Belman equation

LTI

$\dot{x}(t) = a(x(t), u(t))$

t_f is fixed

$u \in U$

$$J = h(x(t_f)) + \int_0^{t_f} g(x(t), u(t))\,dt$$

N equally spade time increments in the interval

$0 \leq t \leq t_f$

$$\frac{x(t+\Delta t) - x(t)}{\Delta t} \approx a(x(t), u(t))$$

short handed notation
K

$$x(t+\Delta t) = x(t) + \Delta t\, a(x(t), u(t))$$

$$x(K+1) = \underbrace{x(k) + \Delta t\, a(x(k), u(k))} \rightarrow a_D(x(k), u(k))$$

for simplifying the notation. It is assumed the system is LTI

$$J = h(x(N\Delta t)) + \int_0^{\Delta t} g\,dt + \int_{\Delta t}^{2\Delta t} g(\cdot)\,dt + \cdots + \int_{(N-1)\Delta t}^{N\Delta t} g(\cdot)\,dt$$

$$J \approx h(x(N)) + \Delta t \sum_{K=0}^{N-1} g(x(k), u(k)),$$

$$J = h(x(N)) + \sum_{K=0}^{N-1} g_D(x(k), u(k))$$

$J_{NN}(x(N)) \triangleq h(x(N)) \rightarrow$ Cost of reaching the final state value $x(N)$

interval last — state last corresponding to N.

$$J_{N-1,N}(x(N-1), u(N-1)) \triangleq g_D(x(N-1), u(N-1)) + h(x(N))$$

$$= g_D(\cdot) + \underline{J_{NN}(x(N))} \rightarrow J_{NN}(a_D(x(N-1), u(N-1)))$$

$u^*(t) = f(x(t), t)$

$$J^*_{N-1,N}(x(N-1)) \triangleq \min_{u(N-1)} \{ g_D(x(N-1), u(N-1)) + J_{NN}(a_D(x(N-1), u(N-1)))$$

$U(N-1)$ depends on $x(N-1)$

$$J_{N-2,N}(x(N-2), u(N-2), U(N-1)) = g_D(x(N-2), U(N-2)) +$$

$$g_D(x(N-1), U(N-1)) + \underbrace{h(x(N))} \longrightarrow J_{N-1,N}(x(N-1), U(N-1))$$

$$J^*_{N-1\},N}(x(N-2)) \triangleq \min_{U(N-2),u(N-1)} \{ g_D(x(N-2), U(N-2)) + J_{N-1,N}(x(N-1), u(N-1)) \}$$

According to the principle of optimality, the initial state $x(N-2)$ and initial decision $U(N-2)$, the remaining decision $U(N-1)$ must be optimal

$$J^*_{N-2,N}(x(N-2)) = \min_{U(N-2)} \{ g_D(x(N-2), U(N-2)) + J^*_{N-1,N}(x(N-1))$$

$$J^*_{N-2,N}(x(N-2)) = \min_{U(N-2)} \{ g_D(x(N-2), U(N-2)) + J^*_{N-1,N}(a_D(x(N-2), U(N-2))) \}$$

$$J^*_{N-3,N}(x(N-3)) = \min_{U(N-3)} \{ g_D(x(N-3), U(N-3)) + J^*_{N-2,N}(a_D(x(N-3), u(N-3))) \}$$

$$J^*_{N-k,N}(x(N-k)) = \min_{U(N-k), U(N-k+1), \dots, U(N-1)} \{ h(x(N)) + \sum_{k=N-K}^{N-1} g_D(x(k), u(k))$$

$$J^*_{N-K,N}(x(N-K)) = \min_{U(N-K)} \{ g_D(x(N-K), U(N-K)) + J^*_{N-(K-1),N}(a_D(x(N-K), U(N-K))) \}.$$

$$H\left(x(t), u^*(x(t), J_x^*, t), J_x^*, t\right) = \min_{u(t)} H\left(x(t), u(t), J_x^*, t\right)$$

$$J_x^* = \frac{\partial J^*}{\partial x}(x(t), t)$$

HJ

$$\boxed{0 = J_t^*(x(t), t) + H\left(x(t), u^*(x(t), J_x^*, t), J_x^*, t\right)}$$

and

B

$$\boxed{\begin{aligned} J^*_{N-k,N}\left(x(N-k)\right) = \min_{u(N-k)} \Big\{ & g_D\left(x(N-k), u(N-k)\right) + \\ & J^*_{N-(k-1),N}\left(a_D\left(x(N-k), u(N-k)\right)\right)\Big\}. \end{aligned}}$$