ON VARIAIONAL INEQUALITIES

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ABSTRACT

It is well known that the minimum of a linear functional on a convex set in a Hilbert space can be characterized by a variational inequality. The same is proved for a class of differentiable and non-differentiable nonlinear functionals. Representation theorems are proved for nonlinear problems in a Hilbert space. The Riesz-Frechet theorem and the Lax-Milgram lemma can be deduced from these theorems. Techniques based on the contraction mapping theorem are used to prove the existence of a unique solution to a new class of nonlinear variational inequalities. It has been shown that the linearization of the variational inequalities is useful for the regularization approximation. The equivalence of variational and weak formulations of nonlinear boundary value problems is proved. A finite element approximation to the solution of the weak problem in a finite dimensional subspace of the original Hilbert space is defined. Using the concept of pseudo projection, an inequality bounding the error in this approximation over all functions of the subspace is derived. In the study of variational inequalities, it is necessary that in addition to the usual finite dimensional subspace of the Hilbert space, we construct finite dimensional convex subset of the Hilbert space. We note that this finite dimensional convex subset is not necessarily contained in the original convex subset of the original Hilbert space. New inequalities bounding the error in these approximations over the functions of the finite dimensional convex subset are derived for the nonlinear variational inequalities.

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CONTENTS

INTRODUCTION

The Riesz-Fréchet representation theorem implies the existence of continuous linear functionals along with a unique representation for each of them by an inner product in a Hilbert space. The fact that the minimum of a certain quadratic form does exist, which arises in the proof of the Riesz-Fréchet theorem, is fundamental in variational problems.
To be more precise, let H be a real Hilbert space with its dual space H'. If $a\langle u, v\rangle$ is a continuous bilinear form and F is a continuous linear functional on H, then we consider a functional $I[v]$, defined by

$$I[v] \equiv a\langle v, v\rangle - 2F(v), \qquad \text{for all } v \in H. \tag{0.1}$$

Lax and Millgram [3] considered the problem of finding $u \in H$ such that

$$a\langle u, v\rangle = F(v), \qquad \text{for all } v \in H. \tag{0.2}$$

This problem is a natural extension (linear to bilinear), of the Riesz-Fréchet theorem. It turns out for a symmetric bilinear form $a\langle u, v\rangle$, that equations (0.2) are obtained by imposing on the functional $I[v]$ the necessary conditions for obtaining a minimum on H. It has been shown [34] that the variational, (0.1), and weak, (0.2), formulations of linear elliptic boundary value problems are equivalent. Mikhlin [24] also proves that the minimum of a quadratic form on H can be characterized by the weak formulation. This formulation enables estimates to be obtained for the error in Finite element approximations to the solutions of linear elliptic boundary value problems derived using the Galerkin technique.

Naturally a question arises as to whether or not the equivalence of the variational and weak solutions holds on a convex set in a Hilbert space. The fact this is so has been shown by Stampacchia [34], who characterized the minimum of the functional $I[v]$ on a convex subset of H by a variational inequality. The convex subset here usually consists of admissible functions, which in addition satisfy extra auxiliary conditions. Closely related to the variational inequalities are the unilateral problems [17]. These problems mainly arise in mechanics and physics, see [17, 15], where the material structure under consideration is submitted to unilateral constraints. In the absence of the constraints, we have the variational equations.

In 1960, Minty [25] and Zarantonello [48] independently introduced the notion of a monotone operator. Thus it is possible to consider not merely bilinear forms, but also forms linear in only one of the two arguments. Making use of monotone operator theory, Browder [7] considers variational inequalities of the type (5.27) in Banach spaces, which are more general than and include the variational inequalities studied by Lions and Stampacchia [21], as a special case. These inequalities allow us to consider some nonlinear elliptic boundary value problems. Sibony [32, 33] has proved that the minimum of the functional $I[v]$, when $a\langle u, v\rangle$ is a differentiable nonlinear functional, can be characterized by the variational inequalities introduced by Browder.

The variational inequality approach consists in dealing with such inequalities without assuming a priori that the monotone operator involved is the Fréchet derivative of a differentiable nonlinear functional. Brezis and Stampacchia [6], and Brezis [5] used this approch to study the regularity of the solution of the original problem. Recently Falk [16], and Mosco and Strang [28] have derived the error estimates for the approximation of a class of variational inequalities studied by Lions and Stampacchia.

We have noted that all these cases are extensions of the Riesz-Fréchet representation theorem in various directions for the study of a wide class of linear and nonlinear boundary value problems. In all the previous investigations, the emphasis has been mainly on the form $a\langle u, v\rangle$ whilst F, as in (0.1), remains linear.

We consider the problem when F is a nonlinear continuous functional and $a\langle u, v\rangle$ is a bilinear form. We study this problem in this thesis and explore some cases analogous to the Riesz-Fréchet representation theorem and various generalizations. We show that a class of nonlinear boundary value problems can be studies by this technique. In every case, we have shown the relationship between our and the corresponding previous known results. In some cases, the method considered by us simplify the proofs of the previous known problems.

We now summarize some of these new results and the ideas used.

Chapter 1 begins with some preliminary definitions. We prove that for a differentiable nonlinear functional $I[v]$, variational and weak formulations in a Hilbert space are equivalent. The minimum of both differentiable and non-differentiable nonlinear functionals on a convex subset of a Hilbert space is shown to be characterized by a class of nonlinear variational inequalities.

Chapter 2 contains the representation theorems for a class of nonlinear operators in a Hilbert space, which include the representation theorems of Riesz and Fréchet, and Lax and Milgram as special cases. An iterative scheme is given which proves the existence of the representation theorems.

We consider a new class of nonlinear variational inequalities in chapter 3. The technique of Lions and Stampacchia has been used to prove the existence of a unique solution of these variational inequalities. Linearization of these variational inequalities is used to give a method of approximation.

The applications of the abstract theory developed in the previous chapters to nonlinear elliptic boundary value problems are discussed in chapter 4.

In chapter 5, we derive the error bounds for the approximation of nonlinear boundary value problems via the weak formulation using the concept of pseudo projection. The inequalities bounding the error for the approximation of a class of nonlinear variational inequalities have been obtained. A simplified proof of a problem considered by Browder is given.

Chapter 1

Formulation of Variational Inequalities

In this chapter, we prove that the minimum of differentiable and non-differentiable nonlinear functionals can be characterized by variational equations or inequalities over the whole Hilbert space or over a convex subset in the Hilbert space. We show that the formulations considered by us are more general and include as special cases all the corresponding previous formulations.
Let H be a real Hilbert space with its dual H', whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively. The paring between H' and H is denoted by $\langle \cdot, \cdot \rangle$.
We now give some definitions, which will be used later on.

Definition 1.1 A function $F : H \to F$ (where $F = \mathbb{R}$ or $\mathbb{C}$), is a bounded linear functional, if for all $u, v \in H$,
(i). $F(\mu u + \lambda v) = \mu F(u) + \lambda F(v), \qquad$ for all $\mu, \lambda \in F$,
(ii). there exists a constant $\alpha > 0$ such that

$$|F(v)| \leq \alpha \parallel v \parallel, \qquad \text{for all } v \in H.$$

Note that continuity and boundedness are equivalent. A functional, which is not linear, is called nonlinear.

Definition 1.2 Let u, v be two different elements of H. The set of all elements $(1-t)u + tv$, where t assumes all values over the field F is called the straight line through u and v. If $t \in [0, 1]$, we obtain the line segment between u and v. Moreover, a subset M of H is said to be convex, if $(1-t)u + tv \in M$, whenever $u, v \in M$ and for all $t \in [0, 1]$.

From now onwards we denote by M, a convex subset, unless otherwise specified.
A functional F is said to be convex on a convex subset M of H, if, for given $u, v \in M$, $0 \leq t \leq 1$,

$$F(tu + (1-t)v) \leq tF(u) + (1-t)F(v)$$

holds. F is a concave functional, if and only if, $-F$ is convex.

Definition 1.3 A subset M of H is called a subspace, if $\mu u + \lambda v \in M$, whenever $u, v \in M$ and μ, λ are scalars. It is said to be closed, if $u_n \in M$, $u \in H$ such that $u_n \to u$ as $n \to \infty$, i.e., $\lim_{n\to\infty} \| u_n - u \| = 0$, implies that $u \in M$.

Definition 1.4 A bilinear form (functional) on H is a mapping $a : H \times H \to F$ satisfying;
(i). for each fixed element v in H, the mapping $u \to a(u, v)$ is a linear functional on H.
(ii). for each fixed element u in H, the mapping $v \to a(u, v)$ is a linear functional on H.
In case H is a complex Hilbert space, the condition (ii) is replaced by
(ii).* for each fixed u in H, the mapping $v \to a(u, v)$ is a antilinear functional on H, i.e.,

$$a(u, \alpha v) = \bar{\alpha} a(u, v), \qquad \text{for all } \alpha \in F.$$

Further, a bilinear form $a(u, v)$ is said to be symmetric form, if

$$a(u, v) = a(v, u), \qquad \text{for all } u, v \in H.$$

Definition 1.5 A bilinear form $a(u, v)$ on H is said to be positive definite (coercive [21], H-elliptic [41]), if there exists a constant $\rho > 0$ such that

$$a(u, v) \geq \rho \| v \|^2, \qquad \text{for all } v \in H,$$

and positive (non-negative), if

$$a(u, v) \geq 0, \qquad \text{for all } v \in H,$$

which is weaker assumption than positive definiteness. Note that positive definiteness implies positively but not conversely.
A bilinear form $a(u, v)$ is said to be continuous (bounded), if there exist a constant $\mu > 0$ such that

$$|a(u, v)| \leq \mu \| u \| \| v \|, \qquad \text{for all } u, v \in H.$$

The Cauchy-Schwartz inequality holds for the bilinear form $a(u, v)$ and is given by

$$|a(u, v)|^2 \leq a(u, u) a(v, v), \qquad \text{for all } u, v \in H.$$

Lemma 1.1 A bilinear form is continuous with respect to the norm convergence.

Proof. Let $u_n \to u$ and $v_n \to v$. Then these sequences are bounded. Let β be their bound, and $\| u_n \| \leq \beta$.

Now

$$\begin{aligned} |a(u_n, v_n) - a(u, v)| &= |a(u_n, v_n) - a(u_n, v) + a(u_n, v) - a(u, v)| \\ &\leq |a(u_n, v_n - v)| + |a(u_n - u, v)| \\ &\leq \mu\beta \parallel v_n - v \parallel + \mu \parallel u_n - u \parallel \parallel v \parallel . \end{aligned}$$

But $\parallel u_n - u \parallel \to 0$ and $\parallel v_n - v \parallel \to 0$ as $n \to \infty$, and therefore

$$|a(u_n, v_n) - a(u, v)| \to 0,$$

i.e.,

$$a(u_n, v_n) \to a(u, v).$$

□

Definition 1.6 A functional $F : H \to Y$, where H and Y are normed linear spaces, is Fréchet differentiable at $u \in H$, if there is an element $F'(u) \in L[H, Y]$, the space of all linear continuous functionals from H into Y, such that

$$F(u + v) - F(u) = F'(u)v + \epsilon(v), \qquad \text{for all } v \in H,$$

where $\frac{\|\epsilon(v)\|}{\|v\|} \to 0$ as $v \to 0$. It follows that, if $F'(u)$ exists, then

$$\lim_{t \to 0} \frac{F(u + tv) - F(u)}{t} = \langle F'(u), v \rangle, \qquad \text{for all } v \in H.$$

The operator $F' : H \to L[H, Y]$, which assigns $F'(u)$ to u called the Fréchet derivative of F and $\langle F'(u), v \rangle$ is called the Fréchet differential of the functional F in the direction v.

Since we are only interested in the case $Y = \mathbb{R}$ with H, a real Hilbert space, then it is clear [10] that $L[H, Y] = H'$, the dual of H. Thus from the definition of Fréchet derivative, it follows that $F'(u)$ is an element of the space H'.
Moreover, $F'(u)v$ being a continuous linear functional in $v \in H$ can be uniquely represented as a paring between H' and H, that is $F'(u)v = \langle F'(u), v \rangle$. Thus we have

$$\lim_{t \to 0} \frac{F(u + tv) - F(u)}{t} = \langle F'(u), v \rangle, \qquad \text{for all } u, v \in H.$$

We will always use this form of representation for the Fréchet differentiable. The Fréchet derivative $F'(u)$ means that the derivative of the functional F is taken with respect to the argument u.

Remark 1.1 The Fréchet derivative always has its domain in the same space as the original functional. With this interpretation F and F' have the same range space. For more details see Tapia [37], Nashed [29] and Carrol [10].

Remark 1.2 For a linear functional F, we note that, for all $u, v \in H$;

$$\begin{aligned} \langle F'(u), v \rangle &= \lim_{t \to 0} \frac{F(u+tv) - F(u)}{t} \\ &= \lim_{t \to 0} \frac{F(u) + tF(v) - F(u)}{t} \quad F \text{ is linear} \\ &= F(v) \\ &\equiv \langle F, v \rangle, \qquad \text{for all } F \in H'. \end{aligned}$$

Thus we see that for a linear functional F, $\langle F'(u), v \rangle = \langle F, v \rangle$.

This fact will play a very important part in showing the relationship between our results and previous known results for the linear case.
From now onward, differentiable functionals mean Fréchet differentiable functionals unless otherwise specified.

Definition 1.7 The operator $T : M \to H'$ is called antimonotone, if

$$\langle Tu - Tv, u - v \rangle \leq 0, \qquad \text{for all } u, v \in M,$$

or T is antimonotone if and only if $-T$ is monotone [10].

We note from Remark 1.2 that every linear functional is both monotone and antimonotone. We have the following relationship between the antimonotonicity of the Fréchet derivative of a differentiable functional and the functional itself. This result is essentially due to Vainberg [40], but our method of proof is different.

Lemma 1.2 Let F be a real-valued differentiable functional on M. Then F is concave, if and only if, its Fréchet derivative F' is antimonotone.

Proof. Let F be a concave functional, then for all $u, v \in M$ and $t \in [0, 1]$, $tu + (1-t)v \in M$, and we have

$$\begin{aligned} F(v + t(u - v)) &\equiv F(tu + (1-t)v) \\ &\geq tF(u) + (1-t)F(v), \qquad \text{by definition.} \end{aligned}$$

Rearranging and dividing by t, we get

$$\frac{F(v + t(u - v)) - F(v)}{t} \geq F(u) - F(v).$$

Let $t \to 0$. Since F is differentiable, it follows that

$$\langle F'(v), u - v\rangle \geq F(u) - F(v).$$

Similarly, for all $u, v \in M$,

$$\langle F'(u), v - u\rangle \geq F(v) - F(u).$$

By adding, we have

$$\langle F'(v) - F'(u), v - u\rangle \leq 0,$$

showing that F' is antimonotone.

Conversely suppose that the Fréchet derivative F' is antimonotone. Sice F is a differentiable functional, so we have the following representation [40].

$$F(u) - F(v) = \int_0^1 \langle F'(v + s(u - v)), u - v\rangle \mathrm{d}s. \tag{1.1}$$

Consider, for all $u, v \in M$ and $t \in [0, 1]$,

$$\begin{aligned} F(v + t(u - v)) - F(v) &= \int_0^1 t\langle F'(v + ts(u - v)), u - v\rangle \mathrm{d}s \\ &= \int_0^1 t\langle F'(v + ts(u - v)) - F'(v + s(u - v)), u - v\rangle \mathrm{d}s \\ &\quad + \int_0^1 t\langle F'(v + s(u - v)), u - v\rangle \mathrm{d}s. \end{aligned}$$

Let $U = v + s(u - v)$ and $V = v + st(u - v)$, then $U - V = s(1 - t)(u - v)$, and so, since $(1 - t) \geq 0$ for all $t \in [0, 1]$, it follows by the antimonotonicity of F' that

$$0 \leq \langle F'(V) - F'(U), U - V\rangle = s(1 - t)\langle F'(V) - F'(U), u - v\rangle.$$

Thus we have

$$\begin{aligned} F(v + t(u - v)) - F(v) &\geq \int_0^1 t\langle F'(v + s(u - v)), u - v\rangle \mathrm{d}s \\ &= t\{F(u) - F(v)\}, \quad \text{by (5.51)}. \end{aligned}$$

That is

$$F(v + t(u - v)) \geq tF(u) + (1 - t)F(v),$$

the required result. $\square$

If $a(u,v)$ is a continuous bilinear form on H and F is a real-valued continuous functional, then we consider a functional $I[v]$ defined by

$$I[v] = a(v,v) - 2F(v), \qquad \text{for all } v \in H. \tag{1.2}$$

Many mathematical problems either arise or can be formulated in term of functional of this form. Here one seeks to minimize the functional $I[v]$ defined by (5.52) over a whole space or a convex subset M of H bearing in mind whether the real-valued functional F is linear or not. We point out that the whole theory of variational methods can be based on the minimum of the functional $I[v]$. For the nonlinear form $a(u,v)$ and linear functional F, it has been shown [33] that the minimum of $I[v]$ on H (or $M \subset H$) can be characterized by a variational equation (or inequality). Furthermore, there arise two cases according as F is linear or nonlinear. The case F is linear has been studied by Stampacchia [34] and Temam [38]. We state their result without proof.

Theorem 1.1 If $F \in H'$ and $a(u,v)$ is a positive, symmetric and continuous bilinear form on H, then the function $u \in H$ minimizes $I[v]$, defined by (5.52), if and only if

$$a(u,v) = \langle F, v\rangle \qquad \text{for all } v \in H. \tag{1.3}$$

A result analogous to Theorem 1.1 is now established for the case when F is a nonlinear differentiable functional.

Theorem 1.2 Let $a(u,v)$ be a positive, summetric and continuous bilinear form on H. If the Fréchet derivative F' of a nonlinear functional F defined on H exists and is antimonotone, then the function $u \in H$ minimizes $I[v]$ if and only if

$$a(u,v) = \langle F'(u), v\rangle \qquad \text{for all } v \in H. \tag{1.4}$$

Proof. Let u minimize $I[v]$, then for all $\lambda \in \mathbb{R}$ and $w \in H$, $v = u + \lambda w \in H$,

$$I[u] \leq I[u + \lambda w].$$

Thus from (5.52), it follows that

$$\begin{aligned} a(u,u) - 2F(u) &\leq a(u+\lambda w, u+\lambda w) - 2F(u+\lambda w) \\ &= a(u,u) + \lambda 2a(u,w) + \lambda^2 a(w,w) - 2F(u+\lambda w), \end{aligned}$$

and so

$$a(u,w) \geq \frac{F(u+\lambda w) - F(u)}{\lambda} - \frac{\lambda}{2} a(w,w). \tag{1.5}$$

Let $\lambda \to 0$. Since F is differentiable, we have

$$a(u,w) \geq \langle F'(u), w\rangle \qquad \text{for all } w \in H.$$

For now replacing w by $-w \in H$, we have

$$a(u,w) \leq \langle F'(u), w\rangle \qquad \text{for all } w \in H.$$

Thus we obtain

$$a(u,w) = \langle F'(u), w\rangle \qquad \text{for all } w \in H.$$

We shall use this device several times in this thesis.
For the converse, if $u \in H$ satisfies (5.57), then using the positivity of $a(u, v)$, we have

$$\begin{aligned} I[u] - I[v] &= a(u,u) - 2F(u) - a(v,v) - 2F(v) \\ &= a(u,u-v) - a(v,v-u) - 2F(u) + 2F(v) \\ &= 2a(u,u-v) - 2a(v-u,v-u) - 2F(u) + 2F(v) \\ &\leq 2\langle F'(u), u-v\rangle + 2F(v) - 2F(u), \qquad \text{by (5.57).} \end{aligned}$$

Since F' is antimonotone, hence by Lemma 1.2, it is concave. Thus

$$a\langle F'(u), u-v\rangle + 2F(v) - 2F(u) \leq 0, \qquad \text{for all } u, v \in H,$$

so that

$$I[u] \leq I[v].$$

Note that for the case when F is a linear functional, it follows that the result of Theorem 1.2 is exactly that given in Temam [38]. In this case it is seen from (1.5) that

$$a(u,w) + \frac{\lambda}{2} a(w,w) \geq F(w), \qquad \text{for all } w \in H,$$

so that as $\lambda \to 0$, we get

$$a(u,w) = \langle F, w\rangle, \qquad \text{for all } w \in H.$$

(See, e.g. Zlamal on P. 62 of Whiteman [45]). This could alternatively follow from Remark 1.2. □

For the special case, $a(u, v) = (u, v)$, we have the following functional $I_1[v]$, defined by

$$I_1[v] = (v,v) - 2F(v), \qquad \text{for all } v \in H.$$

We again consider the following two cases;

(i). For the linear case, one can easily show that for each linear continuous functional F on H, the function $u \in H$ minimizes $I_1[v]$ if and only if

$$(u, v) = \langle F, v \rangle, \qquad \text{for all } v \in H.$$

Thus we have the variational character of the Riesz-Fréchet representation theorem, see Theorem 5.10.

For the nonlinear differential functional F, we have the following result.

Remark 1.3 If the Fréchet derivative F' of a nonlinear differential functional F exists and is antimonotone, then the function $u \in H$ minimizes $I_1[v]$ if and only if

$$(u, v) = \langle F'(u), v \rangle, \qquad \text{for all } v \in H,$$

This can be viewed as the variational formulation of the Riesz-Fréchet theorem for a class of differentiable nonlinear functionals.

For minimizing $I[v]$ over a convex set M in H instead of the entire space H, the variational equation turns into inequality. The case when F is linear, has been studied by Stampacchia [34] and Lions-Stampacchia [21], and is known as one of the first and most fundamental of the variational problems. A similar idea has been used by Fichera [17] in the study of unilateral constraint problems in elasticity. We state their result without proof.

Theorem 1.3 Let $a(u, v)$ be a positive definite, symmetric and continuous bilinear form and M a convex subset of H. If $F \in H'$, then the function $u \in M$ minimizes $I[v]$, defined by (5.52), if and only if

$$a(u, v - u) \geq \langle F, v - u \rangle, \qquad \text{for all } v \in M. \tag{1.6}$$

We here study the class when F is a nonlinear functional.

We discuss the following two cases.

(a). F is a differentiable functional.

(b). F is non-differentiable functional, but it can be decomposed as follows;

$$F = F_1 + F_2,$$

where F_1 is a differentiable functional, but F_2 is not. In chapter 4, we give some examples which can be written in this form.

(a).

F IS DIFFERENTIABLE

In this case, it has been shown [30], that the following characterization holds.

Theorem 1.4 If the Fréchet derivative F' of a nonlinear functional F is antimonotone, then under the assumptions of Theorem 1.3, $u \in M$ is a solution of

$$a(u, v-u) \geq \langle F'(u), v-u \rangle, \qquad \text{for all } v \in M, \tag{1.7}$$

if and only if

$$a(u,u) - 2F(u) \leq a(u,v) - 2F(v), \qquad \text{for all } v \in M.$$

First of all one observes that the solution $u \in M$ of (1.7) does not necessarily minimize $I[v]$ on M. The question arises under what conditions the solution u of (1.7) gives the exact minimum of $I[v]$ on M. In an attempt to answer this question, we assume that the bilinear form $a(u, v)$ is positive, which is weaker assumption than the positive definiteness. We prove the following main theorem in this case.

Theorem 1.5 Let $a(u, v)$ be a positive, symmetric, continuous bilinear form and M a convex subset of H. If the Fréchet derivative of F' of F is antimonotone, then $u \in M$ is a solution of (1.7), if and only if, $u \in M$ minimizes $I[v]$, defined by (5.51), on M.

Proof. Let $u \in M$ minimize $I[v]$. Then

$$I[u] \leq I[v], \qquad \text{for all } v \in M.$$

Since M is a convex, so for all $t \in [0, 1]$ and $w \in M$, $v_t = u + t(w - u) \in M$. Setting $v = v_t$, we get

$$I[u] \leq I[v_t], \qquad \text{for all } w \in M.$$

Using (5.51), we obtain

$$a(u,u) - 2F(u) \leq a(u + t(w-u), u + t(w-u)) - 2F(u + t(w-u)).$$

By the symmetry of $a(u, v)$, and dividing by t, we get

$$a(u, w-u) \geq \frac{F(u + t(w-u)) - F(u)}{t} + \frac{1}{2} t a(u-w, w-u).$$

Let $t \to 0$. Since F is differentiable, we get

$$a(u, w-u) \geq \langle F'(u), w-u \rangle, \qquad \text{for all } w \in M,$$

the desired (1.7).

Conversely, let $u \in M$ be a solution of (1.7). We can rewrite (1.7) in the following way,

$$a(v, v-u) - a(u-v, u-v) \geq \langle F'(u), v-u \rangle.$$

Since the bilinear form $a(u, v)$ is positive, we obtain

$$a(v, v-u) \geq \langle F'(u), v-u \rangle,$$

which again can be written in the following form.

$$a(v, v-u) \geq \langle F'(v), v-u \rangle + \langle F'(u) - F'(v), v-u \rangle.$$

The antimonotonicity of F' implies that

$$a(v, v-u) \geq \langle F'(v), v-u \rangle, \qquad \text{for all } v \in M.$$

Setting $v = v_t$, we obtain

$$a(u + t(w-u), w-u) - \langle F'(v_t), w-u \rangle \geq 0, \qquad \text{for all } w \in M. \tag{1.8}$$

Following the technique of Sibony [32], write for all $u, w \in M$,

$$h(t) = ta(u, w-u) + \frac{t^2}{2} a(w-u, w-u) - F(u + t(w-u)).$$

Then, for all $t \in [0, 1]$, there exists

$$h'(t) = a(u, w-u) + ta(w-u, w-u) - \langle F'(v_t), w-u \rangle,$$

so that $h(t) \geq 0$, by (1.8). Thus $h(0) \leq h(1)$ gives us

$$-F(u) \leq a(u, w-u) + \frac{1}{2} a(w-u, w-u) - F(w),$$

i.e.,

$$a(u, u) - 2F(u) \leq a(w, w) - 2F(w),$$

which shows that $u \in M$ is the minimum of $I[v]$. □

We make the following observations.

Remark 1.4 If $F(u) = 0$, then we see that the function u which minimizes $I[v] = a(v, v)$ on $M \subset H$ is also a solution of

$$a(u, v-u) \geq 0, \qquad \text{for all } v \in M,$$

and conversely. This is a well known result [10] in the calculus of variations.

Remark 1.5 If F is a linear functional, then Theorem 1.5 reduces to the following result, which appears to be new in this form.

Theorem 1.6 If $a(u, v)$ is a positive, symmetric, continuous bilinear form and M a convex subset in H, then for each $F \in H'$, $u \in M$ is a solution of (1.6), if and only if, $u \in M$ minimizes $I[v]$ on M.

Proof. The proof follows exactly on the same lines as for Theorem 1.5. $\square$

Remark 1.6 If $M = H$, then (1.7) becomes

$$a(u, v) = \langle F'(u), v \rangle, \qquad \text{for all } v \in H,$$

for replacing v by $\pm v + u$ in (1.7), we have

$$a(u, v) \geq \langle F'(u), v \rangle,$$

and

$$a(u, v) \leq \langle F'(u), v \rangle.$$

Hence it follows that

$$a(u, v) = \langle F'(u), v \rangle, \qquad \text{for all } v \in H.$$

Thus Theorem 1.5 is a generalization of Theorem 1.2.

Remark 1.7 We also note that for some elements v, equality holds in the variational inequality (1.7). This happens, when together with v, $2u - v$ also lies in M. We see this by replacing v by $2u - v$ in (1.7),

$$a(u, v - u) \leq \langle F'(u), v - u \rangle, \qquad \text{for all } v \in M.$$

Comparing it with (1.7), we obtain

$$a(u, v - u) = \langle F'(u), v - u \rangle, \qquad \text{for all } v \in M.$$

Mosco and Strang [28] have pointed out that if u is not an extreme point of the convex subset M, then there are directions in which u is interior to a line segment, and in these directions the equality holds. A similar conclusion has been drawn by Mancino and Stampacchia [23].

(b).

F IS NON-DIFFERENTIABLE

In this case, we show that a similar characterization of $I[v]$, defined in (5.52) holds, which is more general than and includes the inequality (1.7) as a special case.

Theorem 1.7 Let $a(u, v)$ be a positive, symmetric and continuous bilinear form. If $F = F_1 + F_2$, where F_1 is a concave differentiable functional and F_2 is a concave but non-differentiable functional, then $u \in M$ is a solution of

$$a(u, v-u) \geq \langle F_1'(u), v-u\rangle + F_2(v) - F_2(u), \qquad \text{for all } v \in M, \tag{1.9}$$

if and only if $u \in M$ minimizes $I[v]$ on M.

Proof. Let $u \in M$ minimize $I[v]$. Then

$$I[u] \leq I[v], \qquad \text{for all } v \in M.$$

Setting $v = v_t \equiv u + t(w-u) \in M$, for all $w \in M$, $t \in [0,1]$, we get

$$I[u] \leq I[v_t], \qquad \text{for all } w \in M.$$

Using (5.52) and the concavity of F_2, we obtain

$$a(u, w-u) \geq \frac{F_1(u + t(w-u)) - F_1(u)}{t} + F_2(w) - F_2(u) + \frac{1}{2}ta(u-w, w-u).$$

Let $t \to 0$. Since F_1 is differentiable, (1.9) follows.
Conversely, let $u \in M$ be a solution of (1.9). Now using (5.52), we obtain

$$\begin{aligned} I[u] - I[v] &= a(u,u) - a(v,v) + 2F_1(v) - 2F_1(u) + 2F_2(u) - 2F_2(v) \\ &\leq 2\langle F_1'(u), u-v\rangle + 2F_1(v) - 2F_1(u), \end{aligned}$$

as in Theorem 1.2. By assumption F_1 is concave differentiable, so, for all $u, v \in M$, it follows, by definition that

$$\langle F_1'(u), u-v\rangle \leq F_1(u) - F_1(v).$$

Using this inequality, we get

$$I[u] \leq I[v], \qquad \text{for all } v \in M.$$

□

Remark 1.8 For $F_2 = 0$, it is obvious that Theorem 1.7 contains Theorem 1.5.
For $F_1 = 0$, we obtain the mixed type of variational inequality. In this case Theorem 1.7 reduces to:

Theorem 1.8 If F is a concave non-differentiable functional, then a solution $u \in M$ of

$$I[u] \leq I[v]$$

for all $v \in M$, can be characterized by the variational inequality

$$a(u, v - u) \geq F(v) - F(u), \qquad \text{for all } v \in M. \tag{1.10}$$

The variational inequality (1.10) is a special case of variational inequality considered by Sibony [33]. Browder [8] and Mosco [26] observed that the inequality (1.10) unifies the direct and weak formulation of minimum problems, which is also clear from Theorem 1.8. The existence of a solution of mixed type variational inequalities has been considered by Browder [8, 9], Mosco [26] and Carrol [10].

Remark 1.9 If F_1 is a linear functional, then it follows from Remark 1.2 that the minimum $u \in M$ of the functional

$$I[v] = a(v, v) - 2F_1(v) - 2F_2(v),$$

is characterized by the variational inequality

$$a(u, v - u)) \geq \langle F_1, v - u \rangle + F_2(v) - F_2(u), \qquad \text{for all } v \in M. \tag{1.11}$$

Browder [8, 9] has given existence and uniqueness theorems for nonlinear variational inequalities of the type (1.10) in Banach Spaces by the monotone operator theory method.

It is worth mentioning that even the formulation of variational inequalities such as (1.11) has not been considered from the view point of calculus of variations, although many authors have investigated such inequalities in the monotone operator theory.

Chapter 2

Representation Theorems

In this chapter, we give a new proof of the celebrated Lax-Milgram lemma, based on the contraction mapping theorem.
We prove a representation theorem for differentiable nonlinear functionals, which generalizes the representation theorems of Riesz-Fréchet and Lax-Milgram by using the methods of montone operator theory.
An iterative scheme is given similar to Picard's, which enable us to find an approximate solution for a class of nonlinear problems.
We first state the Riesz-Fréchet representation theorem, whose extension is considered for a class of nonlinear differentiable functionals.

Theorem 2.1 [3] For every linear continuous functional F on H, there is a uniquely determined element $u \in H$ such that

$$(u, v) = F(v), \qquad \text{for all } v \in H. \tag{2.1}$$

This theorem enables us to identify the dual space H' of H, where H' is the space of all continuous linear functionals on H. Often it is not convenient to choose such a representation for H', see [29]. In such cases, in terms of pairing $\langle \cdot, \cdot \rangle$ between H' and H, the Reisz-Fréchet theorem can be stated in the following alternative way:
"For every $F \in H'$, there exists a unique $u \in H$ such that

$$(u, v) = \langle F, v \rangle \quad \text{for all } v \in H."$$

Remark 2.1 If $a(u,v)$ is a positive definite and continuous bilinear form, which means that there exist constants $\rho > 0$, $\mu > 0$ such that

$$\rho \parallel u \parallel^2 \leq a(u,v) \leq \mu \parallel u \parallel^2, \qquad \text{for all } u \in H,$$

then we note that
(i).

$$\rho \leq \mu. \tag{2.2}$$

(ii). For each $u \in H$, $a(u,v)$ is a continuous linear functional on H. Hence by Theorem 5.10, there exists an element $Tu \in H'$, see [2, 3], such that

$$a(u,v) = \langle Tu, v\rangle, \qquad \text{for all } v \in H.$$

Now

$$\begin{aligned} \parallel Tu \parallel_{H'} &= \sup_{v \in H} \frac{|\langle Tu, v\rangle|}{\parallel v \parallel} \\ &= \sup_{v \in H} \frac{|a(u,v)|}{\parallel v \parallel} \\ &\leq \mu \parallel u \parallel, \quad \text{by the continuity of } a(u,v). \end{aligned}$$

Also

$$\parallel T \parallel = \sup_{u \in H} \frac{\parallel Tu \parallel}{\parallel v \parallel}.$$

Thus one has

$$\parallel T \parallel \leq \mu. \tag{2.3}$$

Further, we define Λ, a canonical isomorphism from H' onto H by

$$\langle F, v\rangle \equiv (\Lambda F, v), \qquad \text{for all } v \in H, F \in H'. \tag{2.4}$$

Then

$$\parallel \Lambda \parallel_{H'} = \parallel \Lambda^{-1} \parallel_H = 1.$$

The following result is due to Lions-Stampacchia [21]. We include its proof for the sake of completeness.

Lemma 2.1 Let ξ be a number such that $0 < \xi < \frac{2\rho}{\mu^2}$. Then there exists a θ with $0 < \theta < 1$ such that

$$|(u,v) - \xi a(u,v)| \leq \theta \parallel u \parallel \parallel v \parallel, \qquad \text{for all } u, v \in H.$$

Proof. For all $u, v \in H$,

$$\begin{aligned}(u,v) - \xi a(u,v) &= (u,v) - \xi\langle Tu, v\rangle \\ &= (u,v) - \xi(\Lambda Tu, v), \quad \text{by (2.4)} \\ &= (u - \xi\Lambda Tu, v).\end{aligned}$$

Thus

$$|(u,v) - \xi a(u,v)| \leq \| u - \xi\Lambda Tu \| \| v \|, \quad \text{by the cauchy-Schwartz inequality.}$$

But

$$\begin{aligned}\| u - \xi\Lambda Tu \|^2 &= \| u \|^2 + \xi \| T \|^2 \| u \|^2 - 2\xi a(u,u) \\ &\leq \| u \|^2 + \xi^2\mu^2 \| u \|^2 - 2\xi\rho \| u \|^2,\end{aligned}$$

by (2.3) and the positive definiteness of $a(u, v)$.

$$= (1 + \xi^2\mu^2 - 2\xi\rho) \| u \|^2 .$$

Hence

$$|(u,v) - \xi a(u,v)| \leq \theta \| u \| \| v \|, \qquad \text{for all } u, v \in H,$$

where $\theta^2 = 1 + \xi^2\mu^2 - 2\xi\rho < 1$ for $0 < \xi < \frac{2\rho}{\mu^2}$. □

Using Lemma 2.1, we give a new proof of the Lax-Milgram lemma, which is a natural generalization of Theorem 5.10.

Theorem 2.2 [3] Let $a(u, v)$ be a positive definite and continuous bilinear form on H. Then for every $F \in H'$, there exists a unique element $u \in H$ such that

$$a(u,v) = \langle F, v\rangle, \qquad \text{for all } v \in H. \tag{2.5}$$

Furthermore, the map $F \rightarrow u$ is continuous from H' into H. If, in addition, $a(u, v)$ is symmetric and positive, then solving (2.5) is equivalent to finding

$$\min_{v \in H}\{a(v,v) - 2F(v)\},$$

as follows from Theorem 1.1.

Proof. Uniqueness and Continuity:

Given $F_i \in H'$, $i = 1, 2$, let u_1, u_2, be solutions in H of

$$a(u_1, v) = \langle F_1, v\rangle, \qquad \text{for all } v \in H,$$
$$a(u_2, v) = \langle F_2, v\rangle, \qquad \text{for all } v \in H.$$

Thus one has

$$a(u_1 - u_2, v) = \langle F_1 - F_2, v\rangle, \qquad \text{for all } v \in H.$$

Taking v as $(u_1 - u_2)$, we get

$$a(u_1 - u_2, u_1 - u_2) = \langle F_1 - F_2, u_1 - u_2\rangle.$$

Since $a(u, v)$ is a positive definite bilinear form, there exists a constant $\rho > 0$ such that

$$\begin{aligned} \rho \parallel u_1 - u_2 \parallel^2 &\leq a(u_1 - u_2, u_1 - u_2) \\ &= \langle F_1 - F_2, u_1 - u_2\rangle \\ &\leq \parallel F_1 - F_2 \parallel \parallel u_1 - u_2 \parallel, \end{aligned}$$

by the Cauchy-Schwartz inequality. Thus we have

$$\parallel u_1 - u_2 \parallel \leq \rho^{-1} \parallel F_1 - F_2 \parallel,$$

from which the uniqueness and the continuity of u follows.

Existence:

For a fixed ξ as in Lemma 2.1, and $u \in H$, define $\phi(u) \in H'$ by

$$\langle \phi(u), v\rangle = (u, v) - \xi a(u, v) + \xi\langle F, v\rangle, \qquad \text{for all } v \in H,\ F \in H'.$$

By Theorem 5.10, there exists a unique $w \in H$ such that

$$(w, v) = \langle \phi(u), v\rangle, \qquad \text{for all } v \in H,$$

and w is given by

$$w = \Lambda\phi(u) = Tu,$$

which defines a map from H into itself.

Now for all $u_1, u_2 \in H$,

$$\begin{aligned} \parallel Tu_1 - Tu_2 \parallel &= \parallel \Lambda\phi(u_1) - \Lambda\phi(u_2) \parallel \\ &\leq \parallel \phi(u_1) - \phi(u_2) \parallel_{H'} \\ &= \sup_{v \in H} \frac{|\langle \phi(u_1) - \phi(u_2), v\rangle|}{\parallel v \parallel} \\ &\leq \theta \parallel u_1 - u_2 \parallel, \quad \text{by Lemma 2.1.} \end{aligned}$$

Since $\theta < 1$, Tu is a contraction and has a fixed point $Tu = u \in H$, which satisfies

$$\begin{aligned} (u, v) &= \langle \phi(u), v \rangle \\ &= (u, v) - \xi a(u, v) + \xi \langle F, v \rangle. \end{aligned}$$

Thus for $\xi > 0$, we have

$$a(u, v) = \langle F, v \rangle, \qquad \text{for all } v \in H,$$

the desired result. □

Remark 2.2 The continuity of u on f is proved in [21]. The above method of proof throws light on the hidden aspects of the constructive nature of the Lax-Milgram lemma, which are not noted in the literature because the other methods do not depend upon the fixed-point theorem. This method can be used to prove more general representation theorems for monotone operator.

It has been observed by Nashed [29] that often a certain property of a differentiable functional can be deduced from a related property of its derivative. We shall show that Fréchet derivatives play an important role in providing for generalization of a certain class of differentiable nonlinear functionals, which otherwise might not be possible.
We follow the theory of monotone operator methods developed by Minty, Browder and Sibony, which includes the Fréchet derivative as a special case.

Definition 2.1 An operator $T : H \to H'$ is said to be Lipschitz continuous, if there exists a constant $\gamma > 0$ such that

$$\| Tu_1 - Tu_2 \| \leq \gamma \| u_1 - u_2 \|, \qquad \text{for all } u_1, u_2 \in H,$$

and γ is known as the Lipschitz constant.

We assume the following condition.

Condition A:
We assume that $\gamma < \rho$, where ρ is the positive definiteness constant associated with $a(u, v)$ and γ is the Lipschitz constant of the operator A as in (2.6).

We now state and prove a representation theorem for a class of monotone operators on a Hilbert space.

Theorem 2.3 Let $a(u, v)$ be a positive definite and continuous bilinear on H. If A is a Lipschitz continuous and antimonotone operator, and the condition A holds, then there exists a unique $u \in H$ such that

$$a(u, v) = \langle A(u), v\rangle, \qquad \text{for all } v \in H. \tag{2.6}$$

Moreover, if $a(u, v)$ is a symmetric, positive bilinear form and $A(u) = F'(u)$, the Fréchet derivative of F at u, then solving (2.6) is equivalent to finding

$$\min_{v\in H}\{a(v \cdot v) - 2F(v)\},$$

as follows from Theorem 1.2.

We need the following lemma, which itself a generalization of Lemma 2.1.

Lemma 2.2 Let ξ be a number such that $0 < \xi < 2\frac{\rho-\gamma}{\mu^2-\gamma^2}$ and $\xi\gamma < 1$. Then there exists a θ with $0 < \theta < 1$ such that

$$\| \phi(u_1) - \phi(u_2) \| \leq \theta \| u_1 - u_2 \|, \qquad \text{for all } u_1, u_2 \in H,$$

where for $u \in H$, $\phi(u) \in H'$ is defined by

$$\langle \phi(u), v\rangle = (u, v) - \xi a(u, v) + \xi\langle A(u), v\rangle, \qquad \text{for all } v \in H. \tag{2.7}$$

Proof. For all $u_1, u_2 \in H$,

$$\begin{aligned}
\langle \phi(u_1) - \phi(u_2), v\rangle &= (u_1 - u_2, v) - \xi a(u_1 - u_2, v) + \xi\langle A(u_1) - A(u_2), v\rangle, \qquad \text{for all } v \in H. \\
&= (u_1 - u_2, v) - \xi\langle T(u_1 - u_2), v\rangle + \xi\langle A(u_1) - A(u_2), v\rangle \\
&= (u_1 - u_2, v) - \xi(\Lambda T(u_1 - u_2), v) + \xi(\Lambda A(u_1) - \Lambda A(u_2), v), \quad \text{by (2.4)} \\
&= (u_1 - u_2 - \xi\Lambda T(u_1 - u_2), v) + \xi(\Lambda A(u_1) - \Lambda A(u_2), v).
\end{aligned}$$

Thus

$$|\langle \phi(u_1) - \phi(u_2), v\rangle \leq \| u_1 - u_2 - \xi\Lambda T(u_1 - u_2) \| \| v \| + \xi \| A(u_1) - A(u_2) \| \| v \| .$$

Now by (2.2) and the positive definiteness of $a(u, v)$, we have

$$\begin{aligned}
\| u_1 - u_2 - \xi\Lambda T(u_1 - u_2) \|^2 &\leq \| u_1 - u_2 \|^2 + \xi^2 \| T \|^2 \| u_1 - u_2 \|^2 - 2\xi a(u_1 - u_2, u_1 - u_2), \\
&\leq (1 + \xi^2\mu^2 - 2\xi\rho) \| u_1 - u_2 \|^2 .
\end{aligned}$$

Hence

$$\begin{aligned}
|\langle \phi(u_1) - \phi(u_2), v\rangle| &\leq \sqrt{(1 + \xi^2\mu^2 - 2\xi\rho)} \| u_1 - u_2 \| \| v \| + \xi \| A(u_1) - A(u_2) \| \| v \| \\
&\leq \{\sqrt{(1 + \xi^2\mu^2 - 2\xi\rho)} + \xi\gamma\} \| u_1 - u_2 \| \| v \|,
\end{aligned}$$

by Lipschitz continuity of A.

$$= \theta \parallel u_1 - u_2 \parallel \parallel v \parallel,$$

where $\theta = \sqrt{1 + \xi^2\mu^2 - 2\xi\rho} + \gamma\xi < 1$ for $0 < \xi < 2\frac{\rho-\gamma}{\mu^2-\gamma^2}$, and $\xi\gamma < 1$, because $\gamma < \rho$ by condition A.

Thus for all $u_1, u_2 \in H$,

$$\begin{aligned} \parallel \phi(u_1) - \phi(u_2) \parallel_{H'} &= \sup_{v\in H} \frac{|\langle \phi(u_1) - \phi(u_2), v\rangle|}{\parallel v \parallel} \\ &\leq \theta \parallel u_1 - u_2 \parallel . \end{aligned}$$

□

Proof of Theorem 2.3:

Proof. Uniqueness:

Let u_1, u_2 be two solutions in H of

$$a(u_1, v) = \langle A(u_1), v\rangle, \qquad \text{for all } v \in H,$$
$$a(u_2, v) = \langle A(u_2), v\rangle, \qquad \text{for all } v \in H.$$

Thus by subtracting and taking v as $(u_1 - u_2)$, we get

$$a(u_1 - u_2, u_1 - u_2) = \langle A(u_1) - A(u_2), u_1 - u_2\rangle.$$

By the positive definiteness of $a(u, v)$ and the antimonotonicity of A, it follows that there exists $\rho > 0$ such that

$$\begin{aligned} \rho \parallel u_1 - u_2 \parallel^2 &= a(u_1 - u_2, u_1 - u_2) \\ &= \langle A(u_1) - A(u_2), u_1 - u_2\rangle \\ &\leq 0. \end{aligned}$$

Hence $u_1 = u_2$, the uniqueness.

Existence:

For a fixed ξ as in Lemma 2.2 and $u \in H$, define $\phi(u) \in H'$, by (2.7). Thus by Theorem 5.10, there exists a unique $w \in H$ such that

$$(w, v) = \langle \phi(u), v\rangle, \qquad \text{for all } v \in H,$$

and w is given by

$$w = \Lambda\phi(u) = Tu,$$

which defines a map from H into itself. Now as in the proof of the Theorem 2.2, one can show by using Lemma 2.2 that Tu is a contraction and has a fixed point $Tu = u \in H$, which satisfies for $\xi > 0$,

$$a(u, v) = \langle A(u), v\rangle, \qquad \text{for all } v \in H.$$

□

Remark 2.3 In particular if $A(u)$ is the Fréchet derivative of a nonlinear functional F (say), then Theorem 2.3 still holds.

For a nonlinear functional F, we have noted that $\langle A(u), v\rangle = \langle F'(u), v\rangle = \langle F, v\rangle$ for all $v \in H$. In this case the condition A implies that $\rho > 0$ and the Lipschitz constant becomes zero. This it is obvious that Lemma 2.2 includes Lemma 2.1 as a special case. Moreover Theorem 2.3 generalizes the Theorem 2.2.
Furthermore, for the special case $a(u, v) = (u, v)$, Theorem 2.3 reduces to:

Theorem 2.4 If A is a lipschitz continuous and antimonotone operator with Lipschitz constant $\gamma < 1$, then there exist a unique $u \in H$ such that

$$(u, v) = \langle A(u), v\rangle, \qquad \text{for all } v \in H.$$

Proof. The proof can be deduced from Lemma 2.2 and Theorem 2.3 by taking $\rho = \mu = 1$. □

Theorem 2.4 shows that the Reisz-Fréchet theorem also holds for a class of monotone operators on H, which includes the Fréchet derivatives of nonlinear functionals as a special case.
We give another proof of Theorem 2.3 based on iteration scheme similar to Picard's and also derive a bound for the error.
We define the iteration u_n by the following scheme

$$a(u_{n+1}, v) = \langle A(u_n), v\rangle, \qquad \text{for all } v \in H. \tag{2.8}$$

Theorem 2.5 If $a(u, v)$ is a positive definite bilinear form on H and A is a Lipschitz continuous operator such that the condition A holds, then the iteration u_n defined by (2.8) converges strongly to u, the solution of (2.6) in H. Moreover, the bound for the error, for any $u_0 \in H$, is given by

$$\| u_n - u \| \leq \frac{\alpha^n}{1-\alpha} \| u_1 - u_0 \|, \quad \text{for } n = 0, 1, 2, ...$$

where $\alpha = \frac{\gamma}{\rho}$.

Proof. By the positive definiteness of $a(u,v)$, it follows that

$$\begin{aligned}\rho \parallel u_{n+1}-u_n \parallel &\leq a(u_{n+1}-u_n, u_{n+1}-u_n)\\ &= \langle A(u_n)-A(u_{n-1}), u_{n+1}-u_n\rangle, \quad \text{by (2.8)}\\ &\leq \parallel A(u_n)-A(u_{n-1}) \parallel\parallel u_{n+1}-u_n \parallel,\end{aligned}$$

by the cauchy-Schwartz inequality.

$$\leq \gamma \parallel u_n - u_{n-1} \parallel\parallel u_{n+1}-u_n \parallel,$$

by the Lipschitz continuity of A.
Thus

$$\begin{aligned}\parallel u_{n+1}-u_n \parallel &\leq \frac{\gamma}{\rho} \parallel u_n - u_{n-1} \parallel\\ &= \alpha \parallel u_n - u_{n-1} \parallel,\end{aligned}$$

where $\alpha = \frac{\gamma}{\rho} < 1$ by condition A.
Continuing in this way, we obtain

$$\parallel u_{n+1}-u_n \parallel \leq \alpha^n \parallel u_1 - u_0 \parallel .$$

Hence, by the repeated use of the triangle inequality, it follows that

$$\begin{aligned}\parallel u_{n+k}-u_n \parallel &\leq (\alpha^{n+k-1}+\cdots+\alpha^n) \parallel u_1-u_0 \parallel,\\ &\leq \frac{\alpha^n}{1-\alpha} \parallel u_1-u_0 \parallel .\end{aligned}$$

Since $\alpha < 1$, it follows that u_n is a Cauchy sequence and has a limit point such that $u_n \rightarrow u \in H$, the unique solution of (2.6).
Also at the same time it implies that

$$u_n \rightarrow u, \text{ in } H \text{ strongly.}$$

□

Remark 2.4 Theorem 2.5 holds for any general complete normed space. Note that it also shows the existence of a unique solution of (2.6).

Remark 2.5 We note that if $a(u,v)$ is a positive definite bilinear form on H, then from (5.57), it follows that for all $u \in H$,

$$\begin{aligned}\rho \parallel u \parallel^2 &\leq a(u,u) = \langle F'(u), v\rangle\\ &\leq \parallel F'(u) \parallel_{H'}\parallel u \parallel,\end{aligned}$$

by the Cauchy-Schwartz inequality.
Thus

$$\| u \| \leq \frac{1}{\rho} \| F'(u) \|_{H'} .$$

This expresses the continuous dependence of u on the Fréchet derivative $F'(u)$. For the linear functional F, it follows from Remark 1.2 that

$$\| u \| \leq \frac{1}{\rho} \| F \|_{H'} .$$

a well known result, see Strang and Fix ([36], page 16).

Chapter 3

Nonlinear Variational Inequalities

In this chapter, we prove the existence, under certain conditions, of unique solutions of the variational inequalities formulated in Chapter 1.
We have seen that the variational inequalities arise naturally in connection with the minimization of convex functionals subject to certain constraints. This suggests that the variational inequalities should share some of the properties of these functionals. We know that one of the specific features of convexity is that some linearization is always possible. We show that linearization of the variational inequalities is also possible. In fact linearization of the variational inequalities plays a very important part in proving the so called regularization approximation. We show that under certain conditions there does exist a unique solution of a more general variational inequality of which (1.7) is a special case.
Let us consider the following problem.

PROBLEM 1:
Find $u \in M$ such that

$$a(u, v-u) \geq \langle A(u), v-u \rangle, \qquad \text{for all } v \in M, \tag{3.1}$$

where A is a nonlinear operator such that $A(u) \in H'$.
We first define the notion of hemicontinuity which will be used later.

Definition 3.1 The operator $T : M \to H'$ is called hemicontinuous [32], if for all $u, v \in M$, the mapping $t \in [0, 1]$ implies that $\langle T(u + t(v-u)), u-v \rangle$ is continuous.

We now state and prove the main result.

Theorem 3.1 let $a(u, v)$ be a positive definite and continuous bilinear form on H and M a closed convex subset of H. If A is Lipschitz continuous and antimonotone such that the condition A holds, then there exists a unique $u \in M$ such that (5.88) holds.

The following lemmas needed for the proof are essentially due to Mosco [26].

Lemma 3.1 Let M be a convex subset of H. Then, given $z \in H$, we have

$$x = P_M z,$$

if and only if

$$x \in M : (x - z, y - x) \geq 0, \qquad \text{for all } y \in M,$$

where P_M is the projection of H in M.

Such type of formulation of the projection P_M is called the weak characterization of P_M and is very useful for proving that P_M is non-expansive.

Lemma 3.2 P_M is non-expansive, that is

$$\| P_M z_1 - P_M z_2 \| \quad \leq \quad \| z_1 - z_2 \|, \qquad \text{for all } z_1, z_2 \in H.$$

Proof. From Lemma 3.1, it follows that we can consider the weak characterization of $x_i = P_M z_i$, $i = 1, 2$, i.e.,

$$(x_i - z_i, y - x_i) \geq 0, \qquad \text{for all } y \in M.$$

Replacing y by x_{3-i}, $i = 1, 2$ adding, we find that

$$(x_1 - x_2, x_1 - x_2) \leq (z_1 - z_2, x_1 - x_2).$$

Hence, by the Cauchy-Schwartz inequality, we obtain

$$\| x_1 - x_2 \| \quad \leq \quad \| z_1 - z_2 \|,$$

showing that

$$\| P_M z_1 - P_M z_2 \| \quad \leq \quad \| z_1 - z_2 \| \, .$$

□

using the technique of Lions-Stampacchia [21], we now prove Theorem 5.11

Proof of Theorem 5.11.

Proof. Uniqueness:

Its proof is similar to that of Theorem 2.3

Existence:

For a fixed ξ as in Lemma 2.2, and $u \in H$, define $\phi(u) \in H'$ by (2.7). By Lemma 3.1, there exists a unique $w \in M$ such that

$$(w, v - w) \geq \langle \phi(u), v - w \rangle, \qquad \text{for all } v \in M,$$

and w is given by

$$w = P_M \Lambda \phi(u) = Tu,$$

which defines a map from H into M.
Now for all $u_1, u_2 \in H$,

$$\begin{aligned}
\| Tu_1 - Tu_2 \| &= \| P_M \Lambda \phi(u_1) - P_M \Lambda \phi(u_2) \| \\
&\leq \| \Lambda \phi(u_1) - \Lambda \phi(u_2) \|, \quad \text{by Lemma 3.2} \\
&\leq \| \phi(u_1) - \phi(u_2) \| \\
&\leq \theta \| u_1 - u_2 \|, \quad \text{by Lemma 2.2,}
\end{aligned}$$

Since $\theta \leq 1$, Tu is a contraction and has a fixed point $Tu = u$, which belongs to M, a closed convex subset of H and satisfies

$$\begin{aligned}
(u, v - u) &\geq \langle \phi(u), v - u \rangle \\
&= (u, v - u) - \xi \{ a(u, v - u) - \langle A(u), v - u \rangle \}.
\end{aligned}$$

Thus for $\xi > 0$,

$$a(u, v - u) \geq \langle A(u), v - u \rangle, \qquad \text{for all } v \in M,$$

showing that u is a unique solution of problem 1. □

Remarks:

1. It is obvious that if $A(u) = F'(u)$, the existence of a unique solution of the variational inequality (1.6) follows under the assumptions of Theorem 5.11.

2. If $M = H$, then solving (5.88) is equivalent to finding $u \in H$ such that

$$a(u, v) = \langle A(u), v \rangle, \qquad \text{for all } v \in H.$$

   Thus Theorem 5.11 includes Theorem 2.3 as a special case, which itself a generalization of the Lax-Milgram lemma.

3. If F is a linear functional, then from Remark 1.2 follows that $\langle F'(u), v\rangle = \langle F, v\rangle$, for all $v \in H$. Thus it follows that the Lipschitz constant γ is zero and Theorem 5.11 reduces to a theorem of Lions-Stampacchia [21].

Theorem 3.2 Let $a(u, v)$ ba a positive definite and continuous bilinear form on H and M ba a closed subset of H. Then for each given $F \in H'$, there exists a unique $u \in M$ such that

$$a(u, v-u) \geq \langle F, v-u\rangle, \qquad \text{for all } v \in M.$$

Furthermore, the map $F \to u$ is continuous from H' into M.

We now recall that when the functional $I[v]$ as defined in (5.52) can be written as a sum of convex differentiable and non-differentiable functionals namely $a(v, v) - 2F_1(v)$ and $-2F_2(v)$ respectively, then the minimum of $I[v]$ on $M \subset H$ is characterized by the variational inequality

$$u \in M : a(u, v-u) \geq \langle F_1'(u), v-u\rangle + 2F_2(v) - 2F_2(u), \qquad \text{for all } v \in M. \tag{3.2}$$

If $a(u, v)$ is a positive definite continuous bilinear form, then from Remark 5.3, it follows that

$$a(u, v) = \langle Tu, v\rangle, \qquad \text{for all } v \in H. \tag{3.3}$$

By using (5.90). the inequalities (1.7) and (5.89) can be written in the following forms

$$\langle Tu, v-u\rangle \geq \langle F'(u), v-u\rangle, \qquad \text{for all } v \in M. \tag{3.4}$$

and

$$\langle Tu, v-u\rangle \geq \langle F_1'(u), v-u\rangle + F_2(v) - F_2(u), \qquad \text{for all } v \in M. \tag{3.5}$$

We will show that Theorem 5.11 can be used to prove the existence of a unique solution of the variational inequality (3.5). In order to apply Theorem 5.11, we must show that (3.5) can be written in the form of (5.91). We follow the technique of Mosco [27], which is also used by Sibony [33] and Brezis [5].

Definition 3.2 As with a convex functional F defined on a convex set M in a Hilbert space, we define the convex set $[F, M]$ in $H \times \mathbb{R}$ as

$$[F, M] = \{[v, \beta] \in H \times \mathbb{R} : F(v) \leq \beta\},$$

where $H \times \mathbb{R}$ is the direct sum of H and the real line $\mathbb{R}$. The set $[F, M]$ is sometimes called the epigraph of F over M.

For the properties of this set see Mosco [26] and Luenberger [22].
We shall denote by $H \oplus \mathbb{R}$, the space $H \times \mathbb{R}$, normed by

$$\| [v, \beta] \| = \{\| v \|^2 + |\beta|^2\}^{\frac{1}{2}}$$

and identify the dual $(H \oplus \mathbb{R})'$ of $H \oplus \mathbb{R}$ with $H' \oplus \mathbb{R}$, i.e., $(H \oplus \mathbb{R}) = H' \oplus \mathbb{R}$. The paring between $H' \oplus \mathbb{R}$ and $H \oplus \mathbb{R}$ will be denoted by

$$\langle [v, \beta], [v', \beta'] \rangle = \langle v, v' \rangle + \beta\beta'. \tag{3.6}$$

Thus in the epigraph formulation the variational inequality (3.5) can be written in the form;

$$u_1 \in M_1 : \langle T_1 u_1, v_1 - u_1 \rangle \geq \langle F_1'(u_1), v_1 - u_1 \rangle, \qquad \text{for all } v_1 \in M_1, \tag{3.7}$$

where

$$\begin{cases} v_1 = [v, \beta] \in H \oplus \mathbb{R} \\ T_1 u_1 = T_1[u, \alpha] = [Tu, -1] \\ F_1'(u_1) = F_1'[u, \alpha] = [F_1'(u), 0] \\ \text{and} \\ M_1 = \{v_1 : v_1 = [v, \beta] \in H \oplus \mathbb{R}; F_2(v) \leq \beta\} \end{cases} \tag{3.8}$$

Conversely, if $u_1 = [u, \alpha] \in M_1$ satisfies (5.96), then from (5.99), it follows that

$$\langle [Tu, -1], [v, \beta] - [u, \alpha] \rangle \geq \langle [F_1'(u), 0], [v, \beta] - [u, \alpha] \rangle,$$

i.e.,

$$\langle [Tu, -1], [v - u, \beta - \alpha] \rangle \geq \langle [F_1'(u), 0], [v - u, \beta - \alpha] \rangle.$$

Using (5.95), we obtain

$$\langle Tu, v - u \rangle \geq \langle F_1'(u), v - u \rangle + \beta - \alpha,$$

that is, $u \in M$, $\alpha \in \mathbb{R}$ with $F_2(u) = \alpha$ such that

$$\langle Tu, v - u \rangle \geq \langle F_1(u), v - u \rangle + F_2(v) - F_2(u),$$

for all $v \in M$ and $\beta \in \mathbb{R}$ with $\beta \geq F_2(u)$. In otherwords, we have (1.7), since by (5.90), (5.96) can be written as

$$u_1 \in M_1 : a_1(u_1, v_1 - u_1) \geq \langle F_1'(u_1), v_1 - u_1 \rangle, \qquad \text{for all } v_1 \in M_1. \tag{3.9}$$

We have shown that the mixed type of variational inequalities can be written in the form (1.7). We conclude that for $A(u) = F_1'(u_1)$ and $a(u, v) = a_1(u_1, v_1)$, the existence of a unique solution of the variational inequality (5.100) follows under the hypotheses of Theorem 5.11.

We prove the following lemma, which gives the linearization of (5.88) and is used in our method of approximation.

Lemma 3.3 If A is hemicontinuous and antimonotone, then $u \in M$ is a solution of (5.88) if and only if u satisfies

$$a(u, v - u) \geq \langle A(v), v - u \rangle, \qquad \text{for all } v \in M. \tag{3.10}$$

Proof. If for a given u in M, (5.88) holds, then (5.101) follows by the antimonotonicity of A. Conversely, suppose (5.101) holds, then for all $t \in [0, 1]$, and $w \in M$, $v_t \equiv u + t(w - u) \in M$, Since M is a convex subset.
Setting $v = v_t$ in (5.101), we have

$$a(u, w - u) \geq \langle A(v_t), w - u \rangle, \qquad \text{for all } w \in M.$$

Let $t \to 0$. Since A is hemicontinuous, $A(v_t) \to A(u)$. It follows

$$a(u, w - u) \geq \langle A(u), w - u \rangle, \qquad \text{for all } w \in M.$$

$\square$

Similarly if $A(u) = F_1'(u)$ is hemicontinuous and antimonotone, then the following problems are equivalent:
I. $u \in M : a(u, v - u) \geq \langle F_1'(u), v - u \rangle + F_2(v) - F_2(u), \quad \text{for all } v \in M.$
II. $u \in M : a(u, v - u) \geq \langle F_1'(v), v - u \rangle + F_2(v) - F_2(u), \quad \text{for all } v \in M.$
From now on, we suppose that the bilinear form $a(u, v)$ is positive, i.e.,

$$a(v, v) \geq 0, \qquad \text{for all } v \in H. \tag{3.11}$$

Assume that there exists at least one solution $u \in M$ of

$$a(u, v - u) \geq \langle A(u), v - u \rangle, \qquad \text{for all } v \in M, \tag{3.12}$$

and X is the set of all solutions of (5.103).
Finally, let $b(u, v)$ be a positive definite bilinear form on H, that is there exists a constant $\rho > 0$ such that

$$b(v, v) \geq \rho \parallel v \parallel^2, \qquad \text{for all } v \in H. \tag{3.13}$$

First of all, we prove some elementary but important results.

Lemma 3.4 If $a(u,v)$ is a positive bilinear form and $u \in M$, then the inequality (5.88) is equivalent to the inequality

$$a(v, v-u) \geq \langle A(u), v-u \rangle, \qquad \text{for all } v \in M. \tag{3.14}$$

Proof. Let (5.88) holds, then

$$\begin{aligned} a(v, v-u) &\geq \langle A(u), v-u \rangle + a(v-u, v-u) \\ &\geq \langle A(u), v-u \rangle, \quad \text{by (5.102).} \end{aligned}$$

Thus (3.14) holds.
Conversely, let (3.14) holds, then for all $t \in [0,1]$ and $w \in M$, $v_t \equiv u + t(w-u) \in M$. Setting $v = v_t$ in (3.14), it follows that

$$a(u, w-u) + ta(w-u, w-u) \geq \langle A(u), w-u \rangle, , \qquad \text{for all } w \in M.$$

Letting $t \to 0$, (5.88) follows. □

As a consequence of Lemma 3.3 and Lemma 3.4, we have:

Theorem 3.3 If $a(u,v)$ is a positive bilinear form and A is a hemicontinuous and antimonotone operator, then the variational inequality (5.88) is equivalent to

$$a(v, v-u) \geq \langle A(v), v-u \rangle, \qquad \text{for all } v \in M.$$

Similarly under the assumptions of Theorem 3.3, one can show that the following problems are equivalent;
I. $u \in M : a(u, v-u) \geq \langle F_1'(u), v-u \rangle + F_2(v) - F_2(u), \quad \text{for all } v \in M.$
II. $u \in M : a(v, v-u) \geq \langle F_1'(v), v-u \rangle + F_2(v) - F_2(u), \quad \text{for all } v \in M.$

Theorem 3.4 If $b(u,v)$ is a positive definite continuous bilinear form and B is a Lipschitz continuous antimonotone operator with $\gamma < \rho$, then there exists a unique $u_0 \in X$ such that

$$b(u_0, v-u_0) \geq \langle B(u_0), v-u_0 \rangle, \qquad \text{for all } v \in X. \tag{3.15}$$

Proof. Obviously X is closed. In order to prove Theorem 3.4, it is enough to show that X is convex. Since $a(u,v)$ is positive, (5.103) is equivalent to

$$a(v, v-u) \geq \langle A(v), v-u \rangle, \quad \text{by Theorem 3.3.}$$

Now for all $t \in [0,1]$, $u_1, u_2 \in X$,

$$\begin{aligned} a(v, v-u_2-t(u_1-u_2)) &= a(v, v-u_2) - ta(v, u_1-u_2) \\ &= a(v, v-u_2) - ta(v, u_1-v+v-u_2) \\ &= a(v, v-u_2) - ta(v, u_1-v) - ta(v, v-u_2) \\ &= (1-t)a(v, v-u_2) + ta(v, v-u_1) \\ &\geq (1-t)\langle A(v), v-u_2\rangle + t\langle A(v), v-u_1\rangle, \end{aligned}$$

by Theorem 3.3.

Thus for all $t \in [0,1]$, $u_1, u_2 \in X$, $tu_1 + (1-t)u_2 \in X$, which implies that X is a convex subset. Hence by Theorem 5.11, there exists a unique solution $u_0 \in X$ satisfying (5.105). □

Theorem 3.5 Assume that (5.102) and (5.104) hold. If $a(u,v) + \epsilon b(u,v)$ is a continuous bilinear form and A and B both antimonotone Lipschitz continuous with $\gamma < \rho$, then there exists a unique solution $u_\epsilon \in M$ such that

$$a(u_\epsilon, v-u_\epsilon) + \epsilon b(u_\epsilon, v-u_\epsilon) \geq \langle A(u_\epsilon) + \epsilon B(u_\epsilon), v-u_\epsilon\rangle, \qquad \text{for all } v \in M. \tag{3.16}$$

Proof. Since for $\epsilon > 0$, and by (5.102) and (5.104), the continuous bilinear form $a(u,v) + \epsilon b(u,v)$ is positive definite on H, then by Theorem 5.11, there does exist a unique solution $u_\epsilon \in M$ satisfying (5.106). □

Using Lemma 3.3 and the method of Sibony [32] and Lions-Stampacchia [21], we prove that the elements of X can be approximated.

Theorem 3.6 Suppose $A, B : M \to H'$ are both hemicontinuous operators and the assumptions of Theorem 3.4 and Theorem 3.5 hold. If u_0 is the element of X defined by (5.105) satisfying

$$a(u_0, v-u_0) \geq \langle A(u_0), v-u_0\rangle, \qquad \text{for all } v \in X, \tag{3.17}$$

and u_ϵ is the element of M defined by (5.106), then

$$u_\epsilon \to u_0 \text{ strongly in } H \text{ as } \epsilon \to 0.$$

Proof. This is proved in three steps.

(i). u_ϵ is bounded in H.

Setting $v = u_0$ in (5.106) and $v = u_\epsilon$ in (3.17), we get

$$a(u_\epsilon, u_0-u_\epsilon) + \epsilon b(u_\epsilon, u_0-u_\epsilon) \geq \langle A(u_\epsilon) + \epsilon B(u_\epsilon), u_0-u_\epsilon\rangle,$$

and

$$a(u_0, u_\epsilon - u_0) \geq \langle A(u_0), u_\epsilon - u_0 \rangle.$$

By addition of these inequalities, it follows from (5.102) and the antimonotonicity of A that

$$b(u_\epsilon, u_0 - u_\epsilon) \geq \langle B(u_\epsilon), u_0 - u_\epsilon \rangle. \tag{3.18}$$

Since $b(u_0, u_\epsilon)$ is a positive definite bilinear form, there exists a constant $\rho > 0$ such that

$$\begin{aligned} \rho \parallel u_\epsilon \parallel^2 &\leq b(u_\epsilon, u_\epsilon) \\ &\leq b(u_\epsilon, u_0) + \langle B(u_\epsilon), u_\epsilon - u_0 \rangle, \quad \text{by (3.18)}. \end{aligned}$$

It follows that $\parallel u_\epsilon \parallel \leq$ constant, independent of ϵ. Hence there exists a subsequence u_ϵ which converges to ξ, say.

(ii). ξ belongs to X.

Since A and B are both antimonotone operators, then by (5.106) and the application of Lemma 3.3, we get

$$a(u_\epsilon, v - u_\epsilon) + \epsilon b(u_\epsilon, v - u_\epsilon) \geq \langle A(v) + \epsilon B(v), v - u_\epsilon \rangle, \qquad \text{for all } v \in M.$$

Now let $\epsilon \to 0$, then $u_\epsilon \to \xi$ weakly and $\liminf a(u_\epsilon, u_\epsilon) \geq a(\xi, \xi)$, see [21], we have

$$a(\xi, v - \xi) \geq \langle A(v), v - \xi \rangle, \qquad \text{for all } v \in X,$$

which is by Lemma 3.3 equivalent to

$$a(\xi, v - \xi) \geq \langle A(\xi), v - \xi \rangle, \qquad \text{for all } v \in X.$$

Thus $\xi \in X$.

(iii). Finally $u_\epsilon \to \xi$, when $\epsilon \to 0$.

Setting $v = u \in X$ in (5.106), we have

$$a(u_\epsilon, u - u_\epsilon) + \epsilon b(u_\epsilon, u - u_\epsilon) \geq \langle A(u_\epsilon) + \epsilon B(u_\epsilon), u - u_\epsilon \rangle,$$

which is by Lemma 3.3 equivalent to

$$a(u_\epsilon, u - u_\epsilon) + \epsilon b(u_\epsilon, u - u_\epsilon) \geq \langle A(u) + \epsilon B(u), u - u_\epsilon \rangle.$$

Also by setting $v = u_\epsilon \in X$ in (5.103), it follows

$$a(u, u_\epsilon - u) \geq \langle A(u), u_\epsilon - u \rangle.$$

By addition of these inequalities one has

$$a(u_\epsilon - u, u - u_\epsilon) + \epsilon b(u_\epsilon, u - u_\epsilon) \geq \epsilon\langle B(u), u - u_\epsilon\rangle.$$

Using (5.102) and for $\epsilon > 0$, we get

$$b(u_\epsilon, u - u_\epsilon) \geq \langle B(u), u - u_\epsilon\rangle, \qquad \text{for all } u \in X.$$

Letting $\epsilon \to 0$, $u_\epsilon \to \xi$, we have

$$\begin{aligned} b(\xi, u - \xi) &\geq \langle B(u), u - \xi\rangle \\ &\geq \langle B(\xi), u - \xi\rangle \quad \text{by Lemma 3.3.} \end{aligned}$$

Thus $\xi \in X$ is a solution of (5.105) and since the solution is unique, it follows that $\xi = u_0$. Also form (3.18), by the positive definiteness of $b(u_0, u_\epsilon)$, it follows that

$$\begin{aligned} \rho \parallel u_\epsilon - u_0 \parallel^2 &\leq b(u_\epsilon - u_0, u_\epsilon - u_0) \\ &\leq \langle B(u_\epsilon), u_\epsilon - u_0\rangle - b(u_0, u_\epsilon - u_0) \\ &\leq \langle B(u), u_\epsilon - u_0\rangle - b(u_0, u_\epsilon - u_0), \quad \text{by Lemma 3.3,} \end{aligned}$$

which $\to 0$, as $\epsilon \to 0$. Hence it follows that $u_\epsilon \to u_0$ strongly in H. $\square$

We note that when $A(u) = F$ and $B(u) = G$, i.e., for the linear functionals F and G, the Theorem 3.4, 3.5 and 3.6 reduces to the results proved by Lions-Stampacchia [21]. A similar approximation also has been considered by Sibony [32].

Chapter 4

Applications

We now describe some examples of the functional $I[v]$, as defined in (5.52) and the variational inequalities as a motivation to all the problems previously discussed in an abstract framework. It has already been pointed out that these problems arise in the mathematical description of physical and engineering situations.

Notations:

Let Ω be a bounded open subset of $\mathbb{R}^n$, $\partial\Omega$ its boundary and $\bar{\Omega} = \Omega \cup \partial\Omega$, its closure. We use the multi-index notation, i.e.,

$$\begin{aligned} &\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n) \\ &|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n \\ &D^\alpha = \frac{\partial|\alpha|}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}. \end{aligned}$$

Let

$$W_2^m(\Omega) = \{u : u \in L_2(\Omega) \text{ and } D^\alpha u \in L_2(\Omega), \text{ for all } |\alpha| \le m\},$$

with norm

$$\| u \| = \left\{ \int_\Omega \sum_{|\alpha| \le m} |D^\alpha u|^2 \mathrm{d}\Omega \right\}^{\frac{1}{2}}.$$

The space $W_2^m(\Omega)$ with this norm are Hilbert spaces. We use the notation $W_2^m(\Omega) \equiv H^m$. The inner product on H^m is given by

$$(u, v) = \int_\Omega \sum_{\alpha \le m} D^\alpha u D^\alpha v \mathrm{d}\Omega, \qquad \text{for all } u, v \in H^m.$$

We define the subspace $W_2^{0m}(\Omega)$ of $W_2^m(\Omega)$ as

$$W_2^{0m}(\Omega) \equiv H_0^m(\Omega) = \left\{ u \in W_2^m(\Omega) : \frac{\partial u}{\partial n}\bigg|_{\partial\Omega} = 0, \ 0 \le \alpha \le m-1 \right\},$$

where the derivatives are considered in a generalized sense. The dual of H_0^m is denoted by $\bar{H}^m$, i.e., $(H_0^m)' = \bar{H}^m$.

Example 4.1 For simplicity, we shall confine ourselves to second order elliptic equations. The problem is thus to find $u \equiv u(x)$ satisfying

$$\left.\begin{aligned} -\Delta u = f(x,u) \qquad & x \in \Omega \\ u = 0, \qquad\quad & x \in \partial\Omega \end{aligned}\right\}, \tag{4.1}$$

where Δ is a Laplace operator and $f(u) \equiv f(x,u)$ is a real-valued continuous nonlinear function involving the unknown u. For instance, if $f(u) \equiv u^n$, then we have the steady-state diffusion equation with a chemical reaction term [18].

It has been shown [18] and [39] that the functional $I[v]$ associated with (4.1) can be given by

$$I[v] = \int_\Omega \left\{ \left(\frac{\partial v}{\partial x}\right)^2 - 2\int_0^v f(\eta)\mathrm{d}\eta \right\} \mathrm{d}\Omega, \qquad \text{for all } v \in H_0^1. \tag{4.2}$$

Let

$$a(u,v) \equiv \int_\Omega \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\mathrm{d}\Omega, \qquad \text{for all } u, v \in H_0^1, \tag{4.3}$$

and

$$F(u) \equiv \int_\Omega \int_0^u f(\eta)\mathrm{d}\eta\mathrm{d}\Omega, \qquad \text{for all } u \in H_0^1, \tag{4.4}$$

so that (4.2) can be written as

$$I[u] = a(v,v) - 2F(v), \tag{4.5}$$

which is the same functional as that considered in chapter 1. It is obvious that the form $a(u,v)$ is bilinear, positive, and symmetric and $F(u)$ is a differentiable nonlinear functional.

We prove a very important result.

Theorem 4.1 If $f(u)$ is a real-valued continuous differentiable nonlinear function, then

$$\langle F'(u), v \rangle = \int_\Omega f(u)v\mathrm{d}\Omega, \qquad \text{for all } v \in H_0^1. \tag{4.6}$$

where $F'(u)$ is the Fréchet derivative of $F(u)$ defined by (4.4).

Proof. For all $u, v \in H_0^1$ and $t \in [0,1]$, we have

$$F(u+tv) = \int_\Omega \left(\int_0^{u+tv} f(\eta)\mathrm{d}\eta \right) \mathrm{d}\Omega. \tag{4.7}$$

Subtracting (4.4) from (4.7), we obtain

$$\begin{aligned} F(u+tv) - F(u) &= \int_\Omega \left\{ \int_0^{u+tv} f(\eta)\mathrm{d}\eta - \int_0^u f(\eta)\mathrm{d}\eta \right\}\mathrm{d}\Omega \\ &= \int_\Omega \left\{ \int_u^{u+tv} f(\eta)\mathrm{d}\eta \right\}\mathrm{d}\Omega \\ &= \int_\Omega \{(u+tv-u)f(\zeta)\}\mathrm{d}\Omega, \end{aligned}$$

by the integral mean value theorem, where $u \le \zeta \le u+tv$.
Thus

$$\frac{F(u+tv)-F(u)}{t} = \int_\Omega f(\zeta)v\mathrm{d}\Omega.$$

Since $f(u)$ is a continuous function, it follows that $f(\zeta) \to f(u)$ as $t \to 0$. Hence letting $t \to 0$, (4.6) follows from the definition of the Fréchet derivative.
Consequently, the variational solution of (4.1) can then be characterized as the solution of the problem:
find $u \in H_0^1$ such that

$$a(u,v) = \langle F'(u), v\rangle, \qquad \text{for all } v \in H_0^1,$$

as follows from Theorem 1.2. □

In order to prove the existence of u, we assume additionally that the nonlinear function $f(u)$ satisfies the following hypotheses:
(i). for each $c > 0$, there exists a number $\gamma_1(c)$ such that

$$\| f(u) - f(v) \|_{L_2(\Omega)} \;\le\; \gamma_1(c) \| u - v \|_{L_2(\Omega)},$$

for all $x \in \bar{\Omega}$ and $|u| \le c$, $|v| \le c$, $u \ne v$, i.e., $f(u)$ is Lipschitz continuous, see, e.g. [14].
(ii). for all $u, v \in H_0^1$, $(f(u)-f(v))(u-v) \le 0$, i.e., $f(u)$ is antimonotone.
The hypothesis (ii) is fundamental to the existence, uniqueness and approximation of the solution of (4.1). We show that the hypotheses (i) and (ii) imply that the Fréchet derivative $F'(u)$ of F at u is Lipschitz continuous and antimonotone. In fact, we have the following lemmas.

Lemma 4.1 Assume that (i) holds, then there exists $\gamma > 0$ such that for all $u, v \in H_0^1$,

$$\| F'(u) - F'(v) \|_{H^{-1}} < \gamma \| u - v \|_{H_0^1} .$$

Proof. For all $u, v \in H_0^1$, consider

$$\langle F'(u) - F'(v), w \rangle = \int_\Omega \{f(u) - f(v)\} w \mathrm{d}\Omega, \qquad \text{for all } w \in H_0^1.$$

Thus

$$\begin{aligned} |\langle F'(u) - F'(v), w \rangle| &= \left| \int_\Omega \{f(u) - f(v)\} w \mathrm{d}\Omega \right| \\ &\leq \left(\int_\Omega |f(u) - f(v)|^2 \mathrm{d}\Omega \right)^{\frac{1}{2}} \left(\int_\Omega |w|^2 \mathrm{d}\Omega \right)^{\frac{1}{2}}, \end{aligned}$$

by Cauchy-Schwartz inequality.

$$\begin{aligned} &\leq \| f(u) - f(v) \| \quad \| w \| \\ &\leq \gamma \| u - v \|_{H_0^1} \| w \|_{H_0^1}, \end{aligned}$$

by (i) and the Sobolev embedding theorem [2, 3].
But for all $u, v \in H_0^1$,

$$\| F'(u) - F'(v) \|_{H^{-1}} = \sup_{w \in H_0^1} \frac{|\langle F'(u) - F'(v), w \rangle|}{\| w \|}.$$

Hence

$$\| F'(u) - F'(v) \|_{H^{-1}} \leq \gamma \| u - v \|_{H_0^1}, \qquad \text{for all } u, v \in H_0^1,$$

showing that $F'(u)$ is also Lipschitz continuous. □

Lemma 4.2 Assume that (ii) holds, then $F'(u)$, defined by (4.6) is antimonotone.

Proof. For all $u, v, w \in H_0^1$, we have

$$\langle F'(u) - F'(v), w \rangle = \int_\Omega (f(u) - f(v)) w \mathrm{d}\Omega.$$

Taking w as $(u - v) \in H_0^1$, it follows that

$$\begin{aligned} \langle F'(u) - F'(v), u - v \rangle &= \int_\Omega (f(u) - f(v))(u - v) \mathrm{d}\Omega \\ &\leq 0, \qquad \text{by (ii)}, \end{aligned}$$

showing that $F'(u)$ is antimonotone. □

We now show that the bilinear form $a(u, v)$ associated with $-\Delta$, defined by (4.3) is continuous and positive definite.

Lemma 4.3 For all $u, v \in H_0^1$,

$$|a(u,v)| \leq \| u \| \| v \| .$$

Proof. For all $u, v \in H^{0,1}$,

$$\begin{aligned} |a(u,v)| &\leq \int_\Omega \left|\frac{\partial u}{\partial x}\right| \cdot \left|\frac{\partial v}{\partial x}\right| \mathrm{d}\Omega \\ &\leq \left(\int_\Omega \left|\frac{\partial u}{\partial x}\right|^2 \mathrm{d}\Omega\right)^{\frac{1}{2}} \left(\int_\Omega \left|\frac{\partial v}{\partial x}\right|^2 \mathrm{d}\Omega\right)^{\frac{1}{2}}, \end{aligned}$$

by Cauchy-Schwartz inequality.

$$\leq \| u \| \| v \|,$$

showing that the bilinear form $a(u,v)$ is continuous with $\mu = 1$. □

We need the following inequality known as the Friedrich's inequality, whose proof is given in [42].

" There exists a positive constant λ such that

$$\int_\Omega \left|\frac{\partial u}{\partial x}\right|^2 \mathrm{d}\Omega \geq \lambda \int_\Omega |u|^2 \mathrm{d}\Omega, \qquad \text{for all } u \in H_0^1\text{."} \tag{4.8}$$

Lemma 4.4 The bilinear form $a(u,v)$ defined by (4.3) is positive definite, i.e., there exists a constant $\rho > 0$ such that

$$a(v,v) \geq \rho \| v \|^2, \qquad \text{for all } v \in H_0^1.$$

Proof. For all $v \in H_0^1$,

$$\begin{aligned} a(v,v) &= \int_\Omega \left\{\left|\frac{\partial v}{\partial x}\right|^2\right\} \mathrm{d}\Omega \\ &= \int_\Omega \left\{\frac{1}{2}\left|\frac{\partial v}{\partial x}\right|^2 + \frac{1}{2}\left|\frac{\partial v}{\partial x}\right|^2\right\} \mathrm{d}\Omega \\ &\geq \frac{1}{2}\int_\Omega \left\{\left|\frac{\partial v}{\partial x}\right|^2\right\} \mathrm{d}\Omega + \frac{1}{2}\lambda \int_\Omega |v|^2 \mathrm{d}\Omega, \quad \text{by (4.8)} \\ &\geq \min\left\{\frac{1}{2}, \frac{1}{2}\lambda\right\} \left(\int_\Omega \left\{|v|^2 + \left|\frac{\partial v}{\partial x}\right|^2\right\} \mathrm{d}\Omega\right) \\ &= \rho \| v \|_{H_0^1}^2, \end{aligned}$$

where $\rho = \min\{\frac{1}{2}, \frac{1}{2}\lambda\}$, implying that $a(u,v)$ is a positive definite bilinear form. □

We can now prove the existence of a solution of the variational problem associated with (4.1). By Lemma 4.1 and 4.2, $F'(u)$ is Lipschitz continuous antimonotne. Further the bilinear

form $a(u, v)$ associated with (4.1) is also continuous and positive definite with $\mu = 1$, $\rho = \min\{\frac{1}{2}, \frac{1}{2}\lambda\}$ as shown in Lemma 4.3 and 4.4. Thus it follows with $\gamma < \rho = \min\{\frac{1}{2}, \frac{1}{2}\lambda\}$, by Theorem 2.3, that there does exist a unique solution of (4.1).
We also note that when $f(u) \equiv f$, i.e., independent of the unknown u, then the existence and uniqueness of a solution of the variational problem

$$a(u, v) = \langle F, v\rangle, \qquad \text{for all } v \in H_0^1,$$

follow from the Lax-Milgram lemma (Theorem 2.2), as has been shown by many authors.

Example 4.2 We now want to minimize the functional $I[v]$ associated with (4.1) given by (4.2) over the set $M \equiv \{v \in H_0^1 : v \geq \psi \text{ a.e. on } \Omega\}$, $\psi(x)$ is a preassigned function on Ω. The function ψ is known as an obstacle function [20, 26]. The set M is a closed convex subset of H_0^1, see [20]. The minimum of $I[v]$ on M, can then be characterized by the variational inequality:

$$u \in M : a(u, v - u) \geq \langle F'(u), v - u\rangle, \qquad \text{for all } v \in M,$$

as follows from Theorem 1.5.

By a similar argument as in Example 4.1, it can be shown that all the hypotheses of Theorem 5.11 are satisfied, from which the existence and uniqueness of $u \in M$ follow.
If the obstacle functions ψ were absent, i.e., if we have $M = \{v \in H_0^1 : v = 0, \text{on } \Omega\}$, then M is a subspace and the inequality becomes an equality. In this case the condition for a minimum of $I[v]$ would be

$$a(u, v) = \langle F'(u), v\rangle, \qquad \text{for all } v \in H_0^1.$$

the case considered in Example 4.1.
A similar problem which includes the capacity problem consisting of minimizing the functional $I[v]$ over all functions v which are $\geq \psi$ only on a given compact subset E of Ω has been considered by Lions-Stampacchia [21] and Mosco [26].

Example 4.3 (**A Bingham's fluid:**)
In its direct variational formulation, the problem consists in minimizing the non-differentiable functional

$$I[u] = \int_\Omega \left\{ \left(\frac{\partial v}{\partial x}\right)^2 \mathrm{d}\Omega - 2\int_0^v f(\eta)\mathrm{d}\eta \right\} \mathrm{d}\Omega + 2g\int_\Omega |\text{grad } v| \mathrm{d}\Omega, \tag{4.9}$$

where $g > 0$, the plasticity threshold and $f(v) \in L_2(\Omega)$, $F_1(v) = \int_\Omega \{\int_0^v f(\eta)\mathrm{d}\eta\}\mathrm{d}\Omega$, is the continuous decay of the pressure over the closed convex set $M = \{v \in H_0^1 : v \geq \psi \text{ on } \Omega\}$ in H_0^1.

Writing (4.9) as

$$I[v] = a(v,v) - 2F_1(v) - 2F_2(v),$$

where

$$a(u,v) = \int_\Omega \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\mathrm{d}\Omega, \text{ a differentiable functional on } H_0^1.$$

$$F_1(v) = \int_\Omega \left\{\int_0^v f(\eta)\mathrm{d}\eta\right\}\mathrm{d}\Omega, \text{ a nonlinear differentiable functional on } H_0^1,$$

and

$$F_2(v) = -g\int_\Omega |\mathrm{grad} v|\mathrm{d}\Omega, \quad \text{a non-differentiable functional on } H_0^1.$$

Thus we have here a functional arising from a physical and engineering system which can be written as a sum of differentiable and non-differentiable functionals. The minimizing u of $I[v]$ on M can then be characterized by the mixed nonlinear variational inequality

$$a(u, v-u) \geq \langle F_1'(u), v-u\rangle + F_2(v) - F_2(u), \qquad \text{for all } v \in M,$$

as follows from the Theorem 1.7.

If $F_1 \equiv 0$, then the minimizing u is characterized by the mixed variational inequality

$$a(u, v-u) \geq F_2(v) - F_2(u), \qquad \text{for all } v \in M,$$

a case studied for example by Sibony [33], Mosco [26] and Duvaut-Lions [15]. For the physical motivation of this problem, see Duvaut-Lions [15].

Problems involving non-differentiable functionals occur in the theory of elastic bodies subject to unilateral (obstacle) boundary constraints, Fichera [17] and Duvaut-Lions [15].

If $F_2 \equiv 0$, and F_1 is a linear functional, then we have the first and most fundamental of the variational problems introduced and considered by Fichera [17] and Stampacchia [34] in connection with elasticity and capacity problems.

All the problems involving the bilinear form $a(u,v)$, so far considered, can be generalized, see for example Sibony [33].

Chapter 5

Inequalities Bounding the Error

Much attention has been given to the numerical solution of boundary value problems involving nonlinear elliptic partial differential equations, see Ciarlet, Schultz and Varga [12], and Varga [41]. The approach has been that of considering weak form of these problems which involve quasibilinear forms. In this chapter, a method developed in chapters 1,2 and 3 is adopted whereby only a bilinear form is employed.
We are concerned with the numerical solution $u \equiv u(x)$ of problems of the type

$$\left.\begin{array}{rl} L[u] = f(x,u) & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{array}\right\}, \tag{5.1}$$

where $\Omega \subset \mathbb{R}^2$ is a simply connected open domain with boundary $\partial\Omega$, L is a second order linear, positive definite elliptic operator and f is a nonlinear function of unknown u. It is assumed that the function f and the boundary $\partial\Omega$ satisfy certain smoothness conditions, see [1] and [2], which ensure the existence and the uniqueness of the solution u.

First of all, we consider the analogous linear problem of the the type

$$\left.\begin{array}{rl} L[u] = f(x) & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{array}\right\}, \tag{5.2}$$

where f is now a function only of the space variables. In this latter case, the approach is to consider either of two related problems, the solutions of which are the weak solutions of (5.2). The problems equivalent to (5.2) are:

(a). The variational problem:

Find $u \in H_0^1$, which gives the minimum value to the functional associated with (5.2), namely

$$I[v] = a(v,v) - 2\langle f, v\rangle, \qquad \text{for all } v \in H_0^1. \tag{5.3}$$

(b). The weak problem:

Find $u \in H_0^1$ such that

$$a(u,v) = \langle f, v\rangle, \qquad \text{for all } v \in H_0^1. \tag{5.4}$$

where the bilinear form $a(u,v)$ is associated with the operator L, and is in fact

$$\langle Lu, v\rangle = \int_{\Omega} Luv\mathrm{d}\Omega,$$

after integration by parts has been performed.
The equivalence of the variational and weak problems has been shown by Stampacchia [34], see Theorem 1.1.
A technique for the approximate solution of (5.2) is to seek $u_h \in S_h$, where S_h is a finite dimensional subspace if H_0^1, such that u_h solves

$$\min_{v_h \in S_h} (I[v_h]), \tag{5.5}$$

or

$$a(u_h, v_h) = \langle f, v_h\rangle, \qquad \text{for all } v_h \in S_h. \tag{5.6}$$

From (5.4) and (5.6), we obtain

$$a(u - u_h, v_h) = 0, \qquad \text{for all } v_h \in S_h,$$

which shows that u_h is the projection of u on the finite dimensional subspace S_h. If $a(u,v)$ is a positive definite and continuous bilinear form, then it has been shown by many workers including Birkhoff, Schultz and Varga [4], and Zlamal [47] that

$$\| u - u_h \| \leq C \| u - \tilde{u_h} \|, \tag{5.7}$$

where $C > 0$ is a constant independent of h, and $\tilde{u_h} \in S_h$ is an interpolant to u. The inequality (5.7) has the effect of making the problem of finding a bound on $\| u - u_h \|$ one of approximation theory. Bounds on this interpolation error in various situations have been derived by Ciarlet and Raviart [11] and Zlamal [47].

Our aim in this chapter is to show that similar bounds exist for the nonlinear problem (5.1). We have shown in chapters 1 and 4 that the problems equivalent to (5.1) are:

(a$'$). The Variational Problem:

Find $u \in H_0^1$ which gives the minimum value to the functional associated with (5.1), namely

$$I[v] \equiv a(v,v) - 2F(v), \qquad \text{for all } v \in H_0^1. \tag{5.8}$$

(b$'$). The Weak Problem:

Find $u \in H_0^1$ such that

$$a(u,v) = \langle F'(u), v\rangle, \qquad \text{for all } v \in H_0^1, \tag{5.9}$$

where

$$F(v) \equiv \int_{\Omega} \left(\int_0^v f(\eta) \mathrm{d}\eta \right) \mathrm{d}\Omega,$$

and $\langle F'(u), v \rangle$ is the Fréchet differential.

We have shown in chapters 1 and 4 that under certain conditions the variational and the weak problem associated with (5.1) are equivalent.

As in the linear case, approximations u_h to u from a finite dimensional subspace $S_h \subset H_0^1$ are sought. problems (5.8) and (5.9) are therefore reformulated respectively in finite dimensional form as:

Find $u_h \in S_h$ such that

$$I[u_h] \leq I[v_h], \qquad \text{for all } v_h \in S_h, \tag{5.10}$$

and find $u_h \in S_h$ such that

$$a(u_h, v_h) = \langle F'(u_h), v_h \rangle, , \qquad \text{for all } v_h \in S_h. \tag{5.11}$$

The function $u_h \in S_h$ can for example be found from (5.10) using the Ritz method or from (5.11) using the Galerkin technique [44].

Subtracting (5.11) from (5.9), we obtain

$$a(u - u_h, v_h) = \langle F'(u) - F'(u_h), v_h \rangle, \qquad \text{for all } v_h \in S_h. \tag{5.12}$$

This shows that in the nonlinear case, the approximate solution u_h is not the projection of u on the finite dimensional subspace S_h as it was in the linear case. We, therefore, introduce the concept of pseudo projection in order to obtain the inequality bounding the error $(u - u_h)$. Similar projection have been introduced by Dailey and Pierce [14], Wheeler [43], and Pierce and varga [31] to derive the error bounds for $(u - u_h)$.

Define $\bar{u} \in S_h$ to be the pseudo projection of $u \in H_0^1$ by the orthogonality condition

$$a(u - \bar{u}, w_h) = 0, \qquad \text{for all } w_h \in S_h. \tag{5.13}$$

We remark that for the linear case $\bar{u} \in S_h$ is the Galerkin approximation of the solution of (5.4) and hence $\bar{u}$ is the projection of u on $S_h \subset H_0^1$, see [14, 31].

We have the following.

Theorem 5.1 If $a(u, v)$ is a positive definite continuous bilinear form on H_0^1 and $\bar{u} \in S_h$ is defined by (5.13) and $u \in H_0^1$ by (5.9), then

$$\| u - \bar{u} \| \leq \frac{\mu}{\rho} \| u - v_h \|, \qquad \text{for all } v_h \in S_h, \tag{5.14}$$

where μ and ρ are as in Definition 1.5.

Proof. From the positive definiteness of $a(u,v)$, it follows that

$$\begin{aligned}\rho \parallel u-\bar{u} \parallel^2 &\leq a(u-\bar{u}, u-\bar{u}) \\ &= a(u-\bar{u}, u-v_h) + a(u-\bar{u}, v_h-\bar{u}),\end{aligned}$$

for all $v_h \in S_h$.
Since $v_h - \bar{u} \in S_h$, so taking w_h as $v_h - \bar{u}$ in (5.13), we get

$$a(u-\bar{u}, v_h-\bar{u}) = 0.$$

Thus

$$\begin{aligned}\rho \parallel u-\bar{u} \parallel^2 &\leq a(u-\bar{u}, u-v_h) \\ &\leq \mu \parallel u-\bar{u} \parallel\parallel u-v_h \parallel,\end{aligned}$$

by the continuity of $a(u,v)$. Hence we have

$$\parallel u-\bar{u} \parallel \leq \frac{\mu}{\rho} \parallel u-v_h \parallel \qquad \text{for all } v_h \in S_h.$$

□

Theorem 5.2 Assume that the hypotheses of Theorem 5.1 hold. If F' is Lipschitz continuous and antimonotone, then

$$\parallel \bar{u}-u_h \parallel \leq \frac{\gamma}{\rho} \parallel u-\bar{u} \parallel, \tag{5.15}$$

where u_h is defined by (5.11).

Proof. It follows from the positive definiteness of $a(u,v)$ that for all $\bar{u}, u_h \in S_h$,

$$\begin{aligned}\rho \parallel \bar{u}-u_h \parallel^2 &\leq a(\bar{u}-u_h, \bar{u}-u_h) \\ &\leq a(\bar{u}-u_h, \bar{u}-u_h) - \langle F'(\bar{u}) - F'(u_h), \bar{u}-u_h\rangle,\end{aligned}$$

by the antimonotonicity of F',

$$\begin{aligned}&= a(\bar{u}-u, \bar{u}-u_h) + a(u-u_h, \bar{u}-u_h) - \langle F'(\bar{u}) - F'(u_h), \bar{u}-u_h\rangle \\ &= a(u-u_h, \bar{u}-u_h) - \langle F'(\bar{u}) - F'(u_h), \bar{u}-u_h\rangle, \quad \text{by (5.13)} \\ &= a(u-u_h, \bar{u}-u_h) - \langle F'(\bar{u}) - F'(u), \bar{u}-u_h\rangle - \langle F'(u) - F'(u_h), \bar{u}-u_h\rangle \\ &= \langle F'(u) - F'(\bar{u}), \bar{u}-u_h\rangle, \quad \text{by (5.12)} \\ &\leq \parallel F'(u) - F'(\bar{u}) \parallel\parallel \bar{u}-u_h \parallel,\end{aligned}$$

by Cauchy-Schwartz inequality.

$$\leq \gamma \parallel u - \bar{u} \parallel \parallel \bar{u} - u_h \parallel,$$

by the Lipschitz continuity of F'.
Thus we obtain

$$\parallel \bar{u} - u_h \parallel \leq \frac{\gamma}{\rho} \parallel u - \bar{u} \parallel, \text{ the desired (5.15).}$$

□

From Theorem 5.1 and 5.2, we obtain the fundamental inequality for the error $u - u_h$.

Theorem 5.3 Under the assumptions of Theorem 5.2, we get

$$\parallel u - u_h \parallel \leq \frac{\mu}{\rho}\left(1 + \frac{\gamma}{\rho}\right) \parallel u - v_h \parallel, \qquad \text{for all } v_h \in S_h. \tag{5.16}$$

Proof. Let u and u_h be solutions of (5.9) and (5.11) respectively, then

$$\begin{aligned} \parallel u - u_h \parallel &\leq \parallel u - \bar{u} \parallel + \parallel \bar{u} - u_h \parallel, \quad \text{for all } u \in S_h \\ &\leq \left(1 + \frac{\gamma}{\rho}\right) \parallel u - \bar{u} \parallel, \quad \text{by (5.15)} \\ &\leq \frac{\mu}{\rho}\left(1 + \frac{\gamma}{\rho}\right) \parallel u - v_h \parallel, \quad \text{by (5.14).} \end{aligned}$$

□

In particular (5.16) holds for the interpolant $\tilde{u}_h \in S_h$ to u as in the linear case, so that

$$\parallel u - u_h \parallel \leq C_1 \parallel u - \tilde{u}_h \parallel, \tag{5.17}$$

where $C_1 = \frac{\mu}{\rho}(1 + \frac{\gamma}{\rho}) > 0$ is a constant. The inequality (5.17) has the effect of making the problem of finding a bound on $\parallel u - u_h \parallel$ one of approximation theory.

Variational Inequalities:

In this section, we study the general error bound for the nonlinear variational inequalities introduced in chapter 1. The problem is now that of minimizing $I[v]$ associated with (5.1) over a set of admissible function v which satisfy an additional constraint condition such as $v \geq \psi$ for given ψ on the domain Ω. In this case, the solution $u \in M \subset H_0^1$ is characterized by the nonlinear variational inequality

$$a(u, v - u) \geq \langle F'(u), v - u \rangle, \qquad \text{for all } v \in M. \tag{5.18}$$

For the linear elliptic boundary value problem (5.2), this type of variational inequality has been much studied by Lions and Stampacchia [21], Mosco [26], Duvaut and Lions [15], and Brezis [5] in an abstract setting. In a more practical setting Fremond [19] has considered a problem from elasticity concerning the equilibrium position of a solid resting on a rigid support, where v is the displacement. Here the potential energy functional is minimized over the space of admissible displacements. The constraints arises from the contact of the solid and the support. Another application is in plasticity [35], where v represents the stress and ψ is the yield limit. If the yield limit is reached, the physical situation is one of plasticity.
We first state the problem of Fichera [17] and Stampacchia [34], see Theorem 1.3.
Find that function u in the convex set $M = \{v : v \in H_0^1, v \geq \psi \ on \ \Omega\}$, such that

$$a(u, v-u) \geq \langle F, v-u \rangle, \qquad \text{for all } v \in M, \tag{5.19}$$

where $F \in \bar{H}^1$, the dual space of H_0^1.
We are of course interested here in the use of finite element method for the approximation of the solution u of (5.19) over a finite dimensional convex set $M_h \subset H_0^1$. The finite dimensional form of (5.19) is that of finding $u_h \in M_h$ such that

$$a(u_h, v_h - u_h) \geq \langle F, v_h - u_h \rangle, \qquad \text{for all } v_h \in M_h. \tag{5.20}$$

We note that the choice of an approximate convex set M_h is thus a basic additional element of the finite element method of approximation of the variational inequalities, we are dealing with. In the case of an equation. we have to choose the subspace $S_h \subset H_0^1$ only. For the construction of a closed convex subset M_h, the following criteria should be implied.

1. M_h should be "good" approximation to M. Mosco [26] has shown how the trivial choice of M_h can be a bad one.

2. The corresponding discrete problem should be as "easy" to solve as possible.

We know [36] that the usual estimate of the error $(u - u_h)$ in the finite element method for the linear elliptic equation is bases on the inequality

$$\| u - u_h \| \leq \operatorname{dist}(u, s_h),$$

where S_h is the subspace of H_0^1.
Mosco [26], Mosco and Strang [28] have shown that in general for variational inequalities, this estimate is not true.
Bound for the error $(u - u_h)$ have recently been derived, for the linear variational inequality (5.19) and (5.20), by Falk [16].
We state his result without proof.

Theorem 5.4 If u and u_h are solutions of (5.19) and (5.20) respectively, and the mapping $L : H_0^1 \to \bar{H}^1$ is defined for all $u \in H_0^1$ by $a(u, v) = \langle Lu, v\rangle$, where $a(u, v)$ is a positive definite, symmetric and continuous bilinear form on H_0^1, then

$$\| u - u_h \| \leq \left\{ \frac{\mu^2}{\rho^2} \| u - v_h \|^2 + \frac{2}{\rho} \| F - Lu \|_{\bar{H}^1} (\| u - v_h \| + \| u_h - v \|) \right\}^{\frac{1}{2}}, \tag{5.21}$$

for all $v \in M$ and $v_h \in M_h$.

Note that we do not require that $M_h = M \cap S_h$, or even that M_h be contained in M, where S_h is a finite dimensional subspace of H_0^1. We also observe that on account of the constraint Lu is unequal to F in part of $\bar{H}^1$. Removal of this constraint reduces $Lu = F$ throughout $\bar{H}^1$ so that the second term on the right hand side of (5.21) drops out, M_h becomes S_h, a finite dimensional subspace of H_0^1, and again we have the inequality (5.7).
A natural extension to this theory is to the nonlinear variational inequality (5.18). The finite dimensional problem is that of finding $u_h \in M_h$ such that

$$a(u_h, v_h - u_h) \geq \langle F'(u_h), v_h - u_h\rangle, \qquad \text{for all } v_h \in M_h. \tag{5.22}$$

We are interested in deriving the bound for the error $(u - u_h)$. We prove.

Theorem 5.5 Let u and u_h be respectively solutions of (5.18) and (5.22). The mapping $L : H_0^1 \to \bar{H}^1$ is defined for all $v \in H_0^1$ by $a(u, v) = \langle Lu, v\rangle$, where $a(u, v)$ is a positive definite and continuous bilinear form on H_0^1. If the Fréchet derivative F' is Lipschitz continuous, then

$$\begin{aligned} \| u - u_h \| \quad \leq \quad & \left\{ \frac{3\mu^2}{\rho^2} \| u - v_h \|^2 + \frac{3\mu^2}{\rho^2} \| v - u_h \|^2 + \frac{3\gamma^2}{\rho^2} \| v_h - v \|^2 \right. \\ & + \frac{1}{\rho} \| F'(u) - Lu \| \| u - v_h \| + \frac{1}{\rho} \| F'(u_h) - Lu_h \| \| u_h - v \|, \end{aligned} \tag{5.23}$$

for all $v \in M$ and $v_h \in M_h$.

Proof. Since u and u_h are solutions of (5.18) and (5.22), we have

$$a(u, u - v) \leq \langle F'(u), u - v\rangle, \qquad \text{for all } v \in M,$$

and

$$a(u_h, u_h - v_h) \leq \langle F'(u_h), u_h - v_h\rangle, \qquad \text{for all } v_h \in M_h.$$

Adding these inequalities, we obtain

$$\begin{aligned} a(u, u) + a(u_h, u_h) \quad \leq \quad & \langle F'(u), u - v\rangle + \langle F'(u_h), u_h - v_h\rangle \\ & + a(u, v) + a(u_h, v_h). \end{aligned}$$

Subtracting $a(u,u_h)+a(u_h,u)$ from both sides and rearranging terms, we get

$$\begin{aligned}
a(u-u_h,u-u_h) &\leq \langle F'(u),u-v\rangle+\langle F'(u),u_h-v_h\rangle+a(u,v-u_h)+a(u_h,v_h-u)\\
&= \langle F'(u),u-v_h\rangle+\langle F'(u_h),u_h,v\rangle+\langle F'(u)-F'(u_h),v_h-v\rangle\\
&\quad +a(u,v-u_h)+a(u_h,v_h-u)\\
&= \langle F'(u)-Lu,u-v_h\rangle+\langle F'(u_h)-Lu_n,u_h-v\rangle\\
&\quad +\langle F'(u)-F'(u_h),v_h-v\rangle+a(u-u_h,v-u_h\rangle+a(u-u_h,u-v_h).
\end{aligned}$$

By the positive definiteness, continuity of $a(u,v)$ and the Cauchy-Schwartz inequality, it follows that

$$\begin{aligned}
\rho \parallel u-u_h\parallel^2 &\leq \parallel F'(u)-Lu\parallel\parallel u-v_h\parallel+\parallel F'(u_h)-Lu_h\parallel\parallel u_h-v\parallel\\
&\quad +\parallel F'(u)-F'(u_h)\parallel\parallel v_h-v\parallel+\mu\parallel u-u_h\parallel\parallel v-u_h\parallel\\
&\quad +\mu\parallel u-u_h\parallel\parallel u-v_h\parallel\\
&\leq \parallel F'(u)-Lu\parallel\parallel u-v_h\parallel+\parallel F'(u_h)-Lu_h\parallel\parallel u_h-v\parallel\\
&\quad +\gamma\parallel u-u_h\parallel\parallel v_h-v\parallel+\mu\parallel u-u_h\parallel\parallel v-u_h\parallel\\
&\quad +\mu\parallel u-u_h\parallel\parallel u-v_h\parallel,
\end{aligned}$$

by the Lipschitz continuity of F'.
Since the inequality

$$ab\leq\epsilon a^2+\frac{1}{4\epsilon}b^2,$$

holds for positive a,b and any $\epsilon>0$, we have

$$\begin{aligned}
\gamma\parallel u-u_h\parallel\parallel v_h-v\parallel &\leq \frac{\rho}{6}\parallel u-u_h\parallel^2+\frac{3\gamma^2}{2\rho}\parallel v_h-v\parallel^2\\
\mu\parallel u-u_h\parallel\parallel v-u_h\parallel &\leq \frac{\rho}{6}\parallel u-u_h\parallel^2+\frac{3\mu^2}{2\rho}\parallel v-u_h\parallel^2\\
\mu\parallel u-u_h\parallel\parallel u-v_h\parallel &\leq \frac{\rho}{6}\parallel u-u_h\parallel^2+\frac{3\mu^2}{2\rho}\parallel u-v_h\parallel^2.
\end{aligned}$$

By the use of these inequalities, (5.23) follows. □

Note that in the absence of the constraints, we have $Lu=F'(u)$ throughout the whole space $\bar{H}^1$ and the last two terms of the right hand side of (5.23) drop out. In this case, we have the following error bound for the problem (5.9),

$$\parallel u-u_h\parallel\leq\left\{\frac{3\mu^2}{\rho^2}\parallel u-v_h\parallel^2+\frac{3\mu^2}{\rho^2}\parallel v-u_h\parallel^2+\frac{3\gamma^2}{\rho^2}\parallel v_h-v\parallel^2\right\}^{\frac{1}{2}}. \qquad (5.24)$$

Comparing (5.24) with (5.16), we note the differences. The inequality (5.16) was derived by using the concept of pseudo projection whereas (5.24) without it.
We now consider another type of the variational inequality introduced and studied by Browder [7] and Sibony [32]. First, we define the concept of strongly monotonicity.

Definition 5.1 A nonlinear operator $T : M \to H'$ is said to be strongly monotone, if there exists a constant $\tau > 0$ such that

$$\langle Tu - Tv, u - v \rangle \geq \tau \parallel u - v \parallel^2, \qquad \text{for all } u, v \in M, \tag{5.25}$$

and is Lipschitz continuous, if there exists a constant $\lambda > 0$ such that

$$\parallel Tu - Tv \parallel \leq \lambda \parallel u - v \parallel, \qquad \text{for all } u, v \in M, \tag{5.26}$$

We consider the following problem.
Problem 5.1:
Given $F \subset H'$, and T is a stongly monotone Lipschitz continuous operator from M into H', then find $u \in M$ such that

$$\langle Tu, v - u \rangle \geq \langle F, v - u \rangle, \qquad \text{for all } v \in M. \tag{5.27}$$

This problem has been studied by Browder [7] in the Banach spaces. We consider this problem in a Hilbert space H and prove a new theorem using the technique of Lions-Stampacchia developed in chapters 2 and 3. Note that we do not require that the nonlinear operator T is hemicontinuous as was the case in [7] and [13].

Theorem 5.6 Let M be a closed convex subset of H. If T is nonlinear strongly monotone and Lipschitz continuous from M into H', then for each $F \subset H'$, there exists a unique solution $u \in M$ of (5.27). Moreover, the solution depends continuously on F.

We defer the proof until later.
If $M = H$, then the inequality (5.27) reduces to an inequality

$$\langle Tu, v \rangle = \langle F, v \rangle, \qquad \text{for all } v \in H. \tag{5.28}$$

In this case, we have the following.

Theorem 5.7 If T is a strongly monotone and Lipschitz continuous operator, then for each $F \subset H'$, there exists a unique solution of (5.28).

Proof. The proof follows from Theorem 5.6. $\square$

Given a finite dimensional subspace $S_h \subset H$, we consider approximate problem of finding $u_h \subset S_h$ such that

$$\langle Tu_h, v_h \rangle = \langle F, v_h \rangle, \qquad \text{for all } v_h \in S_h. \tag{5.29}$$

From (5.28) and (5.29), we obtain

$$\langle Tu - Tu_h, v_h \rangle = 0, \qquad \text{for all } v_h \in S_h. \tag{5.30}$$

The relation (5.30) is known as the orthogonality condition for the nonlinear case.
We derive the error bound, which is well-known [41] and [13].

Theorem 5.8 If u and u_h are respectively solutions of (5.28) and (5.29), then under the assumptions of Theorem 5.7,

$$\| u - u_h \| \leq \left(1 + \frac{\lambda}{\tau}\right) \| u - v_h \|, \qquad \text{for all } v_h \in S_h, \tag{5.31}$$

where τ, λ are as defined in Definition 5.1.

Proof. Since T is strongly monotone, we have for all $u_h, v_h \in S_h$,

$$\begin{aligned} \tau \| u_h - v_h \|^2 &\leq \langle Tu_h - Tv_h, u_h - v_h \rangle \\ &= \langle Tu - Tv_h, u_h - v_h \rangle, \quad \text{by (5.30)} \\ &\leq \| Tu - Tv_h \| \| u_h - v_h \|, \end{aligned}$$

by the Cauchy-Schwartz inequality.

$$\leq \lambda \| u - v_h \| \| u_h - v_h \|,$$

by the Lipschitz continuity of T.
Thus

$$\| u_h - v_h \| \leq \frac{\lambda}{\tau} \| u - v_h \| . \tag{5.32}$$

We want the error bound for $(u - u_h)$.
Now

$$\begin{aligned} \| u - u_h \| &\leq \| u - v_h \| + \| v_h - u_h \|, \quad \text{for all } v_h \in S_h \\ &\leq \left(1 + \frac{\lambda}{\tau}\right) \| u - v_h \|, \quad \text{by (5.32)}, \end{aligned}$$

the required result. $\square$

In particular (5.31) holds for the interpolant $\tilde{u} \in S_h$ of u. The inequality (5.31) is a generalization of the inequality (5.7). This in fact allows us to treat nonlinear versions of the homogeneous boundary value problems of (5.2).
Our next result is a generalization of the Lions-Stampacchia Lemma 2.1.

Lemma 5.1 *Let T be a nonlinear strongly monotone Lipschitz continuous operator and ζ be a number such that $0 < \zeta < \frac{2\tau}{\lambda^2}$. Then for κ, $0 < \kappa < 1$ and $u_1, u_2 \in H$,*

$$\| \phi(u_1) - \phi(u_2) \| \leq \kappa \| u_1 - u_2 \|,$$

where $\phi(u) \in H'$ is defined, for each $F \in H'$; by

$$\langle \phi(u), v \rangle = (u, v) - \zeta \langle Tu, v \rangle + \zeta \langle F, v \rangle, \qquad \text{for all } v \in H. \tag{5.33}$$

Proof. For all $v, u_1, u_2 \in H$,

$$\begin{aligned} \langle \phi(u_1) - \phi(u_2), v \rangle &= (u_1 - u_2, v) - \zeta \langle Tu_1 - Tu_2, v \rangle \\ &= (u_1 - u_2 - \zeta(\Lambda Tu_1 - \Lambda Tu_2), v), \quad \text{by (2.4)}. \end{aligned}$$

Thus

$$|\langle \phi(u_1) - \phi(u_2), v \rangle| \leq \| u_1 - u_2 - \zeta(\Lambda Tu_1 - \Lambda Tu_2) \| \| v \| .$$

Now by the strongly monotonicity of T,

$$\begin{aligned} \| u_1 - u_2 - \zeta(\Lambda Tu_1 - \Lambda Tu_2) \|^2 &\leq \| u_1 - u_2 \|^2 + \zeta^2 \| Tu_1 - Tu_2 \|^2 \\ &\quad -2\tau\zeta \| u_1 - u_2 \|^2 \\ &\leq (1 + \zeta^2\lambda^2 - 2\tau\zeta) \| u_1 - u_2 \|^2, \end{aligned}$$

by Lipschitz continuity of T.
Hence using the duality norm [11], we obtain as in Lemma 2.2,

$$\| \phi(u_1) - \phi(u_2) \| \leq \kappa \| u_1 - u_2 \|,$$

where $\kappa^2 = 1 + \zeta^2\lambda^2 - 2\tau\zeta < 1$ for $0 < \zeta < \frac{2\tau}{\lambda^2}$. $\square$

Proof of Theorem 5.6.

Proof. (a). Uniqueness and continuity of $u \in M$ follow from Theorem 2.2.
(b). **Existence:**

For a fixed ζ as in Lemma 5.1 and $u \in H$, define $\phi(u) \in H'$, by (5.33). Then by Lemma 3.1, there exists a unique $w \in M$ such that

$$(w, v - w) \geq \langle \phi(u), v - w \rangle, \qquad \text{for all } w \in M,$$

and w is given by

$$w = P_M \Lambda \phi(u) = T_1 u,$$

which defines a map from H into M.
Now for all $u_1, u_2 \in H$,

$$\begin{aligned} \| T_1 u_1 - T_1 u_2 \| &= \| P_M \Lambda \phi(u_1) - P_M \Lambda \phi(u_2) \| \\ &\leq \| \Lambda \phi(u_1) - \Lambda \phi(u_2) \|, \quad \text{by Lemma 3.2} \\ &\leq \| \phi(u_1) - \phi(u_2) \| \\ &\leq \kappa \| u_1 - u_2 \|, \quad \text{by Lemma 5.1.} \end{aligned}$$

Since $\kappa < 1$, $T_1 u$ is a contraction and has a fixed point $T_1 u = u \in M$, a closed convex subset of H. Thus for $\zeta > 0$, we get $u \in M$ satisfying

$$\langle T(u), v - u \rangle \geq \langle F, v - u \rangle, \qquad \text{for all } v \in M.$$

□

Remark 5.1 If T is a linear operator, then Theorem 5.6 reduces to Theorem 3.2.

The finite element methods can be used to find an approximation to the solution u of (5.27) over a convex subset M_h of H. The approximate problem is that of finding $u_h \in M_h$ such that

$$\langle T u_h, v_h - u_h \rangle \geq \langle F, v_h - u_h \rangle, \qquad \text{for all } v_h \in M_h. \tag{5.34}$$

We now derive the general bound for the error $u - u_h$.

Theorem 5.9 If u and u_h are the solutions of (5.27) and (5.34) respectively and T is a strongly monotone Lipschitz continuous operator, then

$$\| u - u_h \| \leq \left\{ \frac{\gamma^2}{\tau^2} \| u - v_h \|^2 + \frac{2}{\tau} \| F - Tu \| \left(\| u_h - v \| + \| u - v_h \| \right) \right\}^{\frac{1}{2}}, \tag{5.35}$$

for all $v \in M$ and $v_h \in M_h$.

Proof. Since u and u_h are the solutions of (5.27) and (5.34) respectively, we have

$$\langle Tu, u-v\rangle \le \langle F, u-v\rangle, \qquad \text{for all } v \in M,$$

and

$$\langle Tu_h, u_h - v_h\rangle \le \langle F, u_h - v_h\rangle, \qquad \text{for all } v_h \in M_h.$$

Adding these inequalities and rearranging, we obtain

$$\langle Tu, u\rangle + \langle Tu_h, u_h\rangle \le \langle F, u - v_h\rangle + \langle F, u_h - v\rangle + \langle Tu, v\rangle + \langle Tu_h, v_h\rangle.$$

Subtracting $\langle Tu, u_h\rangle + \langle Tu_h, u\rangle$ from both sides, we obtain

$$\begin{aligned}\langle Tu - Tu_h, u - u_h\rangle &\le \langle F - Tu, u - v_h\rangle + \langle F - Tu, u_h - v\rangle \\ &\quad + \langle Tu - Tu_h, u - v_h\rangle \\ &\le \| F - Tu \| \| u - v_h \| + \| F - Tu \| \| u_h - v \| \\ &\quad + \| Tu - Tu_h \| \| u - v_h \|,\end{aligned}$$

by the Cauchy-Schwartz inequality.
Since T is a strongly monotone and Lipschitz continuous operator, it follows that

$$\begin{aligned}\tau \| u - u_h \|^2 &\le \lambda \| u - u_h \| \| u - v_h \| + \| F - Tu \| \| u - v_h \| \\ &\quad + \| F - Tu \| \| u_h - v \| .\end{aligned}$$

Now

$$\lambda \| u - u_h \| \| u - v_h \| \le \frac{\tau}{2} \| u - u_h \|^2 + \frac{\lambda^2}{2\tau} \| u - v_h \|^2 .$$

Thus, by using this inequality, we get

$$\| u - u_h \| \le \left\{ \frac{\lambda^2}{\tau^2} \| u - v_h \|^2 + \frac{2}{\tau} \| F - Tu \| \left(\| u - v_h \| + \| u_h - v \| \right) \right\}^{\frac{1}{2}},$$

the required bound. $\square$

Note that for $M = H$, the inequality (5.35) reduces to (5.31). The bound (5.35) is a generalization of Falk's bound. This fact allows us to consider the nonlinear variational inequality (5.27) associated with the nonlinear versions of the homogeneous boundary value problems (5.2).

CONCLUSION

The derivative of the error estimate for the approximation of variational inequality depends on the regularity of the solution u of the variational inequality, we are considering. In the case of linear variational inequality (1.5), Brezis and Stampacchia [6] have shown that if the linear functional $F \in L_2(\Omega)$ and the obstacle function X, (say) is in $H^2(\Omega)$, then $u \in H^2(\Omega)$. The corresponding problem of regularity for the nonlinear variational inequalities studies in chapters 1,3 and 5 is open.
For the linear problem (5.2) subject to the extra condition $u \geq X$ on Ω, under certain conditions of smoothness on $\partial\Omega$, Falk [16] and Mosco and Strang [28], using condition piecewise linear trial functions and triangular elements, have produced $O(h)$ bound on $\| u - u_h \|_{H_0^1}$.
We conjecture that for the nonlinear variational inequalities, the error estimate for $u - u_h$ would be $O(h)$ as in the linear case. In brief, the error analysis for the variational inequalities if far from complete.
The theory developed by us can be extended to treat the elliptic boundary value problems in two or more dimensions in Hilbert spaces and Banach spaces.
Combining (5.88) and (5.27), we get the problem of finding $u \in M$ such that

$$\langle Tu, v - u \rangle \geq \langle A(u), v - u \rangle, \quad \forall v \in M, \tag{5.36}$$

where T and A are both nonlinear operators as defined in Theorem 5.6 and Theorem 5.11 respectively. Note this problem is more general than and includes all the previous variational inequalities so far considered in the literature.
The existence of a unique solution of this inequality follows from Lemma 2.2 and Lemma 5.1 under the assumption of $\tau < \gamma$, where τ and γ are as defined in Definition 5.1 and Definition 2.1.
The corresponding approximate problem is of finding $u_h \in M_h$ such that

$$\langle Tu_h, v_h - u_h \rangle \geq \langle A(u_h), v_h - u_h \rangle, \qquad \text{for all } u_h \in M_h.$$

One can show that in this case, the inequality bounding the error $u - u_h$ is

$$\begin{aligned} \| u - u_h \| \quad \leq \quad & \left\{ \frac{3\lambda^2}{\tau^2} \| u - v_h \|^2 + \frac{3\lambda^2}{\tau^2} \| v - u_h \|^2 + \frac{3\gamma^2}{\tau^2} \| v_h - v \|^2 \right. \\ & \left. + \frac{1}{\tau} \| A(u) - Tu \| \| u - v_h \| + \frac{1}{\tau} \| A(u_h) - Tu_h \| \| u_h - v \| \right\}^{\frac{1}{2}}, \end{aligned}$$

for all $v \in M$ and $v_h \in M_h$, where τ, λ and γ are constants as defined in Definition 5.1 and Definition 2.1.

During the last decade, many problems from optimal control theory [20] and convex programming [23] have been studies via the variational approach. We conclude that, despite the current activity, much clearly remains to be done in this field.

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Some Related Important Problems and Future Research

We now discuss some very important recent important problems, which are inspired amd motiivated by the novel, innovative discussion and analysis based of Variaional Inequalities introduced and investiagted in the **PhD Thesis of M. Aslam Noor, Brunel Universiyt, London, UK, (1975),** see [2]

For simplicity and to convey the mian ideas of these results, we included some recent references. To be more precise, consider the functional

$$J[v] \quad = \quad \langle Tv, v\rangle - 2F(v), \quad \forall v \in H, \tag{5.37}$$

where T is a linear, symmetric and positive operator.
If $\langle Tv, v\rangle = v^2$, then $\langle Tv, v\rangle = v^2$ is a convex function. This implies that the functional $J[v]$ can be viewed as the difference of two convex functions, provided the function $F(v)$ is a convex function. Consequently this problem can be regarded as the minimum of two convex functions and is known as DC-problem. This is an interesting problem. See, for example, Noor et al.[27, 36].
Motivared and inspired by the recent activites in variational inequalities and nonlinear optimizations, we consider the higher order general variational inequalities.

1. Let $T, g, h : H \Longrightarrow H$ be nonlinear operators. Then we consider the problem of finding $u \in M$ such that

   $$\langle Tu, g(v) - h(u)\rangle \geq \langle A(u), g(v) - h(u)\rangle + \zeta\|g(v) - h(u)\|^p, , \quad \forall v \in M, \quad p \geq 0, \tag{5.38}$$

   which is called the higher order extended general variational inequality, where $\zeta \geq 0$ is a papameter.
   For $\zeta = 0$, the problem (5.38) collapses to:

2. Let $T, g, h : H \Longrightarrow H$ be nonlinear operators. Then we consider the problem of finding $u \in M$ such that

   $$\langle Tu, g(v) - h(u)\rangle \geq \langle A(u), g(v) - h(u)\rangle, \quad \forall v \in M, \tag{5.39}$$

   which is called the extended general variational inequality.
   Clealry, for $g = I, \quad h = I$, the prolem (5.39) reduces to the problem (5.36). Clealry the problem (5.39 can be viewed as the extended general variational inequalities involving

the difference of two monotone operators. This fact inspired the researchers to consider the problem of finding the problem of finding $u \in H$ such that

$$0 \in \rho Tu - h(u) - g(u) - \rho N(h(u)), \tag{5.40}$$

where $N(h(u))$ is the normal cone. This fact motivated us to consider the more general variational inclsuion of the type:
Find $u \in H$ such that

$$0 \in \rho Tu - h(u) - g(u) - \rho A(h(u)), \tag{5.41}$$

which is called the extended general variational inclusions, where $A : H \Longrightarrow H$ is the maxial monotone operator. It is an interesting problem to explore the applications of the problem (5.41) invarious fields and develop implementable numerical methods.

3. For $\langle A(u), g(v)-h(u)\rangle = A(u; g(v)-h(u))$, the problem (5.39) reduces to finding $u \in M$ such that

$$\langle Tv, g(v) - h(u)\rangle - A(u; g(v) - h(u)) \geq 0, \quad \forall v \in M, \tag{5.42}$$

which is called the extended general variational hemivariational inequality. Some special cases have been conisdered by Panagiotopoulos [55] in structal analysis.

4. For $A(u) = A|u|$, the problem (5.39) collapses to finding $u \in M$ such that

$$\langle Tv, g(v) - h(u)\rangle - \langle A|u|, g(v) - h(u))\rangle \geq 0, \quad \forall v \in M, \tag{5.43}$$

is called the extended general absolute variational inequalities.

5. For $M = H$, then problem (5.43) is equivalent to finding $u \in H$ such that

$$\langle Tu - A|u|, g(v) - h(u)\rangle = 0, \tag{5.44}$$

which is called the system of extended general absolute value equations,
where, $T \in R^{n,n}, A \in R^{n,n}$.
From (5.44), it follows that

$$Tu - A|u| = b, \tag{5.45}$$

where b is the given data. The system (5.45) is called the system of absolute value equations. For the motivation, formulation, applications, numerical methods and other aspects of the system of absolute value equations, see [26, 28, 31, 33, 34, 43, 48, 49, 50, 51, 56, 58, 59, 60, 61, 62, 63, 64] and the references.

6. For $M^* = \{u \in H : \langle u, g(v)\rangle \geq 0, \quad \forall v \in H : g(v) \in M\}$ is a polar (dual) cone of a cone M in H, the problem (5.39) is equivalent to finding $u \in H$ such that

$$h(u) \in M, \quad Tu - A(u) \in M^* \quad \text{and} \quad \langle Tu - A(u), h(u)\rangle = 0, \tag{5.46}$$

   which is called the strongly extended general nonlinear complementarity problem and appears to a be a new ones. If $A(u) = 0, \quad h = I$, then the problem (5.46) is called the nonlinear complementarity problem, which is mainly due to Karamardian [44]. For the motivation, formulation, applications and numerical methods of the complmentarity theory, see [5, 10, 13, 14, 15, 16, 17, 39, 40, 41, 44, 45, 48, 54] and the references therein.

7. For $A(u) = A|u|$, the problem (5.46) is called the strongly nonlinear absolute complementarity problem and appears to be new one.

We would to point out Lemma 2.2 is the auxilairy principle technqiue, which has been used to studiy the existence of solutions for various classes of variational inequalities as well as to develop a wide class of numerical methods for solving diffferent classes of variational inequalities and related optimization problems. For the conveince of the redaers and completeness, we include some details of the auxilairy principle technqiue for the variational inequality (5.88). For given $u \in M$ satisfying (5.88, consider the problem of finding a unique $w \in M$ such that

$$\rho a(u, v - w) + \langle w - u, v - w\rangle \geq \langle \rho A(u), v - w\rangle, \quad \forall v \in M, \tag{5.47}$$

where $\rho > 0$ is a constant. The problem (5.47) is called the auxilairy variational inequality and can be rewritten as: or given $u \in M$ satisfying (5.88, consider the problem of finding a unique $w \in M$ such that

$$\rho\langle Tu, v - w\rangle + \langle w - u, v - w\rangle \geq \langle \rho A(u), v - w\rangle, \quad \forall v \in M \tag{5.48}$$

which implies that

$$w = P_M[u - \rho(Tu - A(u))], \tag{5.49}$$

where P_M is the projection of H on the closed convex set M.
Clearly on the whole real Hilbert space H, the equation (5.49) is equivalent to:

$$\langle \phi(w), v\rangle - \langle u, v\rangle = -\langle \rho(Tu - A(u)), v\rangle, \quad \forall v \in H. \tag{5.50}$$

which is the exactly the problem (2.7).

It is an interesting problem to explore the applications and develop the numerical methods for the extended general variational inequalities and related optimizaation problems We have included some relevant references to point out that the problems introduced and considered in [1, 2] have influnced and explored several important problems such as: Hemivariational inequalities, general variational inequalities, systems of absolute value equations, complementarity problems and optimizations problems directly/indirectly. It is worh mentioning that all these problems van be viewed as significant generalizations and modifications of Riesz-Frechet representaion theorem and Lax-Milgram Lemma.

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Some Classes of New Systems of Exponentially General Equations

1

Abstract: Some new systems of exponentially general equations are introduced and investigated, which can be used to study the odd-order, non-positive and nonsymmetric exponentially boundary value problems. Some important and interesting results such as Riesz-Frechet representation theorem, Lax-Milgram lemma and system of absolute values equations can be obtained as special cases. It is shown that the system of exponentially general equations is equivalent to nonlinear optimization problem. The auxiliary principle technique is used to prove the existence of a solution to the system of exponentially general equations. This technique is also used to suggest some new iterative methods for solving the system of the exponentially general equations. The convergence analysis of the proposed methods is analyzed. Ideas and techniques of this paper may stimulate further research.

Introduction

It is well known [1, 2, 3] that a linear continuous functional on a Hilbert space can be represented by an inner product as well as by an arbitrary bifunction Lax-Milgram [4]. These results are known as representation theorems and can be viewed as the weak formulation of the initial and boundary value problems. One can easily that the minimum of the functional $I[v]$ on the Hilbert space H

$$I[v] = \langle v, v\rangle - 2f(v), \quad v \in H, \tag{5.51}$$

ie equivalent to finding $u \in H$ such that

$$\langle u, v\rangle = \langle f, v\rangle, \quad \forall v \in H, \tag{5.52}$$

which is known as the Riesz-Frechet representation theorem [1, 2, 3]. This result had played a significant role in the development of various branches of mathematical and engineering sciences and is continue to inspire new ideas and techniques to tackle complicated and complex problems. See[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] and the references therein. From the day of discovery of the representation theorems, many important contributions have been made in this direction. In every case, a new approach and method is applied to generalize some of these results and the ideas they used.
For a symmetric, positive, bilinear $a(.,.)$ and f linear continuous functions, the the minimum of the functional $J[v]$ defined by

$$J[v] = a(v, v) - 2f(v), \quad v \in H, \tag{5.53}$$

[1]**M. A. Noor and K. I Noor, Some Slasses of New Systems of Exponentially General Equations, Advances in Linear Algebra and Matrix Theory, 12(2022)**

can be characterized by

$$a(u,v) = \langle f, v\rangle, \quad \forall v \in H, \tag{5.54}$$

which is known as the Lax-Milgram Lemma [4].
If the function f is nonlinear Frechet differentiable, then the minimum of the functional $J[v]$ defined by (5.53) can be characterized by

$$a(u,v) = \langle f'(u), v\rangle, \quad \forall v \in H, \tag{5.55}$$

where $f'(u)$ is the differential of f. Problem of the type (5.55) is called the general Lax-Milgram Lemma, introduced and studied by Noor [9]. For motivation, formulation, numerical applications, generalizations and novel aspects of the general Lax-Milgram Lemma, see [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16].
Noor and Noor [10] considered the problem of finding $u \in H$ such that

$$a(u,v) = \langle A(u), v\rangle, \quad \forall v \in H, \tag{5.56}$$

where A is the nonlinear operator. Problem (5.56) is also called the general Lax-Milgram Lemma and have been used in finite element analysis of mildly nonlinear boundary value problems [9, 12].
From the problems (5.52), (5.54), (5.55) and (5.56), it is clear that these representation theorems have variational character, the origin of which can be traced back to Euler, Newton and Bernoulli's brothers.
For given nonlinear operators $T, A : H \to H$, consider the problem of finding $u \in H$ such that

$$\langle Tu, v\rangle = \langle A(u), v\rangle, \quad \forall v \in H, \tag{5.57}$$

which is called the general Lax-Milgram Lemma [10]. One can easily show that the problems involving difference of two monotone operators, system of absolute value equations, difference of two convex functions, known as DC-problem and complementarity problems are special cases of the general Lax-Milgram Lemma and Riesz-Frechet representation theorems for different and appropriate choice of the operators. Mangasarian et al[17] considered the systems of absolute value equations. Karamardian [18] had established the equivalence between the complementarity problems and variational inequalities. The equivalence interplay among these different fields enables us to use various techniques, which have been developed for variational inequalities, systems of absolute value problems and complementarity problems for solving the system of general equations and vice versa. For recent numerical methods for solving these different problems, see [4, 5, 7, 9, 10, 12, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47] and the references therein.

It have been observed that only the even order, positive and symmetric boundary value problems can be studied. For odd-order, non-positive and non-symmetric , these representation theorems can not be used. To tackle such problems, the operator may be made positive and symmetric with respect to an arbitrary map. Noor [30, 31, 32, 33, 34] introduced the general variational inequalities, which are used to study the odd-order and nonsymmetric boundary value problems. From the general variational inequalities, we can obtain system of general equations, see Noor [30, 31, 32] and Noor et al [36]. To be more precise, for given nonlinear operators $T, A, g : H \to H$, consider the problem of fining $u \in H$ such that

$$\langle Tu, g(v)\rangle = \langle A(u), g(v)\rangle, \quad \forall v \in H, \tag{5.58}$$

This system of general equations can be viewed as a weak formulation of the non-positive and nonsymmetric odd-order boundary value problems. For more details, see Filippov [27], Noor [30, 31, 32], Noor et al.[36], Petryshin [28],Tonti [29], and the references therein.

Motivated and inspired by ongoing research in these fields, we introduce and study some new systems of exponentially general equations. This new system of exponentially general equations can be viewed as a weak formulation of the non-positive and nonsymmetric exponentially boundary value problems. It is shown that the system of absolute value equations, complementarity problems and Lax-Milgram Lemma can be obtained as special cases. The auxiliary principle technique, which is mainly due to Lions and Stampacchia [39] and Glowinski et al. [40], is used to discuss the existence of a solution for the system of exponentially general equations. This approach is also applied to suggest some hybrid inertial iterative methods for solving the system of general exponentially equations. The convergence analysis of these methods is investigated under weaker conditions.

In Section 2, we introduce new system of exponentially general equations and discuss their applications. It is shown that the Reisz-Frechet representation theorem, Lax-Milgram Lemma and system of absolute value exponentially equations can be obtained as special cases. As an interesting case, a new inner product space is derived. This may be starting point to explore the applications of the general inner product space. It is shown that the exponentially third order boundary value problems can be studied in the general framework of exponentially general equations. In section 3, we use the auxiliary principle technique to discuss the existence of a solution as well as to suggest some iterative methods for solving the general absolute value equations. The convergence criteria of proposed iterative methods is considered under weaker conditions. Our method of proofs is very simple with other techniques. Several new iterative methods for solving the exponentially general equations are obtained as novel applications of the results.

Basic Concepts and Formulation

Let H be a Hilbert space, whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively.

For given operators $L, A, g : H \longrightarrow H$ and a continuous linear functional f, we consider the problem of finding $u \in H$ such that

$$\langle e^{Lu} + e^{A(u)}, g(v) - g(u)\rangle = \langle f, g(v) - g(u)\rangle, \quad \forall v \in H, \tag{5.59}$$

which is called the system of exponentially general equations. We note that the problem (5.59) is equivalent to finding $u \in H$ such that

$$\langle e^{Lu} + e^{A(u)}, g(v) - g(u)\rangle = \langle f, g(v)\rangle, \quad \forall v \in H. \tag{5.60}$$

Several important applications are discussed as special cases of the problems (5.59) and (5.60):

1. . If $A = 0$, then problem (5.59) collapses to finding $u \in H$, such that

$$\langle e^{Lu}, g(v) - g(u)\rangle = \langle f, g(v) - g(u)\rangle, \quad \forall v \in H \tag{5.61}$$

which is called the exponentially general equations.
Systems of the exponentially general equations of type (5.59), (5.60) and (5.61)can be viewed as exponentially general Lax-Milgram lemma. These system of exponentially general equations are called the weak formulations of the odd-order, non-symmetric and non-positive boundary value problems. We use these systems to discuss the unique existence to a solution of the odd-order and nonsymmetric exponentially boundary value problems. This result plays a significant role in the study of function spaces and partial differential equations.

2. . If $g = I$, the identity operator and $\langle e^{Lu}, v\rangle = a(u, v)$, where $a(.,.)$ is bilinear continuous form, then problem (5.60) reduces to finding $u \in H$ such that

$$a(u, v) + \langle e^{A(u)}, v\rangle = \langle f, v\rangle, \quad \forall v \in H, \tag{5.62}$$

which is known as the exponentially general Lax-Milgram Lemma.

3. . If $g = I$, the identity operator, then problem (5.60) reduced to finding $u \in H$ such that

$$\langle e^{Lu} + e^{A(u)}, v\rangle = \langle f, v\rangle, \quad \forall v \in H \tag{5.63}$$

which is known as the weak formulation of the exponentially boundary value problems.

4. . From problem (5.63), we have the problem of finding $u \in H$ such that

$$e^{Lu} + e^{A(u)} = f, \tag{5.64}$$

which is called the system of exponentially equations.

5. . If $e^{Lu} + e^{A(u)} = e^{Lu} + e^{A|u|}$, then problem (5.59) reduces to finding $u \in H$ such that

$$\langle e^{Lu} + e^{A|u|}, g(v) - g(u)\rangle = \langle f, g(v) - g(u)\rangle, \quad \forall v \in H, \tag{5.65}$$

which is called the system of exponentially general absolute value equations. From (5.74), we find $u \in H$ such that

$$e^{Lu} + e^{A|u|} = f, \quad \forall v \in H, \tag{5.66}$$

which is called the system of exponentially absolute value equations.

6. . If $e^{Lu} = \Phi(u), \quad e^{A(u)} = N(u)$, are nonlinear operators, then problem (5.74)reduces to finding $u \in H$ such that

$$\Phi(u) + N|u| = f, \quad \forall v \in H, \tag{5.67}$$

is called the system of absolute value equations. It is known that the system of absolute value equations are equivalent to the complementarity problems. If the involved convex set in the variational inequalities is a convex cone, then variational inequalities are equivalent to the complementarity problems. Consequently, all these problems are equivalent under some suitable conditions. This fascinating interplay among these problems can be used in developing several numerical methods to solve complicated and complex problems.

7. . For $A = 0, L = I$, the problem (5.60) collapses to finding $u \in$ such that

$$\langle u, g(v)\rangle = \langle f, g(v)\rangle, \quad \forall v \in H, \tag{5.68}$$

which is called the general Riesz-Frechet representation theorem.

8. . If $g = I$, then general Riesz-Frechet representation theorem reduces to finding $u \in H$. such that

$$\langle u, v\rangle = \langle f, v\rangle, \quad \forall v \in H, \tag{5.69}$$

which is the celebrated Riesz-Frechet representation theorem, introduced by Riesz [2, 3] and Frechet [1] in 1907, independently. It has been shown by Noor [8, 9] that the Riesz-Frechet representation theorem has a variational character. In fact, $u \in H$ is solution of (5.69), if and only if, $u \in H$ is the minimum of the energy functional

$$I[v] = \langle v, v\rangle - 2\langle f, v\rangle, \quad \forall v \in H. \tag{5.70}$$

It is obvious that the energy function $I[v]$ is a strongly convex functions. Consequently it has a unique minimum $u \in H$. This equivalent formulation can be used to discuss the unique existence of the Riesz-Frechet representation theorem, which can be viewed as novel way.

9. . If $e^{Lu} = \Phi(u), \quad e^{A(u)} = N(u)$, are nonlinear operators, then problem (5.59) reduces to finding $u \in H$, such that

$$\langle \Phi(u) + N(u), g(v) - g(u)\rangle = \langle f, g(v) - g(u)\rangle, \quad \forall v \in H, \tag{5.71}$$

which is called the system of general equations, introduced and studied by Noor [30].

10. . If $g = I$, the identity operator, then (5.71) is equivalent to fining $u \in H$ such that

$$\langle \Phi(u) + N(u), v - u\rangle = \langle f, v - u)\rangle, \quad \forall v \in H, \tag{5.72}$$

is called the generalized Lax-Milgram Lemma, introduced and studied by Noor [30].

Remark 5.2 *For suitable and appropriate choice of the operators T, A, g, one can obtain various classes of new and known classes of problems as application of the problem (5.59). This shows that the system of exponentially general equations is a unified one.*

It is known [27, 28, 29] that, if the operator is not symmetric and non-positive, then it can be made symmetric and positive with respect to an arbitrary operator.

Definition 5.2 *An operator $T : H \longrightarrow H$ with respect to an arbitrary operator $g : H \rightarrow H$ is said to be :*

(a). *exponentially general symmetric , if,*

$$\langle e^{Tu}, g(v)\rangle = \langle g(u), e^{Tv}\rangle, \quad \forall u, v \in H.$$

(b). *exponentially general positive, if,*

$$\langle e^{Tu}, g(u)\rangle \geq 0, \quad \forall u \in H.$$

(c). *exponentially general coercive (g-elliptic), if there exists a constant $\alpha > 0$ such that*

$$\langle e^{Tu}, g(u)\rangle \geq \alpha \|g(u)\|^2, \quad \forall u \in H.$$

Note that exponentially general coercivity implies exponentially general positivity, but the converse is not true. It is also worth mentioning that there are operators which are not exponentially general symmetric but exponentially general positive. On the other hand, there are g-positive , but not g-symmetric operators. Furthermore, it is well-known [27, 28, 29] that, if for a linear operator L, there exists an inverse operator L^{-1} on $R(L)$, the range of L, with $\overline{R(L)} = H$, then one can find an infinite set of auxiliary operators g such that the operator T is both g-symmetric and g-positive.

Remark 5.3 *If $e^{Tu} = I(u)$, the identity operator, then Definition (5.2) reduces to:*

Definition 5.3 *An inner produce $\langle ,.,\rangle$ with respect to an arbitrary operator $g : H \rightarrow H$ is said to be*

(a). *general symmetric , if, $\langle u, g(v)\rangle = \langle g(u), v\rangle, \quad \forall u, v \in H.$*

(b). *general positive, if, $\langle u, g(u)\rangle \geq 0, \quad \forall u \in H.$*

(c). *general positive definite , if there exists a constant $\alpha > 0$ such that*

$$\langle u, g(u)\rangle \geq \alpha \|g(u)\|^2, \quad \forall u \in H.$$

Motivated by the Remark (5.3), we can define the general inner product space with respect to an arbitrary function g such that

1. $\langle u, g(u)\rangle \geq 0, \quad \text{and} \quad \langle u, g(u)\rangle = 0, \quad \Leftrightarrow \quad u = g(u), \quad \forall u, v \in H.$

2. $\langle u, g(v)\rangle = \langle g(u), v\rangle, \quad \forall u, v \in H.$

3. $\langle u, g(v) + g(w)\rangle = \langle u, g(v)\rangle + \langle u, g(w)\rangle, \quad \forall u, v, w \in H.$

Also, we can obtain the result for general inner product spaces, that is,

$$\|u + g(v)\|^2 + \|u - g(v)\|^2 = 2\{\|u\|^2 + \|g(v)\|^2\}, \quad \forall u, v \in H, \tag{5.73}$$

which is known as the parallelogram laws and can be used to characterize the general Hilbert space.

It is interesting problem to consider the completeness of the general inner product spaces and explore their properties in fixed-point theory, differential equations and optimization theory. If the operators L, A are linear, general positive, general symmetric and the operator g is linear,

then the problem (5.59) is equivalent to finding a minimum of the function $I[v]$ on H, where

$$I[v] = \langle e^{Lv} + e^{A(v)}, g(v)\rangle - 2\langle f, g(v)\rangle, \quad \forall v \in H, \tag{5.74}$$

which is a nonlinear programming problem and can be solved using the known techniques of the optimization theory.

We now consider the problem of finding the minimum of the functional $I[v]$, defined by (5.74). For the sake of completeness and to convey the main ideas, we include all the details.

Theorem 5.10 *Let the operators $L, A : H \longrightarrow H$ be linear, exponentially general symmetric and exponentially general positive. If the operator $g : H \longrightarrow H$ is linear, then the function $u \in H$ minimizes the functional $I[v]$, defined by (5.74), if and only if,*

$$\langle e^{Lu} + e^{A(u)}, g(v) - g(u)\rangle = \langle f, g(v) - g(u)\rangle, \quad \forall v \in H. \tag{5.75}$$

Proof. Let $u \in H$ satisfy (5.75). Then, using the exponentially general positivity of the operators L, A, we have

$$\langle e^{Lv} + e^{A(v)}, g(v) - g(u)\rangle \geq \langle f, g(v) - g(u)\rangle, \quad \forall v \in H. \tag{5.76}$$

$\forall u, v \in H, \quad \epsilon \geq 0$, let $v_\epsilon = u + \epsilon(v - u) \in H$. Taking $v = v_\epsilon$ in (5.76) and using the fact that g is linear, we have

$$\langle e^{Lv_\epsilon} + e^{A(v_\epsilon)}, g(v_\epsilon) - g(u)\rangle \geq \langle f, g(v_\epsilon) - g(u)\rangle. \tag{5.77}$$

We now define the function

$$\begin{aligned} h(\epsilon) &= \epsilon\langle e^{Lu} + e^{A(u)}, g(v) - g(u)\rangle + \frac{\epsilon^2}{2}\langle e^{L(v-u)} + e^{A(v-u)}, g(v) - g(u)\rangle \\ &\quad -\epsilon\langle f, g(v) - g(u)\rangle, \end{aligned} \tag{5.78}$$

such that

$$\begin{aligned} h'(\epsilon) &= \langle e^{Lu} + e^{A(u)}, g(v) - g(u)\rangle \\ &\quad +\epsilon\langle e^{L(v-u)} + e^{A(v-u)}, g(v) - g(u)\rangle - \langle f, g(v) - g(u)\rangle \\ &\geq 0, \quad \text{by } (5.77)\ . \end{aligned}$$

Using the g symmetry of L, A we see that $h(\epsilon)$ is an increasing function on $[0,1]$ and so $h(0) \leq h(1)$ gives us

$$\langle e^{Lu} + e^{A(u)}, g(u)\rangle - 2\langle f, g(u) \leq \langle e^{Lv} + e^{A(v)}, g(v)\rangle - 2\langle f, g(v)\rangle,$$

that is,

$$I[u] \leq I[v], \quad \forall v \in H,$$

which shows that $u \in H$ minimizes the functional $I[v]$, defined by (5.74).
Conversely, assume that $u \in H$ is the minimum of $I[v]$, then

$$I[u] \leq I[v], \quad \forall v \in H. \tag{5.79}$$

Taking $v = v_\epsilon \equiv u + \epsilon(v - u) \in H, \forall u, v \in H$ in (5.79), we have

$$I[u] \leq I[v_\epsilon].$$

Using (5.74), g-positivity and the linearity of L, A, we obtain

$$\begin{aligned}\langle e^{Lu} + e^{A(u)}, g(v) - g(u)\rangle \quad &+ \quad \frac{\epsilon}{2}\langle e^{L(g(v)-g(u))} + e^{A(g(v)-g(u))}, g(v) - g(u)\rangle \\ &\geq \quad \langle f, g(v) - g(u)\rangle,\end{aligned}$$

from which, as $\epsilon \longrightarrow 0$, we have

$$\langle e^{Lu} + e^{A(u)}, g(v) - g(u)\rangle \geq \langle f, g(v) - g(u)\rangle, \quad \forall v \in H. \tag{5.80}$$

Replacing $g(v) - g(u)$ by $(g(u) - g(v))$ in inequality (5.80), we have

$$\langle e^{Lu} + e^{A(u)}, g(v) - g(u)\rangle \leq \langle f, g(v) - g(u)\rangle, \quad \forall v \in H. \tag{5.81}$$

From (5.80) and (5.81, it follows that $u \in H$ satisfies

$$\langle e^{Lu} + e^{A(u)}, g(v) - g(u)\rangle = \langle f, g(v) - g(u)\rangle, \quad \forall v \in H, \tag{5.82}$$

the required result (5.75). $\square$

We now show that the third order exponentially boundary value problems can be studied via problem (5.59).

Example 5.1 *Consider the third order exponentially boundary value problem of finding u such that*

$$-e^{\frac{d^3u}{dx^3}} + \frac{dv}{dx} = f(x), \quad \forall x \in [a, b], \tag{5.83}$$

with boundary conditions

$$u(a) = 0, \quad u'(a) = 0, \quad u'(b) = 0, \tag{5.84}$$

where $f(x)$ is a continuous function. This problem can be studied in the general framework of the problem (5.59) To do so, let

$$H = \{u \in H_0^2[a,b] : u(a) = 0, \quad u'(a) = 0, \quad u'(b) = 0\}$$

be a Hilbert space, see [5]. One can easily show that

$$\begin{aligned} & -\int_a^b e^{\frac{d^3v}{dx^3}} \frac{dv}{dx} dx + \int_a^b \frac{dv}{dx}\frac{dv}{dx} dx - \int_a^b f \frac{dv}{dx} dx, \quad \forall \frac{dv}{dx} \in H_0^2[a,b] \\ = & \int_a^b (\frac{d^2v}{dx^2})^2 + \int_a^b \frac{dv}{dx}\frac{dv}{dx} - \int_a^b f \frac{dv}{dx} dx \\ = & \langle e^{Lv}, g(v)\rangle + \langle e^{Au}, g(v)\rangle - \langle f, g(v)\rangle = 0, \end{aligned}$$

from which, it follows that

$$\langle e^{Lv}, g(v)\rangle + \langle e^{Au}, g(v)\rangle = \langle f, g(v)\rangle.$$

This is the weak formulation of the third order exponentially boundary (5.83).

Definition 5.4 *An operator $L : H \longrightarrow H$ is said to be;*
(i). Strongly exponentially monotone, if there exists a constant $\alpha > 0$, such that

$$\langle e^{Lu} - e^{Lv}, u - v\rangle \geq \alpha \|u - v\|^2, \quad \forall u, v \in H.$$

(ii). exponentially Lipschitz continuous, if there exists a constant $\beta > 0$, such that

$$\|e^{Lu} - e^{Lv}\| \leq \beta \|u - v\|, \quad \forall u, v \in H.$$

(iii) exponentially monotone, if

$$\langle e^{Lu} - e^{Lv}, u - v\rangle \geq 0, \quad \forall u, v \in H.$$

(iv) firmly strongly exponentially monotone, if

$$\langle e^{Lu} - e^{Lv}, u - v\rangle \geq \|u - v\|^2, \quad \forall u, v \in H.$$

We remark that, if the operator L is both strongly exponentially monotone with constant $\alpha > 0$ and exponentially Lipschitz continuous with constant $\beta > 0$, respectively, then from (i) and (ii), it follows that $\alpha \leq \beta$.

Itertive Methods and Convergence Criteria

We apply the auxiliary principle technique, the origin of which can be traced back to Lions and Stampacchia [39] and Glowinski et al [40], as developed by Noor [30, 31, 32], Noor et al. [13, 14, 15, 16, 22, 32] and Zhu et al[41]. The main idea of this technique is to consider an auxiliary problem related to the original problem. This way, one defines a mapping connecting the solutions of both problems. To prove the existence of solution of the original problem, it is enough to show that this connecting mapping is a contraction mapping and consequently has a unique solution of the original problem. In recent several inertial type algorithms have been analyzed for solving variational inequalities and optimization problems, which are mainly due to Polyak [46]. These methods help us to improve convergence rate of the iterative methods. In this section, we use the auxiliary principle technique to suggest some new inertial iterative methods for solving the system of exponentially general equations. These inertial methods do not involve the evaluations of the projection methods, resolvent methods and their variant forms. This is advantage of this technique.

We now consider the problem of the uniqueness and existence of the solution of (5.59) using the technique of the auxiliary principle approach, which is subject of our nest result.

Theorem 5.11 *Let the operator L be a strongly exponentially monotone with constant $\alpha > 0$ and exponentially Lipschitz continuous with constant $\beta > 0$, respectively. Let the operator g be firmly strongly monotone and Lipschitz continuous with constant β_1. If the operator A is Lipschitz continuous with constant $\lambda > 0$ and there exists a constant $\rho > 0$ such that*

$$\left|\rho - \frac{\alpha+\nu-1}{\beta^2-\lambda^2}\right| < \frac{\sqrt{(\alpha+\nu-1)^2-(\beta^2-\lambda^2)\nu(2-\nu)}}{\beta^2-\lambda^2}, \tag{5.85}$$

$$\alpha > 1-\nu+\sqrt{(\beta^2-\lambda^2)\nu(2-\nu)}, \quad \rho\lambda < 1-\nu, \quad \nu < 1, \tag{5.86}$$

where

$$\nu = \sqrt{\beta_1^2-1} \tag{5.87}$$

then the problem (5.59) has a solution.

Proof. We use the auxiliary principle technique to prove the existence of a solution of (5.59).
For a given $u \in H$, consider the problem of finding $w \in H$ such that,

$$\langle \rho(e^{Lu}+e^{A(u)}), g(v)-g(w)\rangle + \langle g(w)-g(u), g(v)-g(w)\rangle = \langle \rho f, g(v)-g(w)\rangle, \quad \forall v \in H, \tag{5.88}$$

which is called the auxiliary problem, where $\rho > 0$ is a constant. It is clear that (5.88) defines a mapping w connecting the both problems (5.59) and (5.88). To prove the existence of a solution of (5.59), it is enough to show that the mapping w defined by (5.88) is a contraction mapping.

Let $w_1 \neq w_2 \in H$ (corresponding to $u_1 \neq u_2$) satisfy the auxiliary problem (5.88). Then

$$\begin{aligned}\langle \rho(e^{Lu_1} + e^{A(u_1)}), g(v) - g(w_1)\rangle \quad &+ \quad \langle g(w_1) - g(u_1), g(v) - g(w_1)\rangle \\ &= \langle \rho f, g(v) - g(w_1)\rangle, \quad \forall v \in H, \qquad (5.89)\\ \langle \rho(e^{Lu_2} + e^{A(u_2)}), g(v) - g(w_2)\rangle \quad &+ \quad \langle g(w_2) - g(u_2), g(v) - g(w_2)\rangle \\ &= \langle \rho f, g(v) - g(w_2)\rangle, \quad \forall v \in H. \qquad (5.90)\end{aligned}$$

Taking $v = w_2$ in (5.89) and $v = w_1$ in (5.90) and adding the resultant, we have

$$\begin{aligned}\|g(w_1) - g(w_2)\|^2 &= \langle g(w_1) - g(w_2), g(w_1) - g(w_2)\rangle \\ &= \langle g(u_1) - g(u_2) - \rho(e^{Lu_1} - e^{Lu_2}) \\ &\quad +\rho(e^{A(u_1)} - e^{A(u_2)}), g(w_1) - g(w_2)\rangle. \qquad (5.91)\end{aligned}$$

From (5.91), we have

$$\|g(w_1) - g(w_2)\|^2 \leq \|g(u_1) - g(u_2) - \rho(e^{Lu_1} - e^{Lu_2}) + \rho(e^{A(u_1)} - e^{A(u_2)})\|\|g(w_1) - g(w_2)\|$$

from which, using the exponentially Lipschitz continuity of the operator A with constant $\lambda > 0$, it follows that

$$\begin{aligned}\|w_1 - w_2| &\leq \|g(w_1) - g(w_2)\| \\ &\leq \|g(u_1) - g(u_2) - \rho(e^{Lu_1} - e^{Lu_2})\| + \rho\|e^{A(u_1)} - e^{A(u_2)}\| \\ &\leq \|g(u_1) - g(u_2) - \rho(e^{Lu_1} - e^{Lu_2})\| + \rho\|e^{A(u_1)} - e^{A(u_2)}\| \\ &\leq \|u_1 - u_2 - g(u_1) - g(u_2)\| + \|u_1 - u_2 - \rho(e^{Lu_1} - e^{Lu_2})\| \\ &\quad +\rho\lambda\|u_1 - u_2\|. \qquad (5.92)\end{aligned}$$

Using the strongly exponentially monotonicity and exponentially Lipschitz continuity of the operator L with constants $\alpha > 0$ and $\beta > 0$, we have

$$\begin{aligned}\|u_1 - u_2 - \rho(e^{Lu_1} - e^{Lu_2})\|^2 &= \langle u_1 - u_2 - \rho(e^{Lu_1} - e^{Lu_2}), u_1 - u_2 - \rho(e^{Lu_1} - e^{Lu_2})\rangle \\ &= \langle u_1 - u_2, u_1 - u_2\rangle - 2\rho\langle e^{Lu_1} - e^{Lu_2}, u_1 - u_2\rangle \\ &\quad +\rho^2\langle e^{Lu_1} - e^{Lu_2}, e^{Lu_1} - e^{Lu_2}\rangle \\ &\leq (1 - 2\rho\alpha + \rho^2\beta^2)\|u_1 - u_2\|^2. \qquad (5.93)\end{aligned}$$

Similarly, using the strongly firmly monotonicity and Lipschitz continuity of the operator g with constant β_1, we have

$$\|u_1 - u_2 - (g(u_1) - g(u_2))\|^2 \leq \{\sqrt{\beta_1^2 - 1}\}\|u_1 - u_2\|^2. \tag{5.94}$$

Combining (5.92),(5.93) and (5.94), we have

$$\begin{aligned} \|w_1 - w_2\| &\leq \left(\sqrt{\beta_1^2 - 1} + \rho\lambda + \sqrt{(1 - 2\rho\alpha + \rho^2\beta^2)}\right)\|u_1 - u_2\| \\ &= \theta\|u_1 - u_2\|, \end{aligned} \tag{5.95}$$

where

$$\begin{aligned} \theta &= \left(\sqrt{\beta_1^2 - 1} + \rho\lambda + \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}\right) \\ &= \nu + \rho\lambda + \sqrt{1 - 2\rho\alpha + \beta^2\rho^2}, \end{aligned}$$

and

$$\nu = \sqrt{\beta_1^2 - 1}.$$

From (5.97) and (5.86), it follows that $\theta < 1$, so the mapping w is a contraction mapping and consequently, it has a fixed point $w(u) = u \in H$ satisfying the problem (5.59). $\square$

Remark 5.4 *We point out that the solution of the auxiliary problem (5.88) is equivalent to finding the minimum of the functional* $I[w]$, *where*

$$I[w] = \frac{1}{2}\langle g(w) - g(u), g(w) - g(u)\rangle - \rho\langle e^{Lu} + e^{A(u)} - f, g(w) - g(u)\rangle,$$

which is a differentiable convex functional associated with the inequality (5.88), if the operator g *is differentiable. This alternative formulation can be used to suggest iterative methods for solving the general absolute value equations. This auxiliary functional can be used to find a kind of gap function, whose stationary points solves the problem (5.60).*

We now use the auxiliary principle to suggest some iterative methods for solving the system of exponentially general equations (5.59). It is clear that, if $w = u$, then w is a solution of (5.59). This observation shows that the auxiliary principle technique can be used to propose the following iterative method for solving the system of general equations (5.59).

Algorithm 5.1 *For a given initial value* u_0, *compute the approximate solution* x_{n+1} *by the iterative scheme*

$$\begin{aligned} \langle e^{Lu_n} + e^{A(u_n)}, g(v) - g(u_{n+1})\rangle &+ \langle g(u_{n+1}) - g(u_n), g(v) - g(u_{n+1})\rangle \\ &= \langle f, g(v) - g(u_{n+1})\rangle, \quad \forall v \in H. \end{aligned}$$

We again use the auxiliary principle technique to suggest an implicit method for solving the problem (5.59).

For a given $u \in H$, consider the problem of finding $w \in H$ such that,

$$\langle \rho(e^{L(w+\zeta(u-w))} + e^{A(w+\zeta(u-w)))}), g(v) - g(w)\rangle \quad + \langle g(w) - g(u) + \eta(g(u) - g(u)), g(v) - g(w)\rangle$$
$$= \rho\langle f, g(v) - g(w)\rangle, \quad \forall v \in H, \tag{5.96}$$

which is called the auxiliary problem, where $\eta \geq 0, \zeta \geq 0$, are parameter. We note that the auxiliary problems (5.88) and (5.96) are quite different.

Clearly $w = u \in H$ is a solution of (5.59). This observation allows us to suggest the following iterative method for solving the problem (5.59).

Algorithm 5.2 *For given initial values u_0, u_1, compute the approximate solution x_{n+1} by the iterative scheme*

$$\begin{aligned}
&\langle \rho(e^{L(u_{n+1}+\zeta(u_n-u_{n+1}))} + e^{A(u_{n+1}+\zeta(u_n-u_{n+1}))}) \\
&\quad + g(u_{n+1}) - g(u_n) + \eta(g(u_n) - g(u_{n-1})), g(v) - g(u_{n+1})\rangle \\
&= \langle \rho f, g(v) - g(u_{n+1})\rangle, \forall v \in H,
\end{aligned}$$

which is an inertial implicit method.

If $\zeta = \frac{1}{2}$, then Algorithm 5.2 reduces to

Algorithm 5.3 *For given initial values u_0, u_1, compute the approximate solution x_{n+1} by the iterative scheme*

$$\begin{aligned}
&\langle \rho(e^{L(\frac{u_{n+1}+u_n}{2})} + e^{A(\frac{u_{n+1}+u_n}{2})}) \\
&\quad + g(u_{n+1}) - g(u_n) + \eta(g(u_n) - g(u_{n-1})), g(v) - g(u_{n+1})\rangle \\
&= \langle \rho f, g(v) - g(u_{n+1})\rangle, \forall v \in H,
\end{aligned}$$

which is an inertial mid point implicit method.

From Algorithm 5.2, we can obtain the following iterative method for solving (5.59).

Algorithm 5.4 *For a given initial value u_0, u_1 compute the approximate solution x_{n+1} by the iterative scheme*

$$g(u_{n+1}) = g(u_n) - \eta(g(u_n) - g(u_{n-1}) - \rho(e^{L(u_{n+1}+\zeta(u_n-u_{n+1}))} + e^{A(u_{n+1}+\zeta(u_n-u_{n+1}))} - f).$$

This is a new implicit method for solving the system of exponentially general equations (5.59). To implement the implicit method (5.2) with $\eta(g(u_n) - g(u_{n-1})) = 0$, one uses the explicit method as a predictor and implicit method as a predictor. Consequently, we obtain the two-step method for solving the problem (5.59).

Algorithm 5.5 *For a given initial value u_0, compute the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned}
&\langle \rho e^{Lu_n} + \rho e^{A(u_n)} + g(y_n) - g(u_n), g(v) - g(u_n)\rangle \\
&\quad = \langle \rho f, g(v) - g(y_n)\rangle, \forall v \in H, \\
&\langle \rho e^{L(y_n+\zeta(u_n-y_n)} + \rho e^{A(y_n+\zeta(u_n-y_n))} + g(u_{n+1}) - g(u_n), g(v) - g(y_n)\rangle \\
&= \quad \langle \rho f, g(v) - g(y_n)\rangle, \forall v \in H,
\end{aligned}$$

which is known as two-step iterative method for solving problem (5.59).
Based on the above arguments, we can suggest a new two-step(predictor-corrector) method for solving the system of exponentially general equations (5.59).

Algorithm 5.6 *For a given initial value u_0, compute the approximate solution x_{n+1} by the iterative schemes*

$$\begin{aligned}
g(y_n) &= g(u_n) - \rho(e^{Lu_n} + e^{A(u_n)} - f) \\
g(u_{n+1}) &= g(u_n) - \rho(e^{L(y_n+\zeta(u_n-y_n))} + e^{A(y_n+\zeta(u_n-y_n)))} - f), \quad n = 0, 1, 2...
\end{aligned}$$

We again apply the auxiliary principle technique to suggest another iterative method for solving (5.59).
For a given $u \in H$, consider the problem of finding $w \in H$ such that,

$$\langle \rho(e^{Lw} + e^{A(w))}, g(v) - g(w)\rangle + \langle w - u + \xi(u - u), v - w\rangle = \langle \rho f, g(v) - g(w)\rangle, \quad \forall v \in H, \tag{5.97}$$

which is called the auxiliary problem, where $\rho > 0, \xi > 0$ are constants. It is clear that (5.88) defines a mapping w connecting the both problems (5.59) and (5.97).
If $w = u$, then w is a solution of (5.59). This observation is used to propose the iterative method.

Algorithm 5.7 *For given initial values u_0, u_1 compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}
\langle \rho e^{Lu_{n+1}} + \rho e^{A(u_{n+1})}, g(v) - g(u_{n+1})\rangle \quad + \quad &\langle u_{n+1} - u_n + \xi(u_n - u_{n-1}, v - u_{n+1}\rangle \\
&= \langle \rho f, g(v) - g(u_{n+1})\rangle, \forall v \in H,
\end{aligned} \tag{5.98}$$

which is an inertial implicit method.
If $\xi = 0$, then Algorithm 5.7 reduces to

Algorithm 5.8 *For a given initial value u_0, compute the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}\langle \rho e^{Lu_{n+1}} + \rho e^{A(u_{n+1})}, g(v) - g(u_{n+1})\rangle \quad &+ \quad \langle u_{n+1} - u_n, v - u_{n+1}\rangle \\ &= \langle \rho f, g(v) - g(u_{n+1})\rangle, \forall v \in H, \end{aligned} \tag{5.99}$$

which is an inertial implicit method.

For the convergence analysis of the iterative methods, we need the following concept.

Definition 5.5 *The operator L is said to be pseudo exponentially general monotone with respect to A, if*

$$\begin{aligned}&\langle e^{Lu} + e^{A(u)}, g(v) - g(u)\rangle = \langle f, g(v) - g(u)\rangle, \quad \forall v \in H, \\ &\Rightarrow \\ &\langle e^{Lv} + e^{A(v)}, g(v) - g(u)\rangle \geq \langle f, g(v) - g(u)\rangle, \quad \forall v \in H.\end{aligned}$$

We now consider the convergence analysis of Algorithm 5.8, which is the main motivation of our next result.

Theorem 5.12 *Let $u \in H$ be a solution of problem (5.59) and let u_{n+1} be the approximate solution obtained from Algorithm 5.8. If L is an exponentially monotone operator with respect to $A(.)$, then*

$$\|u_{n+1} - u\|^2 \leq \|u_n - u\|^2 - \|u_{n+1} - u_n\|^2. \tag{5.100}$$

Proof. Let $u \in H : g(u) \in H$ be a solution of (5.59). Then

$$\langle e^{Lu} + e^{A(u)}, g(v) - g(u)\rangle = \langle f, g(v) - g(u)\rangle, \quad \forall v \in H,$$

which implies that

$$\langle e^{Lv} + e^{A(v)}, g(v) - g(u)\rangle \geq \langle f, g(v) - g(u)\rangle, \quad \forall v \in H, \tag{5.101}$$

since the operator L is an exponentially monotone operator with respect to $\lambda|.|$.

Taking $v = u_{n+1}$ in (5.101) and $v = u$ in (5.99), we have

$$\langle e^{Lu_{n+1}} + e^{A(u_{n+1})}, g(u_{n+1}) - g(u)\rangle \geq \langle f, g(u_{n+1}) - g(u)\rangle, \quad \forall v \in H, \tag{5.102}$$

and

$$\begin{aligned}&\langle \rho e^{Lu_{n+1}} + \rho e^{A(u_{n+1})}, g(u) - g(u_{n+1})\rangle + \langle u_{n+1} - u_n, u - u_{n+1}\rangle \\ &= \langle \rho f, g(u) - g(u_{n+1})\rangle, \forall v \in H.\end{aligned} \tag{5.103}$$

From (5.103), we have

$$\begin{aligned}\langle u_{n+1}-u_n, u-u_{n+1}\rangle \quad &\geq \quad \rho\langle(e^{Lu_{n+1}}+e^{A(u_{n+1}))}, g(u_{n+1})-g(u)\rangle \\ &\quad -\rho\langle f, g(u_{n+1})-g(u)\rangle \\ &\geq 0, \end{aligned} \tag{5.104}$$

where we have used (5.102).

Using the relation $2\langle a,b\rangle = \|a+b\|^2-\|a\|^2-\|b\|^2, \quad \forall a,b \in H$, the Cauchy inequality and from (5.104), we have

$$\|u-u_{n+1}\|^2 \leq \|u-u_n\|^2 - \|u_n-u_{n+1}\|^2,$$

which is the required (5.100). □

Theorem 5.13 *Let $\bar{u} \in H$ be a solution of (5.59) and let u_{n+1} be the approximate solution obtained from Algorithm 5.2. Let L be an exponentially monotone operator with respect to A and the operator g is continuous. Then*

$$\lim_{n\to\infty} u_{n+1} = \bar{u}. \tag{5.105}$$

Proof. Let $\bar{u} \in H$ be a solution of (5.59). From ((5.100), it follows that the sequence $\{\|\bar{u}-u_n\|\}$ is noncreasing and consequently the sequence $\{u_n\}$ is bounded. Also, from (5.100), we have

$$\sum_{n=0}^{\infty}\|u_{n+1}-u_n)\|^2 \leq \|u_0-\bar{u}\|^2,$$

which implies that

$$\lim_{n\to\infty}\|u_{n+1}-u_n\| = 0 \Longrightarrow \quad \lim_{n\to\infty}\|u_{n+1}-u_n\| = 0. \tag{5.106}$$

Let $\hat{u}$ be a cluster point of $\{u_n\}$ and the subsequences $\{u_{n_j}\}$ of the sequence $\{u_n\}$ converges to $\bar{u} \in H$. Replacing u_n by u_{n_j}in (5.99), taking the limit as $n_j \to \infty$ and using (5.106), we have

$$\langle e^{L\hat{u}}+e^{A(\hat{u})}, g(v)-g(\hat{u})\rangle = \langle f, g(v)-g(\hat{u})\rangle, \quad \forall v \in H,$$

which shows that $\hat{u} \in H$ satisfies (5.59) and

$$\|u_{n+1}-u_n\|^2 \leq \|u_n-\hat{u}\|^2.$$

From the above inequality, it follows that the sequence $\{u_n\}$ has exactly one cluster point $\hat{u}$ and $\lim_{n\to\infty} u_n = \hat{u}$. $\Box$

The auxiliary principle technique is used to suggest another iterative method for solving (5.59).

For a given $u \in H$, enwinding $w \in H$ such that,

$$\begin{aligned} \langle \rho e^{(L((1-\xi)w+\xi u)} &+ e^{A((1-\xi)w+\xi u))}, g(v) - g((1-\xi)w+\xi u))\rangle \\ &+ \langle g((1-\xi)w+\xi u)) - g(u), g(v) - g((1-\xi)w+\xi u))\rangle \\ &= \langle \rho f, g(v) - g((1-\xi)w+\xi u))\rangle, \quad \forall v \in H, \xi \in [0,1], \end{aligned} \tag{5.107}$$

which is called the auxiliary problem, where $\rho > 0, \xi \geq 0$, are constants. It is clear that (5.88) defines a mapping w connecting the both problems (5.59) and (5.97). Clearly, $w = u$ is a solution of the problem (5.59). This allows us to suggest the inertial iterative method.

Algorithm 5.9 *For a given $u_1, u_2 \in H$, calculate the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned} \langle \rho (e^{L((1-\xi)u_n+\xi u_{n-1})} &+ e^{A((1-\xi)u_n+\xi u_{n-1}))}, g(v) - g((1-\xi)u_n+\xi u_{n_1}))\rangle \\ &+ \langle g((1-\xi)u_n+\xi u_{n-1})) - g(u_n), g(v) - g((1-\xi)u_n+\xi u_{n-1}))\rangle \\ &= \langle \rho f, g(v) - g((1-\xi)u_n+\xi u_{n-1}))\rangle, \quad \forall v \in H, \xi \in [0,1] \end{aligned}$$

This is an inertial type implicit method for solving the problem (5.59), which is equivalent to the following two-step method.

Algorithm 5.10 *For a given $u_1, u_1 \in H$, calculate the approximate solution u_{n+1} by the iterative schemes*

$$\begin{aligned} g(y_n) &= (1-\xi)u_n + \xi u_{n-1}) \\ g(u_{n+1}) &= g(u)_n - \rho(e^{Ly_n} + e^{A((y_n))} - \rho f, \quad n = 1, 2, ..., \quad \xi \in [0,1] \end{aligned}$$

We now use the auxiliary principle technique involving the Bregman function to suggest and analyze the proximal method for solving exponentially general equations (5.59). For the sake of completeness and to convey the main ideas of the Bregman distance functions, we recall the basic concepts and applications.

The Bregman distance function is defined as

$$B(u,w) = E(g(u)) - E(g(w)) - \langle E'(g(w)), g(u) - g(w)\rangle \geq \nu \|g(u) - g(w)\|^2, \tag{5.108}$$

using the strongly general convexity with modulus ν.
The function $B(u,w)$ is called the general Bregman distance function associated with general convex functions.
For $g = I$, we obtain the original Bregman distance function

$$B(u,w) = E(u) - E(w)) - \langle E'(w), u - w\rangle \geq \nu\|u - w\|^2.$$

It is important to emphasize that various types of convex function E gives different Bregman distance.
For a given $u \in H$, find $w \in H$ satisfying the auxiliary system of exponentially general equations (5.59).

$$\begin{aligned}\langle \rho(e^{L(w+\zeta(u-w))} + e^{A(w+\zeta(u-w)))}) \quad &+ \quad E'(g(w)) - E'(g(u)), g(v) - g(w)\rangle \\ &= \langle \rho f, g(v) - g(w)\rangle, \forall v \in H,\end{aligned}$$

where $E'(u)$ is the differential of a strongly general convex function E.
Note that, if $w = u$, then w is a solution of (5.59). Thus, we can suggest the following iterative method for solving (5.59).

Algorithm 5.11 *For a given $u_0 \in H$, calculate the approximate solution by the iterative scheme*

$$\begin{aligned}\langle \rho(e^{L(u_{n+1}+\zeta(u_n-u_{n+1}))} + e^{A(u_{n+1}+\zeta(u_n-u_{n+1}))}) \quad &+ \quad E'(g(u_{n+1}) - E'(g(u_n)), g(v) - g(u_{n+1})\rangle \\ &= \langle \rho f, g(v) - g(u_{n+1})\rangle, \forall v \in H, \quad (5.109)\end{aligned}$$

which is known as the proximal point method.
For $\zeta = 0$ and $\zeta = 1$, Algorithm 5.11 reduces to:

Algorithm 5.12 *For a given $u_0 \in H$, calculate the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}\langle \rho(e^{Lu_{n+1}} + e^{A(u_{n+1})}) \quad &+ \quad E'(g(u_{n+1})) - E'(g(u_n)), g(v) - g(u_{n+1})\rangle \\ &= \quad \langle \rho f, g(v) - g(u_{n+1})\rangle, \forall v \in H,\end{aligned}$$

which is known as the proximal implicit proximal method.

Algorithm 5.13 *For a given $u_0 \in H$, calculate the approximate solution u_{n+1} by the iterative scheme*

$$\begin{aligned}\langle \rho(e^{Lu_n} + e^{A(u_n)}) \quad &+ \quad E'(g(u_{n+1})) - E'(g(u_n)), g(v) - g(u_{n+1})\rangle \\ &= \quad \langle \rho f, g(v) - g(u_{n+1})\rangle, \forall v \in H,\end{aligned}$$

which is an explicit method.
For $\zeta = \frac{1}{2}$, Algorithm 5.11 collapses to:

Algorithm 5.14 *For a given $u_0 \in H$, calculate the approximate solution by the iterative scheme*

$$\begin{aligned}\langle \rho(e^{L(\frac{u_{n+1}+u_n}{2})} + e^{A(\frac{u_{n+1}+u_n}{2})}) \quad &+ \quad E'(g(u_{n+1}) - E'(g(u_n)), g(v) - g(u_{n+1})\rangle \\ &= \langle \rho f, g(v) - g(u_{n+1})\rangle, \forall v \in H, \end{aligned} \tag{5.110}$$

which is known as the mid-point proximal method.

Remark 5.5 *We would like to emphasize that for appropriate choice of the operators L, A, g one can suggest and analyze several new iterative methods for solving system of exponentially general equations and related problems. The implementation and comparison with other techniques need further efforts.*

Conclusion: In this paper, we have considered a new class of system of exponentially general equations involving three operators Several important problems such as system of absolute value equations, complementarity problems, Lax-Milgram Law and Reisz-Frechet representation theorem can be obtained as special cases. It is shown that the third order exponentially boundary value problems can be studied in the framework of general equations. We have used the auxiliary principle technique to study the existence of the unique solution of the system of general equations. Some new hybrid inertial iterative methods are suggested for solving the system of exponentially general equations using the auxiliary principle technique. The convergence analysis of these iterative methods is investigated under suitable conditions. This is a new approach for solving the system of exponentially general equations, see [48, 49, 50, 51]. We have only discussed the theoretical aspects of the proposed methods. The implementation and comparison with other numerical methods is the subject of the future research efforts. We would like to emphasize that the results obtained and discussed in this paper may motivate novel applications and extensions in these areas.

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