HJB 1

$\dot{x}(t) = a(x(t), u(t), t)$ $\qquad J = h(x(t_f), t_f) + \int_{t_0}^{t_f} g(x(\tau), u(\tau); \tau)\, d\tau$

τ dummy variable h and g are specified functions, t_0 and t_f are fixed.

* Imbeding principle *

$$J(x(t), t, U(\tau)) = h(x(t_f), t_f) + \int_{t}^{t_f} g(x(\tau), u(\tau); \tau)\, d\tau$$

$$t \leq \tau \leq t_f$$

$x(t)$ any admissible state value.

$$J^*(x(t), t) = \min_{\substack{U(\tau) \\ t \leq \tau \leq t_f}} \left\{ \int_{t}^{t_f} g(x(\tau), u(\tau), \tau)\, d\tau + h(x(t_f), t_f) \right\}.$$

By subdividing the interval, we obtain

$$J^*(x(t), t) = \min_{\substack{U(\tau) \\ t \leq \tau \leq t_f}} \left\{ \int_{t}^{t+\Delta t} g\, d\tau + \underbrace{\int_{t+\Delta t}^{t_f} g\, d\tau + h(x(t_f), t_f)}_{J^*(x(t+\Delta t), t, \Delta t)} \right\}.$$

According to principle of optimality

$$J^*(x(t), t) = \min_{\substack{U(\tau) \\ t \leq \tau \leq t_f}} \left\{ \int_{t}^{t+\Delta t} g\, d\tau + J^*(x(t+\Delta t), t+\Delta t) \right\}$$

Assumption J^* exist and bounded

$\hookrightarrow$ second partial derivatives of J^* exists and bonded

Taylor series

about the point. $(x(t), t)$

$$P_n(x, y) = \sum_{i=0}^{n} \sum_{j=0}^{n-i} \frac{\frac{\partial^{(i+j)} f(a, b)}{\partial x^i \partial y^j}}{i!\, j!} (x-a)^i (y-b)^j$$

$i=0$

$j=1$ $\qquad = V(a,b) + \frac{\partial V(a,b)}{\partial t}(t-b) + \frac{\partial V(a,b)}{\partial x}(x-a) + \frac{\partial^2 V(a,b)}{\partial x \partial y\, 2!}(x-a)(t-b)$

$i=1$ $\quad j=i=1$

$j=0$

$$J^*(x(t+\Delta t), t, \Delta t) = J^*(x(t),t) + \left[\frac{\partial J^*}{\partial t}(x(t),t)\right]\Delta t + \left[\frac{\partial J^*}{\partial x}(x(t),t)\right]^T \left[x(t+\Delta t) - x(t)\right] + H.O.T\,]$$

$$J^*(x(t),t) = \min_{\substack{U(\tau)\\ t\le\tau\le t+\Delta t}} \left\{ \int_t^{t+\Delta t} g\, d\tau + J^*(x(t),t) + \left[\frac{\partial J^*}{\partial t}(x(t),t)\right]\Delta t + \left[\frac{\partial J^*}{\partial x}(x(t),t)\right]^T \left[x(t+\Delta t) - x(t)\right] + H.O.T \right]$$

Now for small Δt

$$J^*(x(t),t) = \min_{U(t)} \left\{ g(x(t),U(t),t)\Delta t + J^*(x(t),t) + \left[\frac{\partial J^*}{\partial t}(x(t),t)\right]\Delta t + \left[\frac{\partial J^*}{\partial x}(x(t),t)\right]^T \left[\underbrace{x(t+\Delta t) - x(t)}_{a(x(t),U(t),t)\,\Delta t} + \cancel{H.O.T}\right]\right\}.$$

$$\frac{x(t+\Delta t) - x(t)}{\Delta t} = \dot{x}(t)$$

$$0 = \left[\frac{\partial J^*}{\partial t}(x(t),t)\right]\Delta t + \min_{U(t)} \left\{ g(x(t),U(t),t)\Delta t + \left[\frac{\partial J^*}{\partial x}(x(t),t)\right]^T \left[a(x(t),U(t),t)\,\Delta t\right] \right\} + O(\Delta t)$$

$\div\ \Delta t$ and $\Delta t \to 0$

$$0 = \frac{\partial J^*}{\partial t}(x(t),t) + \min_{U(t)} \left\{ g(x(t),U(t),t) + \left[\frac{\partial J^*}{\partial x}(x(t),t)\right]^T \left[a(x(t),U(t),t)\right] \right\}$$

Boundary Value $J^*(x(t_f),t_f) = h(x(t_f),t_f)$.

Also Hamiltonian is $\mathcal{H}(x(t),U(t),J^*_x,t) \triangleq g(x(t),U(t),t) + \left[\frac{\partial J^*}{\partial x}(x(t),t)\right]^T \left[a(x(t),U(t),t)\right]$