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DIFFERENTIAL GAMES WITH MAXIMUM COST

E. N. BARRON*

Department of Mathematical Sciences, Loyola University, Chicago, IL 60626, U.S.A.

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FORMULATION OF THE BASIC PROBLEM AND INTRODUCTION

GIVEN THE dynamics on the interval $[t, T]$

$$\frac{d\xi}{d\tau} = f(\tau, \xi(\tau), \zeta(\tau), \eta(\tau)) \qquad 0 \le t < \tau \le T, \tag{0.1}$$

$$\xi(t) = x \in \mathbb{R}^n, \tag{0.2}$$

and the payoff

$$\mathbb{P}(\eta, \zeta) \equiv \|h(\tau, \xi(\tau), \eta(\tau), \zeta(\tau))\|_{L^\infty[t,T]} = \operatorname*{ess.sup}_{t \le \tau \le T} h(\tau, \xi(\tau), \eta(\tau), \zeta(\tau)). \tag{0.3}$$

We consider the differential game with players η, the maximizer of $\mathbb{P}$, and ζ, the minimizer of $\mathbb{P}$.

Thus, our game differs from the classical problem in that our payoff is measured pointwise in time rather than cumulatively with the form $\int_t^T h$. The motivation is to consider problems in which cumulative cost is not a sufficient criteria. This problem is also a generalization of optimal stopping time problems in which the obstacle (or reward function) can also depend on the controls. We do not stop the process, however. Differential games with payoffs of type (0.3) were also considered by Krasovskii and Subbotin [11] but they considered only the case $h = h(t, x)$, i.e. no explicit control dependence.

The problem with one player, i.e. the optimal control problem, was first considered in [3]. We will draw on many of the ideas of that paper. However, it is also true that the difficulties associated with introducing another player in the classical case are also present here.

Let us now define precisely what we mean by the differential game. We will use the Elliott and Kalton [8] formulation of the theory of differential games, although any of the various approaches could also be used. To this end, we define

$$\mathcal{Z}[t, T] \text{ is the class of measureable functions } \zeta\colon [t, T] \to Z \subset \mathbb{R}^{q_1}$$

and

$$\mathcal{Y}[t, T] \text{ is the class of measureable functions } \eta\colon [t, T] \to Y \subset \mathbb{R}^{q_2}$$

where Y and Z are compact.

Let $\Gamma(t)$ be the set of *strategies* for η (starting at time t), defined as a map $\alpha\colon \mathcal{Z}[t, T] \to \mathcal{Y}[t, T]$ satisfying the nonanticipating condition $\zeta(\tau) = \bar{\zeta}(\tau)$ for a.e. $t \le \tau \le s$, for each

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$t \le s \le T$, implies that $\alpha[\zeta](\tau) = \alpha[\tilde{\zeta}](\tau)$ for a.e. $t \le \tau \le s$. Similarly, let $\Delta(t)$ be the set of nonanticipating strategies $\beta: \mathcal{Y}[t,T] \to \mathcal{Z}[t,T]$ for ζ (starting at time t).

The *upper value of the differential game* associated with (0.1)-(0.3) is defined by the function $V^+: [0,T] \times \mathbb{R}^n \to \mathbb{R}^1$,

$$V^+(t,x) = \sup_{\alpha \in \Gamma(t)} \inf_{\zeta \in \mathcal{Z}[t,T]} \mathbb{P}(\alpha[\zeta], \zeta) = \sup_{\alpha \in \Gamma(t)} \inf_{\zeta \in \mathcal{Z}[t,T]} \|h(\tau, \xi(\tau), \alpha[\zeta](\tau), \zeta(\tau))\|_{L^\infty[t,T]},$$

and the *lower value* is defined by the function $V^-: [0,T] \times \mathbb{R}^n \to \mathbb{R}^1$,

$$V^-(t,x) = \inf_{\beta \in \Delta(t)} \sup_{\eta \in \mathcal{Y}[t,T]} \mathbb{P}(\eta, \beta[\eta]) = \inf_{\beta \in \Delta(t)} \sup_{\eta \in \mathcal{Y}[t,T]} \|h(\tau, \xi(\tau), \eta(\tau), \beta[\eta](\tau))\|_{L^\infty[t,T]}.$$

The trajectory ξ on $[t,T]$ is the solution of (0.1)-(0.2) corresponding to the pair of controls $(\alpha[\zeta], \zeta)$ and $(\eta, \beta[\eta])$, respectively.

We will prove that $V^\pm$ is the unique solution in the viscosity sense [1, 10, 7] of the upper (+) (respectively, lower (−)) Isaacs equation

$$V_t + H^\pm(t,x,V,\nabla_x V) = 0 \qquad \text{in } (0,T) \times \mathbb{R}^n$$

with

$$V^+(T,x) = \min_{z \in Z} \max_{y \in Y} h(T,x,y,z) \qquad \text{if } x \in \mathbb{R}^n,$$

and

$$V^-(T,x) = \max_{y \in Y} \min_{z \in Z} h(T,x,y,z) \qquad \text{if } x \in \mathbb{R}^n,$$

where

$$H^+(t,x,r,q) \equiv \min_{z \in Z(t,x,r)} \max_{y \in Y} q \cdot f(t,x,y,z)$$

$$= +\infty \qquad \text{if } Z(t,x,r) = \{z \in Z \mid \max_{y \in Y} h(t,x,y,z) \le r\} = \varnothing,$$

and

$$H^-(t,x,r,q) \equiv \max_{y \in Y} \min_{z \in Z^y(t,x,r)} \{q \cdot f(t,x,y,z)\}$$

$$= +\infty \qquad \text{if } Z^y(t,x,r) = \{z \in Z \mid h(t,x,y,z) \le r\} = \varnothing.$$

Notice that the set over which the minimum on z is taken depends on $V^\pm$.

We will establish that an equivalent formulation with $H^\pm < \infty$ is

$$\max\{V_t^+ + H^+(t,x,V^+,\nabla_x V^+), \min_{z \in Z} \max_{y \in Y} h(t,x,y,z) - V^+\} = 0$$

for V^+, and

$$\max\{V_t^- + H^-(t,x,V^-,\nabla_x V^+), \max_{y \in Y} \min_{z \in Z} h(t,x,y,z) - V^-\} = 0$$

for V^-, in $(0,T) \times \mathbb{R}^n$.

We obtain from the Isaacs equations and the uniqueness of viscosity solutions the existence of value, i.e. $V^+ = V^-$, under the Isaacs condition $H^+(t,x,r,q) \equiv H^-(t,x,r,q)$.

Also, we will apply this result to obtain an approximation theorem using Lipschitz controls. In particular, we will establish that the viscosity solution $V^{M,L}$ of

$$\max\{V_t^{M,L} + \nabla_x V^{M,L} \cdot f(t,x,y,z) + M|\nabla_y V^{M,L}| - L|\nabla_z V^{M,L}|, h(t,x,y,z) - V^{M,L}\} = 0,$$

if $t \in (0, T)$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^{q_2}$, $z \in \mathbb{R}^{q_1}$, with $V^{M,L}(T, x, y, z) = h(T, x, y, z)$ satisfies

$$\lim_{L \to \infty} \lim_{M \to \infty} V^{M,L}(t, x, y, z) = V^+(t, x) \qquad \text{and} \qquad \lim_{M \to \infty} \lim_{L \to \infty} V^{M,L}(t, x, y, z) = V^-(t, x).$$

An assertion of this type was first proved by Barron for classical games. The proof was considerably simplified in Barron *et al.* [4] using the ideas of viscosity solutions. We also use this approach here.

Finally, the theory of viscosity solutions introduced by Crandall and Lions [7] and extended to include discontinuous Hamiltonians by Ishii [10] is used throughout this paper.

1. BASIC PROPERTIES OF $V^\pm$

Let Lip(G) be the class of all bounded and uniformly Lipschitz continuous functions on G. We make the following assumption regarding f and h which will hold throughout this paper:

(A) f: $[0, T] \times \mathbb{R}^n \times Y \times Z \to \mathbb{R}^n$ is in Lip($[0, T] \times \mathbb{R}^n \times Y \times Z$). The function h: $[0, T] \times \mathbb{R}^n \times Y \times Z \to \mathbb{R}^1$ is also in Lip($[0, T] \times \mathbb{R}^n \times Y \times Z$).

In general, our assumption on f and h is not the weakest possible; in fact, in most cases, just uniformly continuous for f and h, f with linear growth and h bounded is sufficient.

Throughout this paper K will denote a generic constant larger than the bounds on f and h and the Lipschitz constants for f and h. We shall often ignore constants which depend only on the bounds of the functions involved.

PROPOSITION 1.1. $V^\pm \in$ Lip($[0, T] \times \mathbb{R}^n$) and

$$V^+(T, x) = \min_{z \in Z} \max_{y \in Y} h(T, x, y, z), \qquad V^-(T, x) = \max_{y \in Y} \min_{z \in Z} h(T, x, y, z). \tag{1.1}$$

Proof. We only prove the assertion for V^+. Clearly under assumption (A), V^+ is bounded. We will next establish that V^+ is Lipschitz continuous in x.

To this end, fix $0 \le t < T$ and $x_1, x_2 \in \mathbb{R}^n$. Also fix $\zeta \in \mathcal{Z}[t, T]$ and α in $\Gamma(t)$. Let $\xi_1(\cdot)$ be the trajectory corresponding to $(\alpha[\zeta], \zeta)$, with the initial position x_i, $i = 1, 2$:

$$\frac{d\xi_i}{d\tau} = f(\tau, \xi_i(\tau), \alpha[\zeta](\tau), \zeta(\tau)) \qquad 0 \le t \le \tau \le T,$$

$$\xi_i(t) = x_i \in \mathbb{R}^n, \qquad i = 1, 2.$$

Then, by (A) and standard results in o.d.e.s,

$$\max_{t \le \tau \le T} \|\xi_1(\tau) - \xi_2(\tau)\| \le K\|x_1 - x_2\|,$$

so that

$$|h(\tau, \xi_1(\tau), \alpha[\zeta](\tau), \zeta(\tau)) - h(\tau, \xi_2(\tau), \alpha[\zeta](\tau), \zeta(\tau))| \le K\|x_1 - x_2\|,$$

for every $\tau \in [t, T]$. This implies that

$$\big|\|h(\tau, \xi_1(\tau), \alpha[\zeta](\tau), \zeta(\tau))\|_{L^\infty[t,T]} - \|h(\tau, \xi_2(\tau), \alpha[\zeta](\tau), \zeta(\tau))\|_{L^\infty[t,T]}\big| \le K\|x_1 - x_2\|.$$

We conclude that for $0 \le t \le T$ and $x_1, x_2 \in \mathbb{R}^n$,

$$|V^+(t, x_1) - V^+(t, x_2)| \le K\|x_1 - x_2\|. \tag{1.2}$$

Thus V^+ is Lipschitz continuous in x.

To establish that V^+ is Lipschitz continuous in t we may fix $x \in \mathbb{R}^n$. Also, suppose that we fix $0 \leq t_1 < t_2 \leq T$. Given $\varepsilon > 0$, there exists $\tilde{\alpha}_2 \in \Gamma(t_2)$ such that for every $\zeta_2 \in \mathcal{Z}[t_2, T]$ we have

$$V^+(t_2, x) \leq \|h(\tau, \xi_2(\tau), \tilde{\alpha}_2[\zeta_2](\tau), \xi_2(\tau))\|_{L^\infty[t_2, T]} + \varepsilon. \tag{1.3}$$

Given $\zeta_1 \in \mathcal{Z}[t_1, T]$, define the strategy $\alpha_1 \in \Gamma(t_1)$ by $\alpha_1(\zeta_1)(\tau) = y$ if $t_1 \leq \tau < t_2$, and $\alpha_1(\zeta_1)(\tau) = \tilde{\alpha}_2(\zeta_1^2)(\tau)$ if $t_2 \leq \tau \leq T$, where $y \in Y$ is any fixed point, and $\zeta_1^2 \in \mathcal{Z}[t_2, T]$ is the restriction of the control ζ_1 to the interval $[t_2, T]$. Let ξ_i, $i = 1, 2$, be the trajectory, with $\xi_i(t_i) = x$, $i = 1, 2$, corresponding to the controls $(\alpha_1[\zeta_1], \zeta_1)$ and $(\tilde{\alpha}_2[\zeta_1^2], \zeta_1^2)$ on the interval $[t_1, T]$ and $[t_2, T]$, respectively. Then, by standard results in o.d.e.s we have $\|\xi_1(t_2) - x\| \leq K(t_2 - t_1)$. It follows that for every $\tau \in [t_2, T]$ we have

$$\|\xi_1(\tau) - \xi_2(\tau)\| \leq K\|\xi_1(t_2) - x\| \leq K(t_2 - t_1).$$

Consequently, we have for h that

$$|h(\tau, \xi_1(\tau), \alpha_1[\zeta_1](\tau), \zeta_1(\tau)) - h(\tau, \xi_2(\tau), \tilde{\alpha}_2[\zeta_1^2](\tau), \zeta_1^2(\tau))| \leq K(t_2 - t_1)$$

for $\tau \in [t_2, T]$. But then

$$\begin{aligned} \|h(\tau, \xi_1(\tau), \alpha_1[\zeta_1](\tau), \zeta_1(\tau))\|_{L^\infty[t_1, T]} &\geq \|h(\tau, \xi_1(\tau), \alpha_1[\zeta_1](\tau), \zeta_1(\tau))\|_{L^\infty[t_2, T]} \\ &\geq \|h(\tau, \xi_2(\tau), \tilde{\alpha}_2[\zeta_1^2](\tau), \zeta_1^2(\tau))\|_{L^\infty[t_2, T]} - K(t_2 - t_1). \end{aligned} \tag{1.4}$$

Combining (1.3) and (1.4) gives us

$$V^+(t_2, x) \leq \|h(\tau, \xi_1(\tau), \alpha_1[\zeta_1](\tau), \zeta_1(\tau))\|_{L^\infty[t_1, T]} + K(t_2 - t_1) + \varepsilon.$$

This implies that

$$V^+(t_2, x) \leq V^+(t_1, x) + K(t_2 - t_1). \tag{1.5}$$

For the reverse inequality, $\forall\, \varepsilon > 0$, there is $\tilde{\alpha}_1 \in \Gamma(t_1)$ such that

$$V^+(t_1, x) \leq \|h(\tau, \xi_1(\tau), \tilde{\alpha}_1[\zeta_1](\tau), \zeta_1(\tau))\|_{L^\infty[t_1, T]} + \varepsilon \tag{1.6}$$

for every $\zeta_1 \in \mathcal{Z}[t_1, T]$, where ξ_1 is the trajectory on $[t_1, T]$ corresponding to the pair $(\tilde{\alpha}_1[\zeta_1](\tau), \zeta_1(\tau))$, with $\xi_1(t_1) = x$. Let $s\colon [t_1, T] \to [t_2, T]$ be defined by

$$s(\tau) = t_2 + \frac{T - t_2}{T - t_1}(\tau - t_1), \qquad \tau(s) = t_1 + \frac{T - t_1}{T - t_2}(s - t_2).$$

This map is a bijection. Following Berkovitz [6, pp. 181–182], this map sets up a one-to-one correspondence between controls (and also strategies) on $[t_1, T]$ and controls (and strategies) on $[t_2, T]$ in the obvious way. Given $\zeta_2 \in \mathcal{Z}[t_2, T]$, define the strategy $\tilde{\alpha}_2 \in \Gamma(t_2)$ by

$$\tilde{\alpha}_2(\zeta_2)(s) = \tilde{\alpha}_1(\zeta_2^1)(\tau(s)), \qquad \text{where } \zeta_2^1(\tau(s)) \equiv \zeta_2(s).$$

Let ξ_1 be the trajectory on $[t_1, T]$ corresponding to the controls $(\tilde{\alpha}_1[\zeta_2^1](\tau), \zeta_2^1(\tau))$ and let ξ_2 be the trajectory on $[t_2, T]$ corresponding to $(\tilde{\alpha}_2(\zeta_2), \zeta_2)$. Then, c.f. Berkovitz [6, lemma 6.4],

$$\|\xi_1(\tau) - \xi_2(s(\tau))\|_{L^\infty(t_1, T)} \leq K(t_2 - t_1),$$
$$\|\xi_1(\tau(s)) - \xi_2(s)\|_{L^\infty(t_2, T)} \leq K(t_2 - t_1).$$

Using the Lipschitz continuity of h, we obtain that for $t_2 \le s \le T$,

$$|h(\tau(s), \xi_1(\tau(s)), \tilde{\alpha}_1[\zeta_2^1](\tau(s)), \zeta_2^1(\tau(s))) - h(s, \xi_2(s), \tilde{\alpha}_2[\zeta_2](s), \zeta_2(s))| \le K(t_2 - t_1).$$

But then

$$\|h(\tau, \xi_1(\tau), \tilde{\alpha}_1[\zeta_2^1](\tau), \zeta_2^1(\tau))\|_{L^\infty[t_1,T]} \le \|h(s, \xi_2(s), \tilde{\alpha}_2[\zeta_2](s), \zeta_2(s))\|_{L^\infty[t_2,T]} + K(t_2 - t_1).$$

This implies, using (1.6) that

$$V^+(t_1, x) \le V^+(t_2, x) + \varepsilon + K(t_2 - t_1).$$

Combining this with (1.5) we have shown that V^+ is uniformly Lipschitz continuous on $[0,T) \times \mathbb{R}^n$.

We have left to show that

$$\lim_{t \to T} V^+(t,x) = \min_{z \in Z} \max_{y \in Y} h(T,x,y,z). \tag{1.7}$$

To see that (1.7) is true, given any $z \in Z$, for $\zeta \in \mathcal{Z}[t,T]$ with $\zeta(\tau) \equiv z$

$$V^+(t,x) \le \sup_{\alpha \in \Gamma(t)} \|h(\tau, \xi(\tau), \alpha[\xi](\tau), \zeta(\tau))\|_{L^\infty(t,T)} \le K(T-t) + \max_{y \in Y} h(T,x,y,z).$$

We conclude that $\limsup_{t \to T} V^+(t,x) \le \min_{z \in Z} \max_{y \in Y} h(T,x,y,z)$.

For the reverse inequality we have that for every $\zeta \in \mathcal{Z}(t,T]$ and $\alpha^* \in \Gamma(t)$ defined by $\alpha^*[\zeta] \equiv y^*$ where $\max_{y \in Y} h(T,x,y,\zeta(T)) = h(T,x,y^*,\zeta(T))$

$$\begin{aligned}
\|h(\tau, \xi(\tau), \alpha^*[\zeta](\tau), \zeta(\tau))\|_{L^\infty[t,T]} &\ge h(T, \xi(T), \alpha^*[\zeta](T), \zeta(T)) \\
&\ge h(T, x, \alpha^*[\zeta](T), \zeta(T)) - K(T-t) \\
&= \max_{y \in Y} h(T,x,y,\zeta(T)) - K(T-t) \\
&\ge \min_{z \in Z} \max_{y \in Y} h(T,x,y,z) - K(T-t).
\end{aligned}$$

Consequently, $\liminf_{t \to T} V^+(t,x) \ge \min_{z \in Z} \max_{y \in Y} h(T,x,y,z)$ and (1.7) is established. This completes the proof. ■

Next we establish the principle of dynamic programming on which our derivation of the Isaacs equation is based.

THEOREM 1.2. Let $0 \le t < s \le T$, and $x \in \mathbb{R}^n$.

(i) $V^+(t,x) = \sup_{\alpha \in \Gamma(t)} \inf_{\zeta \in \mathcal{Z}(t,s)} \max\{\|h(\tau, \xi(\tau), \alpha[\zeta](\tau), \zeta(\tau))\|_{L^\infty(t,s)}, V^+(s, \xi(s))\}$

(ii) $V^-(t,x) = \inf_{\beta \in \Delta(t)} \sup_{\eta \in \mathcal{Y}(t,s)} \max\{\|h(\tau, \xi(\tau), \eta(\tau), \beta[\eta](\tau))\|_{L^\infty(t,s)}, V^-(s, \xi(s))\}$.

Proof. We will only prove part (ii) since part (i) is entirely similar. The proof is modelled on the classical assertion in [9]. Let $W(t,x)$ denote the right hand side of (ii). Let $\varepsilon > 0$ and $\tilde{\beta} \in \Delta(t)$ satisfy

$$W(t,x) \ge \sup_{\eta \in \mathcal{Y}(t,s)} \max\{\|h(\tau, \tilde{\xi}(\tau), \eta(\tau), \tilde{\beta}[\eta](\tau))\|_{L^\infty(t,s)}, V^-(s, \tilde{\xi}(s))\} - \varepsilon, \tag{1.8}$$

where $\tilde{\xi}$ is the trajectory on $[t, T]$ associated with $(\eta(\tau), \tilde{\beta}[\eta](\tau))$. For any $d \in \mathbb{R}^n$ we have

$$V^-(s, d) = \inf_{\beta \in \Delta(s)} \sup_{\eta \in \mathcal{Y}[s,T]} \|h(\tau, \xi(\tau), \eta(\tau), \beta[\eta](\tau))\|_{L^\infty[s,T]},$$

where $\xi(s) = d$. Thus, we can also find $\beta_d \in \Delta(s)$ so that

$$V^-(s, d) \geq \sup_{\eta \in \mathcal{Y}(s,T)} \|h(\tau, \xi_d(\tau), \eta(\tau), \beta_d[\eta](\tau))\|_{L^\infty(s,T)} - \varepsilon \tag{1.9}$$

where ξ_d is the trajectory on $[s, T]$ corresponding to controls $(\eta, \beta_d[\eta])$, and $\xi_d(s) = d$. Given $\eta \in \mathcal{Y}[t, T]$, define $\delta \in \Delta(t)$ by

$$\delta[\eta](\tau) = \begin{cases} \tilde{\beta}[\eta](\tau) & \text{if } t \leq \tau \leq s \\ \beta_{\xi(s)}[\eta](\tau) & \text{if } s < \tau \leq T. \end{cases}$$

Combining (1.8) and (1.9), for every $\eta \in \mathcal{Y}[t, T]$ we have

$$\begin{aligned} W(t, x) &\geq \max\{\|h(\tau, \xi(\tau), \eta(\tau), \delta[\eta](\tau))\|_{L^\infty(t,s)}, \|h(\tau, \xi(\tau), \eta(\tau), \delta[\eta](\tau))\|_{L^\infty(s,T)}\} - 2\varepsilon \\ &= \|h(\tau, \xi(\tau), \eta(\tau), \delta[\eta](\tau))\|_{L^\infty(t,T)} - 2\varepsilon \end{aligned}$$

where ξ is the trajectory on $[t, T]$ corresponding to controls $(\eta, \delta[\eta])$, and $\xi(t) = x$. Note that, by definition of the controls and uniqueness of the trajectories, $\tilde{\xi}$ and ξ coincide on $[t, s]$.

We then obtain, $W(t, x) + 2\varepsilon \geq \mathbb{P}(\eta, \delta[\eta])$ for every $\eta \in \mathcal{Y}[t, T]$. Consequently, $W(t, x) \geq V^-(t, x)$ if $0 \leq t < T$, $x \in \mathbb{R}^n$.

For the reverse inequality, for $\varepsilon > 0$ there is a $\beta' \in \Delta(t)$, such that

$$V^-(t, x) \geq \sup_{\eta \in \mathcal{Y}(t,T)} \|h(\tau, \xi(\tau), \eta(\tau), \beta'[\eta](\tau))\|_{L^\infty(t,T)} - \varepsilon.$$

For this β', by definition of W,

$$W(t, x) \leq \sup_{\eta \in \mathcal{Y}(t,s)} \max\{\|h(\tau, \xi(\tau), \eta(\tau), \beta'[\eta](\tau))\|_{L^\infty(t,s)}, V^-(s, \xi(s))\}.$$

Consequently, there is $\eta^1 \in \mathcal{Y}[t, s]$ such that

$$W(t, x) \leq \max\{\|h(\tau, \xi^1(\tau), \eta^1(\tau), \beta'[\eta^1](\tau))\|_{L^\infty(t,s)}, V^-(s, \xi^1(s))\} + \varepsilon \tag{1.10}$$

where ξ^1 is the trajectory on $[t, s]$ corresponding to controls $(\eta^1, \beta'[\eta^1])$, and $\xi^1(t) = x$.

Given any $\eta_s \in \mathcal{Y}[s, T]$, let $\tilde{\eta} \in \mathcal{Y}[t, T]$ be

$$\tilde{\eta}(\tau) = \begin{cases} \eta^1(\tau) & \text{if } t \leq \tau \leq s \\ \eta_s(\tau) & \text{if } s < \tau \leq T. \end{cases}$$

Define $\tilde{\beta} \in \Delta(s)$ by $\tilde{\beta}[\eta_s](\tau) = \beta'[\tilde{\eta}](\tau)$ if $s \leq \tau \leq T$. Then, by definition,

$$V^-(s, \xi^1(s)) \leq \sup_{\eta_s \in \mathcal{Y}[s,T]} \|h(\tau, \tilde{\xi}(\tau), \eta_s(\tau), \tilde{\beta}[\eta_s](\tau))\|_{L^\infty[s,T]}$$

where $\tilde{\xi}$ is the trajectory on $[s, T]$ corresponding to controls $(\eta_s, \tilde{\beta}[\eta_s])$ with $\tilde{\xi}(s) = \xi^1(s)$. There exists $\eta^2 \in \mathcal{Y}[s, T]$ so that

$$V^-(s, \xi^1(s)) \leq \|h(\tau, \xi^2(\tau), \eta^2(\tau), \tilde{\beta}[\eta^2](\tau))\|_{L^\infty[s,T]} + \varepsilon. \tag{1.11}$$

Define the control $\eta^* \in \mathcal{Y}[t, T]$ by

$$\eta^*(\tau) = \begin{cases} \eta^1(\tau) & \text{if } t \leq \tau \leq s \\ \eta^2(\tau) & \text{if } s < \tau \leq T. \end{cases}$$

Then, combining (1.10) and (1.11) we get

$$\begin{aligned}
W(t,x) &\le \max\{\|h(\tau,\xi^1(\tau),\eta^1(\tau),\beta'[\eta^1](\tau))\|_{L^\infty(t,s)}, V^-(s,\xi^1(s))\} + \varepsilon \\
&\le \max\{\|h(\tau,\xi^1(\tau),\eta^1(\tau),\beta'[\eta^1](\tau))\|_{L^\infty(t,s)}, \|h(\tau,\xi^2(\tau),\eta^2(\tau),\tilde\beta[\eta^2](\tau))\|_{L^\infty(s,T)}\} + 2\varepsilon \\
&= \max\{\|h(\tau,\xi(\tau),\eta^*(\tau),\beta'[\eta^*](\tau))\|_{L^\infty(t,s)}, \|h(\tau,\xi(\tau),\eta^*(\tau),\beta'[\eta^*](\tau))\|_{L^\infty(s,T)}\} + 2\varepsilon \\
&= \|h(\tau,\xi(\tau),\eta^*(\tau),\beta'[\eta^*](\tau))\|_{L^\infty(t,T)} + 2\varepsilon \\
&\le \sup_{\eta\in\mathcal{Y}(t,T)} \|h(\tau,\xi(\tau),\eta(\tau),\beta'[\eta](\tau))\|_{L^\infty(t,T)} + 2\varepsilon \\
&\le V^-(t,x) + 3\varepsilon.
\end{aligned}$$

Consequently $W(t,x) \le V^-(t,x)$ and the proof is complete. ■

2. THE ISAACS EQUATION FOR $V^\pm$

In this section we derive the Isaacs equations for the upper and lower value functions. To this end define the compact sets

$$Z(t,x,r) = \{z \in Z;\ \max_{y\in Y} h(t,x,y,z) \le r\}, \qquad Z^y(t,x,r) = \{z\in Z;\ h(t,x,y,z)\le r\}.$$

Define the *upper Hamiltonian function* $H^+: [0,T]\times\mathbb{R}^n\times\mathbb{R}^1\times\mathbb{R}^n \to \mathbb{R}^1$ by,

$$H^+(t,x,r,q) \equiv \min_{z\in Z(t,x,r)} \max_{y\in Y}\{f(t,x,y,z)\cdot q\} \qquad (= +\infty \text{ if } Z(t,x,r)=\varnothing)$$

and define the *lower Hamiltonian function* $H^-: [0,T]\times\mathbb{R}^n\times\mathbb{R}^1\times\mathbb{R}^n\to\mathbb{R}^1$ by,

$$H^-(t,x,r,q) \equiv \max_{y\in Y}\min_{z\in Z^y(t,x,r)}\{f(t,x,y,z)\cdot q\} \qquad (= +\infty \text{ if } Z^y(t,x,r)=\varnothing).$$

The next lemma contains some properties of the sets Z, Z^y, and the Hamiltonians $H^\pm$. We will let $B_\delta(\mu)$ denote the ball of radius δ centered at μ.

LEMMA 2.1. (i) If $r\le r'$, then $Z(t,x,r)\subseteq Z(t,x,r')$ and $Z^y(t,x,r)\subseteq Z^y(t,x,r')$ for any $y\in Y$.

(ii) If $r<r'$, $x\in\mathbb{R}^n$, $y\in Y$ and $0\le t<T$, there exists a $\delta=\delta(r,r')>0$ such that

$$Z(\tau,\xi,r)\subseteq Z(t,x,r') \qquad \forall\ (\tau,\xi)\in B_\delta(t,x)$$

and

$$Z^{\tilde y}(\tau,\xi,r)\subseteq Z^y(t,x,r') \qquad \forall\ (\tau,\xi,\tilde y)\in B_\delta(t,x,y).$$

(iii) If $r\le r'$, then $H^\pm(t,x,r,q)\ge H^\pm(t,x,r',q)$.

(iv) If $r<r'$, $x\in\mathbb{R}^n$ and $0\le t\le T$, there exists a $\delta=\delta(r,r')>0$ such that

$$H^\pm(\tau,\xi,r,q)\ge H^\pm(t,x,r',q) \qquad \text{for any } (\tau,\xi)\in B_\delta(t,x).$$

The proof of the lemma is similar to that of lemmas 2.4 and 2.5 in [3] and is therefore omitted. It is important to note that δ in the lemma is independent of y or z.

We will establish in this section that $V^\pm$ are viscosity solutions of an Isaacs equation involving $H^\pm$ as Hamiltonians. Since it is possible that $H^\pm$ are not continuous (or finite) we recall the definition of viscosity solution of a possibly discontinuous Hamilton–Jacobi equation.

Definition 2.2 [10]. Let G be a function on $(0, T) \times \mathbb{R}^n \times \mathbb{R}^1 \times \mathbb{R}^n$, possibly taking the values $+\infty$ or $-\infty$. A continuous function u on $(0, T) \times \mathbb{R}^n$ is a viscosity solution of

$$u_t(t, x) + G(t, x, u(t, x), \nabla_x u(t, x)) = 0,$$

if

(i) for any $\varphi \in C^1((0, T) \times \mathbb{R}^n)$ so that $u - \varphi$ has a minimum at $(\tau, \xi) \in (0, T) \times \mathbb{R}^n$ we have

$$\varphi_t(\tau, \xi) = G_*(\tau, \xi, u(\tau, \xi), \nabla_x \varphi(\tau, \xi)) \leq 0.$$

(That is, u is a viscosity supersolution.)

(ii) for any $\varphi \in C^1((0, T) \times \mathbb{R}^n)$ so that $u - \varphi$ has a maximum at $(\tau, \xi) \in (0, T) \times \mathbb{R}^n$ we have

$$\varphi_t(\tau, \xi) + G^*(\tau, \xi, u(\tau, \xi), \nabla_x \varphi(\tau, \xi)) \geq 0.$$

(That is, u is a viscosity subsolution.)

The upper and lower semicontinuous envelopes of G, G^* and G_*, respectively, are defined by the following functions:

$$G^*(t, x, r, q) \equiv \lim_{\varepsilon \to 0} \sup\{G(\tau, \xi, \rho, m) : |\tau - t| + \|x - \xi\| + |r - \rho| + \|q - m\| \leq \varepsilon\},$$

and

$$G_*(t, x, r, q) \equiv \lim_{\varepsilon \to 0} \inf\{G(\tau, \xi, \rho, m) : |\tau - t| + \|x - \xi\| + |r - \rho| + \|q - m\| \leq \varepsilon\}.$$

Definition 2.2 may be extended to include (possibly) discontinuous viscosity solutions by taking the lower semicontinuous envelope $u_*(t, x) \equiv \liminf_{s \to t, y \to x} u(s, y)$ (respectively, upper semicontinuous envelope $u^*(t, x) \equiv \limsup_{s \to t, y \to x} u(s, y)$) of the locally bounded function u in the definition of viscosity supersolution (respectively, subsolution). We have already proved that $V^\pm$ are continuous so that this is not required here.

In the next lemma we calculate the semicontinuous envelopes of $H^\pm$.

LEMMA 2.3. We have

$$(H^\pm)^*(t, x, r, q) \equiv H^\pm(t, x, r - 0, q), \tag{2.1}$$

and

$$(H^\pm)_*(t, x, r, q) \equiv H^\pm(t, x, r + 0, q). \tag{2.2}$$

The proof of this lemma is a calculation based on lemma 2.1 and can be found for optimal control problems in [3, proposition 2.6].

THEOREM 2.4. (a) V^+ is a viscosity solution of the equation

$$V_t^+(t, x) + H^+(t, x, V^+(t, x), \nabla_x V^+(t, x)) = 0, \tag{2.3}$$

if $t \in (0, T)$, $x \in \mathbb{R}^n$, with

$$V^+(T, x) = \min_{z \in Z} \max_{y \in Y} h(T, x, y, z) \qquad \text{on } \mathbb{R}^n \tag{2.4}$$

(b) V^- is a viscosity solution of the equation

$$V_t^-(t, x) + H^-(t, x, V^-(t, x), \nabla_x V^-(t, x)) = 0, \tag{2.5}$$

if $t \in (0, T)$, $x \in \mathbb{R}^n$, with

$$V^-(T, x) = \max_{y \in Y} \min_{z \in Z} h(T, x, y, z) \qquad \text{on } \mathbb{R}^n. \tag{2.6}$$

Remark. The assumption of Lipschitz continuity of f and h has led to Lipschitz continuity of $V^{\pm}$. Then Rademacher's theorem tells us that $V^{\pm}$ is differentiable almost everywhere. Therefore, under this assumption, $V^{\pm}$ satisfies the corresponding equation almost everywhere. However, if we merely assumed uniform continuity, then $V^{\pm}$ would still satisfy the appropriate equation in the viscosity sense.

Proof. We will only prove part (a) since the proof of (b) is similar.

Let $\varphi \in C^1((0, T) \times \mathbb{R}^n)$ and $(\tau, \xi) \in (0, T) \times \mathbb{R}^n$ a point at which $V^+ - \varphi$ has a (strict) local maximum (of 0). We will show that V^+ is a subsolution of (2.3). Let us suppose, in view of (2.1), that

$$\varphi_t + H^+(\tau, \xi, \varphi(\tau, \xi) - 0, \nabla_x \varphi) < 0 \qquad \text{at } (\tau, \xi). \tag{2.7}$$

We will derive a contradiction. If (2.7) holds, then there is a $\delta > 0$ for which

$$\varphi_t + H^+(\tau, \xi, \varphi(\tau, \xi) - 4\delta, \nabla_x \varphi) < -4\delta \qquad \text{at } (\tau, \xi).$$

There is, therefore, $z^* \in Z(\tau, \xi, \varphi(\tau, \xi) - 4\delta)$ such that

$$\varphi_t(\tau, \xi) + \max_{y \in Y} \{f(\tau, \xi, y, z^*) \cdot \nabla_x \varphi(\tau, \xi)\} < -4\delta. \tag{2.8}$$

Put $\zeta^* \in \mathcal{Z}[\tau, T]$, $\zeta^*(s) \equiv z^*$. Let $\alpha \in \Gamma(\tau)$ be arbitrary and set $\eta(s) = \alpha[\zeta^*](s)$, for $\tau \le s \le T$. Let $x(\cdot)$ be the trajectory associated with (η, ζ^*) on $[\tau, T]$, with $x(\tau) = \xi$. Using (2.8), by continuity, there exists $T - \tau > \sigma > 0$, independent of any controls, such that $(s, x(s)) \in B_\delta(\tau, \xi)$, $\tau \le s \le \tau + \sigma$, and

$$\varphi_t(s, x(s)) + f(s, x(s), \eta(s), \zeta^*(s)) \cdot \nabla_x \varphi(s, x(s)) < -3\delta \qquad \text{for all } \tau \le s \le \tau + \sigma. \tag{2.9}$$

Further, $h(s, x(s), \eta(s), \zeta^*(s)) \le \max_y h(s, x(s), y, \zeta^*(s)) \le \max_y h(\tau, \xi, y, z^*) + \delta$. Consequently,

$$h(s, x(s), \eta(s), \zeta^*(s)) \le \varphi(\tau, \xi) - 3\delta, \qquad \text{if } \tau \le s \le \tau + \sigma. \tag{2.10}$$

We integrate (2.9) from τ to $\tau + \sigma$ to get

$$\varphi(\tau + \sigma, x(\tau + \sigma)) - \varphi(\tau, \xi) \le -3\delta\sigma.$$

Using the facts $V^+ \le \varphi$ and $V^+(\tau, \xi) = \varphi(\tau, \xi)$, we get

$$V^+(\tau + \sigma, x(\tau + \sigma)) - V^+(\tau, \xi) \le \varphi(\tau + \sigma, x(\tau + \sigma)) - \varphi(\tau, \xi) \le -3\delta\sigma,$$

so that

$$V^+(\tau + \sigma, x(\tau + \sigma)) \le V^+(\tau, \xi) - 3\delta\sigma < V^+(\tau, \xi). \tag{2.11}$$

Using (2.10), we see that

$$\|h(s, x(s), \eta(s), \zeta^*(s))\|_{L^\infty(\tau, \tau+\sigma)} \le \varphi(\tau, \xi) - 3\delta < V^+(\tau, \xi). \tag{2.12}$$

We have shown that (2.11) and (2.12) are true for some ζ^*, but for every α. Consequently,

$$V^+(\tau, \xi) > \sup_{\alpha \in \Gamma(\tau)} \inf_{\zeta \in \mathcal{Z}(\tau, \tau+\sigma)} \max\{\|h(s, x(s), \alpha[\zeta](s), \zeta(s))\|_{L^\infty(\tau, \tau+\sigma)}, V^+(\tau + \sigma, x(\tau + \sigma))\}$$

which is a contradiction of theorem 1.2 (i). This proves that V^+ is a viscosity subsolution of (2.3).

Now we let $\varphi \in C^1((0, T) \times \mathbb{R}^n)$ and $(\tau, \xi) \in (0, T) \times \mathbb{R}^n$ a point at which $V^+ - \varphi$ has a (strict) local minimum (of 0). Using (2.2) let us suppose that

$$\varphi_t + H^+(\tau, \xi, \varphi(\tau, \xi) + 0, \nabla_x \varphi) > 0 \qquad \text{at } (\tau, \xi). \tag{2.13}$$

We will again derive a contradiction. There is a $\delta > 0$ for which

$$\varphi_t(\tau, \xi) + H^+(\tau, \xi, \varphi(\tau, \xi) + 4\delta, \nabla_x \varphi) \geq 4\delta \qquad \text{at } (\tau, \xi).$$

By lemma 2.1 (iv) we may assume that

$$\varphi_t(t, x) + H^+(t, x, \varphi(\tau, \xi) + 3\delta, \nabla_x \varphi(t, x)) \geq 3\delta \qquad \text{for any } (t, x) \in B_\delta(\tau, \xi).$$

That is,

$$\varphi_t(t, x) + \min_{z \in Z(t, x, \varphi(\tau, \xi) + 3\delta)} \max_{y \in Y} \{f(t, x, y, z) \cdot \nabla_x \varphi(t, x)\} \geq 3\delta \tag{2.14}$$

for any $(t, x) \in B_\delta(\tau, \xi)$.

Let $\zeta \in \mathcal{Z}[\tau, T]$ be arbitrary and select a $T - \tau > \sigma > 0$, independent of controls, so that $(s, x(s)) \in B_\delta(\tau, \xi)$ if $\tau \leq s \leq \tau + \sigma$, where $x(\cdot)$ is the trajectory associated with the controls ζ and any η, $x(\tau) = \xi$. There are now two possibilities.

Case 1. There is a control $\eta^1 \in \mathcal{Y}[\tau, T]$ such that

$$\|h(s, x(s), \eta^1(s), \zeta(s))\|_{L^\infty(\tau, \tau + \sigma)} > \varphi(\tau, \xi) + 3\delta.$$

In this case, using the fact that $V^+(\tau, \xi) = \varphi(\tau, \xi)$, we obtain

$$\sup_{\alpha \in \Gamma(\tau)} \inf_{\zeta \in \mathcal{Z}[\tau, T]} \|h(s, x(s), \alpha[\zeta](s), \zeta(s))\|_{L^\infty[\tau, \tau + s]} > V^+(\tau, \xi). \tag{2.15}$$

Case 2. For any $\eta \in \mathcal{Y}[\tau, T]$,

$$\|h(s, x(s), \eta(s), \zeta(s))\|_{L^\infty(\tau, \tau + \sigma)} \leq \varphi(\tau, \xi) + 3\delta.$$

Thus, $\zeta(s) \in Z(s, x(s), \varphi(\tau, \xi) + 3\delta)$, if $\tau \leq s \leq \tau + \sigma$. Using (2.14) we get

$$\varphi_t(s, x(s)) + \max_{y \in Y} \{f(s, x(s), y, \zeta(s)) \cdot \nabla_x \varphi(s, x(s))\} \geq 2\delta \qquad \text{for a.e. } \tau \leq s \leq \tau + \sigma. \tag{2.16}$$

As in the proof of lemma 4.3 (b) of [9] we can obtain from (2.16) the existence of a strategy $\alpha \in \Gamma(\tau)$ and $T - \tau > \sigma' > 0$ (σ' independent of controls) such that

$$\varphi_t(s, x(s)) + f(s, x(s), \alpha[\zeta](s), \zeta(s)) \cdot \nabla_x \varphi(s, x(s)) \geq 2\delta \qquad \text{for a.e. } \tau \leq s \leq \tau + \tilde{\sigma}, \tag{2.17}$$

where $x(\cdot)$ is the trajectory associated with $(\alpha[\zeta], \zeta)$ on $[\tau, T]$, $x(\tau) = \xi$, and $(s, x(s)) \in B_\delta(\tau, \xi)$ for all $\tau \leq s \leq \tau + \tilde{\sigma}$, $\tilde{\sigma} = \min(\sigma, \sigma')$. Integrate (2.17) from τ to $\tau + \tilde{\sigma}$ to get

$$2\tilde{\sigma}\delta \leq \varphi(\tau + \tilde{\sigma}, x(\tau + \tilde{\sigma})) - \varphi(\tau, \xi),$$

so that using that fact that $V^+ - \varphi$ has a minimum at (τ, ξ), we obtain

$$V^+(\tau, \xi) < V^+(\tau, \xi) + \tilde{\sigma}\delta \leq V^+(\tau + \tilde{\sigma}, x(\tau + \tilde{\sigma})). \tag{2.18}$$

Then, since ζ was arbitrary,

$$\sup_{\alpha \in \Gamma(\tau)} \inf_{\zeta \in \mathcal{Z}[\tau, T]} V^+(\tau + \sigma, x(\tau + \sigma)) > V^+(\tau, \xi). \tag{2.19}$$

Combining (2.15) and (2.19) we conclude that

$$V^+(\tau, \xi) < \sup_{\alpha \in \Gamma(\tau)} \inf_{\zeta \in Z[\tau, T]} \max\{V^+(\tau + \bar{\sigma}, x(\tau + \bar{\sigma})), \|h(s, x(s), \alpha[\zeta](s), \zeta(s))\|_{L^\infty(\tau, \tau+\bar{\sigma})}\},$$

which contradicts the dynamic programming principle, theorem 1.2 (i). We conclude that V^+ is a viscosity supersolution of (2.3) as well. Since we have established (2.4) in proposition 1.1, we have completed the proof. ■

Our next proposition gives an equivalent form of the Isaacs equations.

PROPOSITION 2.5. The continuous function u^+ (resp. u^-) on $[0, T] \times \mathbb{R}^n$ is a viscosity solution of

$$u_t(t, x) + H^\pm(t, x, u(t, x), \nabla_x u(t, x)) = 0 \tag{2.20}$$

if and only if u^+ (resp. u^-) is a viscosity solution of

$$\max\{u_t(t, x) + H^+(t, x, u(t, x), \nabla_x u(t, x)), \min_{z \in Z} \max_{y \in Y} h(t, x, y, z) - u(t, x)\} = 0, \tag{2.21}$$

$$(\text{resp. } \max\{u_t(t, x) + H^-(t, x, u(t, x), \nabla_x u(t, x)), \max_{y \in Y} \min_{z \in Z} h(t, x, y, z) - u(t, x)\} = 0) \tag{2.22}$$

if $t \in [0, T)$, $x \in \mathbb{R}^n$.

Proof. Let u^- be a viscosity solution of (2.20). Suppose that $\min(u^- - \varphi) = u^-(\tau, \xi) - \varphi(\tau, \xi) = 0$ with $\varphi \in C^1$. Then

$$\varphi_t(\tau, \xi) + H^-(\tau, \xi, \varphi(\tau, \xi) + 0, \nabla_x \varphi(\tau, \xi)) \leq 0.$$

It follows that the set $Z^y(\tau, \xi, u^-(\tau, \xi)) \neq \emptyset$ for every $y \in Y$, so that $\max_{y \in Y} \min_{z \in Z} h(\tau, \xi, y, z) - u(\tau, \xi) \leq 0$. We conclude that

$$\max\{\varphi_t(\tau, \xi) + H^-(\tau, \xi, u^-(\tau, \xi) + 0, \nabla_x \varphi(\tau, \xi)), \max_{y \in Y} \min_{z \in Z} h(\tau, \xi, y, z) - u^-(\tau, \xi)\} \leq 0$$

and hence u^- is a viscosity supersolution of (2.21).

The fact that u^- is a viscosity subsolution of (2.21) is trivial.

Now let u^- be a viscosity solution of (2.21). The fact that u^- is then a viscosity supersolution of (2.20) is immediate so we have only to show that u^- is a viscosity subsolution of (2.20). To see this, suppose $\varphi \in C^1$ with $u^- - \varphi$ attaining a local max $(=0)$ at the point $(\tau, \xi) \in (0, T) \times \mathbb{R}^n$. Since u^- is a viscosity subsolution of (2.21) we have that

$$\max\{\varphi_t(\tau, \xi) + H^-(\tau, \xi, \varphi(\tau, \xi) - 0, \nabla_x \varphi(\tau, \xi)), \max_{y \in Y} \min_{z \in Z} h(\tau, \xi, y, z) - u^-(\tau, \xi)\} \geq 0. \tag{2.23}$$

If $\varphi_t(\tau, \xi) + H^-(\tau, \xi, \varphi(\tau, \xi) - 0, \nabla_x \varphi(\tau, \xi)) < 0$ then we must have that there is some $\delta > 0$ so that $\varphi_t(\tau, \xi) + H^-(\tau, \xi, \varphi(\tau, \xi) - \delta, \nabla_x \varphi(\tau, \xi)) \leq -\delta$. Consequently, for every $y \in Y$ there is $z \in Z^y(\tau, \xi, \varphi(\tau, \xi) - \delta)$ so that $h(\tau, \xi, y, z) \leq \varphi(\tau, \xi) - \delta$. This implies that $\max_{y \in Y} \min_{z \in Z} h(\tau, \xi, y, z) \leq \varphi(\tau, \xi) - \delta < \varphi(\tau, \xi)$. On the other hand, (2.23) tells us that we must have $\max_{y \in Y} \min_{z \in Z} h(\tau, \xi, y, z) - \varphi(\tau, \xi) \geq 0$. This is a contradiction. We conclude that

$$\varphi_t(\tau, \xi) + H^-(\tau, \xi, \varphi(\tau, \xi) - 0, \nabla_x \varphi(\tau, \xi)) \geq 0$$

and so u^- is a viscosity subsolution of (2.20).

The proof for u^+ is similar and is therefore omitted. ■

Remark 2.6. When the function $h = h(t, x)$ is independent of y and z we have a differential game problem with an obstacle and with continuous control. The Isaacs equations is the variational inequality

$$\max\{V_t^+(t,x) + \min_{z\in Z}\max_{y\in Y} \nabla_x V^+(t,x)\cdot f(t,x,y,z),\, h(t,x) - V^+(t,x)\} = 0,$$

for V^+ and

$$\max\{V_t^-(t,x) + \max_{y\in Y}\min_{z\in Z} \nabla_x V^-(t,x)\cdot f(t,x,y,z),\, h(t,x) - V^-(t,x)\} = 0,$$

for V^-.

These variational inequalities are what one would obtain in the problem of optimal stopping with continuous time control as a differential game as well. But here we do not stop the process when we hit the obstacle. We refer to [5, Chapter 4] for a similar problem with stochastic dynamics and to Krasovskii and Subottin [11] where this case is studied for their definition of value and examples are given. Incidentally, uniqueness of solutions (see theorem 3.1 below) now also proves that the Krasovskii-Subbotin value and the (Elliott–Kalton) value in this paper are identical.

3. THE EXISTENCE OF VALUE

In this section we will generalize the so-called Isaacs' minimax condition to yield the existence of value for maximum cost games. The proof will be an immediate consequence of the uniqueness of the viscosity solution of the Isaacs equation.

THEOREM 3.1. There is at most one viscosity solution $u^\pm$ of

$$u_t^\pm(t,x) + H^\pm(t,x,u^\pm(t,x),\nabla_x u^\pm(t,x)) = 0 \tag{3.1}$$

satisfying $u^+(T,x) = \min_{z\in Z}\max_{y\in Y} h(T,x,y,z)$, $u^-(T,x) = \max_{y\in Y}\min_{z\in Z} h(T,x,y,z)$, $x \in \mathbb{R}^n$.

The proof of this result is almost identical to that of theorem 4.2 in [3] so we omit it.

Definition 3.2. The game associated with (0.1)–(0.3) has value if $V^+ = V^-$.

THEOREM 3.3. Let f and h satisfy the Isaacs condition

$$H^+(t,x,r,q) = H^-(t,x,r,q) \qquad \text{on } [0,T]\times\mathbb{R}^n\times\mathbb{R}^1\times\mathbb{R}^n.$$

Then $V^+(t,x) = V^-(t,x)$ on $[0,T]\times\mathbb{R}^n$.

Proof. In view of the uniqueness we need only see that

$$\min_{z\in Z}\max_{y\in Y} h(t,x,y,z) = \max_{y\in Y}\min_{z\in Z} h(t,x,y,z), \qquad x\in\mathbb{R}^n, \qquad 0\le t\le T. \tag{3.2}$$

To see this, if $H^+(t,x,r,q) = H^-(t,x,r,q)$, $\forall\ (t,x,r,q)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^1\times\mathbb{R}^n$, choose $r > \|h\|_{L^\infty([0,T]\times\mathbb{R}^n\times Y\times Z)}$ and, for each (t,x,y,z), let $q\in\mathbb{R}^n$ satisfy $q\cdot f(t,x,y,z) = h(t,x,y,z)$. This can be done if f is not identically zero. Then, by definition, it follows that $Z(t,x,r) \equiv Z$, $Z^y(t,x,r) \equiv Z$, $\forall\ y\in Y$, and so (3.2) is true.

Remark 3.4. Consider the following differential game with payoff

$$\mathbb{P}(\eta, \zeta) \equiv \operatorname{ess} \sup_{t \le \tau \le T} \left(\int_t^\tau g(s, \xi(s), \eta(s), \zeta(s))\, \mathrm{d}s + h(\tau, \xi(\tau), \eta(\tau), \zeta(\tau)) \right), \tag{3.3}$$

and dynamics (0.1) and (0.2). This problem is a generalization in that we are including the running cost $g \in \operatorname{Lip}([0, T]\mathbb{R}^n \times Y \times Z)$, say.

The upper and lower value functions for this problem are given respectively by

$$V^+(t, x) = \sup_{\alpha \in \Gamma(t)} \inf_{\zeta \in \mathcal{Z}[t,T]} \mathbb{P}(\alpha[\zeta], \zeta), \qquad \text{and} \qquad V^-(t, x) = \inf_{\beta \in \Delta(t)} \sup_{\eta \in \mathcal{Y}[t,T]} \mathbb{P}(\eta, \beta[\eta]).$$

Then we have, as in preceding sections.

THEOREM 3.5. (i) $V^\pm \in \operatorname{Lip}([0, T] \times \mathbb{R}^n)$;

(ii) $V^\pm$ is the unique viscosity solution of

$$V_t + H^\pm(t, x, V, \nabla_x V) = 0 \qquad \text{in } (0, T) \times \mathbb{R}^n$$

with

$$V^+(T, x) = \min_{z \in Z} \max_{y \in Y} h(T, x, y, z) \qquad \text{if } x \in \mathbb{R}^n,$$

and

$$V^-(T, x) = \max_{y \in Y} \min_{z \in Z} h(T, x, y, z) \qquad \text{if } x \in \mathbb{R}^n,$$

where $H^+(t, x, r, q) \equiv \min_{z \in Z(t,x,r)} \max_{y \in Y} \{q \cdot f(t, x, y, z) + g(t, x, y, z)\}$ and $= +\infty$ if $Z(t, x, r) = \varnothing$. and $H^-(t, x, r, q) \equiv \max_{y \in Y} \min_{z \in Z^y(t,x,r)} \{q \cdot f(t, x, y, z) + g(t, x, y, z)\}$ and $= +\infty$ if $Z^y(t, x, r) = \varnothing$.

Example 3.6. To see that not every game has value we consider the game with dynamics

$$\frac{\mathrm{d}\xi}{\mathrm{d}\tau} = (\zeta(\tau) - \eta(\tau))^2 \qquad 0 \le t < \tau \le T,$$

$$\xi(\tau) = x \in \mathbb{R}^1,$$

and the payoff

$$\mathbb{P}(\eta, \zeta) \equiv \|\xi(\tau)\|_{L^\infty[t,T]}.$$

We take the control sets $Y = [0, 1]$ and $Z = [0, 1]$.

It is easy to see that $V^+(t, x) = x + \frac{1}{4}(T - t)$ since η can always stay at least distance $\frac{1}{2}$ away from ζ but by choosing $\zeta \equiv \frac{1}{2}$, ζ can guarantee that η cannot get more than $\frac{1}{2}$ distance away. Further, $V^-(t, x) = x$ since ζ can, if η plays first, always choose $\zeta \equiv \eta$ and make $\xi(\tau) \equiv x$. Consequently, this game does not have a value if $t < T$. This game is a simple variant of the classical example of a game without value due to Berkovitz (cf. [8]).

4. GAMES WITH LIPSCHITZ CONTROLS

Let M and L be positive constants. Consider the following differential game with dynamics on $0 \leq t < \tau \leq T$, given by

$$\frac{d\xi}{d\tau} = f(\tau, \xi(\tau), \zeta(\tau), \eta(\tau)) \tag{4.1}$$

$$\frac{d\eta}{d\tau} = u(\tau) \tag{4.2}$$

$$\frac{d\zeta}{d\tau} = v(t) \tag{4.3}$$

$$\xi(t) = x \in \mathbb{R}^n, \qquad \eta(t) = y \in \mathbb{R}^{q_2}, \qquad \zeta(t) = z \in \mathbb{R}^{q_1}, \tag{4.4}$$

and the payoff

$$\mathbb{P}(u, v) \equiv \|h(\tau, \xi(\tau), \eta(\tau), \zeta(\tau))\|_{L^\infty[t,T]}. \tag{4.5}$$

The players are the controls u (the maximizer of $\mathbb{P}$) and v (the minimizer of $\mathbb{P}$) with

$$u: [t, T] \to U^M \equiv \{u \in \mathbb{R}^{q_2};\ |u| \leq M\} \qquad \text{and} \qquad v: [t, T] \to V^L \equiv \{v \in \mathbb{R}^{q_1};\ |v| \leq L\}.$$

Notice that the payoff does not depend directly on the controls u and v.

The equations (4.1)–(4.4) model a differential game in which the actual players are η and ζ but they are restricted to choosing control functions which are uniformly Lipschitz continuous with the fixed Lipschitz contants M (for η) and L (for ζ). Further, they are restricted to start at some known points. The first restriction makes the class of control functions compact. The second restriction is significant in that an instantaneous jump to a more desirable place in the control set is not possible. In the context of classical games, it is shown in [2] that this is sufficient for the existence of value. In fact, only one of the players need be so restricted and value will exist for classical games. It is also true here.

The Hamiltonians for this problem become

$$\begin{aligned} H^+(t, x, y, z, r, (q_1, q_2, q_3)) &\equiv \min_{\{v \in V^L;\, h(t,x,y,z) \leq r\}} \max_{|u| \leq M} \{q_1 \cdot f(t, x, y, z) + q_2 \cdot u + q_3 \cdot v\} \\ &= q_1 \cdot f(t, x, y, z) + M|u| - L|v| \qquad \text{if } r \geq h(t, x, y, z), \\ &= +\infty \qquad \text{if } r < h(t, x, y, z). \\ H^-(t, x, y, z, r, (q_1, q_2, q_3)) &\equiv \max_{|u| \leq M} \min_{\{v \in V^L;\, h(t,x,y,z) \leq r\}} \{q_1 \cdot f(t, x, y, z) + q_2 \cdot u + q_3 \cdot v\} \\ &= q_1 \cdot f(t, x, y, z) + M|u| - L|v| \qquad \text{if } r \geq h(t, x, y, z), \\ &= +\infty \qquad \text{if } r < h(t, x, y, z). \end{aligned}$$

We conclude that $H^-(t,x,y,z,r,(q_1,q_2,q_3)) = H^+(t,x,y,z,r,(q_1,q_2,q_3))$ and so, by uniqueness, the differential game associated with (4.1)–(4.5) has a value for each M and $L > 0$, say

$V^{M,L}(t, x, y, z)$. Furthermore, $V^{M,L}$ satisfies (see remark 2.6)

$$\max\{V_t^{M,L} + \nabla_x V^{M,L} \cdot f(t, x, y, z) + M|\nabla_y V^{M,L}| - L|\nabla_z V^{M,L}|, h(t, x, y, z) - V^{M,L}\} = 0, \tag{4.6}$$

if $t \in (0, T)$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, $z \in \mathbb{R}^q$, and

$$V^{M,L}(T, x, y, z) = h(T, x, y, z). \tag{4.7}$$

We now make the additional assumption that $Y \equiv [0, 1]^{q_2}$ and $Z \equiv [0, 1]^{q_1}$, i.e. Y and Z are the unit cubes in $\mathbb{R}^{q_2}$ and $\mathbb{R}^{q_1}$, respectively. Further, we assume that f and h are periodic as functions of y_j $(1 \le j \le q_2)$ and z_j $(1 \le j \le q_1)$, respectively.

THEOREM 4.1. Under the preceding hypotheses we have

(i) $\lim_{L\to\infty} \lim_{M\to\infty} V^{M,L}(t, x, y, z) = V^+(t, x)$

(ii) $\lim_{M\to\infty} \lim_{L\to\infty} V^{M,L}(t, x, y, z) = V^-(t, x)$,

where $V^\pm$ are the upper and lower value functions of the differential game associated with the dynamics (0.1)–(0.2) and payoff (0.3) with the control sets $Y \equiv [0, 1]^{q_2}$ and $Z \equiv [0, 1]^{q_1}$.

Proof. The proof is similar to that of the classical assertion in [4] so we will only give the pertinent distinct details. Also, we will only do so for part (ii) of the theorem since part (i) is similar. The approach is to show that the iterated limit in (ii) satisfies the same equation as V^-. Uniqueness then allows us to conclude.

First, it is true that $V^{M,L}$ and all of its first derivatives are bounded, independently of M and L. Indeed, the proof is essentially the same as that of lemma 4.1 of [4]. In our case we would use the approximating problem (for V^+)

$$\lambda \Delta V + V_x \cdot f + M|V_y| - L|V_z| + \beta_\varepsilon(\min \max h - V) = 0$$

to bound the derivatives. The derivatives can also be bounded using direct game theory methods as in [2].

Notice that we have monotonicity in M and L in the sense

$$V^{M,L'} \le V^{M,L} \le V^{M',L}, \qquad \text{for } M' \ge M \text{ and } L' \ge L.$$

We start with the fact that there is a bounded, Lipschitz function $V^M = V^M(t, x, z)$, such that, at least on a subsequence, $V^{M,L} \to V^M$ as $L \to \infty$, uniformly.

LEMMA 4.2. $V^M(t, x, y)$ is the unique viscosity solution of

$$\max\{V_t^M(t, x, y) + \min_{Z^y(t,x,V^M)} f(t, x, y, z) \cdot \nabla_x V^M(t, x, y) + M|\nabla_y V^M(t, x, y)|,$$

$$\min_{z \in Z} h(t, x, y, z) - V^M(t, x, y)\} = 0, \tag{4.8}$$

with $V^M(T, x, y) = \min_{z \in Z} h(T, x, y, z)$. V^M is the value of the differential game associated with (4.1)–(4.2) and payoff (4.5) in which ζ is allowed to choose any measurable function in $\mathcal{Z}[t, T]$ and η must be Lipschitz with constant M.

Remark. Equivalently, by taking the min to be $+\infty$ on an empty set,

$$V_t^M(t,x,y) + \min_{Z^y(t,x,V^M)} f(t,x,y,z) \cdot \nabla_x V^M(t,x,y) + M|\nabla_y V^M(t,x,y)| = 0.$$

Proof. We will only prove the first assertion.

Suppose that $V^M - \varphi$ has a local minimum at $(t_0, x_0, y_0) \in (0,T) \times \mathbb{R}^n \times Y$. By uniform convergence, there is a sequence of points $(t_L, x_L, y_L, z_L) \in (0,T) \times \mathbb{R}^n \times Y \times Z$, such that, for large L, $V^{M,L} - \varphi$ has a local minimum at (t_L, x_L, y_L, z_L) and $(t_L, x_L, y_L) \to (t_0, x_0, y_0)$ as $L \to \infty$. Since $V^{M,L}$ is the viscosity solution of (4.6), this implies that at (t_L, x_L, y_L, z_L)

$$\max\{\varphi_t + \nabla_x\varphi \cdot f(t_L,x_L,y_L,z_L) + M|\nabla_y\varphi| - L|\nabla_z\varphi|, h(t_L,x_L,y_L,z_L) - V^{M,L}(t_L,x_L,y_L,z_L)\} \le 0. \tag{4.9}$$

We have then

$$\min_{z\in Z} h(t_L, x_L, y_L, z) \le h(t_L, x_L, y_L, z_L) \le V^{M,L}(t_L, x_L, y_L, z_L).$$

Letting $L \to \infty$, we get

$$\min_{z\in Z} h(t_0, x_0, y_0, z) \le V^M(t_0, x_0, y_0). \tag{4.10}$$

Also, noting that φ is independent of z, (4.9) also tells us that

$$\varphi_t + \nabla_x\varphi \cdot f(t_L,x_L,y_L,z_L) + M|\nabla_y\varphi| \le 0 \qquad \text{at } (t_L, x_L, y_L, z_L). \tag{4.11}$$

Since $\{z_L\}$ lies in a compact set, we may assume that $z_L \to \tilde{z} \in Z$. Let $\varepsilon > 0$ and take L sufficiently large such that

$$|h(t_L, x_L, y_L, z_L) - h(t_0, x_0, y_0, \tilde{z})| < \varepsilon$$

and also

$$0 \le V^{M,L}(t_L, x_L, y_L, z_L) - V^M(t_0, x_0, y_0) < \varepsilon.$$

Then, from (4.9) we have that $V^M(t_0, x_0, y_0) + \varepsilon \ge V^{M,L}(t_L, x_L, y_L, z_L) \ge h(t_L, x_L, y_L, z_L) \ge h(t_0, x_0, y_0, \tilde{z}) - \varepsilon$. Consequently, $\tilde{z} \in Z^{y_0}(t_0, x_0, V^M(t_0, x_0, y_0) + 2\varepsilon)$. Letting $L \to \infty$ in (4.11) we obtain that

$$\varphi_t(t_0,x_0,y_0) + \min_{Z^{y_0}(t_0,x_0,V^M+2\varepsilon)} f(t_0,x_0,y_0,z) \cdot \nabla_x\varphi(t_0,x_0,y_0) + M|\nabla_y\varphi(t_0,x_0,y_0)| \le 0.$$

Since ε was arbitrary,

$$\varphi_t(t_0,x_0,y_0) + \min_{Z^{y_0}(t_0,x_0,V^M+0)} f(t_0,x_0,y_0,z) \cdot \nabla_x\varphi(t_0,x_0,y_0) + M|\nabla_y\varphi(t_0,x_0,y_0)| \le 0.$$

Combining this with (4.10) we have

$$\begin{aligned}\max\{\varphi_t(t_0,x_0,y_0) &+ \min_{Z^{y_0}(t_0,x_0,V^M+0)} f(t_0,x_0,y_0,z) \cdot \nabla_x\varphi(t_0,x_0,y_0) \\ &+ M|\nabla_y\varphi(t_0,x_0,y_0)|, \min_{z\in Z} h(t_0,x_0,y_0,z) - V^M(t_0,x_0,y_0)\} \le 0,\end{aligned}$$

and so V^M is a viscosity subsolution of (4.8).

Suppose that $V^M - \varphi$ has a local maximum at $(t_0, x_0, y_0) \in (0,T) \times \mathbb{R}^n \times Y$, with φ continuously differentiable. Let $z_0 \in \operatorname{int}(Z)$ and choose a smooth function $\gamma: Z \to \mathbb{R}^1$ with $0 \le \gamma \le 1$, $\gamma(z_0) = 1$ and $\gamma(z) < 1$ if $z \ne z_0$, and γ has compact support in $\operatorname{int}(Z)$. We set $\psi \equiv \varphi - \gamma$. Then $V^M - \psi$ has a strict local max at (t_0, x_0, y_0, z_0). By uniform convergence, there is a sequence of points $(t_L, x_L, y_L, z_L) \in (0, T) \times \mathbb{R}^n \times Y \times Z$, such that, for large L, $V^{M,L} - \varphi$ has a local maximum at (t_L, x_L, y_L, z_L) and $(t_L, x_L, y_L, z_L) \to (t_0, x_0, y_0, z_0)$ as $L \to \infty$. Since $V^{M,L}$ is the viscosity solution of (4.6), this implies that at (t_L, x_L, y_L, z_L)

$$\max\{\varphi_t + \nabla_x \varphi \cdot f(t_L, x_L, y_L, z_L) + M|\nabla_y \varphi| - L|\nabla_z \gamma|, h(t_L, x_L, y_L, z_L) - V^{M,L}(t_L, x_L, y_L, z_L)\} \geq 0. \tag{4.12}$$

If $h(t_L, x_L, y_L, z_L) - V^{M,L}(t_L, x_L, y_L, z_L) \geq 0$, then letting $L \to \infty$ we get $h(t_0, x_0, y_0, z_0) \geq V^M(t_0, x_0, y_0)$. Since $z_0 \in Z$ was arbitrary, we obtain

$$\min_{z \in Z} h(t_0, x_0, y_0, z) - V^M(t_0, x_0, y_0) \geq 0. \tag{4.13}$$

If, on the other hand,

$$\varphi_t + \nabla_x \varphi \cdot f(t_L, x_L, y_L, z_L) + M|\nabla_y \varphi| - L|\nabla_z \gamma| \geq 0,$$

then also

$$\varphi_t + \nabla_x \varphi \cdot f(t_L, x_L, y_L, z_L) + M|\nabla_y \varphi| \geq 0 \qquad \text{at } (t_L, x_L, y_L, z_L).$$

Let $L \to \infty$ and we get

$$\varphi_t + \nabla_x \varphi \cdot f(t_0, x_0, y_0, z_0) + M|\nabla_y \varphi| \geq 0 \qquad \text{at } (t_0, x_0, y_0).$$

Therefore,

$$\varphi_t + \min_{z \in Z} \nabla_x \varphi \cdot f(t_0, x_0, y_0, z) + M|\nabla_y \varphi| \geq 0 \qquad \text{at } (t_0, x_0, y_0),$$

and so,

$$\varphi_t + \min_{z \in Z^{y_0}(t_0, x_0, V^M - 0)} \nabla_x \varphi \cdot f(t_0, x_0, y_0, z) + M|\nabla_y \varphi| \geq 0 \qquad \text{at } (t_0, x_0, y_0). \tag{4.14}$$

Combining (4.14) and (4.13) we see that V^M is a viscosity supersolution of (4.8) as well. This completes the proof of the lemma.

We have left to prove the following lemma.

Lemma 4.3. $\lim_{M \to \infty} V^M(t, x, y) = W(t, x)$ and $W \equiv V^-$.

Proof. As mentioned earlier it is not hard to show that the limit in the assertion exists, locally uniformly. We need only show that W satisfies the equation (2.5) (see also (2.22)) for V^-.

Suppose that $W - \varphi$ has a local minimum at $(t_0, x_0) \in (0, T) \times \mathbb{R}^n$. Let $y_0 \in Y$ be arbitrary. As in the proof of lemma 4.2, there is a smooth function $\gamma: Y \to \mathbb{R}^1$, such that, setting $\psi(t, x, y) \equiv \varphi(t, x) + \gamma(y)$, $W - \psi$ has a strict local minimum at (t_0, x_0, y_0). By uniform convergence, there is a sequence of points $(t_M, x_M, y_M) \in (0, T) \times \mathbb{R}^n \times Y$, such that, for large M, $V^M - \varphi$ has a local minimum at (t_M, x_M, y_M) and $(t_M, x_M, y_M) \to (t_0, x_0, y_0)$ as $M \to \infty$. Since V^M is the viscosity solution of (4.8), this implies that at (t_M, x_M, y_M)

$$\max\{\varphi_t + M|\nabla_y \gamma| + \min_{z \in Z^{y_M}(t_M, x_M, V^M + 0)} \nabla_x \varphi \cdot f(t_M, x_M, y_M, z),$$
$$\min_{z \in Z} h(t_M, x_M, y_M, z) - V^M(t_M, x_M, y_M)\} \leq 0. \tag{4.15}$$

We have then

$$\min_{z \in Z} h(t_0, x_0, y_0, z) = \lim_{M \to \infty} \min_{z \in Z} h(t_M, x_M, y_M, z) \leq \lim_{M \to \infty} V^M(t_M, x_M, y_M) = W(t_0, x_0).$$

Therefore, since $y_0 \in Y$ was arbitrary

$$\max_{y \in Y} \min_{z \in Z} h(t_0, x_0, y, z) \leq W(t_0, x_0). \tag{4.16}$$

Also, we obtain from (4.15) noting that $M|\nabla_y \gamma| \geq 0$

$$\varphi_t + \min_{z \in Z^{y_M}(t_M, x_M, V^M+0)} \nabla_x \varphi \cdot f(t_M, x_M, y_M, z) \leq 0 \qquad \text{at } (t_M, x_M, y_M). \tag{4.17}$$

Using the lower semicontinuity of the Hamiltonian in (4.17) (see lemma 2.3 (2.2)) we let $M \to \infty$ to obtain that

$$\varphi_t + \min_{z \in Z^{y_0}(t_0, x_0, W(t_0, x_0)+0)} \nabla_x \varphi \cdot f(t_0, x_0, y_0, z) \leq 0 \qquad \text{at } (t_0, x_0).$$

Since this is true for any $y_0 \in Y$, we see that

$$\varphi_t + \max_{y \in Y} \min_{z \in Z^y(t_0, x_0, W(t_0, x_0)+0)} \nabla_x \varphi \cdot f(t_0, x_0, y, z) \leq 0 \qquad \text{at } (t_0, x_0).$$

Combining this with (4.16) we have

$$\max\{\varphi_t(t_0, x_0, y_0) + \max_{y \in Y} \min_{z \in Z^y(t_0, x_0, W(t_0, x_0)+0)} \nabla_x \varphi \cdot f(t_0, x_0, y, z),$$
$$\max_{y \in Y} \min_{z \in Z} h(t_0, x_0, y, z) - W(t_0, x_0)\} \leq 0,$$

and so W is a viscosity supersolution of (2.5) by proposition 2.5 (2.22).

Next, suppose that $W - \varphi$ has a local maximum at $(t_0, x_0) \in (0, T) \times \mathbb{R}^n$, with φ continuously differentiable. By uniform convergence, there is a sequence of point $(t_M, x_M, y_M) \in (0, T) \times \mathbb{R}^n \times Y$, such that, for large M, $V^M - \varphi$ has a local maximum at (t_M, x_M, y_M) and $(t_M, x_M) \to (t_0, x_0)$ as $M \to \infty$. Since V^M is the viscosity solution of (4.8), this implies that at (t_M, x_M, y_M)

$$\max\{\varphi_t + \min_{z \in Z^{y_M}(t_M, x_M, V^M+0)} \nabla_x \varphi \cdot f(t_M, x_M, y_M, z),$$
$$\min_{z \in Z} h(t_M, x_M, y_M, z) - V^M(t_M, x_M, y_M)\} \geq 0. \tag{4.18}$$

If $\min_{z \in Z} h(t_M, x_M, y_M, z) - V^M(t_M, x_M, y_M) \geq 0$, then

$$\max_{y \in Y} \min_{z \in Z} h(t_M, x_M, y, z) - V^M(t_M, x_M, y_M) \geq 0.$$

Taking $M \to \infty$ we get

$$\max_{y \in Y} \min_{z \in Z} h(t_0, x_0, y, z) - W(t_0, x_0) \geq 0.$$

On the other hand, if

$$\varphi_t + \min_{z \in Z^{y_M}(t_M, x_M, V^M+0)} \nabla_x \varphi \cdot f(t_M, x_M, y_M, z) \geq 0,$$

then

$$\varphi_t + \max_{y \in Y} \min_{Z^y(t_M, x_M, V^M(t_M, x_M)+0)} \nabla_x \varphi \cdot f(t_M, x_M, y, z) \geq 0 \qquad \text{at } (t_M, x_M).$$

Let $M \to \infty$ and use lemma 2.3 to get

$$\varphi_t + \max_{y \in Y} \min_{Z^y(t_0, x_0, W(t_0, x_0)+0)} \nabla_x \varphi \cdot f(t_0, x_0, y, z) \geq 0 \qquad \text{at } (t_0, x_0).$$

We conclude that

$$\max\{\varphi_t(t_0, x_0, y_0) + \max_{y \in Y} \min_{Z^y(t_0, x_0, W(t_0, x_0)+0)} \nabla_x \varphi \cdot f(t_0, x_0, y, z),$$
$$\max_{y \in Y} \min_{z \in Z} h(t_0, x_0, y, z) - W(t_0, x_0)\} \geq 0,$$

and W is a viscosity subsolution of (2.5), i.e. (2.22) as well. This completes the proof of the lemma and theorem 4.1 also. ■

REFERENCES

1. BARLES G. & PERTHAME B., Discontinuous solutions of deterministic optimal stopping time problems, *Math. Modelling Numer. Analysis* **21**, 557-579 (1987).
2. BARRON E. N., Differential games with Lipschitz control functions and fixed initial control positions, *J. diff. Eqns* **26**, 161-180 (1977).
3. BARRON E. N. & ISHII H., The Bellman equation for minimizing the maximum cost, *Nonlinear Analysis* **3**, 1067-1090 (1989).
4. BARRON E. N., EVANS L. C. & JENSEN R., Viscosity solutions of Isaacs' equations and differential games with Lipschitz controls, *J. diff. Eqns* **53**, 213-233 (1984).
5. BENSOUSSAN A. & LIONS, J. L., *Applications of Variational Inequalities in Stochastic Control, Studies in Mathematics and its Applications*, Vol. 12. North Holland, Amsterdam (1982).
6. BERKOVITZ L. D., The existence of value and saddle point in games of fixed duration, *SIAM J. Control Optim.* **23**, 172-196 (1985).
7. CRANDALL M. G. & LIONS P.-L., Viscosity solutions of Hamilton-Jacobi equations, *Trans. Am. math. Soc.* **277**, 1-42 (1983).
8. ELLIOTT R. J. & KALTON N. J., The existence of value in differential games, *Mem. Am. math. Soc.* **126** (1972).
9. EVANS L. C. & SOUGANIDIS P., Differential games and representation formulas for Hamilton-Jacobi equations, *Indiana Univ. math. J.* **33**, 773-795 (1984).
10. ISHII H., Hamilton-Jacobi equations with discontinuous Hamiltonians on arbitrary open sets, *Bull. Fac. Sci. Engng, Chuo Univ.* **28** 33-77 (1985).
11. KRASOVSKII N. N. & SUBBOTIN A. I., *Game-Theoretical Control Problems*. Springer, New York (1988).
12. LIONS P.-L., *Generalized Solutions of Hamilton-Jacobi Equations*. Pitman, London (1982).