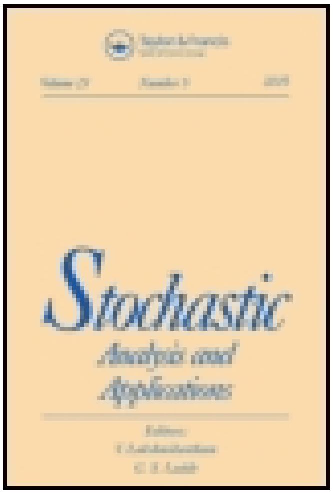

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STOCHASTIC NAGUMO'S VIABILITY THEOREM

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ABSTRACT

This paper is devoted to viability of random set-valued variables by stochastic differential equations, characterized in terms of stochastic tangent sets to random set-valued variables.

1 INTRODUCTION

The main aim of this paper is to extend to the stochastic case Nagumo's Theorem on viability properties of closed subsets with respect to a differential equation. In [2, Aubin & Da Prato], only invariance theorems were presented : we proved that under adequate stochastic tangential conditions, any solution to the stochastic differential equation starting from $\mathcal{K}$, if any, remains in $\mathcal{K}$, whereas in viability theorems, we prove the existence of one solution which is viable in $\mathcal{K}$.

Let us consider a closed subset K of $X := \mathbf{R}^n$ and a stochastic differential equation

$$d\xi = f(\xi(t))dt + g(\xi(t))dW(t)$$

the solution of which is given by the formula

$$\xi(t) = \xi(0) + \int_0^t f(\xi(s))ds + \int_0^t g(\xi(s))dW(s)$$

when one of the following conditions is satisfied:

- f and g are Lipschitz functions
- f and g are uniformly continuous and monotone

We want to characterize the (stochastic) viability property of K with respect to the pair (f, g): for any random variable x in K, there exists a solution ξ to the stochastic differential equation starting at x which is viable in K, in the sense that

$$\forall\, t \in [0, T], \text{ for almost all } \omega \in \Omega, \ \xi_\omega(t) \in K_\omega$$

The construction of approximate viable solutions we provide when f and g are uniformly continuous is interesting for its own sake, and may be useful in other problems.

For that purpose, we use the concept of contingent set to a subset introduced in [2, Aubin & Da Prato]. Let us consider a $\mathcal{F}_t$-random variable $x \in K$.

We define the stochastic contingent set $\mathcal{T}_K(t, x)$ to K at x (with respect to $\mathcal{F}_t$) as the set of pairs (γ, v) of $\mathcal{F}_t$-random variables satisfying the following property: There exist sequences of $h_n > 0$ converging to 0 and of $\mathcal{F}_{t+h_n}$-measurable random variables a^n and b^n such that

$$\begin{cases} i) & \mathbf{E}(\|a^n\|^2) \to 0 \\ ii) & \mathbf{E}(\|b^n\|^2) \to 0 \\ iii) & \mathbf{E}(b^n) = 0 \\ iv) & b^n \text{ is independent of } \mathcal{F}_t \end{cases}$$

and satisfying for almost all $\omega \in \Omega$,

$$\forall\, n \ge 0, \ x_\omega + \gamma_\omega h_n + v_\omega(W_\omega(t + h_n) - W(t)) + h_n a^n_\omega + \sqrt{h_n} b^n_\omega \in K_\omega$$

Then we shall prove in essence that the following conditions are equivalent:

1. — The subset K enjoys the viability property with respect to the pair (f, g)

2. — for every $\mathcal{F}_t$-random variable x viable in K,

$$(f(x), g(x)) \in \mathcal{T}_K(t,x)$$

For instance, this condition means that for every $\mathcal{F}_t$-random variable x viable in K

- $$f(x) \in K \ \& \ g(x) \in K$$

  when K is a vector subspace,

- $$\langle x, g(x)\rangle = 0 \ \& \ \langle x, f(x)\rangle + \frac{1}{2}\|g(x)\|^2 = 0$$

  when K is the unit sphere

- $$\langle x, g(x)\rangle = 0 \ \& \ \langle x, f(x)\rangle + \frac{1}{2}\|g(x)\|^2 \leq 0$$

  when K is the unit ball.

We mention that an elementary calculus of stochastic tangent sets to direct images, inverse images and intersections of closed subsets can be found in [2, Aubin & Da Prato].

2 STOCHASTIC TANGENT SETS

Let us consider a complete probability space $(\Omega, \mathcal{F}, P)$, an increasing family of σ-sub- algebras $\mathcal{F}_t \subset \mathcal{F}$ and a finite dimensional vector-space $X := \mathbf{R}^n$. Moreover $W(t), t \geq 0$ is a real standard Brownian motion such that $W(t)$ is $\mathcal{F}_t$-measurable and $W(t+h) - W(t)$ is independent of $\mathcal{F}_t$ for any $h \geq 0$.

The constraints are defined by closed subsets $K_\omega \subset X$, where the set-valued map

$$K : \omega \in \Omega \mapsto K_\omega \subset X$$

is assumed to be $\mathcal{F}_0$- measurable (which can be regarded as a random set-valued variable).

We denote by $\mathcal{K}$ the subset

$$\mathcal{K} := \{u \in L^2(\Omega, \mathcal{F}, P) \mid \text{for almost all } \omega \in \Omega,\ u_\omega \in K_\omega\}$$

For simplicity, we restrict ourselves to scalar $\mathcal{F}_t$-Wiener processes $W(t)$.

Definition 2.1 (Stochastic Contingent Set) *Let us consider a $\mathcal{F}_t$-random variable $x \in K$ (i.e., a $\mathcal{F}_t$-measurable selection of K).*

We define the stochastic contingent set $\mathcal{T}_K(t,x)$ *to K at x (with respect to $\mathcal{F}_t$) as the set of pairs (γ, v) of $\mathcal{F}_t$-random variables satisfying the following property: For any $\alpha,\ \rho > 0$, there exist $h \in]0,\alpha[$ and $\mathcal{F}_{t+h}$-random variables a^h and b^h such that*

$$\begin{cases} i) & \mathbf{E}(\|a^h\|^2) \leq \rho^2 \\ ii) & \mathbf{E}(\|b^h\|^2) \leq \rho^2 \\ iii) & \mathbf{E}(b^h) \ = \ 0 \\ iv) & b^h \text{ is independent of } \mathcal{F}_t \end{cases} \tag{2.1}$$

and satisfying

$$x + v(W(t+h) - W(t)) + h\gamma + ha^h + \sqrt{h}b^h \ \in \ \mathcal{K} \tag{2.2}$$

We consider the stochastic differential equation

$$d\xi \ = \ f(\xi(t))dt + g(\xi(t))dW(t) \tag{2.3}$$

where f and g are Lipschitz.

We say that a stochastic process $\xi(t)$ defined by

$$\xi(t) \ = \ \xi(0) + \int_0^t f(\xi(s))ds + \int_0^t g(\xi(s))dW(s) \tag{2.4}$$

is a solution to the stochastic differential equation (2.3) if the functions f and g satisfy:

$$\text{for almost all } \omega \in \Omega,\ \ f(\xi(\cdot)) \in L^1(0,T;X) \ \ \& \ \ g(\xi(\cdot)) \in L^2(0,T;X)$$

Definition 2.2 *We shall say that a stochastic process $x(\cdot)$ is* viable in *K if and only if*

$$\forall\, t \in [0,T],\ \ x(t) \ \in \ \mathcal{K} \tag{2.5}$$

i.e., if and only if

$$\forall\, t \in [0,T],\ \text{ for almost all } \omega \in \Omega,\ \ \xi_\omega(t) \ \in \ K_\omega$$

We shall say that K enjoys the (stochastic) viability property *with respect to the pair (f,g) if for any random variable x in K, there exists a solution ξ to the stochastic differential equation starting at x which is* viable in *K.*

3 STOCHASTIC VIABILITY

Theorem 3.1 (Stochastic Viability) *Let K be a closed subset of X. We assume that either*

- *the maps f and g are Lipschitz*
- *the maps f and g are uniformly continuous and monotone in the sense that there exists $\nu \in \mathbf{R}$ such that*

$$\forall\, x, y \in X,\ \ 2\langle f(x) - f(y), x - y\rangle + \|g(x) - g(y)\|^2 \leq \nu \|x - y\|^2,$$

Then the following conditions are equivalent:

1. — From any initial stochastic process $\xi_0 \in \mathcal{K}$ starts a solution to the stochastic differential equation which is viable in $\mathcal{K}$.

2. — for every $\mathcal{F}_t$-random variable x in $\mathcal{K}$,

$$(f(x), g(x)) \in \mathcal{T}_{\mathcal{K}}(t, x) \tag{3.1}$$

We already proved in [2, Aubin & Da Prato] that the condition was necessary. It remains to prove that it is sufficient.

We begin by constructing approximate viable solutions to the stochastic differential equation.

Lemma 3.2 *Let K be a closed subset of X. We assume that the maps f and g are uniformly continuous. Then, for any $\varepsilon > 0$, the set $\mathcal{S}_\varepsilon(\xi_0)$ of stochastic processes $\xi(\cdot)$ on $[0, 1]$ satisfying $\xi(0) = \xi_0$ and*

$$\begin{cases} i) & \forall\, t \in [0,1],\ \mathbf{E}(\boldsymbol{d}^2(\xi(t), \mathcal{K})) \leq \varepsilon^2 \\ ii) & \forall\, t \in [0,1],\ \mathbf{E}\left\|\xi(t) - \xi(0) - \int_0^t f(\xi(s))ds - \int_0^t g(\xi(s))dW(s)\right\|^2 \leq \varepsilon^2 \end{cases} \tag{3.2}$$

is not empty.

Proof — Let us fix $\varepsilon > 0$. Since f and g are uniformly continuous with concave uniform continuity modulus[1] ω, we choose $\eta \in]0, \varepsilon]$ such that

$$\omega(\eta^2) \leq \frac{\varepsilon^2}{4}.$$

[1] Set $\omega(t) = \sup_{\|x-y\|^2 \leq t} \|f(x) - f(y)\|^2$. Then ω is a non decreasing, subadditive continuity modulus of f. One can check that the concave envelope of ω is still a uniform continuity .

We denote by $\mathcal{A}_\varepsilon(\xi_0)$ the set of pairs $(T_\xi, \xi(\cdot))$ where $T_\xi \in [0,1]$ and $\xi(\cdot)$ is a stochastic process satisfying $\xi(0) = \xi_0$ and

$$\begin{cases} i) & \forall\, t \in [0, T_\xi],\ \mathbf{E}\boldsymbol{d}^2(\xi(T_\xi), \mathcal{K}) \ \leq\ \eta^2 T_\xi \\ ii) & \forall\, t \in [0, T_\xi],\ \mathbf{E}\boldsymbol{d}^2(\xi(t), \mathcal{K}) \leq \eta^2 \\ iii) & \forall\, t \in [0, T_\xi],\ \mathbf{E}\left\|\xi(t) - \xi(0) - \int_0^t f(\xi(s))ds - \int_0^t g(\xi(s))dW(s)\right\|^2 \leq \varepsilon^2 \end{cases} \tag{3.3}$$

The set $\mathcal{A}_\varepsilon(\xi)$ is not empty: take $T_\xi = 0$ and $\xi(0) \equiv \xi_0$. It is an inductive set for the order relation

$$(T_{\xi_1}, \xi_1(\cdot)) \ \preceq\ (T_{\xi_2}, \xi_2(\cdot))$$

if and only if

$$T_{\xi_1} \ \leq\ T_{\xi_2} \ \ \&\ \ \xi_2(\cdot)|_{[0,T_{\xi_1}]} \ =\ \xi_1(\cdot)$$

Zorn's Lemma implies that there exists a maximal element $(T_\xi, \xi(\cdot)) \in \mathcal{A}_\varepsilon(\xi_0)$. The Lemma follows from the claim that for such a maximal element, we have $T_\xi = 1$. If not, we shall extend $\xi(\cdot)$ by a stochastic process $\widehat{\xi}(\cdot)$ on an interval $[T_\xi, S_\xi]$ where $S_\xi > T_\xi$, contradicting the maximal character of $(T_\xi, \xi(\cdot))$.

Since K_ω and $\xi_\omega(T_\xi)$ are $\mathcal{F}_{T_\xi}$ measurable, the projection map $\Pi_{K_\omega}(\xi_\omega(T_\xi))$ is also $\mathcal{F}_{T_\xi}$-measurable (see [3, Theorem 8.2.13, p. 317]). Then there exists a $\mathcal{F}_{T_\xi}$-measurable selection $y_\omega \in \Pi_{K_\omega}(\xi_\omega(T_\xi))$, which we call a projection of the random variable $\xi(T_\xi)$ onto the random set-valued variable $\mathcal{K}$. For simplicity, we set $x = \xi(T_\xi)$ and thus choose a projection $y \in \Pi_{\mathcal{K}}(x)$.

We take

$$\rho := \frac{\eta\sqrt{1 - T_\xi}}{2} > 0$$

and we set

$$c^2 \ :=\ \max(\mathbf{E}(\|f(y)\|^2), \mathbf{E}(\|g(y)\|^2)) \ <\ +\infty \tag{3.4}$$

We then introduce

$$\alpha \ :=\ \min\left(\eta, \frac{(1 - T_\xi)\eta^2}{\eta^2 + 4c^2}\right) \ >\ 0$$

which is positive whenever $T_\xi < 1$.

We know that $(f(y), g(y))$ belongs to the stochastic contingent set $\mathcal{T}_{\mathcal{K}}(T_x, y)$: There exist $h_x \in]0, \alpha]$ and $\mathcal{F}_{T_x + h_x}$-random variables a^{h_x} and b^{h_x} such that

$$\begin{cases} i) & \mathbf{E}(\|a^{h_x}\|^2) \leq \rho^2 \\ ii) & \mathbf{E}(\|b^{h_x}\|^2) \leq \rho^2 \\ iii) & \mathbf{E}(b^{h_x}) \ =\ 0 \\ iv) & b^{h_x} \ \text{is independent of } \mathcal{F}_t \end{cases} \tag{3.5}$$

and satisfying

$$y + g(y)(W(T_x + h_x) - W(T_x)) + h_x f(y) + h_x a^h + \sqrt{h_x} b^{h_x} \in \mathcal{K} \tag{3.6}$$

We then set $S_x := T_x + h_x > T_x$ and we define the stochastic process $\widehat{\xi}(t)$ on the interval $[T_x, S_x]$ by

$$\widehat{\xi}(t) \ := \ x + (t - T_x)f(y) + (W(t) - W(T_x))g(y)$$

Therefore, setting $h := t - T_x$,

$$d^2_{\mathcal{K}}(\widehat{\xi}(t)) - d^2_{\mathcal{K}}(\widehat{\xi}(T_x)) \ \leq \ \left\| x - y - ha^h - \sqrt{h} b^h \right\|^2 - \|x - y\|^2 =:$$

$$\|ha^h + \sqrt{h} b^h\|^2 - 2\left\langle x - y, ha^h \right\rangle - 2\left\langle x - y, \sqrt{h} b^h \right\rangle$$

We take the expectation in both sides of this inequality and estimate each term of the right hand-side. First, we use estimate

$$\mathbf{E}(\|ha^h + \sqrt{h} b^h\|^2) \ \leq \ 2h(h\mathbf{E}(\|a^h\|^2) + \mathbf{E}(\|b^h\|^2))$$

because

$$\mathbf{E}\left(\left\| \int_0^t \varphi(s)ds \right\|^2\right) \leq t \int_0^t \mathbf{E}(\|\varphi(s)\|^2)ds$$

and

$$\mathbf{E}\left(\left\| \int_0^t \varphi(s)dW(s) \right\|^2\right) = \int_0^t \mathbf{E}(\|\varphi(s)\|^2)ds$$

Next,

$$\mathbf{E}\left\langle x - y, a^h \right\rangle \ \leq \ \mathbf{E}\left(\|x - y\|^2\right)^{\frac{1}{2}} \left(\mathbf{E}\left(\left\|a^h\right\|^2\right)\right)^{\frac{1}{2}}$$

and we observe that

$$\mathbf{E}\left\langle x - y, \frac{1}{\sqrt{h}} b^h \right\rangle \ = \ 0$$

since b^h is independent of $x - y$ and $\mathbf{E}(b^h) = 0$.

We obtain, by the very choice of ρ,

$$\mathbf{E}(\boldsymbol{d}^2(\widehat{\xi}(S_x), \mathcal{K})) \ = \ \mathbf{E}(\boldsymbol{d}^2(\widehat{\xi}(T_x + h_x), \mathcal{K}))$$

$$\leq \ \mathbf{E}\boldsymbol{d}^2(\widehat{\xi}(T_x), \mathcal{K}) + 2h_x \mathbf{E}\left(\|x - y\|^2\right)^{\frac{1}{2}} \left(\mathbf{E}\left(\left\|a^{h_x}\right\|^2\right)\right)^{\frac{1}{2}}$$

$$+2h_x(h_x\mathbf{E}(\|a^{h_x}\|^2) + \mathbf{E}(\|b^{h_x}\|^2))$$

$$\leq \; \mathbf{E}\boldsymbol{d}^2(\widehat{\xi}(T_x), \mathcal{K}) + h_x \left(\mathbf{E}\,(\|x-y\|^2) + 3\mathbf{E}(\|a^{h_x}\|^2) + \mathbf{E}(\|b^{h_x}\|^2))\right)$$

$$\leq \; \eta^2 T_x + h_x(\eta^2 T_x + 4\rho^2) \;\leq\; \eta^2 T_x + h_x \eta^2 \;=\; \eta^2 S_x$$

by (3.3)i).

Hence $\widehat{\xi}(\cdot)$ satisfies (3.3)i) for S_x.

We observe also that for any $t \in [T_x, S_x]$,

$$\boldsymbol{d}^2_{\mathcal{K}}(\widehat{\xi}(t)) \;\leq\; \|\widehat{\xi}(t) - y\|^2$$

and that

$$\|\widehat{\xi}(t) - y\|^2 \;=\; \|x - y + (t-T_x)f(y) + (W(t) - W(T_x))g(y)\|^2$$

$$= \; \boldsymbol{d}^2_{\mathcal{K}}(x) + 2\langle x - y, (t-T_x)f(y) + (W(t) - W(T_x))g(y)\rangle$$

$$+\|(t-T_x)f(y) + (W(t) - W(T_x))g(y)\|^2$$

By taking the expectations, we obtain

$$\mathbf{E}(\|\widehat{\xi}(t) - y\|^2) - \mathbf{E}(\boldsymbol{d}^2_{\mathcal{K}}(\widehat{\xi}(T_x)))$$

$$\leq \; (t-T_x)(\mathbf{E}(\boldsymbol{d}^2_{\mathcal{K}}(\widehat{\xi}(T_x)) + (1+2(t-T_x))\mathbf{E}(\|f(y)\|^2) + \mathbf{E}(\|g(y)\|^2))$$

Therefore, since $\max(\mathbf{E}(\|f(y)\|^2), \mathbf{E}(\|g(y)\|^2)) = c^2$ by (3.4), we deduce that

$$\mathbf{E}\|\widehat{\xi}(t) - y\|^2 \;\leq\; \eta^2 T_x + (t-T_x)(\eta^2 T_x + 4c^2) \;\leq\; \eta^2 T_x + \alpha(\eta^2 + 4c^2) \;\leq\; \eta^2 \tag{3.7}$$

since, by the choice of α, we have $\alpha(\eta^2 + 4c^2) \leq (1-T_x)\eta^2$. Therefore,

$$\mathbf{E}(\boldsymbol{d}^2_{\mathcal{K}}(\widehat{\xi}(t))) \;\leq\; \mathbf{E}(\|\widehat{\xi}(t) - y\|^2) \;\leq\; \eta^2$$

Hence $\widehat{\xi}(\cdot)$ satisfies (3.3)ii) for S_x.

We also observe that

$$\mathbf{E}\left(\left\|\widehat{\xi}(t) - x - \int_{T_x}^t f(\widehat{\xi}(s))ds - \int_0^t g(\widehat{\xi}(s))dW(s)\right\|^2\right)$$

$$= \; \mathbf{E}\left(\left\|\int_{T_x}^t (f(y) - f(\widehat{\xi}(s)))ds + \int_{T_x}^t (g(y) - g(\widehat{\xi}(s)))ds\right\|^2\right)$$

$$\leq \; 2\Big(\mathbf{E}\left(\int_{T_x}^t \left\|f(y) - f(\widehat{\xi}(s))\right\|^2 ds\right) + \mathbf{E}\left(\int_{T_x}^t \left\|g(y) - g(\widehat{\xi}(s))\right\|^2 ds\right)\Big)$$

Since the functions f and g are uniformly continuous, we deduce from the concavity of the continuous modulus $\omega(\cdot)$ that

$$\mathbf{E}\left(\left\|\widehat{\xi}(t)-x-\int_{T_x}^t f(\widehat{\xi}(s))ds-\int_{T_x}^t g(\widehat{\xi}(s))dW(s)\right\|^2\right)$$

$$\leq 2\left(\mathbf{E}\left(\int_{T_x}^t \omega\left(\left\|y-\widehat{\xi}(s)\right\|^2\right)ds\right)+\mathbf{E}\left(\int_{T_x}^t \omega\left(\left\|y-\widehat{\xi}(s)\right\|^2\right)ds\right)\right)$$

$$\leq 4\mathbf{E}\left(\int_{T_x}^t \omega\left(\left\|y-\widehat{\xi}(s)\right\|^2\right)ds\right) \leq 4\omega\left(\int_{T_x}^t \mathbf{E}\left(\|y-\widehat{\xi}(s)\|^2\right)\right)$$

$$\leq 4\omega(\eta^2) \leq \varepsilon^2.$$

since we have already proved that

$$\mathbf{E}(\|\widehat{\xi}(t)-y\|^2) \leq \eta^2$$

so that $\widehat{\xi}(\cdot)$ satisfies (3.3)iii). Therefore, we have extended the maximal solution $(T_\xi, \xi(\cdot))$ on the interval $[0, S_x]$ and obtained the desired contradiction. Hence the proof of Lemma 3.2 is completed. ∎

It remains now to prove that the limit of the sequence of approximate solutions to a viable stochastic process exists and is a solution to the stochastic differential equation.

Let us choose for every ε an approximate solution ξ_ε which can be written in the form

$$\xi_\varepsilon(t) = \xi_0 + \int_0^t f(\xi_\varepsilon(s))ds + \int_0^t g(\xi_\varepsilon(s))dW(z) + \zeta_\varepsilon(t)$$

where $\sup_{t\in[0,1]} \mathbf{E}(\|\zeta_\varepsilon(t)\|^2) \leq \varepsilon^2$. Then for any $\varepsilon, \eta > 0$,

$$\mathbf{E}\,\|\xi_\varepsilon(t)-\xi_\eta(t)\|^2 \leq$$

$$\mathbf{E}\|\int_0^t(f(\xi_\varepsilon(s))-f(\xi_\eta(s)))ds+\int_0^t(g(\xi_\varepsilon(s))-g(\xi_\eta(s))dW(z)+\zeta_\varepsilon(t)-\zeta_\eta(t)\|^2$$

It follows that

Lipschitz case

$$\mathbf{E}\left(\|\xi_\varepsilon(t)-\xi_\eta(t)\|^2\right) \leq 4l^2\left(\int_0^t \mathbf{E}\left(\|\xi_\varepsilon(s)-\xi_\eta(s)\|^2\right)ds\right)+2(\varepsilon^2+\eta^2)$$

Gronwall's Lemma implies that

$$\mathbf{E}\left(\|\xi_\varepsilon(t) - \xi_\eta(t)\|^2\right) \;\leq\; 2(\varepsilon^2 + \eta^2)e^{4l^2 t}$$

Monotone case We use Ito formula for the function $\|\xi_\varepsilon - \xi_\eta\|^2$ to obtain

$$\mathbf{E}\,\|\xi_\varepsilon(t) - \xi_\eta(t)\|^2$$

$$\leq \mathbf{E}\int_0^t \left(\langle \xi_\varepsilon(s) - \xi_\eta(s), f(\xi_\varepsilon(t)) - f(\xi_\eta(s))\rangle + \|g(\xi_\varepsilon(t)) - g(\xi_\eta(s))\|^2\right) ds$$

$$+2(\varepsilon^2 + \eta^2)$$

$$\leq \; \nu^2 \int_0^t \mathbf{E}(\|\xi_\varepsilon(s) - \xi_\eta(s)\|^2 ds + 2(\varepsilon^2 + \eta^2)$$

Gronwall's Lemma implies that

$$\mathbf{E}\,\|\xi_\varepsilon(t) - \xi_\eta(t)\|^2 \;\leq\; 2(\varepsilon^2 + \eta^2)e^{\nu^2 t}$$

In both cases we deduce that the above Cauchy sequences converge to some $\xi(\cdot)$:

$$\forall\, t \in [0,1], \;\; \lim_{\varepsilon \to 0} \mathbf{E}(\|\xi_\varepsilon(t) - \xi(t)\|^2) \;=\; 0$$

Furthermore, inequalities (3.2)ii) imply that

$$\mathbf{E}\left(d_{\mathcal{K}}^2(\xi(t))\right) \;=\; 0$$

so that the solution is viable in $\mathcal{K}$. ∎.

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