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Maximin, viability and sustainability

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ABSTRACT

The maximin criterion defines the highest utility level that can be sustained in an intergenerational equity perspective. The viability approach makes it possible to characterize all the economic trajectories sustaining a given, not necessarily maximal, utility level. In this paper, we exhibit the strong links between *maximin* and *viability*: we show that the value function of the maximin problem can be obtained in the viability framework, and that the maximin path is a particular viable path. This result allows us to extend the recommendations of the maximin approach beyond optimality, to characterize the sustainability of economic trajectories which differ from the maximin path. Attention is especially paid to non-negative net investment at maximin accounting prices, which is shown to be necessary to maintain the productive capacity of the economy, whether the development path is optimal or not. Our results provide a new theoretical ground to account for sustainability in imperfect economies, based on maximin prices.

1. Introduction

Discounted utility, the main criterion used for intertemporal choice in economics, defines the Net Present Value of the economy (Weitzman, 2003) and provides a theoretical basis of National Net Product index used for national accounting (Weitzman, 1976; Dasgupta and Mäler, 2000; Arrow et al., 2003).

In the Weak Sustainability paradigm *à la Solow* (Neumayer, 2010), "sustainability ... must amount to an injunction to preserve the productive capacity for the indefinite future" (Solow, 1993, p. 163). A challenge to operationalize sustainability is to determine an index to measure it and define sustainability accounting (Cairns, 2008; Dasgupta, 2009). A first attempt to tackle this challenge is to complete the National Net Product by accounting for all sorts of capital stocks and consumptions, including natural resources depreciation and non-market goods, to obtain a "comprehensive" accounting (Repetto et al., 1989; Asheim, 1994; Weitzman and Löfgren, 1997; Cairns, 2003). Such an approach can be applied to imperfect economies (Arrow et al., 2003). This approach is usually developed in the theoretical vein of discounted utility, which has been criticized in the sustainability literature and qualified as a "dictatorship of the present" by Chichilnisky (1996). Alternative criteria have been proposed to deal with sustainability issues (Heal, 1998). If sustainability requires the sustaining of utility for intergenerational equity concerns, a criterion to address this issue can be the *maximin* (Solow, 1974; Cairns and Long, 2006). This criterion emerges from the Rawls (1971) conception of

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justice and equity. It maximizes the utility of the poorest generation (or the minimal utility over time in a continuous time framework). An interesting feature of this approach is that the maximin path, if egalitarian and efficient, satisfies Hartwick's rule (Withagen and Asheim, 1998; Mitra, 2002), which requires the investing of rents from exhaustible resources in reproducible capital to compensate for the depletion of their stocks (Hartwick, 1977). This rule has been generalized, that is to say that a nil net investment is required to keep the total productivity of all stocks constant, and to sustain the consumption or utility (Dixit et al., 1980). This rule, related to the concept of genuine savings, is argued to be a condition for sustainability and a basis for sustainability accounting (Solow, 1986, 1993). Cairns (2008), however, emphasizes that, even if it is optimal to follow Hartwick's investment rule along egalitarian maximin paths, a nil net investment does not imply sustained utility in distorted economies. For instance, Martinet (2007) shows in an example that following Hartwick's investment rule along a constant consumption path at a level different from the maximin may reduce sustainability. As it is not straightforward to compute the sustainability indicators provided by the maximin approach for real economies (Asheim, 1994), its results are difficult to apply. An important theoretical challenge to address sustainability is thus to extend the recommendations of the maximin approach to study the sustainability of economies that are not at the maximin optimum.

In this paper, we propose a framework to extend maximin beyond optimality. This framework is based on the *viability* approach (Aubin, 1991) or weak-invariance approach (Clarke et al., 1995) which characterizes intertemporal dynamic trajectories regarding their consistency with given state and control constraints. Interpreting viability constraints as minimal rights to be guaranteed to all generations, the viability approach can be used to address the sustainability issue (Martinet and Doyen, 2007; Baumgärtner and Quaas, 2009; Martinet, 2011). It has been notably applied to the sustainable management of renewable resources (e.g., Béné et al., 2001; Doyen and Péreau, 2012; Péreau et al., 2012). In most of these viability studies, the so-called *viability kernel* plays a major mathematical role. This set is the set of all initial (economic) states from which start viable (economic) trajectories, i.e., trajectories respecting the given (sustainability) constraints at all times. Therefore, the viability approach can be used to define all the economic trajectories sustaining a specific, not necessarily maximal, utility level. From that point of view, the viability approach provides a relevant tool to study the sustainability of "sub-optimal economies" which differ from the maximin path.

We exhibit the strong links between *maximin* and *viability*. More specifically, we show that the value function of the maximin problem is the solution of a static optimization problem under constraints involving the viability kernel. Maximin trajectories are shown to be particular viable trajectories, and thus inherit viability properties. Our results are given in a general and abstract framework, and are valid for regular and non-regular maximin problems. Particular emphasis is put on the Hamiltonian formulation of the viability problem, that we interpret as a *weak* Hartwick rule. We relate this result to non-negative net investment *at maximin prices*, and describe how it makes it possible to characterize the sustainability of any development path, providing theoretical grounds for sustainability accounting in imperfect economies.

We first present in Section 2 the links between maximin and viability in terms of states and value functions. We then present in Section 3 how these links allow us to characterize maximin trajectories within the viability framework. In Section 4, we discuss the potential use of our framework, which extends maximin with viability, to examine the sustainability of trajectories which are not maximin paths. The implications of our results in terms of sustainability accounting are presented in Section 5. We conclude in Section 6. The appendix gathers mathematical details and the proofs of propositions (Appendix A), along with an illustration of our results to the canonical Dasgupta–Heal–Solow model often used to investigate sustainability issues (Dasgupta and Heal, 1974, 1979; Solow, 1974; Heal, 1998) (Appendix B).

2. Maximin and viability

2.1. A general dynamic economic model

Consider an economy with n capital stocks (e.g., manufactured capital, labor or natural resources) and m economic decision parameters (e.g., consumption, investment or resource extraction). This economy is characterized by the state $X(t) \in \mathbb{R}^n$ and the control $u(t) \in \mathbb{R}^m$. All the economic dynamics are captured by a function $f : \mathbb{R}^n \times \mathbb{R}^m \mapsto \mathbb{R}^n$ which may involve capital dynamics, production functions or natural resource growth functions. This economy is represented by the controlled dynamic system[1]

$$\dot{X}(t) = f(X(t),u(t)), \quad t \in \mathbb{R}_+. \tag{1}$$

At each time t, states and controls have to belong to some admissibility set represented by q inequalities (e.g., positivity of consumption, irreversibility of investment, availability of labor, scarcity of resources)

$$g_i(X(t),u(t)) \geq 0 \quad \text{for } i = 1,\dots,q. \tag{2}$$

Initial economic state at time $t_0 = 0$ is denoted by $X(t_0) = X_0$. We shall denote by $X(\cdot)$ and $u(\cdot)$ state and control trajectories.

[1] We focus on time autonomous problems for the sake of exposition clarity. Focusing on time autonomous problems excludes the possibility of exogenous technical change; but endogenous technical change can be accounted for (see Martinet and Doyen, 2007, Section 3.4.2.2, for an example in the Dasgupta–Heal–Solow model).

2.2. The maximin approach

Consider a specific payoff function $L(X(t),u(t))$ which may depend on states and controls. In economic terms, this payoff captures instantaneous utility, i.e., the utility of the generation living at time t, under the usual assumption in continuous time models that each generation lives for one instant.

The maximin approach (Solow, 1974; Cairns and Long, 2006) aims at maximizing the minimal level of utility over time. In other words, the maximin criterion defines the maximal level of utility that can be sustained given economic endowments, i.e., from the initial economic state X_0. Hence, the maximin value function $V:\mathbb{R}^n \to \mathbb{R}$ is defined by

$$V(X_0) = \sup_{(X(\cdot),u(\cdot))} \left(\inf_{t \in \mathbb{R}_+} L(X(t),u(t)) \right) \quad \text{s.t.} \begin{cases} \dot{X}(t) = f(X(t),u(t)), \\ g_i(X(t),u(t)) \geq 0, \quad i = 1,\ldots,q, \\ X(t_0) = X_0. \end{cases} \tag{3}$$

Whenever the supremum is reached and corresponds to a maximum,[2] this criterion defines an optimal economic trajectory $X^\star(\cdot)$, with associated optimal economic decisions $u^\star(\cdot)$.

When a *regular* maximin path exists (Burmeister and Hammond, 1977; Cairns and Long, 2006), utility is constant over time, i.e., $L(X^\star(t),u^\star(t)) = V(X_0)$ for all $t \in \mathbb{R}_+$. This can be interpreted as intergenerational equity (equality) from a sustainability point of view (Heal, 1998). However, this constant utility should not hide an important feature of the maximin approach: its dynamic nature. In fact, at any point of a regular maximin path, the utility is equal to the maximin value of the associated economic state, i.e., $L(X^\star(t),u^\star(t)) = V(X^\star(t))$. This is because the maximin value is constant along regular maximin paths, that utility is constant.[3] In such a regular case, it turns out that the optimal controls satisfy Hartwick's rule, which, in such an abstract model, requires to keep the total value of net investment nil (Hartwick, 1977, 1978; Dixit et al., 1980; Withagen and Asheim, 1998; Mitra, 2002). Under this rule, natural resources stock depletion is compensated for by reproducible capital accumulation, maintaining total productive capacity (Solow, 1986).

For *non-regular* maximin problems, the utility may increase or decrease over time along maximin paths, the maximin value being non-decreasing over time (Asako, 1980; Cairns and Tian, 2010).

2.3. The viability approach

The viability approach aims at studying the consistency between a dynamic system and a set of so-called *viability constraints*. It determines the conditions for these constraints to be satisfied at all times. These constraints can represent sustainability objectives to be achieved for all generations (Martinet and Doyen, 2007; Martinet, 2011). For example, it is possible to study the viability of an economy under a guaranteed utility constraint $L(X(t),u(t)) \geq L_{\min}$ to be satisfied over time.

The so-called *viability kernel* (Aubin, 1991) plays a major mathematical role in the viability analysis. It is the set of all states $X(t_0)$ such that from any of those states there are admissible decisions resulting in trajectories satisfying the given constraints at all times. Here, we consider a specific viability kernel associated with dynamics (1), inequality constraints (2), and the following viability constraint requiring a guaranteed utility $L_{\min}$ at all times:

$$L(X(t),u(t)) \geq L_{\min}, \quad \forall t \geq t_0. \tag{4}$$

The viability kernel depends on the guaranteed utility $L_{\min}$ as follows:

$$\mathrm{Viab}(L_{\min}) = \left\{ X_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists (X(\cdot),u(\cdot)) \text{ such that } \forall t \geq t_0, \\ \dot{X}(t) = f(X(t),u(t)), \\ X(t_0) = X_0, \\ g_i(X(t),u(t)) \geq 0, i = 1,\ldots,q \\ L(X(t),u(t)) \geq L_{\min}. \end{array} \right\}. \tag{5}$$

From a general mathematical point of view, this domain is a subset (potentially empty) of the state domain $\mathbb{R}^n$. All the states belonging to the viability kernel have a sufficient productive capacity to sustain (at least) a level of utility $L_{\min}$.

The viability kernel captures an irreversibility mechanism. From the very definition of this kernel, it is not possible to sustain $L_{\min}$ from any state lying outside the viability kernel; whatever are the decisions, trajectories leaving the viability kernel will eventually violate the constraints in finite time. This means that a necessary condition to sustain a given guaranteed utility level $L_{\min}$ is that the economic trajectory evolves within the associated viability kernel $\mathrm{Viab}(L_{\min})$.

[2] By convention, we set $V(X_0) = -\infty$ whenever the problem has no solution in the sense that, for any admissible control path $u(\cdot)$, the minimal payoff L is not bounded from below, namely $\inf_{t \in \mathbb{R}_+} L(X(t),u(t)) = -\infty$. Note also that our formulation of the maximin problem as a 'sup inf' problem is more general than the classical 'max min' formulation. In particular, it allows us to account for the cases of non-existent solution, when the supremum is not reached. In such a case, it is possible to define, for practical purpose, a development path sustaining a utility level chosen as close to the supremum as desired.

[3] For a clarifying description of the maximin approach, especially when the maximin path is non-regular, see Cairns and Tian (2010).

The velocity of a viable trajectory is thus inward or tangential to the viability kernel (Aubin, 1991, Theorem 6.1.4). We will use this important property in Section 3.

2.4. Maximin as the optimization of viability

In this section, we characterize the maximin value function $V(.)$, defined by Eq. (3), through a static optimization problem involving the viability kernel defined by Eq. (5). It allows us to interpret maximin as an extreme case of viability.

We start from the following simple proposition.

Proposition 1. *Assume the existence of a maximin optimal solution* $(X^\star(\cdot),u^\star(\cdot))$ *starting from state* X_0. *Then*

$$X_0 \in \mathrm{Viab}(V(X_0)).$$

The proof of the proposition is presented in the appendix.

The interpretation of this simple proposition is that an economic endowment X_0 makes it possible to guarantee a utility level equal to the initial maximin value $V(X_0)$.

We now present the main proposition of the paper, which will be the corner stone of our key results. This proposition states that the maximin value of a state X_0 is the highest level of the viability constraint such that the given state is within the associated viability kernel.

Proposition 2. *For any initial conditions* X_0, *we have*

$$V(X_0) = \sup\,(L_{\min} \,|\, X_0 \in \mathrm{Viab}(L_{\min})).$$

The proof of the proposition is detailed in the appendix.[4]

We interpret this result as follows. We know that a utility level $L_{\min}$ is sustainable from initial state X_0 if and only if X_0 belongs to the viability kernel $\mathrm{Viab}(L_{\min})$. The higher the level of utility $L_{\min}$ to be sustained, the less numerous the initial economic states making it possible to sustain it, i.e., the smaller the viability kernel $\mathrm{Viab}(L_{\min})$. The maximal sustainable utility (maximin value) corresponds to the highest guaranteed utility for which the associated viability kernel contains initial state X_0. Whenever this is possible, the guaranteed utility level $L_{\min}$ is increased until the state X_0 lies on the boundary of the viability kernel. When this is not possible, the state X_0 is located in the interior of the viability kernel.[5] Proposition 2 also obviously means that, from a given X_0, no utility greater than the maximin value $V(X_0)$ can be sustained.

The interpretation of this proposition is quite simple but has powerful implications. It means that the maximin value can be defined within the viability framework using a static optimization problem on the viability kernel. Even if this method is not necessarily simpler than the standard approach to solve maximin problems, it allows us to characterize the maximin path as a particular viable trajectory. Whenever the solution of a given optimization problem can be formulated in terms of a viability kernel, the solution inherits the properties of the kernel. Such properties will allow us to exhibit that maximin paths[6] satisfy a *weak* Hartwick rule, that is interpreted as a non-negative net investment at maximin prices. This property can be extended to consider the sustainability of other (sub-optimal) trajectories, and provides a theoretical ground for sustainability accounting with maximin prices.

3. The maximin path characterized as a viable trajectory

3.1. Maximin trajectory and viability kernels

We first show that the maximin trajectory evolves within the viability kernel associated with the constraint level $V(X_0)$. This result is valid whether the maximin path is regular or not.

Proposition 3. *Assume that there exists a maximin optimal solution* $(X^\star(\cdot),u^\star(\cdot))$ *starting from state* X_0. *Then*

$$X^\star(t) \in \mathrm{Viab}(V(X_0)), \quad \forall t \geq t_0.$$

The proof of the proposition is given in the appendix.

According to Proposition 3, the maximin trajectory remains within the viability kernel associated with the constraint $L(X(t),u(t)) \geq V(X_0)$. In the viability framework, this trajectory is viable for the constraint $L(X(t),u(t)) \geq V(X_0)$. Consequently, it can be characterized using the properties of viable trajectories. In particular, viable trajectories must either evolve within the interior of the kernel $\mathrm{Viab}(V(X_0))$ or on its boundary as illustrated by Fig. 1.

[4] Again, by convention, we set $\sup\,(L_{\min} \,|\, X_0 \in \mathrm{Viab}(L_{\min})) = -\infty$ whenever the problem has no solution in the sense that $X_0 \notin \bigcup_{L_{\min}} \mathrm{Viab}(L_{\min})$.

[5] We shall see in Section 3.3 how the maximin path is non-regular in such a case.

[6] Either regular or non-regular paths, and whether the maximin solution is unique or not.

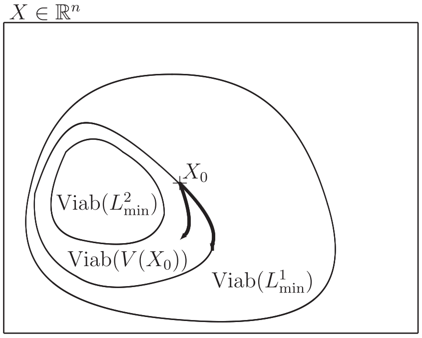

Fig. 1. Viability kernels and maximin trajectories. $L^2_{\min} > V(X_0) > L^1_{\min}$. The higher the viability constraint, the smaller the viability kernel. The maximin trajectories evolve either along the boundary of the kernel $\mathrm{Viab}(V(X_0))$, or inside it.

All the states of the viability kernel $\mathrm{Viab}(V(X_0))$ correspond to combinations of capital stocks that make it possible to sustain utility $V(X_0)$ over time. This means that, as the maximin trajectory stays in the viability kernel, the productive capacity of the economy, as defined by Solow (1993), is maintained. This has sound connections with the idea to maintain the total capital productivity constant and with Hartwick's investment rule (Hartwick, 1977; Dixit et al., 1980) or, more generally, with nonnegative net investment.

3.2. *Hamiltonian formulation of the viability problem: a weak Hartwick rule*

Viable trajectories evolve inside the viability kernel or along its boundary. A trajectory starting from a point of the boundary is thus locally inward or tangential to the viability kernel (Aubin, 1991). It is possible to characterize these trajectories within the viability framework using a Hamiltonian formulation and normal cones. For this purpose, we introduce the following Hamiltonian:

$$H(X,u,p) = \langle p, f(X,u) \rangle = \sum_{i=1}^{n} p_i f_i(X,u).$$

Following Aubin (1991, Definition 3.2.2 and Theorem 6.1.4), it turns out that the *viable decisions* $u(t)$ associated with a viable trajectory starting from $X(t) \in \mathrm{Viab}(L_{\min})$ are solutions of a specific Hamilton–Jacobi–Bellman inequality[7]

$$H(X(t),u(t),p(t)) \geq 0, \quad \forall p(t) \in -N_{\mathrm{Viab}(L_{\min})}(X(t)), \tag{6}$$

where $N_{\mathrm{Viab}(L_{\min})}(X)$ is the normal cone to set $\mathrm{Viab}(L_{\min})$ at state X. The interpretation of the notion of normal cone is the following. The normal to a smooth surface is a vector that is perpendicular to the tangential hyperplane of that surface. In this smooth case, the normal cone is the half-line generated by this normal (see Fig. 2). When the surface is non-smooth, the set of normals becomes a cone, as captured by Fig. A1 in the appendix.[8] Inequality (6) implies that the trajectory generated by $u(t)$ is tangential or inward to the surface at X. When X lies strictly in the set $\mathrm{Viab}(L_{\min})$, the normal cone is reduced to vector 0, and condition (6) is satisfied for any control. When the guaranteed utility threshold coincides with the maximin value, namely $L_{\min} = V(X)$, these normal cones are directly connected to the derivative $V_x(X)$ of the maximin value function by the relation

$$V_x(X) \in -N_{\mathrm{Viab}(V(X))}(X). \tag{7}$$

Such relation is derived from Proposition 2 linking maximin value $V(X)$ and the viability kernels $\mathrm{Viab}(L_{\min})$. The proof is given in Appendix A.5. The intuition, captured by Fig. 2, is that the viability kernels $\mathrm{Viab}(V(X))$ are the level sets of the maximin value V. Such a relation can be adapted in a non-smooth geometrical context using generalized gradients.

Consequently, from condition (6), we can derive a Hamilton–Jacobi condition for the maximin value function V. This is the purpose of Proposition 4. To achieve this, we first need to assume that the dynamic system (f,g,L) is regular enough in the sense defined below through assumptions H. We use the following convenient notation $A(X)$ for the control constraints

[7] Traditionally, the Hamiltonian condition is written as a non-positive Hamiltonian for all the vectors of the Normal cone, i.e., $H(X(t),u(t),p(t)) \leq 0$, $\forall p(t) \in N_{\mathrm{Viab}(L_{\min})}(X(t))$. This characterizes trajectories that do not exit the set $\mathrm{Viab}(L_{\min})$. We, however, reverse the inequality to emphasize the economic interpretation of the Hamiltonian. From an economic point of view, the vectors p in the normal cone are normal to a feasibility set, and are homogeneous to prices, up to the sign. The Hamiltonian condition can thus be interpreted as non-negative total investment, accounted at some prices.

[8] More detailed definitions and main properties of this normal cone can be found for instance in Aubin and Frankowska (1990) and Rockafellar and Wets (1998), and are recalled in Appendix A.4.

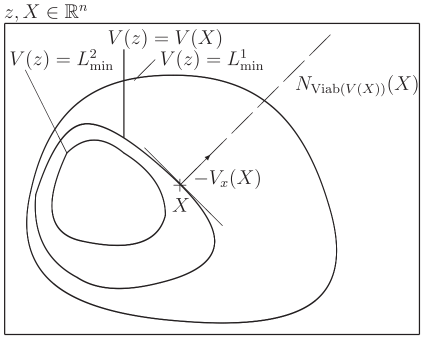

Fig. 2. The viability kernels Viab($L_{\min}$) are the level sets of the maximin value V. On the boundary of the kernel where $L_{\min} = V(X)$, the normal cone $N_{\mathrm{Viab}(V(X))}(X)$ reduces to the cone spanned by the gradient $-V_x(X)$.

(2) and payoff constraint (4), especially when the viability constraint equals the maximin value, i.e., $L_{\min} = V(X)$

$$A(X) = \{u \,|\, L(X,u) \geq V(X),\ g_i(X,u) \geq 0\}. \tag{8}$$

Assumption H.

H1 The functions f, g and L are continuous.
H2 The set of admissible velocities $F(X) = \{f(X,u) \,|\, u \in A(X)\}$ is convex for any state X.[9]
H3 The growth of f is bounded in the following sense.[10]

$$\exists c > 0 \quad \text{such that} \sup_{u \in A(X)} \frac{\|f(X,u)\|}{1+\|X\|} \leq c.$$

Proposition 4. *Under assumption H, for any X such that $V(X) > -\infty$, the maximin value function V is a solution of the following Hamilton Jacobi inequality*:

$$\sup_{u \in A(x)} H(X,u,V_x(X)) \geq 0. \tag{9}$$

Moreover, optimal controls are solutions of the Hamilton–Jacobi–Bellman inequality

$$u^\star(X) \in \operatorname{Arg}\max_{u \in A(x)} H(X,u,V_x(X)). \tag{10}$$

The proposition is proved in the appendix.

We interpret this Hamiltonian condition as a *weak* Hartwick rule: consider any maximin trajectory $X^\star(\cdot)$ associated with optimal controls $u^\star(\cdot)$.[11] Assuming that maximin value function V is continuously differentiable,[12] we deduce from previous Hamiltonian assertion captured by Eq. (6) that

$$\frac{d}{dt}V(X^\star(t)) = \left\langle V_x(X^\star(t)), \frac{d}{dt}X^\star(t) \right\rangle = \left\langle V_x(X^\star(t)), f(X^\star(t),u^\star(t)) \right\rangle = H(X^\star(t),u^\star(t),V_x(X^\star(t))) \geq 0. \tag{11}$$

Condition (11) means that, along any maximin path $X^\star(t)$, the time derivative of the maximin value function remains nonnegative. In other words, net investment computed at maximin prices is nonnegative, and the total productivity is non-decreasing.[13]

[9] Typically, this holds true when dynamics f is affine with respect to control u in the sense that $f(X,u) = a(X)+b(X)u$ and when L and g are convex functions with respect to control u.

[10] This is to avoid too explosive dynamics. Typically, this holds true when the derivative f_x is uniformly bounded for any control u, and the control constraints set $A(X)$ is bounded.

[11] The result remains valid for every maximin paths when there is not uniqueness.

[12] The result is also valid for states in which the maximin value is not differentiable, using the general condition (6), contingent derivatives, and normal cones, as described in the appendix.

[13] The Hamiltonian condition (6) is in fact stronger than this statement. Indeed, the "investment" condition has to be satisfied for all the vectors in the normal cone. For smooth cases, the normal cone is reduced to vectors proportional to the negative of the maximin prices (i.e., the partial derivatives of the maximin value with respect to the stocks), and the conditions (6) and (11) are equivalent. For non-smooth cases, the normal cone is not reduced to the negative of maximin prices. For example, in the one-sector Ramsey model (or the fishery model), at the golden rule capital stock (the MSY stock) the maximin price of capital (fish) is nil, and all decisions, including overconsumption (overfishing) decisions, are associated to a non-negative investment at the maximin price. The normal cone is, however, not reduced to zero, and is $\mathbb{R}^-$. Condition (6) implies that investment accounted at all non-negative price have to be non-negative, i.e., that the stock is not allowed to decline. This rules out unsustainable decisions.

An important remark is that the Hamiltonian condition (6) is valid for all the trajectories that remain in the viability kernel $\text{Viab}(V(x))$, and not only for the maximin paths. The *weak Hartwick rule* interpretation of viable trajectories can thus be extended to any path staying in the viability kernel, and is not limited to maximin paths. Section 4 builds on this rule to characterize the sustainability of any economic trajectory. We would like, however, to briefly depart from our main line, and give some comments on regular and non-regular maximin paths.

3.3. On non-regular maximin paths

Of interest is the fact that condition (11) is obtained without strong regularity assumptions on the maximin value function. Having an upper semi-continuous value function is sufficient, and the result is thus valid for problems in which other constraints g are binding. In such non regular cases, the HJB formulation relies on the use of contingent derivatives and generalized gradients. These notions are defined in Appendix A.4. In particular, our result holds true whether the maximin path is regular or not.

Even if our results hold true indistinctly for both type of problems, the viability characterization of maximin paths allows us to characterize the regularity (or non-regularity) of maximin problem in a way which has never been discussed in the literature. In particular, we can distinguish two types of non-regular maximin problems: in Section 2.2, we emphasized that regular maximin paths are such that $dV(X^\star(t))/dt = 0$ (Hartwick's rule) *and* $L(X^\star(t),u^\star(t)) = V(X^\star(t))$. We shall describe in which way the maximin problem is non-regular when one or the other of these conditions are not satisfied.

For regular maximin problems, Hartwick's rule is satisfied and the maximin value stays constant along the maximin path. This means that the Hamiltonian condition (11) is binding and thus that regular maximin paths remain on the boundary of the viability kernel: at each time, the derivative of the trajectory is tangential to the kernel. Moreover, regular maximin paths evolve along states with the same maximin value. In other words, all the states of the path have the same long-run productivity potential. These particular economic states have been interpreted as *capital valuation contours* by Burmeister and Hammond (1977). We can thus argue that the boundary of viability kernels correspond to these *capital valuation contours* for regular problems.

For non-regular maximin paths, Eq. (11) allows us to state that the maximin value is also non-decreasing. We can distinguish between two contrasted cases. On the one hand, for some t^* we may have $dV(X^\star(t^*))/dt = 0$, along with $L(X^\star(t^*),u^\star(t^*)) > V(X^\star(t^*))$. On the other hand, we may have $(d/dt)V(X^\star(t^*)) > 0$ with $L(X^\star(t^*),u^\star(t^*)) = V(X^\star(t^*))$ for some t^*. Each case corresponds to a special case of non-regularity which has been described in the literature.

The first case corresponds to non-regular paths characterized by "redundant" capital stocks, as described by Asako (1980). At some level, the stocks do not contribute to the maximin value.[14] For these redundant stocks levels, one has $V_x(X) = 0$. This is the case for the simple fishery model when the fish stock is above the Maximum Sustainable Yield stock, or for the Ramsey model when the capital stock is above the golden rule level of capital. In the viability framework, such non-regular maximin paths correspond to the case in which the initial state lies in the interior of the viability kernel associated with the maximin value. If the maximin value is "globally bounded" and the initial state of the economy is in the "redundant part," it is not be possible to increase the viability constraint up to the point the initial state is on the boundary of the viability kernel.[15] The maximin path then starts from the inside of the kernel and evolves strictly within it along the non-regular part of the path. The maximin value stays constant (Eq. (11)), even if the current utility is higher than the maximin value (so as the fishery catches above the MSY or the consumption in the Ramsey model above the golden rule). In such non-regular problems, the utility may be higher than the maximin value during a period of time (without reducing it) but will eventually decrease toward this maximin value.

By contrast, the second case corresponds to non-regular paths characterized by "locally bounded" utility, as described by Cairns and Tian (2010). When some constraints do not allow to smooth utility over time (e.g., labor availability in Cairns and Tian, 2010), utility may be increasing along a maximin path. Investment, in terms of the weak Hartwick rule (11), is positive, and the maximin value is increasing (so is the utility). In terms of viability, this means that such non-regular maximin paths start from the boundary of the viability kernel, but their velocity is inward (not tangential) to the kernel.

To sum up, when characterizing the maximin solution within the viability framework, if the initial state X_0 is not located on the boundary of the viability kernel $\text{Viab}(V(X_0))$, the maximin problem is non-regular of the "redundant stock type." If the initial stock belongs to the boundary of the viability kernel, but an admissibility constraint is binding and limits the current level of utility, one may have the other type of non-regularity. To our knowledge, these two types of non-regularity have never been formally distinguished before.

4. Characterizing the (un)sustainability of economic development paths

We have shown that the maximin path of a problem and more generally the maximin value function can be characterized using the viability approach, by maximizing the level of the viability constraint on minimal utility. We argue

[14] This occurs in particular when the maximin value $V(x)$ is "globally bounded" $\mathbb{R}^n$.

[15] In fact, the viability kernel does not retract continuously when the viability constraint increases, and the states with redundant stocks are on no kernel frontier. Formally, there are some states X being strictly within $\text{Viab}(V(X))$ but $\forall L_{\min} > V(X), \quad X \notin \text{Viab}(L_{\min})$. If the maximin value is globally bounded, the viability kernel reduces to the empty set once the viability constraint exceeds the upper limit of the maximin value.

that extending the maximin approach with viability makes it possible to study the sustainability of trajectories which differ from maximin paths. These trajectories are sub-optimal with respect to the maximin criterion in the sense that the sustained level of utility is lower than the maximin value. There are several reasons for a given trajectory to deviate from the maximin path. The most obvious one is when the maximin path is not pursued as an economic development path. Another particular case of such sub-optimal economy is studied by Martinet (2007) who examined how the maximin value function evolves along a constant consumption path which follows the Hartwick investment rule when consumption is different from the maximin consumption.

We shall consider two levels of knowledge of the trajectory under study. First we quickly examine the case in which the whole trajectory is known. Second, we consider the more interesting case in which only the current economic state and decisions are known.

4.1. Sustainability of a path

Consider a whole economic trajectory $(X(\cdot),u(\cdot))$, starting from $X(t_0)=X_0$ and defined by arbitrary open-loop controls or by a given feedback rule as in Arrow et al. (2003) and Vouvaki and Xepapadeas (2008). Assume that this trajectory deviates from the maximin path in the sense that

$$\exists\ T \text{ such that } L(X(T),u(T)) < V(X_0).$$

This trajectory does not sustain the maximin level of utility. An easy way to characterize the sustainability of this trajectory is to consider the minimal level of utility over time, i.e., the utility level which is actually sustained

$$\underline{L} = \inf_{t \geq t_0} L(X(t),u(t)).$$

In the viability framework, such a trajectory is contained in the viability kernel Viab($\underline{L}$) associated with this sustained level of utility. Although this sustained level is sub-optimal with respect to the maximin criterion (as $\underline{L} < V(X_0)$), it is well characterized in the viability framework.

4.2. Sustainability of the current state and decisions

If the whole economic trajectory is unknown, our framework still allows us to assess the sustainability of current decisions u_0. To know if current utility $L(X_0,u_0)$ is sustainable, we can examine the location of the economic state X_0 with respect to the viability kernel Viab($L(X_0,u_0)$). This provides a first, global condition for sustainability

$$\textit{Global condition for sustainability}: X_0 \in \text{Viab}(L(X_0,u_0)). \tag{12}$$

However, even in the favorable case where X_0 belongs to Viab($L(X_0,u_0)$), one may require the trajectory to remain (locally) in the viability kernel Viab($V(X_0)$), to maintain the productive capacity of the economy. This means that a marginal condition relying on the Hamiltonian condition (6) is relevant to characterize the sustainability of the economy, and in particular of economic decisions on investment. This provides a second, local condition for sustainability

$$\textit{Local condition for sustainability}: H(X_0,u_0,V_x(X_0)) \geq 0. \tag{13}$$

The intuition underlying such sustainability characterization is that the function $t \to V(X(t))$ is locally non-decreasing at time t_0 as in condition (11) or equivalently that the *weak* Hartwick rule (9) holds true.

In such a framework, we can discuss the three following cases depending on the current utility level $L(X_0,u_0)$:

- $L(X_0,u_0) > V(X_0)$. In this case, the current utility is higher than the maximin value. This case has been discussed in the literature and qualified as unsustainability by Pezzey (1997).[16] From Proposition 2, we deduce that $X_0 \notin \text{Viab}(L(X_0,u_0))$. In other words, the economy is faced with unsustainability of first kind: the current economic state does not make it possible to sustain the current utility.[17]
- $L(X_0,u_0) = V(X_0)$. In this case, the current utility is equal to the maximin value and consequently, the first or global condition (12) for sustainability is satisfied since, from Proposition 1, $X_0 \in \text{Viab}(V(X_0)) = \text{Viab}(L(X_0,u_0))$. However, the second or local condition of sustainability (13) related to the weak Hartwick rule may not hold depending on whether

[16] Pezzey (1997) describes three different constraints (using our notations, but Pezzey's terminology)
(1) If $L(X(t),u(t)) \leq V(X(t))$, $\forall t$, development is said to be sustainable.
(2) If $(dL(X(t),u(t))/dt) \geq 0$, $\forall t$, development is said to be *sustained*.
(3) If $L(X(t),u(t)) \geq L_{\min}$, development is said to be *survivable*. This last constraint corresponds to a viability constraint.

[17] Note that such unsustainability situation can also occur for some non-regular (optimal) maximin paths where the utility $L(X_0,u_0^\star)$ may be greater than the maximin value for some time. This is the case for example for the simple fishery economy, if the fish stock is larger than the Maximum Sustainable Yield Stock, or for the simple Ramsey model. In this case, the utility will eventually decrease to the maximin value after some finite time.

decisions u_0 is some optimal maximin feedback $u^\star(X_0)$ or not. Hence, regarding sustainability, we have to distinguish between two cases:

- ○ If the second or local condition of sustainability holds true, then from property (7) linking the derivative $V_x(X_0)$ of the maximin value function and the normal cone $N_{\mathrm{Viab}(V(X_0))}(X_0)$, we deduce that

$$\langle f(X_0,u_0),p\rangle \geq 0, \quad \forall p \in -N_{\mathrm{Viab}(V(X_0))}(X_0). \tag{14}$$

  This means equivalently that the velocity $f(X_0,u_0)$ is tangential or inward to the set Viab($V(X_0)$). In other words, the trajectory starting from state X_0 with current decision u_0 remains within the viability kernel Viab($V(X_0)$) at least during a short period of time.
- ○ By contrast, if u_0 does not comply with the second or local condition of sustainability, then $H(X_0,u_0,V_x(X_0))<0$. Using similar reasonings with normals, we deduce that the velocity $f(X_0,u_0)$ is outward to the set Viab($V(X_0)$). In other words, the trajectory starting from state X_0 with current decision u_0 leaves the viability kernel Viab($V(X_0)$). In such a negative case, the total productivity of the economy (the maximin value) locally decreases and the trajectory leaves the viability kernel Viab($V(X_0)$).

- $L(X_0,u_0)<V(X_0)$. In this case, the first condition for sustainability is satisfied as $X_0 \in \mathrm{Viab}(V(X_0)) \subset \mathrm{Viab}(L(X_0,u_0))$, and current utility can be sustained as it is lower than the maximin value. Nevertheless, as in the previous case, depending on current decision u_0 and in particular on investment, the trajectory may either stay within the viability kernel Viab($V(X_0)$) or leave it. In the former case, the maximin value does not decrease, which corresponds to a non-negative net investment at maximin prices, satisfying our second condition for sustainability. In the latter case, net investment computed at maximin prices is negative and the maximin value decreases.

5. A step toward sustainability accounting

In this section, we explain how the previous results could be used to define a framework for sustainability accounting along any development trajectory.

"A sustainable path … is one that *allows* every future generation *the option* of being as well off as its predecessors" (Solow, 1993, p. 163; our emphasize). Such a definition does not assume that utility *must* be non-decreasing, but rather considers the possibility to sustain utility. Sustainability accounting representing such a concern should thus focus on the preservation of the productive capacity of the economy.

Based on the results of this paper, we argue that studying the sustainability of an economy in the viability framework is equivalent to considering that a path is sustainable if simultaneously its current utility is no greater than the maximin level (first, global condition for sustainability) and our weak Hartwick rule applies (second, local condition for sustainability) such that the maximin value is non-decreasing at the current time. This characterizes altogether the current state, decision and dynamics of the economy. In particular, note that the maximin value, i.e., the productive capacity of the economy, may decrease even if utility is lower than its level. The first condition alone is not sufficient.

According to our analysis, what matters is the time derivative of the maximin value, and not that of the discounted utility. The difference is very important in terms of accounting. Unsustainability occurs if net investment *at maximin accounting prices* (local shadow values) is negative. Sustainability accounting should thus be based on maximin prices, which are related to the long-run productivity and sustainability of the economy, rather than on discounted utility prices which are related to the discounted, short-run utility. This result has important consequences in terms of sustainability accounting for imperfect economies. Arrow et al. (2003) study this issue in the discounted utility framework. They define accounting prices as the derivative of the Net Present Value of the economic trajectory generated by a given resource allocation mechanism (RAM thereafter) (Arrow et al., 2003, definition 5). This requires to assume that the RAM is known, which amounts to assume that the decision maker knows all the decision rules future generations will apply (and the way these decision rules evolve when the RAM is not stationary over time). Defining these accounting prices requires to integrate the Net Present Value for each problem, and to take the derivative of this value with respect to all stocks. This is not an easy task, and it can be done analytically only for simple RAMs (as illustrated in Arrow et al., 2003, Sections 3–13).

Asheim (2007) generalizes the concept of genuine savings to any dynamic welfare function satisfying a property of "independent future" (which encompasses the discounted utility and the maximin cases). As emphasized by Cairns (2008), changing the objective function modifies the concept of value used, and thus the shadow values of the problem. Our results advocate for the use of maximin shadow values to account for sustainability, along any trajectory, even when a maximin objective is not pursued and the maximin "prices" are not used to implement a sustainable path. Even if these prices are very different from the market prices, they can be computed as the derivative of the maximin value with respect to all stocks. On the one hand, our results are consistent with some of the conclusions of Arrow et al. (2003): sustainability accounting prices are different from market prices (even corrected for environmental externalities). Moreover, the accounting prices are given by the derivatives of a value function with respect to the capital stocks. This corresponds to the general concept of genuine savings of Asheim (2007). On the other hand, our results differ from the conclusions of Arrow et al. (2003) as we do not consider the same objective function. They assume that sustainability requires maintaining discounted utility, while we find that the maximin value should be used as an indicator.

The maximin accounting prices may be harder to assess in practice than the discounted utility accounting prices for particular RAMs. Note, however, that accounting investment at the maximin shadow values requires no assumption on the actual development path or on the RAM of the society. The maximin accounting prices are defined for a *potential* path depending only on the current state of the economy, and not on the actual RAM.[18] Another practical consequence of our results can be drawn from the links between the maximin and viability approaches. We have seen that the maximin shadow values are related to the normal cones of the viability kernel. As there are algorithms to compute viability kernels, this framework provides tools to determine maximin prices numerically. In particular, it is possible to examine if a given local trajectory (i.e., an implementation of given decisions on the current economic state) stays locally in the viability kernel.

6. Conclusion

The maximin criterion defines the maximal utility level that can be sustained in an intergenerational equity perspective. Nevertheless, the results of the maximin approach are obtained in an optimality context, and are valid along the maximin path only. The sustainability indicators it provides are difficult to compute apart from the maximin path. An important challenge to address the sustainability issue in real economies is thus to extend the results of the maximin approach beyond optimality to study the sustainability of economic trajectories that differ from the maximin path.

In this paper, we propose to extend the maximin approach using the viability approach. The viability approach studies the consistency between a dynamic system and given constraints. It makes it possible, for instance, to define all the economic trajectories sustaining a given, not necessarily maximal, utility level. We exhibit the strong links between the *maximin* criterion and the *viability* approach. In particular, we show that the value function of the maximin problem is the solution of a static optimization problem under constraints, involving the so-called viability kernel defined in the viability approach.

On the one hand, our results emphasize the relevance of the viability approach to address the sustainability issue and deal with intergenerational equity concerns. In particular, viability defines decisions that satisfy the sustainability constraints now, and maintain the capability of the economy to satisfy these constraints in the future. From this point of view, the viability approach is consistent with the Brundtland definition of sustainability characterizing sustainable development as development "that meets the needs of the present without compromising the ability of future generations to meet their own needs."

On the other hand, our results point out that extending the maximin approach with the viability approach provides a framework to study the sustainability of any (sub-optimal) economic trajectory. Taking the maximin as an objective may result in a constant utility, which may keep society poor if the initial maximin value is low. We argue that, even if maximin is not taken as an objective, the maximin value function can be used as an indicator of sustainability along any trajectory, including sub-optimal ones. The viability approach then offers a framework to study the sustainability of development paths, characterized by the preservation of the productive capacity of the economy. This has important theoretical implications for sustainability accounting.

Our results open interesting research opportunities. The more challenging one may be to define "supporting prices" in the viability framework, and compute them in practice. These prices would correspond to the local maximin shadow values, which have to be used to determine if net investment maintains the productive capacity of the economy. This would provide a framework for the practice of sustainability accounting.

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Appendix A. Mathematical details and proofs

A.1. Proof of Proposition 1

Assume the existence of a maximin optimal solution $(X^\star(\cdot),u^\star(\cdot))$ starting from state X_0 at time t_0. Then

$$V(X_0) = \inf_{t \geq t_0} L(X^\star(t),u^\star(t)).$$

Consequently

$$L(X^\star(t),u^\star(t)) \geq V(X_0),\ \forall t \geq t_0.$$

In other words, state X_0 belongs to the viability kernel $\mathrm{Viab}(V(X_0))$.

[18] Note that in our framework, if the RAM is known, it is possible to fully characterize the sustainability of the path, as described in Section 4.1. If the RAM is unknown, and only the current decisions are observed, we can characterize sustainability locally with net investment at maximin accounting prices, as described in Section 4.2. From this point of view, our framework is more flexible than that of Arrow et al. (2003) as no assumption is needed on the actual RAM. However, to define the maximin prices, one needs to solve the maximin problem, even if the maximin path is not implemented.

A.2. Proof of Proposition 2

The degenerated case where $V(X_0) = -\infty$ corresponds to a situation where the problem has no solution in the sense that, for any admissible control path $u(\cdot)$, the minimal payoff L is not bounded from below, namely

$$\inf_{t \in \mathbb{R}_+} L(X(t),u(t)) = -\infty.$$

This means equivalently that

$$\forall L_{\min},\ X_0 \notin \mathrm{Viab}(L_{\min}).$$

By using our convention for supremum, this indicates that

$$\sup\,(L_{\min} \mid X_0 \in \mathrm{Viab}(L_{\min})) = -\infty$$

The proposition thus holds in this extreme case.

In the general case where $V(X_0) > -\infty$, we proceed in two steps. First, consider the initial time t_0 and the initial state X_0.

From previous existence comments, we know that there exists some $L_{\min}$ such that $X_0 \in \mathrm{Viab}(L_{\min})$. From the very definition of the viability kernel, this implies the existence of a feasible path $(\widehat{X}(t),\widehat{u}(t))$ such that

$$\begin{cases} \dot{\widehat{X}}(t) = f(\widehat{X}(t),\widehat{u}(t)), \\ \widehat{X}(t_0) = X_0, \\ g_i(\widehat{X}(t),\widehat{u}(t)) \geq 0 \qquad \text{for} \quad i = 1,\ldots,q, \\ L(\widehat{X}(t),\widehat{u}(t)) \geq L_{\min}, \quad \forall t \geq t_0. \end{cases} \tag{A.1}$$

We thus have $\inf_t L(\widehat{X}(t),\widehat{u}(t)) \geq L_{\min}$ and we deduce

$$V(X_0) = \sup_{(X(\cdot),u(\cdot))\ \text{satisfying}\ (A.1)} \inf_t L(X(t),u(t)) \geq L_{\min}.$$

Since the inequality holds for any $L_{\min}$ such that $X_0 \in \mathrm{Viab}(L_{\min})$, this leads to

$$V(X_0) \geq \sup\,(L_{\min} \mid X_0 \in \mathrm{Viab}(L_{\min})).$$

Conversely, from the definition of the value function $V(X_0)$, for any $n \in \mathbb{N}$, there exists an admissible (i.e., satisfying (A.1)) and maximizing sequence $(X_n^\star(\cdot),u_n^\star(\cdot))$ in the sense that

$$V(X_0) \geq \inf_t L(X_n^\star(t),u_n^\star(t)) \geq V(X_0) - \frac{1}{n}.$$

This implies that $X_0 \in \mathrm{Viab}(V(X_0) - (1/n))$, which leads to

$$V(X_0) - \frac{1}{n} \leq \sup\,(L_{\min} | X_0 \in \mathrm{Viab}(L_{\min})),$$

and finally, letting n converges toward $+\infty$, we get

$$V(X_0) \leq \sup\,(L_{\min} \mid X_0 \in \mathrm{Viab}(L_{\min})).$$

Hence the equality holds true.

A.3. Proof of Proposition 3

Assume the existence of a maximin optimal solution $(X^\star(\cdot),u^\star(\cdot))$ starting from state X_0 at time t_0. Consider the translated path

$$\tilde{X}(s) = X^\star(s - t_0 + t),\ \tilde{u}(s) = u^\star(s - t_0 + t),\ s \geq t_0.$$

It is straightforward to prove that

$$\tilde{X}(t_0) = X^\star(t),$$

and that, for any $s \geq t_0$

$$\begin{cases} \dot{\tilde{X}}(s) = f(\tilde{X}(s),\tilde{u}(s)), \\ g_i(\tilde{X}(s),\tilde{u}(s)) \geq 0, \\ L(\tilde{X}(s),\tilde{u}(s)) \geq V(X_0). \end{cases}$$

Therefore, $X^\star(t)$ belongs to the viability kernel $\mathrm{Viab}(V(X_0))$.

A.4. Normal cones and generalized gradients at a glimpse

Normal cones: Given a closed convex set M, the normal cone to the set M at the point $X \in M$ is given by

$$N_M(X) = \{p \,|\, \langle p, v \rangle \leq 0,\ \forall v \in M - X\}.$$

Of course, if X belongs to the interior of M, the normal cone is reduced to zero. It is more informative on the boundary of the set. Typically, consider the simple case where M is a smooth convex manifold characterized by an inequality

$$M = \{X \in \mathbb{R}^n \text{ such that } a(X) \geq 0\},$$

where the function a is a continuously differentiable function (typically $\mathcal{C}^1$) with derivative $a_x(X) \neq 0$. Then, on the boundary of the set M where $a(X) = 0$, the normal cone $N_M(X)$ to M at point X reduces to the cone spanned by the marginal value $-a_x$ and reads

$$N_M(X) = \{p \in \mathbb{R}^n,\ p = -\lambda a_x,\ \lambda \geq 0\}.$$

More generally, when the set M is non-smooth or non-convex, we can extend previous ideas using tangent cones, as illustrated in Fig. A1. See for instance Aubin and Frankowska (1990) and Rockafellar and Wets (1998) for details. The tangent cone $T_M(X)$ is defined as a generalization of $M - X$ for non-(necessarily) convex sets

$$T_M(X) = \limsup_{h \to 0^+} \frac{M - X}{h}.$$

The normal cone $N_M(X)$ turns out to be the (negative) polar cone of the tangent cone as follows:

$$N_M(X) = (T_M(X))^{\ominus} = \{p \,|\, \langle p, v \rangle \leq 0,\ \forall v \in T_M(X)\}. \tag{A.2}$$

Contingent derivatives and generalized gradients: The (upper) contingent derivative $DV(X)(v)$ of a function V at a point X in the direction v is defined by

$$DV(X)(v) = \limsup_{h \to 0^+, v' \to v} \frac{V(X + hv') - V(X)}{h}.$$

Such contingent derivatives coincide with usual derivatives when the function V is smooth. In particular, if V is differentiable, we have

$$DV(X)(v) = \langle V_x(X), v \rangle.$$

These contingent derivatives can also be defined for only semi-continuous functions, which can occur for the value functions of optimal control problems under constraints. The connections between tangent cones and contingent derivatives $DV(X)(v)$ are strong since it has been proved (Aubin and Frankowska, 1990) that

$$\mathrm{Hyp}(DV(X)) = T_{\mathrm{Hyp}(V)}(X, V(X)), \tag{A.3}$$

where the hypograph $\mathrm{Hyp}(V)$ of the function V is defined by

$$\mathrm{Hyp}(V) = \{(X, y),\ y \leq V(X))\}. \tag{A.4}$$

The generalized gradient $\partial V(X)$ can be defined through the derivative DV as follows:

$$p \in \partial V(X) \Leftrightarrow (-p, 1) \in (\mathrm{Hyp}(DV(X)))^{\ominus}. \tag{A.5}$$

We can define similar derivatives using epigraph instead of hypograph.

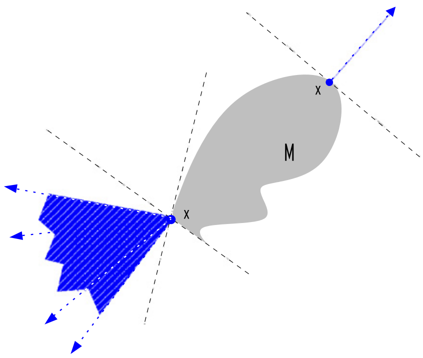

Fig. A1. Two illustrations of normal cones $N_M(X)$ to a set M at a point X. Top: the regular case where $N_M(X)$ is a half-line. Bottom: for non-regular cases, the normals form a cone.

A.5. Proof of inclusion (7)

Consider any p in the generalized gradient $\partial V(X)$ of the maximin value function. Using the relation (A.5) between the generalized gradient and the contingent cone to the epigraph, we deduce that

$$(-p,1) \in (T_{\mathrm{Hyp}(V)}(X,V(X)))^{\ominus}.$$

We now use the Proposition 2 and $V(X) = \sup(y,\ X \in \mathrm{Viab}(y))$ to claim that

$$\mathrm{Viab}(V(X)) \times \{V(X)\} \subset \mathrm{Hyp}(V).$$

Consequently, using properties of tangent cones, we obtain

$$(-p,1) \in (T_{\mathrm{Viab}(V(X))\times\{V(X)\}}(X,V(X)))^{\ominus} = T_{\mathrm{Viab}(V(X))}(X)^{\ominus} \times T_{\{V(X)\}}(V(X)))^{\ominus}.$$

By Definition A.2, this yields

$$(-p,1) \in N_{\mathrm{Viab}(V(X))}(X) \times N_{\{V(X)\}}(V(X)).$$

Since the normal cone to a singleton is the whole real space $\mathbb{R}$, we infer that

$$-p \in N_{\mathrm{Viab}(V(X))}(X).$$

We deduce the desired inclusion

$$\partial V(X) \subset -N_{\mathrm{Viab}(V(X))}(X).$$

A.6. Proof of Proposition 4

We aim at proving that the hypograph $\mathrm{Hyp}(V)$ of the maximin value function as defined in (A.4) is viable for the augmented dynamics

$$\dot{X} = f(X,u),\ \dot{y} = 0,\ L(X,u) \ge y,\ g_i(X,u) \ge 0.$$

We proceed in several steps.

First step: First pick up some $(X,y) \in \mathrm{Hyp}(V)$.

We first prove that the supremum involved in the maximin value $V(X)$ is reached and is actually a maximum. Indeed, from the very definition of $V(X)$, we can write, for any $n \in N$, that there exists an admissible (satisfying (A.1)) and maximizing sequence $(X_n^\star(\cdot),u_n^\star(\cdot))$ in the sense that

$$V(X) \ge \inf_t L(X_n^\star(t),u_n^\star(t)) \ge V(X) - \frac{1}{n}. \tag{A.6}$$

From the whole assumption H, we apply Aubin (1991, Theorem 3.5.2) and use compacity arguments on the solutions $(X_n^\star(\cdot),u_n^\star(\cdot))$ of the controlled dynamic system. Hence we claim the existence of a sub-sequence $(X_n(\cdot),u_n(\cdot))$ which converges in the pointwise topology to $(X^\star(\cdot),u^\star(\cdot))$ and such that $\dot{X}^\star(t) = f(X^\star(t),u^\star(t))$ and $X^\star(t_0) = X$. Furthermore, since g is continuous from assumption H1, passing to the limit with n implies that $g(x^\star(t),u^\star(t)) \ge 0$. Similarly, since L is continuous from assumption H1, we deduce from (A.6) that

$$L(X^\star(t),u^\star(t)) \ge V(X),\ \forall t \ge t_0.$$

Therefore, $\inf_{t\ge t_0} L(X^\star(t),u^\star(t)) \ge V(X)$ and $(X^\star(\cdot),u^\star(\cdot))$ is an optimal solution of the maximin program.

Now set the sequence $y(t) = y$. Since $y \le V(X)$, the augmented path $(X(\cdot),y(\cdot),u(\cdot))$ is also a solution of

$$\dot{X}(t) = f(X(t),u(t)),\ \dot{y} = 0,\ L(X(t),u(t)) \ge y(t),\ g_i(X(t),u(t)) \ge 0$$

starting from (X,y) at time t. This implies in particular that, for any time $t \ge t_0$, we have $(X(t),y(t)) \in \mathrm{Hyp}(V)$. In other words, the hypograph $\mathrm{Hyp}(V)$ is viable for the augmented dynamics.

Second step: Now, we use a variational characterization for the viability of $\mathrm{Hyp}(V)$. From the viability theorem (Aubin, 1991, Theorem 6.1.4) relying on inward tangential conditions and the whole assumption H, we deduce that

$$\forall (X,y) \in \mathrm{Hyp}(V), \exists u \quad \text{s.t. } L(X,u) \ge y,\ g_i(X,u) \ge 0,$$

$$(f(X,u),0) \in T_{\mathrm{Hyp}(V)}(X,y),$$

where $T_{\mathrm{Hyp}(V)}(X,y)$ stands for the tangent cone of the set $\mathrm{Hyp}(V)$ at point (X,y). Such assertion is really informative on the boundary of $\mathrm{Hyp}(V)$ where $y = V(X)$. Moreover, from relation (A.3), we know that

$$T_{\mathrm{Hyp}(V)}(X,V(X)) = \mathrm{Hyp}(DV(X)),$$

where $DV(X)$ means the contingent derivative of V at X. We deduce that

$$\forall X \in \mathrm{Dom}(V), \exists\ u \quad \text{such that } L(X,u) \ge V(X),\ g(X,u) \ge 0,$$

$$DV(X)(f(X,u)) \ge 0.$$

Using the Hamiltonian $H(X,u,p) = \langle p, f(X,u) \rangle$, we conclude to obtain

$$\forall X \in \text{Dom}(V), \quad \sup_{u \in A(X)} H(X,u,V_x(X)) \geq 0. \tag{A.7}$$

where $V_x(X)$ stands for the elements of the generalized gradient $\partial V(X)$.

Appendix B. Illustration in the Dasgupta–Heal–Solow model

To illustrate the results of the paper, we study a canonical model often used to examine the sustainability issue in exhaustible resource economics: the Dasgupta–Heal–Solow model (Dasgupta and Heal, 1974, 1979; Solow, 1974; Heal, 1998).

B.1. A consumption–production economy with a non-renewable resource

Consider an intertemporal resource allocation model with a manufactured capital stock $K(t)$ and nonrenewable natural resources $S(t)$. The rate of extraction of natural resources is $r(t)$. Natural resources and capital are used to produce a composite good with a Cobb–Douglas technology represented by the production function $f(K,r) = K^{\alpha} r^{\beta}$, with $\beta < \alpha \leq 1$. The production can either be invested to accumulate capital ($\dot{K}$) or consumed (c).

The economy, represented by state (K,S) and control (c,r), is subject to dynamics

$$\begin{cases} \dot{K}(t) = K(t)^{\alpha} r(t)^{\beta} - c(t), \\ \dot{S}(t) = -r(t). \end{cases} \tag{B.1}$$

We assume that both natural resources and capital stocks must remain non-negative, that the extraction $r(t)$ is irreversible, and that consumption cannot exceed the production level (investment is irreversible), which imply the following admissibility constraints:

$$\begin{cases} 0 \leq K(t), \quad 0 \leq S(t), \quad 0 \leq r(t), \\ 0 \leq K(t)^{\alpha} r(t)^{\beta} - c(t). \end{cases} \tag{B.2}$$

B.2. The maximin approach

This model has been studied within the maximin framework by Solow (1974) and Dasgupta and Heal (1979). The purpose was to determine the maximal sustainable consumption in an economy with an essential nonrenewable resource. This objective reflects an intergenerational equity concern.[19] The problem reads

$$V(S_0,K_0) = \max \begin{cases} c(\cdot), r(\cdot) \\ \text{satisfying (B.1) and (B.2)} \\ S(t_0) = S_0,\ K(t_0) = K_0. \end{cases} \min_{t \in \mathbb{R}_+} c(t) \tag{B.3}$$

According to Solow's result (Solow, 1974, p. 39), the maximal sustainable consumption in this model, which is also the maximin value function for the initial state (K_0,S_0), is

$$V(S_0,K_0) = (1-\beta)(S_0(\alpha-\beta))^{\beta/(1-\beta)} K_0^{(\alpha-\beta)/(1-\beta)}. \tag{B.4}$$

The associated maximin path is regular, and is characterized by a constant consumption $c^{\star}(t) = V(S_0,K_0)$. The path evolves through states which have the same "maximin value," and these states constitute what Burmeister and Hammond (1977) named a capital valuation contour. At each time, one has $c^{\star}(t) = V(K^{\star}(t),S^{\star}(t)) = V(S_0,K_0)$. It is made possible only because the depletion of the resource stock (or more precisely the decreasing extraction and use of the natural resource) is compensated for by the capital accumulation at an adequate level, which is defined by the Hartwick rule (Hartwick, 1977). In this model, the Hartwick rule reads $\dot{K} = rf'_r = \beta K^{\alpha} r^{\beta}$. A part β of the production is invested. The consumption is a part $(1-\beta)$ of the production, and is constant. The production level is constant.[20] The maximin path is unique, efficient and, of course, optimal with respect to the maximin criterion. It maximizes the sustained level of consumption.

B.3. The viability approach

The sustainability of the DHS model has been studied within the viability framework by Martinet and Doyen (2007). To address the sustainability issue in this model, the viability approach is based on a guaranteed consumption level $c_{\min}$, that

[19] The problem could be stated in utilitarian terms, assuming an increasing utility function, without modifying the results. However, to be consistent with the Solow's analysis, we assume that the objective is to sustain the consumption level.

[20] See Solow (1974), Hartwick (1977), Dasgupta and Heal (1979) and Cairns and Long (2006) for details of the maximin solution in this model.

has to be sustained over time

$$0 < c_{\min} \leq c(t). \tag{B.5}$$

The viability kernel then corresponds to the set

$$\mathrm{Viab}(c_{\min}) = \left\{ (S_0, K_0) \;\middle|\; \begin{array}{l} \exists \text{ decisions } (c(\cdot), r(\cdot)) \text{ and states } (S(\cdot), K(\cdot)) \\ \text{starting from } (S_0, K_0) \text{ satisfying dynamics } (B.1) \\ \text{and constraints (B.2) and (B.5) at any time } t \in \mathbb{R}_+ \end{array} \right\}.$$

The viability kernel of this problem is given by the following expression (Martinet and Doyen, 2007, Proposition 3)[21]:

$$\mathrm{Viab}(c_{\min}) = \{(S, K) \text{ such that } S \geq \mathcal{S}(K, c_{\min})\},$$

where $\mathcal{S} : \mathbb{R}^2 \to \mathbb{R}$ is a function defined by

$$\mathcal{S}(K, c_{\min}) = \frac{1}{\alpha - \beta} \left(\frac{c_{\min}}{1 - \beta} \right)^{(1-\beta)/\beta} K^{(\beta - \alpha)/\beta}. \tag{B.6}$$

According to Eq. (B.6), a sustainability condition linking resource and capital stocks is required. Resource stock S has to be larger than a threshold $\mathcal{S}(K, c_{\min})$ depending on capital stock K and guaranteed consumption $c_{\min}$. The higher the capital stock (*ceteris paribus*), the lower the threshold. The higher the guaranteed consumption (*ceteris paribus*), the higher the threshold. From the very definition of the viability kernel, if the initial state is not within the viability kernel, it is not possible to sustain the consumption level $c_{\min}$ over an infinite time horizon. On the contrary, from any state within the viability kernel, it is possible to sustain $c_{\min}$. In other words, it is possible to sustain a given level of consumption $c_{\min}$ only by staying within the associated viability kernel $\mathrm{Viab}(c_{\min})$. The viability approach thus defines the conditions to sustain a consumption level, without maximizing this level. As a consequence, there is not necessarily a unique sustainable path starting from a given initial state. Any path satisfying the viability constraint (sustaining the minimal consumption level) is viable. Relevant decisions consist in maintaining the state within the viability kernel. From any state which is strictly within the viability kernel (not on the border), every admissible control is relevant, i.e., viable decisions must belong to the set $\mathcal{C}(K, S) = \{(r, c) | c_{\min} \leq c \leq K^{\alpha} r^{\beta}\}$. In particular, sustainable paths are not reduced to constant consumption paths, and the extraction does not necessarily satisfy the Hartwick rule (Martinet and Doyen, 2007). However, on the boundary of the viability kernel, i.e., if $S = \mathcal{S}(K, c_{\min})$, a specific path must be followed to ensure that the velocities $(\dot{K}, \dot{S})$ are tangent to the viability kernel. Applying the Hamiltonian characterization (6), the viable feedbacks $u^* = (r^*, c^*)$ are the solution of the following Hamilton–Jacobi–Bellman inequality:

$$\begin{cases} \max\limits_{\substack{(r,c) \\ K^{\alpha} r^{\beta} \geq c \geq c_{\min} \\ r \geq 0,}} H(S, K, r, c, p_1, p_2) \geq 0, \end{cases}$$

with the Hamiltonian defined by

$$H(S, K, r, c, p_1, p_2) = -p_1 r + p_2 (K^{\alpha} r^{\beta} - c).$$

Specific computations detailed in Martinet and Doyen (2007) imply that, on the boundary of the viability kernel, viable feedback decisions are reduced to

$$r^*(S, K) = \left(\frac{c_{\min}}{1 - \beta} \right)^{1/\beta} K^{-\alpha/\beta} \quad \text{and} \quad c^*(S, K) = c_{\min}.$$

When the initial state is on the boundary of the viability kernel, there is only one viable path, characterized by a constant consumption and Hartwick's investment rule.[22]

B.4. Maximin as the optimization of viability

To present the link between the maximin framework and the viability approach in the particular DHS model, we show that Proposition 2 is satisfied in this illustrative model. For this purpose, we compute the maximum viability constraint $c_{\min}$ for which a given initial state (K_0, S_0) still belongs to the associated viability kernel $\mathrm{Viab}(c_{\min})$. We denote this level $c^+_{\min}(K_0, S_0)$

$$c^+_{\min}(K_0, S_0) = \max\, (c_{\min} | (K_0, S_0) \in \mathrm{Viab}(c_{\min}))$$

This "maximum viability constraint" solves the equation

$$\mathcal{S}(K_0, c^+_{\min}) = S_0,$$

[21] The result presented here is a special case of the result in Martinet and Doyen (2007), with no resource preservation constraint, i.e., $S_{\min} = 0$.

[22] One has $r^* = (\frac{c^*}{1-\beta})^{\frac{1}{\beta}} K^{-\frac{\alpha}{\beta}}$ or equivalently $c^* = (1-\beta) K^{\alpha} r^{*\beta}$ which is also the Hartwick investment rule, i.e., $\dot{K} = \beta K^{\alpha} r^{\beta} = r f'_r$.

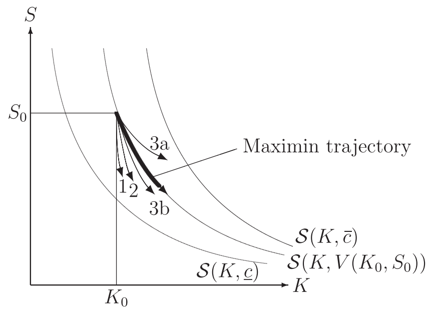

Fig. B1. Viability kernels (Viab($\bar{c}$) $\subset$ Viab($V(K_0,S_0)$) $\subset$ Viab($\underline{c}$), with $\bar{c} > V(K_0,S_0) > \underline{c}$) are, respectively, the Epigraph of curves $\mathcal{S}(K,\bar{c})$, $\mathcal{S}(K,V(K_0,S_0))$, and $\mathcal{S}(K,\underline{c})$. The maximin trajectory and four sub-optimal trajectories are represented. Trajectories 1, 2 and 3b are unsustainable. The maximin trajectory and trajectory 3a are sustainable.

where $\mathcal{S}(K_0,c_{\min})$ is given by Eq. (B.6). We obtain

$$c^+_{\min} = (1-\beta)(S_0(\alpha-\beta))^{\beta/(1-\beta)} K_0^{(\alpha-\beta)/(1-\beta)}, \tag{B.7}$$

which means, according to Eq. (B.4), that $c^+_{\min} = V(S_0,K_0)$.

The boundary of the viability kernel Viab($c^+_{\min}(K_0,S_0)$) thus represents the capital valuation contour associated with the maximin path starting from (K_0,S_0).

B.5. Graphical illustration

To illustrate how this framework makes it possible to characterize the sustainability of economic trajectories, we use a graphical representation. Fig. B1 represents three viability kernels (Viab($\bar{c}$) $\subset$ Viab($V(K_0,S_0)$) $\subset$ Viab($\underline{c}$), with $\underline{c} < V(K_0,S_0) < \bar{c}$) and five trajectories starting from a given initial state (K_0,S_0): the maximin trajectory and four trajectories illustrating the sub-optimal cases are described in Section 4.2, p. 16.

The various trajectories are interpreted as follows:

- *Maximin:* The maximin trajectory follows the boundary of the viability kernel associated to the viability constraint $c(t) \geq c^+(K_0,S_0) = V(K_0,S_0)$, i.e., $\mathcal{S}(K,c^+(K_0,S_0))$, and thus sustains the maximin value.
- *Traj* 1: Trajectory 1 is characterized by a consumption larger than the maximin value, $c_0 = \bar{c} > V(K_0,S_0)$; the economy is faced with the first kind of unsustainability: the current state is not within the viability kernel of current consumption, i.e., $(K_0,S_0) \notin$ Viab($\bar{c}$). Moreover, the trajectory is leaving Viab($V(K_0,S_0)$). The maximin value decreases as net investment at maximin prices is negative.
- *Traj* 2: Trajectory 2 has a consumption equal to the maximin level, $c_0 = V(K_0,S_0)$, but extraction and investment are different from the maximin decisions. The second, local condition for sustainability is not satisfied. The trajectory leaves the viability kernel Viab($V(K_0,S_0)$), which means that net investment at maximin prices is negative and the maximin value decreases.
- *Traj* 3: The two remaining trajectories (3a and 3b) have a consumption lower than the maximin value, $c_0 = \underline{c} < V(K_0,S_0)$. These two trajectories satisfy the first, global condition for sustainability as $(K_0,S_0) \in$ Viab($\underline{c}$). Extraction and investment differ between the two trajectories:
  - 3a: Trajectory 3a has higher investment, resulting in a positive net investment at maximin shadow values. The trajectory is entering the viability kernel Viab($V(K_0,S_0)$), which means that the maximin value increases. This trajectory thus also satisfies the second, local condition for sustainability.
  - 3b: Trajectory 3b is investing less. Net investment at maximin shadow values is negative, i.e., $H(S_0,K_0,r_0,c_0,V_K,V_S) < 0$ and weak Hartwick's rule is violated. The trajectory is leaving the viability kernel Viab($V(K_0,S_0)$). Maximin value decreases. This trajectory does not satisfy the second, local condition for sustainability.

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