Viability Theory: New Directions

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Viability Theory.
New Directions

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With 147 figures

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Chapter 1
Overview and Organization

Viability theory designs and develops mathematical and algorithmic methods for investigating the *adaptation to viability constraints of evolutions governed by complex systems under uncertainty* that are found in many domains involving living beings, from biological evolution to economics, from environmental sciences to financial markets, from control theory and robotics to cognitive sciences. It involves interdisciplinary investigations spanning fields that have traditionally developed in isolation.

The purpose of this book is to present an initiation to applications of viability theory, explaining and motivating the main concepts and illustrating them with numerous numerical examples taken from various fields.

Viability Theory. New Directions plays the role of a second edition of *Viability Theory*, [18, Aubin] (1991), presenting advances occurred in set-valued analysis and viability theory during the two decades following the publication of the series of monographs: *Differential Inclusions. Set-Valued Maps and Viability Theory*, [25, Aubin & Cellina] (1984), *Set-valued Analysis*, [27, Aubin & Frankowska] (1990), *Analyse qualitative*, [85, Dordan] (1995), *Neural Networks and Qualitative Physics: A Viability Approach*, [21, Aubin] (1996), *Dynamic Economic Theory: A Viability Approach*, [22, Aubin] (1997), *Mutational, Morphological Analysis: Tools for Shape Regulation and Morphogenesis*, [23, Aubin] (2000), *Mutational Analysis*, [150, Lorenz] (2010) and *Sustainable Management of Natural Resources*, [77, De Lara & Doyen].

The monograph *La mort du devin, l'émergence du démiurge. Essai sur la contingence et la viabilité des systèmes*, [24, Aubin] (2010), divulges vernacularly the motivations, concepts, theorems and applications found in this book. Its English version, *The Demise of the Seer, the Rise of the Demiurge. Essay on contingency, viability and inertia of systems*, is under preparation.

However, several issues presented in the first edition of *Viability Theory*, [18, Aubin] are not covered in this second edition for lack of place. They concern Haddad's viability theorems for functional differential inclusions where both the dynamics and the constraints depend on the history (or path) of the evolution and the Shi Shuzhong viability theorems dealing with partial differential evolution equation (of parabolic type) in Sobolev spaces, as well as fuzzy control systems and constraints, and, above all, differential (or dynamic) games. A sizable monograph on tychastic and stochastic viability and, for instance, their applications to finance, would be needed to deal with uncertainty issues where the actor has no power on the choice of the uncertain parameters, taking over the problems treated in this book in the worst case (tychastic approach) or in average (stochastic approach).

We have chosen an outline, which is increasing with respect to mathematical technical difficulty, relegating to the end the proofs of the main Viability and Invariance Theorems (see Chapter 19, p.789).

The proofs of the theorems presented in *Set-valued analysis* [27, Aubin & Frankowska] (1990) and in convex analysis (see *Optima and Equilibria*, [19, Aubin]), are not duplicated but referred to. An appendix, *Set-Valued Analysis at a Glance* (18, p. 733) provides without proofs the statements of the main results of set-valued analysis used in these monographs.

1.1 Motivations

1.1.1 Chance and Necessity

The purpose of viability "theory" (in the sense of a sequence [*theôria*, procession] of mathematical tools sharing a common background, and not necessarily an attempt to explain something [*theôrein*, to observe]) is to attempt to answer directly the question of dynamic adaptation of uncertain evolutionary systems to environments defined by constraints, that we called viability constraints for obvious reasons. Hence the name of this body of mathematical results developed since the end of the 1970's that needed to forge a differential calculus of set-valued maps (set-valued analysis), differential inclusions and differential calculus in metric spaces (mutational analysis). These results, how imperfect they might be to answer this challenge, have at least been motivated by social and biological sciences, even though constrained and shaped by the mathematical training of their authors.

It is by now a consensus that the evolution of many variables describing systems, organizations, networks arising in biology and human and social sciences do not evolve in a deterministic way, not even always in a stochastic way as it is usually understood, but evolve with a Darwinian flavor.

Viability theory started in 1976 by translating mathematically the title

$$\boxed{\begin{array}{ccc} \textit{Chance} & \textit{and} & \textit{Necessity} \\ \Updownarrow & & \Updownarrow \\ x'(t) \in F(x(t)) & \& & x(t) \in K \end{array}}$$

of the famous 1973 book by *Jacques Monod*, *Chance and Necessity* (see [164, Monod]), taken from an (apocryphical?) quotation of Democritus who held that *"the whole universe is but the fruit of two qualities, chance and necessity"*.

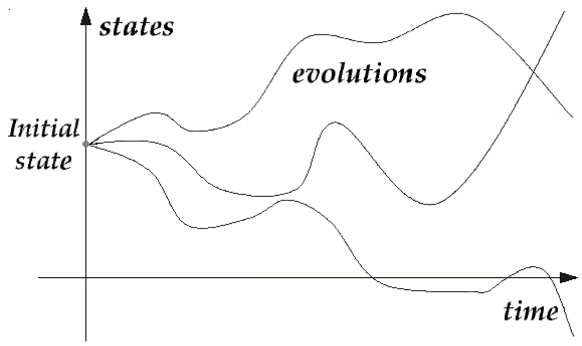

Fig. 1.1 The mathematical translation of "chance".
*The mathematical translation of "**chance**" is the differential inclusion $x'(t) \in F(x(t))$, which is a type of evolutionary engine (called an* evolutionary system*) associating with any initial state x the subset $\mathcal{S}(x)$ of evolutions starting at x and governed by the differential inclusion above. The figure displays evolutions starting from a give initial state, which are functions from time (in abscissas) to the state space (ordinates).*

The system is said to be *deterministic* if for any initial state x, $\mathcal{S}(x)$ is made of one and only one evolution, whereas "contingent uncertainty" happens when the subset $\mathcal{S}(x)$ of evolutions contains more than one evolution for at least one initial state. "Contingence is a non-necessity, it is a characteristic attribute of freedom", wrote *Gottfried Leibniz*.

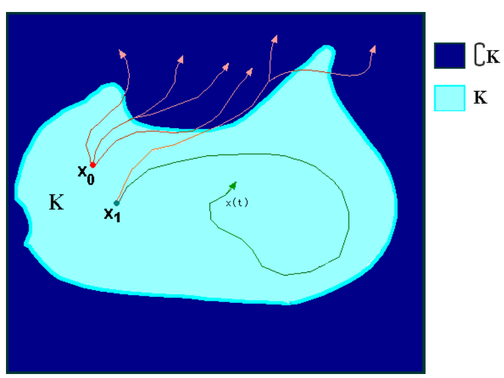

Fig. 1.2 The mathematical translation of "necessity".
*The mathematical translation of "**necessity**" is the requirement that for all $t \geq 0$, $x(t) \in K$, meaning that at each instant, "viability constraints" are*

satisfied by the state of the system. The figure represents the state space as the plane, and the environment defined as a subset. It shows two initial states, one, x_0 *from which all evolutions violate the constraints in finite time, the other one* x_1*, from which starts one viable evolution and another one which is not viable.*

One purpose of viability theory is to attempt to answer directly the question that some economists, biologists or engineers ask: *"Complex organizations, systems and networks, yes, but for what purpose?"* The answer we suggest: *"to adapt to the environment."*

This is the case in economics when we have to adapt to scarcity constraints, balances between supply and demand, and many other constraints.

This is also the case in biology, since Claude Bernard's "constance du milieu intérieur" and Walter Cannon's "homeostasis". This is naturally the case in ecology and environmental studies.

This is equally the case in control theory and, in particular, in robotics, when the state of the system must evolve while avoiding obstacles forever or until they reach a target.

In summary, *the environment is described by viability constraints* of various types, a word encompassing polysemous concepts as *stability, confinement, homeostasis, adaptation*, etc., expressing the idea that some variables must obey some constraints (representing physical, social, biological and economic constraints, etc.) that can never be violated. So, viability theory started as *the confrontation of evolutionary systems governing evolutions and viability constraints* that such evolutions must obey.

In the same time, controls, subsets of controls, in engineering, regulons (regulatory controls) such as prices, messages, coalitions of actors, connectionnist operators in biological and social sciences, which parameterize evolutionary systems, do evolve: *Their evolution must be consistent with the constraints, and the targets or objectives they must reach in finite or prescribed time*. The aim of viability theory is to provide the "regulation maps" associating with any state the (possibly empty) subset of controls or regulons governing viable evolutions.

Together with the selection of evolutions governed by teleological objectives, mathematically translated by intertemporal optimality criteria as in optimal control, viability theory offers other selection mechanisms by requiring evolutions to obey several forms of *viability* requirements. In social and biological sciences, intertemporal optimization can be replaced by *myopic, opportunistic, conservative and lazy* selection mechanisms of viable evolutions that involve present knowledge, sometimes the knowledge of the history (or the path) of the evolution, instead of anticipations or knowledge of the future (whenever the evolution of these systems cannot be reproduced experimentally). Other forms of uncertainty do not obey statistical laws, but take also into account unforeseeable rare events (tyches, or perturbations, disturbances) that must be avoided at all costs (precautionary principle[1]). These systems can be regulated by using regulation (or cybernetical) controls that have to be chosen as feedbacks for guaranteeing the viability of constraints and/or the capturability of targets and objectives, possibly against perturbations played by "Nature", which we call *tyches*.

However, there is no reason why collective constraints are satisfied at each instant by evolutions under uncertainty governed by evolutionary systems. This leads us to the study of *how to correct either the dynamics, and/or the constraints* in order to restore viability. This may allow us to provide an explanation of the formation and the evolution of controls and regulons through regulation or adjustment laws that can be designed (and computed) to insure viability, as well as other procedures, such as using *impulses* (evolutions with infinite velocity) governed by other systems, or by regulating the evolution of the environment.

Presented in such an evolutionary perspective, this approach of (complex) evolutionary systems departs from main stream modelling by a direct approach:

1 [**Direct Approach.**] It consists in studying properties of evolutions governed by an evolutionary system: gather the larger number of properties of evolutions starting from each initial state. It may be an information both costly and useless, since our human brains cannot handle simultaneously too many observations and concepts.

Moreover, it may happen that evolutions starting from a given initial state satisfy properties which are lost by evolutions starting from another initial state, even close to it (sensitivity analysis) or governed by (stability analysis).

Viability theory rather uses instead an *inverse approach*:

2 [**Inverse Approach.**] A set of prescribed properties of evolutions being given, study the (possibly empty) subsets of initial states from which

1. starts at least one evolution governed by the evolutionary system satisfying the prescribed properties,
2. all evolutions starting from it satisfy these prescribed properties.

These two subsets coincide whenever the evolutionary system is deterministic.

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[1] stating that one should limit, bound or even forbid potential dangerous actions, without waiting for a scientific proof of their hazardous consequences, whatever the economic cost.

Stationarity, periodicity and asymptotic behavior are examples of classical properties motivated by physical sciences which have been extensively studied.

We thus have to add to this list of classical properties other ones, such as concepts of *viability* of an environment, of *capturability* of a target in finite time, and of other concepts combining properties of this type.

1.1.2 Motivating applications

For dealing with these issues, one needs *"dedicated"* concepts and formal tools, algorithms and mathematical techniques motivated by complex systems evolving under uncertainty. For instance, and without going into details, we can mention systems sharing common features:

1. ***Systems designed by human brains*** in the sense that agents, actors, decision-makers act on the evolutionary system, as in engineering. ***Control theory and differential games***, conveniently revisited, provide numerous metaphors and tools for grasping viability questions. Problems in *control design, stability, reachability, intertemporal optimality, tracking of evolutions, observability, identification and set-valued estimation*, etc., can be formulated in terms of viability and capturability concepts investigated in this book.
   Some technological systems such as robots of all types, from drones, unmanned underwater vehicles, etc., to animats (artificial animals, a contraction of anima-materials) need "embedded systems" implementations *autonomous* enough to regulate viability/capturability problems by adequate regulation (feedback) control laws. Viability theory provides algorithms for computing the feedback laws by modular and portable software flexible enough for integrating new problems when they appear (hybrid systems, dynamical games, etc.).
2. ***Systems observed by human brains***, are more difficult to understand since human beings did not design or construct them. Human beings live, think, are involved in socio-economic interactions, but struggle for grasping why and how they do it, at least, why. This happens for instance the following fields:

   - ***economics***, where the viability constraints are the scarcity constraints among many other ones. We can replace the fundamental Walrasian model of resource allocations by decentralized dynamical model in which the role of the controls is played by the prices or other economic decentralizing messages (as well as coalitions of consumers, interest rates, and so forth). The regulation law can be interpreted as the behavior of Adam Smith's invisible hand choosing the prices as a function of allocations of commodities,

- ***finance***, where shares of assets of a portfolio play the role of controls for guaranteing that the values of the portfolio remains above a given time/price dependent function at each instant until the exercise time (horizon), whatever the prices and their growth rates taken between evolving bounds,
- ***dynamical connectionnist networks and/or dynamical cooperative games***, where coalitions of players may play the role of controls: each coalition acts on the environment by changing it through dynamical systems. The viability constraints are given by the architecture of the network allowed to evolve,
- ***population genetics***, where the viability constraints are the ecological constraints, the state describes the phenotype and the controls are genotypes or fitness matrices.
- ***sociological sciences***, where a society can be interpreted as a set of individuals subjected to viability constraints. Such constraints correspond to what is necessary for the survival of the social organization. Laws and other cultural codes are then devised to provide each individual with psychological and economical means of survival as well as guidelines for avoiding conflicts. Subsets of cultural codes (regarded as cultures) play the role of regulation parameters.
- ***cognitive sciences***, in which, at least at one level of investigation, the variables describe the sensory-motor activities of the cognitive system, while the controls translate into what could be called conceptual controls (which are the synaptic matrices in neural networks.)

Theoretical results about the ways of thinking described above are useful for the understanding of non teleological evolutions, of inertia principle, of emergence of new regulons when viability is at stakes, of the role of different types of uncertainties (contingent, tychastic or stochastic), the (re)designing of regulatory institutions (regulated markets when political convention must exist for global purpose, mediation or metamediation of all types, including law, social conflicts, institutions for sustainable development, etc.). And progressively, when more data gathered by these institutions will be available, qualitative (and sometimes quantitative) prescriptions of viability theory may be useful.

1.1.3 Motivations of Viability Theory from Living Systems

Are social and biological systems sufficiently similar to systems currently studied in mathematics, physics, computer sciences or engineering? *Eugene Wigner*'s considerations on *the unreasonable effectiveness of mathematics in the natural sciences* [216, Wigner] are even more relevant in life sciences.

For many centuries, human minds used their potential "mathematical capabilities" to describe and share their "mathematical perceptions" of the world. This mathematical capability of human brains is assumed to be analogous to the language capability. Each child coming to this world uses this specific capability in social interaction with other people to join at each instant an (evolving) consensus on the perception of their world by learning their mother tongue (and few others before this capability fades away with age). We suggest the same phenomenon happens with mathematics. They play the "mathematical role" of *metaphors* that language uses for allowing us to understand a new phenomenon by metaphors comparing it with previously "understood phenomena". Before it exploded recently in a Babel skyscraper, this "mathematical father tongue" was quite consensual and perceived as universal. This is this very universality which makes mathematics so fascinating, deriving mathematical theories or tools motivated by one field to apply them to several other ones. However, apparently, because up to now, the mathematical "father tongue" was mainly shaped by "simple" physical problems of the inert part of the environment, letting aside, with few exceptions, the living world. For good reasons. Fundamental simple principles, such as the *Pierre de Fermat*'s "variational principle", including *Isaac Newton*'s law thanks to *Maupertuis*'s least action principle, derived explanations of complex phenomena from simple principles, as *Ockham*'s razor prescribes: This "law of parsimony" states that an explanation of any phenomenon should make as few assumptions as possible, and to choose among competing theories the one that postulates the fewest concepts. This is the result of an "abstraction process", which is the (poor) capability of human brains that select among the perceptions of the world the few ones from which they may derive many logically or mathematically many other ones. *Simplifying complexity* should be the purpose of an emerging science of complexity, if such a science will emerge beyond its present fashionable status.

So physics, which could be defined as the part of the cultural and physical environment which is understandable by mathematical metaphors, has not yet, in our opinion, encapsulated the mathematical metaphors of living systems, from organic molecules to social systems, made of human brains controlling social activities. The reason seems to be that the adequate mathematical tongue does not yet exist. And the challenge is that before creating it, the present one has to be forgotten, de-constructed. This is quite impossible because mathematicians have been educated *in the same way* all over the world, depriving mathematics from the Darwinian evolution which has operated on languages. This uniformity is the strength and the weakness of present day mathematics: its universality is partial. The only possibility to perceive mathematically living systems wool remain a dream: to gather in secluded convents young children with good mathematical capability, but little training in the present mathematics, under the supervision or guidance of economists or biologists without mathematical training. They possibly could come up with new mathematical languages unknown to us providing the long expected *unreasonable effectiveness of mathematics in the social and biological sciences.*

Even the concept of natural number is oversimplifying, by putting in a same equivalence class so several different sets, erasing their *qualitative* properties or hiding them behind their *quantitative* ones. Numbers, next measurements, and then, statistics, how helpful they are for understanding and controlling the physical part of the environment, may be a drawback to address the qualitative aspects of our world, left to plain language for the quest of elucidation. We may have to return to the origins and explore new "qualitative" routes, without overusing the mathematics that our ancestors accumulated so far and bequeathed to us.

Meanwhile, we are left with this paradox: "simple" physical phenomena are explained by more and more sophisticated and abstract mathematics, whereas "complex" phenomena of living systems use, most of the time, relatively rudimentary mathematical tools. For instance, the mathematical tools used so far did not answer the facts that, for instance,

1. economic evolution is *never* at equilibrium (stationary state),
2. and thus, there were no need that this evolution converges to it, in a stable or unstable way,
3. that elementary cognitive sciences cannot accept the rationality assumption of human brains,
4. and even more that they can be reduced to utility functions, the existence of which was already questioned by *Henri Poincaré* when he wrote to *Léon Walras* that "*Satisfaction is thus a magnitude, but not a measurable magnitude*" (numbers are not sufficient to grasp satisfaction),
5. that uncertainty can be mathematically captured only by probabilities (numbers, again),
6. that chaos, defined as a property of deterministic system, is not fit to represent a nondeterministic behavior of living systems which struggle to remain as stable (and thus, "non chaotic") as possible,
7. that intertemporal optimality, a creation of the human brain to explain some physical phenomena, is not the only creation of Nature, (in the sense that "Nature" created it only through human brains!),
8. that those human brains should complement it by another and more recent principle, *adaptation of transient evolutions to environments*,
9. and so on.

These epistemological considerations are developed in *La mort du devin, l'émergence du démiurge. Essai sur la contingence et la viabilité des systèmes* (*The Demise of the Seer, the Rise of the Demiurge. Essay on contingency, viability and inertia of systems*).

1.1.4 Applications of Viability Theory to... Mathematics

It is time to cross the interdisciplinary gap and to confront and hopefully to merge points of view rooted in different disciplines.

After years of study of various problems of different types, motivated from robotics (and animat theory), game theory, economics, neuro-sciences, biological evolution and, unexpectedly, from financial mathematics, these few relevant features common to all these problems were uncovered, after noticing the common features of the proofs and algorithms.

This history is a kind of *mathematical striptease*, the modern version of what Parmenides and the pre-Socratic Greeks called *a-letheia*, the discovering, un-veiling of the world that surrounds us. This is exactly the drive to "abstraction", isolating, in a given perspective, the relevant information in each concept and investigate the interplay between them. Indeed, one by one, slowly and very shyly, the required properties of the control system were taken away (see Section 18.8, p. 785 for a brief illustration of the Graal of the Ultimate Derivative).

Mathematics, thanks to its abstraction power by isolating only few key features of a class of problems, can help to bridge these barriers as long as it proposes new methods motivated by these new problems instead of applying the classical ones only motivated until now by physical sciences. Paradoxically, the very fact that the mathematical tools useful for social and biological sciences are and have to be quite sophisticated impairs their acceptance by many social scientists, economists and biologists, and the gap menaces to widen.

Hence, viability theory designs and devises a mathematical tool-box universal enough to be efficient in many apparently different problems. Furthermore, using methods that are rooted neither in linear system theory nor in differential geometry, the results

1. hold true for *nonlinear systems*,
2. are *global* instead of being local,
3. and allow an *algorithmic treatment* without loss of information due to the treatment of classical equivalent problems (systems of first-order partial differential equations for instance).

Although viability theory has been designed and developed for studying the evolutions of uncertain systems confronted to viability constraints arising in socioeconomic and biological sciences, as well as in control theory, it had also been used as a mathematical tool for deriving purely mathematical results. These tools enrich the panoply of those diverse and ingenious techniques born out of the pioneering works of Lyapunov and Poincaré more than one century ago. Most of them were motivated by physics and mechanics, not necessarily designed to adaptation problems to environmental or viability constraints. These concepts and theorems provide deep insights in the behavior of complex evolutions that even simple deterministic dynamical systems provide.

Not only the viability/capturability problems are important in themselves in the fields we have mentioned, but it happens that many other important concepts of control theory and mathematics can be formulated in terms of viability kernels or capture basins under auxiliary systems. Although they were designed to replace the static concept of equilibrium by the dynamical concept of viability kernel, to offer an alternative to optimal control problems by introducing the concepts of inertia functions, heavy evolutions and punctuated equilibrium, it happens that nevertheless, the viability results

1. provides new insights and results in the study of Julia sets and Fatou dusts, fractals and Lorenz attractors, all concepts closely related to viability kernels;
2. are useful for providing the final version of the inverse function theorem for set-valued maps;
3. offer new theorems on the existence of equilibria and the study of their asymptotic properties, even for determinist systems, where the concepts of attractors are closely related to the concept of viability kernels;
4. provide tools to study several types of first-order partial differential equations, conservation laws and Hamilton-Jacobi-Bellman equations, following a long series of articles by Hélène Frankowska who pioneered the use of viability techniques, and thus to solve many intertemporal optimization problems.

Actually, the role played by the Viability Theorem in dynamical and evolutionary systems is analogous to the one played by the Brouwer Fixed Theorems in static nonlinear analysis. Numerous static problems solved by the equivalent statements and consequences of this Brouwer Theorem in nonlinear analysis can be reformulated in an evolutionary framework and solved by using viability theory.

The miraculous universality of mathematics is once again illustrated by the fact that some viability tools, inspired by the evolutionary questions raised in life sciences, also became, in a totally unpredicted way, tools to be added to the existing panoply for solving engineering and purely mathematical problems well outside their initial motivation.

1.2 Main Concepts of Viability Theory

Viability theory incorporates some mathematical features of uncertainty without statistical regularity, deals not only with optimality but also with viability and *decisions taken at the appropriate time.* Viability techniques are also *geometric* in nature, but they do not require smoothness properties usually assumed in differential geometry. They not only deal with asymptotic behavior, but also and mainly with *transient* evolutions and capturability of targets in finite or prescribed time. They are *global* instead of local, and truly *nonlinear* since they bypass linearization techniques of the dynamics around equilibria, for instance. They bring other insights to the decipherability of complex, paradoxical and strange dynamical behaviors by providing other types of mathematical results and algorithms. And above all, they have been motivated by dynamical systems arising in issues involving living beings, as well as networks of systems (or organizations, organisms).

In a nutshell, viability theory investigates evolutions

1. in *continuous time, discrete time, or a "hybrid"* of the two when *impulses* are involved,
2. constrained to *adapt* to an environment,
3. evolving under *contingent and/or tychastic uncertainty*
4. using for this purpose *controls*, *regulons* (regulation controls), subsets of regulons, and in the case of networks, connectionnist matrices,
5. *regulated* by *feedback laws* (static or dynamic) that are then "computed" according to given principles, such as the *inertia principle*, intertemporal optimization, etc.,
6. co-evolving with their environment (*mutational and morphological viability*),
7. and corrected by introducing adequate controls (*viability multipliers*) when viability or capturability is at stakes.

1.2.1 Viability Kernels and Capture Basins under Regulated Systems

We begin by defining set-valued maps (see Definition 18.2.1, p. 739):

3 ***[Set-Valued Map]*** *A* set-valued *map $F : X \rightsquigarrow Y$ associates with any $x \in X$ a subset $F(x) \subset Y$ (which may be the empty set $\emptyset$). It is a (single-valued) map $f := F : X \mapsto Y$ if for any x, $F(x) := \{y\}$ is reduced to a single element y. The symbol "$\rightsquigarrow$" denotes set-valued maps whereas the classical symbol "$\mapsto$" denotes single-valued maps.*
The graph $\mathrm{Graph}(F)$ *of a set-valued map F is the set of pairs $(x, y) \in X \times Y$ satisfying $y \in F(x)$. If $f := F : X \mapsto Y$ is a single-valued map, it coincides with the usual concept of graph. The* inverse F^{-1} *of F is the set-valued map from Y to X defined by*

$$x \in F^{-1}(y) \iff y \in F(x) \iff (x, y) \in \mathrm{Graph}(F)$$

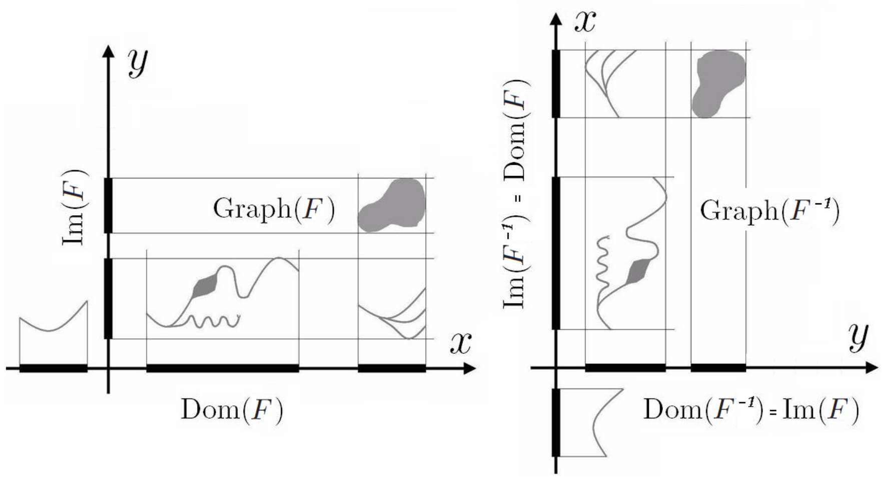

Fig. 1.3 Graphs of a set-valued map and of the inverse map.

The main examples of evolutionary systems, those "engines" providing evolutions, are associated with differential inclusions, which are multi-valued or set-valued differential equations:

> **4** *[**Evolutionary Systems**] Let* $X := \mathbb{R}^d$ *be a finite dimensional space, regarded as the* state space*, and* $F : X \rightsquigarrow X$ *be a set-valued map associating with any state* $x \in X$ *the set* $F(x) \subset X$ *of* velocities *available at state* x*. It defines the* differential inclusion $x'(t) \in F(x(t))$ *(boiling down to an ordinary differential equation whenever* F *is single-valued). An* evolution $x(\cdot) : t \in [0, \infty[\mapsto x(t) \in \mathbb{R}^d$ *is function of time taking its values in a vector space* $\mathbb{R}^d$*. Let* $\mathcal{C}(0, +\infty; X)$ *denote the space of* continuous evolutions *in the state space* X*. The* evolutionary system $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, +\infty; X)$ *maps any initial state* $x \in X$ *to the set* $\mathcal{S}(x)$ *of evolutions* $x(\cdot)$ *starting from* x *and governed by differential inclusion* $x'(t) \in F(x(t))$*. It is* deterministic *if the evolutionary system* $\mathcal{S}$ *is single-valued and* non deterministic *if it is set-valued.*

The main examples of differential inclusions are provided by

> **5** *[**Parameterized Systems**] Let* $\mathcal{U} := \mathbb{R}^c$ *be a space of parameters. A parameterized system is made of two "boxes":*
> *1 - The* "input-output box" *associating with any evolution* $u(\cdot)$ *of the parameter (*input*) the evolution governed by differential equation* $x'(t) =$

$f(x(t), u(t))$ *starting from an initial state* (open loop),
2 - The non deterministic "output-input box"*, associating with any state a subset* $U(x)$ *of parameters (*output*).*
It defines the set-valued map F *associating with any* x *the subset* $F(x) := \{f(x,u)\}_{u \in U(x)}$ *of velocities parameterized by* $u \in U(x)$*. The associated evolutionary system* $\mathcal{S}$ *maps any initial state* x *to the set* $\mathcal{S}(x)$ *of evolutions* $x(\cdot)$ *starting from* x *(*$x(0) = x$*) and governed by*

$$x'(t) \;=\; f(x(t), u(t)) \text{ where } u(t) \;\in\; U(x(t)) \tag{1.1}$$

or, equivalently, to differential inclusion $x'(t) \in F(x(t))$.

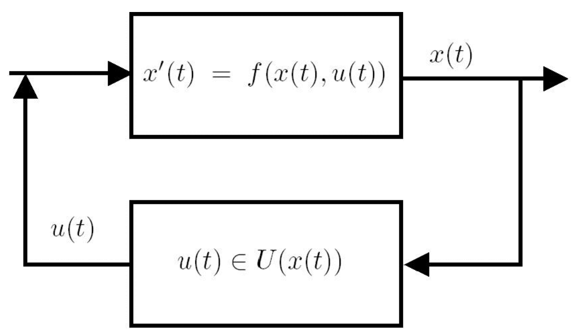

Fig. 1.4 Parameterized Systems.
The input-output and output-input boxes of a parameterized systems are depicted in this diagram: the controlled dynamical system at the input-output level, the "cybernetical" map imposing state-dependent constraints on the control at the output-input level.

The parameters range over a state-dependent *"cybernetic" map* $U : x \rightsquigarrow U(x)$, providing the system opportunities to adapt at each state to viability constraints (often, as slowly as possible) and/or to regulate intertemporal optimal evolutions.

The nature of the parameters differs according to the problems and to questions asked: They can be

- "controls", whenever a controller or a decision maker "pilots" the system by choosing the controls, as in engineering,
- "regulons" or regulatory parameters in those natural systems where no identified or consensual agent acts on the system,
- "tyches" or disturbances, perturbations under which nobody has any control.

We postpone to Chapter 2, p. 55 more details on these issues.

To be more explicit, we have to introduce formal definitions describing the concepts of viability kernel of an environment and capturability of a target under a evolutionary system.

6 *[**Viability and Capturability**] If a subset $K \subset \mathbb{R}^d$ is regarded as an* environment *(defined by* viability constraints*), an evolution $x(\cdot)$ is said to be* viable in the environment $K \subset \mathbb{R}^d$ on an interval $[0, T[$ *(where $T \leq +\infty$) if for every time $t \in [0, T[$, $x(t)$ belongs to K.*
If a subset $C \subset K$ is regarded as a target, an evolution $x(\cdot)$ captures C *if there exists a finite time T such that the evolution is viable in K on the interval $[0, T[$ until it reaches the target at $x(T) \in C$ at time T.* *See Definition 2.1.3, p. 61.*

Viability and capturability are the main properties of evolutions that are investigated in this book.

1.2.1.1 Viability kernels

We begin by introducing the *viability kernel of an environment* under an evolutionary system associated with a nonlinear parameterized system.

7 *[**Viability Kernel**] Let K be an environment and $\mathcal{S}$ an evolutionary system. The* viability kernel of K under the evolutionary system $\mathcal{S}$ *is the set* $\mathrm{Viab}_{\mathcal{S}}(K)$ *of initial states $x \in K$ from which starts at least one evolution $x(\cdot) \in \mathcal{S}(x)$ viable in K for all times $t \geq 0$:*

$$\mathrm{Viab}_{\mathcal{S}}(K) := \{x_0 \in K \mid \exists x(\cdot) \in \mathcal{S}(x_0) \;\; \textit{such that} \; \forall t \geq 0, \; x(t) \in K\}$$

Two extreme situations deserve to be singled out: The environment is said to be

1. viable *under $\mathcal{S}$ if it is equal to its viability kernel:* $\mathrm{Viab}_{\mathcal{S}}(K) = K$,
2. a repeller *under $\mathcal{S}$ if its viability kernel is empty:* $\mathrm{Viab}_{\mathcal{S}}(K) = \emptyset$.

It is equivalent to say that all evolutions starting from a state belonging to the complement of viability kernel in K leave the environment *in finite time.*

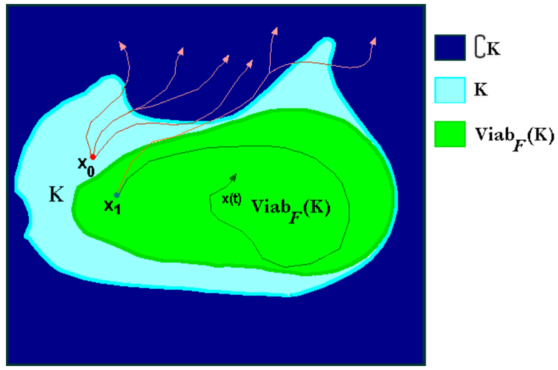

Fig. 1.5 Illustration of a Viability Kernel.
From a point x_1 in the viability kernel of the environment K starts ***at least*** *one evolution viable in K forever.* ***All*** *evolutions starting from $x_0 \in K$ outside the viability kernel leave K in finite time.*

Hence, the viability kernel plays the role of a *viabilimeter*, the *"size"* of which measuring the degree of viability of an environment, so to speak.

1.2.1.2 Capture Basins

8 ***[Capture Basin Viable in an Environment]*** *Let K be an environment, $C \subset K$ be a target and $\mathcal{S}$ an evolutionary system. The* capture basin of C (viable in K) under the evolutionary system $\mathcal{S}$ *is the set* $\mathrm{Capt}_{\mathcal{S}}(K, C)$ *of initial states $x \in K$ from which starts at least one evolution $x(\cdot) \in \mathcal{S}(x)$ viable in K on $[0, T[$ until the finite time T when the evolution reaches the target at $x(T) \in C$.*

For simplicity, we shall speak of capture basin without explicit mention of the environment whenever there is no risk of confusion.

It is equivalent to say that, starting from a state belonging to the complement in K of the capture basin, **all** evolutions remain outside the target until they leave the environment K.

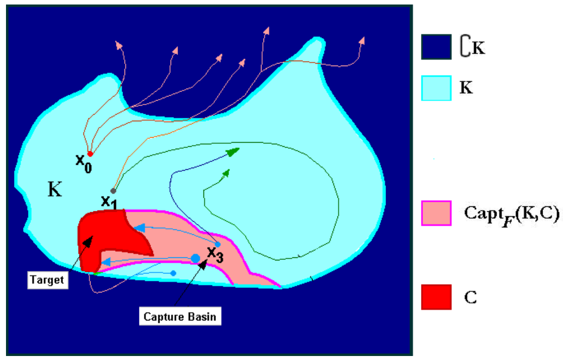

Fig. 1.6 Figure of a Capture Basin.
From a state x_3 in the capture basin of the target C viable in the environment K starts ***at least*** *one evolution viable in K until it reaches C in finite time.* ***All*** *evolutions starting from $x_1 \in K$ outside the capture basin remain outside the target C forever or until they leave K.*

The concept of capture basin of a target requires that at least one evolution reaches the target in finite time, and *not only asymptotically*, as it is usually studied with concepts of *attractors* since the pioneering works of *Alexandr Lyapunov* going back to 1892.

1.2.1.3 Designing Feedbacks

The theorems characterizing the viability kernel of an environment of the capture basin of a target provide also a regulation map $x \rightsquigarrow R(x) \subset U(x)$ regulating evolutions viable in K:

> **9** *[**Regulation Map**] Let us consider control system*
>
> $$\begin{cases} (i) \;\; x'(t) = f(x(t), u(t)) \\ (ii) \; u(t) \in U(x(t)) \end{cases}$$
>
> *and an environment K. A set-valued map $x \rightsquigarrow R(x) \subset U(x)$ is called a* regulation map governing viable evolutions *if the viability kernel of K is invariant under the control system*
>
> $$\begin{cases} (i) \;\; x'(t) = f(x(t), u(t)) \\ (ii) \; u(t) \in R(x(t)) \end{cases}$$

Actually, we are looking for single-valued regulation maps governing viable evolutions, which are usually called feedbacks:

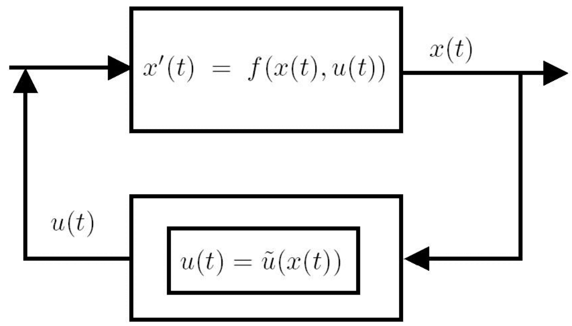

Fig. 1.7 Feedbacks.
The single-valued maps $x \mapsto \widetilde{u}(x)$ *are called the* feedbacks *(or servomechanisms, closed loop controls, etc.) allowing to pilot evolutions by using controls of the form* $u(t) := \widetilde{u}(x(t))$ *in system* $\mathcal{S}$ *defined by (1.1), p. 14:* $x'(t) = f(x(t), u(t))$ where $u(t) \in U(x(t))$. *Knowing such a feedback, the evolution is governed by ordinary differential equation* $x'(t) = f(x(t), \widetilde{u}(x(t)))$.

Hence, knowing the regulation map R, *viable feedbacks* are single-valued regulation maps.

Building viable feedbacks, combining prescribed feedbacks to govern viable evolutions, etc., are issues investigated in Chapter 11, p. 455.

1.2.2 Viability Kernel and Capture Basin Algorithms

These kernels and basins can be approximated by discrete analog methods and computed numerically thanks to *Viability Kernel and Capture Basin Algorithms*. Indeed, all examples shown in this introductory chapter have been computed using the software packages based on these algorithms, developed by LASTRE (Laboratoire d'Applications des Systèmes Tychastiques Régulés). These algorithms compute not only the viability kernel of an environment or the capture basin of a target, but also the regulation map and viable feedbacks regulating evolutions viable in the environment (until reaching the target if any). By opposition to *"shooting methods"* using classical solvers of differential equations, these *"viability algorithms"* allow evolutions governed by these feedbacks to remain viable. Shooting methods do not provide the corrections needed to keep the solutions viable in given sets. This is one of the reasons why shooting methods cannot compute evolutions viable in attractors, or in stable manifolds, in fractals of Julia sets, even in the case of discrete and deterministic systems, although the theory states that they should remain viable in such states. Viability algorithms allow us to govern evolutions satisfying these viability requirements: They are designed to do so.

1.2.3 Restoring Viability

There is no reason why an arbitrary subset K should be viable under a control system. The introduction of the concept of viability kernel does not exhaust the problem of restoring viability. One can imagine several other methods for this purpose:

1. Keep the constraints and change initial dynamics by introducing regulons that are "viability multipliers";
2. Keep the constraints change the initial conditions by introducing a *reset map* Φ mapping any state of K to a (possibly empty) set $\Phi(x) \subset X$ of new "initialized states" (*impulse control*);
3. Keep the same dynamics and let the set of constraints evolve according to *mutational equations* (see *Mutational and morphological analysis: tools for shape regulation and morphogenesis*, [23, Aubin] and *Mutational Analysis*, [150, Lorenz]).

Introductions to the two first approaches, viability multipliers and impulse systems, are presented in Chapter 12, p.505.

1.3 Examples of Simple Viability Problems

In order to support our claim that viability problems are more or less hidden in a variety of disciplines, we have selected a short list of very simple examples which can be solved in terms of viability kernels and capture basins developed in this monograph.

> **10** ***Heritage.*** *Whenever the solution to a given problem can be formulated in terms of viability kernel, capture basin, or any of the other combinations of them, this solution inherits the properties of kernels and basins gathered in this book. In particular, the solution can be computed by the Viability Kernel and Capture Basin Algorithms.*

Among these problems, some of them can be formulated directly in terms of viability kernels or capture basins. We begin our gallery by those ones.

However, many other problems are viability problems in disguise. They require some mathematical treatment to uncover them, by introducing auxiliary environments and auxiliary targets under auxiliary systems. After some accidental discoveries and further development, unexpected and unsuspected examples of viability kernels and capture basins have emerged in optimal control theory. Optimal control problems of various types appear in engineering, social sciences, economics and other fields. They have been investigated with methods born out of the study of calculus of variations by Hamilton and Jacobi, adapted to differential games and control problems by Isaacs and Bellman (under the name *"dynamical programming"*). The ultimate aim is to provide regulation laws allowing us to pilot optimal evolutions. They are derived from the derivatives of the *"value function"*, which is classically a solution to the Hamilton-Jacobi-Bellman partial differential equations in the absence of viability constraints on the states. It happens that these functions can be characterized by means of viability kernels of auxiliary environments or of capture basins of auxiliary targets under auxiliary dynamical systems.

The same happy events happened also in the field of systems of first-order partial differential equations. These examples are investigated in depth in Chapter 4, p. 139, Chapter 6, p. 213, Chapter 16, p. 651 and Chapter 17, p. 701. They are graphs of solutions of systems of partial differential equations of first order or of Hamilton-Jacobi-Bellman partial differential equations of various types arising in optimal control theory. The feature common to those problems is that environments and targets are either graphs of maps from a vector space to another or epigraphs of extended functions.

All examples which are presented have been computed using the Viability Kernel or Capture Basin Algorithms, and represent benchmark examples which can now be solved in a standard manner using viability algorithms.

1.3.1 Engineering Applications

A fundamental problem of engineering is "obstacle avoidance", which appears in numerous application fields.

1.3.1.1 Rallying a Target while Avoiding Obstacles

Chapter 5, p. 193, *Avoiding Skylla and Charybdis*, illustrates several concepts on the same Zermelo navigation problems of ships aiming to rally a harbor while avoiding obstacles called (Skylla and Charybdis in our illustration, in reference to the Homer *Odyssey*). The environment is the see, the target the harbor. The chapter presents the computations of

1. the domain of the *minimal length function* (see Definition 4.3.1, p.154) which is contained in the viability kernel,

2. the complement of the viability kernel which is the domain of the *exit function*, measuring the largest survival time before the ship sinks (see Definition 4.2.3, p.149);
3. the *capture basin of the harbor*, which is the domain of the *minimal time function*;
4. the *attraction basin*, which is the domain of the *Lyapunov function*;
5. the *controllability basin*, which is the domain of the value function of the optimal control problem minimizing the intertemporal (square of the) velocity.

For each of these examples, the graph of the feedback maps associating with each state the velocity and the steering direction of the ship are computed, producing evolutions (viable with minimal length, non viable *persistent evolutions*, *minimal time evolutions* reaching the harbor in finite time, *Lyapunov evolutions* asymptotically converging to the harbor and *optimal evolutions* minimizing the intertemporal criterion). In each case, examples of trajectories of such evolutions are displayed.

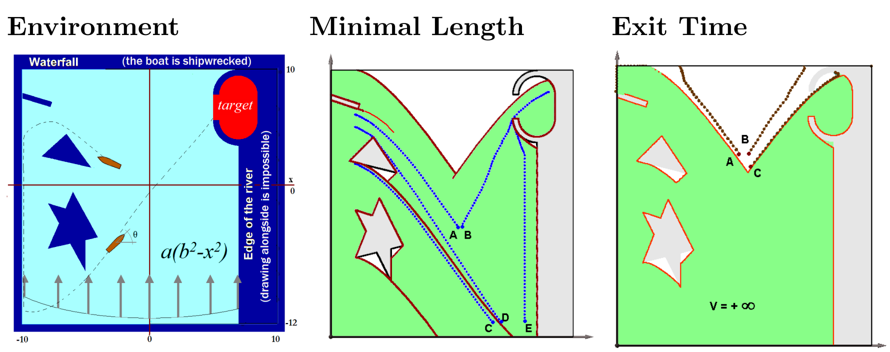

Fig. 1.8 Minimal Length and Exit Time Functions.
The left figure describes the irregular environment (the complement of the two obstacles and the dyke in the square) and the harbor, regarded as the target. The wind blows stronger in the center than near the banks, where it is weak enough to allow the boat to sail south. The figure in the center displays the viability kernel of the environment, which is the domain of the minimal length functions. The viability kernel is empty north of the waterfall and south of the two mains obstacles and of the harbor on the east. The feedback map governs the evolution of minimal length evolutions which converge to equilibria. The presence of obstacles implies three south-east to north-west discontinuities. For instance, trajectories starting from C and D go north and south of the obstacle. The set of equilibria is not connected: some of them are located in a vertical line near the west bank, the other ones on a vertical line in the harbor. The trajectory of the evolution starting from A sails to an equilibrium near the west bank and from B sails to an

equilibrium in the harbor. The figure on the right provides trajectories of non viable evolutions which are persistent in the sense that they maximize their exit time from the environment. The ones starting from A and B exit through the waterfall, the one starting from C exit through the harbor. The complement of the viability kernel is the domain of the exit time function.

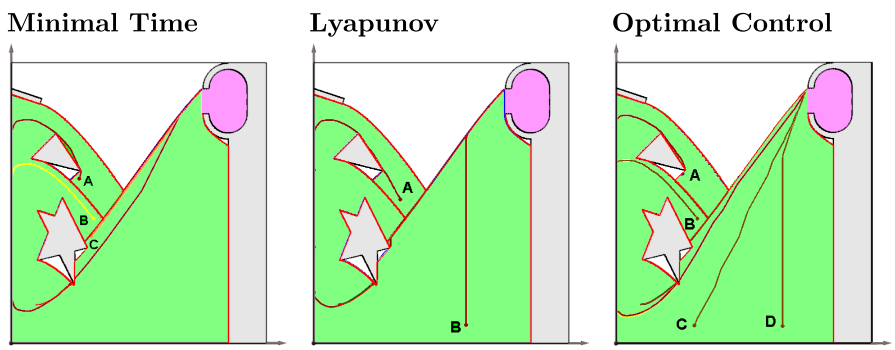

Fig. 1.9 Minimal Time, Lyapunov and Optimal Value Functions.
These three figures display the domains of the minimal time function, which is the capture basin of the harbor, of the Lyapunov function, which is the attraction basin and of the value function of an optimal control problem (minimizing the cumulated squares of the velocity). These domains are empty north of the waterfall and south of the two mains obstacles and of the harbor on the east. The presence of obstacles induces three lines of discontinuity, delimitating evolutions sailing north of the northern obstacle, between the two obstacles and south or east of the southern obstacle, as it is shown for the evolutions starting from A, B and C in the capture basin, domain of the minimal time function. The trajectories of the evolutions starting from A and B are the same after they leave the left bank for reaching the harbor.

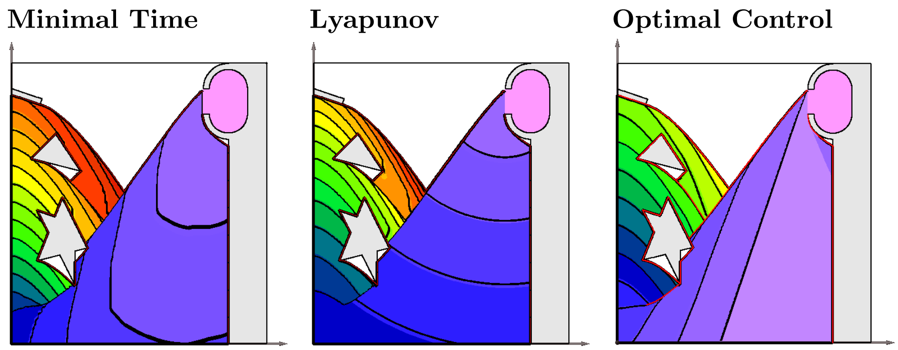

Fig. 1.10 Isolines of the minimal time, Lyapunov and value functions.
These figures display the isolines of these functions on their respective domains, indicating in the color scale the value of these functions. For the minimal time function, the level curves provide the (minimum) time needed to reach the target.

1.3.1.2 Autonomous Navigation in an Urban Network

Viability concepts have not only be simulated for a large variety of problems, but been implemented in field experiments. This is the case of obstacle avoidance on which an experimental robot (Pioneer 3AT of *activmedia robotics*) has been programmed to reach a target from any point of its capture basin (see Section 3.1, p.119). The environment is a road network, on which a target to be reached in minimal time has been assigned. Knowing the dynamics of the pioneer robot, both the capture basin and, above all, the feedback map, have been computed. The graph of the feedback (the command card) has been embedded in the navigation unit of the robot. Two sensors (odometers and GPS) tell the robot were it is, but the robot does not use sensors locating the obstacles nor the target.

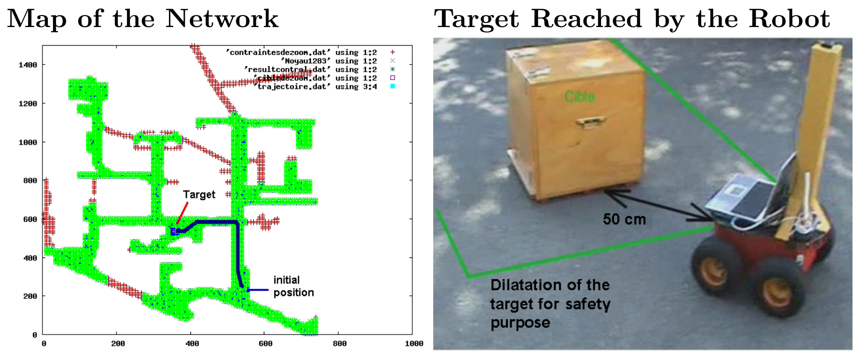

Fig. 1.11 Experiment of the Viability Algorithm for Autonomous Navigation.
The left figure displays the trajectory of the robot in the map of the urban network. The right figure is a close up photograph of the robot near the target. Despite the lack of precision of the GPS, the trajectory of the robot followed exactly the one which was simulated.

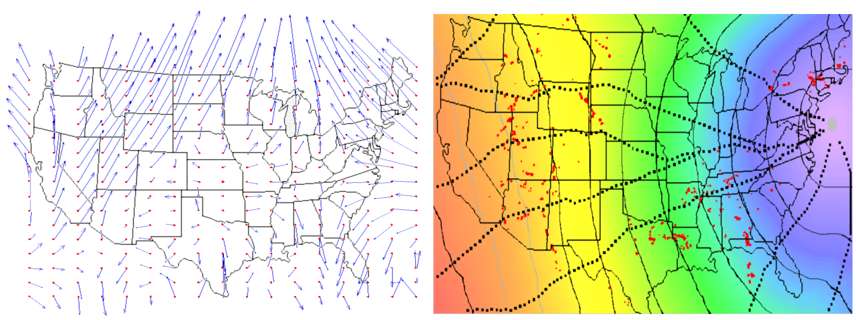

Fig. 1.12 Wind-optimal High Altitude Routes.
The left figure shows the wind-field over the continental US (source: NOAA). The right figure displays the wind-optimal routes avoiding turbulence to reach the New York airport in minimal time.

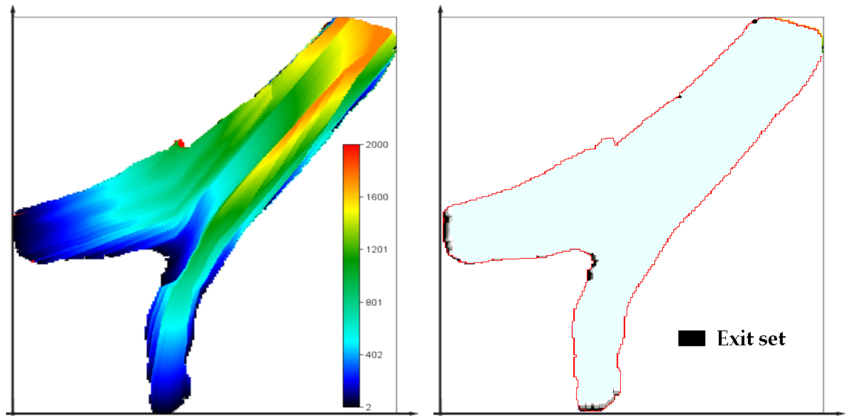

Fig. 1.13 Persistent Evolutions of Drifters and their Exit Sets.
The figure displays the exit time function and the exit sets of drifters in the Georgianna Slough of the San Francisco Bay Area (see Figure 1.12, p.24). The exit sets (right subfigure) are the places of the boundary of the domain of interest where these drifters can be recovered. The level-curves of the exit time function are indicated in the left subfigure. In this example, drifters evolve according to currents (see Section 3.3, p.132).

1.3.1.3 Safety Envelopes for Landing of Plane

The state variables are the velocity V, the flight path angle γ and the altitude z of a plane. We refer to Section 3.2, p. 122 for the details concerning the flight dynamics. The environment is the "flight envelope" taking into account the fight constraints and the target is the "zero altitude" subset of the flight envelope, called the *"touch down envelope"*.

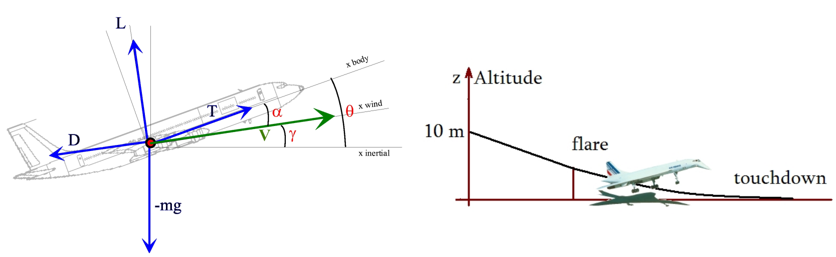

Fig. 1.14 Model of the touch down manoeuver of an aircraft.

In order to ensure safety in the last phase of landing, the airplane must stay within a "flight envelope" which plays the role of environment.

The *Touch Down Envelope* is the set of flight parameters at which it is safe to touch the ground. The *touch down safety envelope* is the set of altitude, velocities and path angles from which one can reach the touch down envelope. The touch down safety envelope can then be computed with the viability algorithms (for the technical characteristics of a DC9-30) since it is a capture basin:

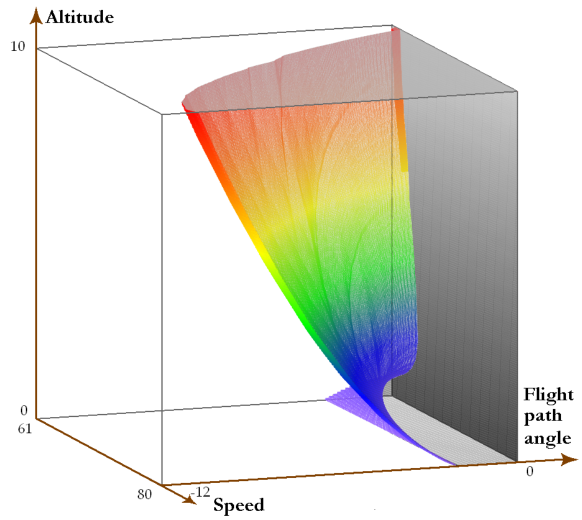

Fig. 1.15 The "Touch Down Safety Envelope.
The 3-d boundary of the safety envelope is displayed. The colorimetric scale indicates the altitude.
This topic is developed in Section 3.2, p. 122.

1.3.2 Environmental Illustrations

1.3.2.1 Management of Renewable Resources

This example is fully developed in Section 9.1, p. 335 and Section 7.2, p. 276.

The management of renewable resources requires both dynamics governing the renewable resource (fishes) and the economics (fisheries), and biological and economic viability constraints..

The following classical example (called Verhulst-Schaeffer example) assumes that the evolution of the population of a fish species $x(t)$ is driven by the logistic Verhlust differential equation $x'(t) = rx(t)\left(1 - \frac{x(t)}{b}\right)$ (see Section 7.1, p. 262) and that its growth rate is depleted proportionally to the fishing activity $y(t)$: $x'(t) = r(t)y(t)$ (see Section 7.2, p. 276). The controls are the velocities $v(t) = y'(t)$ of the fishing activity (see (7.6), p. 277).

The ecological[2] environment is made of number of fishes above a certain threshold in order to survive (on the right of the black area on Figure 7.8). The economic environment is the subset of the ecological environment, since, without renewable biological resources there is no economic activity whatsoever. In this illustration, the economic environment is the subset above the hyperbola: fishing activity is not viable (grey area) outside of it.

We shall prove that the economic environment is partitioned into three zones:

- *Zone* (1), where economic activity is consistent with the ecological viability: it is the ecological and economical paradise;
- *Zone* (2), an economical purgatory, where economic activity will eventually disappear, but can revive later since enough fishes survive (see the trajectory of the evolution starting form A);
- *Zone* (3), the ecological and economical hell, where economic activity leads both to the eventual bankruptcy of fisheries and eventual extinction of fishes, like the sardines in Monterey after an intensive fishery activity during the 1920-1940s before disappearing in the 1950s together with Cannery Row.

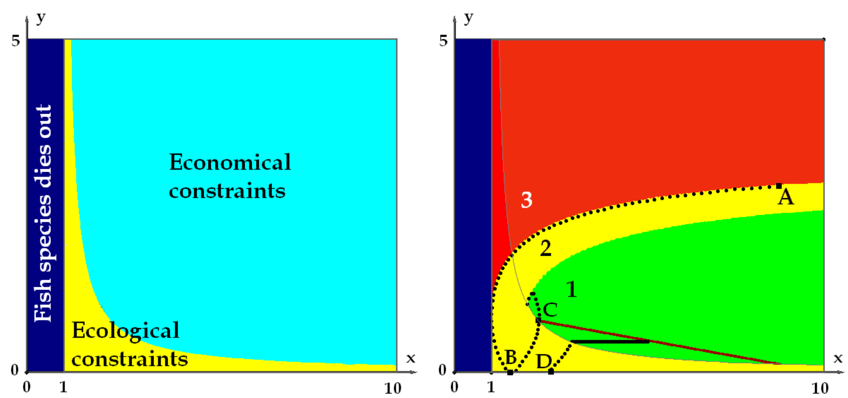

Fig. 1.16 Permanence Basin of a Sub-environment.
Zone (1) *is the* viability kernel *of the economic environment and* Zone (2) *is the* permanence kernel *(see Definition 9.1.3, p.337) of the economical) environment, which is viable subset of the ecological environment. In* Zone (3), *economic activity leads both to the bankruptcy of fisheries and extinction of fishes.*

[2] the root *"eco"* comes from the classical Greek *"oiko"*, meaning house.

We shall see in Section 7.2.3, p. 283 that the concept of crisis function measures the shortest time spent outside the target, equal to zero in the viability kernel, finite in the permanence kernel and infinite outside of the permanence kernel: See Figure 7.8, p. 283.

1.3.2.2 Climate-Economic Coupling and the Transition Cost Function

This example is a metaphor of Green House Gas emissions. For stylizing the problems and providing a three-dimensional illustration, we begin by isolating three variables defining the state:

- the concentration $x(t)$ of greenhouse gases,
- the short-term pollution rate $y(t) \in \mathbb{R}_+$,
- generated by economic activity summarized by a macro-economic variable $z(t) \in \mathbb{R}_+$ representing the overall economic activity producing emissions of pollutants.

The control represents an economic policy taken here as the velocity of the economic activity.

The ultimate constraint is the simple limit b on the concentration of the greenhouse gases: $x(t) \in [0, b]$.

We assume known the dynamic governing the evolution of the concentration of greenhouse gases, the emission of pollutants and of economic activity, depending, in the last analysis, on the economic policy, describing how one can slow or increase the economic activity. The question asked is how to compute the transitions cost (measured, for instance, by the intensity of the economic policy). Namely, transition cost is defined as the largest intensity of the economic activities consistent with the bound b on concentration of greenhouse gases. The transition cost function is the smallest transaction cost over the intensities of economic policies. This is a crucial information to know what would be the price to pay for keeping pollution under a given threshold. Knowing this function, we can derive, for each amount of transition cost, what will be the three-dimensional subset of triples (concentration-emission-economic activity) the transition costs of which are smaller than or equal to this amount.

We provide these subsets (called level-sets) for four amounts c, including the most interesting one, $c = 0$, for an example of dynamical system:

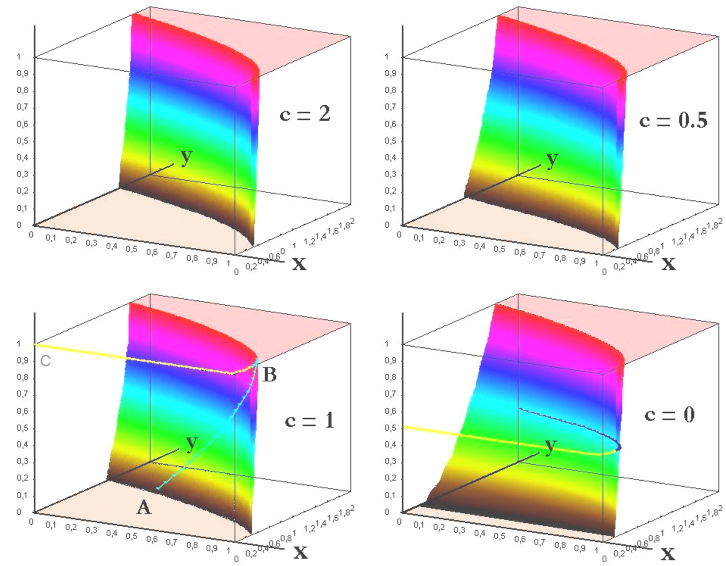

Fig. 1.17 Level-sets of the Transition Function.
We provide the level-sets of the transition function for the amounts $c = 0,\ 0.5,\ 1\ \&\ 2$. *The trajectories of some evolutions are displayed (see The Coupling of Climate and Economic Dynamics: Essays on Integrated Assessment, [113, Haurie & Viguier]).*

Since this analysis involves macroeconomic variables which are not really defined and the dynamics of which are not known, this illustration has only a qualitative and metaphoric value, yet, instructive enough.

1.3.3 Strange Attractors and Fractals

1.3.3.1 Lorenz Attractors

We consider the celebrated Lorenz system (see Section 9.2, p. 363), known for the "chaotic" behavior of some of its evolutions and the "strange" properties of its attractor. *The classical representations of the Lorenz attractor are not the mathematical attractor*, though, because "shooting methods" commonly used to approximate the attractor by taking the union of the trajectories computed for a finite number of iterations (computers do not experiment infinity). Theorem 9.2.12, p.371 states that the Lorenz attractor is contained in the backward viability kernel (the viability kernel under the Lorenz system with the opposite sign). If the solution starts outside the backward viability theorem, then it cannot reach the attractor in finite time, but only approaches it asymptotically (see Figure 9.12, p.363). Hence, the first task is to study and compute this backward viability kernel containing the attractor:

However, the attractor being invariant, evolutions starting from the attractor are viable in it. Unfortunately, the usual *shooting methods* do not provide a good approximation of the Lorenz attractor. Even when one element the attractor is known, the extreme sensitivity of the Lorenz system on initial conditions forbid the usual *shooting methods* to govern evolutions viable in the attractor.

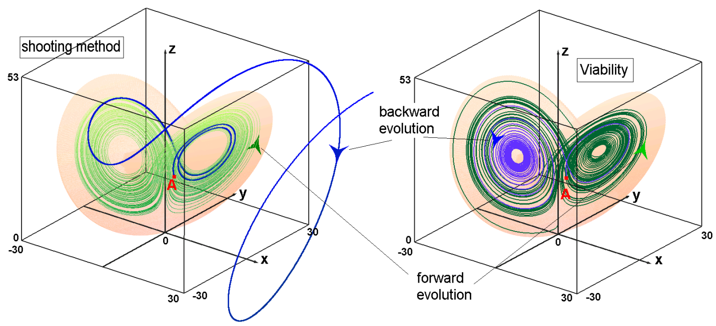

Fig. 1.18 Viable Evolutions on the Backward Viability Kernel.
Knowing that the attractor is contained in the backward viability kernel computed by the viability algorithm and is forward invariant, the viability algorithm provides evolutions viable in this attractor, such as the one displayed in the figure on the left. It "corrects" the plain shooting method using a finite-difference approximation of the Lorenz systems (such as the Runge-Kutta method used in this example) which are too sensitive to the round-up errors and "tame" viable evolutions (which loose their wild behavior on the attractor), as can be shown in the left figure. The viability kernel algorithm providing a feedback governing viable evolution, the right figure displays an evolution arriving at A and starting from A. Contrary to the shooting method, this evolution is viable.

The Lorenz system has fascinated mathematicians, who are interested to the evolutions converging to the three equilibria of this system when t goes to ∞ or $-\infty$. They are called "clines" and studied in Section 9.1.8, p.358. The following figure displays the menagerie of clines of the Lorenz system:

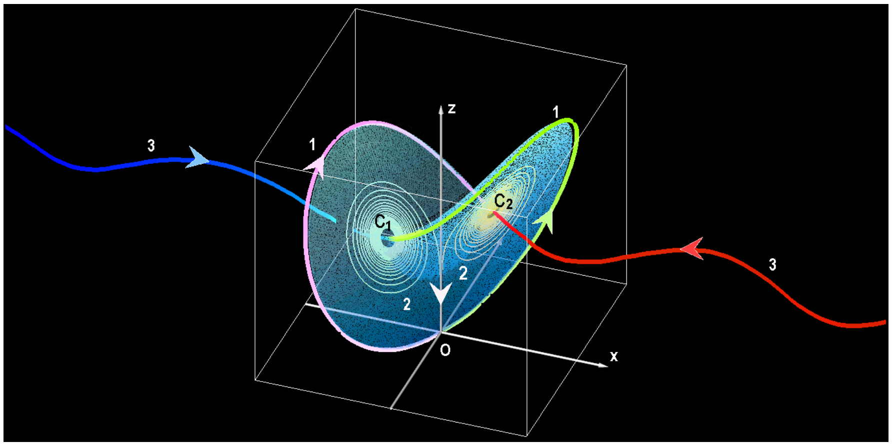

Fig. 1.19 The set of heteroclines and exoclines for the Lorenz system. *Heteroclines 1 are particular evolutions connecting the local stable manifold of e_1 or e_2 and the local unstable manifold around the origin 0 while heteroclines 2 are particular evolutions connecting the local unstable manifold of e_1 or e_2 and the local stable manifold around the origin 0. Heteroclines 1 are obtained by computing the backward viability kernel of the complement of a neighborhood of e_1 (or e_2). This excludes all evolutions which fluctuate and pass close to the e_1 (or e_2). Heteroclines 2 are obtained by computing the backward viability kernel of the forward viability kernel of the half cube K_1 (or K_2). This excludes all evolutions which fluctuate between K_1 (or K_2). Exoclines 3 arrive to the stable manifolds of e_1 or e_2 which do not belong the backward viability kernel.*

1.3.3.2 Fluctuations under the Lorenz system

Another feature of the wild behavior of evolutions governed by the Lorenz system is their "fluctuating property": most of them flip back and forth from one area to another. This is not the case of all of them, though, since the evolutions starting from equilibria remain in this state, and therefore, do not fluctuate. We shall introduce the concept of *fluctuation basin* (see Definition 9.1.1, p.336), which is the set of initial states from which starts a fluctuating evolution. It can be formulated in terms of viability kernels and capture basins, and thus computed with a viability algorithm.

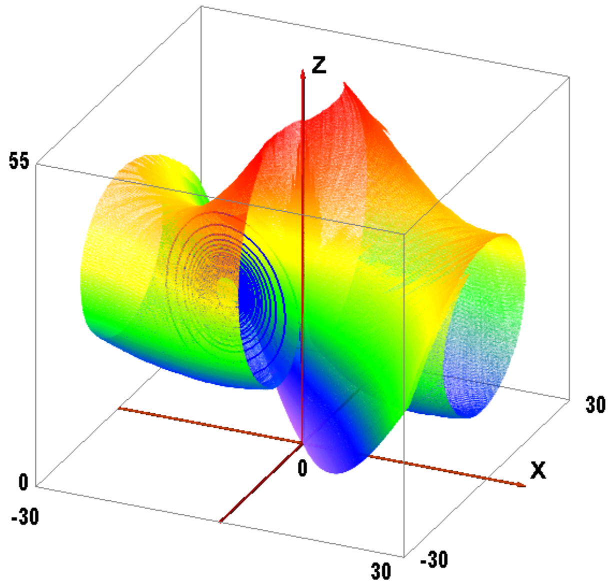

Fig. 1.20 Fluctuation Basin of the Lorenz System.
The figure displays the (very small) complement *of the fluctuation: outside of it, evolutions flip back and forth from one half-cube to the other. The complement of the fluctuation basin is viable.*

1.3.3.3 Fractals properties of some Viability Kernels

Viability kernels of compact sets under a class of discrete set-valued dynamical systems are Cantor sets having fractal dimensions, as it is explained in Section 2.8.4, p. 91. Hence they can be computed with the Viability Kernel Algorithm, which provides an exact computation up to the pixel, instead of approximations as it is usually obtained by "shooting methods".

This allows us to revisit some classical examples, among which we quote here the *Julia sets*, studied in depth for more than a century. They are (the boundaries of) the subsets of initial complex numbers z such that their iterates $z_{n+1} = z_n^2 + u$ (or, equivalently, setting $z = x + iy$ ad $u = a + ib$, $(x_{n+1}, y_{n+1}) = (x_{n+1}^2 - y_{n+1}^2 + a, 2x_{n+1}y_{n+1} + b))$, remain in a given ball. As it is clear from this definition, Julia sets are (boundaries) of viability kernels, depending on the parameter $u = a + ib$.

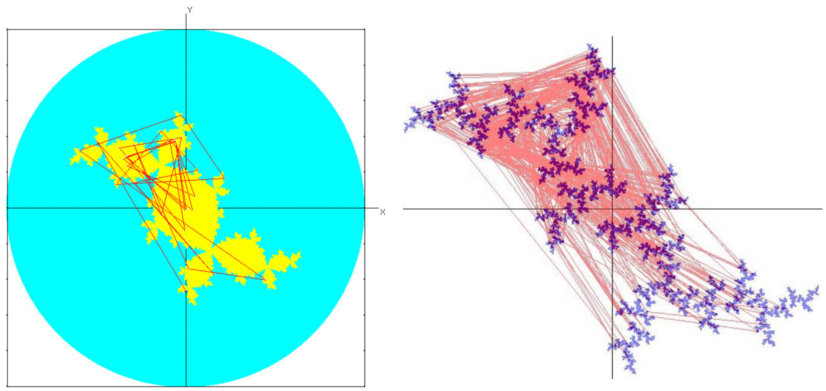

Fig. 1.21 Julia Sets and Fractals.
Unlike "shooting methods", the viability kernel algorithm provides the exact viability kernels and the viable iterates for two values of the parameter u. On the left, the viability kernel as a non-empty interior and its boundary, the Julia set, *has fractal properties. The second example, called* Fatou dust, *having an empty interior, coincides with its boundary, and thus is a Julia set. The discrete evolutions are not governed by the standard dynamics, which are sensitive to round-up errors, but by the corrected one, which allows the discrete evolutions to be viable on the Julia sets.*

Actually, viability kernel algorithms provide also the Julia set, which is the boundary of the viability kernel, equal to another boundary kernel thanks to Theorem 9.1.18, p. 353:

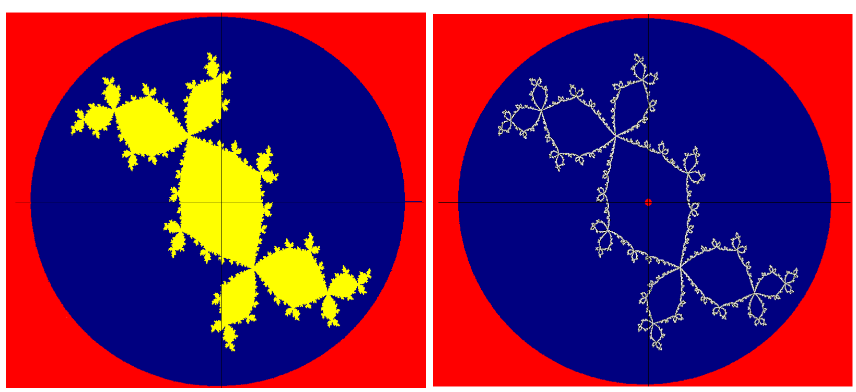

Fig. 1.22 Julia sets and Filled-in Julia sets.
The left figure reproduces the left figure of Figure 2.4, which is the filled-in Julia set, *the boundary of which is the* Julia set *(see Definition 2.8.6,*

p.88). We shall prove that this boundary is itself a viability kernel (see Theorem 9.1.18, p. 353), which can then be computed by the viability kernel algorithm.

1.3.3.4 Computation of Roots of Nonlinear Equations

The set of all roots of a nonlinear equation located in a given set A can be formulated as the viability kernel of an auxiliary set under an auxiliary differential equation (see Section 4.3, p. 154), and thus, can be computed with the Viability Kernel Algorithm. We illustrate this for the map $f : \mathbb{R}^3 \mapsto \mathbb{R}$ defined by

$$f(x, y, z) = (|x - \sin z| + |y - \cos z|) \cdot (|x + \sin z| + |y + \cos z|)$$

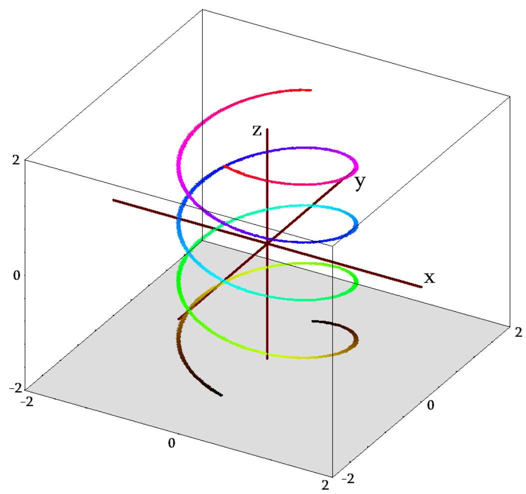

Fig. 1.23 The Set of Roots of Nonlinear Equations.
The set of roots of the equation

$$(|x - \sin z| + |y - \cos z|) \cdot (|x + \sin z| + |y + \cos z|) \; = \; 0$$

is a double helix. It is recovered in this example by using the viability kernel algorithm.

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