Proceedings of the American Control Conference
San Diego, California • June 1999

Generalized Lyapunov and Invariant Set Theorems for Nonlinear Dynamical Systems

VijaySekhar Chellaboina, Alexander Leonessa, and Wassim M. Haddad

School of Aerospace Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0150

Abstract

In this paper we develop generalized Lyapunov and invariant set theorems for nonlinear dynamical systems wherein all regularity assumptions on the Lyapunov function and the system dynamics are removed. In particular, local and global stability theorems are given using lower semicontinuous Lyapunov functions. Furthermore, generalized invariant set theorems are derived wherein system trajectories converge to a union of largest invariant sets contained in intersections over finite intervals of the closure of generalized Lyapunov level surfaces. The proposed results provide transparent generalizations to standard Lyapunov and invariant set theorems.

1. Introduction

One of the most basic issues in system theory is stability of dynamical systems. The most complete contribution to the stability analysis of nonlinear dynamical systems is due to Lyapunov [14]. Lyapunov's results, along with the Barbashin-Krasovskii-LaSalle invariance principle [4,10,11], provide a powerful framework for analyzing the stability of nonlinear dynamical systems as well as designing feedback controllers which guarantee closed-loop system stability. In particular, Lyapunov's direct method can provide local and global stability conclusions of an equilibrium point of a nonlinear dynamical system if a smooth (C^1) positive-definite function of the nonlinear system states (Lyapunov function) can be constructed for which its time rate of change due to perturbations in a neighborhood of the system's equilibrium is always negative or zero, with strict negative definiteness ensuring asymptotic stability. Alternatively, using the Barbashin-Krasovskii-LaSalle invariance principle [4,10,11] the positive definite condition on the Lyapunov function as well as the strict negative definiteness condition on the Lyapunov derivative can be relaxed while assuring asymptotic stability. In particular, if a smooth function defined on a compact invariant set with respect to the nonlinear dynamical system can be constructed whose derivative along the system's trajectories is negative semidefinite and no system trajectories can stay indefinitely at points where the function's derivative identically vanishes, then the system's equilibrium is asymptotically stable.

Most Lyapunov stability and invariant set theorems presented in the literature require that the Lyapunov function candidate for a nonlinear dynamical system be a C^1 function with a negative-definite derivative (see [8,9,19] and the numerous references therein). This is due to the fact that the majority of the dynamical systems considered are systems possessing continuous motions and hence Lyapunov theorems provide stability conditions that do not require knowledge of the system trajectories [8,9,19]. However, in light of the increasingly complex nature of dynamical systems such as biological systems [13], hybrid systems [20], sampled-data systems [7], discrete-event systems [16], gain scheduled systems [12,15,17], constrained mechanical systems [2], and impulsive systems [3], system discontinuities as well as discontinuous motions arise naturally. In the case of discontinuous motions, standard Lyapunov and invariant set theorems are in general not applicable. Alternatively, in the case of discontinuous system dynamics with continuous motions standard Lyapunov theory is applicable, however, it might be simpler to construct discontinuous "Lyapunov" functions to establish system stability. For example, in gain scheduling control it is not uncommon to use several different controllers designed over several fixed operating points covering the system's operating range and to switch between them over this range. Even though for each operating range one can construct a C^1 Lyapunov function, to show closed-loop system stability over the whole system operating envelope for a given switching control strategy, a *generalized* Lyapunov function involving combinations of the Lyapunov functions for each operating range can be constructed [12,15,17]. However, in this case, it can be shown that the generalized Lyapunov function is non-smooth and non-continuous [12,15,17].

In this paper we develop generalized Lyapunov and invariant set theorems for nonlinear dynamical systems wherein all regularity assumptions on the Lyapunov function and the system dynamics are removed. In particular, local and global stability theorems are presented using generalized Lyapunov functions that are lower semicontinuous. Furthermore, generalized invariant set theorems are derived wherein system trajectories converge to a union of largest invariant sets contained in intersections over finite intervals of the closure of generalized Lyapunov level surfaces. In the case where the generalized Lyapunov function is taken to be a C^1 function, our results collapse to the standard Lyapunov stability and invariant set theorems. Finally, we note that lower semicontinuous Lyapunov functions are not new to this paper. Specifically, lower semicontinuous Lyapunov functions have been considered in [1] in the context of viability theory and differential inclusions. However, the present formulation provides invariant set stability theorem generalizations not considered in [1].

2. Mathematical Preliminaries

In this section we establish definitions, notation, and two key results used later in the paper. Let $\mathbb{R}$ denote the set of real numbers and let $\mathbb{R}^n$ denote the set of $n \times 1$ real column vectors. Furthermore, let $\partial\mathcal{S}$, $\overset{\circ}{\mathcal{S}}$, and $\overline{\mathcal{S}}$ denote the boundary, the interior, and the closure of the set $\mathcal{S} \subset \mathbb{R}^n$, respectively. Let $\|\cdot\|$ denote the Euclidean vector norm, let $\mathcal{B}_\varepsilon(\alpha)$, $\alpha \in \mathbb{R}^n$, $\varepsilon > 0$, denote the open ball centered at α with radius ε, that is, $\mathcal{B}_\varepsilon(\alpha) \triangleq \{x \in \mathbb{R}^n : \|x - \alpha\| < \varepsilon\}$, and let $V'(x)$ denote the Fréchet derivative of V at x. Finally, let C^0 denote the set of continuous functions and C^n denote the set of functions

This research was supported in part by the National Science Foundation under Grant ECS-9496249 and the Air Force Office of Scientific Research under Grant F49620-96-1-0125.

0-7803-4990-6/99 $10.00 © 1999 AACC

with n-continuous derivatives.

In this paper we consider the general nonlinear dynamical system

$$\dot{x}(t) = f(x(t)), \qquad x(0) = x_0, \qquad t \geq 0, \qquad (1)$$

where $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$ is the system state vector, $\mathcal{D}$ is an open set, $0 \in \mathcal{D}$, $f : \mathcal{D} \to \mathbb{R}^n$, and $f(0) = 0$. A function $x : \mathcal{I} \to \mathcal{D}$ is said to be a solution to (1) on the interval $\mathcal{I} \subseteq \mathbb{R}$ if $x(t)$ satisfies (1) for all $t \in \mathcal{I}$. Note that we do not assume any regularity conditions on the function $f(\cdot)$. However, we do assume that $f(\cdot)$ is such that the solution $x(t)$, $t \geq 0$, to (1) is well-defined on the time interval $\mathcal{I} = [0, \infty)$. That is, we assume that for every $y \in \mathcal{D}$ there exists a unique solution $x(\cdot)$ of (1) defined on $[0,\infty)$ satisfying $x(0) = y$. Furthermore, we assume that all the solutions $x(t)$, $t \geq 0$, to (1) are continuous functions of the initial conditions $x_0 \in \mathcal{D}$. The nonlinear system (1) can denote a nonlinear *feedback* control system in closed-loop configuration. The following definition introduces three types of stability and attractivity corresponding to the zero solution $x(t) \equiv 0$ of (1).

Definition 2.1. The zero solution $x(t) \equiv 0$ to (1) is *Lyapunov stable* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $\|x(0)\| < \delta$, then $\|x(t)\| < \varepsilon$, $t \geq 0$. The zero solution $x(t) \equiv 0$ to (1) is *attractive* if there exists $\delta > 0$ such that if $\|x(0)\| < \delta$, then $\lim_{t\to\infty} x(t) = 0$. The zero solution $x(t) \equiv 0$ to (1) is *asymptotically stable* if it is Lyapunov stable and attractive. The zero solution $x(t) \equiv 0$ to (1) is *globally asymptotically stable* if it is Lyapunov stable and for all $x(0) \in \mathbb{R}^n$, $\lim_{t\to\infty} x(t) = 0$. Finally, the zero solution $x(t) \equiv 0$ to (1) is *unstable* if it is not Lyapunov stable.

Next, we introduce several definitions and two key results that are necessary for the main results of this paper.

Definition 2.2. The *trajectory* $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $t \geq 0$, of (1) denotes the solution to (1) corresponding to the initial condition $x(0) = x_0$ evaluated at time t. The trajectory $x(t)$, $t \geq 0$, of (1) is *bounded* if there exists $\gamma > 0$ such $\|x(t)\| < \gamma$, $t \geq 0$.

Definition 2.3. A set $\mathcal{M} \subset \mathcal{D} \subseteq \mathbb{R}^n$ is an *invariant set* for the nonlinear dynamical system (1) if $x(0) \in \mathcal{M}$ implies that $x(t) \in \mathcal{M}$ for all $t \geq 0$.

Definition 2.4. $p \in \overline{\mathcal{D}} \subseteq \mathbb{R}^n$ is a *positive limit point* of the trajectory $x(t)$, $t \geq 0$, if for all $\varepsilon > 0$ and finite time $T > 0$ there exists $t > T$ such that $\|x(t) - p\| < \varepsilon$. The set of all positive limit points of $x(t)$, $t \geq 0$, is the *positive limit set* $\mathcal{P}^+_{x_0}$ of $x(t)$, $t \geq 0$.

Note that $\|x(t) - p\| < \varepsilon$ for all $\varepsilon > 0$ and $t > T > 0$ is equivalent to the existence of a sequence $\{t_n\}_{n=0}^{\infty}$, with $t_n \to \infty$ as $n \to \infty$, such that $x(t) \to p$ as $n \to \infty$.

The following result on positive limit sets is fundamental and forms the basis for all later developments.

Lemma 2.1 [9]. Suppose the solution $x(t)$, $t \geq 0$, to (1) corresponding to an initial condition $x(0) = x_0$ is bounded. Then the positive limit set $\mathcal{P}^+_{x_0}$ of $x(t)$, $t \geq 0$, is a nonempty, compact, connected invariant set. Furthermore, $x(t) \to \mathcal{P}^+_{x_0}$ as $t \to \infty$.

Remark 2.1. It is important to note that Lemma 2.1 holds for time-invariant nonlinear dynamical systems (1) possessing unique solutions forward in time with the solutions being continuous functions of the initial conditions. More generally, letting $s(\cdot, x_0)$ denote the solution of a dynamical system with initial condition $x(0) = x_0$, Lemma 2.1 holds if $s(t+\tau, x_0) = s(t, s(\tau, x_0))$, $t, \tau \geq 0$, and $s(\cdot, x_0)$ is a continuous function of $x_0 \in \mathcal{D}$. Finally, Lemma 2.1 also holds for all discrete-time, time-invariant nonlinear dynamical systems that have (unique) solutions which are continuously dependent on the system initial conditions.

Remark 2.2. If $f(\cdot)$ is Lipschitz continuous on $\mathcal{D}$ then there exists a unique solution to (1) and hence the required semi-group property $s(t+\tau, x_0) = s(t, s(\tau, x_0))$, $t, \tau \geq 0$, and the continuity of $s(t, \cdot)$ on $\mathcal{D}$, $t \geq 0$, hold. Alternatively, uniqueness of solutions in forward time along with the continuity of $f(\cdot)$ ensure that the solutions to (1) satisfy the semi-group property and are continuous functions of the initial conditions $x_0 \in \mathcal{D}$ even when $f(\cdot)$ is not Lipschitz continuous on $\mathcal{D}$ (see [5, Theorem 4.3, p. 59]). More generally, $f(\cdot)$ need not be continuous. In particular, if $f(\cdot)$ is discontinuous but bounded and $x(\cdot)$ is the unique solution to (1) in the sense of Filippov [6], then the required semi-group property along with the continuous dependence of solutions on initial conditions hold [6].

Next, we present a key theorem due to Weierstrass involving lower semicontinuous functions on compact sets. For the statement of the result the following definition is needed.

Definition 2.5. A function $V : \mathcal{D} \to \mathbb{R}$ is *lower semicontinuous* on $\mathcal{D}$ if for every sequence $\{x_n\}_{n=0}^{\infty} \subset \mathcal{D}$ such that $\lim_{n\to\infty} x_n = x$, $V(x) \leq \liminf_{n\to\infty} V(x_n)$.

Note that a function $V : \mathcal{D} \to \mathbb{R}$ is lower semicontinuous at $x \in \mathcal{D}$ if and only if for each $\alpha \in \mathbb{R}$ the set $\{x \in \mathcal{D} : V(x) > \alpha\}$ is open. Equivalently, a bounded function $V : \mathcal{D} \to \mathbb{R}$ is lower semicontinuous at $x \in \mathcal{D}$ if and only if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\|x - y\| < \delta$, $y \in \mathcal{D}$, implies $V(x) - V(y) \leq \varepsilon$.

Theorem 2.1 [18]. Suppose $\mathcal{D}_{\rm c} \subset \mathcal{D}$ is compact and $V : \mathcal{D}_{\rm c} \to \mathbb{R}$ is lower semicontinuous. Then there exists $x^* \in \mathcal{D}_{\rm c}$ such that $V(x^*) \leq V(x)$, $x \in \mathcal{D}_{\rm c}$.

Finally, the following definitions are used later in the paper.

Definition 2.6. Let $\mathcal{Q} \subseteq \mathcal{D}$ and let $V : \mathcal{Q} \to \mathbb{R}$. For $\alpha \in \mathbb{R}$, the set $V^{-1}(\alpha) \triangleq \{x \in \mathcal{Q} : V(x) = \alpha\}$ is called the *α-level set*. For $\alpha, \beta \in \mathbb{R}$, $\alpha \leq \beta$, the set $V^{-1}([\alpha, \beta]) \triangleq \{x \in \mathcal{Q} : \alpha \leq V(x) \leq \beta\}$ is called the *$[\alpha, \beta]$-sublevel set.*

Definition 2.7. A function $V : \mathcal{D} \to \mathbb{R}$ is *positive-definite* on $\mathcal{D}$ if $V(0) = 0$ and $V(x) > 0$, $x \in \mathcal{D}$, $x \neq 0$. Furthermore, a function $V : \mathbb{R}^n \to \mathbb{R}$ is *radially unbounded* if $V(x) \to \infty$ as $\|x\| \to \infty$.

3. Generalized Stability Theorems

As discussed in the introduction, most Lyapunov stability theorems presented in the literature require that the Lyapunov function candidate for a nonlinear dynamical system be a C^1 function with a negative-definite derivative along the system trajectories. In this section, we present several generalized stability theorems where we relax both of these assumptions while guaranteeing local and global stability of a nonlinear dynamical system. The following result gives sufficient conditions for Lyapunov stability of a nonlinear dynamical system.

Theorem 3.1. Consider the nonlinear dynamical system (1) and let $x(t)$, $t \geq 0$, denote the solution to (1). Assume that there exists a lower semicontinuous positive-definite function $V : \mathcal{D} \to \mathbb{R}$ such that $V(\cdot)$ is continuous at the origin and $V(x(t)) \leq V(x(\tau))$, $0 \leq \tau \leq t$. Then the zero solution $x(t) \equiv 0$ to (1) is Lyapunov stable.

Proof. Let $\varepsilon > 0$ be such that $\mathcal{B}_\varepsilon(0) \subset \mathcal{D}$. Since $\partial\mathcal{B}_\varepsilon(0)$ is compact and $V(x)$, $x \in \mathcal{D}$, is lower semicontinuous it follows from Theorem 2.1 that there exists $\alpha = \min_{x \in \partial\mathcal{B}_\varepsilon(0)} V(x)$. Note $\alpha > 0$ since $0 \notin \partial\mathcal{B}_\varepsilon(0)$ and $V(x) > 0$, $x \in \mathcal{D}$, $x \neq 0$. Next, since $V(0) = 0$ and $V(\cdot)$ is continuous at the origin it follows that there exists $\delta \in (0, \varepsilon)$ such that $V(x) < \alpha$, $x \in \mathcal{B}_\delta(0)$. Now, it follows that for all $x(0) \in \mathcal{B}_\delta(0)$,

$$V(x(t)) \leq V(x(0)) < \alpha, \qquad t \geq 0,$$

which, since $V(x) \geq \alpha$, $x \in \partial\mathcal{B}_\varepsilon(0)$, implies that $x(t) \notin \partial\mathcal{B}_\varepsilon(0)$, $t \geq 0$. Hence, for all $\varepsilon > 0$ such that $\mathcal{B}_\varepsilon(0) \subset \mathcal{D}$ there exists $\delta > 0$ such that if $\|x(0)\| < \delta$, then $\|x(t)\| < \varepsilon$, $t \geq 0$, which proves Lyapunov stability. $\square$

A lower semicontinuous positive-definite function $V(\cdot)$ which is continuous at the origin is called a *generalized Lyapunov function candidate* for the nonlinear dynamical system (1). If, additionally, $V(\cdot)$ satisfies $V(x(t)) \leq V(x(\tau))$, $0 \leq \tau \leq t$, $V(\cdot)$ is called a *generalized Lyapunov function* for the nonlinear dynamical system (1). Note that in the case where the function $V(\cdot)$ is C^1 on $\mathcal{D}$ in Theorem 3.1, it follows that $V(x(t)) \leq V(x(\tau))$, for all $t \geq \tau \geq 0$, is equivalent to $\dot{V}(x) \triangleq V'(x)f(x) \leq 0$, $x \in \mathcal{D}$. In this case, Theorem 3.1 specializes to the standard Lyapunov stability theorem [8,9].

Next, we generalize the Barbashin-Krasovskii-LaSalle invariant set theorems [4,9–11] to the case in which the function $V(\cdot)$ is lower semicontinuous.

Theorem 3.2. Consider the nonlinear dynamical system (1), let $x(t)$, $t \geq 0$, denote the solution to (1), and let $\mathcal{D}_c \subset \mathcal{D}$ be a compact invariant set with respect to (1). Assume that there exists a lower semicontinuous function $V : \mathcal{D}_c \to \mathbb{R}$ such that $V(x(t)) \leq V(x(\tau))$, $0 \leq \tau \leq t$, for all $x_0 \in \mathcal{D}_c$. Let $\mathcal{R}_\gamma \triangleq \bigcap_{c > \gamma} \overline{V^{-1}([\gamma, c])}$, $\gamma \in \mathbb{R}$, and let $\mathcal{M}_\gamma$ be the largest invariant set contained in $\mathcal{R}_\gamma$. If $x_0 \in \mathcal{D}_c$, then $x(t) \to \mathcal{M} \triangleq \bigcup_{\gamma \in \mathbb{R}} \mathcal{M}_\gamma$ as $t \to \infty$.

Proof. Let $x(t)$, $t \geq 0$, be the solution to (1) with $x_0 \in \mathcal{D}_c$. Since $V(\cdot)$ is lower semicontinuous on the compact set $\mathcal{D}_c$, there exists $\beta \in \mathbb{R}$ such that $V(x) \geq \beta$, $x \in \mathcal{D}_c$. Hence, since $V(x(t))$, $t \geq 0$, is nonincreasing, $\gamma_{x_0} \triangleq \lim_{t \to \infty} V(x(t))$, $x_0 \in \mathcal{D}_c$, exists. Now, for all $p \in \mathcal{P}^+_{x_0}$ there exists an increasing unbounded sequence $\{t_n\}_{n=0}^\infty$, with $t_0 = 0$, such that $x(t_n) \to p$ as $n \to \infty$. Next, since $V(x(t_n))$, $n \geq 0$, is nonincreasing it follows that for all $N \geq 0$, $\gamma_{x_0} \leq V(x(t_n)) \leq V(x(t_N))$, $n \geq N$, or, equivalently, since $\mathcal{D}_c$ is invariant, $x(t_n) \in V^{-1}([\gamma_{x_0}, V(x(t_N))])$, $n \geq N$. Now, since $\lim_{n \to \infty} x(t_n) = p$ it follows that $p \in \overline{V^{-1}([\gamma_{x_0}, V(x(t_n))])}$, $n \geq 0$. Furthermore, since $\lim_{n \to \infty} V(x(t_n)) = \gamma_{x_0}$ it follows that for every $c > \gamma_{x_0}$, there exists $n \geq 0$ such that $\gamma_{x_0} \leq V(x(t_n)) \leq c$ which implies that for every $c > \gamma_{x_0}$, $p \in \overline{V^{-1}([\gamma_{x_0}, c])}$. Hence, $p \in \mathcal{R}_{\gamma_{x_0}}$ which implies that $\mathcal{P}^+_{x_0} \subseteq \mathcal{R}_{\gamma_{x_0}}$. Now, since $\mathcal{D}_c$ is compact and invariant it follows that the solution $x(t)$, $t \geq 0$, to (1) is bounded for all $x_0 \in \mathcal{D}_c$ and hence it follows from Lemma 2.1 that $\mathcal{P}^+_{x_0}$ is a nonempty compact invariant set which further implies that $\mathcal{P}^+_{x_0}$ is a subset of the largest invariant set contained in $\mathcal{R}_{\gamma_{x_0}}$, that is, $\mathcal{P}^+_{x_0} \subseteq \mathcal{M}_{\gamma_{x_0}}$. Hence, for all $x_0 \in \mathcal{D}_c$, $\mathcal{P}^+_{x_0} \subseteq \mathcal{M}$. Finally, since $x(t) \to \mathcal{P}^+_{x_0}$ as $t \to \infty$ it follows that $x(t) \to \mathcal{M}$ as $t \to \infty$. $\square$

Remark 3.1. If in Theorem 3.2 $\mathcal{M}$ contains no other trajectory other than the trivial trajectory $x(t) \equiv 0$, then the zero solution $x(t) \equiv 0$ to (1) is attractive and $\mathcal{D}_c$ is a subset of the domain of attraction.

Remark 3.2. Note that if $V : \mathcal{D}_c \to \mathbb{R}$ is a lower semicontinuous function such that all the conditions of Theorem 3.2 are satisfied, then for every $x_0 \in \mathcal{D}_c$ there exists $\gamma_{x_0} \leq V(x_0)$ such that $\mathcal{P}^+_{x_0} \subseteq \mathcal{M}_{\gamma_{x_0}} \subseteq \mathcal{M}$.

It is important to note that as in standard Lyapunov and invariant set theorems involving C^1 functions, Theorem 3.2 allows one to characterize the invariant set $\mathcal{M}$ *without* knowledge of the system trajectories $x(t)$, $t \geq 0$. Similar remarks hold for the rest of the theorems in this section. The following corollary to Theorem 3.2 presents sufficient conditions that guarantee local asymptotic stability of the nonlinear dynamical system (1).

Corollary 3.1. Consider the nonlinear dynamical system (1), let $x(t)$, $t \geq 0$, denote the solution to (1), and let $\mathcal{D}_c \subset \mathcal{D}$ with $0 \in \overset{\circ}{\mathcal{D}}_c$ be a compact invariant set with respect to (1). Assume that there exists a lower semicontinuous positive-definite function $V : \mathcal{D}_c \to \mathbb{R}$ such that $V(\cdot)$ is continuous at the origin and $V(x(t)) \leq V(x(\tau))$, $0 \leq \tau \leq t$, for all $x_0 \in \mathcal{D}_c$. Let $\mathcal{R}_\gamma \triangleq \bigcap_{c > \gamma} \overline{V^{-1}([\gamma, c])}$, $\gamma \geq 0$, and let $\mathcal{M}_\gamma$ be the largest invariant set contained in $\mathcal{R}_\gamma$. Furthermore, assume $\mathcal{M} \triangleq \bigcup_{\gamma \geq 0} \mathcal{M}_\gamma$ contains no trajectory other than the trivial trajectory $x(t) \equiv 0$. Then the zero solution $x(t) \equiv 0$ to (1) is asymptotically stable and $\mathcal{D}_c$ is a subset of the domain of attraction of (1).

Proof. The result is a direct consequence of Theorems 3.1 and 3.2. $\square$

Next, we specialize Theorem 3.2 to the Barbashin-Krasovskii-LaSalle invariant set theorem wherein $V(\cdot)$ is a C^1 function.

Corollary 3.2. Consider the nonlinear dynamical system (1), assume $\mathcal{D}_c \subset \mathcal{D}$ is a compact invariant set with respect to (1), and assume that there exists a C^1 function $V : \mathcal{D}_c \to \mathbb{R}$ such that $V'(x)f(x) \leq 0$, $x \in \mathcal{D}_c$. Let $\mathcal{R} \triangleq \{x \in \mathcal{D}_c : V'(x)f(x) = 0\}$ and let $\mathcal{M}$ be the largest invariant set contained in $\mathcal{R}$. If $x_0 \in \mathcal{D}_c$, then $x(t) \to \mathcal{M}$ as $t \to \infty$.

Proof. The result follows from Theorem 3.2. Specifically, since $V'(x)f(x) \le 0$, $x \in \mathcal{D}_c$, it follows that

$$V(x(t)) - V(x(\tau)) = \int_\tau^t V'(x(s))f(x(s))\mathrm{d}s \le 0, \quad t \ge \tau,$$

and hence $V(x(t)) \le V(x(\tau))$, $t \ge \tau$. Now, since $V(\cdot)$ is C^1 it follows that $\mathcal{R}_\gamma = V^{-1}(\gamma)$, $\gamma \in \mathbb{R}$. In this case, it follows from Theorem 3.2 and Remark 3.2 that for every $x_0 \in \mathcal{D}_c$ there exists $\gamma_{x_0} \in \mathbb{R}$ such that $\mathcal{P}^+_{x_0} \subseteq \mathcal{M}_{\gamma_{x_0}}$, where $\mathcal{M}_{\gamma_{x_0}}$ is the largest invariant set contained in $\mathcal{R}_{\gamma_{x_0}} = V^{-1}(\gamma_{x_0})$ which implies that $V(x) = \gamma_{x_0}$, $x \in \mathcal{P}^+_{x_0}$. Hence, since $\mathcal{M}_{\gamma_{x_0}}$ is an invariant set it follows that for all $x(0) \in \mathcal{M}_{\gamma_{x_0}}$, $x(t) \in \mathcal{M}_{\gamma_{x_0}}$, $t \ge 0$, and thus $\dot{V}(x(0)) \triangleq \frac{\mathrm{d}V(x(t))}{\mathrm{d}t}\Big|_{t=0} = V'(x(0))f(x(0)) = 0$, which implies the $\mathcal{M}_{\gamma_{x_0}}$ is contained in $\mathcal{M}$ which is the largest invariant set contained in $\mathcal{R}$. Hence, since $x(t) \to \mathcal{P}^+_{x_0} \subseteq \mathcal{M}$ as $t \to \infty$, it follows that $x(t) \to \mathcal{M}$ as $t \to \infty$. □

Next, we sharpen the results of Theorem 3.2 by providing a refined construction of the invariant set $\mathcal{M}$. In particular, we show that the system trajectories converge to a union of largest invariant sets contained in intersections over the largest limit value of $V(\cdot)$ at the origin of the closure of generalized Lyapunov surfaces. First, however, the following key lemma is needed.

Lemma 3.1. Let $\mathcal{Q} \subseteq \mathbb{R}^n$ and let $V : \mathcal{Q} \to \mathbb{R}$. Define $\mathcal{R}_\gamma \triangleq \bigcap_{c>\gamma} \overline{V^{-1}([\gamma, c])}$, $\gamma \in \mathbb{R}$, and let $\gamma_0 \triangleq \limsup_{x\to 0} V(x)$. If $0 \in \mathcal{R}_\gamma$ for some $\gamma \in \mathbb{R}$, then $\gamma \le \gamma_0$.

Proof. If $0 \in \mathcal{R}_\gamma$ for $\gamma \in \mathbb{R}$, then there exists a sequence $\{x_n\}_{n=0}^\infty \subset \mathcal{R}_\gamma$ such that $\lim_{n\to\infty} x_n = 0$. Now, since $\gamma_0 = \limsup_{x\to 0} V(x)$, it follows that $\limsup_{n\to\infty} V(x_n) \le \gamma_0$. Next, note that $x_n \in \overline{V^{-1}([\gamma, c])}$, $c > \gamma$, $n = 0, 1, \ldots$, which implies that $V(x_n) \ge \gamma$, $n = 0, 1, \ldots$. Thus, using the fact that $\limsup_{n\to\infty} V(x_n) \le \gamma_0$ it follows that $\gamma \le \gamma_0$. □

Remark 3.3. Note that if in Lemma 3.1 $V(\cdot)$ is continuous at the origin then $\gamma_0 = V(0)$.

Theorem 3.3. Consider the nonlinear dynamical system (1), let $x(t)$, $t \ge 0$, denote the solution to (1), and let $\mathcal{D}_c \subset \mathcal{D}$ with $0 \in \overset{\circ}{\mathcal{D}}_c$ be a compact invariant set with respect to (1). Assume that there exists a lower semicontinuous positive-definite function $V : \mathcal{D}_c \to \mathbb{R}$ such that $V(x(t)) \le V(x(\tau))$, $0 \le \tau \le t$, for all $x_0 \in \mathcal{D}_c$. Furthermore, assume that for all $x_0 \in \mathcal{D}_c$, $x_0 \ne 0$, there exists an increasing unbounded sequence $\{t_n\}_{n=0}^\infty$, with $t_0 = 0$, such that $V(x(t_{n+1})) < V(x(t_n))$, $n = 0, 1, \ldots$. Define $\mathcal{R}_\gamma \triangleq \bigcap_{c>\gamma} \overline{V^{-1}([\gamma, c])}$, $\gamma \ge 0$, and let $\mathcal{M}_\gamma$ be the largest invariant set contained in $\mathcal{R}_\gamma$. If $x_0 \in \mathcal{D}_c$, then $x(t) \to \hat{\mathcal{M}} \triangleq \bigcup_{\gamma\in\mathcal{G}} \mathcal{M}_\gamma$ as $t \to \infty$, where $\mathcal{G} \triangleq \{\gamma \in [0, \gamma_0] : 0 \in \mathcal{R}_\gamma\}$ and $\gamma_0 \triangleq \limsup_{x\to 0} V(x)$. If, in addition, $V(\cdot)$ is continuous at the origin then the zero solution $x(t) \equiv 0$ to (1) is locally asymptotically stable and $\mathcal{D}_c$ is a subset of the domain of attraction.

Proof. It follows from Theorem 3.2, Remark 3.2, and the fact that $V(\cdot)$ is positive definite that, for every $x_0 \in \mathcal{D}_c$, there exists $\gamma_{x_0} \ge 0$ such that $\mathcal{P}^+_{x_0} \subseteq \mathcal{M}_{\gamma_{x_0}} \subseteq \mathcal{R}_{\gamma_{x_0}}$. Furthermore, since all solutions $x(t)$, $t \ge 0$, to (1) are bounded it follows from Lemma 2.1 that $\mathcal{P}^+_{x_0}$ is a nonempty, compact, invariant set. Now, ad absurdum, suppose $0 \notin \mathcal{P}^+_{x_0}$. Since $V(\cdot)$ is lower semicontinuous it follows from Theorem 2.1 that $\alpha \triangleq \min_{x\in\mathcal{P}^+_{x_0}} V(x)$ exists. Furthermore, there exists $\hat{x} \in \mathcal{P}^+_{x_0}$ such that $V(\hat{x}) = \alpha$. Now, with $x(0) = \hat{x} \ne 0$ it follows that there exists an increasing unbounded sequence $\{t_n\}_{n=0}^\infty$, with $t_0 = 0$, such that $V(x(t_{n+1})) < V(x(t_n))$, $n = 0, 1, \ldots$, which implies that there exists $t > 0$ such that $V(x(t)) < \alpha$ which further implies that $x(t) \notin \mathcal{P}^+_{x_0}$ contradicting the fact that $\mathcal{P}^+_{x_0}$ is an invariant set. Hence, $0 \in \mathcal{P}^+(x_0) \subseteq \mathcal{R}_{\gamma_{x_0}}$ which, using Lemma 3.1, implies that $\gamma_{x_0} \le \gamma_0$ for all $x_0 \in \mathcal{D}_c$, which further implies that $\mathcal{P}^+_{x_0} \subseteq \hat{\mathcal{M}}$. Now, since $x(t) \to \mathcal{P}^+_{x_0} \subseteq \hat{\mathcal{M}}$ as $t \to \infty$ it follows that $x(t) \to \hat{\mathcal{M}}$ as $t \to \infty$.

Finally, if $V(\cdot)$ is continuous at the origin then Lyapunov stability follows from Theorem 3.1. Furthermore, in this case, $\gamma_0 = V(0) = 0$ which implies that $\hat{\mathcal{M}} \equiv \{0\}$. Hence, $x(t) \to 0$ as $t \to \infty$ for all $x_0 \in \mathcal{D}_c$ establishing local asymptotic stability with a subset of the domain of attraction given by $\mathcal{D}_c$. □

In all the above results we explicitly assumed that there exists a compact invariant set $\mathcal{D}_c \subset \mathcal{D}$ of (1). Next, we provide a result that does not require the existence of such $\mathcal{D}_c$.

Theorem 3.4. Consider the nonlinear dynamical system (1) and let $x(t)$, $t \ge 0$, denote the solution to (1). Assume that there exists a lower semicontinuous positive-definite function $V : \mathbb{R}^n \to \mathbb{R}$ such that $V(x(t)) \le V(x(\tau))$, $0 \le \tau \le t$. Let $\mathcal{R}_\gamma \triangleq \bigcap_{c>\gamma} \overline{V^{-1}([\gamma, c])}$, $\gamma \ge 0$, and let $\mathcal{M}_\gamma$ be the largest invariant set contained in $\mathcal{R}_\gamma$. Then all solutions $x(t)$, $t \ge 0$, to (1) that are bounded approach $\mathcal{M} \triangleq \bigcup_{\gamma\ge 0} \mathcal{M}_\gamma$ as $t \to \infty$. If, in addition, for all $x_0 \in \mathbb{R}^n$, $x_0 \ne 0$, there exists an increasing unbounded sequence $\{t_n\}_{n=0}^\infty$, with $t_0 = 0$, such that $V(x(t_{n+1})) < V(x(t_n))$, $n = 0, 1, \ldots$, then all solutions $x(t)$, $t \ge 0$, to (1) that are bounded approach $\hat{\mathcal{M}} \triangleq \bigcup_{\gamma\in\mathcal{G}} \mathcal{M}_\gamma$, where $\mathcal{G} \triangleq \{\gamma \in [0, \gamma_0] : 0 \in \mathcal{R}_\gamma\}$ and $\gamma_0 \triangleq \limsup_{x\to 0} V(x)$.

Proof. The proof is a direct consequence of Theorems 3.2 and 3.3 with $\mathcal{D}_c$ given by the union of all bounded trajectories of (1). □

Next, we present a generalized global invariant set theorem for guaranteeing global attraction and global asymptotic stability of a nonlinear dynamical system.

Theorem 3.5. Consider the nonlinear dynamical system (1) with $\mathcal{D} = \mathbb{R}^n$ and let $x(t)$, $t \ge 0$, denote the solution to (1). Assume that there exists a lower semicontinuous, radially unbounded positive-definite function

$V : \mathbb{R}^n \to \mathbb{R}$ such that $V(x(t)) \leq V(x(\tau))$, $0 \leq \tau \leq t$. Define $\mathcal{R}_\gamma \triangleq \bigcap_{c>\gamma} \overline{V^{-1}([\gamma, c])}$, $\gamma \geq 0$, and let $\mathcal{M}_\gamma$ be the largest invariant set contained in $\mathcal{R}_\gamma$. Then for all $x_0 \in \mathbb{R}^n$, $x(t) \to \mathcal{M} \triangleq \bigcup_{\gamma \geq 0} \mathcal{M}_\gamma$, as $t \to \infty$. If, in addition, for all $x_0 \in \mathbb{R}^n$, $x_0 \neq 0$, there exists an increasing unbounded sequence $\{t_n\}_{n=0}^{\infty}$, with $t_0 = 0$, such that $V(x(t_{n+1})) < V(x(t_n))$, $n = 0, 1, \ldots$, then $x(t) \to \hat{\mathcal{M}} \triangleq \bigcup_{\gamma \in \mathcal{G}} \mathcal{M}_\gamma$ as $t \to \infty$, where $\mathcal{G} \triangleq \{\gamma \in [0, \gamma_0] : 0 \in \mathcal{R}_\gamma\}$ and $\gamma_0 \triangleq \limsup_{x \to 0} V(x)$. Finally, if $V(\cdot)$ is continuous at the origin then the zero solution $x(t) \equiv 0$ to (1) is globally asymptotically stable.

Proof. Note that since $V(x) \to \infty$ as $\|x\| \to \infty$ it follows that for every $\beta > 0$ there exists $r > 0$ such that $V(x) > \beta$ for all $x \notin \mathcal{B}_r(0)$, or, equivalently, $V^{-1}([0, \beta]) \subseteq \overline{\mathcal{B}}_r(0)$ which implies that $V^{-1}([0, \beta])$ is bounded for all $\beta > 0$. Hence, for all $x_0 \in \mathbb{R}^n$, $V^{-1}([0, \beta_{x_0}])$ is bounded where $\beta_{x_0} \triangleq V(x_0)$. Furthermore, since $V(\cdot)$ is a positive-definite lower semicontinuous function it follows that $V^{-1}([0, \beta_{x_0}])$ is closed and since $V(x(t))$, $t \geq 0$, is nonincreasing it follows that $V^{-1}([0, \beta_{x_0}])$ is an invariant set. Hence, for every $x_0 \in \mathbb{R}^n$, $V^{-1}([0, \beta_{x_0}])$ is a compact invariant set. Now, with $\mathcal{D}_c = V^{-1}([0, \beta_{x_0}])$ it follows from Theorem 3.2 and Remark 3.2 that there exists $0 \leq \gamma_{x_0} \leq \beta_{x_0}$ such that $\mathcal{P}_{x_0}^{+} \subseteq \mathcal{M}_{\gamma_{x_0}} \subseteq \mathcal{R}_{\gamma_{x_0}}$ which implies that $x(t) \to \mathcal{M}$ as $t \to \infty$. If, in addition, for all $x_0 \in \mathbb{R}^n$, $x_0 \neq 0$, there exists an increasing unbounded sequence $\{t_n\}_{n=0}^{\infty}$, with $t_0 = 0$, such that $V(x(t_{n+1})) < V(x(t_n))$, $n = 0, 1, \ldots$ holds then it follows from Theorem 3.3 that $x(t) \to \hat{\mathcal{M}}$ as $t \to \infty$.

Finally, if $V(\cdot)$ is continuous at the origin then Lyapunov stability follows from Theorem 3.1. Furthermore, in this case, $\gamma_0 = V(0) = 0$ which implies that $\hat{\mathcal{M}} \equiv \{0\}$. Hence, $x(t) \to 0$ as $t \to \infty$ establishing global asymptotic stability. $\square$

Remark 3.4. If in Theorems 3.3 and 3.5 the function $V(\cdot)$ is C^1 on $\mathcal{D}_c$ and $\mathbb{R}^n$, respectively, and $V'(x)f(x) < 0$, $x \in \mathbb{R}^n$, $x \neq 0$, then every increasing unbounded sequence $\{t_n\}_{n=0}^{\infty}$, with $t_0 = 0$, is such that $V(x(t_{n+1})) < V(x(t_n))$, $n = 0, 1, \ldots$. In this case Theorems 3.3 and 3.5 specialize to the standard Lyapunov stability theorems for local and global asymptotic stability, respectively.

Remark 3.5. Note that the results in this paper also hold for nonlinear discrete-time dynamical systems described by time-invariant difference equations whose (unique) solutions are continuous functions of the initial conditions. Specifically, in this case all of the above results and proofs proceed exactly as in the continuous-time case by replacing $t \in [0, \infty)$ with $k \in \mathcal{N}$, where $\mathcal{N}$ is the set of nonnegative integers.

4. Conclusion

Generalized Lyapunov and invariant set stability theorems for nonlinear dynamical systems were developed. In particular, local and global stability theorems were presented using generalized lower semicontinuous Lyapunov functions providing a transparent generalization of standard Lyapunov and invariant set theorems. Finally, the proposed approach was demonstrated on a nonlinear simultaneous stabilization problem involving a nonlinear switching controller.

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