HJB $$\frac{\partial V(x,t)}{\partial t} + \min\left\{\frac{\partial V(x,t)}{\partial x}\cdot F(x,u) + C(x,u)\right\} = 0$$

$V(x,T) = D(x)$ terminal Condition

$f(x,u) = Ax + Bu$

$C(x,u) = x^T Q x + u^T R u$

$V(x,t) = x^T P x$

$$\frac{\partial V(x,t)}{\partial t} = x^T \frac{\partial P(t)}{\partial t} x$$

$$\frac{\partial V(x,t)}{\partial x} = 2 P(t) x$$

$$H = \frac{\partial V(x,t)}{\partial x} \cdot F(x,u) + C(x,u)$$

$$H = \underbrace{(2P(t)x)^T (Ax + Bu)}_{\text{scaler}} + x^T Q x + u^T R u$$

$$H = 2x^T P (Ax + Bu) + x^T Q x + u^T R u$$

$$0 = \frac{\partial H}{\partial u} \Rightarrow u = -R^{-1} B^T P(t) x \qquad \frac{\partial H}{\partial u} = 2B^T P(t) x + 2Ru = 0$$

$(-R^{-1}B^T P x)^T R(-R^{-1}B^T P x)$

$+ x^T P B R^{-1} \cancel{R} R B^T P x$

$x^T P B R^{-1} B^T P x$

$x^T P [B R^{-1} B^T] P x$

$$H = 2x^T P A x - 2x^T P B R^{-1} B^T P x + x^T Q x + x^T P [\cancel{BR^{-1}} B^T] P x$$

$$H = x^T [2PA - PBR^{-1}B^T P + Q] x$$

$$H + \frac{\partial V(x,t)}{\partial t} = 0$$

$$x^T \frac{\partial P(t)}{\partial t} x + x^T \left(2P(t)A - P(t)BR^{-1}B^T P(t)\, Q\right) x = 0$$

$$\frac{\partial P(t)}{\partial t} + 2P(t)A - PBR^{-1}B^T P\, Q = 0$$

for algebraic Riccati equation we assume that $\frac{\partial P(t)}{\partial t} = 0$

$$2PA - PBR^{-1}B^T P + Q = 0$$

$$PA + A^T P - PBR^{-1}B^T P + Q = 0 \checkmark$$

NaGumo's theorem

Cauchy problem

boundary condition

and initial condition

$$x'(t) = f(t, \overset{n}{\cancel{x}}(t))$$

$$x(0) = 0,$$

where $\omega \in \mathcal{L}$ strictly increasing functions

$\omega : [0, \infty) \to [0, \infty)$

$\omega(0) = 0$

$$\int_0^r \frac{\omega(s)}{s}\, ds \leqslant r$$

$r > 0.$

where $a > 0$ and $f : [0, a] \times \mathbb{R}^n \to \mathbb{R}^n$

$$|f(t, x) - f(t, y)| \leqslant \frac{|x - y|}{t}$$

$t \in (0, a]$ $\quad x, y \in \mathbb{R}^n$ with $|x|, |y| \leqslant M$ $\quad M > 0.$

uniqueness holds if $f : [0, a] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous with

$$\frac{f(t, x)}{u'(t)} \to 0$$

$t \downarrow 0$ uniformly in $|x| \leqslant M$ for some $M > 0$ and satisfies

$$|f(t, x) - f(t, y)| \leqslant \frac{u'(t)}{u(t)}\, \omega(|x - y|)$$

$t \in (0, a]$

$x, y \in \mathbb{R}^n$

$|x|, |y| \leqslant M$

$u \to$ absolutley continuous function on $[0, a]$

find $U(x)$ as function defined on half line $x > x_0$ satisfying a first ~~order~~ ordinary differential equation

$$\frac{du}{dx} = f(x, u)$$

$$u(x_0) = u_0$$

in geometrical term means that considering the family of integral curves of equation (1) in the (x, u) -plane, one wishes to find the curve passing through the point (x_0, u_0).

$f(x) = x$ ✓ Continuity of $f(x, y)$

if f and $\frac{\partial f}{\partial y}$ are continuous in a region around (x_0, y_0)

$\frac{x^2}{2} + C$ $\quad \frac{\partial f}{\partial y} = 0$ ✓

$\frac{\partial f}{\partial x} = 1$ ✓

Nagumo's theorem makes sure we have a unique solution in Cauchy's problem with ensuring the function f in the specific interval is continuous.

and satisfying this condition $|f(t,x) - f(t,y)| \leq \frac{1}{t} |x-y|$

$$\frac{|f(t,x) - f(t,y)|}{|x-y|} \leq \frac{1}{t} \quad t \in (0, a]$$