1.$$

3) For each of the following matrices, compute its reduced row echelon form, its pivot column decomposition, and bases for its column space and null space.

(a) $$\begin{bmatrix} -1 & -2 & 2 & 3 & 3 \\ 3 & 6 & 1 & 5 & -1 \\ 2 & 4 & -1 & 0 & 2 \end{bmatrix} \xrightarrow{-R_1} \sim \begin{bmatrix} 1 & 2 & -2 & -3 & -3 \\ 3 & 6 & 1 & 5 & -1 \\ 2 & 4 & -1 & 0 & 2 \end{bmatrix} \begin{matrix} R_2 - 3R_1 \\ R_3 - 2R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & -2 & -3 & -3 \\ 0 & 0 & 7 & 14 & 8 \\ 0 & 0 & 3 & 6 & 8 \end{bmatrix} \xrightarrow{\frac{1}{7}R_2} \sim \begin{bmatrix} 1 & 2 & -2 & -3 & -3 \\ 0 & 0 & 1 & 2 & 8/7 \\ 0 & 0 & 3 & 6 & 8 \end{bmatrix} \xrightarrow{R_3 - 3R_2} \sim \begin{bmatrix} 1 & 2 & -2 & -3 & -3 \\ 0 & 0 & 1 & 2 & 8/7 \\ 0 & 0 & 0 & 0 & 32/7 \end{bmatrix} \frac{7}{32}R_3$$

$$\sim \begin{bmatrix} 1 & 2 & -2 & -3 & -3 \\ 0 & 0 & 1 & 2 & 8/7 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_2 - \frac{8}{7}R_3 \\ R_1 + 3R_3 \end{matrix} \sim \begin{bmatrix} 1 & 2 & -2 & -3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 + 2R_2} \sim \begin{array}{c} \begin{bmatrix} \underline{1} & 2 & 0 & 1 & 0 \\ 0 & 0 & \underline{1} & 2 & 0 \\ 0 & 0 & 0 & 0 & \underline{1} \end{bmatrix} \\ \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 \end{matrix} \end{array} = \operatorname{rref}(A)$$

$A = PR,\quad P = \begin{bmatrix} -1 & 2 & 3 \\ 3 & 1 & -1 \\ 2 & -1 & 2 \end{bmatrix},\ R = \begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

basis for col A : span $\left\{ \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \right\}$

basis for null A : let $Ax = 0$. Then

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} x_1 + 2x_2 + x_4 &= 0 \\ x_3 + 2x_4 &= 0 \Rightarrow x_3 = -2x_4 \\ x_5 &= 0 \end{aligned}$$

$$x_1 = -2x_2 - x_4 \Rightarrow x = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

$\therefore$ basis for null A = span $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}$.

(b) $$\begin{bmatrix} 1 & -1 & 2 & 1 & 3 \\ 1 & 0 & 1 & 1 & 1 \\ 3 & 2 & 2 & 3 & 2 \end{bmatrix} \underset{\sim}{\overset{R_2 - R_1}{R_3 - 3R_1}} \begin{bmatrix} 1 & -1 & 2 & 1 & 3 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & 5 & -4 & 0 & -7 \end{bmatrix} R_3 - 5R_2$$

$$\sim \begin{bmatrix} 1 & -1 & 2 & 1 & 3 \\ 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 3 \end{bmatrix} \underset{\sim}{\overset{R_2 + R_3}{R_1 - 2R_3}} \begin{bmatrix} 1 & -1 & 0 & 1 & -3 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 3 \end{bmatrix} \underset{\sim}{R_1 + R_2} \begin{bmatrix} \underline{1} & 0 & 0 & 1 & -2 \\ 0 & \underline{1} & 0 & 0 & 1 \\ 0 & 0 & \underline{1} & 0 & 3 \end{bmatrix}$$

$A = PR,\ P = \begin{bmatrix} 1 & -1 & 2 \\ 1 & 0 & 1 \\ 3 & 2 & 2 \end{bmatrix},\ R = \begin{bmatrix} 1 & 0 & 0 & 1 & -2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 3 \end{bmatrix}$

basis for col A : span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$

basis for null A : $\begin{bmatrix} 1 & 0 & 0 & 1 & -2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow x_1 + x_4 - 2x_5 = 0 \Rightarrow x_1 = -x_4 + 2x_5$

$x_2 + x_5 = 0 \Rightarrow x_2 = -x_5$

$x_3 + 3x_5 = 0 \Rightarrow x_3 = -3x_5$

$\Rightarrow x = x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 2 \\ -1 \\ -3 \\ 0 \\ 1 \end{bmatrix}$

$\therefore$ basis for null A = span $\left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ -3 \\ 0 \\ 1 \end{bmatrix} \right\}$.

4) let

$$A = \begin{bmatrix} 1 & 2 & 3 & ? & ? & 6 \\ 6 & 5 & 4 & ? & ? & 1 \\ 2 & 3 & 4 & ? & ? & 1 \\ 1 & 6 & 5 & ? & ? & 2 \end{bmatrix} \quad \text{and} \quad E = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -2 & 0 \\ 0 & 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

in which E is the reduced row echelon form of A. Find the missing columns of A and the pivot column decomposition of A.

By matrix E, matrix A has pivots in the first, second, third, and sixth columns. Thus, the pivot-column decomposition of A is

$$P = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 6 & 5 & 4 & 1 \\ 2 & 3 & 4 & 1 \\ 1 & 6 & 5 & 2 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -2 & 0 \\ 0 & 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since $A = PR$, we can multiply P and R to find A and its missing columns:

$$PR = \begin{bmatrix} 1 & 2 & 3 & 6 \\ 6 & 5 & 4 & 1 \\ 2 & 3 & 4 & 1 \\ 1 & 6 & 5 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -2 & 0 \\ 0 & 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \\ 2 & 3 & 4 & 5 & 6 & 1 \\ 1 & 6 & 5 & 4 & 3 & 2 \end{bmatrix}.$$

Hence, the missing columns of A are $\begin{bmatrix} 4 \\ 3 \\ 5 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ 6 \\ 3 \end{bmatrix}$.

5) Let $A = PR$ be a pivot column decomposition, in which

$$A = \begin{array}{c} \begin{bmatrix} 4 & 8 & 1 & 6 & 1 & 0 & 9 & 1 \\ 1 & 2 & 0 & 1 & 1 & 1 & 5 & 0 \\ 2 & ? & 1 & ? & 6 & 1 & 8 & 1 \\ 1 & 2 & 2 & 5 & 1 & 0 & ? & 1 \\ 0 & 0 & 1 & 2 & 2 & 1 & ? & 6 \\ 1 & 2 & 0 & 1 & 1 & 2 & 8 & 1 \end{bmatrix} \\ \begin{array}{cccccccc} a_1 & & a_3 & & a_5 & a_6 & & a_8 \end{array} \end{array} \text{ and } R = \begin{array}{c} \begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\ \begin{array}{cccccccc} r_1 & r_2 & r_3 & r_4 & r_5 & r_6 & r_7 & r_8 \end{array} \end{array}.$$

Find P and the missing entries of A.

By matrix R, $P = \begin{bmatrix} 4 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 6 & 1 & 1 \\ 1 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 6 \\ 1 & 0 & 1 & 2 & 1 \end{bmatrix}$.

$A = PR \Rightarrow$

$$A = \begin{bmatrix} 4 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 2 & 1 & 6 & 1 & 1 \\ 1 & 2 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 6 \\ 1 & 0 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 8 & 1 & 6 & 1 & 0 & 9 & 1 \\ 1 & 2 & 0 & 1 & 1 & 1 & 5 & 0 \\ 2 & 4 & 1 & 4 & 6 & 1 & 8 & 1 \\ 1 & 2 & 2 & 5 & 1 & 0 & 4 & 1 \\ 0 & 0 & 1 & 2 & 2 & 1 & 4 & 6 \\ 1 & 2 & 0 & 1 & 1 & 2 & 8 & 1 \end{bmatrix}.$$

The missing entries are circled.

6) Suppose that $A \in M_{6 \times 4}$ is row equivalent to

$$B = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 3 & 2 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for null A.

let $Bx = 0$: $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 3 & 2 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, now solve for x :

$\Rightarrow x_1 + x_4 = 0 \qquad \Rightarrow x_1 = -x_4 \Rightarrow x_1 = 0$

$2x_2 + 3x_3 + 2x_4 = 0 \qquad \Rightarrow 2x_2 = -3x_3 \quad \Rightarrow x_2 = \frac{-3}{2} x_3$

$4x_4 = 0 \Rightarrow x_4 = 0$

$$x = x_3 \begin{bmatrix} 0 \\ -3/2 \\ 1 \\ 0 \end{bmatrix}$$

basis for null A : span $\left\{ \begin{bmatrix} 0 \\ -3/2 \\ 1 \\ 0 \end{bmatrix} \right\}$.

(b) What is dimnull A^T ?

By the rank-nullity theorem,

$$\text{rank}(A^T) + \dim \text{null}(A^T) = \#\text{ rows of } A.$$

The number of rows of A is 6, and the rank of A^T is 3.

Hence, $\dim \text{null}(A^T) = 3$.

7) Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be defined by

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_2 + x_3 \\ x_3 - x_1 \\ 0 \end{bmatrix}$$

(a) Show that T is linear.

$$T(cx) = T\left(c\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = T\left(\begin{bmatrix} cx_1 \\ cx_2 \\ cx_3 \end{bmatrix}\right) = \begin{bmatrix} cx_1 - cx_2 \\ cx_2 + cx_3 \\ cx_3 - cx_1 \\ 0 \end{bmatrix} = c\begin{bmatrix} x_1 - x_2 \\ x_2 + x_3 \\ x_3 - x_1 \\ 0 \end{bmatrix} = c \cdot T(x)$$

$$T(x+y) = T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_3 + y_3 \end{bmatrix}\right) = \begin{bmatrix} (x_1+y_1) - (x_2+y_2) \\ (x_2+y_2) + (x_3+y_3) \\ (x_3+y_3) - (x_1+y_1) \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} (x_1-x_2) + (y_1-y_2) \\ (x_2+x_3) + (y_2+y_3) \\ (x_3-x_1) + (y_3-y_1) \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_2 - x_3 \\ x_3 - x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} y_1 - y_2 \\ y_2 - y_3 \\ y_3 - y_1 \\ 0 \end{bmatrix} = T(x) + T(y)$$

(b) Let $\mathcal{E} = I_3$, $\alpha = \begin{bmatrix}1\\0\\9\\1\end{bmatrix}, \begin{bmatrix}3\\4\\5\\0\end{bmatrix}, \begin{bmatrix}2\\4\\-4\\2\end{bmatrix}, \begin{bmatrix}5\\8\\10\\0\end{bmatrix}$

Show that α is a basis of $\mathbb{R}^4$ and find ${}_\alpha[T]_\mathcal{E}$.

First, we show that α is linearly independent. Let a_1, a_2, a_3, a_4 denote each vector in α, respectively. Then

$$c_1 a_1 + c_2 a_2 + c_3 a_3 + c_4 a_4 = 0$$

$$\Rightarrow c_1 \begin{bmatrix}1\\0\\9\\1\end{bmatrix} + c_2 \begin{bmatrix}3\\4\\5\\0\end{bmatrix} + c_3 \begin{bmatrix}2\\4\\-4\\2\end{bmatrix} + c_4 \begin{bmatrix}5\\8\\10\\0\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

$$\Rightarrow \begin{aligned} c_1 + 3c_2 + 2c_3 + 5c_4 &= 0 \\ 4c_2 + 4c_3 + 8c_4 &= 0 \Rightarrow 4c_2 = -4c_3 - 8c_4 \Rightarrow c_2 = -c_3 - 2c_4 \\ 9c_1 + 5c_2 - 4c_3 + 10c_4 &= 0 \\ c_1 + 2c_3 &= 0 \Rightarrow c_1 = -2c_3 \end{aligned}$$

$$\Rightarrow -2c_3 + 3(-c_3 - 2c_4) + 2c_3 + 5c_4 = 0 \Rightarrow c_4 = -3c_3$$

$$\Rightarrow 9(-2c_3) + 5(-c_3 - 2c_4) - 4c_3 + 10(-3c_3) = 0 \Rightarrow -27c_3 = 0 \Rightarrow c_3 = 0$$

$$\Rightarrow c_1 = -2(0) = 0, \quad c_4 = -3(0) = 0, \quad c_2 = -0 - 2(0) = 0$$

Hence, the only solution to the linear combination is $c_1 = c_2 = c_3 = c_4 = 0$. Thus, each vector in α is linearly independent. By the Dimension Theorem, since α is a list of 4 linearly independent vectors, and $\dim \mathbb{R}^4 = 4$, α is a basis for $\mathbb{R}^4$.

$${}_\alpha[T]_\mathcal{E} = \left[\left[T\begin{bmatrix}1\\0\\0\end{bmatrix}\right]_\alpha \left[T\begin{bmatrix}0\\1\\0\end{bmatrix}\right]_\alpha \left[T\begin{bmatrix}0\\0\\1\end{bmatrix}\right]_\alpha \right]$$

$$= \alpha^{-1}\left[[T\begin{bmatrix}1\\0\\0\end{bmatrix}]\ [T\begin{bmatrix}0\\1\\0\end{bmatrix}]\ [T\begin{bmatrix}0\\0\\1\end{bmatrix}] \right]$$

$$T\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\-1\\0\end{bmatrix} \quad T\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}-1\\1\\0\\0\end{bmatrix} \quad T\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\1\\0\end{bmatrix}$$

$$\alpha^{-1} = \begin{bmatrix}1&3&2&5\\0&4&4&8\\9&5&-4&10\\1&0&2&0\end{bmatrix}^{-1} = \begin{bmatrix}0&-5/54&2/27&1/3\\2&-55/54&-5/27&-1/3\\0&5/108&-1/27&1/3\\-1&11/18&1/9&0\end{bmatrix}$$

$${}_{\alpha}[T]_{\varepsilon} = \begin{bmatrix}0&-5/54&2/27&1/3\\2&-55/54&-5/27&-1/3\\0&5/108&-1/27&1/3\\-1&11/18&1/9&0\end{bmatrix}\begin{bmatrix}1&-1&0\\0&1&1\\-1&0&1\\0&0&0\end{bmatrix} = \begin{bmatrix}-2/27&-5/54&-1/54\\59/27&-163/54&-65/54\\1/27&5/108&1/108\\-10/9&29/18&13/18\end{bmatrix}$$

(c) Let $\phi = \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}4\\8\\0\end{bmatrix}, \begin{bmatrix}2\\0\\5\end{bmatrix}$, $\varphi = \begin{bmatrix}2\\0\\-3\\0\end{bmatrix}, \begin{bmatrix}0\\3\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\-1\end{bmatrix}, \begin{bmatrix}2\\3\\-1\\2\end{bmatrix}$

Show that ϕ and φ are bases of $\mathbb{R}^3$ and $\mathbb{R}^4$, respectively and find ${}_{\varphi}[T]_{\phi}$

Claim: ϕ is linearly independent

Let the matrix form of ϕ be given by $C = \begin{bmatrix}1&4&2\\2&8&0\\3&0&5\end{bmatrix}$. Then

$$\det C = 1(40-0) - 4(10-0) + 2(0-24) = 40-40-48 = -48 \neq 0.$$

Hence, the columns of C are linearly independent. Thus, ϕ is linearly independent. By the dimension theorem, since $\dim \mathbb{R}^3 = 3$, and

ϕ has 3 linearly independent columns, ϕ is a basis for $\mathbb{R}^3$.

Claim: φ is linearly independent

Let the matrix form of φ be given by $D = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 3 & 0 & 3 \\ -3 & 1 & 1 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$.

We will find the row-echelon form of D.

$$\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 3 & 0 & 3 \\ -3 & 1 & 1 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{matrix} \tfrac{1}{2}R_1 \\ \tfrac{1}{3}R_2 \\ -R_4 \\ \sim \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ -3 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \end{bmatrix} \overset{R_3+3R_1}{\sim} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -2 \end{bmatrix} \overset{R_3-R_2}{\sim} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

$$\overset{R_4-R_3}{\sim} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix} \overset{-\frac{1}{3}R_4}{\sim} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

In this form, we see that D has a pivot in all four columns.

Hence, the columns of D are linearly independent. Therefore, by the Dimension Theorem, φ is a basis for $\mathbb{R}^4$.

$$[T]_{\phi}^{\varphi} = \left[\left[T\begin{bmatrix}1\\2\\3\end{bmatrix}\right]_{\varphi} \left[T\begin{bmatrix}4\\8\\0\end{bmatrix}\right]_{\varphi} \left[T\begin{bmatrix}2\\0\\5\end{bmatrix}\right]_{\varphi} \right]$$

$$= D^{-1}\left[T\begin{bmatrix}1\\2\\3\end{bmatrix} \quad T\begin{bmatrix}4\\8\\0\end{bmatrix} \quad T\begin{bmatrix}2\\0\\5\end{bmatrix} \right]$$

$$T\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}-1\\5\\2\\0\end{bmatrix} \qquad T\begin{bmatrix}4\\8\\0\end{bmatrix} = \begin{bmatrix}-4\\8\\-4\\0\end{bmatrix} \qquad T\begin{bmatrix}2\\0\\5\end{bmatrix} = \begin{bmatrix}2\\5\\3\\0\end{bmatrix}$$

$$D^{-1} = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 3 & 0 & 3 \\ -3 & 1 & 1 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 1/9 & -1/3 & -1/3 \\ -1/2 & 4/9 & -1/3 & -1/3 \\ 1 & -2/9 & 2/3 & -1/3 \\ 1/2 & -1/9 & 1/3 & 1/3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 1/9 & -1/3 & -1/3 \\ -1/2 & 4/9 & -1/3 & -1/3 \\ 1 & -2/9 & 2/3 & -1/3 \\ 1/2 & -1/9 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} -1 & -4 & 2 \\ 5 & 8 & 5 \\ 2 & -4 & 3 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1/9 & 20/9 & -4/9 \\ 37/18 & 62/9 & 2/9 \\ -7/9 & -76/9 & 26/9 \\ -7/18 & -38/9 & 13/9 \end{bmatrix}$$

(d) Express ${}_{\alpha}[T]_{\varepsilon}$ in terms of ${}_{\varphi}[T]_{\phi}$, and express ${}_{\varphi}[T]_{\phi}$ in terms of ${}_{\alpha}[T]_{\varepsilon}$.

$${}_{\alpha}[T]_{\varepsilon} = {}_{\alpha}[I]_{\varphi}\,{}_{\varphi}[T]_{\phi}\,{}_{\phi}[I]_{\varepsilon}$$

$${}_{\alpha}[I]_{\varphi} = \left[\left[\begin{bmatrix} 2 \\ 0 \\ -3 \\ 0 \end{bmatrix}\right]_{\alpha} \left[\begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \end{bmatrix}\right]_{\alpha} \left[\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}\right]_{\alpha} \left[\begin{bmatrix} 2 \\ 3 \\ -1 \\ 2 \end{bmatrix}\right]_{\alpha} \right] = \alpha^{-1}\varphi$$

$$\alpha^{-1}\varphi = \begin{bmatrix} 0 & -5/54 & 2/27 & 1/3 \\ 2 & -55/54 & -5/27 & -1/3 \\ 0 & 5/108 & -1/27 & 1/3 \\ -1 & 11/18 & 1/9 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 3 & 0 & 3 \\ -3 & 1 & 1 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -2/9 & -11/54 & -7/27 & 17/54 \\ 41/9 & -175/54 & 4/27 & 25/54 \\ 1/9 & 11/108 & -10/27 & 91/108 \\ -7/3 & 35/18 & 1/9 & -5/18 \end{bmatrix}$$

$${}_{\phi}[I]_{\varepsilon} = \phi^{-1}\varepsilon \quad ; \quad \phi^{-1} = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 0 \\ 3 & 0 & 5 \end{bmatrix}^{-1} = \begin{bmatrix} -5/6 & 5/12 & 1/3 \\ 5/24 & 1/48 & -1/12 \\ 1/2 & -1/4 & 0 \end{bmatrix}$$

$$\phi^{-1}\varepsilon = \begin{bmatrix} -5/6 & 5/12 & 1/3 \\ 5/24 & 1/48 & -1/12 \\ 1/2 & -1/4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -5/6 & 5/12 & 1/3 \\ 5/24 & 1/48 & -1/12 \\ 1/2 & -1/4 & 0 \end{bmatrix}$$

$\therefore$

$${}_{\alpha}[T]_{\varepsilon} = \begin{bmatrix} -2/9 & -11/54 & -7/27 & 17/54 \\ 41/9 & -175/54 & 4/27 & 25/54 \\ 1/9 & 11/108 & -10/27 & 91/108 \\ -7/3 & 35/18 & 1/9 & -5/18 \end{bmatrix} \begin{bmatrix} -1/9 & 20/9 & -4/9 \\ 37/18 & 62/9 & 2/9 \\ -7/9 & -76/9 & 26/9 \\ -7/18 & -38/9 & 13/9 \end{bmatrix} \begin{bmatrix} -5/6 & 5/12 & 1/3 \\ 5/24 & 1/48 & -1/12 \\ 1/2 & -1/4 & 0 \end{bmatrix}$$

$${}_{\varphi}[T]_{\phi} = {}_{\varphi}[I]_{\alpha}\;{}_{\alpha}[T]_{\varepsilon}\;{}_{\varepsilon}[I]_{\phi}$$

$${}_{\varphi}[I]_{\alpha} = \left({}_{\alpha}[I]_{\varphi}\right)^{-1} = \begin{bmatrix} -2/9 & -11/54 & -7/27 & 17/54 \\ 41/9 & -175/54 & 4/27 & 25/54 \\ 1/9 & 11/108 & -10/27 & 91/108 \\ -7/3 & 35/18 & 1/9 & -5/18 \end{bmatrix}^{-1} = \begin{bmatrix} -10/3 & -11/9 & 10/9 & -22/9 \\ -23/6 & -25/18 & 13/9 & -41/18 \\ 20/3 & 49/9 & -20/9 & 89/9 \\ 23/6 & 49/18 & -1/9 & 89/18 \end{bmatrix}$$

$${}_{\varepsilon}[I]_{\phi} = \left({}_{\phi}[I]_{\varepsilon}\right)^{-1} = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 0 \\ 3 & 0 & 5 \end{bmatrix}$$

$\therefore$

$${}_{\varphi}[T]_{\phi} = \begin{bmatrix} -10/3 & -11/9 & 10/9 & -22/9 \\ -23/6 & -25/18 & 13/9 & -41/18 \\ 20/3 & 49/9 & -20/9 & 89/9 \\ 23/6 & 49/18 & -1/9 & 89/18 \end{bmatrix} \begin{bmatrix} -2/27 & -5/54 & -1/54 \\ 59/27 & -163/54 & -65/54 \\ 4/27 & 5/108 & 1/108 \\ -10/9 & 29/18 & 13/18 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 0 \\ 3 & 0 & 5 \end{bmatrix}$$

(e) Find ${}_{\varphi}[T]_{\varepsilon}$ and ${}_{\alpha}[T]_{\phi}$, and express ${}_{\varphi}[T]_{\varepsilon}$ in terms of ${}_{\alpha}[T]_{\phi}$, as well as ${}_{\alpha}[T]_{\phi}$ in terms of ${}_{\varphi}[T]_{\varepsilon}$.

$${}_{\varphi}[T]_{\varepsilon} = \left[[T(e_1)]_{\varphi} \;\; [T(e_2)]_{\varphi} \;\; [T(e_3)]_{\varphi} \right] = \varphi^{-1}\left[T[e_1] \;\; T[e_2] \;\; T[e_3] \right]$$

$$T\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \qquad T\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad T\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\varphi^{-1} = \begin{bmatrix} 0 & 1/9 & -1/3 & -1/3 \\ -1/2 & 4/9 & -1/3 & -1/3 \\ 1 & -2/9 & 2/3 & -1/3 \\ 1/2 & -1/9 & 1/3 & 1/3 \end{bmatrix}$$

$$\varphi^{-1}\left[T[e_1] \;\; T[e_2] \;\; T[e_3] \right] = \begin{bmatrix} 0 & 1/9 & -1/3 & -1/3 \\ -1/2 & 4/9 & -1/3 & -1/3 \\ 1 & -2/9 & 2/3 & -1/3 \\ 1/2 & -1/9 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/9 & -2/9 \\ -1/6 & 17/18 & 1/9 \\ 1/3 & -11/9 & 4/9 \\ 1/6 & -11/18 & 2/9 \end{bmatrix}$$

$${}_{\alpha}[T]_{\phi} = \left[[T(c_1)]_{\alpha} \ [T(c_2)]_{\alpha} \ [T(c_3)]_{\alpha} \right] = \alpha^{-1}\left[[Tc_1] \ [Tc_2] \ [Tc_3] \right]$$

$$T\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 5 \\ 2 \\ 0 \end{bmatrix} \quad T\begin{bmatrix} 4 \\ 8 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 8 \\ -4 \\ 0 \end{bmatrix} \quad T\begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 3 \\ 0 \end{bmatrix}$$

$$\alpha^{-1} = \begin{bmatrix} 0 & -5/54 & 2/27 & 1/3 \\ 2 & -55/54 & -5/27 & -1/3 \\ 0 & 5/108 & -1/27 & 1/3 \\ -1 & 11/18 & 1/9 & 0 \end{bmatrix}$$

$$\alpha^{-1}\left[[Tc_1][Tc_2][Tc_3] \right] = \begin{bmatrix} 0 & -5/54 & 2/27 & 1/3 \\ 2 & -55/54 & -5/27 & -1/3 \\ 0 & 5/108 & -1/27 & 1/3 \\ -1 & 11/18 & 1/9 & 0 \end{bmatrix} \begin{bmatrix} -1 & -4 & 2 \\ 5 & 8 & 5 \\ 2 & -4 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -17/54 & -28/27 & -13/54 \\ -403/54 & -416/27 & -89/54 \\ 17/108 & 14/27 & 13/108 \\ 77/18 & 76/9 & 25/18 \end{bmatrix}$$

$${}_{\varphi}[T]_{\varepsilon} = {}_{\varphi}[I]_{\alpha} \ {}_{\alpha}[T]_{\phi} \ {}_{\phi}[I]_{\varepsilon}$$

$${}_{\varphi}[I]_{\alpha} = \begin{bmatrix} -10/3 & -11/9 & 10/9 & -22/9 \\ -23/6 & -25/18 & 13/9 & -41/18 \\ 20/3 & 49/9 & -20/9 & 89/9 \\ 23/6 & 49/18 & -1/9 & 89/18 \end{bmatrix} \qquad {}_{\phi}[I]_{\varepsilon} = \begin{bmatrix} -5/6 & 5/12 & 1/3 \\ 5/24 & 1/48 & -1/12 \\ 1/2 & -1/4 & 0 \end{bmatrix}$$

$${}_{\varphi}[T]_{\varepsilon} = \begin{bmatrix} -10/3 & -11/9 & 10/9 & -22/9 \\ -23/6 & -25/18 & 13/9 & -41/18 \\ 20/3 & 49/9 & -20/9 & 89/9 \\ 23/6 & 49/18 & -1/9 & 89/18 \end{bmatrix} \begin{bmatrix} -17/54 & -28/27 & -13/54 \\ -403/54 & -416/27 & -89/54 \\ 17/108 & 14/27 & 13/108 \\ 77/18 & 76/9 & 25/18 \end{bmatrix} \begin{bmatrix} -5/6 & 5/12 & 1/3 \\ 5/24 & 1/48 & -1/12 \\ 1/2 & -1/4 & 0 \end{bmatrix}$$

$$[T]_{\alpha\phi} = [I]_{\alpha\gamma} [T]_{\gamma\varepsilon} [I]_{\varepsilon\phi}$$

$$_{\alpha}[I]_{\gamma} = \begin{bmatrix} -2/9 & -11/54 & -7/27 & 17/54 \\ 41/9 & -175/54 & 4/27 & 25/54 \\ 1/9 & 11/108 & -10/27 & 91/108 \\ -7/3 & 35/18 & 1/9 & -5/18 \end{bmatrix} \qquad _{\varepsilon}[I]_{\phi} = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 0 \\ 3 & 0 & 5 \end{bmatrix}$$

$$_{\alpha}[T]_{\phi} = \begin{bmatrix} -2/9 & -11/54 & -7/27 & 17/54 \\ 41/9 & -175/54 & 4/27 & 25/54 \\ 1/9 & 11/108 & -10/27 & 91/108 \\ -7/3 & 35/18 & 1/9 & -5/18 \end{bmatrix} \begin{bmatrix} 1/3 & 1/9 & -2/9 \\ -1/6 & 17/18 & 1/9 \\ 1/3 & -11/9 & 4/9 \\ 1/6 & -11/18 & 2/9 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 0 \\ 3 & 0 & 5 \end{bmatrix}$$

8) Let $V = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 9 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \\ 0 \end{bmatrix} \right)$.

(a) Construct a non-zero 4×4 matrix B such that $\text{null}\, B = V$.

Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ 9 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 0 \end{bmatrix}$. We will extend V into a basis for $\mathbb{R}^4$ by adding vectors u_1, u_2, such that $\alpha = v_1, v_2, u_1, u_2$. Let ε denote the standard basis for $\mathbb{R}^4$, $\varepsilon = e_1, e_2, e_3, e_4$. Now, we want $Bv_1 = 0$, $Bv_2 = 0$, $Bu_1 = e_1$, $Bu_2 = e_2$. We will construct a change of basis matrix:

$$_{\varepsilon}[B]_{\alpha} = [\,[Bv_1]_{\varepsilon}\ [Bv_2]_{\varepsilon}\ [Bu_1]_{\varepsilon}\ [Bu_2]_{\varepsilon}\,]$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$ Now, we know that

$_{\varepsilon}[B]_{\varepsilon} = {}_{\varepsilon}[I]_{\varepsilon}\ {}_{\varepsilon}[B]_{\alpha}\ {}_{\alpha}[I]_{\varepsilon}$. We find $_{\varepsilon}[I]_{\varepsilon}$ and

${}_{\alpha}[I]_{\alpha}$: ${}_{\varepsilon}[I]_{\varepsilon} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

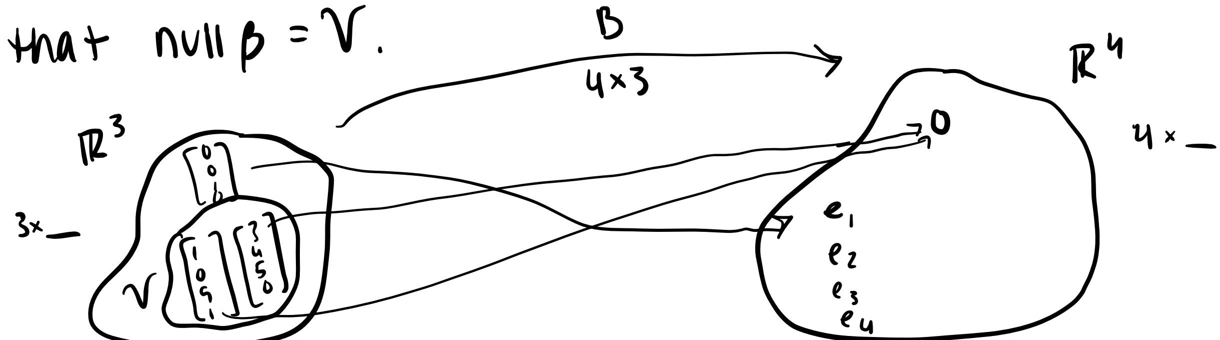
$${}_{\varepsilon}[I]_{\alpha} = [v_1 \ \ v_2 \ \ u_1 \ \ u_2] = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 4 & 0 & 1 \\ 9 & 5 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow {}_{\alpha}[I]_{\varepsilon} = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 4 & 0 & 1 \\ 9 & 5 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1/5 & -9/5 \\ 1 & 0 & -3/5 & 22/5 \\ 0 & 1 & -4/5 & 36/5 \end{bmatrix}.$$

$$\therefore \ {}_{\varepsilon}[B]_{\varepsilon} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1/5 & -9/5 \\ 1 & 0 & -3/5 & 22/5 \\ 0 & 1 & -4/5 & 36/5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -3/5 & 22/5 \\ 0 & 1 & -4/5 & 36/5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Use the method learned in class to construct a 4×3 matrix B such that null $B = V$.



$v_1 = \begin{bmatrix} 1 \\ 0 \\ 9 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 3 \\ 4 \\ 5 \\ 0 \end{bmatrix}$, $u_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $B\begin{bmatrix} 1 \\ 0 \\ 9 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $B\begin{bmatrix} 3 \\ 4 \\ 5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $B\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\alpha = v_1, v_2, u_1$

9) Let $V = \text{span}\left(\begin{bmatrix}1\\6\\9\\-1\\0\end{bmatrix}, \begin{bmatrix}3\\4\\5\\9\\0\end{bmatrix}, \begin{bmatrix}4\\10\\14\\8\\0\end{bmatrix}\right)$.

(a) Construct a non-zero 5×5 matrix B such that the solution set of homogeneous equations $Bx = 0$ is equal to V.

Since $\begin{bmatrix}4\\10\\14\\8\\0\end{bmatrix}$ is a linear combination of $\begin{bmatrix}1\\6\\9\\-1\\0\end{bmatrix}$ and $\begin{bmatrix}3\\4\\5\\9\\0\end{bmatrix}$, by a previous theorem, we can omit it from the span and V remains the same.

Hence, $V = \text{span}\left(\begin{bmatrix}1\\6\\9\\-1\\0\end{bmatrix}, \begin{bmatrix}3\\4\\5\\9\\0\end{bmatrix}\right)$. We add vectors u_1, u_2, u_3, such that

$$[B]_{\alpha}^{\varepsilon} = \begin{bmatrix}0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$. Now, $[B]_{\varepsilon}^{\varepsilon} = [I]_{\varepsilon}^{\varepsilon}[B]_{\alpha}^{\varepsilon}[I]_{\varepsilon}^{\alpha}$.

$$[I]_{\varepsilon}^{\varepsilon} = \begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}, \quad [I]_{\varepsilon}^{\alpha} = \begin{bmatrix}1&3&0&0&0\\6&4&0&0&0\\9&5&1&0&0\\-1&9&0&1&0\\0&0&0&0&1\end{bmatrix}^{-1} = \begin{bmatrix}-2/7&3/14&0&0&0\\3/7&-1/14&0&0&0\\3/7&-11/7&1&0&0\\-29/7&6/7&0&1&0\\0&0&0&0&1\end{bmatrix}$$

Hence,

$$[B]_{\varepsilon}^{\varepsilon} = \begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}\begin{bmatrix}0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}\begin{bmatrix}-2/7&3/14&0&0&0\\3/7&-1/14&0&0&0\\3/7&-11/7&1&0&0\\-29/7&6/7&0&1&0\\0&0&0&0&1\end{bmatrix}$$

$$= \begin{bmatrix}3/7&-11/7&1&0&0\\-29/7&6/7&0&1&0\\0&0&0&0&1\\0&0&0&0&0\\0&0&0&0&0\end{bmatrix}$$

SABELLA SAULINO

HW 8

1) i) Let $X = [X_1 \;\; X_2] \in M_{m\times n}$, in which $X_1 \in M_{m\times n_1}$, $X_2 \in M_{m\times n_2}$, and $n_1 + n_2 = n$. Compute X^TX and XX^T.

$$X^TX = \underset{(n\times m)}{\begin{bmatrix} X_1^T \\ X_2^T \end{bmatrix}} \underset{(m\times n)}{[X_1 \;\; X_2]} = \underset{(n\times n)}{\begin{bmatrix} X_1^TX_1 & X_1^TX_2 \\ X_2^TX_1 & X_2^TX_2 \end{bmatrix}}$$

$$XX^T = \underset{(m\times n)}{[X_1 \;\; X_2]} \underset{(n\times m)}{\begin{bmatrix} X_1^T \\ X_2^T \end{bmatrix}} = \underset{(m\times m)}{\left[X_1X_1^T + X_2X_2^T \right]}$$

ii) Let $A = \left[\begin{array}{c|cc} -1 & 2 & 3 \\ \hline 2 & -3 & 1 \\ 3 & 1 & -2 \end{array}\right] = \left[\begin{array}{cc|c} -1 & 2 & 3 \\ \hline 2 & -3 & 1 \\ 3 & 1 & -2 \end{array}\right]$ and $B = \begin{bmatrix} 1 & 3 & -2 \\ 3 & -2 & 1 \\ -2 & 1 & 3 \end{bmatrix}$.

a) Compute AB in two different ways by partitioning B conformally with each presentation of A and then performing block-matrix multiplication.

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \Rightarrow A_{11} = [-1] \quad A_{12} = [2 \;\; 3] \quad A_{21} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad A_{22} = \begin{bmatrix} -3 & 1 \\ 1 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \Rightarrow B_{11} = [1] \quad B_{12} = [3 \;\; -2] \quad B_{21} = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \quad B_{22} = \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix} \quad \left(B = \left[\begin{array}{c|cc} 1 & 3 & -2 \\ \hline 3 & -2 & 1 \\ -2 & 1 & 3 \end{array}\right] \right)$$

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

$$A_{11}B_{11} + A_{12}B_{21} = [-1][1] + [2 \;\; 3]\begin{bmatrix} 3 \\ -2 \end{bmatrix} = [-1] + [0] = [-1]$$

$$A_{11}B_{12} + A_{12}B_{22} = [-1][3\ \ -2] + [2\ \ 3]\begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix} = [-3\ \ 2] + [-1\ \ 11]$$
$$= [-4\ \ 13]$$

$$A_{21}B_{11} + A_{22}B_{21} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}[1] + \begin{bmatrix} -3 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -11 \\ 7 \end{bmatrix} = \begin{bmatrix} -9 \\ 10 \end{bmatrix}$$

$$A_{21}B_{12} + A_{22}B_{22} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}[3\ \ -2] + \begin{bmatrix} -3 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 6 & -4 \\ 9 & -6 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ -4 & -5 \end{bmatrix}$$
$$= \begin{bmatrix} 13 & -4 \\ 5 & -11 \end{bmatrix}$$

$$AB = \begin{bmatrix} [-1] & [-4\ \ 13] \\ \begin{bmatrix} -9 \\ 10 \end{bmatrix} & \begin{bmatrix} 13 & -4 \\ 5 & -11 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} -1 & -4 & 13 \\ -9 & 13 & -4 \\ 10 & 5 & -11 \end{bmatrix}$$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \Rightarrow A_{11} = [-1\ \ 2] \quad A_{12} = [3]$$
$$A_{21} = \begin{bmatrix} 2 & -3 \\ 3 & 1 \end{bmatrix} \quad A_{22} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \Rightarrow B_{11} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad B_{12} = \begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} \quad \left(B = \left[\begin{array}{c|cc} 1 & 3 & -2 \\ 3 & -2 & 1 \\ \hline -2 & 1 & 3 \end{array}\right] \right)$$
$$B_{21} = [-2] \quad B_{22} = [1\ \ 3]$$

$$AB = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{bmatrix}$$

$$A_{11}B_{11} + A_{12}B_{21} = [-1\ \ 2]\begin{bmatrix} 1 \\ 3 \end{bmatrix} + [3][-2] = [5] + [-6] = [-1]$$

$$A_{11}B_{12} + A_{12}B_{22} = [-1\ \ 2]\begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} + [3][1\ \ 3] = [-7\ \ 4] + [3\ \ 9]$$
$$= [-4\ \ 13]$$

$$A_{21}B_{11} + A_{22}B_{21} = \begin{bmatrix} 2 & -3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}[-2] = \begin{bmatrix} -7 \\ 6 \end{bmatrix} + \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -9 \\ 10 \end{bmatrix}$$

$$A_{21}B_{12} + A_{22}B_{22} = \begin{bmatrix} 2 & -3 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}\begin{bmatrix} 1 & 3 \end{bmatrix} = \begin{bmatrix} 12 & -7 \\ 7 & -5 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -2 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & -4 \\ 5 & -11 \end{bmatrix}$$

$$AB = \begin{bmatrix} [-1] & [-4 \;\; 13] \\ \begin{bmatrix} -9 \\ 10 \end{bmatrix} & \begin{bmatrix} 13 & -4 \\ 5 & -11 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} -1 & -4 & 13 \\ -9 & 13 & -4 \\ 10 & 5 & -11 \end{bmatrix}$$

b) Compute AB in the standard manner, and verify that all answers agree.

$$AB = \begin{bmatrix} -1 & 2 & 3 \\ 2 & -3 & 1 \\ 3 & 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 3 & -2 \\ 3 & -2 & 1 \\ -2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -1+6-6 & -3-4+3 & 2+2+9 \\ 2-9-2 & 6+6+1 & -4-3+3 \\ 3+3+4 & 9-2-2 & -6+1-6 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -4 & 13 \\ -9 & 13 & -4 \\ 10 & 5 & -11 \end{bmatrix}$$ ✓✓ All answers agree

iii) let $A = \left[\begin{array}{c|cc|cc} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ \hline a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ \hline a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{array}\right]$ and $B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ b_{41} & b_{42} & b_{43} \\ b_{51} & b_{52} & b_{53} \end{bmatrix}$

How should B be partitioned to compute AB w/ block matrices?

We have $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$, where A_{11}, A_{31} are 1×1, $A_{12}, A_{13}, A_{32}, A_{33}$ are 1×2, A_{21} is 2×1, and A_{22}, A_{23} are 2×2. Also, the block partition of A is 3×3, so the block partition of B will be $3\times K$, where $K \in \{1, 2, \dots\}$.

let $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix}$. Then $AB = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \\ B_{31} & B_{32} \end{bmatrix}$

$= \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31} & A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32} \\ A_{21}B_{11} + A_{22}B_{21} + A_{23}B_{31} & A_{21}B_{12} + A_{22}B_{22} + A_{23}B_{32} \\ A_{31}B_{11} + A_{32}B_{21} + A_{33}B_{31} & A_{31}B_{12} + A_{32}B_{22} + A_{33}B_{32} \end{bmatrix}$. Based on our partition of A, we have that B_{11}, B_{12} are $1 \times k$, and $B_{21}, B_{31}, B_{22}, B_{32}$ are $2 \times k$. Now,

$A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31} \Rightarrow B_{11}, B_{21}, B_{31}$ have same # cols

$A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32} \Rightarrow B_{12}, B_{22}, B_{32}$ have same # cols

Thus, B should be partitioned as

$$B = \left[\begin{array}{c|cc} b_{11} & b_{12} & b_{13} \\ \hline b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \\ \hline b_{41} & b_{42} & b_{43} \\ b_{51} & b_{52} & b_{53} \end{array}\right].$$

2) let $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 3 & -1 & 0 \\ 4 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix}$.

a) Show that A and B are full column rank matrices.

We will do so by row reducing A and B to show that both have linearly independent columns (pivot in each column):

i)

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 3 & -1 & 0 \\ 4 & 1 & 1 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - 4R_1 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & -3 \\ 0 & 1 & -3 \end{bmatrix} \begin{matrix} R_3 + R_2 \\ R_4 - R_2 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & -4 \end{bmatrix} R_4 - 2R_2 \sim \begin{bmatrix} \underline{1} & 0 & 1 \\ 0 & \underline{1} & 1 \\ 0 & 0 & \underline{1} \\ 0 & 0 & 0 \end{bmatrix}$$

Each column contains a pivot. Hence, A has full column rank.

ii)
$$B = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 \leftrightarrow R_1} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow{R_4 - R_1} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{R_3 - R_2} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Each column contains a pivot, thus B has full column rank.

b) Extend A and B into basis matrices of $\mathbb{R}^4$.

Let $u_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, and $[A \mid u_1] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 \\ 3 & -1 & 0 & 0 \\ 4 & 1 & 1 & 1 \end{bmatrix}$. Now, we know

$\operatorname{rref}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, hence, $\operatorname{rref}[A \mid u_1] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. There is a pivot in each column. Therefore the columns of $[A \mid u_1]$ are linearly independent. Also, dim A = 4, and so by the Dimension Theorem, $[A \mid u_1]$ is a basis matrix of $\mathbb{R}^4$.

Similarly, we can construct $[B \mid u_1]$ and get the same result, since $\operatorname{rref}(B) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ as well. Hence, $[B \mid u_1]$ is also a basis matrix of $\mathbb{R}^4$.

c) Use (b) to construct an invertible matrix C such that $A = CB$.

From $A = CB$, we have

$$AB^{-1} = CBB^{-1}$$

$$\Rightarrow AB^{-1} = C .$$

Let us find B^{-1}, where $B = \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$:

$$\left[\begin{array}{cccc|cccc} 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \overset{R_3 \leftrightarrow R_1}{\sim} \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \overset{R_4 - R_1}{\sim}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 \end{array}\right] \overset{R_3 - R_2}{\sim} \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & -1 & 1 \end{array}\right] \overset{R_4 - \frac{1}{2}R_3}{\sim}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1/2 & 1/2 & -1 & 1 \end{array}\right] \overset{-\frac{1}{2}R_3}{\sim} \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1/2 & 1/2 & -1 & 1 \end{array}\right] \begin{array}{l} R_2 - R_3 \\ R_1 - 2R_3 \end{array}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1/2 & 1/2 & -1 & 1 \end{array}\right] \Rightarrow B^{-1} = \begin{bmatrix} 1 & -1 & 1 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ -1/2 & 1/2 & 0 & 0 \\ -1/2 & 1/2 & -1 & 1 \end{bmatrix}$$

Now, $AB^{-1} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 \\ 3 & -1 & 0 & 0 \\ 4 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ -1/2 & 1/2 & 0 & 0 \\ -1/2 & 1/2 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 5/2 & -7/2 & 3 & 0 \\ 7/2 & -5/2 & 3 & 1 \end{bmatrix} = C$

Check that $A = CB$:

$$CB = \begin{bmatrix} 1/2 & -1/2 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 5/2 & -7/2 & 3 & 0 \\ 7/2 & -5/2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & 1 & 3 & 0 \\ 3 & -1 & 0 & 0 \\ 4 & 1 & 1 & 1 \end{bmatrix} = A.$$

Thus we have found C such that $A = CB$.

3) Let $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & -1 \\ 3 & 3 & 0 & 3 \\ 4 & 5 & 1 & 3 \end{bmatrix}$

a) Find the rank r of A.

Row Reduce:

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & -1 \\ 3 & 3 & 0 & 3 \\ 4 & 5 & 1 & 3 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - 4R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & -3 & -3 & 3 \\ 0 & -3 & -3 & 3 \end{bmatrix} \begin{matrix} R_3 + 3R_2 \\ R_4 + 3R_2 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

There are 2 pivot columns, so rank $A = 2$.

b) Factor A into $A = B \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right] C$, where B and C are 4×4 invertible matrices

$$r = 2, \text{ so } \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We now compute the full-rank factorization of A; $A = XY$, by row reducing to rref(A). The matrix X is made up of the original pivot columns of A, while Y is the reduced pivot rows:

$$\begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & -1 \\ 3 & 3 & 0 & 3 \\ 4 & 5 & 1 & 3 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - 4R_1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & -3 & -3 & 3 \\ 0 & -3 & -3 & 3 \end{bmatrix} \begin{matrix} R_3 + 3R_2 \\ R_4 + 3R_2 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_1 - 2R_2$$

$$\sim \begin{bmatrix} \underline{1} & 0 & -1 & 2 \\ 0 & \underline{1} & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow X = \begin{bmatrix} 1 & 2 \\ 2 & 5 \\ 3 & 3 \\ 4 & 5 \end{bmatrix}, \; Y = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

Now, we extend X to $B = [X \mid U] = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 3 & 3 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{bmatrix}$ and

$$C = \left[\begin{array}{c} y \\ \hline v \end{array}\right] = \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 3 & 3 & 1 & 0 \\ 4 & 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 4 & 5 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 2 & 5 & 3 & -1 \\ 3 & 3 & 0 & 3 \\ 4 & 5 & 1 & 3 \end{bmatrix} \checkmark$$

c) Use this factorization to solve $Ax = b$, where $b = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}$.

$$Ax = b, \quad A = B \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right] C, \quad b = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$

$$\Rightarrow \left(B \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right] C\right) x = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} \quad \Rightarrow \quad B \left[\begin{array}{c|c} I_r & \\ \hline & \end{array}\right] (Cx) = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$

$$\text{let } Cx = y \quad \Rightarrow \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$\Rightarrow B \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 4 & 5 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 1 \\ 2 & 5 & 0 & 0 & -1 \\ 3 & 3 & 0 & 0 & 0 \\ 4 & 5 & 0 & 0 & 2 \end{array}\right] \begin{array}{l} R_2 - 2R_1 \\ R_3 - 3R_1 \\ R_4 - 4R_1 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & -3 & 0 & 0 & -3 \\ 0 & -3 & 0 & 0 & -2 \end{array}\right] \begin{array}{l} R_3 + 3R_2 \\ R_4 + 3R_2 \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & -12 \\ 0 & 0 & 0 & 0 & -11 \end{array}\right]$$

As shown, there is no solution to $Ax=b$.

4) Compute the rref(A), where $A = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 3 & -1 & 0 \\ 4 & 1 & 1 \end{bmatrix}$.

a) For each elementary row operation applied in your row reduction toward rref(A), construct the corresponding elementary matrix.

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 3 \\ 3 & -1 & 0 \\ 4 & 1 & 1 \end{bmatrix} \overset{R_2-2R_1}{\sim} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 3 & -1 & 0 \\ 4 & 1 & 1 \end{bmatrix} \overset{R_3-3R_1}{\sim} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & -3 \\ 4 & 1 & 1 \end{bmatrix} \overset{R_4-4R_1}{\sim}$$

$$E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -4 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & -3 \\ 0 & 1 & -3 \end{bmatrix} \overset{R_3+R_2}{\sim} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \\ 0 & 1 & -3 \end{bmatrix} \overset{R_4-R_2}{\sim} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & -4 \end{bmatrix} \overset{R_4-2R_3}{\sim} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \; -\tfrac{1}{2}R_3$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad E_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \quad E_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{bmatrix} \quad E_7 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \overset{R_2-R_3}{\sim} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \overset{R_1-R_3}{\sim} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \text{rref}(A)$$

$$E_8 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad E_9 = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b) Express rref(A) as a product of these elementary matrices and the matrix A (see the file on elem. matrices & row reduction in modules on canvas).

$$E_9 E_8 E_7 E_6 E_5 E_4 E_3 E_2 E_1 A = E_9 E_8 E_7 E_6 E_5 E_4 E_3 E_2 [A_1] =$$
$$E_9 E_8 E_7 E_6 E_5 E_4 E_3 [A_2] = E_9 E_8 E_7 E_6 E_5 E_4 [A_3] =$$
$$E_9 E_8 E_7 E_6 E_5 [A_4] = E_9 E_8 E_7 E_6 [A_5] = E_9 E_8 E_7 [A_6] =$$
$$E_9 E_8 [A_7] = E_9 [A_8] = \text{rref } A$$

5) Let $A \in M_n(\mathbb{F})$. Prove that if $\text{rref}([A \mid I]) = [I \mid B]$, then $B = A^{-1}$.

<u>Proof</u>. Suppose $\text{rref}([A \mid I]) = [I \mid B]$. Then $[I \mid B]$ is formed by row-reducing $[A \mid I]$ to echelon form. For each row reduction, we form a matrix E_K, $K \in \{1, 2, \dots, p\}$ (p is the total number of row reductions needed) such that $E_K [A_{k-1} \mid I_{k-1}]$ is equivalent to performing the specific row reduction on $[A \mid I]$. Thus, we can write

$$\text{rref}([A \mid I]) = E_n E_{n-1} \cdot \dots \cdot E_1 [A \mid I] = [I \mid B].$$

Let $P = E_n \cdot E_{n-1} \cdot \dots \cdot E_1$. Then

$$P[A \mid I] = [I \mid B]$$
$$\Rightarrow [PA \mid PI] = [I \mid B]$$
$$\Rightarrow PA = I, \quad PI = B \Rightarrow P = B$$

$\Rightarrow BA = I$

$\Rightarrow BAA^{-1} = IA^{-1} \Rightarrow B = A^{-1}$. ■

6) Let $A = \left[\begin{array}{c|c} A_1 & 0 \\ \hline 0 & A_2 \end{array}\right]$, where $A \in M_n$ and A_1, A_2 are square matrices. Prove that A is invertible if and only if A_1 and A_2 are invertible.

Proof. ($\Rightarrow$) Suppose A is invertible. Then $\det(A) \neq 0$, where $\det(A)$ is given by

$$\det(A) = \det\left(\left[\begin{array}{c|c} A_1 & \\ \hline & A_2 \end{array}\right]\right) = \det(A_1)\det(A_2).$$

Suppose that either $\det(A_1) = 0$ or $\det(A_2) = 0$. Then

$$\det(A) = 0 \cdot \det(A_2) = 0$$

or

$$\det(A) = \det(A_1) \cdot 0 = 0,$$

which contradicts the fact that A is invertible. Hence, both A_1 and A_2 must have non-zero determinants. Thus, if A is invertible, then A_1 and A_2 are also invertible.

($\Leftarrow$) Suppose now that A_1 and A_2 are invertible. Then $\det(A_1) \neq 0$, $\det(A_2) \neq 0$. Now, we have

$$\det(A) = \det(A_1)\det(A_2)$$

where $\det(A)$ is the product of two non-zero numbers. Thus, $\det(A) \neq 0$, and therefore A is also invertible. $\blacksquare$

HW 7

SABELLA :)

1) Let $A \in M_n$. Show that $T: M_n \to M_n$ defined by $T(A) = AX - XA$ is a linear transformation.

$$T(cA) = (cA)X - X(cA) = cAX - XcA = c(AX) - c(XA)$$

$$= c(AX - XA) = cT(A)$$

$$T(A+B) = (A+B)X - X(A+B) = (AX+BX) - (XA+XB)$$

$$= (AX - XA) + (BX - XB) = T(A) + T(B)$$

2) Let $n \geq 1$, and let $\beta = v_1, v_2, \ldots, v_n$ and $\gamma = w_1, \ldots w_n$ be bases of $\mathbb{F}^n$. Define the $n \times n$ matrices $V = [v_1 \; v_2 \; \cdots \; v_n]$ and $W = [w_1 \; w_2 \; \cdots \; w_n]$. Let $S = {}_\gamma[I]_\beta$ be the β-γ change-of-basis matrix.

(a) Explain why $V = WS$ and deduce that $S = W^{-1}V$.

We have that ${}_\gamma[I]_\beta = \left[[v_1]_\gamma \; [v_2]_\gamma \; \cdots \; [v_n]_\gamma \right]$, where $[v_i]_\gamma$ for $i \in \{1, 2, \ldots, n\}$ expresses v_i as a linear combination of the basis γ:

$$v_i = c_1 w_1 + c_2 w_2 + \cdots + c_n w_n .$$

We solve for each $[v_i]_\gamma = \begin{bmatrix} c_{1i} \\ \vdots \\ c_{ni} \end{bmatrix}$, then form ${}_\gamma[I]_\beta = S$, where

$[v_i]_\gamma = \begin{bmatrix} c_{1i} \\ \vdots \\ c_{ni} \end{bmatrix}$ is placed in the ith column. Hence, S is the matrix of coefficients that, multiplied by the columns of W $(w_1, \dots, w_n)$, give the columns of V $(v_1, \dots, v_n)$. Hence,

$$V = WS.$$

Now, multiplying $V = WS$ by W^{-1} on the left of both sides, we have

$$W^{-1}V = W^{-1}WS$$

$$\Rightarrow W^{-1}V = IS \qquad \Rightarrow S = W^{-1}V.$$

(b) Why is ${}_\beta[I]_\gamma = V^{-1}W$?

From Theorem 2.5.7, we have that ${}_\gamma[I]_\beta^{-1} = {}_\beta[I]_\gamma$. Hence, using part (a), we have

$$V = W\,{}_\gamma[I]_\beta$$

$$\Rightarrow V\,{}_\gamma[I]_\beta^{-1} = W\,{}_\gamma[I]_\beta\,{}_\gamma[I]_\beta^{-1}$$

$$\Rightarrow V\,{}_\gamma[I]_\beta^{-1} = W$$

$$\Rightarrow V^{-1}V\,{}_\gamma[I]_\beta^{-1} = V^{-1}W$$

$$\Rightarrow {}_\gamma[I]_\beta^{-1} = V^{-1}W$$

$$\Rightarrow {}_\beta[I]_\gamma = V^{-1}W.$$

6) Let $n = 3$, let $\beta = \ell_1, \ell_2, \ell_3$ be the Lagrange basis for the nodes $-1, 0, 1$, and let $\gamma = z^2, z, 1$. Use (2.7.10) and (2.7.11) to compute the entries of

(a) ${}_{\beta}[I]_{\gamma}$

By (2.7.11), we have ${}_{\beta}[I]_{\gamma} = \begin{bmatrix} z_1^2 & z_1 & 1 \\ z_2^2 & z_2 & 1 \\ z_3^2 & z_3 & 1 \end{bmatrix}$.

We have that $z_1 = -1$, $z_2 = 0$, $z_3 = 1$. Hence,

$$_\beta[I]_\gamma = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

(b) $_\gamma[I]_\beta$

By (2.7.10), we have $$_\gamma[I]_\beta = \begin{bmatrix} \frac{1}{(z_1-z_2)(z_1-z_3)} & \frac{1}{(z_2-z_1)(z_2-z_3)} & \frac{1}{(z_3-z_1)(z_3-z_2)} \\ \frac{-(z_2+z_3)}{(z_1-z_2)(z_1-z_3)} & \frac{-(z_1+z_3)}{(z_2-z_1)(z_2-z_3)} & \frac{-(z_1+z_2)}{(z_3-z_1)(z_3-z_2)} \\ \frac{z_2 z_3}{(z_1-z_2)(z_1-z_3)} & \frac{z_1 z_3}{(z_2-z_1)(z_2-z_3)} & \frac{z_1 z_2}{(z_3-z_1)(z_3-z_2)} \end{bmatrix}$$

$-1, 0, 1$

Hence, $$_\gamma[I]_\beta = \begin{bmatrix} \frac{1}{(-1)(-2)} & \frac{1}{(1)(-1)} & \frac{1}{(2)(1)} \\ \frac{-(1)}{(-1)(-2)} & 0 & \frac{-(-1)}{(2)(1)} \\ 0 & \frac{(-1)(1)}{(1)(-1)} & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & -1 & 1/2 \\ -1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}.$$

(c) $_\beta[I]_\gamma \, _\gamma[I]_\beta$

$$_\beta[I]_\gamma \, _\gamma[I]_\beta = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & -1 & 1/2 \\ -1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(d) $_\gamma[I]_\beta \, _\beta[I]_\gamma$

$$_\gamma[I]_\beta \, _\beta[I]_\gamma = \begin{bmatrix} 1/2 & -1 & 1/2 \\ -1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

7) Let $a, b, c, d \in \mathbb{C}$. Show that there is a $p \in \mathcal{P}_2$ such that $p(-1) = a$, $p(0) = b$, $p(1) = c$, and $p(2) = d$ if and only if $a - 3b + 3c - d = 0$.

($\Rightarrow$) If there exists some $p \in \mathcal{P}_2$ such that $p(-1) = a$, $p(0) = b$, $p(1) = c$, and $p(2) = d$, then $a - 3b + 3c - d = 0$.

Let $p = k_2 x^2 + k_1 x + k_0 \in \mathcal{P}_2$. Then

$$p(-1) = a \Rightarrow k_2(-1)^2 + k_1(-1) + k_0 = a \Rightarrow k_2 - k_1 = a - b \quad (1)$$

$$p(0) = b \Rightarrow k_2(0)^2 + k_1(0) + k_0 = b \Rightarrow k_0 = b \quad (2)$$

$$p(1) = c \Rightarrow k_2(1)^2 + k_1(1) + k_0 = c \Rightarrow k_2 + k_1 = c - b \quad (3)$$

$$p(2) = d \Rightarrow k_2(2)^2 + k_1(2) + k_0 = d \Rightarrow 4k_2 + 2k_1 = d - b \quad (4)$$

Adding (1) and (3), we have

$$2k_2 = a + c - 2b \Rightarrow k_2 = \frac{1}{2}a + \frac{1}{2}c - b. \quad (5)$$

Substituting (5) in (1), we have

$$\frac{1}{2}a + \frac{1}{2}c - b - k_1 = a - b \Rightarrow k_1 = \frac{1}{2}c - \frac{1}{2}a \quad (6)$$

Substituting (5) and (6) in (4), we have

$$4\left(\frac{1}{2}a + \frac{1}{2}c - b\right) + 2\left(\frac{1}{2}c - \frac{1}{2}a\right) + b = d$$

$$\Rightarrow 2a + 2c - 4b + c - a + b - d = 0 \Rightarrow a - 3b + 3c - d = 0.$$

Hence shown.

$(\Leftarrow)$ If $a-3b+3c-d=0$, then there exists some $p \in \mathcal{P}_2$ such that $p(-1)=a$, $p(0)=b$, $p(1)=c$, and $p(2)=d$.

Let $a-3b+3c-d=0$. We know that any $p \in \mathcal{P}_2$ has the general form $p(x)=k_2x^2+k_1x+k_0$, where $k_0, k_1, k_2 \in \mathbb{C}$ are constants. We have

$$a-3b+3c-d=0 \implies \begin{aligned} d &= a-3b+3c, \\ a &= 3b-3c+d, \\ b &= \tfrac{1}{3}a+c-\tfrac{1}{3}a, \\ c &= -\tfrac{1}{3}a+b+\tfrac{1}{3}d. \end{aligned}$$

So,

$$p(-1)=a \implies k_2(-1)^2+k_1(-1)+k_0 = 3b-3c+d \implies \quad (1)$$

$$p(0)=b \implies k_2(0)^2+k_1(0)+k_0 = \tfrac{1}{3}a+c-\tfrac{1}{3}d \implies k_0 = \tfrac{1}{3}a+c-\tfrac{1}{3}d$$

$$p(1)=c \implies k_2(1)^2+k_1(1)+k_0 = -\tfrac{1}{3}a+b+\tfrac{1}{3}d \quad (2)$$

$$p(2)=d \implies k_2(2)^2+k_1(2)+k_0 = a-3b+3c \quad (3)$$

$$(1) \implies k_2-k_1+\tfrac{1}{3}a+c-\tfrac{1}{3}d = 3b-3c+d$$

$$\implies k_2 = k_1-\tfrac{1}{3}a+3b-4c+\tfrac{4}{3}d \quad (4)$$

$$(2)+(4) \implies \left(k_1-\tfrac{1}{3}a+3b-4c+\tfrac{4}{3}d\right)+k_1+\left(\tfrac{1}{3}a+c-\tfrac{1}{3}d\right) = -\tfrac{1}{3}a+b+\tfrac{1}{3}d$$

$$\implies 2k_1 = -\tfrac{1}{3}a-2b+3c-\tfrac{2}{3}d \implies k_1 = -\tfrac{1}{6}a-b+\tfrac{3}{2}c-\tfrac{1}{3}d \quad (5)$$

$$\implies k_2 = -\tfrac{1}{6}a-b+\tfrac{3}{2}c-\tfrac{1}{3}d-\tfrac{1}{3}a+3b-4c+\tfrac{4}{3}d$$

$$\implies k_2 = -\tfrac{1}{2}a+2b-\tfrac{5}{2}c+d$$

(3) $4(-\frac{1}{2}a + 2b - \frac{5}{2}c + d) + 2(\frac{-1}{6}a - b + \frac{3}{2}c - \frac{1}{3}d) + \frac{1}{3}a + c - \frac{1}{3}d = a - 3b + 3c$

$-\cancel{2a} + \cancel{8b} - \cancel{10c} + \cancel{4d} - \cancel{\frac{1}{3}a} - \cancel{2b} + \cancel{3c} - \cancel{\frac{2}{3}d} + \cancel{\frac{1}{3}a} \cancel{+c} - \cancel{\frac{1}{3}d} = a - 3b + 3c$

$-\cancel{2a} + \cancel{6b} - \cancel{6c} + \cancel{3d} = a - 3b + 3c$
$\quad\quad +2a - 6b + 6c - 3d$

$$3a - 9b + 9c - 3d = 0 \implies a - 3b + 3c - d = 0.$$

Hence, there exists some $p = k_2x^2 + k_1x + k_0 \in \mathcal{P}_2$ such that $p(-1) = a$, $p(0) = b$, $p(1) = c$, and $p(2) = d$.

8) Consider the Vandermonde Matrix

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

associated with the nodes $-1, 0, 1$; see (2.7.11). Use Example 2.7.13 to determine its inverse. Check your answer.

By (2.7.13), we have

$\ell_1(z) = \frac{1}{2}z^2 - \frac{1}{2}z$, $\ell_2(z) = 1 - z^2$, and $\ell_3(z) = \frac{1}{2}z^2 + \frac{1}{2}z$

These are the Lagrange basis polynomials corresponding to the nodes $z_1 = -1$, $z_2 = 0$, and $z_3 = 1$. In matrix form, we have

$$\begin{bmatrix} 1/2 & -1 & 1/2 \\ -1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}.$$

This matrix expresses $\beta = \ell_1, \ell_2, \ell_3$ (the columns) as a linear combination of $\gamma = z^2, z, 1$ (the columns). Hence,

$$\left(\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 1/2 & -1 & 1/2 \\ -1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}.$$

Let us check our answer by ensuring that

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & -1 & 1/2 \\ -1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & -1 & 1/2 \\ -1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

We have

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & -1 & 1/2 \\ -1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1/2+1/2 & -1+1 & 1/2-1/2 \\ 0 & 1 & 0 \\ 1/2-1/2 & -1+1 & 1/2+1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

$$\begin{bmatrix} 1/2 & -1 & 1/2 \\ -1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1/2+1/2 & -1/2+1/2 & 1/2-1+1/2 \\ -1/2+1/2 & 1/2+1/2 & -1/2+1/2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark$$

HW 6

SABELLA SAULINO

1) Let $\alpha = (1, t)$ and $\beta = (2t-1, 2t+1)$.

(a) Show that α and β are bases for $\mathbb{P}_1$.

$\alpha = (1, t)$ is the standard basis for $\mathbb{P}_1$.

$\beta = (2t-1, 2t+1)$ we put the entries of the list β into matrix form and find determinant:

$$\beta = \begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix} \quad \det(\beta) = ad - bc = (2)(-1) - (2)(1)$$
$$= -2 - 2 = -4 \neq 0$$

Hence, the entries in β are linearly independent.

By the Dimension Theorem, since β is LI and has 2 entries, and $\dim \mathbb{P}_1 = 2$, β is a basis for $\mathbb{P}_1$.

(b) Find the transition matrix from α to β.

$${}_\beta[I]_\alpha = \beta^{-1}\alpha \qquad \beta^{-1} = \frac{1}{-4}\begin{bmatrix} 2 & -1 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} -1/2 & 1/4 \\ 1/2 & 1/4 \end{bmatrix}$$

$$\beta^{-1}\alpha = \begin{bmatrix} -1/2 & 1/4 \\ 1/2 & 1/4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & 1/4 \\ 1/2 & 1/4 \end{bmatrix}$$

(c) Find the change-of-coordinates matrix from α to β.

$$[v]_\beta = {}_\beta[I]_\alpha [v]_\alpha \qquad \alpha\, {}_\alpha[I]_\beta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1/2 & 1/4 \\ 1/2 & 1/4 \end{bmatrix} = \begin{bmatrix} -1/2 & 1/4 \\ 1/2 & 1/4 \end{bmatrix}$$

$$\beta = \alpha\, {}_\alpha[I]_\beta$$

(d) Find the transition matrix from β to α

$$[I]_{\alpha}^{\beta} = \alpha^{-1}\beta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix}$$

(e) Find the change-of-coordinates matrix from β to α

$$[v]_\alpha = [I]_{\alpha}^{\beta}[v]_\beta$$

$$\alpha = \beta\, [I]_{\alpha}^{\beta} = \begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 2 & 6 \end{bmatrix}$$

2) $\alpha = (1, t, t^2)$ and $\beta = (1, 1+t, 1+t+t^2)$.

(a) Show that α and β are bases of $\mathbb{P}_2$.

$\alpha = 1, t, t^2$ is the standard basis of $\mathbb{P}_2$. Let B be the basis matrix corresponding to β. Then

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$ We have $\det(B) = 1(1-0) - 1\cancel{(0-0)} + 1\cancel{(0-0)}$
$= 1 \neq 0$.

Since $\det(B) \neq 0$, the columns of B are linearly independent. By the dimension Theorem, since $\dim \mathbb{P}_2 = 3$ and β is made up of 3 linearly independent vectors, β is also a basis for $\mathbb{P}_2$.

(b) Find $[I]_{\alpha}^{\beta}$ and $[I]_{\beta}^{\alpha}$

$$[I]_{\alpha}^{\beta} = \alpha^{-1}\beta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underset{\beta}{[I]}_{\alpha} = \left(\underset{\alpha}{[I]}_{\beta}\right)^{-1} = \beta^{-1} = \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{matrix} R_2 - R_3 \\ R_1 - R_3 \\ \\ \end{matrix} \sim$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \overset{R_1 - R_2}{\sim} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

3) let α and β be bases for $\mathbb{R}^n$ and let A, B be their corresponding basis matrices. Show that $\underset{\alpha}{[I]}_{\beta}$ can be determined by computing rref$([A \mid B])$

$[A \mid B]$ augments A with B, hence, by row reducing, we are finding the solutions to $AX = B$. The matrix X is $n \times n$ (since A and B are $n \times n$) and its columns represent the scalars which take the columns of B to be a linear combination of the rows of A. The columns of A represent the basis α, and the columns of B represent the basis β. Hence, by solving $Ax = B$, we are solving for the solution to $\alpha \underset{\alpha}{[I]}_{\beta} = \beta$, where we write β as a linear combination of α:

$$\underset{\alpha}{[I]}_{\beta} = \left[[b_1]_\alpha \; \dots \; [b_n]_\alpha \right] \quad (b_1, \dots, b_n \text{ are the columns of } B)$$

where $[b_1]_\alpha = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$, where $c_1, \dots, c_n$ are the solutions to $b_1 = c_1 a_1 + \dots + c_n a_n$

$\vdots \qquad\qquad \vdots$

$[b_n]_\alpha = \begin{bmatrix} k_1 \\ \vdots \\ \vdots \end{bmatrix} \quad b_n = k_1 a_1 + \dots + k_n a_n$

$[\,k_n\,]$,

Hence, $_{\alpha}[I]_{\beta} = \begin{bmatrix} c_1 & \cdots & k_1 \\ \vdots & & \vdots \\ c_n & & k \end{bmatrix}$. Thus, the solution to $Ax = B$ is equivalent to $_{\alpha}[I]_{\beta}$. Therefore, computing $\mathrm{rref}([A \mid B])$, we find $([I \mid {}_{\alpha}[I]_{\beta}])$.

4) let $\alpha = \left(\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 1 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 4 \\ 1 & 1 \end{bmatrix} \right)$,

and $\beta = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$.

(a) Show that α and β are bases for $\mathbb{R}^{2\times 2}$.

We will show that α is linearly independent.

$$c_1 \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + c_3 \begin{bmatrix} 2 & 2 \\ 1 & 4 \end{bmatrix} + c_4 \begin{bmatrix} 0 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow -c_1 + c_2 + 2c_3 = 0$

$2c_2 + 2c_3 + 4c_4 = 0 \qquad \Rightarrow 2c_2 + 2c_4 = 0 \Rightarrow c_2 = -c_4$

$c_3 + c_4 = 0 \quad \Rightarrow c_3 = -c_4$

$2c_1 + c_2 + 4c_3 + c_4 = 0 \qquad -c_1 - c_4 - 2c_4 = 0 \quad \Rightarrow c_1 = -3c_4$

$2(-3c_4) - c_4 - 4c_4 + c_4 = 0 \quad \Rightarrow -10c_4 = 0 \quad \Rightarrow c_4 = 0$

$\Rightarrow c_1 = 0, \; c_2 = 0, \; c_3 = 0$

The only solution to $c_1A_1 + c_2A_2 + c_3A_3 + c_4A_4 = 0$ is $c_1 = c_2 = c_3 = c_4 = 0$. Hence, $\alpha = (A_1, A_2, A_3, A_4)$ is linearly independent.

By the Dimension Theorem, since $\dim \mathbb{R}^{2\times 2} = 4$, and α contains 4 linearly independent matrices, α is a basis for $\mathbb{R}^{2\times 2}$.

$$\beta = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$$ is the standard basis for $\mathbb{R}^{2\times 2}$.

(b) Let $M = \begin{bmatrix} 1 & -2 \\ 7 & 8 \end{bmatrix}$. Find ${}_\alpha[M]_\alpha$ and ${}_\beta[M]_\beta$.

Let B_1, B_2, B_3, B_4 be the respective matrices in β.

Let A_1, A_2, A_3, A_4 be the respective matrices in α.

$${}_\beta[M]_\beta = \big[[MB_1]_\beta \; [MB_2]_\beta \; [MB_3]_\beta \; [MB_4]_\beta \big]$$

$$MB_1 = \begin{bmatrix} 1 & -2 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 7 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 7 \\ 0 \end{bmatrix}$$

$$MB_2 = \begin{bmatrix} 1 & -2 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 7 \end{bmatrix}$$

$$MB_3 = \begin{bmatrix} 1 & -2 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 8 & 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 8 \\ 0 \end{bmatrix}$$

$$MB_4 = \begin{bmatrix} 1 & -2 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \\ 8 \end{bmatrix}$$

$${}_\beta[M]_\beta = \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ 7 & 0 & 8 & 0 \\ 0 & 7 & 0 & 8 \end{bmatrix}$$

$$[M]_{\alpha\alpha} = \left[[MA_1]_\alpha \ [MA_2]_\alpha \ [MA_3]_\alpha \ [MA_4]_\alpha \right]$$

$$MA_1 = \begin{bmatrix} 1 & -2 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -1 & -4 \\ -7 & 16 \end{bmatrix}$$

$$MA_2 = \begin{bmatrix} 1 & -2 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 7 & 22 \end{bmatrix}$$

$$MA_3 = \begin{bmatrix} 1 & -2 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 0 & -6 \\ 22 & 46 \end{bmatrix}$$

$$MA_4 = \begin{bmatrix} 1 & -2 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} 0 & 4 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ 8 & 36 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -4 \\ -7 & 16 \end{bmatrix} = a\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} + b\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + c\begin{bmatrix} 2 & 2 \\ 1 & 4 \end{bmatrix} + d\begin{bmatrix} 0 & 4 \\ 1 & 1 \end{bmatrix}$$

$\Rightarrow$ $-a + b + 2c = -1$

$2b + 2c + 4d = -4$ $\qquad \Rightarrow 2b + 2(-7-d) + 4d = -4$

$c + d = -7$ $\Rightarrow c = -7 - d$ $\qquad \Rightarrow 2b + 2d = 10$

$\qquad \Rightarrow b = 5 - d$

$2a + b + 4c + d = 16$

$-a + 5 - d + 2(-7-d) = -1$ $\Rightarrow -a - 3d = 8$ $\Rightarrow a = -8 - 3d$

$2(-8-3d) + 5 - d + 4(-7-d) + d = 16$ $\Rightarrow -10d - 39 = 16$

$\Rightarrow -10d = 55$ $\Rightarrow d = -55/10$ $\Rightarrow d = -11/2$

$\Rightarrow a = -8 - 3(-11/2)$ $\Rightarrow a = 17/2$

$b = 5 - (-11/2)$ $\Rightarrow b = 21/2$

$c = -7 - (-11/2)$ $\Rightarrow c = -3/2$

$$[MA_1]_\alpha = \begin{bmatrix} 17/2 \\ 21/2 \\ -3/2 \\ -11/2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 7 & 22 \end{bmatrix} \Rightarrow \begin{aligned} -a+b+2c &= 1 \\ 2b+2c+4d &= 0 \\ c+d &= 7 \quad \Rightarrow c = 7-d \\ 2a+b+4c+d &= 22 \end{aligned}$$

$\Rightarrow 2b+2(7-d)+4d = 0 \Rightarrow 2b+2d = -14 \Rightarrow b = -7-d$

$-a-7-d+2(7-d) = 1 \Rightarrow -a-3d = -6 \Rightarrow a = 6-3d$

$2(6-3d)-7-d+4(7-d)+d = 22 \Rightarrow -10d = -18 \Rightarrow d = 9/5$

$a = 6-3(9/5) \Rightarrow a = 3/5$

$b = -7 - 9/5 \Rightarrow b = -44/5$

$c = 7 - 9/5 \Rightarrow c = 26/5$

$$[MA_2]_\alpha = \begin{bmatrix} 3/5 \\ -44/5 \\ 26/5 \\ 9/5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -6 \\ 22 & 46 \end{bmatrix} \Rightarrow \begin{aligned} -a+b+2c &= 0 \\ 2b+2c+4d &= -6 \\ c+d &= 22 \quad \Rightarrow c = 22-d \\ 2a+b+4c+d &= 46 \end{aligned}$$

$2b+2(22-d)+4d = -6 \Rightarrow 2b+2d = -50 \Rightarrow b = -25-d$

$-a-25-d+2(22-d) = 0 \Rightarrow -a-3d+19 = 0 \Rightarrow a = 19-3d$

$2(19-3d)-25-d+4(22-d)+d = 46 \Rightarrow -10d = -55 \Rightarrow d = 11/2$

$a = 19-3(11/2) \Rightarrow a = 5/2$

$b = -25 - 11/2 \Rightarrow b = -61/2$

$c = 22 - 11/2 \Rightarrow c = 33/2$

$$[MA_3]_\alpha = \begin{bmatrix} 5/2 \\ -61/2 \\ 33/2 \\ 11/2 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 \\ 8 & 36 \end{bmatrix} \Rightarrow \begin{aligned} -a+b+2c &= -2 \\ 2b+2c+4d &= 2 \\ c+d &= 8 \quad \Rightarrow c = 8-d \end{aligned}$$

$2a + b + 4c + d = 36$

$2b + 2(8-d) + 4d = 2 \Rightarrow 2b + 2d = -14 \Rightarrow b = -7 - d$

$-a - 7 - d + 2(8-d) = -2 \Rightarrow -a - 3d = -11 \Rightarrow a = 11 - 3d$

$2(11-3d) - 7 - d + 4(8-d) + d = 36 \Rightarrow -10d = -11 \Rightarrow d = 11/10$

$a = 11 - 3(11/10) \Rightarrow a = 77/10$

$b = -7 - 11/10 \Rightarrow b = -81/10$

$c = 8 - 11/10 \Rightarrow c = 69/10$

$$[MA_4]_\alpha = \begin{bmatrix} 77/10 \\ -81/10 \\ 69/10 \\ 11/10 \end{bmatrix}$$

$${}_\alpha[M]_\alpha = \begin{bmatrix} 17/2 & 3/5 & 5/2 & 77/10 \\ 21/2 & -44/5 & -61/2 & -81/10 \\ -3/2 & 26/5 & 33/2 & 69/10 \\ -11/2 & 9/5 & 11/2 & 11/10 \end{bmatrix}$$

(c) Find the change of coordinates matrix from α to β

$${}_\beta[I]_\alpha = \Big[[A_1]_\beta \;\; [A_2]_\beta \;\; [A_3]_\beta \;\; [A_4]_\beta \Big]$$

$$A_1 = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 2 & 2 \\ 1 & 4 \end{bmatrix} \quad A_4 = \begin{bmatrix} 0 & 4 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[A_1]_\beta = \begin{bmatrix} -1 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

Each $[A_i]_\beta$ is just A_i in column form (for $i \in \{1,2,3,4\}$).

$$[A_2]_\beta = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \qquad [A_3]_\beta = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 4 \end{bmatrix} \qquad [A_4]_\beta = \begin{bmatrix} 0 \\ 4 \\ 1 \\ 1 \end{bmatrix}$$

$$_\beta[I]_\alpha = \begin{bmatrix} -1 & 1 & 2 & 0 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 4 & 1 \end{bmatrix}$$

a) use the change of coordinates matrix from α to β to relate $_\alpha[M]_\alpha$ and $_\beta[M]_\beta$

We have that $_\alpha[M]_\alpha = {_\alpha[I]_\beta}\, {_\beta[M]_\beta}\, {_\beta[I]_\alpha}$

$$_\alpha[I]_\beta = \left(_\beta[I]_\alpha\right)^{-1} = \begin{bmatrix} -1 & 1 & 2 & 0 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 4 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} -2/5 & 1/20 & -1/2 & 3/10 \\ 1/5 & 7/20 & -3/2 & 1/10 \\ 1/5 & -3/20 & 1/2 & 1/10 \\ -1/5 & 3/20 & 1/2 & -1/10 \end{bmatrix}$$

Check that $_\alpha[I]_\beta\, {_\beta[M]_\beta}\, {_\beta[I]_\alpha} = {_\alpha[M]_\alpha}$:

$$\begin{bmatrix} -2/5 & 1/20 & -1/2 & 3/10 \\ 1/5 & 7/20 & -3/2 & 1/10 \\ 1/5 & -3/20 & 1/2 & 1/10 \\ -1/5 & 3/20 & 1/2 & -1/10 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ 7 & 0 & 8 & 0 \\ 0 & 7 & 0 & 8 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 & 0 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 4 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} -39/10 & 43/20 & -16/5 & 23/10 \\ -103/10 & 21/20 & -62/5 & 1/10 \\ 37/10 & 11/20 & 18/5 & 11/10 \\ 33/10 & -11/20 & 22/5 & -11/10 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 & 0 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 4 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 17/2 & 3/5 & 5/2 & 77/10 \\ 21/2 & -44/5 & -61/2 & -81/10 \\ -3/2 & 26/5 & 33/2 & 69/10 \\ -11/2 & 9/5 & 11/2 & 11/10 \end{bmatrix} = {}_{\alpha}[M]_{\alpha}$$

Hence, ${}_{\alpha}[M]_{\alpha} = {}_{\alpha}[I]_{\beta}\,{}_{\beta}[M]_{\beta}\,{}_{\beta}[I]_{\alpha}$. Rearranging this formula,

we have ${}_{\beta}[M]_{\beta} = {}_{\beta}[I]_{\alpha}\,{}_{\alpha}[M]_{\alpha}\,{}_{\alpha}[I]_{\beta}$. We check:

$$\begin{bmatrix} -1 & 1 & 2 & 0 \\ 0 & 2 & 2 & 4 \\ 0 & 0 & 1 & 1 \\ 2 & 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} 17/2 & 3/5 & 5/2 & 77/10 \\ 21/2 & -44/5 & -61/2 & -81/10 \\ -3/2 & 26/5 & 33/2 & 69/10 \\ -11/2 & 9/5 & 11/2 & 11/10 \end{bmatrix} \begin{bmatrix} -2/5 & 1/20 & -1/2 & 3/10 \\ 1/5 & 7/20 & -3/2 & 1/10 \\ 1/5 & -3/20 & 1/2 & 1/10 \\ -1/5 & 3/20 & 1/2 & -1/10 \end{bmatrix} =$$

$$\begin{bmatrix} -1 & 1 & 0 & -2 \\ -4 & 0 & -6 & 2 \\ -7 & 7 & 22 & 8 \\ 16 & 22 & 46 & 36 \end{bmatrix} \begin{bmatrix} -2/5 & 1/20 & -1/2 & 3/10 \\ 1/5 & 7/20 & -3/2 & 1/10 \\ 1/5 & -3/20 & 1/2 & 1/10 \\ -1/5 & 3/20 & 1/2 & -1/10 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & -2 \\ 7 & 0 & 8 & 0 \\ 0 & 7 & 0 & 8 \end{bmatrix} = {}_{\beta}[M]_{\beta}$$

6) Let $\mathcal{E}_1 = (e_1, e_2, e_3)$ be the standard basis for $\mathbb{R}^3$, and let $\mathcal{E}_2 = (e_1, e_2)$ be the standard basis for $\mathbb{R}^2$.

Let $\alpha = (u_1, u_2, u_3)$, where $u_1 = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}$, $u_2 = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$, $u_3 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$;

and let $\beta = (v_1, v_2)$, where $v_1 = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -8 \\ 10 \end{bmatrix}$.

For each of the linear transformations L in (a) - (d) of Problem 13, find ${}_{\mathcal{E}_2}[L]_{\mathcal{E}_1}$, ${}_{\beta}[L]_{\alpha}$, and ${}_{\mathcal{E}_2}[L]_{\alpha}$.

(a) $L : \mathbb{R}^3 \to \mathbb{R}^2$, $L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -x_1 + x_2 \\ 0 \end{bmatrix}$

$${}_{\mathcal{E}_2}[L]_{\mathcal{E}_1} = \left[[L(e_1)]_{\mathcal{E}_2} \;\; [L(e_2)]_{\mathcal{E}_2} \;\; [L(e_3)]_{\mathcal{E}_2} \right]$$

$$L(e_1) = L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \left[\begin{bmatrix} -1 \\ 0 \end{bmatrix}\right]_{\mathcal{E}_2} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$L(e_2) = L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \left[\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]_{\mathcal{E}_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$L(e_3) = L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \left[\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right]_{\mathcal{E}_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$${}_{\mathcal{E}_2}[L]_{\mathcal{E}_1} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$${}_{\beta}[L]_{\alpha} = \left[[L(u_1)]_{\beta} \;\; [L(u_2)]_{\beta} \;\; [L(u_3)]_{\beta} \right]$$

$$L(v_1) = L\left(\begin{bmatrix}1\\2\\-3\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix} \qquad \begin{bmatrix}1\\0\end{bmatrix} = a\begin{bmatrix}7\\8\end{bmatrix} + b\begin{bmatrix}-8\\10\end{bmatrix}$$

$$\Rightarrow \begin{matrix} 7a - 8b = 1 \\ 8a + 10b = 0 \end{matrix} \Rightarrow 8a = -10b \Rightarrow a = -5/4\, b$$

$$7(-5/4\, b) - 8b = 1 \Rightarrow \frac{-67}{4} b = 1 \Rightarrow b = -4/67$$

$$a = \frac{-5}{4}\left(\frac{-4}{67}\right) = 5/67 \qquad [L(v_1)]_\beta = \begin{bmatrix}5/67\\-4/67\end{bmatrix}$$

$$L(v_2) = L\left(\begin{bmatrix}0\\1\\4\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix} \qquad \left[\begin{bmatrix}1\\0\end{bmatrix}\right]_\beta = \begin{bmatrix}5/67\\-4/67\end{bmatrix}$$

$$L(v_3) = L\left(\begin{bmatrix}1\\3\\0\end{bmatrix}\right) = \begin{bmatrix}2\\0\end{bmatrix} \Rightarrow \begin{matrix} 7a - 8b = 2 \\ 8a + 10b = 0 \end{matrix} \Rightarrow a = -5/4\, b$$

$$7(-5/4\, b) - 8b = 2 \Rightarrow \frac{-67}{4} b = 2 \Rightarrow b = -8/67$$

$$a = -5/4\,(-8/67) \Rightarrow a = 10/67 \qquad [L(v_3)]_\beta = \begin{bmatrix}10/67\\-8/67\end{bmatrix}$$

$$_\beta[L]_\alpha = \frac{1}{67}\begin{bmatrix}5 & 5 & 10\\-4 & -4 & -8\end{bmatrix}$$

$$_{\varepsilon_2}[L]_\alpha = \left[[L(v_1)]_{\varepsilon_2} \; [L(v_2)]_{\varepsilon_2} \; [L(v_3)]_{\varepsilon_3} \right]$$

$$L(v_1) = L\left(\begin{bmatrix}1\\2\\-3\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix} \qquad [L(v_1)]_{\varepsilon_2} = \begin{bmatrix}1\\0\end{bmatrix}$$

$$L(v_2) = L\left(\begin{bmatrix}0\\1\\4\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix} \qquad [L(v_2)]_{\varepsilon_2} = \begin{bmatrix}1\\0\end{bmatrix}$$

$$L(v_3) = L\left(\begin{bmatrix}1\\3\\0\end{bmatrix}\right) = \begin{bmatrix}2\\0\end{bmatrix} \qquad [L(v_3)]_{\varepsilon_2} = \begin{bmatrix}2\\0\end{bmatrix}$$

$$_{\varepsilon_2}[L]_{\alpha} = \begin{bmatrix}1 & 1 & 2\\0 & 0 & 0\end{bmatrix}$$

(b) $$L\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}-x_1\\x_2\end{bmatrix}$$

$$_{\varepsilon_2}[L]_{\varepsilon_1} = \left[[L(e_1)]_{\varepsilon_2} \quad [L(e_2)]_{\varepsilon_2} \quad [L(e_3)]_{\varepsilon_2}\right]$$

$$L(e_1) = L\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}-1\\0\end{bmatrix} \qquad \left[\begin{bmatrix}-1\\0\end{bmatrix}\right]_{\varepsilon_2} = \begin{bmatrix}-1\\0\end{bmatrix}$$

$$L(e_2) = L\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix} \qquad \left[\begin{bmatrix}0\\1\end{bmatrix}\right]_{\varepsilon_2} = \begin{bmatrix}0\\1\end{bmatrix}$$

$$L(e_3) = L\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\0\end{bmatrix} \qquad \left[\begin{bmatrix}0\\0\end{bmatrix}\right]_{\varepsilon_2} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$_{\varepsilon_2}[L]_{\varepsilon_1} = \begin{bmatrix}-1 & 0 & 0\\0 & 1 & 0\end{bmatrix}$$

$$_{\beta}[L]_{\alpha} = \left[[L(v_1)]_{\beta} \quad [L(v_2)]_{\beta} \quad [L(v_3)]_{\beta}\right]$$

$$L(v_1) = L\left(\begin{bmatrix}1\\2\\-3\end{bmatrix}\right) = \begin{bmatrix}-1\\2\end{bmatrix} \qquad \begin{bmatrix}-1\\2\end{bmatrix} = a\begin{bmatrix}7\\8\end{bmatrix} + b\begin{bmatrix}-8\\10\end{bmatrix}$$

$\Rightarrow \begin{array}{l} 7a - 8b = -1 \\ 8a + 10b = 2 \end{array}$ $\Rightarrow 7a = 8b - 1 \Rightarrow a = 8/7\, b - 1/7$

$\Rightarrow 8(8/7\, b - 1/7) + 10b = 2 \Rightarrow \frac{64}{7} b + 10b = \frac{22}{7}$

$\Rightarrow \frac{134}{7} b = \frac{22}{7} \Rightarrow b = 22/134 \Rightarrow b = 11/67$

$a = 8/7(11/67) - 1/7 = 3/67$ $\quad [L(v_1)]_\beta = \begin{bmatrix} 3/67 \\ 11/67 \end{bmatrix}$

$L(v_2) = L\left(\begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{array}{l} 7a - 8b = 0 \\ 8a + 10b = 1 \end{array} \Rightarrow a = 8/7\, b$

$\Rightarrow 8(8/7\, b) + 10b = 1 \Rightarrow 134/7\, b = 1 \Rightarrow b = 7/134$

$a = 8/7(7/134) \Rightarrow a = 8/134$ $\quad [L(v_2)]_\beta = \begin{bmatrix} 8/134 \\ 7/134 \end{bmatrix}$

$L(v_3) = L\left(\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \Rightarrow \begin{array}{l} 7a - 8b = -1 \\ 8a + 10b = 3 \end{array} \Rightarrow 8a = 3 - 10b$

$\Rightarrow a = 3/8 - \frac{5}{4} b$

$7(3/8 - 5/4\, b) - 8b = -1 \Rightarrow -\frac{67}{4} b = \frac{-29}{8} \Rightarrow b = 29/134$

$a = 3/8 - 5/4(29/134) \Rightarrow a = 7/67$ $\quad [L(v_3)]_\beta = \begin{bmatrix} 7/67 \\ 29/134 \end{bmatrix}$

$_\beta[L]_\alpha = \begin{bmatrix} 3/67 & 8/134 & 7/67 \\ 11/67 & 7/134 & 29/134 \end{bmatrix}$

$_{\varepsilon_2}[L]_\alpha = \left[[L(v_1)]_{\varepsilon_2} \; [L(v_2)]_{\varepsilon_2} \; [L(v_3)]_{\varepsilon_3} \right]$

$L(v_1) = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ $\quad [L(v_1)]_{\varepsilon_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$L(v_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad [L(v_2)]_{\mathcal{E}_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$L(v_3) = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \quad [L(v_3)]_{\mathcal{E}_2} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} \quad {}_{\mathcal{E}_2}[L]_{\alpha} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix}$$

(c) $$L = \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} -x_1 - x_2 \\ 3x_3 - x_2 \end{bmatrix}$$

$${}_{\mathcal{E}_2}[L]_{\mathcal{E}_1} = \left[[L(e_1)]_{\mathcal{E}_2} \quad [L(e_2)]_{\mathcal{E}_2} \quad [L(e_3)]_{\mathcal{E}_2} \right]$$

$$L(e_1) = L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -1 - 0 \\ 3(0) - 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$L(e_2) = L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} -0 - 1 \\ 3(0) - 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$L(e_3) = L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -0 - 0 \\ 3 - 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$${}_{\mathcal{E}_2}[L]_{\mathcal{E}_1} = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & 3 \end{bmatrix}$$

$${}_{\beta}[L]_{\alpha} = \left[[L(v_1)]_{\beta} \quad [L(v_2)]_{\beta} \quad [L(v_3)]_{\beta} \right]$$

$$L(v_1) = L\left(\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \right) = \begin{bmatrix} -1 - 2 \\ 3(-3) - 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -11 \end{bmatrix} = a \begin{bmatrix} 7 \\ 8 \end{bmatrix} + b \begin{bmatrix} -8 \\ 10 \end{bmatrix}$$

$$\Rightarrow \begin{matrix} 7a - 8b = -3 \\ 8a + 10b = -11 \end{matrix} \quad \Rightarrow 7a = 8b - 3 \quad \Rightarrow a = 8/7\, b - 3/7$$

$$8(8/7\, b - 3/7) + 10b = -11 \Rightarrow 134/7\, b = -53/7 \Rightarrow b = -53/134$$

$a = 8/7(-53/134) - 3/7 \Rightarrow a = -59/67 \qquad [L(v_1)]_\beta = \begin{bmatrix} -59/67 \\ -53/134 \end{bmatrix}$

$L(v_2) = L\left(\begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} -0-1 \\ 3(4)-1 \end{bmatrix} = \begin{bmatrix} -1 \\ 11 \end{bmatrix} \Rightarrow \begin{matrix} 7a - 8b = -1 \\ 8a + 10b = 11 \end{matrix} \Rightarrow 7a = 8b - 1 \Rightarrow a = \frac{8}{7}b - \frac{1}{7}$

$8(8/7\, b - 1/7) + 10b = 11 \Rightarrow 134/7\, b = 85/7 \Rightarrow b = 85/134$

$a = 8/7(85/134) - 1/7 \Rightarrow a = 39/67 \qquad [L(v_2)]_\beta = \begin{bmatrix} 39/67 \\ 85/134 \end{bmatrix}$

$L(v_3) = L\left(\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1-3 \\ 3(0)-3 \end{bmatrix} = \begin{bmatrix} -4 \\ -3 \end{bmatrix} \Rightarrow \begin{matrix} 7a - 8b = -4 \\ 8a + 10b = -3 \end{matrix} \Rightarrow 7a = 8b - 4 \Rightarrow a = \frac{8}{7}b - 4/7$

$8(8/7\, b - 4/7) + 10b = -3 \Rightarrow 134/7\, b = 11/7 \Rightarrow b = 11/134$

$a = \frac{8}{7}(11/134) - 4/7 \Rightarrow a = -32/67 \qquad [L(v_3)]_\beta = \begin{bmatrix} -32/67 \\ 11/134 \end{bmatrix}$

$${}_\beta[L]_\alpha = \begin{bmatrix} -59/67 & 39/67 & -32/67 \\ -53/134 & 85/134 & 11/134 \end{bmatrix}$$

$${}_{\varepsilon_2}[L]_\alpha = \Big[[L(v_1)]_{\varepsilon_2} \quad [L(v_2)]_{\varepsilon_2} \quad [L(v_3)]_{\varepsilon_3} \Big]$$

$L(v_1) = \begin{bmatrix} -3 \\ -11 \end{bmatrix} \qquad [L(v_1)]_{\varepsilon_2} = \begin{bmatrix} -3 \\ -11 \end{bmatrix}$

$L(v_2) = \begin{bmatrix} -1 \\ 11 \end{bmatrix} \qquad [L(v_2)]_{\varepsilon_2} = \begin{bmatrix} -1 \\ 11 \end{bmatrix}$

$L(v_3) = \begin{bmatrix} -4 \\ -3 \end{bmatrix} \qquad [L(v_3)]_{\varepsilon_2} = \begin{bmatrix} -4 \\ -3 \end{bmatrix} \qquad {}_{\varepsilon_2}[L]_\alpha = \begin{bmatrix} -3 & -1 & -4 \\ -11 & 11 & -3 \end{bmatrix}$

d) $L\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} -2x_1 - x_2 + x_3 \\ -3x_1 - 5x_2 + 7x_3 \end{bmatrix}$

$_{\mathcal{E}_2}[L]_{\mathcal{E}_1} = \left[[L(e_1)]_{\mathcal{E}_2} \quad [L(e_2)]_{\mathcal{E}_2} \quad [L(e_3)]_{\mathcal{E}_2} \right]$

$L(e_1) = L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2 - 0 + 0 \\ -3 - 0 + 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -3 \end{bmatrix} \qquad \left[\begin{bmatrix} -2 \\ -3 \end{bmatrix}\right]_{\mathcal{E}_2} = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$

$L(e_2) = L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 - 1 + 0 \\ 0 - 5 + 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -5 \end{bmatrix} \qquad \left[\begin{bmatrix} -1 \\ -5 \end{bmatrix}\right]_{\mathcal{E}_2} = \begin{bmatrix} -1 \\ -5 \end{bmatrix}$

$L(e_3) = L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 - 0 + 1 \\ 0 - 0 + 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \end{bmatrix} \qquad \left[\begin{bmatrix} 1 \\ 7 \end{bmatrix}\right]_{\mathcal{E}_2} = \begin{bmatrix} 1 \\ 7 \end{bmatrix}$

$$_{\mathcal{E}_2}[L]_{\mathcal{E}_1} = \begin{bmatrix} -2 & -1 & 1 \\ -3 & -5 & 7 \end{bmatrix}$$

$_{\beta}[L]_{\alpha} = \left[[L(v_1)]_{\beta} \quad [L(v_2)]_{\beta} \quad [L(v_3)]_{\beta} \right]$

$L(v_1) = L\left(\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}\right) = \begin{bmatrix} -2 - 2 - 3 \\ -3 - 10 - 21 \end{bmatrix} = \begin{bmatrix} -7 \\ -34 \end{bmatrix} = a\begin{bmatrix} 7 \\ 8 \end{bmatrix} + b\begin{bmatrix} -8 \\ 10 \end{bmatrix}$

$\Rightarrow 7a - 8b = -7 \quad \Rightarrow 7a = 8b - 7 \quad \Rightarrow a = \frac{8}{7}b - 1$

$8a + 10b = -34$

$\Rightarrow 8(8/7\, b - 1) + 10b = -34 \quad \Rightarrow 134/7\, b = -26 \quad \Rightarrow b = -91/67$

$a = 8/7(-91/67) - 1 \quad \Rightarrow a = -171/67 \qquad [L(v_1)]_{\beta} = \begin{bmatrix} -171/67 \\ -91/67 \end{bmatrix}$

$$L(v_2) = L\left(\begin{bmatrix}0\\1\\4\end{bmatrix}\right) = \begin{bmatrix}0-1+4\\0-5+28\end{bmatrix} = \begin{bmatrix}3\\23\end{bmatrix} \Rightarrow \begin{matrix}7a-8b=3\\8a+10b=23\end{matrix}$$

$$\Rightarrow 7a = 8b+3 \Rightarrow a = \frac{8}{7}b + 3/7$$

$$8(8/7\,b + 3/7) + 10b = 23 \quad \Rightarrow 134/7\,b = 137/7 \quad \Rightarrow b = 137/134$$

$$a = 8/7(137/134) + 3/7 \quad \Rightarrow a = 107/67 \qquad [L(v_2)]_\beta = \begin{bmatrix}107/67\\137/134\end{bmatrix}$$

$$L(v_3) = L\left(\begin{bmatrix}1\\3\\0\end{bmatrix}\right) = \begin{bmatrix}-2-3+0\\-3-15+0\end{bmatrix} = \begin{bmatrix}-5\\-18\end{bmatrix} \Rightarrow \begin{matrix}7a-8b=-5\\8a+10b=-18\end{matrix}$$

$$\Rightarrow 7a = 8b-5 \Rightarrow a = 8/7\,b - 5/7$$

$$8(8/7\,b - 5/7) + 10b = -18 \quad \Rightarrow 134/7\,b = -86/7 \quad \Rightarrow b = -86/134$$

$$a = 8/7(-86/134) - 5/7 \quad \Rightarrow a = -97/67 \qquad [L(v_3)]_\beta = \begin{bmatrix}-97/67\\-86/134\end{bmatrix}$$

$$_\beta[L]_\alpha = \begin{bmatrix}-171/67 & 107/67 & -97/67\\-91/67 & 137/134 & -86/134\end{bmatrix}$$

$$_{\varepsilon_2}[L]_\alpha = \left[\,[L(v_1)]_{\varepsilon_2} \quad [L(v_2)]_{\varepsilon_2} \quad [L(v_3)]_{\varepsilon_3}\right]$$

$$L(v_1) = \begin{bmatrix}-7\\-34\end{bmatrix} \qquad [L(v_1)]_{\varepsilon_2} = \begin{bmatrix}-7\\-34\end{bmatrix}$$

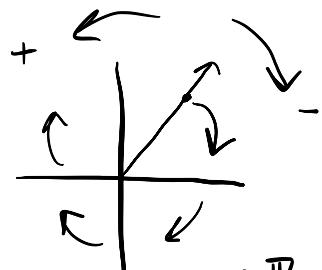
$$L(v_2) = \begin{bmatrix}3\\23\end{bmatrix} \qquad [L(v_2)]_{\varepsilon_2} = \begin{bmatrix}3\\23\end{bmatrix}$$

$$L(v_3) = \begin{bmatrix}-5\\-18\end{bmatrix} \qquad [L(v_3)]_{\varepsilon_2} = \begin{bmatrix}-5\\-18\end{bmatrix} \qquad {_{\varepsilon_2}[L]_\alpha} = \begin{bmatrix}-7 & 3 & -5\\-34 & 23 & -18\end{bmatrix}$$

9) Find the standard matrix representation of each of the following linear operators.

(a) L rotates v by $\pi/4$ in the clockwise direction.

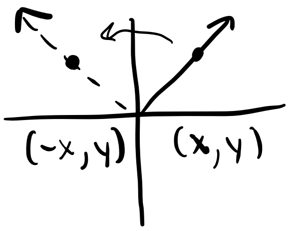
Standard matrix rotation representation : $R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$



Rotating by $\pi/4$ in the clockwise direction means rotating by $-\pi/4$

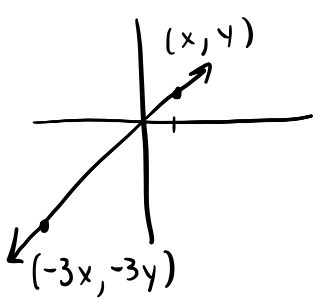
$$R(-\pi/4) = \begin{bmatrix} \cos(-\pi/4) & -\sin(-\pi/4) \\ \sin(-\pi/4) & \cos(-\pi/4) \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = L$$

(b) L reflects v with respect to the y-axis.



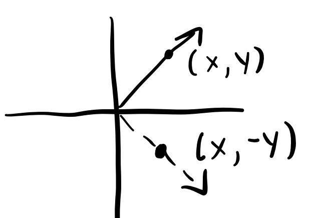
Reflection Representation : $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = L$

(c) L triples the length of v in the opposite direction of v.



$$\begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -3v_1 \\ -3v_2 \end{bmatrix} \qquad L = \begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$$

(d) L reflects v with respect to the line $y = x$.



$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

10) Let $L : \mathbb{P}_3 \to \mathbb{P}_4$ be the transformation defined by

$L(p(t)) = \int_0^t p(x)\,dx$.

(a) Show that L is linear.

$L(k\,p(t)) = \int_0^t k\,p(x)\,dx = k\int_0^t p(x)\,dx = k\,L(p(t))$

$L(p(t)+q(t)) = \int_0^t (p(x)+q(x))\,dx = \int_0^t p(x)\,dx + q(x)\,dx =$

$\int_0^t p(x)\,dx + \int_0^t q(x)\,dx = L(p(t)) + L(q(t))$

(b) Let $\alpha = (1, t, t^2, t^3)$ and $\beta = (1, t, t^2, t^3, t^4)$. Find ${}_\beta[L]_\alpha$

${}_\beta[L]_\alpha = \left[[L(1)]_\beta \ [L(t)]_\beta \ [L(t^2)]_\beta \ [L(t^3)]_\beta \right]$

$L(1) = \int_0^t 1\,dx = x\Big]_0^t = t = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$L(t) = \int_0^t x\,dx = \frac{1}{2}x^2\Big]_0^t = \frac{1}{2}t^2 = \begin{bmatrix} 0 \\ 0 \\ 1/2 \\ 0 \\ 0 \end{bmatrix}$

${}_\beta[L]_\alpha = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/4 \end{bmatrix}$

$L(t^2) = \int_0^t x^2\,dx = \frac{1}{3}x^3\Big]_0^t = \frac{1}{3}t^3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/3 \\ 0 \end{bmatrix}$

$$L(t^3) = \int_0^t x^3\,dx = \frac{1}{4}x^4\Big]_0^t = \frac{1}{4}t^4 = \begin{bmatrix}0\\0\\0\\0\\1/4\end{bmatrix}$$

(c) Let $\gamma = (1, 1+t, 1+t+t^2, 1+t+t^2+t^3)$ and $\delta = (1, -1+t, 1-t+t^2, -1+t-t^2+t^3, t^4)$. Find ${}_\delta[L]_\gamma$

$${}_\delta[L]_\gamma = \left[[L(1)]_\delta \; [L(1+t)]_\delta \; [L(1+t+t^2)]_\delta \; [1+t+t^2+t^3]_\delta \right].$$

$$L(1) = \int_0^t 1\,dx = x\Big]_0^t = t = \begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix} = a\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix} + b\begin{bmatrix}-1\\1\\0\\0\\0\end{bmatrix} + c\begin{bmatrix}1\\-1\\1\\0\\0\end{bmatrix} + d\begin{bmatrix}-1\\1\\-1\\1\\0\end{bmatrix} + e\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}$$

$\Rightarrow a-b+c-d=0$

$b-c+d=1 \quad \Rightarrow b-0+0=1 \Rightarrow b=1$

$c-d=0 \Rightarrow c=0 \quad \Rightarrow a-1+0-0=0 \Rightarrow a=1$

$d=0$

$e=0$

$$[L(1)]_\delta = \begin{bmatrix}1\\1\\0\\0\\0\end{bmatrix}$$

$$L(1+t) = \int_0^t 1+x\,dx = x+\frac{1}{2}x^2\Big]_0^t = t+\frac{1}{2}t^2 = \begin{bmatrix}0\\1\\1/2\\0\\0\end{bmatrix}$$

$\Rightarrow a-b+c-d=0$

$b-c+d=1 \quad \Rightarrow b-1/2=1 \Rightarrow b=3/2$

$c-d=1/2 \Rightarrow c=1/2 \quad \Rightarrow a-3/2+1/2=0$

$d=0 \quad \Rightarrow a=1$

$e=0$

$$[L(1+t)]_\delta = \begin{bmatrix}1\\3/2\\1/2\\0\\0\end{bmatrix}$$

$$L(1+t+t^2) = \int_0^t 1+x+x^2\,dx = x+\tfrac{1}{2}x^2+\tfrac{1}{3}x^3\Big]_0^t = t+\tfrac{1}{2}t^2+\tfrac{1}{3}t^3 = \begin{bmatrix}0\\1\\1/2\\1/3\\0\end{bmatrix}$$

$$\Rightarrow \begin{aligned} a-b+c-d&=0\\ b-c+d&=1\\ c-d&=1/2\\ d&=1/3\\ e&=0\end{aligned}$$

$\Rightarrow b-5/6+1/3=1 \Rightarrow b=3/2$

$\Rightarrow c-1/3=1/2 \Rightarrow c=5/6$

$a-3/2+5/6-1/3=0$
$\Rightarrow a=1$

$$[L(1+t+t^2)]_\delta = \begin{bmatrix}1\\3/2\\5/6\\1/3\\0\end{bmatrix}$$

$$L(1+t+t^2+t^3) = \int_0^t 1+x+x^2+x^3\,dx = t+\tfrac{1}{2}t^2+\tfrac{1}{3}t^3+\tfrac{1}{4}t^4 = \begin{bmatrix}0\\1\\1/2\\1/3\\1/4\end{bmatrix}$$

$$\Rightarrow \begin{aligned} a-b+c-d&=0\\ b-c+d&=1\\ c-d&=1/2\\ d&=1/3\\ e&=1/4\end{aligned}$$

$\Rightarrow a=1$

$\Rightarrow b=3/2$

$\Rightarrow c=5/6$

$$[L(1+t+t^2+t^3)]_\delta = \begin{bmatrix}1\\3/2\\5/6\\1/3\\1/4\end{bmatrix}$$

$$_\delta[L]_\gamma = \begin{bmatrix}1&1&1&1\\1&3/2&3/2&3/2\\0&1/2&5/6&5/6\\0&0&1/3&1/3\\0&0&0&1/4\end{bmatrix}$$

11) Let $L: \mathbb{P}_2 \to \mathbb{R}^2$ be the transformation defined by $L(p) = \begin{bmatrix}\int_0^1 p(t)\,dt\\ p'(0)\end{bmatrix}$

(a) Show that L is linear

$$L(c\cdot p) = \begin{bmatrix}\int_0^1 c\cdot p(t)\,dt\\ c\cdot p'(0)\end{bmatrix} = \begin{bmatrix}c\int_0^1 p(t)\,dt\\ c\cdot p'(0)\end{bmatrix} = c\begin{bmatrix}\int_0^1 p(t)\,dt\\ p'(0)\end{bmatrix} = c\cdot L(p)$$

$$L(p+q) = \begin{bmatrix} \int_0^1 (p(t)+q(t))\,dt \\ (p+q)'(0) \end{bmatrix} = \begin{bmatrix} \int_0^1 p(t)\,dt + \int_0^1 q(t)\,dt \\ p'(0)+q'(0) \end{bmatrix}$$

$$= \begin{bmatrix} \int_0^1 p(t)\,dt \\ p'(0) \end{bmatrix} + \begin{bmatrix} \int_0^1 q(t)\,dt \\ q'(0) \end{bmatrix} = L(p) + L(q)$$

(b) let $\alpha = (1, t, t^2)$ and $\beta = (v_1, v_2)$, where $v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

(i) Find ${}_\beta[L]_\alpha$

$${}_\beta[L]_\alpha = \begin{bmatrix} [L(1)]_\beta & [L(t)]_\beta & [L(t^2)]_\beta \end{bmatrix}$$

$$L(1) = \begin{bmatrix} \int_0^1 1\,dt \\ \frac{d}{dt} 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a\begin{bmatrix} -1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{array}{r} -\cancel{a}+b = 1 \\ + \quad \cancel{a}+b = 0 \\ \hline 2b = 1 \end{array} \Rightarrow b = 1/2$$

$a + 1/2 = 0 \Rightarrow a = -1/2$

$$[L(1)]_\beta = \begin{bmatrix} -1/2 \\ 1/2 \end{bmatrix}$$

$$L(t) = \begin{bmatrix} \int_0^1 t\,dt \\ \frac{d}{dt} t \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \Rightarrow \begin{array}{r} -\cancel{a}+b = 1/2 \\ + \quad \cancel{a}+b = 1 \\ \hline 2b = 3/2 \end{array} \Rightarrow b = 3/4$$

$a = 1 - 3/4 \Rightarrow a = 1/4$

$$[L(t)]_\beta = \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix}$$

$$L(t^2) = \begin{bmatrix} \int_0^1 t^2\,dt \\ \frac{d}{dt} t^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3}(1)^3 \\ 2(0) \end{bmatrix} = \begin{bmatrix} 1/3 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{r} -\cancel{a}+b = 1/3 \\ + \quad \cancel{a}+b = 0 \\ \hline 2b = 1/3 \end{array} \Rightarrow b = 1/6$$

$a = -1/6$

$$[L(t^2)]_\beta = \begin{bmatrix} -1/6 \\ 1/6 \end{bmatrix}$$

$$_\beta[L]_\alpha = \begin{bmatrix} -1/2 & 1/4 & -1/6 \\ 1/2 & 3/4 & 1/6 \end{bmatrix}$$

(ii) Let $p(t) = 3 - 3t + 3/5\, t^2$. Find $[p]_\alpha$ and $[L(p)]_\beta$.

$$p = \begin{bmatrix} 3 \\ -3 \\ 3/5 \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \begin{matrix} a = 3 \\ b = -3 \\ c = 3/5 \end{matrix} \qquad [p]_\alpha = \begin{bmatrix} 3 \\ -3 \\ 3/5 \end{bmatrix}$$

$$L(p) = \begin{bmatrix} \int_0^1 3 - 3t + 3/5\, t^2\, dt \\ \frac{6}{5}(0) - 3 \end{bmatrix} = \begin{bmatrix} 17/10 \\ -3 \end{bmatrix} = a\begin{bmatrix} -1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{array}{r} -\cancel{a} + b = 17/10 \\ + \cancel{a} + b = -3 \\ \hline 2b = -13/10 \end{array}$$

$$\Rightarrow b = -13/20$$

$$a = -3 + 13/20 \Rightarrow a = -47/20 \qquad [L(p)]_\beta = \begin{bmatrix} -47/20 \\ -13/20 \end{bmatrix}$$

(iii) verify that $L(p) = \beta\, {}_\beta[L]_\alpha [p]_\alpha$

We have $\beta = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$, $_\beta[L]_\alpha = \begin{bmatrix} -1/2 & 1/4 & -1/6 \\ 1/2 & 3/4 & 1/6 \end{bmatrix}$, $[p]_\alpha = \begin{bmatrix} 3 \\ -3 \\ 3/5 \end{bmatrix}$

$$\beta\, {}_\beta[L]_\alpha [p]_\alpha = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1/2 & 1/4 & -1/6 \\ 1/2 & 3/4 & 1/6 \end{bmatrix}\begin{bmatrix} 3 \\ -3 \\ 3/5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1/2 & 1/3 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 3 \\ -3 \\ 3/5 \end{bmatrix} = \begin{bmatrix} 17/10 \\ -3 \end{bmatrix} = L(p)$$

(iv) verify that $[L(p)]_\beta = {}_\beta[L]_\alpha [p]_\alpha$.

$$_{\beta}[L]_{\alpha}[p]_{\alpha} = \begin{bmatrix} -1/2 & 1/4 & -1/6 \\ 1/2 & 3/4 & 1/6 \end{bmatrix} \begin{bmatrix} 3 \\ -3 \\ 3/5 \end{bmatrix} = \begin{bmatrix} -47/20 \\ -13/20 \end{bmatrix} = [L(p)]_{\beta}$$

12) Let $L : \mathbb{P}_3 \to \mathbb{P}_3$ be the transformation defined by $L(p) = t\,p'(t)$.

(a) Show that L is linear.

$$L(cp) = t\,cp'(t) = c\,tp'(t) = c\,L(p)$$

$$L(p+q) = t\,(p+q)'(t) = t\,(p'(t) + q'(t)) = tp'(t) + tq'(t)$$
$$= L(p) + L(q)$$

(b) Let $\alpha = (1, 3t, 2t^2)$ and $\beta = (3, t-1, 2(t-1)^2)$.

Find $_{\alpha}[L]_{\alpha}$ and $_{\beta}[L]_{\beta}$.

$$_{\alpha}[L]_{\alpha} = \left[[L(1)]_{\alpha} \;\; [L(3t)]_{\alpha} \;\; [L(2t^2)]_{\alpha} \right]$$

$$L(1) = 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} + c \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \Rightarrow \begin{matrix} a = 0 \\ b = 0 \\ c = 0 \end{matrix} \quad [L(1)]_{\alpha} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$L(3t) = t\,(3t)' = 3t = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} a = 0 \\ 3b = 3 \Rightarrow b = 1 \\ 2c = 0 \quad c = 0 \end{matrix} \quad [L(3t)]_{\alpha} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$L(2t^2) = t(2t^2)' = 4t^2 = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \Rightarrow \begin{matrix} a = 0 \\ b = 0 \\ 2c = 4 \Rightarrow c = 2 \end{matrix} \quad [L(2t^2)]_{\alpha} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$[L]_{\alpha}^{\alpha} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$[L]_{\beta}^{\beta} = \left[[L(3)]_\beta \quad [L(t-1)]_\beta \quad [L(2(t-1)^2)]_\beta \right]$$

$$L(3) = 0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = a\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 2 \\ -4 \\ 2 \end{bmatrix} \Rightarrow \begin{matrix} a=0 \\ b=0 \\ c=0 \end{matrix} \quad [L(3)]_\beta = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$L(t-1) = t(t-1)' = t = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} 3a-b+2c=0 & \\ b-4c=1 & \Rightarrow b=1 \\ 2c=0 & \Rightarrow c=0 \end{matrix}$$

$$3a-1=0 \Rightarrow 3a=1 \Rightarrow a=1/3 \qquad [L(t-1)]_\beta = \begin{bmatrix} 1/3 \\ 1 \\ 0 \end{bmatrix}$$

$$L(2(t-1)^2) = t(2(t-1)^2)' = t(4t-4) = 4t^2-4t = \begin{bmatrix} 0 \\ -4 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{matrix} 3a-b+2c=0 & \\ b-4c=-4 & \Rightarrow b-8=-4 \Rightarrow b=4 \\ 2c=4 \Rightarrow c=2 & \end{matrix} \qquad [L(2(t-1)^2)]_\beta = \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}$$

$$3a-\cancel{4}+\cancel{4}=0 \Rightarrow a=0$$

$$[L]_{\beta}^{\beta} = \begin{bmatrix} 0 & 1/3 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

(c) Relate $[L]_{\alpha}^{\alpha}$ and $[L]_{\beta}^{\beta}$ through a suitable transition matrix.

We have $[L]_{\alpha}^{\alpha} = [I]_{\alpha}^{\beta} [L]_{\beta}^{\beta} [I]_{\beta}^{\alpha}$. Let us find $[I]_{\alpha}^{\beta}$.

$$[I]_{\alpha}^{\beta} = \left[[3]_\alpha \quad [t-1]_\alpha \quad [2(t-1)^2]_\alpha \right]$$

$$\begin{bmatrix}3\\0\\0\end{bmatrix} = a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}0\\3\\0\end{bmatrix} + c\begin{bmatrix}0\\0\\2\end{bmatrix} \Rightarrow \begin{matrix}a=3\\b=0\\c=0\end{matrix} \quad [\,3\,]_\alpha = \begin{bmatrix}3\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}-1\\1\\0\end{bmatrix} \Rightarrow \begin{matrix}a=-1\\3b=1 \Rightarrow b=1/3\\2c=0 \Rightarrow c=0\end{matrix} \qquad [\,t-1\,]_\alpha = \begin{bmatrix}-1\\1/3\\0\end{bmatrix}$$

$$\begin{bmatrix}2\\-4\\2\end{bmatrix} \Rightarrow \begin{matrix}a=2\\3b=-4 \Rightarrow b=-4/3\\2c=2 \Rightarrow c=1\end{matrix} \qquad [\,2(t-1)^2\,]_\alpha = \begin{bmatrix}2\\-4/3\\1\end{bmatrix}$$

$${}_\alpha[I]_\beta = \begin{bmatrix}3 & -1 & 2\\0 & 1/3 & -4/3\\0 & 0 & 1\end{bmatrix}$$

$${}_\beta[I]_\alpha = \left({}_\alpha[I]_\beta\right)^{-1} = \begin{bmatrix}3 & -1 & 2\\0 & 1/3 & -4/3\\0 & 0 & 1\end{bmatrix}^{-1} = \begin{bmatrix}1/3 & 1 & 2/3\\0 & 3 & 4\\0 & 0 & 1\end{bmatrix}$$

$${}_\alpha[I]_\beta\,{}_\beta[L]_\beta\,{}_\beta[I]_\alpha = \begin{bmatrix}3 & -1 & 2\\0 & 1/3 & -4/3\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 1/3 & 0\\0 & 1 & 4\\0 & 0 & 2\end{bmatrix}\begin{bmatrix}1/3 & 1 & 2/3\\0 & 3 & 4\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 2\end{bmatrix} = {}_\alpha[L]_\alpha$$

$${}_\beta[I]_\alpha\,{}_\alpha[L]_\alpha\,{}_\alpha[I]_\beta = \begin{bmatrix}1/3 & 1 & 2/3\\0 & 3 & 4\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\0 & 1 & 0\\0 & 0 & 2\end{bmatrix}\begin{bmatrix}3 & -1 & 2\\0 & 1/3 & -4/3\\0 & 0 & 1\end{bmatrix} = \begin{bmatrix}0 & 1/3 & 0\\0 & 1 & 4\\0 & 0 & 2\end{bmatrix} = {}_\beta[L]_\beta$$

HW 5

SABELLA SAVINO

1) For each of the following bases $\alpha = (v_1, v_2)$, find the $\alpha - \epsilon$ change of basis matrix ${}_\epsilon[I]_\alpha$, where $\epsilon = (e_1, e_2)$ in the standard basis of $\mathbb{R}^2$.

(a) $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ ... $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$${}_\epsilon[I]_\alpha = \left[[v_1]_\epsilon \ [v_2]_\epsilon \right] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

(b) $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$

$${}_\epsilon[I]_\alpha = \left[[v_1]_\epsilon \ [v_2]_\epsilon \right] = \begin{bmatrix} 1 & -2 \\ 2 & 6 \end{bmatrix}$$

(c) $v_1 = \begin{bmatrix} -1 \\ 2/3 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1/2 \\ 3 \end{bmatrix}$

$${}_\epsilon[I]_\alpha = \left[[v_1]_\epsilon \ [v_2]_\epsilon \right] = \begin{bmatrix} -1 & -1/2 \\ 2/3 & 3 \end{bmatrix}$$

2) For each of the bases $\alpha = (v_1, v_2)$ of (a)-(c), find $[e_1]_\alpha$ and $[e_2]_\alpha$.

(a) $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} a-b \\ a+b \end{bmatrix}$$

$$\begin{aligned} &a-b=1 \\ +\ &a+b=0 \\ \hline &2a=1 \end{aligned} \qquad a = 1/2$$

$(\frac{1}{2}) + b = 0 \Rightarrow b = -1/2$

$$[e_1]_\alpha = \begin{bmatrix} 1/2 \\ -1/2 \end{bmatrix}$$

$$e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a-b \\ a+b \end{bmatrix}$$

$$\begin{aligned} &a-b=0 \\ +\ &a+b=1 \\ \hline &2a=1 \end{aligned} \qquad a = 1/2 \qquad (\tfrac{1}{2}) + b = 1 \qquad b = 1/2$$

$$[e_2]_\alpha = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

(b) $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a\begin{bmatrix} 1 \\ 2 \end{bmatrix} + b\begin{bmatrix} -2 \\ 6 \end{bmatrix} = \begin{bmatrix} a-2b \\ 2a+6b \end{bmatrix} \qquad \begin{aligned} a-2b&=1 \\ 2a+6b&=0 \end{aligned}$$

$$\Rightarrow -\ \begin{aligned} &\cancel{2a}-4b=2 \\ &\cancel{2a}+6b=0 \\ \hline &-10b=2 \end{aligned} \qquad b = \frac{2}{-10} = \frac{-1}{5} \qquad a - 2(\tfrac{-1}{5}) = 1 \Rightarrow a + \frac{2}{5} = 1$$

$\Rightarrow a = 3/5$

$$[e_1]_\alpha = \begin{bmatrix} 3/5 \\ -1/5 \end{bmatrix}$$

$$e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a-2b \\ 2a+6b \end{bmatrix} \qquad \begin{aligned} a-2b&=0 \\ 2a+6b&=1 \end{aligned}$$

$$\Rightarrow -\ \begin{aligned} &\cancel{2a}-4b=0 \\ &\cancel{2a}+6b=1 \\ \hline &-10b=-1 \end{aligned} \qquad b = 1/10$$

$a = 2b \Rightarrow a = 2/10 = 1/5$

$$[e_2]_\alpha = \begin{bmatrix} 1/5 \\ 1/10 \end{bmatrix}$$

(c) $v_1 = \begin{bmatrix} -1 \\ 2/3 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1/2 \\ 3 \end{bmatrix}$

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a\begin{bmatrix} -1 \\ 2/3 \end{bmatrix} + b\begin{bmatrix} -1/2 \\ 3 \end{bmatrix} = \begin{bmatrix} -a - 1/2\, b \\ 2/3\, a + 3b \end{bmatrix} \qquad \begin{matrix} -a - 1/2\, b = 1 \\ 2/3\, a + 3b = 0 \end{matrix}$$

$$\begin{array}{r} -\cancel{2a} - b = 2 \\ + \quad \cancel{2a} + 9b = 0 \\ \hline 8b = 2 \end{array} \qquad \Rightarrow b = \frac{2}{8} \qquad \Rightarrow b = 1/4$$

$$-a - 1/2\,(1/4) = 1$$

$$\Rightarrow -a - 1/8 = 1$$

$$\Rightarrow -a = 9/8 \Rightarrow a = -9/8$$

$$[e_1]_\alpha = \begin{bmatrix} -9/8 \\ 1/4 \end{bmatrix}$$

$$e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -a - 1/2\, b \\ 2/3\, a + 3b \end{bmatrix} \qquad \begin{matrix} -a - 1/2\, b = 0 \\ 2/3\, a + 3b = 1 \end{matrix} \quad \Rightarrow \begin{array}{r} -\cancel{2a} - b = 0 \\ + \quad \cancel{2a} + 9b = 3 \\ \hline 8b = 3 \end{array}$$

$$b = 3/8 \qquad -a - 1/2\,(3/8) = 0 \quad \Rightarrow -a - \frac{3}{16} = 0 \quad \Rightarrow a = -3/16$$

$$[e_2]_\alpha = \begin{bmatrix} -3/16 \\ 3/8 \end{bmatrix}$$

3) For each of the bases $\alpha = (v_1, v_2)$ of (a) - (c) in Problem 1, find $\epsilon - \alpha$ change of basis matrix ${}_\alpha[I]_\epsilon$

(a) $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$${}_\alpha[I]_\epsilon = \begin{bmatrix} [e_1]_\alpha & [e_1]_\alpha \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

(b) $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$

$$\underset{\alpha\ \ \epsilon}{[I]} = \begin{bmatrix} [e_1]_\alpha & [e_1]_\alpha \end{bmatrix} = \begin{bmatrix} 3/5 & 1/5 \\ -1/5 & 1/10 \end{bmatrix}$$

(c) $v_1 = \begin{bmatrix} -1 \\ 2/3 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1/2 \\ 3 \end{bmatrix}$

$$\underset{\alpha\ \ \epsilon}{[I]} = \begin{bmatrix} [e_1]_\alpha & [e_1]_\alpha \end{bmatrix} = \begin{bmatrix} -9/8 & -3/16 \\ 1/4 & 3/8 \end{bmatrix}$$

4) For each of the bases $\alpha = (v_1, v_2)$ of (a) - (c) in Problem 1, find $[v]_\alpha$, where $v = \begin{bmatrix} -1/2 \\ \sqrt{3} \end{bmatrix}$.

(a) $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$v = \begin{bmatrix} -1/2 \\ \sqrt{3} \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} a-b \\ a+b \end{bmatrix} \Rightarrow \begin{array}{r} a-b = -1/2 \\ + \quad a+b = \sqrt{3} \\ \hline 2a = -1/2 + \sqrt{3} \end{array}$$

$$\Rightarrow 2a = \frac{-1+2\sqrt{3}}{2} \Rightarrow a = \frac{-1+2\sqrt{3}}{4}$$

$$\left(\frac{-1+2\sqrt{3}}{4}\right) - b = \frac{-1}{2} \Rightarrow b = \left(\frac{-1+2\sqrt{3}}{4}\right) + \frac{1}{2} = \frac{1+2\sqrt{3}}{4}$$

$$[v]_\alpha = \begin{bmatrix} \frac{-1+2\sqrt{3}}{4} \\ \frac{1+2\sqrt{3}}{4} \end{bmatrix}$$

(b) $v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $v_2 = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$

$$v = \begin{bmatrix} -1/2 \\ \sqrt{3} \end{bmatrix} = a\begin{bmatrix} 1 \\ 2 \end{bmatrix} + b\begin{bmatrix} -2 \\ 6 \end{bmatrix} = \begin{bmatrix} a-2b \\ 2a+6b \end{bmatrix} \Rightarrow \begin{array}{r} a-2b = -1/2 \\ 2a+6b = \sqrt{3} \end{array}$$

$$\Rightarrow \begin{array}{r} -\cancel{2}a - 4b = -1 \\ - \quad 2\cancel{a} + 6b = \sqrt{3} \\ \hline -10b = -1-\sqrt{3} \end{array} \quad \Rightarrow \quad b = \frac{1+\sqrt{3}}{10}$$

$$a - 2\left(\frac{1+\sqrt{3}}{10}\right) = -\tfrac{1}{2} \quad \Rightarrow \quad a = -\tfrac{1}{2} + \frac{1+\sqrt{3}}{5} \quad \Rightarrow \quad a = \frac{-3+2\sqrt{3}}{5}$$

$$[v]_\alpha = \begin{bmatrix} \frac{1+\sqrt{3}}{10} \\ \frac{-3+2\sqrt{3}}{10} \end{bmatrix}$$

(c) $v_1 = \begin{bmatrix} -1 \\ 2/3 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1/2 \\ 3 \end{bmatrix}$

$$v = \begin{bmatrix} -1/2 \\ \sqrt{3} \end{bmatrix} = a \begin{bmatrix} -1 \\ 2/3 \end{bmatrix} + b \begin{bmatrix} -1/2 \\ 3 \end{bmatrix} = \begin{bmatrix} -a - \frac{1}{2}b \\ \frac{2}{3}a + 3b \end{bmatrix} \Rightarrow \begin{array}{l} -a - \frac{1}{2}b = -\frac{1}{2} \\ \frac{2}{3}a + 3b = \sqrt{3} \end{array}$$

$$\Rightarrow \begin{array}{r} -\cancel{2}a - b = -1 \\ + \quad 2\cancel{a} + 9b = 3\sqrt{3} \\ \hline 8b = -1+3\sqrt{3} \end{array} \quad \Rightarrow \quad b = \frac{-1+3\sqrt{3}}{8}$$

$$-2a - \left(\frac{-1+3\sqrt{3}}{8}\right) = -1 \quad \Rightarrow \quad -2a = \frac{-9+3\sqrt{3}}{8} \quad \Rightarrow \quad a = \frac{9-3\sqrt{3}}{16}$$

$$[v]_\alpha = \begin{bmatrix} \frac{-1+3\sqrt{3}}{8} \\ \frac{9-3\sqrt{3}}{16} \end{bmatrix}$$

5) let $v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$. For each of the bases $\alpha = (v_1, v_2)$ of (a) - (c), find $[v_1]_\alpha$ and $[v_2]_\alpha$.

(a) $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$v_1 = \begin{bmatrix}3\\2\end{bmatrix} = a\begin{bmatrix}1\\1\end{bmatrix} + b\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}a-b\\a+b\end{bmatrix} \Rightarrow \begin{array}{l} a \cancel{-b} = 3 \\ + \; a \cancel{+b} = 2 \\ \hline 2a = 5 \Rightarrow a = 5/2 \end{array}$$

$$a+b=2 \Rightarrow b = 2-a \Rightarrow b = 2 - 5/2 \Rightarrow b = -1/2$$

$$[v_1]_\alpha = \begin{bmatrix}5/2\\-1/2\end{bmatrix}$$

$$v_2 = \begin{bmatrix}-4\\3\end{bmatrix} = \begin{bmatrix}a-b\\a+b\end{bmatrix} \Rightarrow \begin{array}{l} a \cancel{-b} = -4 \\ + \; a \cancel{+b} = 3 \\ \hline 2a = -1 \end{array} \Rightarrow a = -1/2$$

$$a-b=-4 \Rightarrow b = a+4 \Rightarrow b = -1/2 + 4 \Rightarrow b = 7/2$$

$$[v_2]_\alpha = \begin{bmatrix}-1/2\\7/2\end{bmatrix}$$

(b) $u_1 = \begin{bmatrix}1\\2\end{bmatrix}$, $u_2 = \begin{bmatrix}-2\\6\end{bmatrix}$

$$v_1 = \begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}a-2b\\2a+6b\end{bmatrix} \Rightarrow \begin{array}{l} a-2b=3 \\ 2a+6b=2 \end{array} \Rightarrow \begin{array}{l} \cancel{2a} - 4b = 6 \\ - \; \cancel{2a} + 6b = 2 \\ \hline -10b = 4 \\ \Rightarrow b = -2/5 \end{array}$$

$$a - 2(-2/5) = 3 \Rightarrow a = 3 - 4/5 \Rightarrow a = 11/5$$

$$[v_1]_\alpha = \begin{bmatrix}11/5\\-2/5\end{bmatrix}$$

$$v_2 = \begin{bmatrix}-4\\3\end{bmatrix} = \begin{bmatrix}a-2b\\2a+6b\end{bmatrix} \Rightarrow \begin{array}{l} a-2b=-4 \\ 2a+6b=3 \end{array} \Rightarrow \begin{array}{l} \cancel{2a} - 4b = -8 \\ - \; \cancel{2a} + 6b = 3 \\ \hline -10b = -11 \\ \Rightarrow b = 11/10 \end{array}$$

$$a - 2(11/10) = -4 \Rightarrow a = -4 + 11/5 \Rightarrow a = -9/5$$

$$[v_2]_\alpha = \begin{bmatrix}-9/5\\11/10\end{bmatrix}$$

c) $v_1 = \begin{bmatrix} -1 \\ 2/3 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1/2 \\ 3 \end{bmatrix}$

$v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -a - 1/2\, b \\ 2/3\, a + 3b \end{bmatrix} \Rightarrow \begin{matrix} -a - 1/2\, b = 3 \\ 2/3\, a + 3b = 2 \end{matrix} \Rightarrow + \begin{matrix} -2a - b = 6 \\ 2a + 9b = 6 \\ \hline 8b = 12 \end{matrix}$

$-a - 1/2\,(3/2) = 3 \Rightarrow -a = 3 + 3/4 \Rightarrow a = -15/4 \qquad \Rightarrow b = 3/2$

$[v_1]_\alpha = \begin{bmatrix} -15/4 \\ 3/2 \end{bmatrix}$

$v_2 = \begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -a - 1/2\, b \\ 2/3\, a + 3b \end{bmatrix} \Rightarrow \begin{matrix} -a - 1/2\, b = -4 \\ 2/3 a + 3b = 3 \end{matrix} \Rightarrow + \begin{matrix} -\cancel{2a} - b = -8 \\ \cancel{2a} + 9b = 9 \\ \hline 8b = 1 \Rightarrow b = 1/8 \end{matrix}$

$-a - 1/2\,(1/8) = -4 \Rightarrow -a = -4 + 1/16 \Rightarrow a = 63/16$

$[v_2]_\alpha = \begin{bmatrix} 63/16 \\ 1/8 \end{bmatrix}$

b) let $v_1 = \begin{bmatrix} -3 \\ 12 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -5 \\ 8 \end{bmatrix}$. For each of the bases $\alpha = (v_1, v_2)$ of (a)-(c) in P1, find $[v_1]_\alpha$ and $[v_2]_\alpha$.

(a) $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$v_1 = \begin{bmatrix} -3 \\ 12 \end{bmatrix} = \begin{bmatrix} a - b \\ a + b \end{bmatrix} \Rightarrow \begin{matrix} a - b = -3 \\ a + b = 12 \end{matrix} \Rightarrow 2a = 9 \Rightarrow a = 9/2$

$9/2 - b = -3 \Rightarrow b = 9/2 + 3 \Rightarrow b = 15/2$

$[v_1]_\alpha = \begin{bmatrix} 9/2 \\ 15/2 \end{bmatrix}$

$$v_2 = \begin{bmatrix} -5 \\ 8 \end{bmatrix} = \begin{bmatrix} a-b \\ a+b \end{bmatrix} \Rightarrow \begin{matrix} a-b=-5 \\ a+b=8 \end{matrix} \Rightarrow 2a = 3 \Rightarrow a = 3/2$$

$$3/2 - b = -5 \Rightarrow b = 3/2 + 5 \Rightarrow b = 13/2$$

$$[v_2]_\alpha = \begin{bmatrix} 3/2 \\ 13/2 \end{bmatrix}$$

(b) $u_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $u_2 = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$

$$v_1 = \begin{bmatrix} -3 \\ 12 \end{bmatrix} = \begin{bmatrix} a - 2b \\ 2a + 6b \end{bmatrix} \Rightarrow \begin{matrix} a - 2b = -3 \\ 2a + 6b = 12 \end{matrix} \Rightarrow \begin{matrix} \cancel{2}a - 4b = -6 \\ -\ \cancel{2a} + 6b = 12 \\ \hline -10b = -18 \end{matrix} \Rightarrow b = 9/5$$

$$a - 2(9/5) = -3 \Rightarrow a = -3 + 18/5 = 3/5$$

$$[v_1]_\alpha = \begin{bmatrix} 3/5 \\ 9/5 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -5 \\ 8 \end{bmatrix} \Rightarrow \begin{matrix} a - 2b = -5 \\ 2a + 6b = 8 \end{matrix} \Rightarrow \begin{matrix} \cancel{2a} - 4b = -10 \\ -\ \cancel{2a} + 6b = 8 \end{matrix} \Rightarrow -10b = -18 \Rightarrow b = 9/5$$

$$a - 2(9/5) = -5 \Rightarrow a = -5 + 18/5 \Rightarrow a = -7/5 \qquad [v_2]_\alpha = \begin{bmatrix} -7/5 \\ 9/5 \end{bmatrix}$$

(c) $u_1 = \begin{bmatrix} -1 \\ 2/3 \end{bmatrix}$, $u_2 = \begin{bmatrix} -1/2 \\ 3 \end{bmatrix}$

$$v_1 = \begin{bmatrix} -3 \\ 12 \end{bmatrix} = \begin{bmatrix} -a - 1/2\, b \\ 2/3\, a + 3b \end{bmatrix} \Rightarrow \begin{matrix} -a - 1/2\, b = -3 \\ 2/3\, a + 3b = 12 \end{matrix} \Rightarrow \begin{matrix} -\cancel{2}a - b = -6 \\ +\ \cancel{2a} + 9b = 36 \\ \hline 8b = 30 \end{matrix} \Rightarrow b = 15/4$$

$$-a - \frac{1}{2}\left(\frac{15}{4}\right) = -3 \Rightarrow a = \frac{-15}{8} + 3 \Rightarrow a = 9/8$$

$$[v_1]_\alpha = \begin{bmatrix} 9/8 \\ 15/4 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} -5 \\ 8 \end{bmatrix} \Rightarrow \begin{matrix} -a - 1/2\, b = -5 \\ 2/3\, a + 3b = 8 \end{matrix} \Rightarrow \begin{matrix} -\cancel{2}a - b = -10 \\ \cancel{2a} + 9b = 24 \end{matrix} \Rightarrow 8b = 14 \Rightarrow b = 7/4$$

$$-a - 1/2(7/4) = -5 \Rightarrow a = \frac{-7}{8} + 5 \Rightarrow a = 33/8$$

$$[v_2]_\alpha = \begin{bmatrix} 33/8 \\ 7/4 \end{bmatrix}$$

7) let $\alpha = (v_1, v_2)$, where $v_1 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$.

let $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $y = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $z = \begin{bmatrix} 0 \\ 7/9 \end{bmatrix}$. Find $[x]_\alpha$, $[y]_\alpha$, $[z]_\alpha$.

$[x]_\alpha$:

$x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = a\begin{bmatrix} 5 \\ 3 \end{bmatrix} + b\begin{bmatrix} -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 5a - 3b \\ 3a + 2b \end{bmatrix} \Rightarrow \begin{matrix} (5a - 3b = 1) \times 2 \\ (3a + 2b = 1) \times 3 \end{matrix}$

$\Rightarrow \begin{matrix} 10a - 6b = 2 \\ + \; 9a + 6b = 3 \end{matrix} \Rightarrow 19a = 5 \Rightarrow a = 5/19$ $\qquad [x]_\alpha = \begin{bmatrix} 5/19 \\ 2/19 \end{bmatrix}$

$5(5/19) - 3b = 1 \Rightarrow -3b = 1 - 25/19 \Rightarrow b = 2/19$

$[y]_\alpha$:

$y = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5a - 3b \\ 3a + 2b \end{bmatrix} \Rightarrow \begin{matrix} (5a - 3b = 1)\, 2 \\ (3a + 2b = -1)\, 3 \end{matrix} \Rightarrow \begin{matrix} 10a - 6b = 2 \\ + \; 9a + 6b = -3 \\ \hline 19a = -1 \end{matrix} \Rightarrow a = -1/19$

$3(-1/19) + 2b = -1 \Rightarrow 2b = -1 + 3/19 \Rightarrow b = -8/19$ $\qquad [y]_\alpha = \begin{bmatrix} -1/19 \\ -8/19 \end{bmatrix}$

$[z]_\alpha$:

$z = \begin{bmatrix} 0 \\ 7/9 \end{bmatrix} \Rightarrow \begin{matrix} (5a - 3b = 0)\, 2 \\ (3a + 2b = 7/9)\, 3 \end{matrix} \Rightarrow \begin{matrix} 10a \cancel{-6b} = 0 \\ + \; 9a + \cancel{6b} = 7/3 \end{matrix} \Rightarrow 19a = \frac{7}{3} \Rightarrow a = \frac{7}{57}$

$5a = 3b \Rightarrow 5\left(\frac{7}{57}\right) = 3b \Rightarrow b = \frac{35}{171}$ $\qquad [z]_\alpha = \begin{bmatrix} 7/57 \\ 35/171 \end{bmatrix}$

8) let $u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $u_3 = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$, $v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.

let $\alpha = (u_1, u_2, u_3)$ and $\beta = (v_1, v_2, v_3)$.

(a) Show that α and β are bases for $\mathbb{R}^3$.

Claim: α, β are each linearly independent

We will show the claim is true by finding the determinant of α and β in matrix form.

$$\underset{(\text{matrix})}{\alpha} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & -1 \end{bmatrix} \quad \det(\alpha) = 1(-2-9) - 1(-1-3) + 2(3-2)$$
$$= (-11) - (-4) + (2) = -5 \neq 0$$

Since $\det(\alpha) \neq 0$, the columns v_1, v_2, v_3 are linearly independent.

$$\underset{(\text{matrix})}{\beta} = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 1 & 0 & -1 \end{bmatrix} \quad \det(\beta) = 1(-2-0) - 1(1-1) + 2(0-2)$$
$$= (-2) - (0) + (-4) = -6 \neq 0$$

Since $\det(\beta) \neq 0$, the columns v_1, v_2, v_3 are linearly independent.

Now, since α and β each contain 3 linearly independent vectors, and $\mathbb{R}^3$ is 3-dimensional, by the Dimension Theorem, both α and β are bases for $\mathbb{R}^3$.

(b) Find ${}_{\alpha}[I]_{\beta}$ and ${}_{\beta}[I]_{\alpha}$.

$${}_{\alpha}[I]_{\beta} = \left[[v_1]_\alpha \ [v_2]_\alpha \ [v_3]_\alpha \right]$$

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} a + b + 2c \\ a + 2b + 3c \\ a + 3b - c \end{bmatrix}$$

$$\Rightarrow \begin{aligned} a + b + 2c &= 1 \\ a + 2b + 3c &= -1 \\ a + 3b - c &= 1 \end{aligned}$$

$a = 1 - 2c - b$

$\Rightarrow (1 - 2c - b) + 2b + 3c = -1$

$\Rightarrow 1 + b + c = -1 \Rightarrow c = -2 - b$

$a + 3b - (-2 - b) = 1 \Rightarrow a + 3b + 2 + b = 1 \Rightarrow a + 4b = -1$

$\Rightarrow a = -1 - 4b$

$(-1 - 4b) + b + 2(-2 - b) = 1 \Rightarrow -5b = 6 \Rightarrow b = -6/5$

$a = -1 - 4(-6/5) \Rightarrow a = 19/5$

$c = -2 - (-6/5) \Rightarrow c = -4/5$

$$[v_1]_\alpha = \begin{bmatrix} 19/5 \\ -6/5 \\ -4/5 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} a + b + 2c &= 1 \\ a + 2b + 3c &= 2 \\ a + 3b - c &= 0 \end{aligned} \Rightarrow a = -3b + c$$

$(-3b + c) + b + 2c = 1 \Rightarrow -2b + 3c = 1 \Rightarrow c = 1/3 + 2/3 b$

$a + 2b + 3(1/3 + 2/3 b) = 2 \Rightarrow a + 4b + 1 = 2 \Rightarrow a = 1 - 4b$

$(1 - 4b) + 3b - (1/3 + 2/3 b) = 0 \Rightarrow 2/3 - 5/3 b = 0 \Rightarrow b = 2/5$

$a = 1 - 4(2/5) \Rightarrow a = -3/5$

$c = -3/5 + 3(2/5) \Rightarrow c = 3/5$

$$[v_2]_\alpha = \begin{bmatrix} -3/5 \\ 2/5 \\ 3/5 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \Rightarrow \begin{aligned} a + b + 2c &= 2 \\ a + 2b + 3c &= 1 \\ a + 3b - c &= -1 \end{aligned}$$

$\Rightarrow a = 2 - b - 2c$

$\Rightarrow (2 - b - 2c) + 2b + 3c = 1$

$\Rightarrow b + c = -1 \Rightarrow c = -1 - b$

$a + 3b - (-1 - b) = -1 \Rightarrow a + 4b = -2 \Rightarrow a = -2 - 4b$

$(-2-4b)+b+2(-1-b)=2 \Rightarrow 5b=-6 \Rightarrow b=-6/5$

$a=-2-4(-6/5) \Rightarrow a=14/5$

$c=-1-b \Rightarrow c=1/5$

$[v_3]_\alpha = \begin{bmatrix} 14/5 \\ -6/5 \\ 1/5 \end{bmatrix}$

$${}_\alpha[I]_\beta = \begin{bmatrix} 19/5 & -3/5 & 14/5 \\ -6/5 & 2/5 & -6/5 \\ -4/5 & 3/5 & 1/5 \end{bmatrix}$$

$${}_\beta[I]_\alpha = \begin{bmatrix} [u_1]_\beta & [u_2]_\beta & [u_3]_\beta \end{bmatrix}$$

$u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = a\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + c\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \Rightarrow \begin{aligned} a+b+2c&=1 \\ -a+2b+c&=1 \\ a+0b-c&=1 \end{aligned}$

$\Rightarrow a=1+c$

$(1+c)+b+2c=1 \Rightarrow b+3c=0 \Rightarrow b=-3c$

$-(1+c)+2(-3c)+c=1 \Rightarrow -6c=2 \Rightarrow c=-1/3$

$a=1+(-1/3) \Rightarrow a=2/3$

$b=-3c \Rightarrow b=1$

$[u_1]_\beta = \begin{bmatrix} 2/3 \\ 1 \\ -1/3 \end{bmatrix}$

$u_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Rightarrow \begin{aligned} a+b+2c&=1 \\ -a+2b+c&=2 \\ a+0b-c&=3 \end{aligned} \Rightarrow a=3+c$

$(3+c)+b+2c=1$
$\Rightarrow b=-2-3c$

$-(3+c)+2(-2-3c)+c=2 \Rightarrow -6c=9 \Rightarrow c=-3/2$

$a=3-3/2 \Rightarrow a=3/2$

$b=-2-3(-3/2) \Rightarrow b=5/2$

$[u_2]_\beta = \begin{bmatrix} 3/2 \\ 5/2 \\ -3/2 \end{bmatrix}$

$u_3 = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \Rightarrow \begin{aligned} a+b+2c&=2 \\ -a+2b+c&=3 \\ a+0b-c&=-1 \end{aligned} \Rightarrow a=-1+c$

$(-1+c)+b+2c=2$
$\Rightarrow b=3-3c$

$-(-1+c)+2(3-3c)+c=3 \Rightarrow -6c=-4 \Rightarrow c=2/3$

$a=-1+2/3 \Rightarrow a=-1/3$

$b=3-3(2/3) \Rightarrow b=1$

$$[v_3]_\beta = \begin{bmatrix} -1/3 \\ 1 \\ 2/3 \end{bmatrix}$$

$${}_\beta[I]_\alpha = \begin{bmatrix} 2/3 & 3/2 & -1/3 \\ 1 & 5/2 & 1 \\ -1/3 & -3/2 & 2/3 \end{bmatrix}$$

c) verify that ${}_\alpha[I]_\beta^{-1} = {}_\beta[I]_\alpha$.

$${}_\alpha[I]_\beta^{-1} = \begin{bmatrix} 19/5 & -3/5 & 14/5 \\ -6/5 & 2/5 & -6/5 \\ -4/5 & 3/5 & 1/5 \end{bmatrix}^{-1} = \frac{1}{|{}_\alpha[I]_\beta|} \begin{bmatrix} a_{22}a_{33}-a_{23}a_{32} & \cdots \\ a_{23}a_{31}-a_{21}a_{33} & \cdots \\ a_{21}a_{32}-a_{22}a_{31} & \cdots \end{bmatrix}$$

$$\det({}_\alpha[I]_\beta) = \frac{19}{5}\left(\frac{2}{25}+\frac{18}{25}\right)+\frac{3}{5}\left(\frac{-6}{25}-\frac{24}{25}\right)+\frac{14}{5}\left(\frac{-18}{25}+\frac{8}{25}\right)$$

$$= \frac{19}{5}\left(\frac{20}{25}\right)+\frac{3}{5}\left(\frac{-30}{25}\right)+\frac{14}{5}\left(\frac{-10}{25}\right)$$

$$= \frac{76}{25}-\frac{18}{25}-\frac{28}{25} \qquad = \frac{30}{25} = \frac{6}{5}$$

$$\frac{5}{6}\begin{bmatrix} 4/5 & 9/5 & -2/5 \\ 6/5 & 3 & 6/5 \\ -2/5 & -9/5 & 4/5 \end{bmatrix} = \begin{bmatrix} 2/3 & 3/2 & -1/3 \\ 1 & 5/2 & 1 \\ -1/3 & -3/2 & 2/3 \end{bmatrix} = {}_\beta[I]_\alpha$$

2) Create a basis different from the standard basis for $M_{2\times3}$. Choose a nonzero vector in $M_{2\times3}$ (your choice). Find the coordinates of this vector with respect to your basis.

Let's begin by using the standard basis for $M_{2\times3}$ to construct a new set of six linearly independent matrices:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 7 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 5 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & -4 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix}.$$

We will call this list α. Since the matrices in α are linearly independent, and α has size 6, α is a basis for $M_{2\times3}$.

Now, lets choose $v = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ and find $[v]_\alpha$.

$$v = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$= a\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 7 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 & 0 \\ 5 & 0 & 0 \end{bmatrix} + e\begin{bmatrix} 0 & 0 & 0 \\ 0 & -4 & 0 \end{bmatrix} + f\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$\Rightarrow 2a = 1 \quad \Rightarrow a = 1/2$

$\Rightarrow 7b = 2 \quad \Rightarrow b = 2/7$

$\Rightarrow -3c = 3 \quad \Rightarrow c = -1$

$\Rightarrow 5d = 4 \quad \Rightarrow d = 4/5$

$\Rightarrow -4e = 5 \quad \Rightarrow e = -5/4$

$\Rightarrow 8f = 6 \quad \Rightarrow f = 3/4$

$$\Rightarrow [v]_\alpha = \begin{bmatrix} 1/2 \\ 2/7 \\ -1 \\ 4/5 \\ -5/4 \\ 3/4 \end{bmatrix}$$

3) Create a basis different from the standard basis and different from the basis you selected in Problem 2 for $M_{2\times3}$. Find the coordinates of this vector with respect to this basis.

Ensuring linear independence, we have

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

We will call this list β. Since the matrices in β are linearly independent, and β has size 6, β is a basis for $M_{2\times3}$.

$$v = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} =$$

$$a\begin{bmatrix}1&1&0\\0&0&0\end{bmatrix}+b\begin{bmatrix}1&0&1\\0&0&0\end{bmatrix}+c\begin{bmatrix}0&1&1\\0&0&0\end{bmatrix}+d\begin{bmatrix}1&0&0\\1&0&0\end{bmatrix}+e\begin{bmatrix}0&1&0\\0&1&0\end{bmatrix}+f\begin{bmatrix}0&0&1\\0&0&1\end{bmatrix}$$

$\Rightarrow$

(i) $1 = a+b$

(ii) $2 = a+c+e$

(iii) $3 = b+c+f$

(iv) $4 = d$ $\quad\Rightarrow\quad d = 4,\ e = 5,\ f = 6$

(v) $5 = e$ $\quad\Rightarrow$ (iii) $3 = b + c + b$

(vi) $6 = f$ $\quad\Rightarrow c = -3-b$

$\Rightarrow$ (ii) $2 = a + (-3-b) + 5 \quad\Rightarrow\quad a-b = 0 \quad\Rightarrow\quad a = b$

$\Rightarrow$ (i) $1 = a+b \quad\Rightarrow\quad 1 = 2b \quad\Rightarrow\quad b = 1/2,\ a = 1/2$

$c = -3-b \quad\Rightarrow\quad c = -3-1/2 \quad\Rightarrow\quad c = -7/2$

$$[v]_\beta = \begin{bmatrix}1/2\\1/2\\-7/2\\4\\5\\6\end{bmatrix}$$

4) a) Find the change-of-basis matrix from the basis in Problem 2 to the basis in problem 3.

Want to find: ${}_\beta[I]_\alpha = \begin{bmatrix}[a_1]_\beta & [a_2]_\beta & \dots & [a_6]_\beta\end{bmatrix}$

$$\alpha = \underbrace{\begin{bmatrix}2&0&0\\0&0&0\end{bmatrix}}_{a_1},\ \underbrace{\begin{bmatrix}0&7&0\\0&0&0\end{bmatrix}}_{a_2},\ \underbrace{\begin{bmatrix}0&0&-3\\0&0&0\end{bmatrix}}_{a_3},\ \underbrace{\begin{bmatrix}0&0&0\\5&0&0\end{bmatrix}}_{a_4},\ \underbrace{\begin{bmatrix}0&0&0\\0&-4&0\end{bmatrix}}_{a_5},\ \underbrace{\begin{bmatrix}0&0&0\\0&0&8\end{bmatrix}}_{a_6}$$

$$\beta = \underbrace{\begin{bmatrix}1&1&0\\0&0&0\end{bmatrix}}_{b_1},\ \underbrace{\begin{bmatrix}1&0&1\\0&0&0\end{bmatrix}}_{b_2},\ \underbrace{\begin{bmatrix}0&1&1\\0&0&0\end{bmatrix}}_{b_3},\ \underbrace{\begin{bmatrix}1&0&0\\1&0&0\end{bmatrix}}_{b_4},\ \underbrace{\begin{bmatrix}0&1&0\\0&1&0\end{bmatrix}}_{b_5},\ \underbrace{\begin{bmatrix}0&0&1\\0&0&1\end{bmatrix}}_{b_6}$$

$$a_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} +$$

$$x_4 \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} + x_6 \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow$ (i) $2 = x_1 + x_2 + x_4$

(ii) $0 = x_1 + x_3 + x_5$

(iii) $0 = x_2 + x_3 + x_6$

(iv) $0 = x_4 \Rightarrow x_4 = 0$

(v) $0 = x_5 \Rightarrow x_5 = 0$

(vi) $0 = x_6 \Rightarrow x_6 = 0$

(iii) $x_2 + x_3 = 0 \Rightarrow x_2 = -x_3$

(ii) $x_1 + x_3 = 0 \Rightarrow x_1 = -x_3$

(i) $(-x_3) + (-x_3) = 2$

$\Rightarrow -2x_3 = 2 \Rightarrow x_3 = -1$

$\Rightarrow x_2 = 1, \; x_1 = 1$

$$[a_1]_\beta = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$a_2 = \begin{bmatrix} 0 & 7 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ (i) $0 = x_1 + x_2 + x_4 \Rightarrow x_2 = -x_1$

(ii) $7 = x_1 + x_3 + x_5 \qquad \Rightarrow x_1 + x_3 = 7$

(iii) $0 = x_2 + x_3 + x_6 \Rightarrow x_2 = -x_3 \qquad \Rightarrow -x_1 - x_3 = -7$

(iv) $0 = x_4 \Rightarrow x_4 = 0 \qquad \Rightarrow 2x_2 = -7$

(v) $0 = x_5 \Rightarrow x_5 = 0 \qquad \Rightarrow x_2 = -7/2$

vi) $0 = x_6 \Rightarrow x_6 = 0 \qquad \Rightarrow x_1 = 7/2 \quad x_3 = 7/2$

$$[a_2]_\beta = \begin{bmatrix} 7/2 \\ -7/2 \\ 7/2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$a_3 = \begin{bmatrix} 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[a_3]_\beta = \begin{bmatrix} 3/2 \\ -3/2 \\ -3/2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ (i) $0 = x_1 + x_2 + x_4 \Rightarrow x_1 = -x_2$

(ii) $0 = x_1 + x_3 + x_5 \Rightarrow x_1 = -x_3$

(iii) $-3 = x_2 + x_3 + x_6 \Rightarrow x_2 + x_3 = -3$

(iv) $0 = x_4 \Rightarrow x_4 = 0$ $\quad \Rightarrow -x_2 - x_3 = 3$

(v) $0 = x_5 \Rightarrow x_5 = 0$ $\quad \Rightarrow 2x_1 = 3$

vi) $0 = x_6 \Rightarrow x_6 = 0$ $\quad \Rightarrow x_1 = 3/2$

$\Rightarrow x_2 = -3/2 \quad x_3 = -3/2$

$$a_4 = \begin{bmatrix} 0 & 0 & 0 \\ 5 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ (i) $0 = x_1 + x_2 + x_4 \Rightarrow x_1 + x_2 = -5 \Rightarrow x_1 = -5 - x_2$

(ii) $0 = x_1 + x_3 + x_5 \Rightarrow (-5 - x_2) + x_3 = 0 \Rightarrow x_3 = 5 + x_2$

(iii) $0 = x_2 + x_3 + x_6 \Rightarrow x_2 + 5 + x_2 = 0 \Rightarrow x_2 = -5/2$

(iv) $5 = x_4 \Rightarrow x_4 = 5$ $\quad \Rightarrow x_1 = -5 + 5/2 \Rightarrow x_1 = -5/2$

(v) $0 = x_5 \Rightarrow x_5 = 0$ $\quad \Rightarrow x_3 = 5 - 5/2 \Rightarrow x_3 = 5/2$

vi) $0 = x_6 \Rightarrow x_6 = 0$

$$[a_4]_\beta = \begin{bmatrix} -5/2 \\ -5/2 \\ 5/2 \\ 5 \\ 0 \\ 0 \end{bmatrix}$$

$$a_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -4 & 0 \end{bmatrix}$$

$\Rightarrow$ (i) $0 = x_1 + x_2 + x_4 \Rightarrow x_1 + x_2 = 0 \Rightarrow x_2 = -x_1$

(i)

(ii) $0 = x_1 + x_3 + x_5 \Rightarrow x_1 + x_3 = 4 \Rightarrow x_3 = 4 - x_1$

(iii) $0 = x_2 + x_3 + x_6 \Rightarrow (-x_1) + (4 - x_1) = 0 \Rightarrow x_1 = 2$

(iv) $0 = x_4 \Rightarrow x_4 = 0 \qquad \Rightarrow x_2 = -2$

(v) $-4 = x_5 \Rightarrow x_5 = -4 \qquad \Rightarrow x_3 = 2$

vi) $0 = x_6 \Rightarrow x_6 = 0$

$$[a_5]_\beta = \begin{bmatrix} 2 \\ -2 \\ 2 \\ 0 \\ -4 \\ 0 \end{bmatrix}$$

$$a_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$

$\Rightarrow$

(i) $0 = x_1 + x_2 + x_4 \Rightarrow x_1 + x_2 = 0 \Rightarrow x_1 = -x_2$

(ii) $0 = x_1 + x_3 + x_5$

(iii) $0 = x_2 + x_3 + x_6 \Rightarrow x_2 + x_3 = -8 \Rightarrow x_3 = -8 - x_2$

(iv) $0 = x_4 \Rightarrow x_4 = 0 \qquad$ (ii) $-x_2 - 8 - x_2 = 0 \Rightarrow x_2 = -4$

(v) $0 = x_5 \Rightarrow x_5 = 0 \qquad \Rightarrow x_1 = 4$

vi) $8 = x_6 \Rightarrow x_6 = 8 \qquad \Rightarrow x_3 = -4$

$$[a_6]_\beta = \begin{bmatrix} 4 \\ -4 \\ -4 \\ 0 \\ 0 \\ 8 \end{bmatrix}$$

$$_\beta[I]_\alpha = \begin{bmatrix} 1 & 7/2 & 3/2 & -5/2 & 2 & 4 \\ 1 & -7/2 & -3/2 & -5/2 & -2 & -4 \\ -1 & 7/2 & -3/2 & 5/2 & 2 & -4 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8 \end{bmatrix}$$

b) Use this matrix to express the coordinates you found in Problem 2 in terms of the coordinates in Problem 3.

Coordinates Formula: $[v]_\alpha = {}_\alpha[I]_\beta [v]_\beta$

$$ {}_\alpha[I]_\beta = {}_\beta[I]_\alpha^{-1} = \begin{bmatrix} 1/2 & 1/2 & 0 & 1/2 & 0 & 0 \\ 1/7 & 0 & 1/7 & 0 & -1/7 & 0 \\ 0 & -1/3 & -1/3 & 0 & 0 & -1/3 \\ 0 & 0 & 0 & 1/5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/8 \end{bmatrix} $$

${}_\alpha[I]_\beta [v]_\beta =$

$$ \begin{bmatrix} 1/2 & 1/2 & 0 & 1/2 & 0 & 0 \\ 1/7 & 0 & 1/7 & 0 & -1/7 & 0 \\ 0 & -1/3 & -1/3 & 0 & 0 & -1/3 \\ 0 & 0 & 0 & 1/5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/8 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \\ -7/2 \\ 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 2/7 \\ -1 \\ 4/5 \\ -5/4 \\ 3/4 \end{bmatrix} = [v]_\alpha $$

c) Use this matrix to express the coordinates you found in Problem 3 in terms of the coordinates in Problem 2

Coordinates Formula: $[v]_\beta = {}_\beta[I]_\alpha [v]_\alpha$

${}_\beta[I]_\alpha [v]_\alpha =$

$$\begin{bmatrix} 1 & 7/2 & 3/2 & -5/2 & 2 & 4 \\ 1 & -7/2 & -3/2 & -5/2 & -2 & -4 \\ -1 & 7/2 & -3/2 & 5/2 & 2 & -4 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8 \end{bmatrix} \underbrace{\begin{bmatrix} 1/2 \\ 2/7 \\ -1 \\ 4/5 \\ -5/4 \\ 3/4 \end{bmatrix}}_{[v]_\alpha} = \begin{bmatrix} 1/2 \\ 1/2 \\ -7/2 \\ 4 \\ 5 \\ 6 \end{bmatrix} = [v]_\beta$$