Math 170B: Homework 6
(Due date: Friday, Mar. 7, 11:59 pm)

1. (+4 pts) Use the Taylor's theorem to derive the error term for approximating $f''(x)$ with the formula

$$\frac{f(x+h)+f(x-h)-2f(x)}{h^2}$$

(HINT: Write $f(x+h)=f(x)+hf'(x)+\frac{h^2}{2}f''(x)+\frac{h^3}{3!}f'''(x)+\frac{h^4}{4!}f^{(4)}(\xi)$, and similarly for $f(x-h)$.)

2. (+6 pts) Write MATLAB codes to apply all three formulas (forward difference, backward difference, central difference) to approximate the derivative of e^x at $x=1$. Generate three error plots corresponding to the three formulas. What do you observe?

3. (+10 pts) In calculus, we can show that

$$e=\lim_{h\to 0}\left(\frac{2+h}{2-h}\right)^{\frac{1}{h}}$$

   (a) Use your calculators to compute approximations to e by the formula $N_1(h)=\left(\frac{2+h}{2-h}\right)^{\frac{1}{h}}$, for $h=0.4, 0.2$ and 0.1.

   (b) Assume that $e=N_1(h)+k_1h+k_2h^2+k_3h^3+\cdots$. Use Richardson extrapolation to compute an $\mathcal{O}(h^3)$ approximation to e with $h=0.4$. Do you think this assumption on the expansion formula is correct?

   (c) Show that $N_1(h)=N_1(-h)$.

   (d) Use part (c) to show that $k_1=k_3=k_5=\cdots=0$ in the formula. Therefore, the formula is reduced to

$$e=N_1(h)+k_2h^2+k_4h^4+k_6h^6\ldots$$

   (e) Use the results of part (d) and Richardson extrapolation to compute an $\mathcal{O}(h^6)$ approximation to e with $h=0.4$.

NOTE: For all calculations in (a), (b) and (e), use enough digits of precision for each number to be able to see the differences of all numbers. You may also compare those numbers with e to see whether Richardson extrapolation helps you get better approximations.

1. Use the Taylor's Theorem to derive the error term for approximating $f''(x)$ with the formula

$$\frac{f(x+h)+f(x-h)-2f(x)}{h^2}$$

(Hint: write $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{(4)}(\xi)$, and similarly for $f(x-h)$.)

We have

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f^{(4)}(\xi_1),$$

$$\xi_1 \in (x, x+h)$$

and

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f^{(4)}(\xi_2),$$

$$\xi_2 \in (x-h, x)$$

Adding the two, we have

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + \frac{h^4}{24}f^{(4)}(\xi_1) + \frac{h^4}{24}f^{(4)}(\xi_2)$$

Subtract $2f(x)$ (as in the numerator)

$$f(x+h) + f(x-h) - 2f(x) = h^2 f''(x) + \frac{h^4}{24}\left[f^{(4)}(\xi_1) + f^{(4)}(\xi_2)\right]$$

Divide by h^2:

$$\frac{f(x+h)+f(x-h)-2f(x)}{h^2} = f''(x) + \frac{h^2}{24}\left[f^{(4)}(\xi_1) + f^{(4)}(\xi_2)\right]$$

So, $$e(h) = \left| \frac{f(x+h)+f(x-h)-2f(x)}{h^2} - f''(x) \right| = \left| \frac{h^2}{24}\left(f^{(4)}(\xi_1) + f^{(4)}(\xi_2)\right) \right|$$

By the triangle inequality,

$$e(h) \le \frac{h^2}{24}\left(|f^{(4)}(\xi_1)| + |f^{(4)}(\xi_2)|\right)$$

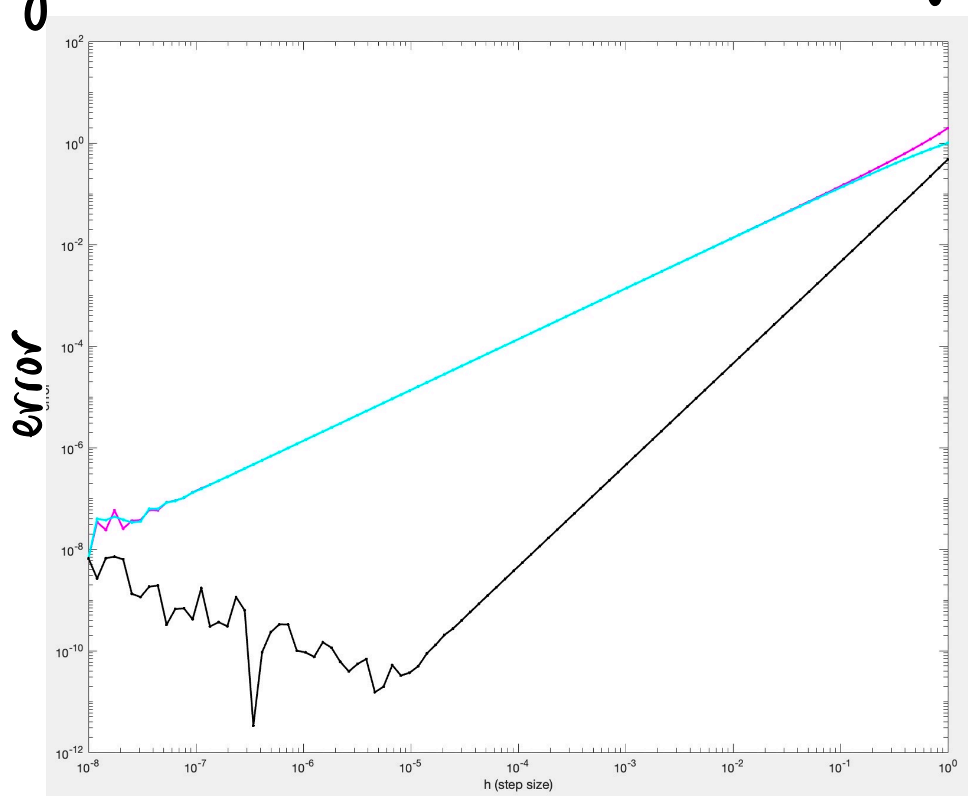
$\exists\, \delta > 0$ such that

$$\sup |f^{(4)}(y)| \le M \;, \quad y \in [x-\delta,\ x+\delta]$$

Hence,

$$e(h) \le \frac{h^2}{24}(M+M) = \frac{h^2 M}{12} = O(h^2).$$

2) Write MATLAB codes to apply all three formulas (forward difference, backward difference, central difference) to approximate the derivative of e^x at $x=1$. Generate three error plots corresponding to the three formulas. What do you observe?



It is clear from the plot that the Central Difference approximation (shown in black) gives by far the smallest error compared to the Forward (shown in magenta) and Backward (shown in cyan) approximations, whose errors are almost the same. This makes sense, since the error of the Central Difference formula is bounded by $O(h^2)$, while the error of the Forward/Backward formulas is bounded by $O(h)$, and so when the step size decreases, the error decreases much more rapidly for the Central Difference plot in relation to the Forward/Backward plots. When plotting with log-log, the order of the error bound becomes the slope, and we can see here that the slope of the Forward/Backward curves look to be 1 and the slope of the Central curve looks to be 2. Additionally, the Central plot is affected by rounding error much faster than the Forward/Backward plots, around $h = 10^{-5}$, which is likely because of rounding errors again, since the formula for Central Difference has to subtract numbers much closer together and divides by a larger value, $2h$. All three curves seem to be affected by rounding errors when h is very small, which makes sense.

3. In calculus, we can show that

$$e = \lim_{h \to 0} \left(\frac{2+h}{2-h} \right)^{1/h}$$

(a) Use your calculators to compute approximations to e by the formula $N_1(h) = \left(\frac{2+h}{2-h} \right)^{1/h}$, for $h = 0.4, 0.2,$ and 0.1.

$$N_1(0.4) = 2.75567596$$

$$N_1(0.2) = 2.72741282$$

$$N_1(0.1) = 2.72055141$$

(b) Assume that $e = N_1(h) + K_1 h + K_2 h^2 + K_3 h^3 + \cdots$. Use Richardson extrapolation to compute an $O(h^3)$ approximation to e with $h = 0.4$. Do you think this assumption on the expansion formula is correct?

Suppose $e = N_1(h) + K_1 h + K_2 h^2 + K_3 h^3 + \cdots$

We have that $N_1(h) = \left(\frac{2+h}{2-h} \right)^{1/h}$. Let $t = 2$. Then

$$N_2(h) = \frac{2N_1(h/2) - N_1(h)}{2^1 - 1} = 2N_1(h/2) - N_1(h)$$

$$N_3(h) = \frac{2^2 N_2(h/2) - N_2(h)}{2^2 - 1} = \frac{4N_2(h/2) - N_2(h)}{3}$$

Now, $N_1(0.4) = 2.75567596$

$N_1(0.2) = 2.72741282$

$N_1(0.1) = 2.72055141$

$\Rightarrow \quad N_2(0.4) = 2(2.72741282) - 2.75567596$
$= 2.69914968$

$N_2(0.2) = 2(2.72055141) - 2.72741282$
$= 2.71369$

$$\Rightarrow \quad N_3(0.4) = \frac{4(2.71369) - 2.69914968}{3}$$
$= 2.71853677$

Check: $|N_1(0.4) - e| = |2.75567596 - 2.71828182\ldots|$
$= 0.03739413$

$|N_2(0.4) - e| = |2.69914968 - 2.71828182\ldots|$
$= 0.01913214$

$|N_3(0.4) - e| = |2.71853677 - 2.71828182\ldots|$
$= 0.00025494$

Although error decreases with increased n, I do not believe that the assumption is correct, since

$$\ln(N_1(h)) = \ln\left(\frac{2+h}{2-h}\right)^{1/h}$$

$$\text{(By Taylor Expansion)} = \frac{1}{h}\left(\frac{h}{2} + \frac{h^3}{24} + \frac{h^5}{160} + \cdots\right)$$

$$\Rightarrow \quad N_1(h) = e^{1/2}\left(1 + \frac{h^2}{24} + \frac{h^4}{160} + \cdots\right)$$

Taylor's Theorem shows that the true expansion does not include any odd h terms.

(c) Show that $N_1(h) = N_1(-h)$.

$$N_1(-h) = \left(\frac{2-h}{2+h}\right)^{-1/h} = \frac{1}{\left(\frac{2-h}{2+h}\right)^{1/h}} = \left(\frac{2+h}{2-h}\right)^{1/h} = N_1(h)$$

(d) Use part (c) to show that $K_1 = K_3 = K_5 = \ldots = 0$ in the formula. Therefore, the formula is reduced to

$$e = N_1(h) + K_2h^2 + K_4h^4 + K_6h^6 + \ldots$$

We have that $e =$

$$N_1(h) + K_1h + K_2h^2 + K_3h^3 + \ldots = N_1(-h) - K_1(-h) - K_2(-h)^2 - K_3(-h)^3 + \ldots$$
$$= N_1(h) - K_1(-h) - K_2h^2 - K_3(-h)^3 + \ldots$$

$N_1(h) = N_1(-h)$, and even orders are all equivalent, so

$$K_1h + K_3h^3 + \ldots = -K_1(-h) - K_3(-h)^3$$

The only values of $K_1, K_3, K_5, \ldots$ that satisfy $K_1 = -K_1$, $K_3 = -K_3$, etc. are $K_1 = K_3 = K_5 = \ldots = 0$. Hence, the formula is reduced to

$$e = N_1(h) + K_2h^2 + K_4h^4 + K_6h^6 + \ldots$$

(e) Use the results of part (d) and Richardson extrapolation to compute an $O(h^6)$ approximation with $h = 0.4$.

$$N_2(h) = \frac{4N_1(h/2) - N_1(h)}{3} \quad (\sim O(h^4))$$

$$N_3(h) = \frac{16N_2(h/2) - N_2(h)}{15} \quad (\sim O(h^6))$$

$$N_2(0.4) = \frac{4N_1(0.2) - N_1(0.4)}{3} = \frac{4(2.72741282) - (2.75567596)}{3}$$

$$= 2.71799$$

$$N_2(0.2) = \frac{4N_1(0.1) - N_1(0.2)}{3} = \frac{4(2.72055141) - (2.72741282)}{3}$$

$$= 2.71826427$$

$$N_3(0.4) = \frac{16N_2(0.2) - N_2(0.4)}{15} = \frac{16(2.71826427) - (2.71799)}{15}$$

$$= 2.71828243$$

$$|N_3(0.4) - e| = 0.0000006$$

Much closer approximation than in part (b).

Math 170B: Homework 5
(Due date: Friday, Feb. 21, 11:59 pm)

1. (+4 pts) Determine whether this is a quadratic spline function:

$$f(x) = \begin{cases} x & x \in (-\infty, 1] \\ -\frac{1}{2}(2-x)^2 + \frac{3}{2} & x \in [1,2] \\ \frac{3}{2} & x \in [2, \infty) \end{cases}$$

   Is $f(x)$ a cubic spline function? Explain your reasons.

2. (+4 pts) Use induction to show the following two statements for the B-splines $\{B_i^k(x)\}$ for $k \geq 0$:

   (a) For all $i \in \mathbb{Z}$ if $x \in (t_i, t_{i+k+1})$ then $B_i^k(x) > 0$.

   (b) $\sum_{i \in \mathbb{Z}} B_i^k(x) \equiv 1$.

3. (+4 pts) Show that for any $\tau \in \mathbb{R}$ and $k \geq 1$,

$$(\tau - x)^k = \sum_{i \in \mathbb{Z}} B_i^k(x)\psi_{i,k}(\tau) \quad \forall\, x \in \mathbb{R},$$

   where

$$\psi_{i,k}(\tau) := \prod_{r=i+1}^{i+k} (\tau - t_r)$$

   and $\{t_i\}_{i \in \mathbb{Z}}$ is the set of knots of the B-splines $\{B_i^k(x)\}$ with $t_{-\infty} < \cdots < t_0 < \cdots < t_\infty$.
   $\Big($HINT: First show that for all $k \geq 1$ we have $\sum_{i \in \mathbb{Z}} B_i^k(x)\psi_{i,k}(\tau) = (\tau - x)\sum_{i \in \mathbb{Z}} B_i^{k-1}(x)\psi_{i,k-1}(\tau)$. Then apply this equality recursively and use the fact that $\psi_{i,0} \equiv 1.\Big)$
   Further, show that for any $0 < j \leq k$:

$$\frac{(\tau - x)^{k-j}}{(k-j)!} = \sum_{i \in \mathbb{Z}} B_i^k(x) \frac{\psi_{i,k}^{(j)}(\tau)}{k!}$$

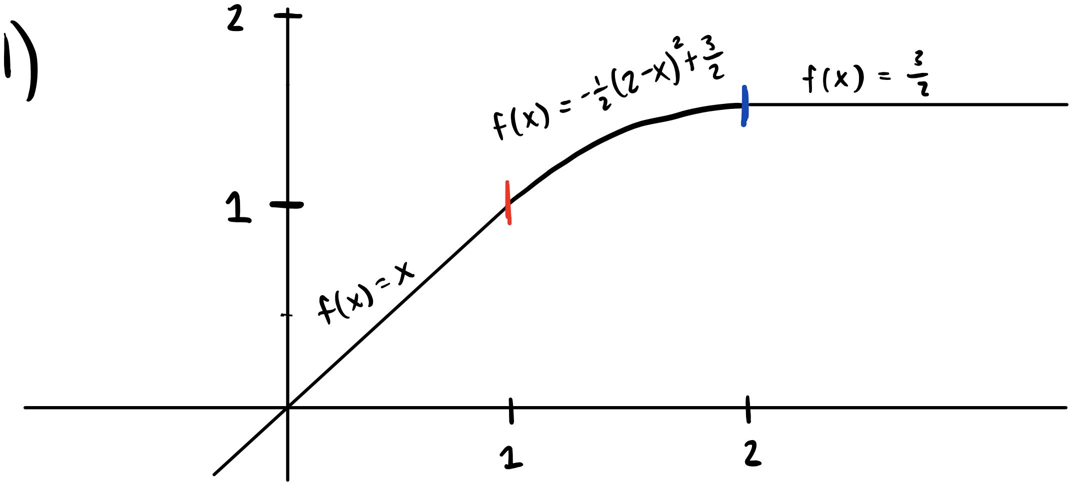
   where $\psi_{i,k}^{(j)}(\tau)$ is the j-th derivative of $\psi_{i,k}(\tau)$ with respect to τ. Note that by using the above result and the Taylor series expansion, any polynomial $p(x)$ of degree $< k$ can be written as a linear combination of $\{B_i^k(x)\}_{i \in \mathbb{Z}}$.

4. (+4 pts) Find a natural cubic spline function whose knots are $-1, 0, 1$ and that takes these values:

x	-1	0	1
y	5	7	9

5. (+4 pts) Show that the matrix $\{B_j^k(x_i)\}_{i,j}$ that arises in B-spline interpolation is banded or diagonally dominant. Specifically, if $B_j^k(x_j) \neq 0$ and $x_j < x_{j+1}$ for each j, then each row and column of the matrix contains at most $2k+1$ nonzero elements.

1)



Check quadratic:

(i) on each interval $[t_{i-1}, t_i)$, S is polynomial of degree $\leq k$

(ii) S has a continuous $(k-1)$st derivative on $[t_0, t_n]$

(i)

on $[-\infty, 1)$, $\deg(f) = 1 < 2$ ✓

on $[1, 2)$, $\deg(f) = 2 = 2$ ✓

on $[2, \infty)$, $\deg(f) = 0 < 2$ ✓

(ii) $k = 2$, so we need cont. 1st derivatives on $[t_0, t_n]$

$$\left.\begin{array}{l} t_0 = 1: \quad f(1) = 1, \quad f'(x) = 1 \Rightarrow f'(1) = 1 \\ f(1) = -\frac{1}{2}(2-1)^2 + \frac{3}{2} = 1, \quad f'(1) = 2 - 1 = 1 \end{array}\right\} \text{continuous}$$

$t_1 = 2$: $f(2) = -\frac{1}{2}(2-2)^2 + \frac{3}{2} = \frac{3}{2}$, $f'(2) = 2 - 2 = 0$

$f(2) = \frac{3}{2}$, $f'(2) = 0$ continuous

Yes, $f(x)$ is a quadratic spline function, since it is continuous piecewise polynomial of degree at most 2 with continuous derivatives up to the first order.

Check cubic:

(ii) $k=3$, must be cont. up to 2nd derivative

$$\left.\begin{aligned} t_0=1: \quad f''(x)=0 &, \quad f''(1)=0 \\ f''(x)=-1 &, \quad f''(1)=-1 \end{aligned}\right\} \text{ not continuous}$$

No, $f(x)$ is not a cubic spline function, since it does not have continuous derivatives up to the 2nd order.

2) $\{B_i^k(x)\}$, $k \geq 0$

(a) For all $i \in \mathbb{Z}$ if $x \in (t_i, t_{i+k+1})$ then $B_i^k(x) > 0$.

Base case: $k=0$,

$$\Rightarrow B_i^k(x) = B_i^{(0)}(x) = \begin{cases} 1 &, x \in (t_i, t_{i+1}) \\ 0 &, \text{otherwise} \end{cases}$$

Hence the base case holds.

Induction Hypothesis: Assume for all $k \geq 0$, $B_i^k(x) > 0$.

Inductive step: Using recurrence relation for B-splines,

$$B_i^{k+1}(x) = \left(\frac{x-t_i}{t_{i+k+1}-t_i}\right) B_i^k(x) + \left(1 - \left(\frac{x-t_{i+1}}{t_{i+k+2}-t_{i+1}}\right)\right) B_{i+1}^k(x)$$

$$= \left(\frac{x-t_i}{t_{i+k+1}-t_i}\right) B_i^k(x) + \left(\frac{t_{i+k+2}-x}{t_{i+k+2}-t_{i+1}}\right) B_{i+1}^k(x),$$

where $B_i^k(x) > 0$, $B_{i+1}^k(x) > 0$ by induction hypothesis, since

$x \in (t_i, t_{i+k+2})$. Also, since $x \in (t_i, t_{i+k+2})$, the coefficients $\left(\frac{x-t_i}{t_{i+k+1}-t_i}\right)$, $\left(\frac{t_{i+k+2}-x}{t_{i+k+2}-t_{i+1}}\right)$ are positive. Thus, $B_i^{k+1}(x) > 0$.

There fore, for all $i \in \mathbb{Z}$, $x \in (t_i, t_{i+k+1})$, $B_i^k(x) > 0$ for any k.

b) $\sum_{i \in \mathbb{Z}} B_i^k(x) \equiv 1$

Base case: $k = 0$, $B_i^{(0)}(x) = \begin{cases} 1, & x \in (t_i, t_{i+1}) \\ 0, & \text{otherwise} \end{cases}$

$$\Rightarrow \sum_{i \in \mathbb{Z}} B_i^0(x) = 1$$

Induction Hyp: Assume for $k \geq 0$, $\sum_{i \in \mathbb{Z}} B_i^k(x) = 1$.

Inductive step: Using recurrence relation for B-splines,

$$B_i^{k+1}(x) = \left(\frac{x-t_i}{t_{i+k+1}-t_i}\right) B_i^k(x) + \left(\frac{t_{i+k+2}-x}{t_{i+k+2}-t_{i+1}}\right) B_{i+1}^k(x)$$

Take sum on both sides:

$$\sum_{i \in \mathbb{Z}} B_i^{k+1}(x) = \sum_{i \in \mathbb{Z}} \left(\frac{x-t_i}{t_{i+k+1}-t_i}\right) B_i^k(x) + \sum_{i \in \mathbb{Z}} \left(\frac{t_{i+k+2}-x}{t_{i+k+2}-t_{i+1}}\right) B_{i+1}^k(x)$$

plug in $t_i = U_i + s \Rightarrow$

$$\sum_{i \in \mathbb{Z}} B_i^{k+1}(x) = \sum_{i \in \mathbb{Z}} \left(\frac{x-(U_i+s)}{U_{i+k+1}-U_i}\right) B_i^k(x) + \sum_{i \in \mathbb{Z}} \left(\frac{U_{i+k+2}-x+s}{U_{i+k+2}-U_{i+1}}\right) B_{i+1}^k(x)$$

As i sums over all integers, $(i+1 \rightarrow i)$ (index shifting)

$\Rightarrow$

$$\sum_{i\in\mathbb{Z}} B_i^{k+1}(x) = \sum_{i\in\mathbb{Z}} \left(\frac{x-(v_i+s)}{v_{i+k+1}-v_i}\right) B_i^k(x) + \sum_{i\in\mathbb{Z}} \left(\frac{v_{i+k+1}-x+s}{v_{i+k+1}-v_i}\right) B_i^k(x)$$

Combine sums:

$$\sum_{i\in\mathbb{Z}} B_i^{k+1}(x) = \sum_{i\in\mathbb{Z}} \frac{B_i^k(x)}{v_{i+k+1}-v_i}\left[\cancel{x} - v_i - \cancel{s} + v_{i+k+1} - \cancel{x} + \cancel{s}\right]$$

$$= \sum_{i\in\mathbb{Z}} \frac{B_i^k(x)}{v_{i+k+1}-v_i}(v_{i+k+1}-v_i)$$

$$= \sum_{i\in\mathbb{Z}} B_i^k(x) = 1 \quad \text{by inductive hypothesis.}$$

Hence, for all $k \geq 0$, $\sum_{i\in\mathbb{Z}} B_i^k(x) \equiv 1$.

3) any $\tau \in \mathbb{R}$ and $k \geq 1$:

$$(\tau - x)^k = \sum_{i\in\mathbb{Z}} B_i^k(x)\,\psi_{i,k}(\tau) \qquad \forall x \in \mathbb{R},$$

$$\psi_{i,k}(\tau) := \prod_{r=i+1}^{i+k}(\tau - t_r),$$

$\{t_i\}_{i\in\mathbb{Z}}$ is set of knots of B-splines $\{B_i^k(x)\}$, $t_{-\infty} < \cdots < t_0 < \cdots t_\infty$

Base case: $k=1$: $\sum_{i\in\mathbb{Z}} B_i^1(x) \prod_{r=i+1}^{i+1}(\tau - t_{i+1})$

where $B_i^1(x) = \begin{cases} 0 & , \; x < t_i \\ \dfrac{x-t_i}{t_{i+1}-t_i} & , \; t_i \leq x < t_{i+1} \end{cases}$

$$\begin{cases} \dfrac{t_{i+1}-x}{t_{i+2}-t_{i+1}} & , \quad t_{i+1} \le x < t_{i+2} \\ 0 & , \quad x \ge t_{0+2} \end{cases}$$

and $\sum_{i \in \mathbb{Z}} B_i^1(x) = 1 \quad \forall x \in \mathbb{R}$

$$\Rightarrow \sum_{i \in \mathbb{Z}} B_i^1(x) \prod_{i+1}^{i+1} (\tau - t_{i+1})$$

$$= B_i^1(x)(\tau - t_{i+1}) + B_{i-1}^1(x)(\tau - t_i)$$

$$= \frac{x - t_i}{t_{i+1}-t_i}(\tau - t_{i+1}) + \left(\frac{t_{i+1}-x}{t_{i+1}-t_i}\right)(\tau - t_i)$$

$$= \frac{(x-t_i)(\tau - t_{i+1}) + (t_{i+1}-x)(\tau - t_i)}{t_{i+1}-t_i}$$

$$= \frac{\cancel{x\tau} - x t_{i+1} - t_i \tau + \cancel{t_i t_{i+1}} + t_{i+1}\tau - \cancel{t_{i+1} t_i} - \cancel{x\tau} + x t_i}{t_{i+1}-t_i}$$

$$= \frac{-x t_{i+1} - t_i \tau + t_{i+1}\tau + x t_i}{t_{i+1}-t_i}$$

$$= \frac{(\tau - x)(t_{i+1}-t_i)}{t_{i+1}-t_i} = \tau - x$$

Hence the base case holds.

Inductive Hyp: Assume for $k \geq 1$, $(\tau - x)^k = \sum_{i \in \mathbb{Z}} B_i^k(x)\, \psi_{i,k}(\tau)$.

Inductive Step: Using recursive formula for B-splines:

$$B_i^{k+1}(x) = \left(\frac{x - t_i}{t_{i+k+1} - t_i}\right) B_i^k(x) + \left(\frac{t_{i+k+2} - x}{t_{i+k+2} - t_{i+1}}\right) B_{i+1}^k(x)$$

$$B_i^{k+1}(x) = \left(\frac{x - t_i}{t_{i+k+1} - t_i} B_i^k(x) + \frac{t_{i+k+2} - x}{t_{i+k+2} - t_{i+1}} B_{i+1}^k(x)\right) \psi_{i,k+1}(\tau)$$

$$\sum_{i \in \mathbb{Z}} B_i^{k+1}(x)\, \psi_{i,k+1}(\tau) = \sum_{i \in \mathbb{Z}} \frac{x - t_i}{t_{i+k+1} - t_i} B_i^k(x)\, \psi_{i,k}(\tau) + \sum_{i \in \mathbb{Z}} \frac{t_{i+k+2} - x}{t_{i+k+2} - t_{i+1}} B_{i+1}^k(x) \left(\psi_{i,k+1}(\tau)\right)$$

Using recursion,

$$\psi_{i,k+1}(\tau) = (\tau - t_{i+k+1})\, \psi_{i,k}(\tau)$$

$\Rightarrow$

$$\sum_{i \in \mathbb{Z}} B_i^{k+1}(x)\, \psi_{i,k+1}(\tau) = \sum_{i \in \mathbb{Z}} B_i^{k+1}(x)\, (\tau - t_{i+k+1})\, \psi_{i,k}(\tau)$$

By our hypothesis,

$$\sum_{i \in \mathbb{Z}} B_i^k(x)\, \psi_{i,k}(\tau) = (\tau - x)^k$$

$\Rightarrow$

$$\sum_{i \in \mathbb{Z}} B_i^{k+1}(x)\, \psi_{i,k+1}(\tau) = (\tau - x)^{k+1}.$$

4) $S_i(x) = a_i + b_i(x-t_i) + c_i(x-t_i)^2 + d_i(x-t_i)^3$

For $i=0$ to $n-1$, let $h_i = t_{i+1} - t_i$

$$S(x) = \begin{cases} S_0(x), & x \in [-1,0] \\ S_1(x), & x \in [0,1] \end{cases}$$

constraints : $S_0(-1)=5$, $S_0(0)=7$, $S_1(0)=7$, $S_1(1)=9$

$S_0'(0) = S_1'(0)$, $S_0''(0) = S_1''(0)$

$S_0(-1) = 0$, $S_1''(1) = 0$

On $[t_{-1}, t_0] \Rightarrow S_0(x) = a_0 + b_0(x+1) + c_0(x+1)^2 + d_0(x+1)^3$

$S_1(x) = a_1 + b_1 x + c_1 x^2 + d_1 x^3$

$S_0(-1) = a_0 = y_0 = 5 \quad \forall i = 0, \dots, n-1$

$S_0(0) = 7 \Rightarrow 7 = 5 + b_0(1) + c_0(1)^2 + a_0(1)^3$

$\Rightarrow 2 = b_0 + c_0 + d_0$

$S_1(0) = a_1 = y_1 = 7 \quad \forall i = 0, \dots, n-1$

$S_1(1) = 9 \Rightarrow 9 = 7 + b_1(1) + c_1(1) + d_1(1)$

$\Rightarrow 2 = b_1 + c_1 + d_1$

$S_0'(x) = b_0 + 2c_0(x+1) + 3d_0(x+1)^2$

$S_1'(x) = b_1 + 2c_1 x + 3d_1 x^2$

$$S_0'(0) = b_0 + 2c_0 + 3d_0 = b_1 + 2c_1(0) + 3d_1(0)^2 = S_1'(0)$$

$$\Rightarrow \quad b_0 + 2c_0 + 3d_0 = b_1$$

$$S_0''(x) = 2c_0 + 6d_0(x+1)$$

$$S_1''(x) = 2c_1 + 6d_1 x$$

$$S_0''(0) = 2c_0 + 6d_0 = 2c_1 = S_1''(0)$$

$$\Rightarrow \quad 2c_0 + 6d_0 = 2c_1 \Rightarrow c_1 = c_0 + 3d_0$$

$$S_0''(-1) = 2c_0 + 6d_0(0) = 2c_0 = 0 \Rightarrow c_0 = 0$$

$$S_1''(1) = 2c_1 + 6d_1(1) = 2c_1 + 6d_1 = 0 \Rightarrow 2c_1 = -6d_1$$

We have

$$2 = b_0 + c_0 + d_0$$

$$2 = b_1 + c_1 + d_1$$

$$b_1 = b_0 + 2c_0 + 3d_0$$

$$c_1 = c_0 + 3d_0$$

$$c_0 = 0$$

$$c_1 = -3d_1$$

$$\Rightarrow \quad c_1 = 3d_0 = -3d_1 \Rightarrow d_0 = -d_1$$

$$2 = b_0 + d_0 \Rightarrow b_0 = 2 - d_0$$

$$2 = b_1 - 3d_1 + d_1 \Rightarrow b_1 = 2 + 2d_1$$

$b_1 = b_0 + 3d_0$

$\Rightarrow \left.\begin{array}{l} 2 = b_0 - d_1 \\ 2 = b_1 - 2d_1 \end{array}\right\} + = 4 = b_0 + b_1 - 3d_1$

$\Rightarrow -3d_1 = 4 - b_0 - b_1$

$b_1 = b_0 - 3d_1$

$\Rightarrow b_1 = b_0 + (4 - b_0 - b_1)$

$= 4 - b_1 \Rightarrow 2b_1 = 4 \Rightarrow b_1 = 2$

$\Rightarrow 2 = 2 + 2d_1 \Rightarrow d_1 = 0 \Rightarrow d_0 = 0$

$= 2 = b_0 - 0 \Rightarrow b_0 = 2$

$c_1 = 3d_0 = 0$

$\Rightarrow S_0(x) = 5 + 2(x+1)$

$S_1(x) = 7 + 2x$

5) we know $B_j^0(x) = \begin{cases} 1 & x \in (t_i, t_{i+k+1}) \\ 0 & \text{other} \end{cases}$,

and $B_j^k(x) = \dfrac{x - t_j}{t_{j+k} - t_j} B_j^k(x) + \dfrac{t_{j+k+1} - x}{t_{j+k+1} - t_{j+1}} B_{j+1}^{k-1}(x)$

$\{B_j^k(x_i)\}_{i,j} \Rightarrow$ i rows $= x_i$, j cols $= B_j^k(x_i)$

For recurrence, x defined on (t_j, t_{j+k+1})

$\Rightarrow 2(k+1)-1$ defined values $\Rightarrow 2k+1$ defined values at **MOST** in each column/row

Math 170B: Homework 4
(Due date: Friday, Feb. 14, 11:59 pm)

1. (+8 pts) Let $f(x) = \frac{1}{(1+25x^2)}$ for $x \in \mathbb{R}$. Implement the Newton divided difference algorithm (Page 332 in the textbook, section 6.2 with subtitle "Algorithm for Divided Difference") with nested multiplication on MATLAB to find the interpolating polynomial $p_n(x)$ for $f(x)$ on the interval $[-1, 1]$. The algorithm must be implemented for $n = 5, 10, 15, 19$ points and for each case the set of data points $\{x_i\}_{i=0}^n$ must be equally spaced, i.e., $x_i = -1 + \frac{2i}{n}$ for $0 \le i \le n$. Generate the interpolating polynomials for the 4 cases along with the function $f(x)$ on the same plot for the interval $[-1, 1]$. State your observation from these plots. Do the interpolating polynomials violate the interpolation error theorem? Justify your answer.

2. (+4 pts) Prove that if f is a polynomial of degree k then for $n > k$

$$f[x_0, x_1, \cdots, x_n] = 0.$$

3. (+4 pts) Use the Newton divided difference method to obtain a quartic polynomial that takes these values:

x	0	1	2
$p(x)$	2	-4	44
$p'(x)$	-9	4	

4. (+4 pts) Answer the following questions based on polynomial interpolation:

   (a) (+2 pts) Let $\{x_i\}_{i=0}^n$ be a set of distinct data points,

$$H_{2n+1}(x) = \sum_{j=0}^{n} f(x_j) H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j) \hat{H}_{n,j}(x)$$

   where

$$H_{n,j}(x) = \Big(1 - 2(x - x_j)\ell'_{n,j}(x_j)\Big)\ell^2_{n,j}(x),$$
$$\hat{H}_{n,j}(x) = (x - x_j)\ell^2_{n,j}(x),$$

   and

$$\ell_{n,j}(x) = \prod_{i=0;\ i \ne j}^{n} \frac{x - x_i}{x_j - x_i}$$

   is the Lagrange polynomial of degree n for x_j. Without computing the derivative of $\ell_{n,j}(x)$ show that for all $k \in \{0, 1, \cdots, n\}$:

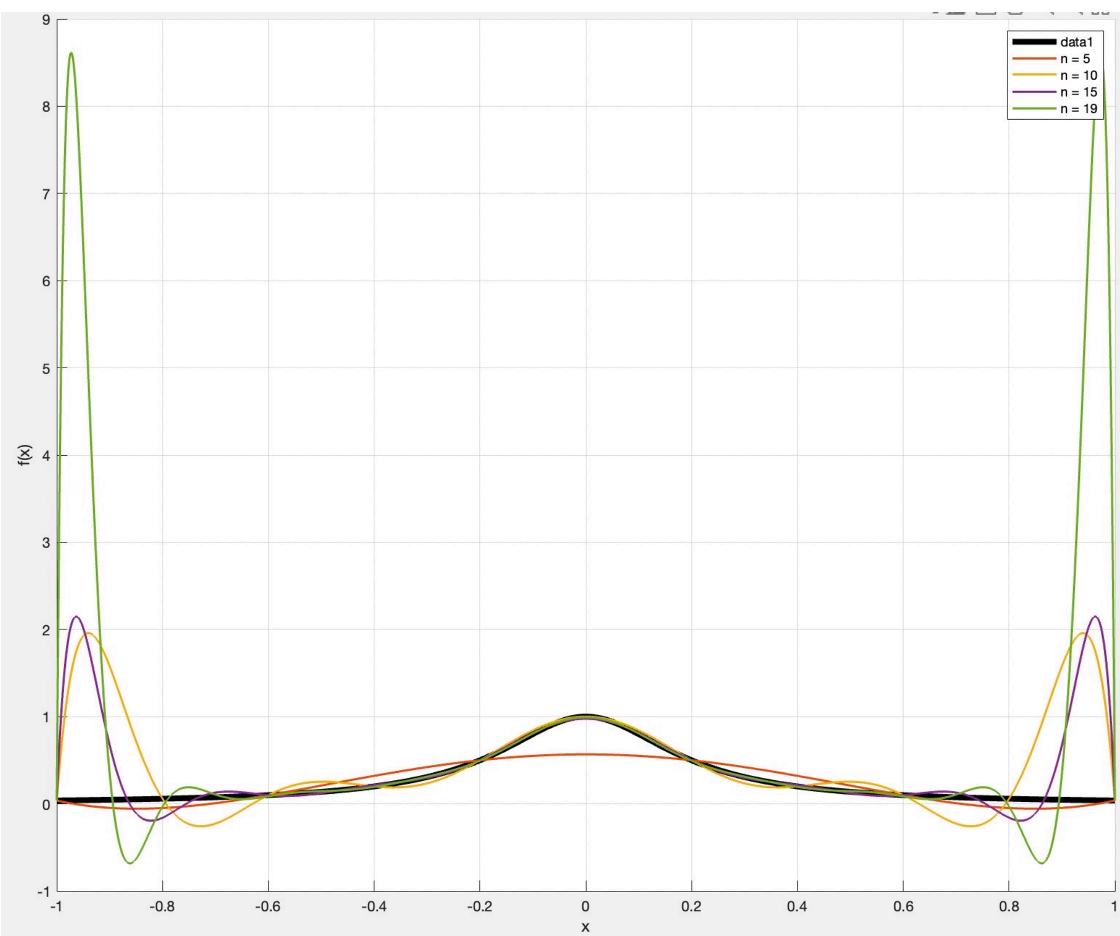
$$H'_{2n+1}(x_k) = f'(x_k).$$

   (b) (+2 pts) What condition will have to be placed on the nodes x_0 and x_1 if the interpolation problem

$$p(x_i) = c_{i0}, \quad p''(x_i) = c_{i2}, \quad i \in \{0, 1\}$$

   is to be solvable by a cubic polynomial (for arbitrary c_{ij})? (HINT: Write the constraints as a system of linear equations and find its Vandermonde matrix.)

1) Let $f(x) = \frac{1}{(1+25x^2)}$ for $x \in \mathbb{R}$. Implement the Newton divided difference algorithm (Page 332 in textbook, section 6.2 with subtitle "Algorithm for divided difference") with nested multiplication on MATLAB to find the interpolating polynomial $p_n(x)$ for $f(x)$ on the interval $[-1, 1]$. The algorithm must be implemented for $n = 5, 10, 15, 19$ points and for each case the set of data points $\{x_i\}_{i=0}^{n}$ must be equally spaced, ie. $x_i = -1 + \frac{2i}{n}$ for $0 \le i \le n$. Generate the interpolating polynomials for the 4 cases along with the function $f(x)$ on the same plot for the interval $[-1,1]$. State your observation from these plots. Do the interpolating polynomials violate the interpolation error theorem? Justify your answer.



Interpolation Error Theorem: Let $f \in C^{n+1}([-1,1])$, $\{x_0, \ldots, x_n\}$ are <u>distinct</u> data points. Then $\forall x \in [-1,1]$, $\exists \xi_x \in [-1,1]$ s.t.

$$f(x) - p_n(x) = \frac{f^{(n+1)}(\xi_x)}{(n+1)!} \left(\prod_{j=0}^{n} (x - x_j) \right),$$

where the term $\frac{f^{(n+1)}(\xi_x)}{(n+1)!}$ corresponds to the highest order derivative of the function and the term $\prod_{j=0}^{n} (x - x_j)$ corresponds to the difference in x-value for each $\{x_j\}_{j=0}^{n}$ where $x_j = -1 + \frac{2j}{n}$ for $0 \leq j \leq n$. From our plot, we can observe that lower values of n interpolate with some accuracy over the entire interval, while higher values interpolate very accurately when closer to the middle of the interval ($x=0$), but as we estimate closer to the endpoints, the interpolated polynomial begins oscillating. This leads to large error near the endpoints, which is likely because of the equally spaced data points taken into account in the $\prod_{j=0}^{n} (x - x_j)$ term. Although the error is greatly amplified at the endpoints, the interpolating polynomials do not violate the theorem, since we can explain this error using the term $\prod_{j=0}^{n} (x - x_j)$ and the fact that each x_j is evenly (but not accurately) spaced.

2) Prove that if f is a polynomial of degree k, then for $n > k$

$$f[x_0, x_1, \cdots, x_n] = 0.$$

<u>Proof</u>. By the Mean Value Theorem for Divided Difference,

$$f[x_0, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}, \quad \xi \in \left(\min_{0 \le i \le n} x_i, \max_{0 \le i \le n} x_i\right).$$

We have that f is a degree k polynomial with $k < n$. Hence, for all $x \in \mathbb{R}$,

$$f^{(k)}(x) = 0$$

$$\Rightarrow f[x_0, x_1, \ldots, x_n] = 0.$$

3) Use the Newton divided difference method to obtain a quartic polynomial that takes these values:

x	0	1	2
$p(x)$	2	-4	44
$p'(x)$	-9	4	

We have 5 roots, $x_0 = 0$, $x_1 = 0$, $x_2 = 1$, $x_3 = 1$, $x_4 = 2$. We would like to obtain a 4th order polynomial

$$P_4(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^2(x-1) + c_4 x^2(x-1)^2.$$

From the chart of values, we have $p(0) = 2$. Hence, $c_0 = 2$. Now, to find c_1, we have

$$c_1 = p[x_0, x_1] = p[0,0] = p'(0) = -9.$$

For c_2, we have

$$c_2 = p[x_0, x_1, x_2] = p[0,0,1]$$

$$= \frac{p[0,0] - p[0,1]}{0-1}$$

$$= 9 + p[0,1]$$

$$p[0,1] = p(1) - p(0) = -4 - 2 = -6$$

$$\Rightarrow \quad = 9 - 6 = 3$$

For c_3:

$$c_3 = \frac{p[0,0,1] - p[0,1,1]}{0-1} = \frac{3 - p[0,1,1]}{-1}$$

$$p[0,1,1] = \frac{p[0,1] - p[1,1]}{0-1} = -p[0,1] + p'(1)$$

We previously found that $p[0,1] = -6$, and from chart $p'(1) = 4$

$$\Rightarrow p[0,1,1] = 6 + 4 = 10$$

$$\Rightarrow c_3 = \frac{3 - (10)}{-1} = 7$$

Finally, for c_4:

$$c_4 = p[x_0, x_1, x_2, x_3, x_4] = p[0,0,1,1,2]$$

$$= \frac{p[0,1,1] - p[0,1,1,2]}{0-2}$$

$$= \frac{7 - p[0,1,1,2]}{-2}$$

$$p[0,1,1,2] = \frac{p[0,1,1] - p[1,1,2]}{-2}$$

$$= \frac{p[0,1,1] - (p[1,2] - p[1,1])}{-2}$$

We previously found that $p[0,1,1] = 10$, and $p[1,1] = p'(1) = 4$

$$\Rightarrow p[0,1,1,2] = \frac{10 - (p[1,2] - 4)}{-2}$$

$$p[1,2] = p(2) - p(1) = 44 - (-4) = 48$$

$$\Rightarrow p[0,1,1,2] = \frac{10-44}{-2} = \frac{-34}{-2} = 17$$

$$\Rightarrow c_4 = p[0,0,1,1,2] = \frac{7-17}{-2} = \frac{-10}{-2} = 5$$

final polynomial:

$$p_4(x) = 2 - 9x + 3x^2 + 7x^2(x-1) + 5x^2(x-1)^2$$

4) Answer the following questions based on polynomial interpolation:

(a) Let $\{x_i\}_{i=0}^{n}$ be a set of distinct data points,

$$H_{2n+1}(x) = \sum_{j=0}^{n} f(x_j)\, H_{n,j}(x) + \sum_{j=0}^{n} f'(x_j)\, \hat{H}_{n,j}(x)$$

where

$$H_{n,j}(x) = \left(1 - 2(x - x_j)\, \ell'_{n,j}(x_j)\right) \ell^2_{n,j}(x),$$

$$\hat{H}_{n,j}(x) = (x - x_j)\, \ell^2_{n,j}(x),$$

and

$$\ell_{n,j}(x) = \prod_{i=0;\, i \neq j}^{n} \frac{x - x_i}{x_j - x_i}$$

is the Lagrange polynomial of degree n for x_j. Without computing the derivative of $\ell_{n,j}(x)$, show that for all $k \in \{0, 1, \ldots, n\}$:

$$H'_{2n+1}(x_k) = f'(x_k).$$

We have that

$$\ell_{n,j}(x_k) = \begin{cases} 1, & k = j \\ 0, & k \neq j \end{cases} = \delta_{jk}$$

and that

$$H_{2n+1}(x) = \sum_{j=0}^{n} G_j(x), \quad \text{where}$$

$$G_j(x) = f(x_j)\, H_{n,j}(x) + f(x_j)\, \hat{H}_{n,j}(x)$$

$$\Rightarrow \quad H'_{2n+1}(x) = \sum_{j=0}^{n} G'_j(x), \qquad \text{where}$$

$$G'_j(x) = f(x_j)\, H'_{n,j}(x) + f'(x_j)\, \hat{H}'_{n,j}(x)$$

Differentiate $H_{n,j}(x)$:

$$\left.\begin{aligned} H'_{n,j}(x) &= -2\,\ell'_{n,j}(x_j)\,\ell^2_{n,j}(x) \\ &\quad + \left(1 - 2(x-x_j)\,\ell'_{n,j}(x_j)\right) 2\,\ell_{n,j}(x)\,\ell'_{n,j}(x) \end{aligned}\right\} \mathrm{I}$$

Differentiate $\hat{H}_{n,j}(x)$:

$$\left.\hat{H}'_{n,j}(x) = \ell^2_{n,j}(x) + 2(x-x_j)\,\ell_{n,j}(x)\,\ell'_{n,j}(x) \right\} \mathrm{II}$$

Plug back into $G'_j(x)$:

$$G'_j(x) = f(x_j)\Big[-2\,\ell'_{n,j}(x_j)\,\ell^2_{n,j}(x) + \left(1-2(x-x_j)\,\ell'_{n,j}(x_j)\right) 2\,\ell_{n,j}(x)\,\ell'_{n,j}(x)\Big] + f'(x_j)\Big[\ell^2_{n,j}(x) + 2(x-x_j)\,\ell_{n,j}(x)\,\ell'_{n,j}(x)\Big]$$

Now, if $x = x_j$,

$$\mathrm{I} = -2\,\ell'_{n,j}(x_j)\,\underbrace{\ell^2_{n,j}(x_j)}_{=1} + \underbrace{\left(1 - \underbrace{2(x_j - x_j)\,\ell'_{n,j}(x_j)}_{=0}\right)}_{=1} 2\,\underbrace{\ell_{n,j}(x_j)}_{=1}\,\ell'_{n,j}(x_j)$$

$$= -2\ell'_{n,j}(x_j) + 2\ell'_{n,j}(x_j) = 0$$

$$\mathrm{II} = \underbrace{\ell^2_{n,j}(x_j)}_{=1} + \underbrace{2(x_j - x_j)\ell_{n,j}(x_j)\ell'_{n,j}(x_j)}_{=0} = 1$$

$$\Rightarrow G'_j(x) = f(x_j)(0) + f'(x_j)(1) = f'(x_j)$$

Similarly, if $x = x_k$, $k \neq j$,

$$\mathrm{I} = -2\ell'_{n,j}(x_j)\underbrace{\ell^2_{n,j}(x_k)}_{=0} + (1 - 2(x_k - x_j)\ell'_{n,j}(x_j))\underbrace{2\ell_{n,j}(x_k)}_{=0}\ell'_{n,j}(x_k)$$

$$= -2\ell'_{n,j}(x_j)(0) + (0)\ell'_{n,j}(x_k) = 0$$

$$\mathrm{II} = \underbrace{\ell^2_{n,j}(x_k)}_{=0} + \underbrace{2(x_k - x_j)\ell_{n,j}(x_k)\ell'_{n,j}(x_k)}_{=0} = 0$$

$$\Rightarrow G'_j(x_j) = f(x_j)(0) + f'(x_j)(0) = 0$$

Thus, we have

$$G'_j(x_j) = \begin{cases} f'(x_j), & j = k \\ 0, & j \neq k \end{cases},$$

where $H'_{2n+1}(x) = \sum_{j=0}^{n} G'_j(x)$. Hence, for all $k \in \{0, 1, \dots, n\}$,

$$H'_{2n+1}(x) = f'(x_j).$$

(b) What condition will have to be placed on the nodes x_0 and x_1 if the interpolation problem

$$p(x_i) = c_{i0}, \quad p''(x_i) = c_{i2}, \quad i \in \{0,1\}$$

is to be solvable by a cubic polynomial (for arbitrary c_{ij})? (HINT: Write the constraints as a system of linear equations and find its Vandermonde matrix.)

Assume we have cubic polynomial

$$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 .$$

Our constraints are

$$p(x_0) = a_0 + a_1 x_0 + a_2 x_0^2 + a_3 x_0^3 = A$$

$$p(x_1) = a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^3 = B$$

$$2a_2 + 6a_3 x_0 = C$$

$$2a_2 + 6a_3 x_1 = D$$

and the Vandermonde Matrix for this system is

$$\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 0 & 0 & 2 & 6x_0 \\ 0 & 0 & 2 & 6x_1 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \\ D \end{bmatrix},$$

We need the leftmost matrix (denote M) to be invertible so that we can solve for each coefficient a_0, a_1, a_2, a_3. To show that M is invertible, we will show that its determinant is nonzero.

Looking at M, we see that it is a block matrix that can be partitioned as $\left[\begin{array}{c|c} M_1 & M_2 \\ \hline 0 & M_3 \end{array}\right]$. By another theorem, the determinant of this matrix is

$$\det \left[\begin{array}{c|c} M_1 & M_2 \\ \hline 0 & M_3 \end{array}\right] = \det(M_1) \cdot \det(M_3)$$

$$\Rightarrow \det(M) = (x_1 - x_0) \cdot (12x_1 - 12x_0)$$

$$= 12(x_1 - x_0)^2 .$$

Now, the value of $12(x_1 - x_0)^2$ is always nonzero if and only if $x_1 \neq x_0$. Hence, the condition that must be placed on the nodes x_0, x_1 is that $x_0 \neq x_1$ (ie., x_0 and x_1 are distinct).

Math 170B: Homework 3
(Due date: Friday, Feb. 7, 11:59 pm)

1. (+4 pts) Starting with the initialization of $(0, 1)$ perform two iterations of Newton's method (without computer) on the system of non-linear equations:

$$4x_1^2 - x_2^2 = 0$$
$$4x_1x_2^2 - x_1 = 1$$

2. (+4 pts) Suppose $\{x_n\}_{n=0}^{\infty}$ is a sequence generated by the secant method on the function $f \in \mathcal{C}^2(\mathbb{R})$ and $\lim_{n\to\infty} x_n = p$. Show that if $f'(p) \neq 0$ then p is a zero of the function f.

3. (+4 pts) Let $f(x) = -x^3 - \cos(x)$ where $x \in \mathbb{R}$. Implement the secant method on MATLAB with an initialization of $x_0 = -1$, $x_1 = 0$, maximum iterations = 30 and a stopping criteria of $\epsilon = \delta = 10^{-8}$. Output your final iterate. Your submission must have the code as well as the plot of x_n as a function of n.

4. (+8 pts) Answer the following questions on fixed point iteration:

   (a) (+4 pts) Let $\{x_n\}_{n=0}^{\infty}$ be a real valued sequence that satisfies

   $$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n} \quad \text{for all } n \geq 0$$

   with $x_0 > 0$. Use the contractive mapping theorem to show that $\{x_n\}_{n=0}^{\infty}$ converges to $\sqrt{2}$ whenever $x_0 > \sqrt{2}$.

   (b) (+4 pts) Show that $g(x) = \pi + 0.5\sin(\frac{x}{2})$ has a unique fixed point on the interval $[0, 2\pi]$. Does the fixed point iteration for $g(x)$ converge if it starts at $x_0 = 1$? Also, if the fixed point iteration converges then what is the order of convergence?

HW 3 SABELLA SAULINO

1) $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

We have $4x_1^2 - x_2^2 = 0$, $x^{(0)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$4x_1 x_2^2 - x_1 = 1$,

$\Rightarrow f_1(x_1, x_2) = 4x_1^2 - x_2^2 = 0$

$\Rightarrow f_2(x_1, x_2) - 4x_1 x_2^2 - x_1 - 1 = 0$

Using Newton's Method,

$$x^{(k+1)} = x^{(k)} - \left[F'(x^{(k)})\right]^{-1} F(x^{(k)})$$

$$F'(x^{(k)}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 8x_1 & -2x_2 \\ 4x_2^2 - 1 & 8x_1 x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{bmatrix} = \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \end{bmatrix} - \left(\begin{bmatrix} 8x_1^{(0)} & -2x_2^{(0)} \\ 4x_2^{(0)^2} - 1 & 8x_1^{(0)} x_2^{(0)} \end{bmatrix}\right)^{-1} \begin{bmatrix} 4x_1^{(0)^2} - x_2^{(0)^2} \\ 4x_1^{(0)} x_2^{(0)^2} - x_1 - 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 8(0) & -2(1) \\ 4(1)^2 - 1 & 8(0)(1) \end{bmatrix}\right)^{-1} \begin{bmatrix} 4(0)^2 - (1)^2 \\ 4(0)(1)^2 - 0 - 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \left(\frac{1}{0+6} \begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix} \right) \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 0 & 1/3 \\ -1/2 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -1/3 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/2 \end{bmatrix}$$

$$\Rightarrow x^{(1)} = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/2 \end{bmatrix}$$

2nd iteration: $x^{(k+1)} = x^{(k)} - [F'(x^{(k)})]^{-1} F(x^{(k)})$

$$\begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \end{bmatrix} = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{bmatrix} = \left(\begin{bmatrix} 8x_1^{(1)} & -2x_2^{(1)} \\ 4x_2^{(1)^2} - 1 & 8x_1^{(1)} x_2^{(1)} \end{bmatrix} \right)^{-1} \begin{bmatrix} 4x_1^{(1)2} - x_2^{(1)^2} \\ 4x_1^{(1)} x_2^{(1)^2} - x_1^{(1)} - 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 \\ 1/2 \end{bmatrix} - \left(\begin{bmatrix} 8(1/3) & -2(1/2) \\ 4(1/2)^2 - 1 & 8(1/3)(1/2) \end{bmatrix} \right)^{-1} \begin{bmatrix} 4(1/3)^2 - (1/2)^2 \\ 4(1/3)(1/2)^2 - (1/3) - 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 \\ 1/2 \end{bmatrix} - \left(\frac{1}{(32/9 - (0))} \begin{bmatrix} 4/3 & 1 \\ 0 & 8/3 \end{bmatrix} \right) \begin{bmatrix} 7/36 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 \\ 1/2 \end{bmatrix} - \frac{9}{32} \begin{bmatrix} 4/3 & 1 \\ 0 & 8/3 \end{bmatrix} \begin{bmatrix} 7/36 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 \\ 1/2 \end{bmatrix} - \begin{bmatrix} 3/8 & 9/32 \\ 0 & 3/4 \end{bmatrix} \begin{bmatrix} 7/36 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 \\ 1/2 \end{bmatrix} - \begin{bmatrix} 7/97 - 9/32 \\ 0 - 3/4 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 \\ 1/2 \end{bmatrix} - \begin{bmatrix} -5/24 \\ -3/4 \end{bmatrix} = \begin{bmatrix} 13/24 \\ 5/4 \end{bmatrix}$$

$$\Rightarrow x^{(2)} = \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \end{bmatrix} = \begin{bmatrix} 13/24 \\ 5/4 \end{bmatrix} \approx \begin{bmatrix} 0.54 \\ 1.25 \end{bmatrix}$$

2) $$x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}, \quad \lim_{n\to\infty} x_n = p$$

Take limit on both sides

$$\lim_{n\to\infty} (x_{n+1}) = \lim_{n\to\infty} \left(x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} \right)$$

As $n \to \infty$, $\quad x_{n-1} = x_n = x_{n+1} = p$

$$\Rightarrow \quad p = p - \lim_{n\to\infty} \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$$\Rightarrow \lim_{n\to\infty} \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})} = 0$$

Note: Taylor expansion of $F(a) - F(b) = F'(\xi)(a-b)$, $\xi \in [a,b]$

$$\Rightarrow f(x_n) - f(x_{n-1}) = f'(\xi)(x_n - x_{n-1}), \quad \xi \in [x_n, x_{n-1}]$$

$$\Rightarrow \lim_{n\to\infty} \frac{f(x_n)\cancel{(x_n - x_{n-1})}}{f'(\xi)\cancel{(x_n - x_{n-1})}} = 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{f(x_n)}{f'(\xi)} = 0 \quad \left(\text{plug in } \lim_{n\to\infty} x_n = p \ \& \ \xi \in [x_n, x_{n-1}]\right)$$

$$\Rightarrow \lim_{n\to\infty} \frac{f(p)}{f'(p)} = 0 \quad (f'(p) \neq 0)$$

$$\Rightarrow f(p) = 0 \quad \Rightarrow p \text{ is a zero of } f$$

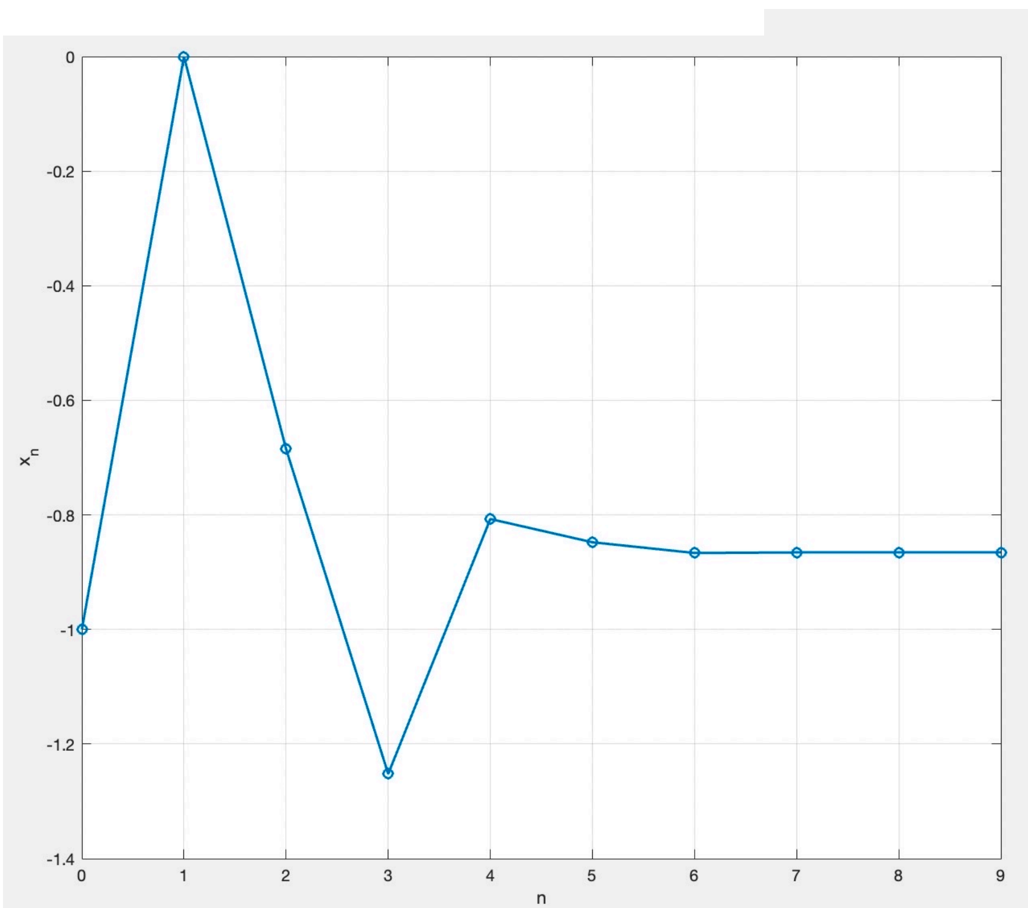
3)

```
f = @(x) -x.^3-cos(x);
x0 = -1;
x1 = 0;
max_it = 30;
eps = 10^-8;
delta = 10^-8;
x_vals = [x0, x1];
f_vals = [f(x0), f(x1)];

for n = 2:max_it
    xn = x1 - f(x1) * (x1 - x0) / (f(x1) - f(x0));
    x_vals = [x_vals, xn];
    f_vals = [f_vals, f(xn)];

    if abs(xn - x1) < eps || abs(f(xn)) < delta
        fprintf('Final iterate: x = %.8f \n', xn);
        fprintf('Number of iterations: %d \n', n)
        break;
    end
    x0 = x1;
    x1 = xn;
end


figure;
plot(0:length(x_vals)-1, x_vals, '-o', 'LineWidth', 1.5);
xlabel('n');
ylabel('x_n');
grid on;
```



4) (a) $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n} \quad \forall n \geq 0 \quad , x_0 > 0$

$$\Rightarrow F(x) = \frac{x}{2} + \frac{1}{x} = x$$

$$\Rightarrow \frac{1}{x} = \frac{x}{2}$$

$$\Rightarrow 1 = \frac{1}{2}x^2 \Rightarrow 2 = x^2 \Rightarrow x = \sqrt{2}$$

$\sqrt{2}$ is the fixed pt of F

WTS: $|F(x) - F(y)| \leq \lambda |x-y|, \quad 0 < \lambda < 1$

$$|F(x) - F(y)| = \left|\left(\frac{x}{2} + \frac{1}{x}\right) - \left(\frac{y}{2} + \frac{1}{y}\right)\right|$$

$$= \cancel{\left|\frac{1}{2}(x-y) + \frac{1}{x} - \frac{1}{y}\right|}$$

let's use MVT: $|F(x) - F(y)| = |F'(c)||x-y|, c \in [x,y]$

$$F'(x) = \frac{1}{2} + \frac{1}{x^2}$$

when $x_0 > \sqrt{2} \Rightarrow x_0^2 > 2 \Rightarrow \frac{1}{x_0^2} < \frac{1}{2}$

$$\Rightarrow |F'(x)| = \left|\frac{1}{2} + \frac{1}{x^2}\right| > 0$$

$$\Rightarrow |F'(c)||x-y| = \left(\frac{1}{2+c^2}\right)|x-y|, \quad c \in [x,y], c > \sqrt{2}$$

$$\Rightarrow |F(x) - F(y)| \leq \frac{1}{(2+c^2)}|x-y| \leq \frac{1}{2}|x-y|, \lambda = \frac{1}{2}$$

$\Rightarrow$ $\{x_n\}$ converges to $\sqrt{2}$ by thm, when $x_0 > \sqrt{2}$

(b) $g(x) = \pi + \frac{1}{2}\sin\left(\frac{x}{2}\right) = x$, $x \in [0, 2\pi]$

$g'(x) = \frac{1}{4}\cos\left(\frac{x}{2}\right)$

we know that $|\cos(\xi)| \le 1 \quad \forall\, \xi$

$\Rightarrow$ $|g'(x)| \le \frac{1}{4}$

Math 170B: Homework 2
(Due date: Friday, Jan. 24, 11:59 pm)

1. (+4 pts) Let $\{x_n\}_{n=0}^{\infty}$ be a real valued sequence that satisfies

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{x_n}\right) \quad \text{for all } n \geq 0$$

with $x_0 = 2$.

- Using induction hypothesis, show that $x_n \geq 1$ for all $n \geq 0$.
- Using the previous part, show that $\{x_n\}_{n=0}^{\infty}$ is a monotonically decreasing sequence. (HINT: Use the facts that $x_n \geq 1$ for all $n \geq 0$ from previous part and x_{n+1} is the mid-point of $\frac{1}{x_n}$ and x_n for all $n \geq 0$.)
- Use the monotone convergence theorem (stated below) to show that the sequence $\{x_n\}_{n=0}^{\infty}$ converges and $\lim_{n\to\infty} x_n = 1$.
- Show that the sequence $\{x_n\}_{n=0}^{\infty}$ converges quadratically to 1, i.e.,

$$|x_{n+1} - 1| \leq C|x_n - 1|^2 \quad \text{for all } n \geq 0$$

for some constant $C \geq 0$.

(**Monotone convergence theorem**: A sequence $\{x_n\}_{n=0}^{\infty}$ that is monotonically non-increasing (or monotonically non-decreasing) and is bounded below (or bounded above respectively) converges to a limit point.)

2. (+4 pts) Let $f(x) = \cos(\pi x)$ for $x \in \mathbb{R}$. Use the bisection method (without computer) on the interval $[-2, 3.1]$ and determine the root r of $f(x)$ that the sequence $\{x_n\}_{n=0}^{\infty}$ generated by the bisection method converges to. Also find an upper bound on the number of iterations required to reach within an accuracy of 10^{-8} of this root.

3. (+4 pts) Let $\{x_n\}_{n=0}^{\infty}$ be a real valued sequence defined as follows:

$$x_{n+1} = x_n - \frac{\tan(x_n) - 1}{\sec^2(x_n)} \quad \text{for all } n \geq 0$$

with $x_0 = \frac{\pi}{3}$.

- Relate the given sequence to the Newton's method for some function $f(x)$.
- Given the sequence converges, find $\lim_{n\to\infty} x_n$.
- Now suppose the initialization $x_0 \in \mathbb{R}$ is arbitrary. Can the global convergence theorem for Newton's method be applied here to claim convergence of the sequence $\{x_n\}_{n=0}^{\infty}$? Justify your answer.

4. (+8 pts) Use MATLAB to find $\sqrt[3]{2}$ by the following two methods:

- Implement the bisection method on the interval $[1, 2]$ with a stopping criteria of $\epsilon = \delta = 10^{-8}$ and maximum iterations $= 30$. Plot the sequence of mid-points $\{c_n\}$ generated from the method as a function of n.
- Implement the Newton's method with $x_0 = 2$, a stopping criteria of $\epsilon = \delta = 10^{-8}$ and maximum iterations $= 30$. Plot the sequence $\{x_n\}$ generated from the method as a function of n.

Note that your submission must have the MATLAB codes from the two methods along with the respective plots. State your inference by comparing the two plots (HINT: In both methods, we are solving for the real root of the function $f(x) = x^3 - 2$).

1) Let $\{x_n\}_{n=0}^{\infty}$ be a real valued sequence that satisfies

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{x_n}\right) \text{ for all } n \geq 0$$

with $x_0 = 2$.

- Using induction hypothesis, show that $x_n \geq 1$ for all $n \geq 0$.

<u>Proof</u>. We will proceed using induction.

<u>Base case</u>: When $n=0$, $x_n = x_0 = 2 > 1$. Thus, the base case holds.

<u>Inductive Hypothesis</u>: Assume that for all $n=k$, $x_k \geq 1$ for all $k \geq 0$.

<u>Inductive step</u>: We have that $x_k \geq 1$, and thus by rearranging the term x_k we have $\frac{1}{x_k} \leq 1$. Since the minimum value of x_k is 1, we have

$$x_k + \frac{1}{x_k} \geq 1 + \frac{1}{1} = 2 .$$

Plugging this into our given formula, we have

$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{1}{x_k}\right)$$

$$\Rightarrow \quad x_{k+1} \geq \frac{1}{2}(2) \quad \Rightarrow x_{k+1} \geq 1 .$$

Hence, for all $k \geq 0$, if $x_k \geq 1$, then $x_{k+1} \geq 1$. Therefore, $x_n \geq 1$ for all $n \geq 0$.

- Using the previous part, show that $\{x_n\}_{n=0}^{\infty}$ is a monotonically decreasing sequence. (HINT: Use the facts that $x_n \geq 1$ for all $n \geq 0$ from previous part and x_{n+1} is the midpoint of $\frac{1}{x_n}$ and x_n for all $n \geq 0$.)

WTS: $x_{n+1} \leq x_n$ for all $n \geq 0$

From the previous part, we have

$$x_n \geq 1 \implies \frac{1}{x_n} \leq 1 \quad \text{for all } n \geq 0$$

This implies that $\frac{1}{x_n} \leq x_n$.

Now, from our given formula for x_{n+1}, we can deduce that x_{n+1} is the midpoint of $\frac{1}{x_n}$ and x_n for all $n \geq 0$. Since $\frac{1}{x_n} \leq x_n$, the value of x_{n+1} cannot be greater than x_n or less than $\frac{1}{x_n}$, but it can be equal to either when $x_n = 1$. Thus,

$$\frac{1}{x_n} \leq x_{n+1} \leq x_n$$

$$\implies x_{n+1} \leq x_n .$$

Therefore, $\{x_n\}_{n=0}^{\infty}$ is a monotonically decreasing sequence. Note that since $x_n \geq 1$, the bound for the decreasing sequence is 1.

- Use the monotone convergence theorem (stated below) to show that the sequence $\{x_n\}_{n=0}^{\infty}$ converges and $\lim_{n \to \infty} x_n = 1$.

MCT: A sequence $\{x_n\}_{n=0}^{\infty}$ that is monotonically non-increasing (or monotonically non-decreasing) and is bounded below (or bounded above respectively) converges to a limit point.

In the previous part, we found that $\{x_n\}_{n=0}^{\infty}$ is monotonically non-increasing and is bounded below by 1. Hence, $\{x_n\}_{n=0}^{\infty}$ converges to a limit point. We have

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{1}{x_n}\right) .$$

As we let $n \to \infty$, adding one to n becomes negligible. So, let $n \to \infty$, and suppose the sequence converges to x^* for some $x^* \geq 1$. Then

$$x^* = \frac{1}{2}\left(x^* + \frac{1}{x^*}\right)$$

$$\Rightarrow 2x^* = x^* + \frac{1}{x^*}$$

$$\Rightarrow x^* = \frac{1}{x^*}$$

$$\Rightarrow (x^*)^2 = 1$$

$$\Rightarrow x^* = \pm 1$$

Since $x^* \geq 1$, the solution is $\lim_{n \to \infty} x_n = 1$.

- Show that the sequence $\{x_n\}_{n=0}^{\infty}$ converges quadratically to 1, ie.,

$$|x_{n+1} - 1| \leq C|x_n - 1|^2 \text{ for all } n \geq 0$$

for some constant $C \geq 0$.

We have $|x_{n+1} - 1| = \left| \frac{1}{2}\left(x_n + \frac{1}{x_n}\right) - 1 \right|$

$$= \left| \frac{1}{2}\left(x_n + \frac{1}{x_n} - 2\right) \right| .$$

Claim: $\left(x_n + \frac{1}{x_n} - 2\right) \leq (x_n - 1)^2$.

Proof. Start by expanding $(x_n - 1)^2$ to $x_n^2 - 2x_n + 1$.

Then we have

$$x_n + \frac{1}{x_n} - 2 \leq x_n^2 - 2x_n + 1$$

multiply by x_n on both sides,

$$x_n^2 + 1 - 2x_n \leq x_n^3 - 2x_n^2 + x_n$$

and rearrange the inequality:

$$x_n^3 - 3x_n^2 + 3x_n + 1 \geq 0$$

$$\Rightarrow (x_n - 1)^3 \geq 0$$

$$\Rightarrow x_n - 1 \geq 0 \quad \Rightarrow x_n \geq 1 .$$

We already have the criterion $x_n \geq 1$, hence the claim is true. Thus,

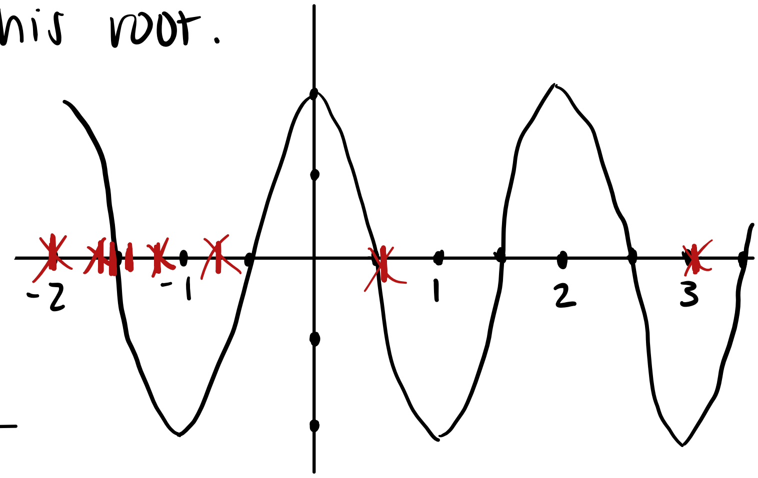
$$\left| \frac{1}{2}\left(x_n + \frac{1}{x_n} - 2\right) \right| \leq \left| \frac{1}{2}(x_n - 1)^2 \right|$$

$$\Rightarrow |x_{n+1} - 1| \leq \frac{1}{2} |x_n - 1|^2 \quad \text{for all } n \geq 0.$$

Therefore the sequence converges quadratically to 1.

2) let $f(x) = \cos(\pi x)$ for $x \in \mathbb{R}$. Use the bisection method (without computer) on the interval $[-2, 3.1]$ and determine the root r of $f(x)$ that the sequence $\{x_n\}_{n=0}^{\infty}$ generated by the bisection method converges to. Also find an upper bound on the number of iterations required to reach within an accuracy of 10^{-8} of this root.

Find a zero of $f(x) = \cos(\pi x)$ in $[-2, 3.1]$ within 10^{-8}



	a	b	c	f(c)
n = 1	-2	3.1	0.55	-0.156
n = 2	-2	0.55	-0.725	-0.649
n = 3	-2	-0.725	-1.3625	-0.418
n = 4	-2	-1.3625	-1.68125	0.53913
n = 5	-1.68125	-1.3625	-1.522	0.0687
n = 6	-1.522	-1.3625	-1.442	-0.1804
n = 7	-1.522	-1.442	-1.4821	-0.0563
n = 8	-1.522	-1.4821	-1.502	0.00618

$$\frac{3.1 - 2}{2} = 0.55$$

$f(a)f(c) < 0 \Rightarrow (-2, 0.55)$

$$\frac{0.55 - 2}{2} = -0.725$$

$f(a)f(c) < 0 \Rightarrow (-2, -0.725)$

$$\frac{-0.725 - 2}{2} = -1.3625$$

$f(a)f(c) < 0 \Rightarrow (-2, -1.3625)$

$$\frac{-1.3625 - 2}{2} = -1.68125$$

$f(b)f(c) < 0 \Rightarrow (-1.68125, -1.3625)$

$$\frac{-1.3625 - 1.68125}{2} = -1.522$$

$f(b)f(c) < 0 \Rightarrow (-1.522, -1.3625)$

$$\frac{-1.522 - 1.3625}{2} = -1.44225$$

$f(a)f(c) < 0 \Rightarrow (-1.522, -1.442)$

$$\frac{-1.522 - 1.44225}{2} = -1.4821$$

$f(a)f(c) < 0 \Rightarrow (-1.522, -1.4821)$

$$\frac{-1.522 - 1.4821}{2} = -1.502$$

$r = -1.5$, $f(-1.5) = \cos(-1.5\pi) = 0$

$$\frac{|3.1 - 2|}{2^{N+1}} < 10^{-8} \quad \text{solve for } 2^{N+1}$$

$\Rightarrow 0.55 \times 10^8 < 2^{N+1} \Rightarrow N + 1 = \log_2(0.55 \times 10^8)$

$\Rightarrow N + 1 \approx 25.7129 \Rightarrow n = 25$.

3) Let $\{x_n\}_{n=0}^{\infty}$ be a real-valued sequence defined as follows:

$$x_{n+1} = x_n - \frac{\tan(x_n) - 1}{\sec^2(x_n)} \quad \text{for all } n \geq 0$$

with $x_0 = \frac{\pi}{3}$.

- Relate the given sequence to Newton's Method for some function $f(x)$.

Formula for Newton's Method: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Hence $f(x) = \tan(x) - 1$, $f'(x) = \sec^2(x)$

- Given the sequence converges, find $\lim_{n\to\infty} x_n$.

Note that as $n \to \infty$, $x_{n+1} = x_n$.

Suppose $\lim_{n\to\infty} x_n = L$. Then taking the limit of both sides of

$$x_{n+1} = x_n - \frac{\tan(x_n) - 1}{\sec^2(x_n)},$$

we have

$$L = L - \frac{\tan(L) - 1}{\sec^2(L)}$$

$$\Rightarrow \frac{\tan(L) - 1}{\sec^2(L)} = 0$$

$$\Rightarrow \tan(L) - 1 =$$

$$\Rightarrow \tan(L) = 1 \quad \Rightarrow L = \frac{\pi}{4} + c\pi$$

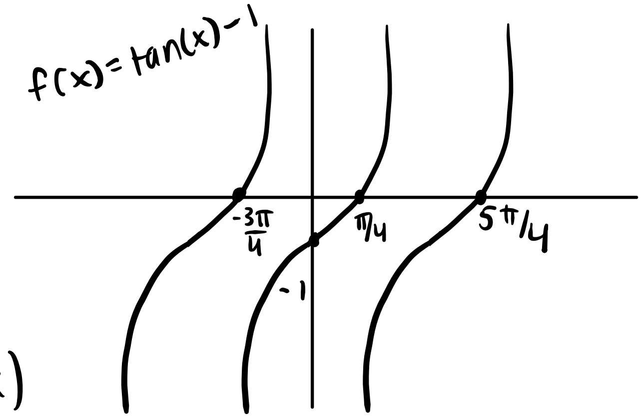
$$x_0 = \frac{\pi}{3} \Rightarrow \lim_{n\to\infty} x_n = \frac{\pi}{4}$$

• Now suppose the initialization $x_0 \in \mathbb{R}$ is arbitrary. Can the global convergence theorem for Newton's method be applied here to claim convergence of the sequence $\{x_n\}_{n=0}^{\infty}$? Justify your answer.

$$f(x) = \tan(x) - 1$$

$$f'(x) = \sec^2(x)$$

$$f''(x) = 2\sec^2(x)\tan(x)$$



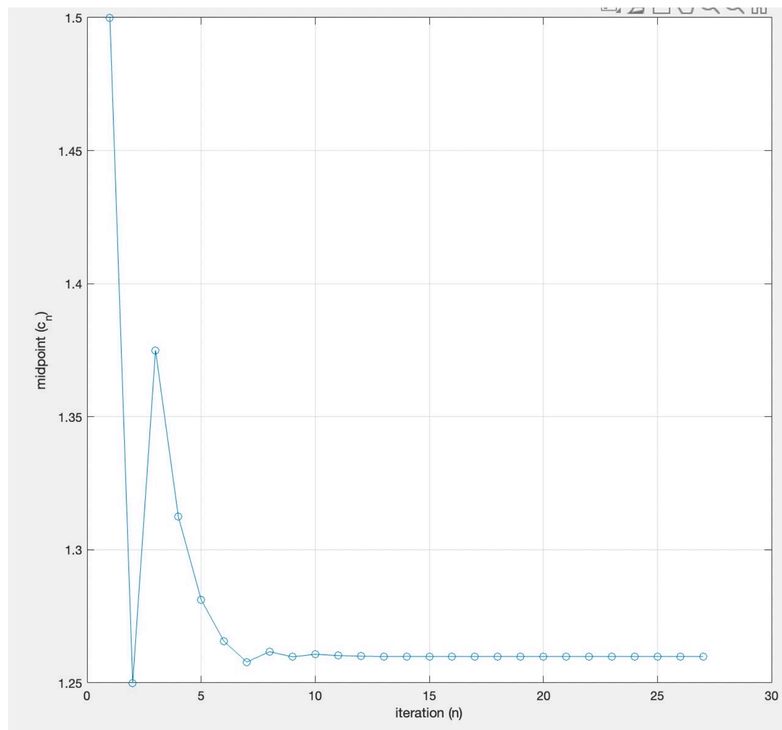
vertical asympotes at $\pi/2$, $3\pi/2$, $5\pi/2$, etc

In order to apply the theorem, $f(x)$ must be globally continuous, but there are asymptotes at $x = \frac{\pi}{2} + c\pi$ where the function is not continuous. Hence, the global convergence theorem for Newton's method cannot be applied to claim convergence of the sequence.

4) Use MATLAB to find $\sqrt[3]{2}$ by the following two methods:

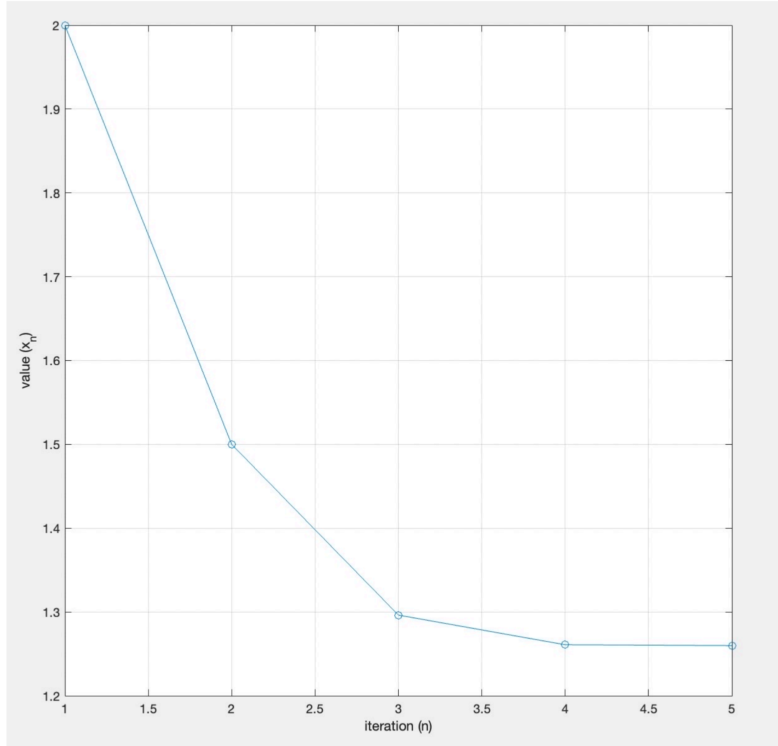
- Implement the bisection method on the interval $[1,2]$ with a stopping criteria of $\varepsilon = \delta = 10^{-8}$ and maximum iterations $= 30$. Plot the sequence of mid-points $\{c_n\}$ generated from the method as a function of n.

```
function c = bisectionsearch(a, b, M, delta, eps)
f = @(x) x^3-2;
u = f(a);
v = f(b);
e = b-a;
g = u*v;
if (g > 0)
  error('IVT cannot be applied');
end
midpoints = [];
    for n = 1 : M
        e = e/2;
        c = a+e;
        w = f(c);
        midpoints = [midpoints, c];
        if abs(e) < delta || abs(w) < eps
            break;
        end
        if sign(w) ~= sign(u)
            b = c;
            v = w;
        else
            a = c;
            u = w;
        end
    end
    figure;
    plot(1:length(midpoints), midpoints, '-o');
    xlabel('iteration (n)');
    ylabel('midpoint (c_n)');
    grid on;
end
```



- Implement the Newton's Method with $x_0 = 2$, a stopping criteria of $\varepsilon = \delta = 10^{-8}$ and maximum iterations $= 30$. Plot the sequence $\{x_n\}$ generated from the method as a function of n.

```
function x = newtonsearch(x0, M, delta, eps)
f = @(x) x^3-2;
y = f(x0);
if abs(y) < eps
    error('root found')
end
g = @(x) 3*x^2;
xn = [];
for n = 1 : M
    xn = [xn, x0];
    x = x0 - f(x0)/g(x0);
    y = f(x);
    if abs(x - x0) < delta || abs(y) < eps
        break;
    else
        x0 = x;
    end
end
figure;
    plot(1:length(xn), xn, '-o');
    xlabel('iteration (n)');
    ylabel('value (x_n)');
    grid on;
end
```



Note that your submission must have the MATLAB codes from the two methods along with the respective plots. State your inference by comparing the two plots (HINT: In both methods, we are solving for the real root of the function $f(x) = x^3 - 2$).

By comparing the two plots, we can see that Newton's method took a significantly smaller number of iterations to approximate the root within the stopping criteria. I can infer that Newton's method converges faster that Newton's method converges faster than the bisection method.