Differential Games and Representation Formulas for Solutions of Hamilton-Jacobi-Isaacs Equations

L. C. EVANS & P. E. SOUGANIDIS

1. Introduction

Recent work by the authors and others has demonstrated the connections between the dynamic programming approach to two-person, zero-sum differential games and the new notion of "viscosity" solutions of Hamilton-Jacobi PDE, introduced in Crandall-Lions [8]. The formal relationships here were observed by Isaacs in the early 1950's (cf. [18]): he showed that if the values of various differential games are regular enough, then they solve certain first order PDE with "max-min" or "min-max" type nonlinearity (the Isaacs equations). The problem here is that usually the value functions are not sufficiently smooth to make sense of these PDE in any obvious way. Many later papers in the subject have worked around this difficulty: see especially Fleming [13], [14], Friedman [15], [16], Elliott-Kalton [9]–[11], Krassovski-Subbotin [20], Subbotin [26], etc., and the references therein.

Recently, however, M. Crandall and P. L. Lions [8] have discovered a new notion of weak or so-called "viscosity" solution for Hamilton-Jacobi equations, and, most importantly, have proved uniqueness of such a solution in a wide variety of circumstances. This concept was reconsidered and simplified in part by Crandall, Evans, Lions [7], whose approach we follow below. Additionally, Lions in his new book [21] has made the fundamental observation that the dynamic programming optimality condition for the value in differential control theory problems implies that this value function is the viscosity solution of the associated Hamilton-Jacobi-Bellman PDE: see [21, pages 53–54] for more explanation. Some related papers are Lions [23], Lions-Nisio [24], Capuzzo Dolcetta-Evans [5], Barles [2], Capuzzo Dolcetta [4], Capuzzo Dolcetta-Ishii [6], etc.

The foregoing considerations turn out to extend to differential game theory, where additional complications arise with respect to the definition of the value functions. Nevertheless the basic idea is still valid, that the dynamic programming optimality conditions imply that the values are viscosity solutions of appropriate PDE. See Souganidis [27] for a demonstration of this based on both the Fleming and the Friedman definitions of upper and lower values for a differential game,

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and Barron-Evans-Jensen [3] for a different proof for the Friedman definition. Some similar results are to be found in P. L. Lions [22].

The present paper represents a simplification of this previous work. The new approach here is to define the values of the differential game following Elliott-Kalton [9]–[11] (cf. Roxin [25]) rather than Fleming or Friedman. This results in a great simplification in the statements and proofs, as the definitions are explicit and do not entail any kind of approximations.

The appropriate terminology is introduced in §2. In §3 we reproduce (and simplify a bit) Elliott and Kalton's proof of the optimality conditions and of the Lipschitz continuity of the upper and lower value functions. Then in §4 we prove that the value functions are the (unique) viscosity solutions of the appropriate Isaacs equations; our demonstration of this owes a lot to previous papers (especially [3] and [27]), but is essentially simpler in many ways.

The remainder of the paper is devoted to some applications. First, in §5 we discuss (cf. Fleming [14]) how to write a fairly arbitrary Hamilton-Jacobi equation as the upper Isaacs equation for some differential game, so that the viscosity solution is this upper value. The consequence is a kind of representation formula for the solution of the original, fully nonlinear first-order PDE. We thereafter in §7 employ this representation formula to prove results about the level sets of solutions to Hamilton-Jacobi equations with homogeneous Hamiltonians; these equations we motivate in §6 with a discussion of geometric optics and Huygens' principle. Part of the point of this application is to show that the game theory methods provide mathematically rigorous and relatively simple procedures for justifying various formal calculations concerning Hamilton-Jacobi equations. Roughly speaking, the trajectories for the differential game serve as "generalized characteristics" existing in the large.

We should note also that our hypotheses throughout are almost always stronger than is really necessary, since we wish to display the methods in the clearest setting. The interested reader should consult Ishii [19] for some extensions of our results to differential game problems under much weaker hypotheses.

We conclude by recording here the relevant definition of viscosity solutions, from [7], [8], [3].

Assume $H: [0,T] \times \mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}$ is continuous, and $g: \mathbf{R}^m \to \mathbf{R}^m$ is bounded, uniformly continuous. A bounded, uniformly continuous function $u: [0,T] \times \mathbf{R}^m \to \mathbf{R}^m$ is called a *viscosity solution* of the Hamilton-Jacobi equation

$$\text{(HJ)} \begin{cases} (1.1) \quad u_t + H(t,x,Du) = 0 & \text{in } (0,T) \times \mathbf{R}^m \\ (1.2) \quad u(T,x) = g(x) & \text{in } \mathbf{R}^m \end{cases}$$

provided (1.2) holds and for each $\phi \in C^1((0,T) \times \mathbf{R}^m)$

(a) if $u - \phi$ attains a local maximum at $(t_0,x_0) \in (0,T) \times \mathbf{R}^m$, then

$$\phi_t(t_0,x_0) + H(t_0,x_0,D\phi(t_0,x_0)) \geq 0, \tag{1.3}$$

and

(b) if $u - \phi$ attains a local minimum at $(t_0,x_0) \in (0,T) \times \mathbf{R}^m$, then

(1.4) $$\phi_t(t_0,x_0) + H(t_0,x_0,D\phi(t_0,x_0)) \leq 0.$$

See [7] for a proof that if u is a viscosity solution of (HJ) and if u is differentiable at some point (t_0,x_0), then

$$u_t(t_0,x_0) + H(t_0,x_0,Du(t_0,x_0)) = 0.$$

Remark. We have described here the appropriate definition for the terminal value problem (1.1), (1.2); this is, as we shall see, the kind of PDE arising in game theory applications. A viscosity solution of the initial value problem (1.1),

(1.2)′ $$u(x,0) = g(x) \qquad \text{in } \mathbf{R}^m,$$

is defined by reversing the inequalities in (1.3), (1.4).

2. Terminology

We mostly adopt here the notation of Elliott-Kalton [9].

(A) Definition of the differential game. Fix $T > t \geq 0$, $x \in \mathbf{R}^m$ and consider the differential equation

$$\text{(ODE)}\begin{cases} \dot{x}(s) = f(s,x(s),y(s),z(s)) & t \leq s \leq T \\ x(t) = x. \end{cases}$$

Here

$$y:[t,T] \to Y$$

and

$$z:[t,T] \to Z$$

are given measurable functions (called the *controls* employed by players I and II, respectively) and $Y \subset \mathbf{R}^k$, $Z \subset \mathbf{R}^\ell$ are given compact sets. We assume

$$f:[0,T] \times \mathbf{R}^m \times Y \times Z \to \mathbf{R}^m$$

is uniformly continuous, with

(2.1) $$\begin{cases} |f(t,x,y,z)| \leq C_1 \\ |f(t,x,y,z) - f(t,\hat{x},y,z)| \leq C_1|x - \hat{x}| \end{cases}$$

for some constant C_1 and all $0 \leq t \leq T$, $x, \hat{x} \in \mathbf{R}^m$, $y \in Y$, $z \in Z$. The (unique) solution $x(\cdot)$ of (ODE) is the *response* of the system to the controls $y(\cdot)$, $z(\cdot)$.

Associated with (ODE) is the *payoff* functional

$$\text{(P) } P(y,z) = P_{t,x}(y(\cdot),z(\cdot)) = \int_t^T h(s,x(s),y(s),z(s))ds + g(x(T)),$$

where $g:\mathbf{R}^m \to \mathbf{R}$ satisfies

$$(2.2)\qquad \begin{cases} |g(x)| \le C_2 \\ |g(x) - g(\hat{x})| \le C_2|x - \hat{x}|, \end{cases}$$

and $h:[0,T] \times \mathbf{R}^m \times Y \times Z \to \mathbf{R}$ is uniformly continuous, with

$$(2.3)\qquad \begin{cases} |h(t,x,y,z)| \le C_3 \\ |h(t,x,y,z) - h(t,\hat{x},y,z)| \le C_3|x - \hat{x}| \end{cases}$$

for constants C_2, C_3 and all $0 \le t \le T$, $x, \hat{x} \in \mathbf{R}^m$, $y \in Y$, $z \in Z$. The goal of player I is to maximize P and the goal of player II is to minimize P.

(B) The upper and lower values. Set

$$M(t) \equiv \{y : [t,T] \to Y \mid y \text{ measurable}\}$$
$$N(t) \equiv \{z : [t,T] \to Z \mid z \text{ measurable}\};$$

these are the sets of all controls for I and II, respectively. We will henceforth identify any two controls which agree a.e.

Following now Varaiva [29], Roxin [25] and Elliott-Kalton [9] we define any mapping

$$\alpha : N(t) \to M(t)$$

to be a *strategy for I* (beginning at time t) provided for each $t \le s \le T$ and $z, \hat{z} \in N(t)$:

$$(2.4)\qquad \begin{cases} z(\tau) = \hat{z}(\tau) \text{ for a.e. } \quad t \le \tau \le s \\ \text{implies } \alpha[z](\tau) = \alpha[\hat{z}](\tau) \text{ for a.e. } \quad t \le \tau \le s. \end{cases}$$

Similarly a mapping

$$\beta : M(t) \to N(t)$$

is a *strategy for II* (beginning at time t) provided for each $t \le s \le T$ and $y, \hat{y} \in M(t)$:

$$(2.5)\qquad \begin{cases} y(\tau) = \hat{y}(\tau) \text{ for a.e. } \quad t \le \tau \le s \\ \text{implies } \beta[y](\tau) = \beta[\hat{y}](\tau) \text{ for a.e. } \quad t \le \tau \le s. \end{cases}$$

Denote by $\Gamma(t)$ the set of all strategies for I and by $\Delta(t)$ the set of all strategies for II, beginning at time t.

Finally define

$$(2.6)\qquad \begin{cases} V(t,x) \equiv \inf\limits_{\beta \in \Delta(t)} \sup\limits_{y \in M(t)} P(y,\beta[y]) \\ \qquad = \inf\limits_{\beta \in \Delta(t)} \sup\limits_{y \in M(t)} \left\{ \int_t^T h(s,x(s),y(s),\beta[y](s))ds + g(x(T)) \right\}, \end{cases}$$

$x(\cdot)$ solving (ODE) for $y(\cdot)$ and $z(\cdot) = \beta[y](\cdot)$. Analogously set

$$(2.7)\quad \begin{cases} U(t,x) \equiv \sup\limits_{\alpha\in\Gamma(t)} \inf\limits_{z\in N(t)} P(\alpha[z],z) \\ \qquad = \sup\limits_{\alpha\in\Gamma(t)} \inf\limits_{z\in N(t)} \left\{ \int_t^T h(s,x(s),\alpha[z](s),z(s))ds + g(x(T)) \right\}, \end{cases}$$

$x(\cdot)$ solving (ODE) with $z(\cdot)$ and $y(\cdot) = \alpha[z](\cdot)$.

We call V the *lower value* and U the *upper value* of the differential game (ODE), (P). Our goal is to show that V and U solve certain nonlinear PDE (in the viscosity sense).

3. Properties of the Upper and Lower Values

The results in this section are proved in Elliott-Kalton [10]. We reproduce and simplify slightly their arguments for the reader's convenience.

Theorem 3.1. *For each* $0 \le t < t + \sigma \le T$ *and* $x \in \mathbf{R}^m$

$$(3.1)\quad V(t,x) = \inf_{\beta\in\Delta(t)} \sup_{y\in M(t)} \left\{ \int_t^{t+\sigma} h(s,x(s),y(s),\beta[y](s))ds + V(t+\sigma,x(t+\sigma)) \right\},$$

and

$$(3.2)\quad U(t,x) = \sup_{\alpha\in\Gamma(t)} \inf_{z\in N(t)} \left\{ \int_t^{t+\sigma} h(s,x(s),\alpha[z](s),z(s))ds + U(t+\sigma,x(t+\sigma)) \right\}.$$

These are the *dynamic programming optimality conditions.*

Remark. In (3.1) and (3.2), as elsewhere below, we implicitly mean $x(\cdot)$ to solve (ODE) with the appropriate controls $y(\cdot)$ and $z(\cdot)$.

Proof. We prove (3.1) only, as the proof of (3.2) is similar. Set

$$(3.3)\quad W(t,x) \equiv \inf_{\beta\in\Delta(t)} \sup_{y\in M(t)} \left\{ \int_t^{t+\sigma} h(s,x(s),y(s),\beta[y](s))ds + V(t+\sigma,x(t+\sigma)) \right\}$$

and fix $\varepsilon > 0$. Then there exists $\delta \in \Delta(t)$ such that

$$(3.4)\quad W(t,x) \ge \sup_{y\in M(t)} \left\{ \int_t^{t+\sigma} h(s,x(s),\delta[y](s))ds + V(t+\sigma,x(t+\sigma)) \right\} - \varepsilon.$$

Also, for each $w \in \mathbf{R}^m$

$$V(t+\sigma,w) = \inf_{\beta\in\Delta(t+\sigma)} \sup_{y\in M(t+\sigma)} \left\{ \int_{t+\sigma}^T h(s,x(s),y(s),\beta[y](s))ds + g(x(T)) \right\},$$

$x(\cdot)$ solving (ODE) on $(t+\sigma,T)$, with the initial condition $x(t+\sigma) = w$. Thus there exists $\delta_w \in \Delta(t+\sigma)$ for which

$$(3.5)\quad V(t+\sigma,w) \ge \sup_{y\in M(t+\sigma)} \left\{ \int_{t+\sigma}^T h(s,x(s),y(s),\delta_w[y](s))ds + g(x(T)) \right\} - \varepsilon.$$

Define $\beta \in \Delta(t)$ in this way: for each $y \in M(t)$ set

$$\beta[y](s) \equiv \begin{cases} \delta[y](s) & t \leq s \leq t + \sigma \\ \delta_{x(t+\sigma)}[y](s) & t + \sigma < s \leq T. \end{cases}$$

Consequently for any $y \in M(t)$, (3.4) and (3.5) imply

$$W(t,x) \geq \int_t^T h(s,x(s),y(s),\beta[y](s))ds + g(x(T)) - 2\varepsilon;$$

so that

$$\sup_{y \in M(t)} \left\{ \int_t^T h(s,x(s),y(s),\beta[y](s))ds + g(x(T)) \right\} \leq W(t,x) + 2\varepsilon.$$

Hence

$$V(t,x) \leq W(t,x) + 2\varepsilon. \tag{3.6}$$

On the other hand there exists $\beta \in \Delta(t)$ for which

$$V(t,x) \geq \sup_{y \in M(t)} \left\{ \int_t^T h(s,x(s),y(s),\beta[y](s))ds + g(x(T)) \right\} - \varepsilon. \tag{3.7}$$

Then

$$W(t,x) \leq \sup_{y \in M(t)} \left\{ \int_t^{t+\sigma} h(s,x(s),y(s),\beta[y](s))ds + V(t + \sigma,x(t + \sigma)) \right\},$$

and consequently there exists $y_1 \in M(t)$ such that

$$W(t,x) \leq \int_t^{t+\sigma} h(s,x(s),y_1(s),\beta[y_1](s))ds + V(t + \sigma,x(t + \sigma)) + \varepsilon. \tag{3.8}$$

Now for each $y \in M(t + \sigma)$ define $\tilde{y} \in M(t)$ by

$$\tilde{y}(s) \equiv \begin{cases} y_1(s) & t \leq s < t + \sigma \\ y(s) & t + \sigma \leq s \leq T \end{cases}$$

and then define $\underset{\sim}{\beta} \in \Delta\,(t + \sigma)$ by

$$\underset{\sim}{\beta}[y](s) \equiv \beta[\tilde{y}](s) \qquad (t + \sigma \leq s \leq T).$$

Hence

$$V(t + \sigma,x(t + \sigma)) \leq \sup_{y \in M(t+\sigma)} \left\{ \int_{t+\sigma}^T h(s,x(s),y(s),\underset{\sim}{\beta}[y](s))ds + g(x(T)) \right\}$$

and so there exists $y_2 \in M(t + \sigma)$ for which

$$(3.9)\quad V(t+\sigma,x(t+\sigma)) \leq \int_{t+\sigma}^{T} h(s,x(s),y_2(s),\underset{\sim}{\beta}[y_2](s))ds + g(x(T)) + \varepsilon.$$

Define $y \in M(t)$ by

$$y(s) \equiv \begin{cases} y_1(s) & t \leq s < t+\sigma \\ y_2(s) & t+\sigma \leq s \leq T. \end{cases}$$

Then (3.8) and (3.9) yield

$$(3.10)\qquad W(t,x) \leq \int_{t}^{T} h(s,x(s),y(s),\beta[y](s))ds + g(x(T)) + 2\varepsilon,$$

and so (3.7) implies

$$W(t,x) \leq V(t,x) + 3\varepsilon.$$

This and (3.6) complete the proof. □

Next we examine the boundedness and continuity of the value functions:

Theorem 3.2. *There exists a constant C_4 such that*

$$(3.11)\qquad |V(t,x)|, |U(t,x)| \leq C_4$$

$$(3.12)\qquad |V(t,x) - V(\hat{t},\hat{x})|, |U(t,x) - U(\hat{t},\hat{x})| \leq C_4(|t-\hat{t}| + |x-\hat{x}|)$$

for all $0 \leq t,\ \hat{t} \leq T,\ x,\ \hat{x} \in \mathbf{R}^m$.

Proof. We give the proof for U only since similar arguments work for V. First, owing to (2.2) and (2.3) we have

$$|P(y,z)| \leq TC_3 + C_2$$

for all $y(\cdot) \in M(t)$, $z(\cdot) \in N(t)$. This at once implies estimate (3.11) for U.

To prove (3.12) for V let us first choose $x_1, x_2 \in \mathbf{R}^m$, $0 \leq t_1 \leq t_2 \leq T$. Pick $\varepsilon > 0$ and then select $\alpha \in \Gamma(t_1)$ so that

$$(3.13)\qquad U(t_1,x_1) \leq \inf_{z \in N(t_1)} P(\alpha[z],z) + \varepsilon.$$

Pick some $z_0 \in Z$, and then define for any $z \in N(t_2)$

$$\tilde{z} \in N(t_1)$$

by

$$\tilde{z}(s) = \begin{cases} z_0 & t_1 \leq s < t_2 \\ z(s) & t_2 \leq s \leq T. \end{cases}$$

Now define $\underset{\sim}{\alpha} \in \Gamma(t_2)$ by setting for each $z \in N(t_2)$

$$\underset{\sim}{\alpha}[z] \equiv \alpha[\tilde{z}] \qquad (t_2 \leq s \leq T).$$

Finally select $z \in N(t_2)$ so that

$$U(t_2,x_2) \geq P(\underset{\sim}{\alpha}[z],z) - \varepsilon. \tag{3.14}$$

According to (3.13)

$$U(t_1,x_1) \leq P(\alpha[\tilde{z}],\tilde{z}) + \varepsilon. \tag{3.15}$$

Now let $x_1(\cdot)$ solve

$$\begin{cases} \dfrac{dx_1(s)}{ds} = f(s,x(s),\alpha[\tilde{z}](s),\tilde{z}(s)) & (t_1 < s < T) \\ x_1(t_1) = x_1 \end{cases}$$

and let $x_2(\cdot)$ solve

$$\begin{cases} \dfrac{dx_2(s)}{ds} = f(s,x(s),\underset{\sim}{\alpha}[z](s),z(s)) & (t_2 < s < T) \\ x_2(t_2) = x_2. \end{cases}$$

We have

$$|x_1(t_2) - x_1| \leq C_1|t_1 - t_2|.$$

Furthermore, since $z = \tilde{z}$ and $\underset{\sim}{\alpha}[z] = \alpha[\tilde{z}]$ on (t_2,T),

$$|x_1(s) - x_2(s)| \leq C|x_1(t_2) - x_2| \leq C(|t_1 - t_2| + |x_1 - x_2|) \qquad (t_2 \leq s \leq T). \tag{3.16}$$

Thus (3.14) and (3.15) imply

$$\begin{aligned} U(t_1,x_1) - U(t_2,x_2) & \\ &\leq P(\alpha[\tilde{z}],\tilde{z}) - P(\underset{\sim}{\alpha}[z],z) + 2\varepsilon \\ &= \int_{t_1}^{t_2} h(s,x_1(s),\alpha[\tilde{z}](s),\tilde{z}(s))ds \\ &\quad + \int_{t_2}^{T} h(s,x_1(s),\underset{\sim}{\alpha}[z](s),z(s)) - h(s,x_2(s),\underset{\sim}{\alpha}[z](s),z(s))ds \\ &\quad + g(x_1(T)) - g(x_2(T)) + 2\varepsilon \\ &\leq C(|t_1 - t_2| + |x_1 - x_2|) + 2\varepsilon, \end{aligned} \tag{3.17}$$

by (2.1)-(2.3) and (3.16).

On the other hand let us select $\alpha \in \Gamma(t_2)$ such that

$$U(t_2,x_2) \leq \inf_{z \in N(t_2)} P(\alpha[z],z) + \varepsilon. \tag{3.18}$$

For each $z \in N(t_1)$ define $\underset{\sim}{z} \in N(t_2)$ by

$$\underset{\sim}{z}(s) = z(s) \qquad (t_2 \leq s < T).$$

Fix any $y_0 \in Y$ and then define $\tilde{\alpha} \in \Gamma(t_1)$ by

$$\tilde{\alpha}[z] = \begin{cases} y_0 & t_1 \leq s \leq t_2 \\ \alpha[\underset{\sim}{z}] & t_2 \leq s \leq T. \end{cases}$$

Now choose $z \in N(t_1)$ so that

$$U(t_1,x_1) \geq P(\tilde{\alpha}[z],z) - \varepsilon. \tag{3.19}$$

According to (3.18)

$$U(t_2,x_2) \leq P(\alpha[\underset{\sim}{z}],\underset{\sim}{z}) + \varepsilon. \tag{3.20}$$

Let $x_1(\cdot)$ solve

$$\begin{cases} \dfrac{dx_1(s)}{ds} = f(s,x_1(s),\alpha[z](s),z(s)) & (t_1 < s < T) \\ x_1(t_1) = x_1 \end{cases}$$

and let $x_2(\cdot)$ solve

$$\begin{cases} \dfrac{dx_2(s)}{ds} = f(s,x_2(s),\alpha[\underset{\sim}{z}](s),\underset{\sim}{z}(s)) & (t_2 < s < T) \\ x_2(t_2) = x_2. \end{cases}$$

As above $|x_1(t_2) - x_1| \leq C_1|t_1 - t_2|$; and since $z = \underset{\sim}{z}$, $\tilde{\alpha}[z] = \alpha[\underset{\sim}{z}]$ on (t_2,T),

$$|x_1(s) - x_2(s)| \leq C\,|x_1(t_2) - x_2| \leq C\,(|t_1 - t_2| + |x_1 - x_2|) \qquad (t_2 \leq s < T). \tag{3.21}$$

Therefore (3.18) and (3.20) imply

$$\begin{aligned} U(t_2,x_2) - U(t_1,x_1) &\leq P(\alpha[\underset{\sim}{z}],\underset{\sim}{z}) - P(\tilde{\alpha}[z],z) + 2\varepsilon \\ &= -\int_{t_1}^{t_2} h(s,x_1(s),\tilde{\alpha}[z](s),z(s))ds \\ &\quad + \int_{t_2}^{T} h(s,x_2(s),\alpha[\underset{\sim}{z}](s),\underset{\sim}{z})(s)) - h(s,x_1(s),\alpha[\underset{\sim}{z}](s),\underset{\sim}{z}(s))ds \\ &\quad + g(x_2(T)) - g(x_1(T)) + 2\varepsilon \\ &\leq C(|t_1 - t_2| + |x_1 - x_2|) + 2\varepsilon, \end{aligned}$$

by (2.1)-(2.3) and (3.21).

This and (3.17) prove estimate (3.12) for U. □

4. Viscosity Solutions of Isaacs' Equations

Next is the observation that the dynamic programming optimality conditions imply U and V to be viscosity solutions of certain PDE.

Theorem 4.1. *(a). U is the viscosity solution of the upper Isaacs equation*

$$(\mathrm{I})^{+}\begin{cases} U_t + H^{+}(t,x,DU) = 0 & (0 \le t \le T, x \in \mathbf{R}^m) \\ U(T,x) = g(x) & (x \in \mathbf{R}^m), \end{cases}$$

where

$$H^{+}(t,x,p) \equiv \min_{z\in Z} \max_{y\in Y} \{f(t,x,y,z) \cdot p + h(t,x,y,z)\}$$

is the upper Hamiltonian.

(b). V is the viscosity solution of the lower Isaacs equation

$$(\mathrm{I})^{-}\begin{cases} V_t + H^{-}(t,x,DV) = 0 & (0 \le t \le T, x \in \mathbf{R}^m) \\ V(T,x) = g(x) & (x \in \mathbf{R}^m), \end{cases}$$

where

$$H^{-}(t,x,p) \equiv \max_{y\in Y} \min_{z\in Z} \{f(t,x,y,z) \cdot p + h(t,x,y,z)\}$$

is the lower Hamiltonian.

Corollary 4.2. *(i) $V \le U$ $(0 \le t \le T, x \in \mathbf{R}^m)$*
(ii) If for all $0 \le t \le T$, $x, p \in \mathbf{R}^m$

$$H^{+}(t,x,p) = H^{-}(t,x,p), \qquad \textit{(minimax condition)}$$

then

$$U \equiv V.$$

The corollary follows from the standard comparison and uniqueness theorems for viscosity solutions: see [7], [8], [21], [27].

Proof of Theorem 4.1. We prove assertion (a) only.

Let $\phi \in C^1((0,T) \times \mathbf{R}^m)$ and suppose $U - \phi$ attains a local maximum at $(t_0,x_0) \in (0,T) \times \mathbf{R}^m$. We must prove

$$\phi_t(t_0,x_0) + H^{+}(t_0,x_0,D\phi(t_0,x_0)) \ge 0. \tag{4.1}$$

Should this fail, there would exist some $\theta > 0$ so that

$$\phi_t(t_0,x_0) + H^{+}(t_0,x_0,D\phi(t_0,x_0)) \le -\theta < 0. \tag{4.2}$$

According to Lemma 4.3 (a) (stated and proved below) this implies that for each sufficiently small $\sigma > 0$ and all $\alpha \in \Gamma(t_0)$

$$(4.3)\quad \int_{t_0}^{t_0+\sigma} h(s,x(s),\alpha[z](s),z(s)) + f(s,x(s),\alpha[z](s),z(s)) \cdot D\phi(s,x(s)) + \phi_t(s,x(s))ds \le \frac{-\sigma\theta}{2}$$

for some $z \in N(t_0)$. Thus

$$(4.4)\quad \sup_{\alpha\in\Gamma(t_0)} \inf_{z\in N(t_0)} \left\{ \int_{t_0}^{t_0+\sigma} h(s,x(s),\alpha[z](s),z(s)) + f(s,x(s),\alpha[z](s),z(s)) \cdot D\phi(s,x(s)) + \phi_t(s,x(s))ds \right\} \le \frac{-\sigma\theta}{2}.$$

However Theorem 3.1 states

$$(4.5)\quad U(t_0,x_0) = \sup_{\alpha\in\Gamma(t_0)} \inf_{z\in N(t_0)} \left\{ \int_{t_0}^{t_0+\sigma} h(s,x(s),\alpha[z](s),z(s))ds + U(t_0+\sigma,x(t_0+\sigma)) \right\}.$$

Since $U - \phi$ has a local maximum at (t_0,x_0), we have for σ small enough that

$$(4.6)\quad U(t_0,x_0) - \phi(t_0,x_0) \ge U(t_0+\sigma,x(t_0+\sigma)) - \phi(t_0+\sigma,x(t_0+\sigma))$$

where $x(\cdot)$ solves (ODE) on $(t_0,t_0+\sigma)$ for any $y(\cdot)$, $z(\cdot)$, with the initial condition $x(t_0) = x_0$. Now (4.5) and (4.6) give

$$(4.7)\quad \sup_{\alpha\in\Gamma(t_0)} \inf_{z\in N(t)} \left\{ \int_{t_0}^{t_0+\sigma} h(s,x(s),\alpha[z](s),z(s))ds + \phi(t_0+\sigma,x(t_0+\sigma)) - \phi(t_0,x_0) \right\} \ge 0.$$

But

$$(4.8)\quad \phi(t_0+\sigma,x(t_0+\sigma)) - \phi(t_0,x_0)) = \int_{t_0}^{t_0+\sigma} f(s,x(s),\alpha[z](s),z(s)) \cdot D\phi(s,x(s)) + \phi_t(s,x(s))ds;$$

and so (4.7) contradicts (4.4). Thus (4.1) must in fact be valid.

Next, suppose $U - \phi$ has a local minimum at $(t_0,x_0) \in (0,T) \times \mathbf{R}^m$. We must demonstrate

$$(4.9)\quad \phi_t(t_0,x_0) + H^+(t_0,x_0,D\phi(t_0,x_0)) \le 0$$

and so will assume the contrary that

$$(4.10)\quad \phi_t(t_0,x_0) + H^+(t_0,x_0,D\phi(t_0,x_0)) \ge \theta > 0$$

for some constant $\theta > 0$. Then Lemma 4.3(b) asserts that there exists for all sufficiently small $\sigma > 0$ some $\alpha \in \Gamma(t_0)$ such that

$$(4.11)\quad \int_{t_0}^{t_0+\sigma} h(s,x(s),\alpha[z](s),z(s)) + f(s,x(s),\alpha[z](s),z(s)) \cdot D\phi(s,x(s)) + \phi_t(s,x(s))ds \geq \frac{\sigma\theta}{2}$$

for all $z \in N(t_0)$. Consequently

$$(4.12)\quad \sup_{\alpha\in\Gamma(t_0)} \inf_{z\in N(t_0)} \left\{ \int_{t_0}^{t_0+\sigma} h(s,x(s),\alpha[z](s),z(s)) + f(s,x(s),\alpha[z](s),z(s)) \cdot D\phi(s,x(s)) + \phi_t(s,x(s))ds \right\} \geq \frac{\sigma\theta}{2}.$$

But since $U - \phi$ has a local minimum at (t_0,x_0), we have for small enough $\sigma > 0$ that

$$U(t_0,x_0) - \phi(t_0,x_0) \leq U(t_0 + \sigma,x(t_0 + \sigma)) - \phi(t_0 + \sigma,x(t_0 + \sigma)),$$

$x(\cdot)$ solving (ODE) on $(t_0,t_0 + \sigma)$ for any $y(\cdot)$, $z(\cdot)$, with the initial condition $x(t_0) = x_0$. This and (4.5) imply

$$\sup_{\alpha\in\Gamma(t_0)} \inf_{z\in N(t_0)} \left\{ \int_{t_0}^{t_0+\sigma} h(s,x(s),\alpha[z](s),z(s))ds + \phi(t_0 + \sigma,x(t_0 + \sigma)) - \phi(t_0,x_0) \right\} \leq 0.$$

Recalling now (4.8), we see that this contradicts (4.12), and thus (4.9) must hold. □

Lemma 4.3. *Assume ϕ is C^1.*

(a). If ϕ satisfies (4.2), then for all sufficiently small $\sigma > 0$ there exists $z \in N(t_0)$ such that (4.3) holds for all $\alpha \in \Gamma(t_0)$.

(b). If ϕ satisfies (4.10), then for all sufficiently small $\sigma > 0$ there exists $\alpha \in \Gamma(t_0)$ such that (4.11) holds for all $z \in N(t_0)$.

Proof. Set

$$\Lambda(t,x,y,z) \equiv \phi_t(t,x) + f(t,x,y,z) \cdot D\phi(t,x) + h(t,x,y,z).$$

(a). According to (4.2)

$$\min_{z\in Z} \max_{y\in Y} \Lambda(t_0,x_0,y,z) \leq -\theta < 0.$$

Hence there exists some $z^* \in Z$ such that

$$\max_{y \in Y} \Lambda(t_0, x_0, y, z^*) \leq -\theta.$$

Since Λ is uniformly continuous, we also have

$$\max_{y \in Y} \Lambda(s, x(s), y, z^*) \leq -\frac{\theta}{2}$$

provided $t_0 \leq s \leq t_0 + \sigma$ (for any small $\sigma > 0$) and $x(\cdot)$ solves (ODE) on $(t_0, t_0 + \sigma)$ for any $y(\cdot)$, $z(\cdot)$, with the initial condition $x(t_0) = x_0$. Hence for $z(\cdot) \equiv z^*$ and any $\alpha \in \Gamma(t_0)$

$$\phi_t(s,x(s)) + f(s,x(s),\alpha[z](s),z(s)) \cdot D\phi(s,x(s)) + h(s,x(s),\alpha[z](s),z(s)) \leq \frac{-\theta}{2}$$

for $t_0 \leq s \leq t_0 + \sigma$. Integrate this from t_0 to $t_0 + \sigma$ to obtain (4.3).

(b). Inequality (4.10) reads

$$\min_{z \in Z} \max_{y \in Y} \Lambda(t_0, x_0, y, z) \geq \theta > 0.$$

Hence for each $z \in Z$ there exists $y = y(z) \in Y$ such that

$$\Lambda(t_0, x_0, y, z) \geq \theta.$$

Since Λ is uniformly continuous we have in fact

$$\Lambda(t_0, x_0, y, \zeta) \geq \frac{3\theta}{4}$$

for all $\zeta \in B(z,r) \cap Z$ and some $r = r(z) > 0$. Because Z is compact there exist finitely many distinct points $z_1, \ldots, z_n \in Z$, $y_1, \ldots, y_n \in Y$, and $r_1, \ldots, r_n > 0$ such that

$$Z \subset \bigcup_{i=1}^{n} B(z_i, r_i)$$

and

$$\Lambda(t_0, x_0, y_i, \zeta) \geq \frac{3\theta}{4} \qquad \text{for } \zeta \in B(z_i, r_i).$$

Define

$$\phi : Z \to Y$$

by setting

$$\phi(z) = y_k$$

if

$$z \in B(y_k, r_k) \setminus \bigcup_{i=1}^{k-1} B(y_i, r_i) \qquad (k = 1, \ldots, n).$$

Thus

$$\Lambda(t_0,x_0,\phi(z),z) \geq \frac{3\theta}{4}$$

for all $z \in Z$. Since Λ is uniformly continuous we therefore have for each sufficiently small $\sigma > 0$

$$\Lambda(s,x(s),\phi(z),z) \geq \frac{\theta}{2} \tag{4.13}$$

for all $z \in Z$, $t_0 \leq s \leq t_0 + \sigma$, and any solution $x(\cdot)$ of (ODE) on $(t_0,t_0 + \sigma)$ for any $y(\cdot)$, $z(\cdot)$, with initial condition $x(t_0) = x_0$.

Finally define $\alpha \in \Gamma(t_0)$ in this way:

$$\alpha[z](s) = \phi(z(s))$$

for each $z \in N(t_0)$, $t_0 \leq s \leq T$. Owing to (4.13)

$$\Lambda(s,x(s),\alpha[z](s),z(s)) \geq \frac{\theta}{2} \qquad (t_0 \leq s \leq t_0 + \sigma)$$

for each $z \in N(t_0)$. Integrate this inequality from t_0 to $t_0 + \sigma$ to arrive at (4.11). □

5. Representation of Solutions of Hamilton-Jacobi Equations

We next employ the theory from §2–4 to derive a representation formula for the viscosity solution of

$$\begin{cases} u_t + H(t,x,Du) = 0 \\ u(0,x) = g(x). \end{cases} \qquad (x \in \mathbf{R}^m,\ 0 < t < T) \tag{5.1}$$

Here $g: \mathbf{R}^m \to \mathbf{R}$ and $H: [0,T] \times \mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}$ satisfy

$$\begin{cases} |g(x)| \leq C_5 \\ |g(x) - g(\hat{x})| \leq C_5|x - \hat{x}| \end{cases} \tag{5.2}$$

and

$$\begin{cases} |H(t,x,0)| \leq C_5 \\ |H(t,x,p) - H(\hat{t},\hat{x},\hat{p})| \leq C_5(|t - \hat{t}| + |x - \hat{x}| + |p - \hat{p}|) \end{cases} \tag{5.3}$$

for some constant C_5 and all $0 \leq t$, $\hat{t} \leq T$, x, $\hat{x}$, p, $\hat{p} \in \mathbf{R}^m$.

Then results of Crandall-Lions [8], Lions [21], and Souganidis [27], [28] imply the existence of a unique viscosity solution u of (5.1), with

$$\begin{cases} |u(t,x)| \leq C_6 \\ |u(t,x) - u(\hat{t},\hat{x})| \leq C_6(|t - \hat{t}| + |x - \hat{x}|) \end{cases} \tag{5.4}$$

for some constant C_6.

First we write H as the max-min of appropriate affine functions:

Lemma 5.1. *For each* $0 \le t \le T$, $x \in \mathbf{R}^m$ *and constant* $\Lambda > 0$,

$$H(t,x,p) = \max_{z \in Z} \min_{y \in Y} \{f(y) \cdot p + h(t,x,y,z)\} \tag{5.5}$$

if $|p| \le \Lambda$, *where*

$$\begin{cases} Y = B(0,1) \subset \mathbf{R}^m \\ Z = B(0,\Lambda) \subset \mathbf{R}^m \\ f(y) = C_5 y \\ h(t,x,y,z) = H(t,x,z) - C_5 y \cdot z. \end{cases} \tag{5.6}$$

Proof. Since

$$H(t,x,z) - H(t,x,p) \le C_5 |p - z| \qquad (z \in \mathbf{R}^m),$$

we have for $|p| \le \Lambda$

$$\begin{aligned} H(t,x,p) &= \max_{z \in Z} \{H(t,x,z) - C_5 |p - z|\} \\ &= \max_{z \in Z} \min_{y \in Y} \{H(t,x,z) + C_5 y \cdot (p - z)\}. \end{aligned}$$ □

Remark. See Fleming [14, pages 996–1000] or Evans [12] for other, more complicated ways of writing a nonlinear function as the max-min (or min-max) of affine mappings.

Note that f and h satisfy (2.1) and (2.3), respectively. Now set

$$\hat{H}(t,x,p) = \max_{z \in Z} \min_{y \in Y} \{f(y) \cdot p + h(t,x,y,z)\} \qquad (p \in \mathbf{R}^m)$$

for $\Lambda = C_6$ from (5.4), Y, Z, f, h from (5.6). Then

$$H(t,x,p) = \hat{H}(t,x,p) \qquad \text{provided } |p| \le C_6.$$

As u satisfies (5.4) it follows from the theory in [8] that u is also the unique viscosity solution of

$$\begin{cases} u_t + \hat{H}(t,x,Du) = 0 \\ u(x,0) = g(x). \end{cases} \qquad (x \in \mathbf{R}^m, 0 < t < T)$$

Hence

$$v(t,x) \equiv u(T - t,x) \tag{5.7}$$

is the viscosity solution of

$$\begin{cases} v_t + H^+(t,x,Dv) = 0 \\ v(T,x) = g(x) \end{cases} \qquad (x \in \mathbf{R}^m, 0 < t < T)$$

for

$$H^+(t,x,p) = \min_{z\in Z}\max_{y\in Y}\{-f(y)\cdot p - h(T-t,x,y,z)\}.$$

Thus the developments in §2-4 imply

$$v(t,x) = U(t,x) = \sup_{\alpha\in\Gamma(t)}\inf_{z\in N(t)}\left\{-\int_t^T h(T-s,x(s),\alpha[z](s),z(s))ds + g(x(T))\right\},$$

where $x(\cdot)$ solves

$$\begin{cases} \dot{x}(s) = -f(y(s)) = -C_5 y(s) & (t < s < T) \\ x(t) = x \end{cases}$$

for $y(\cdot) = \alpha[z]$; that is

$$x(s) = x - C_5\int_t^s \alpha[z](r)dr \qquad (t < s < T).$$

Recall now (5.7) to complete the proof of

Theorem 5.2. *We have for each* $0 \le t \le T$ *and* $x \in \mathbf{R}^m$,

$$(5.8)\quad u(t,x) = \sup_{\alpha\in\Gamma(T-t)}\inf_{z\in N(T-t)}\left\{-\int_{T-t}^T h(T-s,x(s),\alpha[z](s),z(s))ds + g(x(T))\right\},$$

where for each $z \in N(T-t)$ *and* $y = \alpha[z] \in M(T-t)$, $x(\cdot)$ *solves*

$$(5.9)\qquad \begin{cases} \dot{x}(s) = -C_5 y(s) & T-t < s < T \\ x(T-t) = x. \end{cases}$$

Remark. A formula analogous to (5.8) obtains for any choices of Y, Z, f and h for which (5.5) holds (even if $f = f(t,x,y,z)$). The representation we have taken has particularly simple dynamics: note that player II can affect only the running cost h.

An easy application is the following domain-of-dependence assertion.

Corollary 5.3. (cf. [8]). *Assume* H *satisfies (5.3) and that*

$$g,\hat{g}: \mathbf{R}^m \to \mathbf{R}$$

satisfy (5.2). Suppose also that u *is the viscosity solution of (5.1) and* $\hat{u}$ *is the viscosity solution of*

$$(5.1)'\qquad \begin{cases} \hat{u}_t + H(t,x,D\hat{u}) = 0 \\ \hat{u}(0,x) = \hat{g}(x). \end{cases} \qquad (x \in \mathbf{R}^m, 0 < t < T)$$

Fix $x \in \mathbf{R}^m$, $0 < t \le T$. *Then if*

$$g \equiv \hat{g} \qquad \text{on } B(x,tC_5)$$

we have

$$u(x,t) = \hat{u}(x,t).$$

Proof. By Theorem 5.2

$$\hat{u}(t,x) = \sup_{\alpha\in\Gamma(T-t)} \inf_{z\in N(T-t)} \left\{ -\int_{T-t}^{T} h(T-s,x(s),\alpha[z](s),z(s))ds + \hat{g}(x(T)) \right\}$$

where for Γ, N, h, etc. as above and for each $z \in N(T-t)$, $y = \alpha[z] \in M(T-t)$, x solves (5.9). But then

$$|x(T) - x| \leq tC_5$$

and so

$$\hat{g}(x(T)) = g(x(T)).$$

Thus

$$\hat{u}(t,x) = \sup_{\alpha\in\Gamma(T-t)} \inf_{z\in N(T-t)} \left\{ -\int_{T-t}^{T} h(T-s,x(s),\alpha[z](s),z(s))ds + g(x(T)) \right\}$$

$$= u(t,x), \qquad \text{by Theorem 5.2 again.} \qquad \square$$

For an application in §6, §7 we will require a modification of (5.5), (5.6) in the case that $H(t,x,\cdot)$ is positively homogeneous of degree one.

Lemma 5.4. *Suppose in addition to (5.3) that*

$$H(t,x,\lambda p) = \lambda H(t,x,p) \qquad (0 \leq t \leq T, x, p \in \mathbf{R}^m, \lambda \geq 0).$$

Then there exist compact sets $Y \subset \mathbf{R}^{2m}$, $Z \subset \mathbf{R}^{2m}$ *and*

$$f: [0,T] \times \mathbf{R}^m \times Y \times Z \to \mathbf{R}^m$$

satisfying (2.1) such that

$$H(t,x,p) = \max_{z\in Z} \min_{y\in Y} \{f(t,x,y,z)\cdot p\}$$

for all $0 \leq t \leq T$, x, $p \in \mathbf{R}^m$.

Proof. If $|\eta| = 1$, then according to Lemma 5.1.

$$H(t,x,\eta) = \max_{z_1\in Z_1} \min_{y_1\in Y_1} \{f(y_1)\cdot\eta + h(t,x,y_1,z_1)\}$$

for

$$\begin{cases} Y_1 = Z_1 = B(0,1) \subset \mathbf{R}^m \\ f(y_1) = C_5 y_1 \\ h(t,x,y_1,z_1) = H(t,x,z_1) - C_5 y_1 \cdot z_1 . \end{cases}$$

Thus for any $p \neq 0$

$$\begin{aligned} H(t,x,p) &= |p|\,H\left(t,x,\frac{p}{|p|}\right) \\ &= \max_{z_1\in Z_1}\min_{y_1\in Y_1}\{f(y_1)\cdot p + h(t,x,y_1,z_1)|p|\}. \end{aligned}$$

Choose $C_7 > 0$ such that

$$|h| \le C_7$$

for all $0 \le t \le T$, $x \in \mathbf{R}^m$, $z_1 \in Z_1$, $y_1 \in Y_1$. Then

$$\begin{aligned} H(t,x,p) &= \max_{z_1\in Z_1}\min_{y_1\in Y_1}\{f(y_1)\cdot p + C_7|p| + (h(t,x,y_1,z_1) - C_7)|p|\} \\ &= \max_{z_1\in Z_1}\min_{y_1\in Y_1}\max_{z_2\in Z_1}\min_{y_2\in Y_1}\{f(y_1)\cdot p + C_7 z_2\cdot p + (h(t,x,y_1,z_1) - C_7)y_2\cdot p\} \\ &= \max_{z\in Z}\min_{y\in Y}\{f(t,x,y,z)\cdot p\} \end{aligned}$$

where

$$\begin{cases} Y = Z = B(0,1)\times B(0,1) \subset \mathbf{R}^{2m} \\ z = (z_1,z_2),\ y = (y_1,y_2) \\ f(t,x,y,z) = f(y_1) + C_7 z_2 + (h(t,x,y_1,z_1) - C_7)y_2 \\ \qquad\qquad = C_5 y_1 + C_7 z_2 + (H(t,x,z_1) - C_5 y_1\cdot z_1 - C_7)y_2. \end{cases}$$

Note that the interchanging of $\min_{y_1\in Y_1}$ and $\max_{z_2\in Z_1}$ above is valid. □

6. Propagation of Disturbances and Huygens' Principle

As an application of the representation formulas developed in §5 we will discuss in the next section the level sets of solutions of Hamilton-Jacobi equations with Hamiltonians positively homogeneous of degree one. The following considerations—adapted directly from Gelfand-Fomin [17, pages 208–217] and Arnold [1, pages 248–258]—provide motivation.

Regard $\mathbf{R}^m$ as a heterogeneous, nonisotropic medium, comprised of points at each moment in either an "excited" or a "rest" state. Once any given point x is excited by a disturbance propagating in the medium, it thereafter remains excited and so itself serves as a source for further disturbances emanating from it. We wish to describe mathematically the evolution of the disturbances from a given excited set.

For this let $L(x,z)$ denote the reciprocal of the speed of the disturbance leaving x in the direction $z \in S^{m-1}$. Extend L to be positively homogeneous of degree one and set

$$I(x) \equiv \{z \in \mathbf{R}^m : L(x,z) = 1\};$$

$I(x)$ is the *indicatrix* of L at x. We will assume this to be the smooth boundary of an open, strictly convex set. We consider also the *figuratrix*

$$F(x) \equiv \{p = D_z L(x,z) : z \in I(x)\}.$$

Next define the *Hamiltonian H* so that

$$\begin{cases} H(x,p) = 1 & \text{if } p \in F(x) \\ H(x,\cdot) \text{ is positively homogeneous of degree one.} \end{cases}$$

This is the standard Hamiltonian for the parametric Lagrangian L (see Young [30, pages 50–51]), and the reader should check that

$$H(x,p) = \sup\{z \cdot p : z \in I(x)\}. \tag{6.1}$$

Next suppose Γ_0 denotes the set of points excited initially and $\Gamma_t \supseteq \Gamma_0$ the set of points excited at time $t > 0$. We introduce a function $u : \mathbf{R}^+ \times \mathbf{R}^m \to \mathbf{R}$ such that

$$\Gamma_t = \{x : u(t,x) > 0\} \tag{6.2}$$

and

$$\Sigma_t = \{x : u(t,x) = 0\} = \partial\Gamma_t \tag{6.3}$$

for each $t \geq 0$; here Σ_t is the *wave front* at time t. We will show heuristically that u solves a Hamilton-Jacobi equation.

To see this, fix any $t > 0$, $x \in \Sigma_t$, and $0 < \Delta t < t$. According to *Huygens' principle* Σ_t is the envelope of the wave fronts emanating from points in $\Sigma_{t-\Delta t}$: see [1, page 250]. Thus there exists $y \in \Sigma_{t-\Delta t}$ such that $y + \Delta t I(y)$ is—up to error terms of order $o(\Delta t)$—tangent to Σ_t at x. So for some $z \in I(y)$,

$$y + (\Delta t)z \text{ is (approximately) equal to } x$$

and

$$p \equiv -Du(t,x) \text{ is (approximately) normal to } y + \Delta t I(y) \text{ at } x.$$

Consequently

$$H(x,p) = p \cdot z + o(1) \qquad \text{as } \Delta t \to 0. \tag{6.4}$$

On the other hand

$$o(\Delta t) = u(t - \Delta t, x - (\Delta t)z) - u(t,x) = (-\Delta t)(u_t(t,x) + Du(t,x) \cdot z) + o(\Delta t)$$

and so

$$u_t(t,x) = p \cdot z + o(1) \qquad \text{as } \Delta t \to 0.$$

This and (6.4) give

$$u_t + \tilde{H}(x,Du) = 0 \tag{6.5}$$

for

$$\tilde{H}(x,p) \equiv -H(x,-p).$$

Note that the reasoning here works just as well on the sets $\{u = \alpha\}$ for each real number α. Thus (6.5) holds in all of $\mathbf{R}^n \times (0,T)$.

In (6.5) we have derived the required Hamilton-Jacobi equation for u; therefore, in principle, to find the excited sets Γ_t we need only find some function $g:\mathbf{R}^m \to \mathbf{R}$ such that

$$\Gamma_0 = \{x : g(x) > 0\} \tag{6.6}$$

and then solve (6.5) subject to the initial condition

$$u(x,0) = g(x) \qquad (x \in \mathbf{R}^m). \tag{6.7}$$

The sets Γ_t are then given by (6.2).

However, in addition to the obvious objection that (6.5), (6.7) will in general have no smooth solution for large time, it is not immediately clear that our calculation of $\Gamma_t = \{x : u(t,x) > 0\}$ is independent of the choice of g. As we will see in §7 below, a formal calculation using characteristics indicates that Γ_t does indeed only depend upon g's satisfying (6.6) and not on the particular choice of this function. Nevertheless a rigorous proof cannot use characteristics (which need not exist in the large) and will instead rely upon our game theoretic representation formulas for the viscosity solution of (6.5), (6.7).

Remark. For the case at hand $H(x,\cdot)$ is convex and so control theory, rather than game theory, techniques will work. A point of the next section is therefore that the homogeneity and not the convexity of $H(x,\cdot)$ is the crucial property. The reader should also note in the above context that Huygens' principle is a version of the optimality principle in dynamic programming.

7. Level Sets

Motivated by considerations in §6 we now prove

Theorem 7.1. *Let $H:\mathbf{R}^m \times \mathbf{R}^m \to \mathbf{R}$ be uniformly Lipschitz and positively homogeneous of degree 1 in its second argument. Assume g, $\hat{g}$ are bounded, uniformly Lipschitz and are positive on the same set; that is,*

$$\{x \in \mathbf{R}^m : g(x) > 0\} = \{x \in \mathbf{R}^m : \hat{g}(x) > 0\}. \tag{7.1}$$

Suppose u, $\hat{u}$ are the viscosity solutions of, respectively,

$$\begin{cases} u_t + H(x,Du) = 0 & (t > 0, x \in \mathbf{R}^m) \\ u(0,x) = g(x) \end{cases} \tag{7.2}$$

and

$$\begin{cases} \hat{u}_t + H(x,D\hat{u}) = 0 & (t > 0, x \in \mathbf{R}^m) \\ \hat{u}(0,x) = \hat{g}(x). \end{cases} \tag{7.3}$$

Then for each $T > 0$

$$\{x \in \mathbf{R}^m : u(T,x) > 0\} = \{x \in \mathbf{R}^m : \hat{u}(T,x) > 0\}. \tag{7.4}$$

Note that we do not require H to be convex in p, and that "0" in (7.1), (7.4) can be replaced by any real number.

Formal proof of (7.4). For heuristic purposes we begin with a formal proof of (7.4) under the additional assumptions

$$\begin{cases} H \in C^1 \quad \text{for } p \neq 0,\ u,\ \hat{u} \in C^2, \\ \Sigma_0 \equiv \partial\{g > 0\} = \partial\{\hat{g} > 0\} \text{ is a smooth manifold}, \\ Dg, D\hat{g} \neq 0 \quad \text{on } \Sigma_0. \end{cases} \tag{7.5}$$

Consider first (7.2), and for each $x_0 \in \mathbf{R}^m$ define the *characteristics* x, p: $[0,\infty) \to \mathbf{R}^m$ as follows:

$$\begin{cases} \dot{x}(t) = D_pH(x(t),p(t)), & x(0) = x_0 \\ \dot{p}(t) = -D_xH(x(t),p(t)), & p(0) = p_0, \end{cases} \tag{7.6}$$

for $p_0 \equiv Dg(x_0)$. Since u is C^2, we have

$$p(t) = Du(t,x(t)) \qquad (t > 0)$$

and

$$u(t,x(t)) = g(x_0) + \int_0^t H(x(s),p(s)) - p(s) \cdot D_pH(x(s),p(s))ds.$$

But

$$H \equiv p \cdot D_pH \tag{7.7}$$

since H is homogeneous of degree one; consequently

$$u(t,x(t)) \equiv g(x_0) \qquad (t \geq 0). \tag{7.8}$$

In particular

$$u(t,x(t)) \equiv 0 \qquad \text{if } x_0 \in \Sigma_0. \tag{7.9}$$

We next claim that

$$x(\cdot) \text{ depends only on } x_0 \text{ and } \eta_0 \equiv \frac{p_0}{|p_0|} = \frac{Dg(x_0)}{|Dg(x_0)|}. \tag{7.10}$$

To see this set

$$\eta(t) \equiv \frac{p(t)}{|p(t)|} \qquad (t > 0)$$

and compute

$$\begin{aligned}\dot{\eta} &= \frac{\dot{p}}{|p|} - \frac{(p\cdot\dot{p})p}{|p|^3}\\ &= \frac{-D_xH(x,p)}{|p|} + \frac{(p\cdot D_xH(x,p))p}{|p|^3}\\ &= -D_xH(x,\eta) + (\eta\cdot D_xH(x,\eta))\eta, \qquad (t>0)\end{aligned}$$

since H and therefore D_xH are homogeneous of degree one. On the other hand D_pH is homogeneous of degree zero and so

$$\dot{x} = D_pH(x,p) = D_pH(x,\eta).$$

Thus

$$\begin{cases}\dot{x} = D_pH(x,\eta), & x(0) = x_0\\ \dot{\eta} = -D_xH(x,\eta) + (\eta\cdot D_xH(x,\eta))\eta, & \eta(0) = \eta_0;\end{cases} \tag{7.11}$$

this proves (7.10).

Finally let $\hat{x},\hat{p}$: $[0,\infty] \to \mathbf{R}^m$ be the characteristics for $\hat{u}$:

$$\begin{cases}\dot{\hat{x}} = D_pH(\hat{x},\hat{p}), & \hat{x}(0) = x_0\\ \dot{\hat{p}} = -D_xH(\hat{x},\hat{p}), & \hat{p}(0) = p_0,\end{cases} \tag{7.12}$$

where

$$\hat{p}_0 = D\hat{g}(x_0).$$

As above $\hat{x}(\cdot)$ depends only on

$$\hat{\eta}_0 = \frac{\hat{p}_0}{|\hat{p}_0|} = \frac{D\hat{g}(x_0)}{|D\hat{g}(x_0)|}.$$

Hence if $x_0 \in \Sigma_0$, $\eta_0 = \hat{\eta}_0$; and thus

$$x(t) = \hat{x}(t) \qquad (t \geq 0).$$

Since therefore $\hat{u}(t,x(t)) = 0$ and since both u and $\hat{u}$ are constant along characteristics, we have

$$\{x \in \mathbf{R}^m : u(t,x) = 0\} = \{x \in \mathbf{R}^m : \hat{u}(t,x) = 0\} \qquad (t \geq 0).$$

This completes the formal proof of (7.4). □

A rigorous proof along the lines above seems unlikely, as the solutions u, $\hat{u}$ are generally not even C^1, the characteristics may cross, p or $\hat{p}$ may equal zero, etc. Instead we use the game theoretic representation of the solution afforded by Theorem 5.1. Here we regard the (approximate) optimal trajectories as being (approximate) generalized characteristics.

Proof of Theorem 7.1. According to Lemma 5.4

$$H(x,p) = \max_{z\in Z}\min_{y\in Y}\{f(x,y,z)\cdot p\} \qquad (p,x \in \mathbf{R}^m)$$

for appropriate compact sets Y, Z, and f satisfying (2.1). Thus u is the viscosity solution of

$$(7.13)\qquad \begin{cases} u_t + \max_{z\in Z}\min_{y\in Y}\{f(x,y,z)\cdot Du\} = 0 \\ u(x,0) = g(x). \end{cases}$$

Fix any $T > 0$ and set

$$U(t,x) \equiv u(T-t,x) \qquad (0 \le t \le T, x \in \mathbf{R}^m);$$

then U is the viscosity solution of

$$\begin{cases} U_t + \min_{z\in Z}\max_{y\in Y}\{-f(x,y,z)\cdot DU\} = 0 \\ U(T,x) = g(x). \end{cases}$$

Thus, by the uniqueness of viscosity solutions,

$$U(t,x) = \sup_{\alpha\in\Gamma(t)}\inf_{z\in N(t)}\{g(x(T))\},$$

where

$$(7.14)\qquad \begin{cases} \dot{x}(s) &= -f(x(s),\alpha[z](s),z(s)) \qquad (t<s<T) \\ x(t) &= x. \end{cases}$$

Similarly define

$$\hat{U}(t,x) = \hat{u}(T-t,x) \qquad (0 \le t \le T, x \in \mathbf{R}^m),$$

so that

$$\hat{U}(t,x) = \sup_{\alpha\in\Gamma(t)}\inf_{z\in N(t)}\{\hat{g}(x(T))\},$$

$x(\cdot)$ solving (7.14).

Next assume

$$(7.15)\qquad u(T,x_0) > 0;$$

then

$$U(0,x_0) = \sup_{\alpha\in\Gamma(0)}\inf_{z\in N(0)}\{g(x(T))\} > 0.$$

Fix

$$0 < 2\varepsilon < U(0,x_0)$$

and then choose $\alpha \in \Gamma(0)$ such that

$$(7.16)\qquad \inf_{z\in N(0)}\{g(x(T))\} > \varepsilon,$$

$x(\cdot)$ solving

$$\text{(7.17)} \qquad \begin{cases} \dot{x}(s) = -f(x(s),\alpha[z](s),z(s)) & (0 < s < T) \\ x(0) = x_0 . \end{cases}$$

Thus for any $z \in N(0)$,

$$x(T) \subseteq \{\hat{g} > \sigma\}$$

for some $\sigma = \sigma(\varepsilon) > 0$. Consequently

$$\inf_{z \in N(0)} \{\hat{g}(x(T))\} \geq \sigma,$$

$x(\cdot)$ solving (7.17). Therefore

$$\hat{u}(T,x_0) = \hat{U}(0,x_0) = \sup_{\alpha \in \Gamma(0)} \inf_{z \in N(0)} \{g(x(T))\} > 0.$$

We have proved $u(T,x_0) > 0$ implies $\hat{u}(T,x_0) > 0$, and the opposite implication follows from interchanging u and $\hat{u}$ in the argument above. This proves (7.4). □

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Evans: University of Maryland—College Park, MD 20742
Souganidis: Brown University—Providence, RI 02912

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