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Viability theory: an applied mathematics tool for achieving dynamic systems' sustainability

Abstract Sustainability is an issue of paramount importance, as scientists and politicians seek to understand what it means, practically and conceptually, to be sustainable. This paper's aim is to introduce viability theory, a relatively young branch of mathematics which provides a conceptual framework that is very well suited to such problems. Viability theory can be used to answer important questions about the sustainability of systems, including those studied in macroeconomics, and can be used to determine sustainable policies for their management. The principal analytical tool of viability theory is the viability kernel which describes the set of all state-space points in a constrained system starting from which it is possible to remain within the system's constraints indefinitely. Although, in some circumstances, kernel determination can be performed analytically, most practical results in viability theory rely on graphical approximations of viability kernels, which for nonlinear and high-dimensional problems can only be approached numerically. This paper provides an outline of the core concepts of viability theory and an overview of the numerical approaches available for computing approximate viability kernels. VIKAASA, a specialised software application developed by the authors and designed to compute such approximate viability kernels is presented along-side examples of viability theory in action in the spheres of bio-economics and macroeconomics.

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1. Introduction In this paper we are concerned with problems occurring in economics, environmental sciences, and their intersection, generically termed "sustainability problems". Such problems arise in any situation where interaction with a dynamic, changing environment can potentially lead to catastrophic outcomes. Throughout this paper we will make use of two models with which the authors have some familiarity, in order to demonstrate this idea. In macroeconomics we will consider the task of a central bank

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seeking policy rules that prevent run-away hyper-inflation and spiraling unemployment [39, 7, 26, 27, 25, 22, 23]. In ecology (or ecological economics), we will look at how a regulator might restrict the activities of a fishing fleet in such a way that fishing remains profitable and fish stocks are not exhausted [11, 13, 43]. Both models have in common that a "solution" involves examining the compatibility of the systems' dynamics (i.e. macro-economic dynamics and fish population dynamics, respectively) and the geometry of the set of outcomes deemed sustainable, in order to ensure that a given state of affairs, which may appear satisfactory at the outset, does not evolve into a catastrophe subsequently. It is this generalised criterion of long-term acceptability that, in the authors' opinion, best identifies a problem as a "sustainability problem", and it is this class of problems that we wish to consider in this paper.

The aim of this paper is to convince the reader that *viability theory* [1], a relatively young area of continuous optimisation, provides the right framework for modelling and solution (in the sense described above) of these sorts of problems. Viability theory has been explicitly developed for the purpose of analysing dynamic systems which face constraints, making it a perfect fit for considering problems of sustainability. Broadly, viability theory will be of interest to those wishing to ascertain *positive* (or *modal*) information about the *sustainability* of the dynamic systems they are studying. In particular, a solution to a viability theory problem can answer the following questions:

1. whether a system will be able to sustain itself according to the given sustainability criteria over some particular time-frame; and

2. what the necessary conditions are for sustainability; i.e., for which initial states the system has the possibility of sustaining itself, and for which it does not.

Furthermore, when systems are susceptible to control by a regulator, viability theory is also of interest to those wishing to determine *normative* rules. Specifically, it can establish:

3. policies that can be pursued to *guarantee* the sustainability of the system;

4. policies that can be used to *improve* the sustainability of the system; and

5. other policy objectives that are *compatible* with the sustainability of the system.

Items 1 and 2 are the core concerns of viability theory. If from a given system state there is an evolution which is possible according to what is known about the system's dynamics and which sustains the system within the imposed bounds, then that system state is considered to be *viable*. Conversely, where there is no conceivable way for the system to remain within those bounds when starting from a given state, then that state is said to be *non-viable*.[1] These categories should be simple enough to be grasped in a straight-forward manner by analysts interested in sustainability.

Identification of states as viable or non-viable is achieved in viability theory by computing the *viability kernel* – the largest closed subset of points in the constraint set for which all points are viable. Once computed[2], we have a comprehensive answer to item 2 – in order for a state to be sustainable, it must be within the viability kernel (refer to Figure 1 shown later on page 106).

Traditionally, normative problems of the sort outlined in items 3-5 have been solved by finding closed-loop (feedback) rules which stabilise the system around some equilibrium state. This can be done for instance using optimal control theory: a loss function which provides a goodness-ranking for each system state is specified, and provided the system's dynamics can be modelled as either deterministic or stochastic processes, optimal control theory will provide a control strategy that is (usually) unique and scores best ("optimally") on the adopted optimisation criteria. If a regulator knows what needs be optimised, applying the optimal strategy is the unique way to control the dynamic system. Such rules have been used to solve the fisheries problem outlined above [41].[3] Another avenue is to find a so-called "simple policy rule" such as the Taylor Rule in macroeconomics [19, 45, 7], which still targets some "optimal" state (inflation stability in the case of the Taylor rule), but in a way that is not subject to any inter-temporal optimality criteria.[4]

Whilst the above approaches are by no means incompatible with a viability analysis, viability theory introduces additional "viable" control strategies, based around the concept of the viability kernel. The rationale for these strategies is straight-forward: unless the system is in danger of travelling from

[1] By focussing on possibility rather than on certainty, it is emphasised that (except for where the system is completely deterministic) the identification of a system state as viable does not mean that the system is guaranteed to remain within its constraints – whether it does or not may be contingent on a regulator's actions, or it may depend on some other source of uncertainty. Unlike in optimal control, in viability theory, regulation is a source of uncertainty, because commitment by the regulator to a particular "closed-loop" control policy is not assumed. This is why differential inclusions, see Section 2.1, can be used uniformly in viability theory to model control problems, as well as parameter uncertainty, etc. Nor does it need to be considered likely, as viability does not presuppose that anything is known about the likelihood of any particular evolution.

[2] So, mathematically, viability theory is concerned with *invariant maps*.

[3] See [9] for a more detailed account of the history of fisheries models. Other natural resources also have a long history of mathematical models, e.g. see [20] for the classic forestry models.

[4] From a control theory point of view, this is a *P regulator*.

a viable to a non-viable state any control will be viable. It can be argued that such an approach is consistent with "satisficing" behaviour,[5] under which agents are believed to employ strategies that are "good enough" in that they satisfy *normative* constraints and/or those imposed by reality (*modal*).[6] Under this view, it can thus be argued that viability provides a better fit for the real concerns of regulators than optimal control does – instead of requiring a (potentially complex) loss function, a viable control strategy needs only the system's viability kernel, which can be computed directly from the system's dynamics and the sustainability constraints that have been imposed.

Thus, regardless of whether the regulator is simply interested in identifying what states are sustainable in a deterministic system, determining whether a control strategy ("optimally" or "simply" derived) results in a viable system, or discovering sustainable ways of controlling a system, viability theory has the potential of responding to the *behaviourists'* challenges and of providing valuable insights into compatibility between the system's dynamics and the constraints' geometry. It is our view that viability theory is an excellent tool for the analysis of sustainability problems. As said before, we hope to convince the reader of this also.

In this paper, as mentioned above, we will make use of two "sustainability" problems to illustrate the theory, one from macroeconomics and one from ecological economics. To give an indication of the breadth of possible applications of viability theory in business[7] however, we list some other applications:

- in [23] the dual to the above monetary policy problem was considered, namely viable taxation policy;
- in [32] a business deciding when and if to adopt a new technology was modelled. Here, a result on some equivalence between a viability problem and a higher-dimensional associated optimisation problem was exploited;
- in [36] the case of replicating portfolios in finance is considered as a viability-with-target-problem in which an investor wishes to arrive in a certain "pay-off" position without violating either share price or share quantity constraints in doing so.

We will proceed by describing the basic set of tools in the viability theorist's tool-box (notably, the viability kernel) and we will give a brief exposition of the core analytical results, including the viability theorem. We will then turn our attention to the computation of the viability kernel, and the various

[5] Neologism used in [42].

[6] For example, interest rates *cannot* be negative in a monetary policy selection problem.

[7] In this paper, we focus on viability theory's relevance to economics and bio-economics and will not comment on its applications in other areas (e.g. military).

algorithms that exist for this. This leads us to our own specially developed viability kernel computation software, called VIKAASA, which we will provide an introduction to. Overall, we aim to show the usefulness of viability theory for sustainability problems rather than to give a comprehensive account of all existing branches of research in viability. For readers who are interested in learning more, we suggest [10].

2. Key components of viability theory

2.1. Differential inclusions One of the central reasons for our interest in viability theory is because it provides a means of analysing both *non-deterministic* and *stochastic* dynamic systems; a feature that we find important in dealing with economic and environmental processes. A system's *non-determinism* is modelled in viability theory principally through the use of *differential inclusions*, which can be thought of as a set-valued equivalent of differential equations [4, 6]. This way of description of a dynamic system is different to using a stochastic differential equation in that no probabilities are assigned to the set members, meaning that no statement about the expected value of the system velocity can be made. Unless stochastic shocks with a given probability distribution perturb the system, any velocity that is possible according to the system's dynamics may eventuate. The differential inclusion

$$\dot{x}(t) \in F(x(t)) \tag{1}$$

states that at $x(t)$ the change in the system's state – its velocity – will be a member of $F(x(t))$, where F is a map from system states to *sets* of possible velocities. Exactly which element from $F(x(t))$ will eventuate is subject to uncertainty which may come from any of the following sources:

i. The system may be controllable by a regulator. In this case, we can re-write (1) as

$$\dot{x}(t) = f(x(t), u(t)) \tag{2}$$

$$u(t) \in U(x(t)) \tag{3}$$

where (2) is a standard parameterised differential equation and (3) states that the control choice $u(t)$ must come from a potentially state-dependent set, $U(x(t))$.

ii. There may be uncertainty about the underlying model dynamics. For instance, it may be that there are a number of proposed differential equations $\{f_1, f_2, \dots f_j\}$ describing the system's evolution. Then, a differential inclusion could be formulated for the system as

$$\dot{x}(t) \in \{f_1(x(t)), f_2(x(t)), \dots f_j(x(t))\}. \tag{4}$$

A variant of this is the case where there is uncertainty about model parameters; i.e.

$$\dot{x}(t) = f(x; \gamma) \tag{5}$$

where $\gamma \in \Gamma$ is a vector of parameters drawn from a range of hypothesised values.

iii. The system may be truly non-deterministic, i.e. not subject to any identifiable regularities. This means that from the analyst's perspective every evolution which satisfies (1) is in some sense equally likely.[8]

iv. Any combination of the above.

Differential inclusions provide an abstraction over all of these possibilities.[9] Whereas the solution to a differential equation is a path of points through system space, the solution concept of a differential inclusion is the set of all possible paths that stay in the state-path constraints' "tube" and also satisfy the usual terminal constraints (if appropriate).

It is important to note that any ordinary differential equation can be represented as a differential inclusion whose value is a set with just one item in it. In other words, differential equations can be considered a special case of differential inclusions, meaning that although viability theory is geared towards differential inclusions, models that make use of differential equations can be analysed just as well.

Economic applications of system's dynamics so-formulated are immediate. For example, in fishery management, the biomass "tomorrow" will be in a cone determined by the apex at the present state and rays corresponding to different fishing strategies. In a monetary policy problem, the next period's inflation will be in a cone with the apex where the current inflation is and whose rays depend on the change in the nominal interest rate. Etc.

2.2. Viability Given a set-valued map $F : K \rightsquigarrow X$, we say that $x_0 \in K \subset X$ is *viable in K under F* if, starting from $x(0) = x_0$,

$$\forall t \in \Theta \begin{cases} x(t) \in K, \\ \dot{x}(t) \in F(x(t)), \end{cases} \tag{6}$$

[8] Controlled experiments are rarely performed in social sciences. Because of this, it may not be "rational" to update one's belief about the likelihood of something occurring in relation to something else, because without a control, there is nothing to show that this was not the result of some other (possibly unobserved) phenomenon. See for instance [33] for more information. Note also that absolute uncertainty does not mean that the analyst *believes* that the system's evolution has a uniform probability distribution; rather, the analyst may have no belief at all about the system's probability distribution.

[9] For sustainability problems, control will almost certainly be one element in the formulation of any differential inclusions, but it may not be the only one.

where $\Theta \equiv [0, \infty)$. In other words, x_0 is viable in K if it possible to remain in K indefinitely.[10] K is called the *constraint set*. It is a closed set representing the viability constraints to be imposed on the system evolving under F.

Formulation (6) allows us to talk about the sustainability of an individual system state. To go from this to considering viable *areas*, viability theory crucially introduces the *viability theorem*, which establishes the relationship between any *closed set* of points D viable under F, and the concept of the *proximal normal*[11] to D at x. This relationship is defined formally in [38][12]; here, we provide a version of the theorem taken from [10] (Theorem 2.3) and [1] (Theorem 3.2.4). Note that it is expressed in terms of a parameterised differential equation; however, as we saw in Section 2.1, this is equivalent to the formulation for differential inclusions. Also note that the core results of viability theory are not expressed in terms of vector spaces, but in more general *spaces*; however, we limit the explication here to vector spaces for simplicity.

THEOREM 2.1 *Assume D is a closed set in $\mathbb{R}^n$. Suppose that $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ is a continuous function, Lipschitz in the first variable; furthermore, for every x we define a set valued map $f(x, U) = \{f(x, u); u \in U\}$, which is supposed to be Lipschitz continuous with convex, compact, nonempty values.*

Then the two following assertions are equivalent:

a.

$$\forall x \in D, \quad \forall p \in \mathcal{NP}_D(x), \quad \min_u \langle f(x, u), p\rangle \leqslant 0 \tag{7}$$

(respectively, $\max_u \langle f(x, u), p\rangle \leqslant 0$*);*

b. $\forall x \in D$, *there exists a function* $u(\cdot) : \Theta \mapsto U$ *such that*

(respectively, for all $u(\cdot) : \Theta \mapsto U$*)*

$$\text{the solution of } \begin{cases} \dot{x}(s) = f(x(s), u(s)) \text{ for almost every s} \\ x(t) = x \end{cases} \tag{8}$$

remains in D.

Notice that the inequality $\min_u \langle f(x, u), p\rangle \leqslant 0$ in (7) means that there *exists* a control for which the system's velocity $\dot{x}$ "points inside" the set D. Respectively, $\max_u \langle f(x, u), p\rangle \leqslant 0$ means that the system's velocity $\dot{x}$ "points

[10] Viability is normally defined in terms of an infinite time horizon, but it is also possible to define $\Theta \equiv [0, T], T \in \mathbb{R}_+$, and talk about finite-time viability.

[11] Here $\mathcal{NP}_D(x)$ denotes the set of *proximal normals* to D at x i.e., the set of $p \in \mathbb{R}^n$ such that the distance of $x + p$ to D is equal to $||p||$.

[12] For existence and characterisation of feedback controls assuring viability see [46].

inside" the set D for *all* controls from U. In other words, and using an alternative viability characterisation based on *contingent cones*,[13] given system's dynamics $f(\cdot,\cdot)$, whenever it is possible to identify a closed set D such that the "velocities" $f(x,\cdot)$ available at $x \in D$ and the directions from the contingent cone at x intersect, then every $x \in D$ must be viable under F. That is, there must exist a trajectory starting from each $x \in D$ that remains in D.

When a (or b) holds we say that D is a *viability domain* (or, respectively, D is an *invariance domain*) for the dynamics F. This introduces the classical notion of the viability (respectively, invariance) domain [3], as opposed to viability domains in problems with *targets* (see [37]). This establishes that in order to determine whether it is possible (or necessary, in the case of invariance maps), given the dynamics in F, for a set of state-space points in some set D to be sustained in D indefintely, it is sufficient to show the existence of inward-pointing trajectories at the boundaries of D.

The existence of a viability domain has different meanings, depending on the nature of F (as enumerated in Section 2.1):

I. For a control problem, the existence of a viability domain D indicates an area for which sufficient control exists to maintain the system within D from any point in D. That is, for each element x_0 in D, there should exist a function (i.e. feedback rule) $g : X \mapsto Y$ that takes an element $x \in X$ and returns a control policy u such that

$$\forall t \in \Theta, g(x(t)) \in U(x(t)) \wedge \dot{x}(t) = f(x(t), g(x(t)) \wedge x(t) \in D \qquad (9)$$

where $x(t)$ is a solution to (2), (3) with $x(0) = x_0$.

II. For problems involving either model uncertainty or non-deterministic systems' dynamics, a viability domain indicates an area of potential stability or sustainability. Here, the dynamics of F do no preclude the system stabilising in D (i.e. never leaving D), but they may not necessarily guarantee it. For this, an invariant map must be sought.

III. Where the above two concepts are intermingled in the same differential inclusion F, the existence of a viability domain means that sufficient control exists to give the system a chance of remaining in D, but it may not necessarily happen.

2.3. The viability kernel — the largest viability domain in the constraint set

[13] Loosely speaking, a contingent cone at $\bar{x} \in \mathsf{fr}D$ is the set of all directions pointing "into" D at $\bar{x}$ that form acute angles with a tangent to D at $\bar{x}$. If D were a disc, then a contingent cone at any point of the circumference would be a half-space. Also, when $\bar{x}$ is an interior point of D, then the the contingent cone for this point is the whole space.

Definition 2.2 Let K be a closed set in $\mathbb{R}^n$. The problem's *viability kernel* for dynamics F, denoted: $\mathcal{V}_F(K)$, is the *largest* possible viability domain under F that is also a subset of K.

The viability kernel will therefore be the set of *all* points that are viable in K under F.[14] Establishing the viability kernel $\mathcal{V}_F(K) \neq \varnothing$ solves the viability problem. That is, "good" – viable – states $x \in \mathcal{V}_F(K)$ are separated from "bad" – non-viable – states $x \notin \mathcal{V}_F(K)$. Where F represents a control problem, as in (2), (3), this has important implications for policy making, in that it can be used to formulate control rules that maintain the system's sustainability, as follows.

Consider a viable point $x \in \mathcal{V}_F(K)$. Let $W(x)$ be a set of controls available at this x. Because $x \in \mathcal{V}_F(K)$ is a member of the viability kernel, then there must exist at least one control in $W(x)$ that keeps the system evolution inside $\mathcal{V}_F(K)$. Let $W_{\mathcal{V}}(x) \subset W(x)$ denote the collection of these controls[15]. With this notation, the following *sustainable* policy rule can be formulated:

$$\text{if } x \in \mathcal{V}_F(K) \text{ then apply an instrument } u \in W_{\mathcal{V}}(x)\,. \tag{10}$$

The above "generic" rule can be decomposed into two normative directives for a given viability problem:

i. in the interior of the viability kernel $\mathcal{V}_F(K) \setminus \mathsf{fr}\,\mathcal{V}_F(K)$ use any control from $W_{\mathcal{V}}(x)$;

ii. when one gets "near" to the boundary and in particular when one is on the boundary of the kernel $\mathsf{fr}\,\mathcal{V}_F(K)$, an extreme instrument, or a specific path, must be followed that takes us away from boundary.[16]

We can say that the above controls are compatible with the *inertia principle* (see [1, 5]), which states that self-sustaining dynamic systems will not alter their controls except when the viability of the system is at stake.[17] We can similarly say that these controls are realisations of a satisficing policy postulated in [42] – so long as viability is not threatened, any control is "good enough".

Figure 1 provides a geometric interpretation of the concepts presented above for a control problem with otherwise deterministic dynamics. The state constraint set K is represented by the yellow (or light shadowed) round shape contained in the state space (where X denotes the state space; here, $X \equiv \mathbb{R}^2$). The viability kernel for the constraint set K, given controls from set $U(x)$ and the system dynamics F, is the brownish (darker) shadowed contour. The solid

[14] Where K itself is a viability domain for F, K will be the viability kernel of itself.

[15] They can be called *regulation maps*, see [2].

[16] Unless a steady state has been reached.

[17] An example of this may be the actual "real world" behaviour of inflation-targetting central banks, which will often avoid changing interest rates for as long as they can.

lines symbolise system evolutions. If an evolution begins inside the viability kernel $\mathcal{V}_F(K)$ then we have sufficient controls to keep it in the constraint set K for $t \in \Theta$.

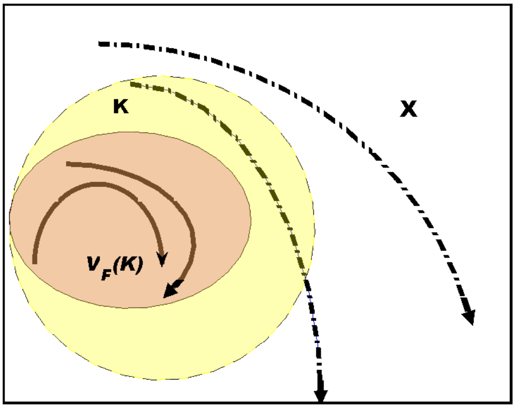


Figure 1: The viable and non viable trajectories.

The system evolution represented by the trajectories that start inside the kernel (solid lines) are *viable* in K *i.e.*, they remain in K. This is not the property of the other trajectories (dash-dotted lines) that start outside the kernel. They leave K in finite time.

2.4. Viability vs. optimality It cannot be over-emphasised that solving a viability theory problem is a different proposition than computing an optimal control.[18]

A crucial difference between these two approaches is that a viability-theory problem's solution explicitly defines the set of acceptable states in K, whereas in the optimisation approach the constraints that define K are usually implicit in the loss function. An immediate upshot of this is that problems modelled using a viability approach do not need to determine utility or loss functions in order to formulate policy rules, and that therefore there is no need to calibrate such functions, which would amount to *subjective* appraisal of which constraints are more important. Instead it is only necessary to determine the bounds of the set K – a potentially much simpler task, given that such bounds (be they normative or modal) are often trivially observable[19].

[18] An optimal control problem can however be a special case of viability if constraints are so rigid that only one path is considered viable.

[19] E.g., the inflation band in New Zealand is legislated.

Furthermore, knowing the viability kernel, which *is* a solution of a viability problem, makes the regulator aware of the locus of states in which the dynamic system can continue to exist, for a given "strength" of implementable controls. Viability is hence a generalisation of stability, rather than optimality. This is so, because the kernel is a closed set and it can be characterised by some measure, which the distance between two states in the kernel will never exceed. More importantly for the regulator, the information about the kernel is sufficient for a realisation of a "satisficing" policy, as opposed to an optimal one. In addition, the rather relaxed approach advocated by directive (i.), page 105, offers the regulator a possibility to strive to achieve other goals (e.g., political), when more than one control $u \in W_{\mathcal{V}}(x)$ is available. These goals might not have been used for the specification of the constraint set K – perhaps they were difficult to specify mathematically or they arose after the viability kernel had been established; or they are considered merely "nice to have" – i.e., "wants" rather than "needs".

3. Numerical delivery Although an analytical characterisation of the viability kernel may in some circumstances be possible [1], most papers published in the economic and bio-economic literature rely on numerical methods to compute their viability kernels. In the subsequent sections we first provide an outline of the simple viability kernel algorithm employed by VIKAASA[20], a specialised MATLAB® application that can compute viability kernel approximations for rectangular constraint sets. We then give a brief summary of some more sophisticated kernel approximation algorithms.

3.1. A method for the determination of viability kernels We will approximate $\mathcal{V}_F(K)$ by looking for solutions to (1); in [18] we can find the base for how to do this. In broad terms, the authors of [18] say that if *an* optimal control problem can be solved from $x(0) \in K$ and $x(t) \in K, \forall t$, where $x(t), \forall t$ is an optimal solution originating from $x(0)$ then $x(0)$ is viable.

VIKAASA (see [28], [29], [30]), is a computational tool based on the above method, which can be used to create approximate viability kernels (actually, viability domains[21]) for the class of viability problems introduced in Section 2.2. The VIKAASA algorithm consists of solving a *truncated* optimal stabilisation (regulation) problem, rather than a general optimization problem as in [18], for each point x in a discretised version of the constraint set, K_δ (see below). For each point in K_δ, VIKAASA assesses whether, when starting from that point, the dynamic evolution of the system can be slowed to a (nearly) steady state without leaving the constraint set in finite time. Those

[20] **V**iability **K**ernel **A**pproximation, **A**nalysis and **S**imulation **A**pplication = VIKAASA, which in Sanscrit विकास means "progress" or "development".

[21] Our method will miss some viable points if they are viable only because the evolutions starting at them are large orbits. In our experiments, we have not encountered points like that.

points that can be brought close enough to such a state are included in the kernel by the algorithm, whilst those that can not are excluded. In Section 4 we present some results from running this algorithm on our two sustainability problems from ecological economics and macroeconomics.

Not withstanding user-friendliness and delivering encouraging results for an array of viability problems, including bio-economic sustainability, in five variables (5D) [43], this algorithm has its limitations. In particular:

i. The algorithm employed by VIKAASA to determine viability kernels suffers heavily from *the curse of dimensionality*: the amount of time required to compute a kernel approximation increases exponentially with the number of dimensions. For instance, with a discretisation of 10 in every dimension, a two-dimensional problem considers 100 points, a three-dimensional problem considers 1,000 points, and a four-dimensional problem considers 10,000 points. This leads to some very long waiting times for "kernel runs" to complete, making it highly desirable to take advantage of the implemented parallel computing option offered in MATLAB® and Octave.

ii. Although any constraint set definable as a subset of a "rectangular" set (hyper-rectangle) can be specified, the underlying algorithm requires that the containing hyper-rectangle be explicitly given as a set of upper- and lower-bounds. That is, $K \subseteq [\underline{x}_1, \overline{x}_1] \times [\underline{x}_2, \overline{x}_2] \times \cdots \times [\underline{x}_n, \overline{x}_n] \equiv \hat{K}$, where $\underline{x}_i$ is the lower bound of the containing rectangle in the i^{th} dimension, and $\overline{x}_i$ is the upper bound.

iii. Problems must be formulated explicity as control problems, using the form described in equations (2) and (3), above. As noted, this problem formulation is generally equivalent to using differential inclusions; however we additionally require that the control set be a constant $U(x) = U, \forall x$.

iv. VIKAASA can only work with *deterministic* autonomous system's dynamics so that any sources of uncertainty must be modelled using the "control metaphor". So, for any given point in the ("meta"-)state space, and any given control choice selected from U, there can only be one system direction $f(x, u)$ and the other directions from $F(x)$ can be selected by a different choice of $u \in U$. Furthermore, it should be noted that the kernel approximation algorithm will perform only as well as the implemented solver's optimisation routine will allow.

In addition to the standard formulation of a viability problem (with the above caveats taken into account), VIKAASA depends on a number of subjectively-determined "settings" or parameters, which control various aspects of how the kernel approximation algorithm works. These parameters include:

A. A *discretisation*, $\delta = [\delta_1, \delta_2, \ldots, \delta_n] \in \mathbb{Z}^n$, which determines the finite subset of $\hat{K}$ to be examined by the algorithm.

For each dimension i in $\hat{K}$, VIKAASA takes a vector of δ_i evenly spaced values, starting at $\underline{x}_i$ and finishing at $\overline{x}_i$. These vectors are then combined using a Cartesian product to make a discretised version of the constraint set's bounding hyper-rectangle, $\hat{K}_\delta \subset \hat{K}$, containing a total of $\prod_{i=1}^{n} \delta_i$ points. The discretised constraint set is then found by taking $K \cap \hat{K}_\delta \equiv K_\delta$. VIKAASA will then consider each of the points in K_δ individually to see whether it is viable or not.

B. A *stopping tolerance*, $\epsilon \in \mathbb{R}_+$ is used as the criterion for "near-steadiness" of the system.

In our truncated regulation, the velocity of the system in state x, subjected to some control u is calculated using the Euclidean norm of the system's velocity at that point, $||\dot{x}|| = \sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \cdots + \dot{x}_n^2}$. Near-steadiness is then said to obtain when this norm is less than what the system would have if its velocity were ϵ in every direction. That is, when $||\dot{x}|| < \sqrt{n \cdot \epsilon^2} = \sqrt{n} \cdot \epsilon$.

C. A *control algorithm*, $u^* : \mathbb{R}^n \to U$, which is a stationary feedback rule, responsible for slowing the system velocity until its norm falls below the stopping tolerance, $\sqrt{n} \cdot \epsilon$. VIKAASA uses this control to choose $u(t) = u^*(x(t))$ at each time realisation. By default a one-step forward-looking numerical norm-minimisation algorithm is used, so that $u^*(x) = \arg\min_u^G\{||f(x + h \cdot f(x, 0), u)||\}$.[22] VIKAASA uses functions provided by MATLAB® and Octave[23] to perform function minimisation. This algorithm is not necessarily suited to all cases, so it may be necessary to consider other algorithms. The choice of a good control algorithm for kernel determination is very important, and is described in more detail in [28].

The above ingredients are used by VIKAASA to approximate viability kernels. For each point x in K_δ, VIKAASA calls a numerical simulation routine, using the above parameters to see if a "near-steady" state can be found. This is done simply by simulating the system's evolution, starting at x to see if when using the control algorithm u^*, the system can be brought to a "near-steady" state without violating the constraints in the process. The approximate viability kernel $\mathcal{V}_F^\delta$ arrived at through this process is constructed as the set of all the points for which this is found to be possible. It should

[22] $\min^G$ refers to the numerical method of function minimisation employed.

[23] In MATLAB®, the principal algorithm used is `fmincon`, available from the Optimization Toolbox. This algorithm is able to numerically solve non-linear constrained optimisation problems using numerical methods. Octave has an equivalent optimisation routine, but at the time of writing, we have yet to provide an interface to this in VIKAASA.

be clear from this that there are some important shortcomings in this algorithm.[24]. For this reason, checking the "goodness" of a viability kernel approximation constructed using VIKAASA is very important. Information on this can be found in [28].

3.2. Other numerical methods

One of the first algorithms for the calculation of viability kernels was proposed in [17] for viability problems with Lipschitzean dynamics. Their algorithm starts with the complete constraint set, and repeatedly "whittles away" at that set. At each pass the algorithm firstly identifies those points on the boundary of the set for which the system's dynamics are such that there are no directions available which are also in the contingent cone of the set at that point. It then eliminates each such point, plus an open ball of points surrounding the point, from the set. The next pass of algorithm is then performed on the closure of the set left over after this elimination. The authors' principle focus is on identifying a radius for the open ball used in elimination, such that over successive passes, the sets will converge to the viability kernel.

In [40], an algorithm is provided for computing viability kernels for "finite-difference inclusions", i.e. a discrete time analogue of differential inclusions

$$x^{n+1} \in G(x^n), \forall n \geqslant 0. \tag{11}$$

Like the algorithm of [17], this algorithm too works by "whittling away" points; in this case, those that exit the set after one discrete-time step. This results in a sequence of closed sets:

$$\begin{array}{lll} K^0 & = & K, \\ K^{n+1} & = & \{x \in K^n | G(x) \cap K^n \neq \varnothing\}. \end{array} \tag{12}$$

It is shown that $K^\infty = \mathcal{V}_G(K)$, the viability kernel for K under G, provided that G is upper-semicontinuous, and K is compact.

This result is then used to present ways of approximating viability kernels for a system with a continuous-time differential inclusion F by providing an approximuation of continuous-time differential inclusions (see (1)) in discrete time:

$$G_\rho = 1 + \rho F. \tag{13}$$

[24]VIKAASA can falsely identify points as viable when they are not, if the stopping tolerance is too "generous". Hence verification of any results with a different grid and/or tighter tolerance is something that should perhaps be considered. Conversely, a check to confirm that points that are deemed non-viable for the selected parameters but are "close" (perhaps adjacent in $\mathcal{V}_F^\delta$) are "really" non-viable could be performed as follows: verify whether a velocity emanating from this point would form an obtuse angle with the normal of viability domain's frontier. For each point where this was the case, the viability domain would then be expanded. This idea is explored in [31].

A method is then presented to combine the viability kernels obtained for different values of ρ; and proofs are given that these combinations provide good approximations for F, provided that certain properties (e.g. F being Lipschitzian) hold.

This method can then be applied successively to a discretised grid of points K_h taken from K, to produce an approximation of the viability kernel. The Saint-Pierre [40] algorithm has the advantage over the previous algorithm mentioned of being more straight-forwardly implementable on a computer, because it is based on discrete approximations of continuous systems. This means that it can be used to produce a graphical depiction of a problem's viability kernel.

In [14, 12] the Saint-Pierre algorithm is built on by replacing the latter's systematic search of the discretised space for viable points with a gradient optimisation procedure using support vector machines (SVMs). It is claimed that this greatly reduces the computation time required, allowing for problems with greater dimensionality to be considered. The authors have implemented their algorithm as a software package; see [21].

4. Viability and VIKAASA in action We will demonstrate the usefulness of viability analysis, and also that of VIKAASA, for the solution of sustainability problems by show-casing two models: one of an environmental-economic process and the other of a monetary policy problem.

4.1. A fisheries example Viability has proven popular with ecologists seeking to model resource use, and the "canonical" model for this can be said to be the fisheries model, which we have already touched on. Versions of this problem have seen numerous treatments with various degrees of complexity, see e.g. [34, 35, 15, 16, 43]. Almost all of them involve two common elements: a stock of fish, for which the basic sustainability criterion is the survival of the stock into the future (perhaps over an infinite time horizon); and a fishing fleet that will only profit from extraction of this stock under certain conditions (when the stock is above a certain level, etc.). The problem therefore becomes one of determining the extent to which fish stock sustainability is compatible with a profitable fisheries industry – whether it is possible for the twin goals of economic and ecological sustainability to be met simultaneously.

In this section we present one of the simpler versions of this problem, based on [9]. This problem has the advantage that it has a two-dimensional state-space, making it easy to display diagramatically, and can be easily extended to complex scenarios. Moreover, because the solution to this problem is already well-established, it serves as a good "acid test" for the capabilities of VIKAASA. As such, we limit our coverage of the model in this section to reproducing the results in [9]. Readers are encouraged to refer to that paper for more information. The elements of the model are as follows:

a. The system is described by of two dynamic variables: fish biomass,

$b(t)$ and effort, $e(t)$. Effort is exerted by the fishing fleet to extract the resource (i.e. fish). This is a fixed fleet-size model, so there is no variation in capital to consider.

A "catchability coefficient" q is defined to determine the quantity of biomass that each unit of effort extracts, relative to the total size of the biomass at the time. Thus, for some biomass level $b(t)$, an effort level of $e(t)$ yields harvest $qe(t)b(t)$ to be subtracted from the resource biomass.

b. Three constraints are given. The first constraint concerns the ecological sustainability of the resource. To this end, a "safe minimum biomass level," $b_{min} > 0$ is specified, below which it is believed that the resource will become extinct. Thus, sustainability requires that $\forall \in \Theta, b(t) \geqslant b_{min}$.

The second constraint concerns the economic sustainability of the fleet, and requires that fishing remain profitable. Profits are given by

$$R(b(t), e(t)) = pqe(t)b(t) - ce(t) - C, \tag{14}$$

where p is the price of a unit of biomass (fixed in this model), and as explained above, $qe(t)b(t)$ is the catch size, making $pqe(t)b(t)$ the revenue gained, whilst C is some fixed cost, and c is a variable cost for each unit of effort. The profitability requirement is then that $R(b, e) > 0$, meaning that revenue must be at least as big as the combination of fixed and variable costs.

The third constraint is not normative, but rather concerns the physical capabilities of the fishing fleet. That is, it is supposed that $\forall t \in \Theta$, $e(t) \in [0, e_{max}]$, where e_{max} is the maximum possible effort exertable by the fleet. Given the fixed size of the fleet, this is the only input into fleet behaviour.

Thus the constraint set is:

$$K = \{(b, e) : b \geqslant b_{min} \wedge pqeb - ce - C \geqslant 0 \wedge e \in [0, e_{max}]\}. \tag{15}$$

It should be noted that this constraint set is *not* closed, as there is no upper limit on b. However, there is an implicit upper limit imposed by the differential inclusion for $b(t)$ (see e below).

c. It is supposed that regulatory instruments can be used to increase or decrease the level of effort exerted by the fleet. Thus, the system can be modelled as having a single scalar control, $u(t) \in \mathbb{R}$, which determines effort variation; $u(t) = \dot{e}(t)$. Given that viability problems concern the existence (or not) of sufficient control, we are here engaged with finding state-space points where effort is at a sustainable level and no variation is required, or where it can be changed (given the available control-set) to become sustainable without any violation of constraints.

d. Effort variation is bounded by $U = [u^-, u^+]$, where $u^- < 0$ and $u^+ > 0$. Thus, where $e(t)$ is too high (entailing imminent extinction), or too low (meaning that fishing is not profitable), it may not be possible to increase or decrease $e(t)$ fast enough (depending on the sizes of u^- and u^+, which determine to speed of changes of $e(t)$) to maintain the viability of the system.

e. Fish population levels $b(t)$ are governed by a *logistic differential equation*,

$$\dot{b}(t) = rb(t)\left(1 - \frac{b(t)}{l}\right) - qe(t)b(t). \tag{16}$$

This equation is based on [41], and is common in models of population growth. The resource grows at a rate proportional to r, up to the "limit carrying capacity," l of the resource's ecosystem, less the size of the catch, $qe(t)b(t)$.

As mentioned in c, the differential inclusion for e is:

$$\dot{e}(t) \in U = [u^-, u^+]. \tag{17}$$

Hence, the *differential inclusion* for the system is

$$\dot{x}(t) \in \{\dot{b}(t)\} \times U. \tag{18}$$

In order to work with this problem numerically, we need to provide values for the various parameters. Here we set these to the following:

- Catchability coefficient, $q = \frac{1}{2}$.
- Cost per unit of effort, $c = 10$.
- Fixed cost, $C = 100$.
- Growth rate, $r = \frac{2}{5}$.
- Limit carrying capacity, $l = 500$.
- Maximum effort, $e_{max} = 1$.
- Maximum effort variation, $c = u^+ = -u^- = \frac{1}{100}$.
- Safe minimum biomass level, $b_{min} = 5$.
- Unit fish price, $p = 8$.

Thus, the differential equations/inclusions become:

$$\dot{b}(t) = \frac{2}{5}b(t)\left(1 - \frac{b(t)}{500}\right) - \frac{1}{2}e(t)b(t) \tag{19}$$

and:

$$\dot{e}(t) \in U = \left[-\frac{1}{100}, \frac{1}{100}\right] \tag{20}$$

Finally, our constraint set is:

$$K = ([5, 500] \times [0, 1]) \cap \{(b, e)|4eb \geqslant 10e + 100\}. \tag{21}$$

This problem can now be fed into VIKAASA. Let us first see what a viability domain for the control policy $u(t) = 0, \forall t$ (called "zero-control" in VIKAASA) would look like. This has the advantage of being much quicker to calculate in VIKAASA. The points identified as viable for a coarse discretisation of $[11, 11]$ are shown in Figure 2a. We can interpolate from these points by drawing a convex hull around the points, as shown in Fiugure 2b. From these diagrams we see that a rounded area at the bottom of the diagam is non-viable. This corresponds to the area where fishing is not profitable, either due to there not being enough fish (top-left), or because too little effort is being expended (bottom-right). When both of these factors are combined, we get the large non-viable area towards the origin.

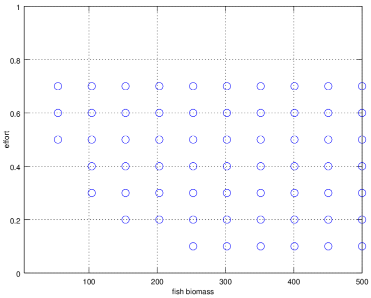
We now switch algorithms to the default norm-minimisation control and re-run the viability kernel approximation algorithm. The difference from doing so is displayed in Figure 2c. From this, we can see that the yellow area at the top of the new viability domain required exertion of non-zero control in order to be viable. That is, the level of effort initially exerted would not have been sustainable in the long-run, but there was sufficient time to reduce effort at the allowed rate before fish stocks fell too low.

Finally, whereas our initial results were obtained for a fairly coarse discretization, in Figure 2d we show the viability domain, again computed using norm-minimisation, but using a much higher discretization of $[50, 50]$ instead. Doing so rounds-out the edges of the area somewhat, bringing additional boundary-points into the area considered viable.

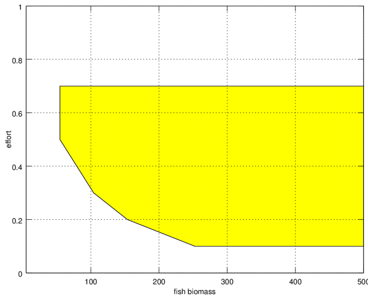
4.2. Satisficing solutions to a monetary policy problem [25]

4.2.1. The system's dynamics. A central bank aims to achieve the maintenance of a few key macroeconomic variables within some bounds. Usually, the bank realises its multiple targets using *optimising* solutions that result from minimisation of the bank's loss function. The loss function usually includes penalties for violating an allowable inflation band, high unemployment and a non-smooth adjustment of interest rates. The solution, which

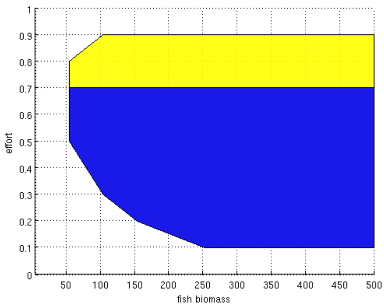
[25] This subsection borrows heavily from [27].



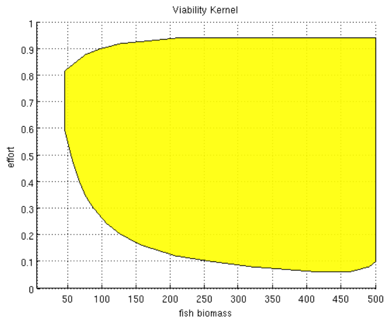
(a) A viability domain for the fisheries problem using $u^*(x) = 0$



(b) An interpolation of the viability domain in 2a



(c) The yellow area represents the additional points determined viable using norm-minimisation, over "zero control" (shown in blue).



(d) An interpolation of the viable points found using norm-minimisation for a higher discretisation.

Figure 2: Viability domains for the fisheries problem.

minimises the loss function, is unique for a given selection of the loss function parameters in that it does not allow for alternative strategies. The bank can also use a simplified rule, such as the Taylor rule that prescribes the rate of interest to set as a function of some subset of the indicators that the central bank concerns itself with. The bank can, obviously, disregard optimisation or the Taylor prescription and set the interest rate *outside* those frameworks.

In Section 2 we discussed how viability theory is, in principle, useful for the problem of keeping state variables within some normative bounds. Our intention now is to exploit this apparent usefulness by applying viability theory to a fairly realistic model of the central bank's problem. In order to do this, we must first develop a state-space model of the economy, see (1).

Suppose a central bank uses a nominal short-term interest rate i_t as an instrument to control inflation π_t and, to a lesser extent, the output gap y_t and, to an even lesser extent, the real exchange rate[26] q_t.

[26]Following [8], we understand *output gap* as the log deviation of actual output from

A model that relates these variables may look as follows.[27]:

$$
\begin{aligned}
y_t &= \delta_1 y_{t-h} - \delta_2\Big(i_{t-h} - E_{t-h}\pi_t\Big) + \delta_3 q_{t-h} + \varepsilon^y_{t-h} && (22)\\
\pi_t &= \phi_0 E_t \pi_{t+h} + (1-\phi_0)\pi_{t-h} + \phi_1\Big(y_t + y_{t-h}\Big) + && (23)\\
&\quad + \mu\Big((1-\phi_0)\Delta q_t - \phi_0 E_t \Delta q_{t+h}\Big) + \varepsilon^\pi_t \\
E_t \Delta q_{t+h} &= \Big(i_t - E_t \pi_{t+h}\Big) + \varepsilon^q_t\,. && (24)
\end{aligned}
$$

In (23) and (24), Δ is the first-difference operator $\Delta x_t = x_t - x_{t-h}$ where x_t can be any of the above variables and $E_{t-h}x_t$ is the variable's expected value formed at time $t-h$.

The following is a continuous-time version of the above model proposed in [27] where y, π, i, q are the time t (dropped for simplicity) expected values of output-gap, inflation, interest rate and exchange rate, respectively.

$$
\begin{aligned}
\dot{y} &= ay - d_2(i-\pi) + d_3 q && (25)\\
\dot{\pi} &= 2py && (26)\\
\dot{q} &= (i-\pi) - \rho u, \quad \rho \geqslant 0\,. && (27)
\end{aligned}
$$

The obtained model (25)-(27) tells us that the expected output gap (see (25)) constitutes a (zero) mean-reverting process driven mainly by real interest rate. The exchange rate affects the competitiveness of domestic goods in the world market and affects the change in output gap: if the domestic currency appreciates (i.e., $q(t)$ diminishes, $d_3 > 0$) then output-gap growth slows down and may become negative.

The expected speed of inflation (see (26)) changes proportionally to the expected output gap *doubled*. Interestingly, in this model, the continuous-time expected inflation-rate (short-term) changes do not depend on the exchange rate differential (as in the discrete time model, see (23)) because the latter tends to zero for short h.

Equation (27) captures the process of currency adjustment to changes of the real interest rate where the coefficient ρ, whose value is obtained in [27] after a supplementary calibration, describes responsiveness of variations

"natural" output. Since it is a log deviation, it shows the magnitude of deviation as a fraction of the natural output. *Inflation* is the CPI inflation rate. *Interest rate* is the short-term nominal interest rate that is used by the central bank as the policy instrument. Inflation rate and interest rates are expressed in fractions rather than as %. Also, as conventionally done in the literature in this area, they are expressed as annualised rates. Finally, we define *exchange rate* $q(t)$ as the *log* ratio of *nominal exchange rate* $\times$ *foreign price index* to *domestic price index*, which can be viewed as an aggregate measure of the strength of a country's currency. If the currency weakens, then $q(t)$ increases. Or, larger $q(t)$ means real depreciation so, that domestic goods become relatively cheaper.

[27] Our model is a version of that described in [8] and [7]. Also see [25] and [24] where we have experimented with another model based on [47] and [44].

of the exchange-rate, to nominal interest-rate adjustments. The first term of (27) reflects long-term exchange-rate behaviour: the exchange rate tends to decrease with positive real interest rate (after an initial spike modelled by $-\rho u$).

The calibrated version of the the above monetary policy model for a "small-open" economy, adopted in this paper is as follows:

$$\dot{y} = -0.2y - 0.5(i - \pi) + 0.2q \tag{28}$$

$$\dot{\pi} = 0.4y \tag{29}$$

$$\dot{q} = (i - \pi) - 0.125u\,. \tag{30}$$

We now need to define the constraint set K, to which we want our economy to be confined.

$$K \equiv \{(y(t), \pi(t), i(t), q(t)) : \tag{31}$$
$$-0.04 \leqslant y(t) \leqslant 0.04,\;\; 0.01 \leqslant \pi(t) \leqslant 0.03,\;\; 0 \leqslant i(t) \leqslant 0.07,\;\; -0.1 \leqslant q(t) \leqslant 0.1\}\,,$$

defines a "box" of constraints, in which output-gap, inflation, interest rate and exchange rate are allowed to move but prohibited from leaving.

Central banks may be even more concerned about managing interest rate volatility rather than constraining the actual rate of interest. This concern is usually modelled by adding $w(i(t) - i(t-h))^2$, $w > 0$ to the central bank's loss function. In continuous time, limiting the interest rate "velocity" $\frac{di}{dt}$ will produce a smooth time profile for $i(t)$. Bearing in mind that the central bank's announcements are usually made monthly and that the typical change, when made, is about $\frac{1}{4}\%$ per announcement, we choose the bank *control* to be $u(t) = \frac{di}{dt} \in [-0.01,\, 0.01]$. Hence, the dynamic system that links the output gap, inflation, the exchange rate and the interest rate augmented by the control constraint now appears as follows:

$$\frac{dy}{dt} = -0.2y - 0.5(i - \pi) + 0.2q \tag{32}$$

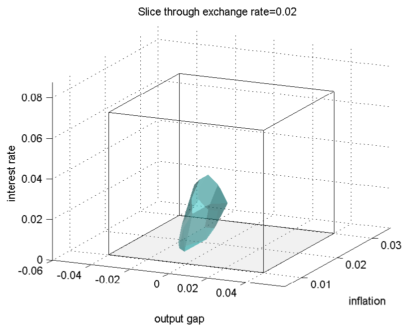
$$\frac{d\pi}{dt} = 0.4y \tag{33}$$

$$\frac{dq}{dt} = (i - \pi) - 0.125u \tag{34}$$

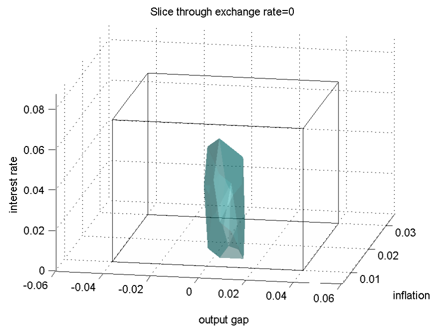
$$\frac{di}{dt} = u \in \mathcal{U} = [-.01,\, .01]\,. \tag{35}$$

The above system of differential equations and a set inclusion define the dynamics of our small open economy for which we want to compute the viability kernel $\mathcal{V}(K)$ (to unburden notation, we have dropped the subindex F on $\mathcal{V}$ because (32) - (35) is the only model we use in this subsection).

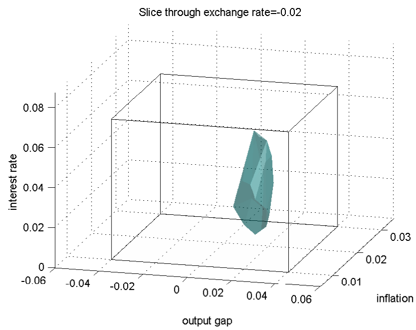
4.2.2. The kernel. We have applied VIKAASA to produce the viability kernel $\mathcal{V}(K)$ for the system's dynamics (32) - (35).



(a)



(b)



(c)

Figure 3: Kernel 3D slices for 3a: $q = 2\%$, 3b: $q = 0\%$ and 3c: $q = -2\%$.

Figure 3 shows three 3D kernel slices ("boulders") that correspond to three exchange rate values from left to right: $q = 0.02$ (undervalued currency), $q = 0$ (neutral currency position) and $q = -0.02$ (overvalued currency). In brief, if an economy is inside a "boulder", then the central bank can control the economy by using an interest rate change $u \in U$, so that the economic variables of interest remain in the constraint set K. If a state is not an interior point of the kernel, the economy is destined to violate the constraint set.

4.2.3. Policy conclusions

The following general guidelines for the central bank governor can be inferred from Figure 3. During a depression, *i.e.*, when the output gap is negative (see the left panel), the economy will be viable, e.g., it will stay clear of a liquidity trap[28], if interest rates are low and currency is slightly undervalued. However, when the economy is booming, i.e., when the output gap is positive (see the right panel), the economy will remain viable, e.g., overheating will be avoided, if interest rates are in the medium range and

[28] This is how the "corner" where output gap is negative and interest rates are almost zero is referred to.

the currency is slightly overvalued. When the output gap is near zero (see the middle panel), there are almost no constraints on interest rate and inflation as long as the exchange rate is close to neutral. We can also formulate several other conclusions in "negative" terms. For example, there are no viable points for an economy with a negative output gap *and* an overvalued currency; somewhat symmetrically, there will be no smooth interest rate adjustments that will keep the economy in K, if the output gap is positive and the currency is overvalued. Furthermore, interest rate ranges that can be applied are sensitive to the other variables, *etc.*

We contend that a monetary policy model based on viability theory, helps to determine both applicable interest-rate adjustment decisions and managerial guidelines regarding which economic states must be avoided (assuming only smooth adjustments are acceptable). Of course, we are not saying that our results are generally applicable to all central banks; rather we are proposing that any central bank could carry out the same sort of analysis that has been done here on a model of their choosing and with their own constraints chosen as appropriate.

We observe in Figure 4 that a "viable" economy, is moved (by a viable control) from the *boundary* point of the kernel slice at $q = -0.0375$[29] to an interior point "well within" the kernel slice at $q \approx 0$. The latter is the larger boulder (also shown in the middle panel of Figure 3). This is achieved by an intensive interest-rate drop. After that, controlling the economy becomes "easy", as advised by directive (i) on page 105 and the bank may try to select policies that could achieve other goals. For example, lower inflation could be achieved by hiking and then dropping the interest rates.

We will add a few more observations and conclusions that are appropriate for the system's dynamics (32)-(35), inherited from [7] and derived in [27].

- The kernel slice shown in the middle panel of Figure 3 (as said, reproduced as the larger boulder in Figure 4), obtained when the local currency is in the neutral position, is the largest of all the slices that we show. It resembles a cylinder centred around $y = 0$ and $\pi = i$ i.e., it contains points that are close to a steady state (obviously, $y = 0$, $\pi = i$, and $q = 0$ define multiple steady states).

- In broad terms, if output gap is (absolutely) large, or the interest rate and inflation rate diverge from one another, then the economy is bound to violate the constraints of K, for $u \in \mathcal{U}$. This conclusion is by-and-large known; what we gain from the analysis of kernel $\mathcal{V}$ are the sizes of deviations from a steady state, which the economy can tolerate before it becomes uncontrollable (for $u \in \mathcal{U}$).

[29] So, some control would have to be applied, otherwise the economy would leave the kernel.

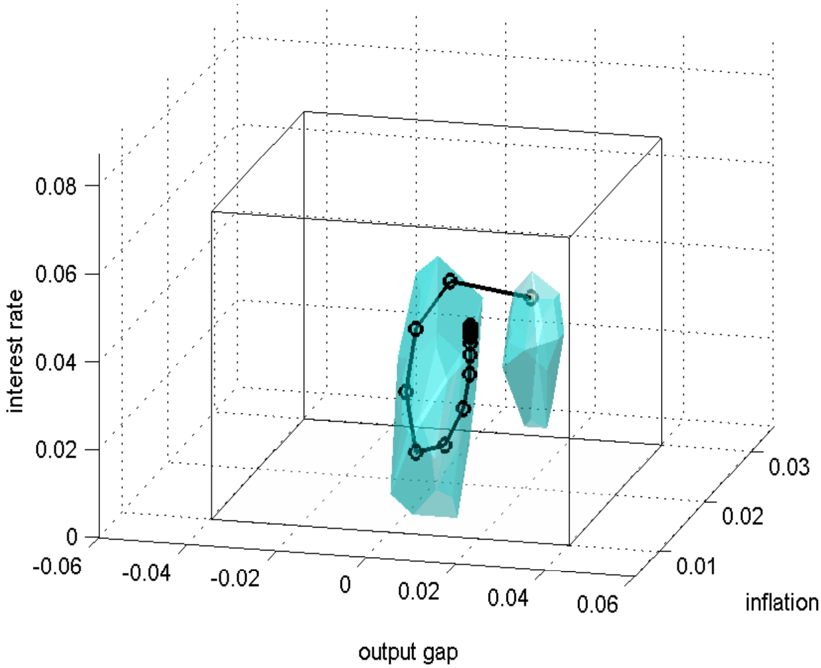


Figure 4: Kernel slices through exchange rates $q = -3.75\%$ (right boulder) and $q \approx 0\%$ (middle boulder).

- Interest rates in excess of 5% should be avoided, as the number of viable points quickly dwindles to zero beyond that level.

- Furthermore, the interest rate has to be "proportional" to inflation, which helps to maintain a low real interest rate. This is evident from Figure 5. Here, we see two slices of the discussed boulder (i.e., $q = 0$): for $i = 1.5\%$ (low interest rate) and 2.5% (high interest rate). We can see that when the interest rate is high, the economy can be viable if inflation is high or medium. In some sense symmetrically, when interest rate is low, the economy can be viable if inflation is low or medium. We say that the economy *can* be viable because this is contingent on the output gap not being too large or too small.

- We also point out that the kernel for a 2%-*over*valued currency (Figure 3 right panel) is visibly larger than for a 2%-*under*valued currency and allows for higher interest rates. The practical implication of this is that for the economy studied, an undervalued currency will be more vulnerable to shocks (smaller kernel), and attempts to realise non-monetary policy goals should be withheld until the economy moves further away from the viability kernel's boundary.

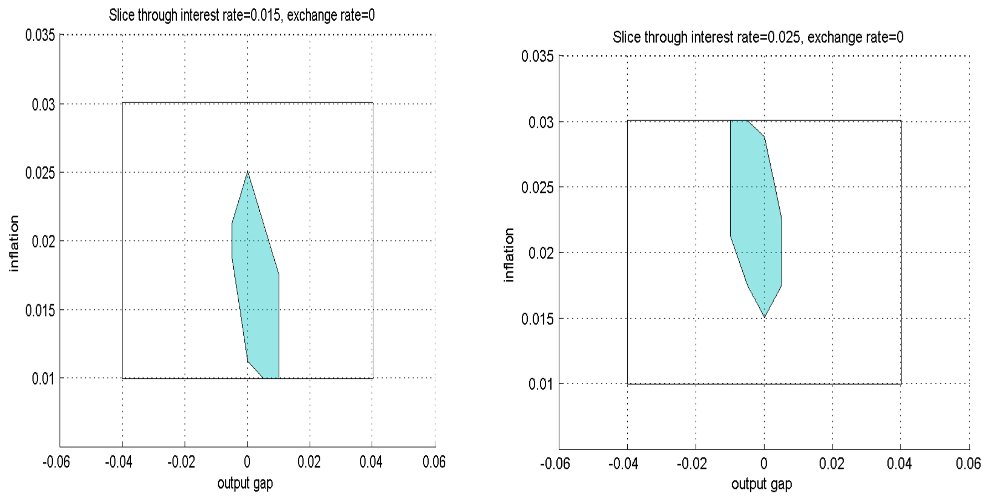


Figure 5: Kernel slices for $q = 0$ and $i = 1.5\%$ (left panel); $q = 0$ and $i = 2.5\%$ (right panel).

The general advice the central bank can draw from the above considerations is that it needs to exert an "extreme" control when the economy is at the kernel boundary. Practically, the bank should implement the full interest-rate drop or increase sufficiently early when the economy is likely to approach one of the kernel boundaries.

5. Future research We propose that more *sustainability* problems, including disease management should be attempted to analyze from the viability-theory point of view. Given that these (ecologic-economic, health care, etc.) problems' models are essentially nonlinear, more research is needed on kernel existence for such models. Last but not least, computational algorithms that lead to kernel determination should be improved as they frequently suffer from the *curse of dimensionality*, which impairs solutions to high-dimensional problems.

6. Concluding remarks We have presented some basic notions of viability theory and shown how this theory can be useful for modeling and solution of sustainability problems.

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Teoria wiabilności: narzędzie matematyki stosowanej do ujawniania stanów zrównoważonego rozwoju

Streszczenie Zrównoważony rozwój jest terminem często używanym lecz naukowcy i politycy nie są zgodni ani co do jego znaczenia, ani jak praktycznie i teoretycznie zapewnić taki rozwój. Niniejsza praca ma na celu wprowadzenie czytelnika do teorii wiabilności[30], stosunkowo młodej gałęzi matematyki ciągłej, której narzędzia nadają się do opisu problemów zrównoważonego rozwoju. W szczególności, teoria wiabilności może być wykorzystana do określania strategii zrównoważonego rozwoju systemów ekonomicznych, w tym makroekonomicznych. Głównym narzędziem analitycznym teorii wiabilności jest jądro wiabilności, którym jest zbiór wszystkich punktów przestrzeni stanu, z jakich mogą się dokonać ewolucje systemu, które nigdy nie przekroczą zadanych z góry ograniczeń. Chociaż w pewnych okolicznościach opis jądra może być otrzymany analitycznie, większość praktycznych rezultatów w teorii wiabilności uzyskuje się przez analizę graficznych przybliżeń jąder wiabilności, które w przypadku nieliniowych i wysokowymiarowych problemów mogą być uzyskane jedynie drogą obliczeniową. Niniejsza praca przedstawia podstawowe pojęcia teorii wiabilności oraz przegląd dostępnych metod numerycznych do obliczania przybliżeń jąder. VIKAASA, specjalistyczne oprogramowanie opracowane przez autorów, umożliwia otrzymywanie takich przybliżeń. W pracy, użycie VIKAASY jest zilustrowane przykładami z zakresu bio- i makroekonomii.

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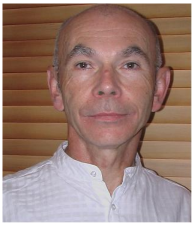
Słowa kluczowe: teoria wiabilności, inkluzje różniczkowe, zrównoważony rozwój.

[30]Po konsultacjach z polonistami (patrz http://poradnia.pwn.pl), postanowiliśmy przetłumaczyć *viability theory* jako *teoria wiabilności.*



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