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VIABILITY SOLUTIONS TO STRUCTURED HAMILTON–JACOBI EQUATIONS UNDER CONSTRAINTS*

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Abstract. Structured Hamilton–Jacobi partial differential equations are Hamilton–Jacobi equations where the time variable is replaced by a vector-valued variable "structuring" the system. It could be the time-age pair (Hamilton–Jacobi–McKendrick equations) or candidates for initial or terminal conditions (Hamilton–Jacobi–Cournot equations) among a manifold of examples. Here, we define the concept of "viability solution" *which always exists and can be computed by viability algorithms*. This is a *constructive* approach allowing us to derive from the tools of viability theory (dealing with sets instead of functions) some known as well as new properties of its solutions. Regarding functions as their epigraphs, we bypass the regularity issues to arrive directly at the concept of Barron–Jensen/Frankowska viscosity solutions. Above all we take into account viability constraints and extend classical boundary conditions to other "internal" conditions. Beyond that, we use the Fenchel–Legendre transform to associate a Lagrangian with the Hamiltonian and uncover an underlying variational problem. It is proved that the viability solution to the structured Hamilton–Jacobi equation is the valuation function of this variational principle, the optimal evolutions of which are regulated by a "regulation map" which is constructed from the viability solution and can be computed by viability algorithms. In other words, the viability solution solves all at once a class of first-order partial differential equations of intertemporal optimization problems and the regulation of their optimal evolutions.

Key words. Hamilton–Jacobi equation, McKendrick equation, Cournot map, variational principle, Lax–Hopf formula, Lax–Oleinik formula, Barron–Jensen/Frankowska solutions, viable-capture basin, viability solution, retroaction, feedback

AMS subject classifications. 49J24, 49L25, 90B20, 58C06

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1. Introduction.

This paper presents the viability approach to a class of Hamilton–Jacobi equations.[1] We assume that the solution depends not only on time but also on "structured" or "causal" variables. They include age-structured Hamilton–Jacobi–McKendrick equations, useful in population dynamics as well as in transport management (the age variable being replaced by the travel time), as well as Hamilton–Jacobi–Cournot equations, where the "structured" or "causal" variable is the initial state of the underlying control system.

In this study, we start with a class of Hamilton–Jacobi equations *the Hamiltonian of which is convex with respect to the gradient of the solution*. The solutions are required to satisfy lower and upper constraints (upper constraints can be derived from boundary conditions, for instance, as well as other internal conditions, and many other conditions). We shall prove that their "solution" is the value function of an intertemporal optimization of evolutions governed by a *hidden* underlying control system, where

1. the hidden controls, called "celerities," range over the state space;

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[1]More details are provided in [11].

2. the optimality criterion involves the Lagrangian associated with the Hamiltonian by the Fenchel transform;
3. the evolutions are governed by a specific class of "epigraphical control systems" involving the epigraph of the Lagrangian;
4. the map regulating optimal evolutions is associated with the gradient of the solution.

We recall that the value function of an optimal control problem is the solution of a nonlinear Hamilton–Jacobi–Bellman problem, but there are Hamilton–Jacobi equations which, a priori, are not derived from a variational problem. The point of this paper is just the opposite approach to uncover a variational problem. Starting with an optimal control problem, the uncovered variational problem does not necessarily coincide with the initial control problem (the "uncovered" control is the velocity of the state involving the "original" control).

Which solution? We define the concept of "viability solution" as *solving all at once the above related problems*. They are "constructive solutions" in the sense that their epigraph is *defined* as the capture basin of adequate epigraphical environment and target. It inherits their properties, which we "translate" into the language of partial differential equations and control theory.

It can be regarded as an "epigraphical miracle." The link between the epigraph of a Lyapunov function and the viability kernel of an epigraph goes back to Chapter 6 of [13], [15], and the link between the epigraph of the minimal time function and the capture basin by Quincampoix (see [33]). Starting in 1987 Frankowska proved in a long series of papers (among which are [25, 26, 27, 28, 29]) the links between the value function as a solution to a Hamilton–Jacobi–Bellman equation and the viability and invariance of the epigraph of the value function, showing that it is a solution both in the sense of "Frankowska solutions" and of viscosity solutions.

We begin by introducing some notations at this stage to explain the problems addressed in this paper.

Let $t \geq 0$ denote the time, $d \in \mathcal{D} \subset \mathbb{R}^m$ the *causal variable*, and $x \in X := \mathbb{R}^n$ the *state variable* of the system. An *extended function* $\mathbf{v} : X \mapsto \mathbb{R} \cup \{+\infty\}$ is a function taking infinite values and concealing constraints in its *domain*

$$\mathrm{Dom}(\mathbf{v}) := \{x \in X \text{ such that } \mathbf{v}(x) < +\infty\}.$$

We introduce

1. a *causal map* $\varphi : d \in \mathbb{R}^m \mapsto \varphi(d) := (\varphi_i(d))_{i=1,\dots,m} \in \mathbb{R}^m$ depending only on the causal variable d (assumed single-valued for simplicity of the exposition);
2. a *Hamiltonian* function

$$(d, x, p) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n \mapsto \ell^\star(d, x; p) \in \mathbb{R} \cup \{+\infty\}$$

   convex with respect to the *"costate" variable* $p \in \mathbb{R}^n$;
3. a *viability constraint* function $(d, x) \mapsto \mathbf{k}(d, x) \in \mathbb{R} \cup \{+\infty\}$ such that $\mathbf{k}(d, x) < +\infty$ implies that $d \in \mathcal{D}$;
4. an *internal condition* function $(d, x) \mapsto \mathbf{c}(d, x) \in \mathbb{R} \cup \{+\infty\}$.
   They satisfy

$$\mathbf{k}(d, x) \leq \mathbf{c}(d, x).$$

The terminology is motivated by the *asymmetric* role played by the two variables d and x, since φ depends *only* on the causal variable d, whereas the Hamiltonian

$\ell^\star$ may depend on *both* causal and state variables. Hijacking the terminology used in population dynamics, we say that the causal variables *structure* the system. The presence of the star in the notation $\ell^\star$ of the Hamiltonian is justified later for deriving the Hamiltonian from the Lagrangian ℓ (see (4)).

The macroscopic approach. The *macroscopic* description of the system requires us to look for the *viable solution* V to the *structured Hamilton–Jacobi equation*

$$\sum_{i=1}^{m} \left\langle \frac{\partial V(d,x)}{\partial d_i}, \varphi_i(d) \right\rangle + \ell^\star \left(d, x; \frac{\partial V(d,x)}{\partial x} \right) = 0 \tag{1}$$

satisfying inequalities

$$\mathbf{k}(d,x) \leq V(d,x) \leq \mathbf{c}(d,x). \tag{2}$$

Examples of structured Hamilton–Jacobi equations.

1. *Hamilton–Jacobi equations.* By taking $d := t \in \mathcal{D} := \mathbb{R}_+$ describing time and $\varphi(t) = 1$, we obtain the usual Hamilton–Jacobi partial differential equation
$$\frac{\partial V(t,x)}{\partial t} + \ell^\star \left(t, x; \frac{\partial V(t,x)}{\partial x} \right) = 0.$$
2. *Hamilton–Jacobi–McKendrick equations.* By taking $d := (t,a) \in \mathcal{D} := \mathbb{R}_+^2$ describing time and age (in population dynamics; see [1, 2, 12, 30, 31, 32, 35, 36], for instance) or travel time (in traffic problems; see [16], for instance) and taking $\varphi(t,a) = (1,1)$, we obtain Hamilton–Jacobi–McKendrick partial differential equations
$$\frac{\partial V(t,a,x)}{\partial t} + \frac{\partial V(t,a,x)}{\partial a} + \ell^\star \left(t, a, x; \frac{\partial V(t,a,x)}{\partial x} \right) = 0.$$
3. Taking $d := (t,b)$ and $\varphi(t,b) := (1, \psi(t,b))$, we obtain the partial differential equation
$$\frac{\partial V(t,b,x)}{\partial t} + \left\langle \frac{\partial V(t,b,x)}{\partial b}, \psi(t,b) \right\rangle + \ell^\star \left(t, b, x; \frac{\partial V(t,b,x)}{\partial x} \right) = 0.$$
4. *Hamilton–Jacobi–Cournot equations.* By taking $\psi(d,\chi) := (\varphi(d), 0)$, we obtain structured Hamilton–Jacobi partial differential equations
$$\sum_{i=1}^{m} \left\langle \frac{\partial V(d,x)}{\partial d_i}, \varphi_i(d) \right\rangle + \ell^\star \left(d, x; \frac{\partial V(d,\chi,x)}{\partial x} \right) = 0$$
structured *also* by *constant parameters* χ. They are solutions to the same partial differential equation (1) but are subjected to conditions
$$\mathbf{k}(d,\chi,x) \leq V(d,\chi,x) \leq \mathbf{c}(d,\chi,x)$$
depending on χ.
An important example is provided by Hamilton–Jacobi–Cournot partial differential equations parameterized by specific parameters $\chi \in \mathbb{R}^n$ regarded as *initial conditions* and requiring that $\mathbf{c}(d,\chi,x) = +\infty$ whenever $x \neq \chi$ (see the end of section 7 for a justification of the terminology). $\square$

Examples of internal and viability functions.

1. An internal condition function $(d, x) \mapsto \mathbf{c}(d, x) \in \mathbb{R} \cup \{+\infty\}$ becomes a *boundary condition* function if $\mathbf{c}(d, x) = +\infty$ whenever $d \in \mathrm{Int}(\mathcal{D})$.
2. Consider the case when the environment in which the state variable x ranges is described by *viability environments* $K(d) \subset X$ of the state variable depending on the structural variable d. This constraint is taken into consideration by *requiring that the constraint function* $\mathbf{k}$ *satisfy* $\mathbf{k}(d, x) = +\infty$ *for all* $x \notin K(d)$ (since this implies automatically that $V(t, x) = +\infty$ whenever $d \notin K(d)$).
   The case "without constraints" is obtained by taking

$$\mathbf{k}(d, x) := \begin{cases} +\infty & \text{if} \quad x \notin K(d), \\ -\infty & \text{if} \quad x \in K(d). \end{cases} \tag{3}$$

   By taking $\mathbf{k}(d, x) = 0$ for all $x \notin K(d)$, we require also that the solution be nonnegative on $K(d)$.

Hamilton–Jacobi equations have been extensively studied (we refer, for instance, to [20, 24], among so many other references on partial differential equation techniques and optimal control theory, and to the recent survey [29] and its bibliography).

This study (which *uses exclusively concepts and theorems of viability theory and convex as well as nonsmooth analysis*) has been motivated by Hamilton–Jacobi equations modeling traffic management (see [11, 10, 7, 3, 22, 23]).

The existence and uniqueness of the Barron–Jensen/Frankowska viscosity solution has been proved independently in [18] with partial differential equation techniques and in [26, 27, 28] showing that the value function of an optimal control problem solves a Hamilton–Jacobi–Bellman equation and studying the regularity of the regulation map.

Here, we define the concept of "viability solution" *which always exists and can be computed by viability algorithms* (see [19], for instance). This is a *constructive* approach allowing us to derive from the tools of viability theory (dealing with sets instead of functions) some known as well as new properties of its solutions. Regarding functions as their epigraphs, we bypass the regularity issues to arrive directly at the concept of Barron–Jensen/Frankowska viscosity solutions. Above all we take into account viability constraints and extend classical boundary conditions to other "internal" conditions.

The variational approach. The link between Hamilton–Jacobi equations and the associated variational problem relies on the *Legendre–Fenchel transform* $(d, x, u) \mapsto \ell(d, x; u)$ of the Hamiltonian defined by

$$\ell(d, x; u) := \sup_p [\langle p, u \rangle - \ell^\star(d, x; p)], \tag{4}$$

the *Lagrangian*. We denote by

$$F(d, x) := \{u \text{ such that } \ell(d, x; u) < +\infty\} \tag{5}$$

the domain of the Lagrangian ℓ. It defines a set-valued map $F : (d, x) \rightsquigarrow F(d, x)$, which will be the right-hand side of the differential inclusion governing the evolutions of the microsystem.

Optimal evolutions achieve the minimum in the *variational principle*

$$V(d, x) := \inf_{x(\cdot)} \left(\mathbf{c}(d(0), x(0)) + \int_0^{t^\sharp} \ell(d(t), x(t), x'(t)) dt \right) \tag{6}$$

among all *viable* evolutions $x(\cdot)$ regulated by the *differential inclusion* $x'(t) \in F(d(t), x(t))$ starting at initial time 0 and arriving at x at terminal time $t \leq t^\sharp$ when $d(t^\sharp) = d$. The function V is called the *valuation function* (and not the classical value function, which depends upon current time t, whereas the valuation function depends upon the terminal time).

They satisfy the *dynamic programming equation* on optimal evolutions: for all $t \in [0, t^\sharp]$,

$$V(d,x) = V(d(t), x(t)) + \int_t^{t^\sharp} \ell(d(\tau), x(\tau), x'(\tau)) d\tau. \tag{7}$$

The microscopic approach. The main purpose of this study is not only to prove that the *viability solution* is the *unique* solution to this partial differential equation in an adequate generalized (weak) sense, but also to *uncover a hidden dual microscopic* equivalent problem allowing us to characterize and *compute* from the solution V the *retroaction map* $(d,x) \rightsquigarrow R(d,x)$ governing *optimal evolutions* of the state. This involves the *microscopic* regulation of structured-viable evolutions $(d(\cdot), x(\cdot))$ satisfying, for any given $d \in \mathcal{D}$ and x and for some $t^\sharp \geq 0$,

$$\begin{cases} \text{(i)} \quad d(t^\sharp) = d \text{ and } x(t^\sharp) = x \ \ \text{(terminal conditions)}, \\ \text{(ii)} \quad x'(t) \in R(d(t), x(t)) \ \ \text{(retroaction law)}. \end{cases} \tag{8}$$

In other words, the macroscopic problem (looking for a macroscopic function V) and microscopic problem (looking for the regulation of evolutions $x(\cdot)$) are "dual."

The viability solution. The links between the microscopic, macroscopic, and variational approaches are due to a "matchmaker," the "viability solution," which

1. coincides with the viable solution to the macroscopic structured Hamilton–Jacobi partial differential equation (1) satisfying the condition (2);
2. provides the *retroaction law* microsystem (8) governing optimal viable evolutions of the state variable to any given terminal state in optimal time;
3. is equal to the valuation function (6) of the associated intertemporal optimization problem.

See Figure 1.

In this introduction we do not describe how the viability solution is defined because it requires viability concepts and tools of viability theory, which will be explained later.

Organization. This study is organized as follows. Section 2 is devoted to the statement of the problem and defines the viability solution to the structured Hamilton–Jacobi problem. The general variational principle is investigated in section 3. Section 4 proves the existence of optimal evolutions and the dynamic programming property. Next, in section 5, we derive from the viability solution the regulation map governing the evolution of optimal viable evolutions arriving at terminal state at optimal time. Section 6 investigates the aggregation problem of "mapping" a Hamilton–Jacobi–Bellman equation in a given vector space into another space through a linear "aggregation operator," which will play an important role in transportation networks. We continue the investigation of this issue in section 7 by looking for the initial states from which optimal evolutions reach the terminal time through the viability solution of the associated Hamilton–Jacobi–Cournot equation. Section 8 provides the particular cases when the Lax–Hopf formula holds true. This study is

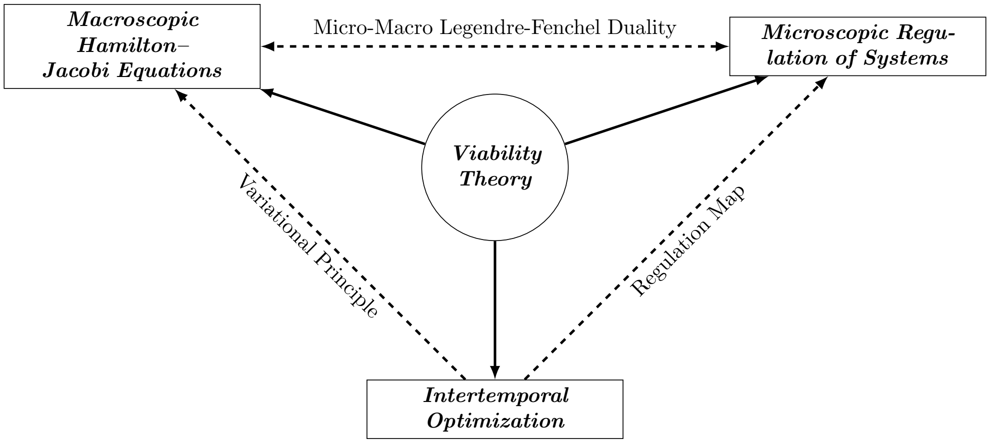

Fig. 1. *From duality to trinity. This diagram describes the three problems under investigation: the macroscopic approach through first-order partial differential equations, the microscopic version dealing with the regulation of an underlying control system, and the intertemporal optimization problem. The links relating optimization problems to Hamilton–Jacobi–Bellman equations and the regulation of control systems have been extensively studied. The tools of viability theory allow us to show that the viability solutions solve these three problems all at once.*

concluded by proving in section 9 that the viability solution is the solution to the structured Hamilton–Jacobi equation under viability constraints.

The study ends with two annexes, one providing the few results of convex analysis needed in this analysis (section 10) and the other, section 11, with a *viability survival kit* summarizing the concepts and results used to make the reading of this study as self-contained as possible.

2. Statement of the problem.

2.1. Lagrangian and Hamiltonian. Let $X := \mathbb{R}^n$ be a finite dimensional vector space. Recall that the *epigraph* $\mathcal{E}p(\mathbf{v})$ of an *extended function* $\mathbf{v} : X \mapsto \mathbb{R} \cup \{+\infty\}$ is the set of pairs $(x, y) \in X \times \mathbb{R}$ such that $\mathbf{v}(x) \leq y$. An extended function is called *lower semicontinuous* if *its epigraph is closed*.

The Lagrangian $\ell : (d, x; u) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n \mapsto \ell(d, x; u) \in \mathbb{R} \cup \{+\infty\}$ is assumed once and for all to be *a nontrivial lower semicontinuous function convex with respect to u*: $\ell(d, x; \cdot) : u \in x \mapsto \ell(d, x; u) \in \mathbb{R} \cup \{+\infty\}$ is convex. Here, x is regarded as the state and u as a *celerity*. For instance, u is a velocity (in mechanics), a transaction (in economics and finance), or a *celerity* in general systems.

We also introduce the *costate* or *dual variable* $p \in \mathbb{R}^{n^\star}$ (the *dual* of $\mathbb{R}^n$, which is isomorphic to $\mathbb{R}^n$, and thus, with the same dimension) and the duality product $\langle p, u \rangle := p(u)$. For instance, p is regarded as a force, a price, or a density, and the duality product $\langle p, u \rangle$ is a power, a value, or a flux, respectively.

The *conjugate function* $\ell^\star(d, x; \cdot)$ is defined on costate variables by

$$\forall\, p \in \mathbb{R}^n, \quad \ell^\star(d, x; p) := \sup_u [\langle p, u \rangle - \ell(d, x; u)].$$

When the function ℓ is regarded as a Lagrangian, then its conjugate can be regarded as a Hamiltonian.

The *conjugate function* $\ell^\star(d,x;\cdot)$ *is always a nontrivial lower semicontinuous convex function* satisfying the *Fenchel inequality*

$$\langle p,u\rangle \leq \ell(d,x;u)+\ell^\star(d,x;p).$$

The biconjugate satisfies $\ell^{\star\star}(d,x;u)\leq \ell(d,x;u)$. The main result of convex analysis states that $\ell^{\star\star}(d,x;\cdot)=\ell(d,x;\cdot)$ *if and only if* $\ell(d,x;\cdot)$ *is convex, lower semicontinuous and nontrivial.*

The *Legendre property* of the *Fenchel transform* $\ell(d,x;\cdot)\mapsto \ell^\star(d,x;\cdot)$ implies that the *subdifferentials* $\partial\ell(u)$ and $\partial\ell^\star(p)$ of the lower semicontinuous convex functions ℓ and $\ell^\star$ are *defined* by the following equivalent conditions:

$$\text{(9)}\qquad \begin{cases} \text{(i)} & \langle p,u\rangle = \ell(d,x;u)+\ell^\star(d,x;p),\\ \text{(ii)} & p\in\partial_u\ell(d,x;u),\\ \text{(iii)} & u\in\partial_p\ell^\star(d,x;p).\end{cases}$$

The two equalities (9)(ii), (iii) describe the Legendre property: the inverse of the subdiffrential map $u\rightsquigarrow \partial_u\ell(d,x;u)$ is the subdiffrential map $p\rightsquigarrow\partial_p\ell^\star(d,x;p)$.

If both functions ℓ or $\ell^\star$ are differentiable in the classical sense, then

$$\begin{cases} \partial_u\ell(d,x;u)=\left\{\dfrac{d}{du}\ell(d,x;u)\right\},\\ \partial_p\ell^\star(d,x;p)=\left\{\dfrac{d}{dp}\ell^\star(d,x;p)\right\}\end{cases}$$

(see [4, 5, 8, 34].

At this point, we have to make assumptions implying viability properties hold true. In our specific settings, we need to make assumptions on either the Lagrangian ℓ or on the Hamiltonian $\ell^\star$ to fit the Marchaud requirement.

Definition 2.1 (Marchaud functions). *We shall say that*

1. *a Lagrangian* $(d,x,u)\mapsto \ell(d,x;u)\in\mathbb{R}\cup\{+\infty\}$ *is* Marchaud *if it is a lower semicontinuous function convex with respect to* u *and if there exists a finite positive constants* $c>0$ *such that*

$$\text{(10)}\qquad \begin{cases} \mathrm{Dom}(\ell(d,x;\cdot))\subset c(\|x\|+\|d\|+1)B \textit{ and is closed},\\ \forall\, u\in\mathrm{Dom}(\ell(d,x;\cdot)),\quad 0\leq \ell(d,x;u)\leq c(\|x\|+\|d\|+1);\end{cases}$$

2. *a Hamiltonian* $(d,x;p)\mapsto \ell^\star(d,x;p)\in\mathbb{R}\cup\{+\infty\}$ *is* Marchaud *if it is convex and lower semicontinuous with respect to* u*, upper semicontinuous with respect to* (d,x)*, and if there exists* $c<0$ *and* c_0 *such that, for all* $p\in\mathbb{R}^n$*,*

$$\begin{cases} \sigma_{\mathrm{Dom}(\ell)}(d,x;p)-c(\|x\|+\|d\|+1)\\ \leq \ell^\star(d,x;p)\leq c(\|x\|+\|d\|+1)\|p\|_\star.\end{cases}$$

Lemma 10.1 states that the Lagrangian ℓ is Marchaud if the Hamiltonian $\ell^\star$ is Marchaud. If the Lagrangian is Marchaud and continuous with respect to (d,x), then the Hamiltonian is Marchaud.

2.2. The viability solution. We denote by $\mathcal{S}(x_0)$ the subset of solutions to differential inclusion $x'(t)\in F(x(t))$ starting at x_0.

Definition 2.2 (viable-capture basin of a target). *Let us consider a differential inclusion* $x'(t)\in F(x)$ *where* $F:\mathbb{R}^n\rightsquigarrow\mathbb{R}^n$*, a subset* K *is regarded as an environment defined by (viability) constraints, and a subset* $C\subset K\subset\mathbb{R}^n$ *as a target.*

The viable-capture basin $\mathrm{Capt}_F(K,C)$ *of C viable in K is the subset of initial states $x_0 \in K$ such that there exist at least one solution $x(\cdot) \in \mathcal{S}(x_0)$ of the differential inclusion $x'(t) \in F(x(t))$ starting at x_0 and one finite time $t^\star$ such that*

$$\begin{cases} \text{(i)} & x(t^\star) \in C \textit{ (capturability)}, \\ \text{(ii)} & \forall\, t \in [0, t^\star], \quad x(t) \in K \textit{ (viability)}. \end{cases}$$

Knowing the Hamiltonian $\ell^\star$, the viability constraint function $\mathbf{k}$, and the internal condition function $\mathbf{c}$, we define the structured Hamilton–Jacobi problem as follows.

DEFINITION 2.3 (structured Hamilton–Jacobi problem). *A function $(d,x) \mapsto V(d,x)$ is said to be a solution to the structured Hamilton–Jacobi equation if*

$$\left\langle \frac{\partial V(d,x)}{\partial d}, \varphi(d) \right\rangle + \ell^\star\left(d, x; \frac{\partial V(d,x)}{\partial x}\right) = 0 \tag{11}$$

and a solution to the structured Hamilton–Jacobi problem if, furthermore, it satisfies the conditions

$$\mathbf{k}(d,x) \le V(d,x) \le \mathbf{c}(d,x). \tag{12}$$

We introduce the *structured characteristic system* defined by

$$\begin{cases} \text{(i)} & \delta'(t) = -\varphi(\delta(t)), \\ \text{(ii)} & (\xi'(t), \eta'(t)) \in -\mathcal{E}p(\ell(\delta(t), \xi(t); \cdot)). \end{cases} \tag{13}$$

The characterization of the solution to the structured Hamilton–Jacobi problem states that its epigraph is the *viable-capture basin* of the epigraph of $\mathbf{c}$, viable in the epigraph of $\mathbf{k}$, under the structured characteristic system defined by (13).

DEFINITION 2.4 (viability solution). *The viability solution V to the structured Hamilton–Jacobi problem* (11)–(12) *is defined by*

$$V(d,x) := \inf_{(d,x,y) \in \mathrm{Capt}_{(13)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c}))} y. \tag{14}$$

It may seem strange at first glance to solve a well-known partial differential equation by a solution of an auxiliary and seemingly artificial viability problem.

Defining (lower semicontinuous) functions through their (closed) epigraphs allows us to treat the functions ℓ, $\mathbf{k}$, $\mathbf{c}$ and the viability solution V as subsets, bypassing and avoiding the pointwise version familiar in classical analysis.

But this allows us to apply results surveyed and summarized in the "viability survival kit" (section 11) that are obtained at the simpler level of set-valued analysis (with much less notation) and are easier to prove. The translation of the properties of viable-capture basins in terms of structured problems provides, without technical difficulties, the properties we shall uncover. The fact that the viability solution is a weak solution of the partial differential equation is based on the fundamental viability and invariance theorems.

The definition of the viability solution does not involve the concept of derivatives, which is a strange way for defining solutions to partial differential equations. Actually, it happens that the solution to the structured Hamilton–Jacobi is not differentiable. This is the case whenever viability constraints are involved: in this case, the most we can require is that the solution be only lower semicontinuous. However, it is possible to give meaning to lower semicontinuous solutions to structured

Hamilton–Jacobi equation (11) by weakening the concepts of gradients in the sense of nonsmooth analysis. The viability solution then becomes a "solution" to this partial differential equation in the sense of the Barron–Jensen/Frankowska viscosity solution (Theorem 9.3).

3. Variational principle.

3.1. Lagrangian microsystems. We shall assume that the causal map φ is Lipschitz (or more generally, monotone) for guaranteeing the *uniqueness* of the solution $\delta(\cdot)$ to differential equation

$$\delta'(t) = -\varphi(\delta(t))$$

starting from any given initial state $d \in \mathcal{D}$.

We assume further that the subset $\mathcal{D}$ is closed and is a *repeller* under $-\varphi$ in the sense that for any $d \in \mathcal{D}$, the evolution $\delta(\cdot)$ leaves $\mathcal{D}$ in finite time

$$\tau^{\sharp}(d) := \inf_{\delta(t) \in \complement \mathcal{D}} t$$

at $\delta(\tau^{\sharp}(d)) \in \partial \mathcal{D}$.

This is the case for usual Hamilton–Jacobi equations when $d := t \in \mathcal{D} := \mathbb{R}_+$ and $\varphi(t) = 1$: in this case $\tau^{\sharp}_{\mathcal{D}}(t) = t$. This is also the case for the Hamilton–Jacobi–McKendrick equations when $d := (t,a) \in \mathcal{D} := \mathbb{R}^2_+$ and $\varphi(t,a) = (1,1)$: in this case $\tau^{\sharp}_{\mathcal{D}}(t,a) = a$ if $t \geq a \geq 0$ and $\tau^{\sharp}_{\mathcal{D}}(t,a) = t$ if $a \geq t \geq 0$.

In summary, whenever we mention d, we attach to it either the unique evolution $t \mapsto \delta(t)$ governed by $\delta'(t) = -\varphi(\delta(t))$ starting from d at initial time 0, or the unique evolution $t \mapsto d(t) := \delta(t^{\sharp} - t)$ governed by $d'(t) = \varphi(d(t))$ and arriving at d at time $t^{\sharp}$, *without mentioning it explicitly.*

We associate with the Lagrangian, the causal map, and the viability constraint function $\mathbf{k}$ the microsystem governing viable evolutions of the state.

DEFINITION 3.1 (microsystem). *Let $F : (d,x) \rightsquigarrow F(d,x)$ be the set-valued map defined by* (5)*, the domain of the Lagrangian ℓ, and denote by $\mathcal{A}_{\mathbf{k}}(t^{\sharp}; d, x)$ the set of evolutions $x(\cdot)$ governed by the system*

$$x'(t) \in F(d(t), x(t)) \tag{15}$$

"viable" in the sense that

$$\sup_{t \in [0,t^{\sharp}]} \mathbf{k}(d(t), x(t)) < +\infty$$

and arriving at x at time $t^{\sharp}$ when $d(t^{\sharp}) = d$.

When the function $\mathbf{k}$ is associated with a viability environment $d \rightsquigarrow K(d)$ by (3)*, the viable evolutions are the ones satisfying*

$$\forall\, t \in [0, t^{\sharp}], \;\; x(t) \in K(d(t)).$$

3.2. The variational principle.

The case of boundary-value problems. We assume first (for simplicity of the formula) that the internal condition is actually a boundary condition: $\mathbf{c}(d,x) = +\infty$ whenever $d \in \mathrm{Int}(\mathcal{D})$.

The *valuation function* of the intertemporal optimization problem is defined by

$$U(d,x) := \inf_{x(\cdot)\in\mathcal{A}_{\mathbf{k}}(\tau^\sharp(d);d,x)} \left(\mathbf{c}(d(0),x(0)) + \int_0^{\tau^\sharp(d)} \ell(d(t),x(t),x'(t))dt \right). \tag{16}$$

THEOREM 3.2 (the viability solution solves the variational problem). *The viability solution V defined by* (14),

$$V(d,x) := \inf_{(d,x,y)\in\mathrm{Capt}_{(13)}(\mathcal{E}p(\mathbf{k}),\mathcal{E}p(\mathbf{c}))} y,$$

is equal to the valuation function U of the variational problem (16).

General case. In the general case, the statement of the intertemporal optimization problem is more intricate and requires further notations.

THEOREM 3.3 (the viability solution solves the variational problem (general case)). *We associate with the function $\mathbf{c}$ the functional $J_{\mathbf{c}}$ defined by*

$$\begin{cases} \mathbf{J}_{\mathbf{c}}(t^\sharp;x(\cdot))(d,x) \\ := \mathbf{c}(d(0),x(0)) + \displaystyle\int_0^{t^\sharp} \ell(d(\tau),x(\tau),x'(\tau))d\tau, \end{cases}$$

with the function $\mathbf{k}$ the functional $I_{\mathbf{k}}$ defined by

$$\begin{cases} \mathbf{I}_{\mathbf{k}}(t^\sharp;x(\cdot))(d,x) \\ := \displaystyle\sup_{s\in[0,t^\sharp]} \left(\mathbf{k}(d(s),x(s)) + \int_s^{t^\sharp} \ell(d(\tau),x(\tau),x'(\tau))d\tau \right), \end{cases}$$

and with both functions $\mathbf{k}$ and $\mathbf{c}$ the functionals defined by

$$\begin{cases} \mathbf{L}_{(\mathbf{k},\mathbf{c})}(t^\sharp;x(\cdot))(d,x) := \max(\mathbf{I}_{\mathbf{k}}(t^\sharp;x(\cdot))(d,x), \mathbf{J}_{\mathbf{c}}(t^\sharp;x(\cdot))(d,x)), \\ \mathbf{M}_{(\mathbf{k},\mathbf{c})}(t^\sharp;d,x) := \inf_{x(\cdot)\in\mathcal{A}_{\mathbf{k}}(t^\sharp;d,x)} \mathbf{L}_{(\mathbf{k},\mathbf{c})}(t^\sharp;x(\cdot))(d,x). \end{cases}$$

Hence the viability solution V is equal to the valuation function U of the variational problem defined by

$$U(d,x) = \inf_{t^\sharp\in[0,\tau^\sharp(d)]} \mathbf{M}_{(\mathbf{k},\mathbf{c})}(t^\sharp;d,x) = \inf_{t^\sharp\in[0,\tau^\sharp(d)]} \inf_{x(\cdot)\in\mathcal{A}_{\mathbf{k}}(t^\sharp;d,x)} \mathbf{L}_{(\mathbf{k},\mathbf{c})}(t^\sharp;x(\cdot))(d,x). \tag{17}$$

Proof. By Definition 2.2 of viable-capture basins, to say that (d,x,y) belongs to the viable-capture basin $\mathrm{Capt}_{(13)}(\mathcal{E}p(\mathbf{k}),\mathcal{E}p(\mathbf{c}))$ means that there exist some $t^\sharp \geq 0$ and a measurable function $\upsilon(\cdot) : [0,t^\sharp] \mapsto \mathrm{Dom}(\ell)$ such that

$$t \in [0,t^\sharp] \mapsto (\delta(t),\xi(t),\eta(t)) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^n$$

is a solution to (13) starting at x viable in the epigraph of $\mathbf{k}$ until time $t^\sharp \le \tau^\sharp(d)$ when it belongs to the epigraph of the function $\mathbf{c}$, where

$$\begin{cases} \text{(i)} & \xi(t) := x - \int_0^t \upsilon(\tau)d\tau, \\ \text{(ii)} & \eta(t) \le \eta_0(t) := y - \int_0^t \ell(\delta(\tau), \xi(\tau), \upsilon(\tau))d\tau. \end{cases}$$

This implies that $t^\sharp \le \tau^\sharp(d)$, that

$$\text{(18)} \quad \begin{cases} \text{(i)} & \mathbf{c}\left(\delta(t^\sharp), \xi(t^\sharp)\right) \le \eta(t^\sharp) \le y - \int_0^{t^\sharp} \ell(\delta(\tau), \xi(\tau), \upsilon(\tau))d\tau = \eta_0(t^\sharp), \\ \text{(ii)} & \forall\, s \in [0, t^\sharp], \\ & \mathbf{k}(\delta(s), \xi(s)) \le \eta(s) \le y - \int_0^s \ell(\delta(\tau), \xi(\tau), \upsilon(\tau))d\tau = \eta_0(s), \end{cases}$$

and, in the particular case of the boundary condition when $\mathbf{c}(d,x) = +\infty$ whenever $d \in \mathrm{Int}(\mathcal{D})$, that $t^\sharp = \tau^\sharp(d)$.

We also deduce that

$$t \in [0, t^\sharp] \mapsto (\delta(t), \xi(t), \eta_0(t))$$

is a solution governed by the differential inclusion

$$\text{(19)} \quad \begin{cases} \text{(i)} & \delta'(t) = -\varphi(\delta(t)), \\ \text{(ii)} & (\xi'(t), \eta'(t)) \in -\mathrm{Graph}(\ell(\delta(t), \xi(t); \cdot)), \end{cases}$$

which can be written in the form

$$\text{(20)} \quad \begin{cases} \text{(i)} & \delta'(t) = -\varphi(\delta(t)), \\ \text{(ii)} & \xi'(t) = -\upsilon(t), \\ \text{(iii)} & \eta'(t) = -\ell(\delta(t), \xi(t); \upsilon(t)), \end{cases}$$

viable in the epigraph of $\mathbf{k}$ until it reaches the epigraph of $\mathbf{c}$ at time $t^\sharp$.

Therefore $\mathrm{Capt}_{(13)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c})) \subset \mathrm{Capt}_{(19)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c}))$. Since $\mathrm{Graph}(\ell) \subset \mathcal{E}p(\ell)$, the converse is true, so that equality

$$\text{(21)} \quad \mathrm{Capt}_{(13)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c})) = \mathrm{Capt}_{(19)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c}))$$

ensues.

Inequalities (18) imply that

$$\sup_{s \in [0, t^\sharp]} \left(\mathbf{k}\left(\delta(s), \xi(s)\right) + \int_0^s \ell(\delta(\tau), \xi(\tau), \upsilon(\tau))d\tau \right) \le y.$$

The target condition implies that

$$\mathbf{c}\left(\delta(t^\sharp), \xi(t^\sharp)\right) + \int_0^{t^\sharp} \ell(\delta(\tau), \xi(\tau), \upsilon(\tau))d\tau \le y.$$

Let us set

$$\overleftarrow{\mathbf{J}}_{\mathbf{c}}(t^\sharp; \upsilon(\cdot))(d, x) := \mathbf{c}(\delta(t^\sharp), \xi(t^\sharp)) + \int_0^{t^\sharp} \ell(\delta(\tau), \xi(\tau), \upsilon(\tau))d\tau$$

and

$$\begin{cases} \overleftarrow{\mathbf{I}}_{\mathbf{k}}(t^\sharp; \upsilon(\cdot))(d,x) \\ := \sup_{s\in[0,t^\sharp]} \left(\mathbf{k}(\delta(s),\xi(s)) + \int_0^s \ell(\delta(\tau),\xi(\tau),\upsilon(\tau))d\tau \right). \end{cases}$$

We posit

$$\begin{cases} \overleftarrow{\mathbf{L}}_{(\mathbf{k},\mathbf{c})}(t^\sharp; \upsilon(\cdot))(d,x) \\ := \max(\overleftarrow{\mathbf{I}}_{\mathbf{k}}(t^\sharp; x(\cdot), \upsilon(\cdot))(d,x), \overleftarrow{\mathbf{J}}_{\mathbf{c}}(t^\sharp; \upsilon(\cdot))(d,x)). \end{cases}$$

We have proved that the viability and capturability conditions imply that there exist $t^\sharp \in [0,\tau^\sharp(d)]$ and $\upsilon(\cdot)$ such that

$$\overleftarrow{\mathbf{L}}_{(\mathbf{k},\mathbf{c})}(t^\sharp; \upsilon(\cdot))(d,x) \le y. \tag{22}$$

Therefore, setting

$$U(d,x) := \inf_{(t^\sharp;\upsilon(\cdot))} \overleftarrow{\mathbf{L}}_{(\mathbf{k},\mathbf{c})}(t^\sharp; \upsilon(\cdot))(d,x),$$

the viability solution $V(d,x) := \inf_{(d,x,y)\in \mathrm{Capt}_{(13)}(\mathcal{E}p(\mathbf{k}),\mathcal{E}p(\mathbf{c}))} y$ defined by (14) satisfies inequality $U(d,x) \le V(d,x)$.

For proving the opposite inequality, take any $\varepsilon > 0$. Then there exist $\upsilon_\varepsilon(\cdot)$ and $t^\sharp_\varepsilon \in [0,\tau^\sharp(d)]$ such that

$$\overleftarrow{\mathbf{L}}_{(\mathbf{k},\mathbf{c})}(t^\sharp_\varepsilon; (\xi_\varepsilon(\cdot),\upsilon_\varepsilon(\cdot)))(d,x) \le U(d,x) + \varepsilon.$$

Setting $\xi_\varepsilon(t) := x - \int_0^t \upsilon_\varepsilon(\tau)d\tau$ and $\eta_\varepsilon(t) := U(d,x) - \int_0^t \ell(\delta(\tau),\xi_\varepsilon(\tau),\upsilon_\varepsilon(\tau))d\tau - \varepsilon$, we observe that $(\delta(\cdot), x_\varepsilon(\cdot), \eta_\varepsilon(\cdot))$ is a solution to (13) starting at $(d,x,U(d,x)-\varepsilon)$, reaching the epigraph of $\mathbf{c}$ at time $t^\sharp_\varepsilon$, and viable in $\mathcal{E}p(\mathbf{k})$ on $[0,t^\sharp_\varepsilon]$. Therefore $(d,x,U(d,x)-\varepsilon)$ belongs to $\mathrm{Capt}_{(13)}(\mathcal{E}p(\mathbf{k}),\mathcal{E}p(\mathbf{c}))$, and thus $U(d,x)-\varepsilon \ge V(d,x)$. Letting ε converge to 0 implies that $U(d,x) \ge V(d,x)$ so that equality ensues.

Finally, setting $d(t) := \delta(t^\sharp - t)$, $x(t) := \xi(t^\sharp - t)$, $y(t) := \eta(t^\sharp - t)$, $u(t) := \upsilon(t^\sharp - t)$, $\mathbf{L}_{(\mathbf{k},\mathbf{c})}(t^\sharp; x(\cdot))(d,x) := \overleftarrow{\mathbf{L}}_{(\mathbf{k},\mathbf{c})}(t^\sharp; \upsilon(\cdot))(d,x)$, etc., we deduce that $x(\cdot) \in \mathcal{A}_{\mathbf{k}}(t^\sharp; d, x)$ is a solution arriving at x at time $t^\sharp$ and starting from $x(0) = \xi(t^\sharp)$ at time $t^\sharp$. Since $x'(t) = -\xi'(t^\sharp - t) = \upsilon(t^\sharp - t) = u(t)$, we infer that

$$\begin{cases} \mathbf{J}_{\mathbf{c}}(t^\sharp; x(\cdot))(d,x) \\ := \mathbf{c}(d(0),x(0)) + \int_0^{t^\sharp} \ell(d(\tau),x(\tau),x'(\tau))d\tau, \\ \mathbf{I}_{\mathbf{k}}(t^\sharp; x(\cdot))(d,x) \\ := \sup_{s\in[0,t^\sharp]} \left(\mathbf{k}(d(s),x(s)) + \int_{t^\sharp - s}^{t^\sharp} \ell(d(\tau),x(\tau),x'(\tau))d\tau \right). \end{cases}$$

Hence we have proved that

$$V(d,x) = \inf_{t^\sharp\in[0,\tau^\sharp(d)]} \inf_{x(\cdot)\in\mathcal{A}_{\mathbf{k}}(t^\sharp;d,x)} \mathbf{L}_{(\mathbf{k},\mathbf{c})}(t^\sharp; x(\cdot))(d,x)$$

is the valuation function of the intertemporal optimization problem. $\square$

Remark. Theorem 3.2 follows from Theorem 3.3 because assumption $\mathbf{c}(d,x) = +\infty$ whenever $d \in \mathrm{Int}(\mathcal{D})$ implies that $t^\sharp = \tau^\sharp(d)$. □

THEOREM 3.4 (continuity properties of the viability solution). *Assume that the Lagrangian is Marchaud. Then the viability solution V defined by* (14) *is lower semicontinuous and its epigraph is equal to the viable-capture basin:*

$$\mathcal{E}p(V) = \mathrm{Capt}_{(13)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c})).$$

Proof. Recalling inequality (10) involved in the definition of the Marchaud Lagrangian, we introduce the set-valued map $\mathcal{F}$ defined by

$$\mathcal{F}(d,x) := \mathcal{E}p(\ell(d,x;\cdot)) \cap (c(\|x\| + \|d\| + 1)B \times [-c(\|x\| + \|d\| + 1), 0]), \tag{23}$$

which has nonempty values, and introduce the system

$$\begin{cases} \text{(i)} & \delta'(t) = -\varphi(d), \\ \text{(ii)} & (\xi'(t), \eta'(t)) \in -\mathcal{F}(\delta(t), \xi(t)). \end{cases} \tag{24}$$

1. Inclusions $\mathrm{Graph}(\ell) \subset \mathcal{F} \subset \mathcal{E}p(\ell)$ imply

$$\mathrm{Capt}_{(19)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c})) \subset \mathrm{Capt}_{(24)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c})) \subset \mathrm{Capt}_{(13)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c}))$$

and equality (21) implies that $\mathrm{Capt}_{(13)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c})) = \mathrm{Capt}_{(19)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c}))$. Hence, these three viable-capture basins coincide:

$$\mathrm{Capt}_{(19)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c})) = \mathrm{Capt}_{(24)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c})) = \mathrm{Capt}_{(13)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c})). \tag{25}$$

2. The differential inclusion (24) is Marchaud. Indeed, the graph of $(d,x) \rightsquigarrow \mathcal{E}p(\ell(d,x;\cdot))$ is closed because it is the epigraph of the Lagrangian ℓ which is lower semicontinuous. Its values are convex since the Lagrangian is convex with respect to u. The set-valued map $\mathcal{F}$ being its intersection with $(d,x) \rightsquigarrow c(\|x\| + \|d\| + 1)B \times [-c(\|x\| + \|d\| + 1), 0]$ has linear growth. Consequently, the intersection $\mathcal{F}$ has a closed graph, convex values, and linear growth; i.e., it is a Marchaud set-valued map (see Definition 11.1).
3. Theorem 11.5, the viable-capture basin $\mathrm{Capt}_{(24)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c}))$, is closed. This implies in particular that $(d, x, V(d,x))$ belongs to this viable-capture basin, which then coincides with the epigraph of V. Being closed, the viability solution is lower semicontinuous. □

A voluminous literature is devoted to regularity theorems providing sufficient conditions for the viability solution to be continuous, Lipschitz, semiconcave, and differentiable in such and such sense.

This study does not deal with the regularity properties of the viability solutions but focuses on the existence of optimal evolutions and on the microsystem regulating them.

4. Viability implies optimality. We deduce now from the viability theorems that there exists an optimal solution to the variational problem (17) and that, actually, all viable evolutions are optimal and satisfy the dynamic programming equations under viability constraints.

4.1. Optimal evolutions.

THEOREM 4.1 (viable and optimal evolutions). *Assume that the Lagrangian is Marchaud. For any* $(d,x) \in \mathrm{Dom}(V)$*, there* exist $t^\sharp \in [0, \tau^\sharp(d)]$ *such that* $d(t^\sharp) = d$ *and one evolution* $\overline{x}(\cdot) \in \mathcal{A}_{\mathbf{k}}(t^\sharp; d, x)$ *such that*

$$V(d,x) = \mathbf{L}_{(\mathbf{k},\mathbf{c})}(t^\sharp; \overline{x}(\cdot))(d,x),$$

thus achieving the minimum in the intertemporal optimization problem.

Proof. Consider *any* solution $t \mapsto (\delta(t), \xi(t), \eta(t))$ to system (13) starting at $(d, x, V(d,x))$ *viable* in $\mathcal{E}p(\mathbf{k})$ until it reaches $\mathcal{E}p(\mathbf{c})$ at some finite time $t^\sharp$. At least one such evolution does exist since $(d, x, V(d,x))$ belongs to viable-capture basin $\mathrm{Capt}_{(13)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c}))$ thanks to Theorem 3.4.

This solution is associated with a control $\upsilon(\cdot)$ satisfying

$$\eta(t) \le \eta_0(t) := V(d,x) - \int_0^t \ell(\delta(\tau), \xi(\tau), \upsilon(\tau)) d\tau.$$

Inequality (22), with $y = V(d,x)$, implies that

$$\overleftarrow{\mathbf{L}}_{(\mathbf{k},\mathbf{c})}(t^\sharp; \upsilon(\cdot))(d,x) \le V(d,x)$$

and thus that $(t^\sharp; \upsilon(\cdot))$ is optimal.

Therefore, setting $d(t) := \delta(t^\sharp - t)$, $x(t) := \xi(t^\sharp - t)$, etc., we infer that there exist $t^\sharp$ and $(d(\cdot), x(\cdot)) \in \mathcal{A}_{\mathbf{k}}(t^\sharp; d, x)$ such that

$$V(d,x) = \overleftarrow{\mathbf{L}}_{(\mathbf{k},\mathbf{c})}(t^\sharp; \upsilon(\cdot))(d,x) = \mathbf{L}_{(\mathbf{k},\mathbf{c})}(t^\sharp; x(\cdot))(d,x)$$

achieves the minimum in the intertemporal optimization problem. ∎

Denoting by

$$\mathbb{T}_{(\mathbf{k},\mathbf{c})}(d,x) := \inf \left\{ t \in [0, \tau^\sharp(d)] \text{ such that } V(d,x) = \mathbf{M}_{(\mathbf{k},\mathbf{c})}(t; d, x) \right\}$$

the *initial time map* and by

$$\mathcal{O}_{\mathbf{k}}(t^\sharp; d, x) := \left\{ x(\cdot) \text{ such that } \mathbf{L}_{(\mathbf{k},\mathbf{c})}(t^\sharp; x(\cdot))(d,x) := \mathbf{M}_{(\mathbf{k},\mathbf{c})}(t; d, x) \right\}$$

the *optimal map*, the search of optimal evolutions can be split into two steps:

1. Take $t^\sharp := \mathbb{T}_{(\mathbf{k},\mathbf{c})}(d,x)$.
2. Choose any evolution $\overline{x}(\cdot) \in \mathcal{O}_{\mathbf{k}}(t^\sharp; d, x)$.

For boundary condition functions $\mathbf{c}$ such that $\mathbf{c}(d,x) = +\infty$ whenever $d \in \mathrm{Int}(\mathcal{D})$, we have $t^\sharp := \tau^\sharp(d)$.

4.2. Dynamic programming under viability constraints. Optimal evolutions satisfy the dynamic programming principle.

THEOREM 4.2 (dynamic programming under viability constraints). *We assume that the Lagrangian* ℓ *is Marchaud and that the function* $\mathbf{k}$ *is continuous on its domain. Consider an optimal evolution* $\overline{x}(\cdot) \in \mathcal{A}_{\mathbf{k}}(t^\sharp; d, x)$*. Let* $\overline{\kappa} \in [0, t^\sharp]$ *be the first time when*

$$\mathbf{k}(d(\overline{\kappa}), \overline{x}(\overline{\kappa})) + \int_{\overline{\kappa}}^{t^\sharp} \ell(d(\tau), \overline{x}(\tau), \overline{x}'(\tau)) d\tau = V(d,x). \tag{26}$$

Set $\kappa^\sharp := \min(t^\sharp, \overline{\kappa})$*. Then* $\overline{x}(\cdot)$ *satisfies the* dynamic programming equation

$$\forall\, t \in [\kappa^\sharp, t^\sharp], \quad V(d(t), \overline{x}(t)) + \int_t^{t^\sharp} \ell(d(\tau), \overline{x}(\tau), \overline{x}'(\tau)) d\tau = V(d,x). \tag{27}$$

In particular, in the case without constraints (when $\mathbf{k}(d,x) = -\infty$*), the dynamic programming equation holds true on the interval* $[0, t^\sharp]$.

Proof. By Theorem 11.2, we know the viable-capture basin $\mathrm{Capt}_{(13)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c}))$ of the epigraph of $\mathbf{c}$ under the auxiliary system (13) is the unique bilateral fixed point

$$\mathrm{Capt}_{(13)}(\mathcal{E}p(V), \mathcal{E}p(\mathbf{c})) = \mathcal{E}p(V) = \mathrm{Capt}_{(13)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(V)).$$

1. Let $(d, x, V(d,x))$ belong to the viable-capture basin $\mathrm{Capt}_{(13)}(\mathcal{E}p(V), \mathcal{E}p(\mathbf{c})) = \mathcal{E}p(V)$ of the epigraph of $\mathbf{c}$ under the auxiliary system (13). There exist $t^\sharp \in [0, \tau^\sharp(d)]$ and $\overline{v}(\cdot)$ such that

$$t \mapsto (\delta(t), \overline{\xi}(t), \overline{\eta}(t))$$

is viable in the epigraph of V until it reaches the epigraph of $\mathbf{c}$ at time $t^\sharp$. Then the proof of Theorem 3.3 implies that

$$\forall \in [0, t^\sharp], \;\; V(\delta(s), \overline{\xi}(s)) + \int_0^s \ell(\delta(\tau), \overline{\xi}(\tau), \overline{v}(\tau)) d\tau \le V(d,x). \tag{28}$$

2. We shall deduce the opposite inequality from the second fixed point property $\mathcal{E}p(V) = \mathrm{Capt}_{(13)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(V))$. The assumption that $\mathcal{D}$ is a repeller under $-\varphi$ implies that $\mathcal{E}p(\mathbf{k})$ is also a repeller under structured characteristic system (13).
We set

$$\complement \mathcal{E}p(V) := \{(d, x, y) \text{ such that } y < V(d,x)\} =: \overset{\circ}{\mathcal{H}yp}\,(V).$$

For any $\varepsilon > 0$, $(d, x, V(d,x) - \varepsilon)$ does not belong to $\mathcal{E}p(V) = \mathrm{Capt}_{(13)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(V))$.
Therefore, for any $v(\cdot)$, there exists $\kappa_\varepsilon \le \tau^\sharp(d)$ such that $(\delta(\kappa_\varepsilon), \xi(\kappa_\varepsilon), \eta(\kappa_\varepsilon))$ reaches $\mathcal{E}p(\mathbf{k})$ and leaves it before eventually reaching $\mathcal{E}p(\mathbf{c})$. Hence κ_ε is defined by

$$\mathbf{k}(\delta(\kappa_\varepsilon), \xi(\kappa_\varepsilon)) + \int_0^{\kappa_\varepsilon} \ell(\delta(\tau), \xi(\tau), v(\tau)) d\tau = V(d,x) - \varepsilon,$$

and, consequently,

$$\forall\, s \in [0, \kappa_\varepsilon], \;\; V(d,x) - \varepsilon \le V(\delta(s), \xi(s)) + \int_0^s \ell(\delta(\tau), \xi(\tau), v(\tau)) d\tau.$$

Since $\kappa_\varepsilon \le \tau^\sharp(d) < +\infty$, a subsequence (again denoted by) κ_ε converges to some $\kappa \le \tau^\sharp(d)$ when $\varepsilon \to 0+$. The function $\mathbf{k}$ being continuous by assumption, we deduce that

$$\mathbf{k}(\delta(\kappa), \xi(\kappa)) + \int_0^{\kappa} \ell(\delta(\tau), \xi(\tau), v(\tau)) d\tau = V(d,x)$$

and that

$$\forall\, s \in [0, \kappa], \;\; V(d,x) \le V(\delta(s), \xi(s)) + \int_0^s \ell(d(\tau), \xi(\tau), v(\tau)) d\tau. \tag{29}$$

This inequality holds for the above viable evolution $(\delta(\cdot), \overline{\xi}(\cdot), \overline{\eta}(\cdot))$ on the interval $[0, t^\sharp]$ so that inequalities (28) and (29) imply that equality

$$\forall\, s \in [0, \min(\kappa, t^\sharp)],\;\; V(d,x) = V(\delta(s), \overline{\xi}(s)) + \int_0^s \ell(d(\tau), \overline{\xi}(\tau), \overline{\upsilon}(\tau))d\tau$$

ensues.

We derive the conclusion (27) by setting $\overline{x}(t) := \overline{\xi}(t^\sharp - t)$ and $\overline{\kappa} := t^\sharp - \kappa$ satisfying (26). ∎

5. Regulation of optimal evolutions. It is not enough to know the existence of optimal evolutions: the question arises of whether we can compute it. For that purpose, in the set-valued map $(d,x) \mapsto F(d,x)$ governing the evolution $x(\cdot)$ through differential inclusion $x'(t) \in F(d(t), x(t))$, we shall carve a regulation map $(d,x) \mapsto R_V(d,x) \subset F(d,x)$ piloting optimal viable evolutions by differential inclusion $x'(t) \in R_V(d(t), x(t))$ until it reaches the terminal state x at optimal time $t^\sharp$.

The regulation map is characterized by the viability solution by a formula which uses the fact that the viability solution V is also the solution of the structured Hamilton–Jacobi inequality, which holds true for the Marchaud Lagrangian.

When V is any differentiable function, the regulation map associated with it is defined by

$$R_V(d,x) := \left\{ \begin{array}{l} \upsilon \in F(d,x) \text{ such that} \\ \left\langle \dfrac{\partial V(d,x)}{\partial d}, \varphi(d) \right\rangle + \left\langle \dfrac{\partial V(d,x)}{\partial x}, \upsilon \right\rangle - \ell(d,x;\upsilon) \geq 0 \end{array} \right\}.$$

Remark (Lax–Oleinik formula). Observe that if the Lagrangian ℓ is Marchaud and if the function V satisfies the Hamilton–Jacobi inequality

$$\left\langle \frac{\partial V(d,x)}{\partial d}, \varphi(d) \right\rangle + \ell\left(d, x; \frac{V(d,x)}{dx}\right) \geq 0,$$

then the Legendre property (9) of the Fenchel transform implies the *generalized Lax–Oleinik formula*

$$\partial \ell \left(d, x; \frac{V(d,x)}{dx}\right) \subset R_V(d,x).$$

Indeed, if we assume further that $R_V(d,x) = \{r(d,x)\}$ is a singleton and that the Lagrangian is differentiable with respect to u, this implies that

$$\frac{\partial V(d,x)}{\partial x} = \frac{d}{du}\ell\left(d, x; r(d,x)\right),$$

which is the *Lax–Oleinik formula*. General formulas (61) relating the regulation map associated with the viability solution and its partial derivatives (or subdifferentials) with respect to state x are obtained under (much) stronger assumptions. They are consequences of the proof that the viability solution is the unique Barron–Jensen/Frankowska viscosity solution of the Hamilton–Jacobi equation. We postpone it to section 9 since we do not need this purely mathematical result for studying the regulation of optimal viable evolutions. ∎

When V is no longer differentiable, we consider its *epiderivative* $D_\uparrow^{\star\star}V(d,x)$ of V defined as

$$\mathcal{E}p(D_\uparrow^{\star\star}V(d,x)) := T^{\star\star}_{\mathcal{E}p(V)}(d, x, V(d,x)),$$

which is a directional derivative $u \mapsto D_{\uparrow}^{\star\star}V(d,x)(\delta,\upsilon) \in \mathbb{R}\cup\{-\infty\}\cup\{+\infty\}$ that is convex and lower semicontinuous instead of being linear (and continuous).

Definition 5.1 (regulation map associated with a function).

$$R_V(d,x) := \left\{\upsilon \in F(d,x) \text{ such that } D_{\uparrow}^{\star\star}V(d,x)(-\varphi(d),-\upsilon)+\ell(d,x;\upsilon)\le 0\right\}. \tag{30}$$

We begin by proving that the regulation map associated with the viability solution has nonempty values.

Theorem 5.2 (regulation map of the viability solution). *If the Lagrangian $(d,p,\upsilon) \rightsquigarrow \ell(d,x;\upsilon)$ is Marchaud, then the viability solution V defined by* (14) *is the smallest lower semicontinuous function satisfying conditions* (12) *and*

$$\inf_{\upsilon\in F(d,x)} \left(D_{\uparrow}^{\star\star}V(d,x)(-\varphi(d),-\upsilon)+\ell(d,x;\upsilon)\right)\le 0$$

(V is the contingent solution introduced by Frankowska) such that, if $V(d,x) < \mathbf{c}(d,x)$, the value $R_V(d,x)$ of the regulation map associated with the viability solution V is not empty.

Proof. Since the Hamiltonian is Marchaud, so is the Lagrangian. Actually, the theorem remains true under the weaker assumption that the Lagrangian ℓ is Marchaud. So is the set-valued map $\mathcal{F}$ defined by (23). By (25), the viable-capture basin $\mathcal{E}p(V) := \mathrm{Capt}_{(13)}(\mathcal{E}p(\mathbf{k}),\mathcal{E}p(\mathbf{c}))$ is equal to $\mathrm{Capt}_{(24)}(\mathcal{E}p(\mathbf{k}),\mathcal{E}p(\mathbf{c})) = \mathrm{Capt}_{(13)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c}))$. It is then the largest closed subset between $\mathcal{E}p(\mathbf{c})$ and $\mathcal{E}p(\mathbf{k})$, locally viable in $\mathcal{E}p(V)\setminus\mathcal{E}p(\mathbf{c})$ thanks to viability Theorem 11.9, which also states that, whenever $(d,x,V(d,x)) \in \mathcal{E}p(V)\setminus\mathcal{E}p(\mathbf{c})$, i.e., whenever $V(d,x) < \mathbf{c}(d,x)$, there exists some $\upsilon \in \mathrm{Dom}(\ell)$ such that

$$(-\varphi(d),-\upsilon,-\ell(d,x;\upsilon)) \in T_{\mathcal{E}p(V)}^{\star\star}(d,x,V(d,x)).$$

Definition 11.8 of the regulation map R for general differential inclusions and Definition 5.1 imply that such a υ belongs to $R_V(d,x)$. ∎

Optimal evolutions do exist thanks to Theorem 3.3. The question asked is how to regulate them. The first answer is provided by the following.

Theorem 5.3 (regulation of optimal evolutions). *If the Lagrangian $(d,p,\upsilon) \rightsquigarrow \ell(d,x;\upsilon)$ is Marchaud, viable optimal evolutions $x(\cdot) \in \mathcal{O}_{\mathbf{k}}(t^\sharp;d,x)$ when $t^\sharp \in \mathbb{T}_{(\mathbf{k},\mathbf{c})}(d,x)$ is the optimal time are regulated by the differential inclusion*

$$\forall\, t\in[0,t^\sharp[,\quad x'(t) \in R_V(d(t),x(t))$$

and satisfy the terminal conditions

$$d(t^\sharp) = d \text{ and } x(t^\sharp) = x.$$

Proof. The proof of Theorem 4.1 states that evolutions $t \mapsto (\delta(t),\xi(t),\eta(t))$ starting from $(d,x,V(d,x)) \in \mathcal{E}p(V)$ viable in $\mathcal{E}p(\mathbf{k})$ until they reach $\mathcal{E}p(\mathbf{c})$ at time $t^\sharp$ are optimal and regulated by

$$\forall\, t\in[0,t^\sharp],\quad \upsilon(t)\in R(\delta(t),\xi(t))$$

thanks to viability Theorem 11.9. By setting $d(t) := \delta(t^\sharp - t)$, $x(t) := \xi(t^\sharp - t)$, and $x'(t) := \upsilon(t^\sharp - t)$, this is equivalent to saying that optimal viable evolutions $x(\cdot) \in \mathcal{O}_{\mathbf{k}}(t^\sharp;d,x)$ are regulated by the differential inclusion

$$x'(t) \in R_V(d(t),x(t))$$

and satisfy the terminal conditions $d(t^\sharp) = d$ and $x(t^\sharp) = x$. $\square$

We also know that

$$\mathbf{c}(d(0), x(0)) + \int_0^{t^\sharp} \ell(d(\tau), x(\tau), x'(\tau))d\tau \leq V(d, x).$$

With the initial causal variable $d(0) := \delta(t^\sharp)$ being known, the initial state $x(0)$ may be derived from the above formula, which requires the knowledge of the evolution $x(\cdot)$ arriving at x and regulated by the regulation map R.

6. Aggregation. In this section, we introduce another state space $\mathbb{R}^q$, with (much) higher dimension $q \geq n$ and a linear operator $A \in \mathcal{L}(\mathbb{R}^q, \mathbb{R}^n)$ and its transpose $A^\star \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^q)$.

This operator can regarded as an aggregration operator generalizing the operator

$$A : x = (x_1, \ldots, x_J) \in \mathbb{R}^q := \prod_{j=1}^{J} \mathbb{R}^{q_j} \mapsto Ax := \sum_{j=1}^{J} x_j \in \mathbb{R}^n.$$

We shall assume throughout this section that the Hamiltonian $(d, p) \mapsto \ell^\star(d; p)$ and the Lagrangian $(d, u) \mapsto \ell(d; u)$ *do not depend upon the state variable* x. We introduce the composed Hamiltonian $m^\star : (d, q) \in \mathbb{R}^m \times \mathbb{R}^n \mapsto m^\star(d, q) \in \mathbb{R} \cup \{+\infty\}$ related to $\ell^\star$ by

$$m^\star(d, q) := \ell^\star(d, A^\star q),$$

and we denote by $m(d, y) := \ell^{\star\star}(d, y)$ its biconjugate.

For instance, if $Ax := \sum_{j=1}^{J} x_j$, then for any $p \in \mathbb{R}^n$,

$$m^\star(d, q) = \sum_{j=1}^{J} \ell_j^\star(d, q).$$

We also introduce two internal condition functions,

1. $\mathbf{c} : (d, x) \in \mathbb{R}^m \times \mathbb{R}^q \mapsto \mathbf{c}(d, x) \in \mathbb{R} \cup \{+\infty\}$,
2. $\mathbf{b} : (d, y) \in \mathbb{R}^m \times \mathbb{R}^n \mapsto \mathbf{b}(d, y) \in \mathbb{R} \cup \{+\infty\}$,

related by

$$\mathbf{b}(d, y) := \inf_{Ax=y} \mathbf{c}(d, x).$$

Theorem 10.2 states that the assumption that there exists a constant $c < +\infty$ such that

(31)
$$\begin{cases} \text{(i)} & \forall\ (d, u) \in \mathrm{Dom}(\ell),\ \forall\ \nu \in \mathbb{R}^n,\ \ \exists\ \mu \in \mathrm{Dom}(D_\uparrow \ell(d, u)) \cap c\|\nu\|B \text{ and } A\nu = \mu, \\ \text{(ii)} & \forall\ (d, x) \in \mathrm{Dom}(\mathbf{c}), \forall\ \nu \in \mathbb{R}^n\ \exists\ \mu \in \mathrm{Dom}(D_\uparrow \mathbf{c}(d, x)) \cap c\|\nu\|B \text{ and } A\nu = \mu \end{cases}$$

implies that the infimum is achieved in formulas

$$\begin{cases} \text{(i)} & m(d, v) = \min_{Au=v} \ell(d, u) \\ & \qquad\qquad \text{is the conjugate function of } m^\star(d, p) := \ell^\star(d, A^\star q), \\ \text{(ii)} & \mathbf{b}(d, y) = \min_{Ax=y} \mathbf{c}(d, x). \end{cases}$$

THEOREM 6.1. *We consider the two structured Hamilton–Jacobi problems*

$$\left\{ \begin{array}{l} \left\langle \frac{\partial V(d,x)}{\partial d}, \varphi(d) \right\rangle + \ell^\star\left(d, \frac{\partial V(d,x)}{\partial x}\right) = 0 \text{ on } \mathbb{R}^m \times \mathbb{R}^q \\ \text{satisfying } V(d,x) \le \mathbf{c}(d,x) \end{array} \right. \tag{32}$$

and

$$\left\{ \begin{array}{l} \left\langle \frac{\partial W(d,y)}{\partial d}, \varphi(d) \right\rangle + \ell^\star\left(d, A^\star \frac{\partial W(d,y)}{\partial y}\right) = 0 \text{ on } \mathbb{R}^m \times \mathbb{R}^n \\ \text{satisfying } W(d,y) \le \mathbf{b}(d,y). \end{array} \right. \tag{33}$$

Assume furthermore that the constraint qualification assumptions (31) *hold true. Then their viability solutions are related by formula*

$$W(d,y) = \inf_{Ax=y} V(d,x). \tag{34}$$

Proof. Let us consider the characteristic systems

$$\left\{ \begin{array}{ll} \text{(i)} & \delta'(t) = -\varphi(\delta(t)), \\ \text{(ii)} & (\eta'(t), \zeta'(t)) \in -\mathcal{E}p(m(\delta(t);\cdot)) = -(\mathbf{1} \times A \times \mathbf{1})\mathcal{E}p(\ell(\delta(t);\cdot)) \end{array} \right. \tag{35}$$

governing the evolution of $\eta(t) \in \mathbb{R}^n$ and

$$\left\{ \begin{array}{ll} \text{(i)} & \delta'(t) = -\varphi(\delta(t)), \\ \text{(ii)} & (\xi'(t), \zeta'(t)) \in -\mathcal{E}p(\ell(\delta(t);\cdot)) \end{array} \right. \tag{36}$$

governing the evolution of $\xi(t) \in \mathbb{R}^q$. We introduce the viable-capture basins defining the epigraphs of the viability solutions

$$\left\{ \begin{array}{ll} \text{(i)} & \text{Capt}_{(35)}(\mathcal{E}p(\mathbf{b})) = \mathcal{E}p(W), \text{ where } \mathcal{E}p(\mathbf{b}) = (\mathbf{1} \times A \times \mathbf{1})\mathcal{E}p(\mathbf{c}), \\ \text{(ii)} & \text{Capt}_{(36)}(\mathcal{E}p(\mathbf{c})) = \mathcal{E}p(V). \end{array} \right. \tag{37}$$

We shall prove that

$$(\mathbf{1} \times A \times \mathbf{1})\text{Capt}_{(36)}(\mathcal{E}p(\mathbf{c})) = \text{Capt}_{(35)}((\mathbf{1} \times A \times \mathbf{1})\mathcal{E}p(\mathbf{c})) = \text{Capt}_{(35)}(\mathcal{E}p(\mathbf{b})).$$

1. *Proof of inequality* $W(d,Ax) \le V(d,x)$. This inequality is always true. Take any $(d,x,V(d,x)) \in \text{Capt}_{(35)}(\mathcal{E}p(\mathbf{c}))$. There exist $t \mapsto \nu(t)$ and $t^\sharp \ge 0$ such that

$$\mathbf{c}\left(\delta(t^\sharp), x - \int_0^{t^\sharp} \nu(t)dt\right) + \int_0^{t^\sharp} \ell(\delta(t);\nu(t))dt \le V(d,x).$$

Since $\mathbf{b}(d,Ax) \le \mathbf{c}(d,x)$ and $m(d,A\nu) \le \ell(d,\nu)$, we infer that setting $\mu(t) := A\nu(t)$, we obtain

$$\mathbf{b}\left(\delta(t^\sharp), Ax - \int_0^{t^\sharp} \mu(t)dt\right) + \int_0^{t^\sharp} m(\delta(t);\mu(t))dt \le V(d,x),$$

and thus, that $(d,Ax,V(d,x))$ belongs to $\text{Capt}_{(35)}(\mathcal{E}p(\mathbf{b})) = \mathcal{E}p(W)$. This implies that $W(d,Ax) \le V(d,x)$ and that

$$(\mathbf{1} \times A \times \mathbf{1})\text{Capt}_{(36)}(\mathcal{E}p(\mathbf{c})) \subset \text{Capt}_{(35)}((\mathbf{1} \times A \times \mathbf{1})\mathcal{E}p(\mathbf{c})).$$

2. *Proof of inequality: for all y, $\exists$ x such that $Ax = y$ and $V(d,x) \le W(d,y)$.* Take any $(d, y, W(d,y)) \in \mathrm{Capt}_{(36)}(\mathcal{E}p(\mathbf{b}))$. There exists an integrable function $t \mapsto \mu(t)$ and $t^\sharp \ge 0$ such that

$$\mathbf{b}\left(\delta(t^\sharp), x - \int_0^{t^\sharp} \mu(t)dt\right) + \int_0^{t^\sharp} m(\delta(t); \mu(t))dt \le W(d,y).$$

Theorem 11.7 and assumption (31)(i) imply that the subset

$$\Phi(d,v) := \{u \text{ such that } Au = v \text{ and } \ell(d;u) \le m(d;v)\}$$

is not empty and that the set-valued map Φ has a closed graph. The measurable selection theorem (see, for instance, Theorem 8.1.13 on page 308 of [14]) implies that we can associate with the measurable function $\mu(\cdot)$ a measurable function $\nu(\cdot)$ such that, for almost all t, $\nu(t) \in \Phi(\delta(t), \mu(t))$. Furthermore, Theorem 11.7 and assumption (31)(ii) imply that we can associate with $y - \int_0^{t^\sharp} \mu(t)dt$ an element z such that $Az = y - \int_0^{t^\sharp} \mu(t)dt$ and $\mathbf{c}(z) = y - \int_0^{t^\sharp} \mu(t)dt$. Setting $x := z + \int_0^{t^\sharp} \nu(t)dt$, we have proved that

$$\begin{cases} \mathbf{c}\left(\delta(t^\sharp), x - \int_0^{t^\sharp} \nu(t)dt\right) + \int_0^{t^\sharp} \ell(\delta(t); \nu(t))dt \\ = \mathbf{b}\left(\delta(t^\sharp), y - \int_0^{t^\sharp} \mu(t)dt\right) + \int_0^{t^\sharp} \ell(\delta(t); \mu(t))dt \le W(d,y). \end{cases}$$

Therefore, $(d, y, W(d,y)) = (\mathbf{1} \times A \times \mathbf{1})(d, x, W(d,y))$, where $(d, x, W(d,y))$ belongs to $\mathrm{Capt}_{(36)}(\mathcal{E}p(\mathbf{c})) = \mathcal{E}p(W)$. Consequently, $(d, y, W(d,y)) \in (\mathbf{1} \times A \times \mathbf{1})\mathcal{E}p(V)$, and thus, $V(d,x) \le W(d,y)$ and

$$\mathrm{Capt}_{(35)}((\mathbf{1} \times A \times \mathbf{1})\mathcal{E}p(\mathbf{c})) \subset (\mathbf{1} \times A \times \mathbf{1})\mathrm{Capt}_{(36)}(\mathcal{E}p(\mathbf{c})).$$

These two inequalities imply that $V(d,x) \le W(d,y) = W(d,Ax) \le V(d,x)$, so that

$$W(d,y) = \min_{Ax=y} V(d,x)$$

and $\mathrm{Capt}_{(35)}(\mathbf{1} \times A \times \mathbf{1})(\mathcal{E}p(\mathbf{c})) = (\mathbf{1} \times A \times \mathbf{1})\mathrm{Capt}_{(36)}(\mathcal{E}p(\mathbf{c}))$. ◻

We now compare the two regulation maps as follows.

THEOREM 6.2. *We posit the assumptions of Theorem* 6.1. *Consider the two regulation maps*

$$R_W(d,y) := \left\{ \begin{array}{l} \mu \in \mathrm{Dom}(m(d,y;\cdot)) \textit{ such that} \\ \left\langle \dfrac{\partial W(d,y)}{\partial d}, \varphi(d) \right\rangle + \left\langle \dfrac{\partial W(d,y)}{\partial y}, \mu \right\rangle - m(d,y;\mu) \ge 0 \end{array} \right\}$$

and

$$R_V(d,x) := \left\{ \begin{array}{l} \nu \in \mathrm{Dom}(\ell(d,x;\cdot)) \textit{ such that} \\ \left\langle \dfrac{\partial V(d,x)}{\partial d}, \varphi(d) \right\rangle + \left\langle \dfrac{\partial V(d,x)}{\partial x}, \nu \right\rangle - \ell(d,x;\nu) \ge 0 \end{array} \right\}.$$

Inclusion $AR_V(d,x) \subset R_V(d,Ax)$ always holds true. If we assume that the solution V satisfies

(38)

$$\begin{cases} \text{(i)} & \forall\ (d,u) \in \mathrm{Dom}(\ell),\ \forall\ \nu \in \mathbb{R}^n,\ \ \exists\ \mu \in \mathrm{Dom}(D_\uparrow \ell(d,u)) \cap c\|\nu\|B \text{ and } A\nu = \mu, \\ \text{(ii)} & \forall\ (d,x) \in \mathrm{Dom}(V), \forall\ \nu \in \mathbb{R}^n\ \exists\ \mu \in \mathrm{Dom}(D_\uparrow V(d,x)) \cap c\|\nu\|B \text{ and } A\nu = \mu, \end{cases}$$

then equality

$$\exists\, x \text{ satisfying } Ax = y \text{ and } R_W(d,y) = AR_V(d;x) \tag{39}$$

ensues.

Proof.

1. *Proof of inclusion* $AR_V(d,x) \subset R_W(d,Ax)$. Let us consider $\nu \in R_V(d,x)$. This means that

$$(-\varphi(d), -\nu, -\ell(d;\nu)) \in T_{\mathcal{E}p(V)}(d,x,V(d,x)).$$

Therefore,

$$\left\{\begin{array}{l} (-\varphi(d), -A\nu, -\ell(d;\nu)) = (\mathbf{1}\times A\times\mathbf{1})(-\varphi(d), -\nu, -\ell(d;\nu)) \\ \in (\mathbf{1}\times A\times\mathbf{1})T_{\mathcal{E}p(V)}(d,x,V(d,x)) \subset T_{(\mathbf{1}\times A\times\mathbf{1})\mathcal{E}p(V)}(d,Ax,V(d,x)) \\ = T_{\mathcal{E}p(W)}(d,y,V(d,x)). \end{array}\right.$$

Since by Proposition 6.1.4 on page 226 of [14] $T_{\mathcal{E}p(W)}(d,y,V(d,x)) = \mathrm{Dom}(D_\uparrow W(d,y)) \times \mathbb{R}$ if $W(d,y) < V(d,x)$ and $T_{\mathcal{E}p(W)}(d,x,V(d,x)) = T_{\mathcal{E}p(W)}(d,y,W(d,y))$ if $W(d,y) = V(d,x)$, we infer that $A\nu$ belongs to $R_W(d,Ax)$.

2. *Proof of inclusion: for all* y, $\exists\, x$ *such that* $Ax{-}y$ *and* $R_W(d,y) \subset AR_V(d,x)$. Let us consider $\mu \in R_W(d,y)$, i.e., satisfying

$$(-\varphi(d), -\mu, -m(d;\mu)) \in T_{\mathcal{E}p(W)}(d,y,W(d,y)).$$

Theorem 11.7 and assumption (38)(ii) imply that there exist x such that $Ax = y$ and $V(d,x) = W(d,y)$.
By applying Theorem 11.7 for the linear operator $\mathbf{1}\times A\times\mathbf{1}$ and the subset $\mathcal{E}p(V)$, we infer that assumption (38)(ii) implies that equality

$$(\mathbf{1}\times A\times\mathbf{1})T_{\mathcal{E}p(V)}(d,x,V(d,x)) = T_{(\mathbf{1}\times A\times\mathbf{1})\mathcal{E}p(V)}(d,Ax,V(d,x)) \tag{40}$$

holds true. Then, there exist ν such that $A\nu = \mu$ and $\ell(d,\nu) = m(d,\mu)$. Consequently,

$$\left\{\begin{array}{l} (-\varphi(d), -\mu, -m(d;\mu)) = (\mathbf{1}\times A\times\mathbf{1})(-\varphi(d), -\nu, -\ell(d;\nu)) \\ \in (\mathbf{1}\times A\times\mathbf{1})T_{\mathcal{E}p(W)}(d,y,W(d,y)) = (\mathbf{1}\times A\times\mathbf{1})T_{\mathcal{E}p(W)}(d,x,V(d,x)). \end{array}\right.$$

Therefore,

$$(-\varphi(d), -\nu, -\ell(d;\nu)) \in T_{\mathcal{E}p(V)}(d,x,V(d,x)),$$

and we infer that ν belongs to $R_V(d,x)$. □

7. Hamilton–Jacobi–Cournot equations. The question we ask here is whether we can obtain beforehand *the initial states* $x(0)$ *and the specific regulation maps* $\widetilde{R}(d,x(0),x)$ *(depending upon* $x(0)$*) driving optimal viable evolutions* $x(\cdot)$ arriving at terminal state x at optimal time $t^\sharp$. This would avoid *computing initial states* $x(0)$ by solving the variational problem or, equivalently, avoid the regulation of *all* optimal evolutions from the terminal state through the regulation map R_V as in Theorem 5.3.

For computing the formerly missing initial conditions, we just introduce an auxiliary parameter $\chi \in \mathbb{R}^n$ which plays the role of *candidate to be an initial condition.* We then extend internal and viability conditions $\widetilde{\mathbf{k}}$ and $\widetilde{\mathbf{c}}$ by setting

$$(41) \qquad \begin{cases} \text{(i)} & \widetilde{\mathbf{k}}(d,\chi,x) := \mathbf{k}(d,x), \\ \text{(ii)} & \widetilde{\mathbf{c}}(d,x,x) := \mathbf{c}(d,x) \text{ and } \widetilde{\mathbf{c}}(d,\chi,x) := +\infty \ \ \chi \neq x. \end{cases}$$

The answer to our question is obtained by computing the viability solution $\widetilde{V}(d,\chi,x)$ to the Hamilton–Jacobi–Cournot partial differential equation

$$(42) \qquad \left\langle \frac{\partial \widetilde{V}(d,\chi,x)}{\partial d}, \varphi(d) \right\rangle + \ell^\star\left(d,x; \frac{\partial \widetilde{V}(d,\chi,x)}{\partial x}\right) = 0$$

satisfying

$$(43) \qquad \widetilde{\mathbf{k}}(d,\chi,x) \leq \widetilde{V}(d,\chi,x) \leq \widetilde{\mathbf{c}}(d,\chi,x).$$

We associate with the viability solution $\widetilde{V}$ of the Hamilton–Jacobi–Cournot equation

1. the *Cournot regulation map* $R_{\widetilde{V}}$ defined by

$$R_{\widetilde{V}}(d,\chi,x) = \left\{ \upsilon \in F(d,x) \text{ such that } D_\uparrow^{\star\star}\widetilde{V}(d,\chi,x)(-\varphi(d),0,-\upsilon) + \mathbf{l}(d,x) \leq 0 \right\};$$

2. the *Cournot map* $\mathbb{C}_{\widetilde{V}}$ defined by

$$(44) \qquad \mathbb{C}_{\widetilde{V}}(d,x) := \left\{ \chi \text{ such that } \widetilde{V}(d,\chi,x) < \infty \right\}.$$

Theorem 7.1 (regulation of optimal evolutions from cournot initial states). *We posit the assumptions of Theorem* 9.2. *For any initial state $\chi \in \mathbb{C}_{\widetilde{V}}(d,x)$ provided by the Cournot map, there exists at least one optimal viable evolution $x(\cdot) \in \mathcal{O}_{\mathbf{k}}(t^\sharp; d,\chi,x)$, where $t^\sharp \in \mathbb{T}_{(\mathbf{k},\mathbf{c})}(d,x)$ starting from initial conditions $x(0) = \chi$ and arriving at $x = x(t^\sharp)$ at optimal $t^\sharp$ when $d(t^\sharp) = d$.*

Optimal evolutions are regulated by the differential inclusion

$$\forall\, t \in [0, t^\sharp[, \quad x'(t) \in R_{\widetilde{V}}(d(t),\chi,x(t)).$$

Proof. Let us consider the viability solution $(d,\chi,x) \mapsto \widetilde{V}(d,\chi,x)$ and the associated regulation map $R_{\widetilde{V}}$.

Its epigraph is the viable-capture basin of $\mathcal{E}p(\widetilde{c})$ viable in $\mathcal{E}p(\widetilde{k})$ under the characteristic system

$$(45) \qquad \begin{cases} \text{(i)} & \delta'(t) = -\varphi(\delta(t)), \\ \text{(ii)} & \chi'(t) = 0, \\ \text{(iii)} & (\xi'(t), \eta'(t)) \in -\mathcal{E}p(\ell(\delta(t),\xi(t);\cdot)). \end{cases}$$

There exist at least some time $t^\sharp$ and one evolution $t \mapsto (\delta(t),\chi,\xi(t),\eta(t))$ starting from $(d,x,\chi,V(d,x)) \in \mathcal{E}p(\widetilde{V})$ viable in $\mathcal{E}p(\widetilde{\mathbf{k}})$ until it reaches $\mathcal{E}p(\widetilde{\mathbf{c}})$ at some time $t^\sharp$. The viability condition implies that $t^\sharp \leq \tau^\sharp(d)$ and that

$$\sup_{s\in[0,t^\sharp]} \left(\mathbf{k}\left(\delta(s),\xi(s)\right) + \int_0^s \ell(\delta(\tau),\xi(\tau),\upsilon(\tau))d\tau \right) \leq \widetilde{V}(d,\chi,x)$$

so that

$$\overleftarrow{\mathbf{I}}_{\mathbf{k}}(t^\sharp; \upsilon(\cdot))(d,x) \le \widetilde{V}(d,\chi,x)$$

and the target condition

$$\widetilde{\mathbf{c}}\left(\delta(t^\sharp),\chi,\xi(t^\sharp)\right) + \int_0^{t^\sharp} \ell(\delta(\tau),\xi(\tau),\upsilon(\tau))d\tau \le \widetilde{V}(d,\chi,x) < +\infty$$

holds true.

Since $\widetilde{\mathbf{c}}(d,\chi,x) := +\infty$ whenever $\chi \neq x$, we infer that $\chi = \xi(t^\sharp)$, that $\chi \in \mathbb{C}_{\widetilde{V}}(d,x)$, and that

$$\mathbf{c}(\delta(t^\sharp),\chi) + \int_0^{t^\sharp} \ell(\delta(\tau),\xi(\tau),\upsilon(\tau))d\tau \le \widetilde{V}(d,\chi,x) < +\infty.$$

Setting

$$\begin{cases} \overleftarrow{\mathbf{J}}_{\mathbf{c}}(t^\sharp;\upsilon(\cdot))(d,\chi,x) \\ := \mathbf{c}(\delta(t^\sharp),\chi) + \displaystyle\int_0^{t^\sharp} \ell(\delta(\tau),\xi(\tau),\upsilon(\tau))d\tau \text{ if } \xi(t^\sharp)=\chi \\ \text{and } +\infty \text{ otherwise,} \end{cases}$$

we infer that

$$\overleftarrow{\mathbf{J}}_{\mathbf{c}}(t^\sharp;\upsilon(\cdot))(d,\chi,x) \le \widetilde{V}(d,\chi,x)$$

and thus that, setting

$$\begin{cases} \overleftarrow{\mathbf{L}}_{(\mathbf{k},\mathbf{c})}(t^\sharp;\upsilon(\cdot))(d,\chi,x) \\ := \max(\overleftarrow{\mathbf{I}}_{\mathbf{k}}(t^\sharp;x(\cdot),\upsilon(\cdot))(d,x), \overleftarrow{\mathbf{J}}_{\mathbf{c}}(t^\sharp;\upsilon(\cdot))(d,\chi,x)), \end{cases}$$

we have proved that

$$\overleftarrow{\mathbf{L}}_{(\mathbf{k},\mathbf{c})}(t^\sharp;\upsilon(\cdot))(d,\chi,x) \le \widetilde{V}(d,\chi,x).$$

The proof of Theorem 3.3 implies that

$$\widetilde{V}(d,\chi,x) = \inf_{(t^\sharp;\upsilon(\cdot))} \overleftarrow{\mathbf{L}}_{(\mathbf{k},\mathbf{c})}(t^\sharp;\upsilon(\cdot))(d,\chi,x)$$

and that the viable evolution is regulated by

$$\upsilon(t) \in R_{\widetilde{V}}(\delta(t),\chi,\xi(t))$$

where, under the assumptions of Theorem 9.2, the regulation map $R_{\widetilde{V}}$ is equal to

$$R_{\widetilde{V}}(d,\chi,x) = \left\{ \upsilon \in F(d,x) \text{ such that } D_{\uparrow}^{\star\star}\widetilde{V}(d,\chi,x)(-\varphi(d),0,-\upsilon) + \mathbf{l}(d,x) \le 0 \right\}.$$

By setting $d(t) := \delta(t^\sharp - t)$, $x(t) := \xi(t^\sharp - t)$ and $x'(t) := \upsilon(t^\sharp - t)$, this is equivalent to saying that that an optimal viable evolution starting from $\chi \in \mathbb{C}_{\widetilde{V}}(d,x)$ is regulated by

$$\upsilon(t) \in R_{\widetilde{V}}(\delta(t),\chi,\xi(t))$$

and satisfies the terminal conditions $d(t^\sharp) = d$ and $x(t^\sharp) = x$ and initial condition $x(0) = \chi$.

Hence, for each initial condition $\chi \in \mathbb{C}_{\widetilde{V}}(d, x)$, there exists an optimal viable evolution starting at χ and arriving at x in optimal time $t^\sharp$. ∎

Consequently, for any pair (d, x),

1. if $\mathbb{C}_{\widetilde{V}}(d, x) = \{\chi\}$ is a singleton, then there exists one optimal viable evolution starting at χ and regulated by $x'(t) \in R_{\widetilde{V}}(d(t), \chi, x(t))$ until it arrives at x at time $t^\sharp$;
2. $\mathbb{C}_{\widetilde{V}}(d, x) = \emptyset$, and no optimal evolution arrive at x;
3. $\mathbb{C}_{\widetilde{V}}(d, x)$ contains several initial states χ. From all $\chi \in \mathbb{C}_{\widetilde{V}}(d, x)$ start optimal viable evolutions regulated by $x'(t) \in R_{\widetilde{V}}(d(t), \chi, x(t))$ until they *collide* at x at time $t^\sharp$.

This is the property which has motivated the terminology of the Cournot map since the time of Antoine Cournot (1801–1877), who suggested to capture one aspect of uncertainty or chance as the collision of several independent causal series.

8. Lax–Hopf formula. In this section we derive the Lax–Hopf formula for the viability solutions of the Hamilton–Jacobi equation under constraints (see [17] for viscosity solutions). This formula is a by-product of the variational principle, which can be proved directly or derived explicitly for a Lax–Hopf formula of the capture basin of a target viable in a closed convex subset under constant closed convex differential inclusions (see Theorem 11.6). The derivatives are not involved in the formulation of Lax–Hopf formula or in its proof.

THEOREM 8.1 (Lax–Hopf formula). *We assume that both φ and ℓ do not depend upon d and x, that the function $\mathbf{c}$ is lower semicontinuous, and that the functions $\mathbf{k}$ and ℓ are convex and lower semicontinuous. Then the viability solution V is equal to the* Lax–Hopf value function

$$V(d, x) = \max\left(\mathbf{k}(d, x), \inf_{t^\sharp \geq 0} \inf_{u \in \mathrm{Dom}(\ell)} \left(\mathbf{c}(d - t^\sharp\varphi, x - t^\sharp u) + t^\sharp \ell(u)\right)\right), \tag{46}$$

which is the marginal function of a static minimization theorem.

The regulation map is the set of elements $u \in \mathrm{Dom}(\ell)$ minimizing this function:

$$\begin{cases} R_V(d, x) = \{u \in \mathrm{Dom}(\ell) \text{ such that} \\ V(d, x) = \max\left(\mathbf{k}(d, x), \left(\mathbf{c}(d - \tau^\sharp(d)\varphi, x - \tau^\sharp(d)u) + \tau^\sharp(d)\ell(u)\right)\right)\}. \end{cases} \tag{47}$$

If the viability solution V and the Lagrangian ℓ are differentiable, and if $u := r(d, x) \in R_V(d, x)$ is the unique minimizer, we obtain the Lax–Oleinik formula

$$\frac{\partial V}{\partial x} = \frac{d}{du}\ell(d, x; r(d, x)). \tag{48}$$

For boundary value problems without constraints, the formula boils down to

$$V(d, x) = \inf_{u \in \mathrm{Dom}(\ell)} \left(\mathbf{c}(d - \tau^\sharp(d)\varphi, x - \tau^\sharp(d)u) + \tau^\sharp(d)\ell(u)\right). \tag{49}$$

Proof. Since the epigraph of $\mathbf{k}$ is convex and since both the map φ and Lagrangian ℓ do not depend on (d, x), the differential inclusion

$$\begin{cases} \text{(i)} & \delta'(t) = -\varphi, \\ \text{(ii)} & (\xi'(t), \eta'(t)) \in -\mathcal{E}p(\ell(\cdot)) \end{cases} \tag{50}$$

can be written in the form

$$(\delta'(t), \xi'(t), \eta'(t)) \in \mathcal{G} := \{-\varphi\} \times -\mathcal{E}p(\ell(\cdot)),$$

where $\mathcal{G}$ is a constant closed convex subset. Lax–Hopf formula (71) of Theorem 11.6 states that if the environment $\mathcal{E}p(\mathbf{k})$ is closed and convex, the target $C \subset K$ is closed, and $\mathcal{G}$ is a closed convex subset, then the viable-capture basin enjoys the *Lax–Hopf formula*

$$\mathrm{Capt}_{\mathcal{G}}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c})) = \mathcal{E}p(\mathbf{k}) \cap (\mathcal{E}p(\mathbf{c}) - \mathbb{R}_+\mathcal{G}).$$

Hence, the epigraph of the valuation function V, defined as the viable-capture basin $\mathrm{Capt}_{(50)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c}))$ under the differential inclusion (50), is equal to

$$(51) \qquad \left\{ \begin{array}{l} \mathrm{Capt}_{(50)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c})) = \mathrm{Capt}_{\mathcal{G}}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c})) \\ = \mathcal{E}p(\mathbf{k}) \cap \left(\mathcal{E}p(\mathbf{c}) - \bigcup_{\lambda \geq 0} \lambda(\{\varphi\} \times \mathcal{E}p(\ell))\right). \end{array} \right.$$

Therefore (d, x, y) belongs to the viable-capture basin if $\mathbf{k}(d, x) \leq y$ and if there exist $(\delta^\star, \xi^\star, \eta^\star) \in \mathcal{E}p(\mathbf{c})$, $t^\sharp > 0$ and $u \in \mathrm{Dom}(\ell)$ such that

$$\mathbf{c}(d - t^\sharp, x - t^\sharp u) = \mathbf{c}(\delta^\star, \xi^\star) \leq \eta \leq y - t^\sharp \ell(u).$$

This means that

$$\max\left(\mathbf{k}(d, x), \inf_{t^\sharp} \inf_{u} \left(\mathbf{c}(t - t^\sharp \varphi, x - t^\sharp u) + t^\sharp \ell(u)\right)\right) = V(d, x),$$

which is the Lax–Hopf formula we were looking for. $\square$

9. Barron–Jensen/Frankowska viscosity solution. This (technical) section is devoted to the proof that the viability solution defined through a viable-capture basin is actually the unique solution to the structured Hamilton–Jacobi equation. This is done by translating the Frankowska property, characterizing a viable-capture basin of a target at the level of differential inclusions, in terms of gradients (or subdifferential) of the viability solutions.

For simplicity of exposition, we begin with the assumption that the viability solution is differentiable, first in the classical case without viability constraint, which easier to formulate, and then under viability constraint, before dropping the differentiability assumption in the proof and proving that the viability solution is the Barron–Jensen/Frankowska viscosity solution (Theorem 9.3).

PROPOSITION 9.1 (Hamilton–Jacobi equation without viability constraints). *We assume, for simplicity, that the viability solution is differentiable. We posit the following assumptions:*

1. *The Lagrangian is Marchaud;*
2. φ, $(d, x, \upsilon) \mapsto \ell(d, x; \upsilon)$, *and the set-valued map* $(d, x) \rightsquigarrow F(d, x)$ *are Lipschitz.*

Then the viability solution is the unique function V solution to the Hamilton–Jacobi equation (11),

$$\text{if } V(d, x) < \mathbf{c}(d, x), \text{ then } \left\langle \frac{\partial V(d, x)}{\partial d}, \varphi(d) \right\rangle + \ell^\star\left(d, x; \frac{\partial V(d, x)}{\partial x}\right) = 0,$$

satisfying conditions $V(d, x) \leq \mathbf{c}(d, x)$.

The regulation map satisfies

$$\textit{if}\, V(d,x) < \mathbf{c}(d,x),\ \textit{then}\ \ R_V(d,x) \subset \partial_p \ell^\star \left(d,x;\frac{\partial V(d,x)}{\partial x}\right).$$

This is a consequence of the following theorem with state constraints.

THEOREM 9.2 (Hamilton–Jacobi equation with viability constraints). *We assume, for simplicity, that the viability solution is differentiable. We posit the following assumptions:*

1. *The Lagrangian is Marchaud;*
2. φ, $(d,x,\upsilon) \mapsto \ell(d,x;\upsilon)$, *and the set-valued map* $(d,x) \rightsquigarrow F(d,x)$ *are Lipschitz;*
3. *the viability constraint function* $\mathbf{k}$ *is differentiable and satisfies*

$$\left\langle \frac{\partial \mathbf{k}(d,x)}{\partial d}, \varphi(d)\right\rangle + \ell^\star\left(d,x;\frac{\partial \mathbf{k}(d,x)}{\partial x}\right) \le 0.$$

Then the viability solution is the unique function V *satisfying conditions* (12)*,*

$$\mathbf{k}(d,x) \le V(d,x) \le \mathbf{c}(d,x),$$

and is the solution to the Hamilton–Jacobi equation (11) *in the sense that*

$$\text{(52)}\qquad \begin{cases} \text{(i)} & \textit{if}\,\mathbf{k}(d,x) < V(d,x) < \mathbf{c}(d,x),\ \textit{then} \\ & \left\langle \dfrac{\partial V(d,x)}{\partial d}, \varphi(d)\right\rangle + \ell^\star\left(d,x;\dfrac{\partial V(d,x)}{\partial x}\right) = 0, \\ \text{(ii)} & \textit{if}\,\mathbf{k}(d,x) = V(d,x) \le \mathbf{c}(d,x),\ \textit{then} \\ & \left\langle \dfrac{\partial V(d,x)}{\partial d}, \varphi(d)\right\rangle + \ell^\star\left(d,x;\dfrac{\partial V(d,x)}{\partial x}\right) \le 0. \end{cases}$$

When $\mathbf{k}(d,x) < V(d,x)$*, the regulation map satisfies*

$$\text{(53)}\qquad \textit{if}\,\mathbf{k}(d,x) < V(d,x) < \mathbf{c}(d,x),\ \textit{then}\ \ R_V(d,x) \subset \partial_p \ell^\star \left(d,x;\frac{\partial V(d,x)}{\partial x}\right).$$

This formula can be written

$$\text{(54)}\qquad \textit{if}\ \ \mathbf{k}(d,x) < V(d,x) < \mathbf{c}(d,x),\ \textit{then}\ \ \frac{\partial V(d,x)}{\partial x} \in \bigcap_{u \in R_V(d,x)} \partial_u \ell\,(d,x;u)$$

and regarded as an extension of the Lax–Oleinik formula *to this general case.*

Proof. Actually, we shall derive the equality in formula (52) from two inequalities; the first is valid when the Lagrangian is Marchaud, and the second is valid under the Lipschitz conditions.

1. Since the Lagrangian is Marchaud, so is the set-valued map $\mathcal{F}$ defined by (23), and by (25), the viable-capture basin $\mathcal{E}p(V) := \mathrm{Capt}_{(13)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c}))$ is equal to $\mathrm{Capt}_{(24)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c})) = \mathrm{Capt}_{(13)}(\mathcal{E}p(\mathbf{k}), \mathcal{E}p(\mathbf{c}))$.
   It is then the largest closed subset between $\mathcal{E}p(\mathbf{c})$ and $\mathcal{E}p(\mathbf{k})$, locally viable in $\mathcal{E}p(V) \setminus \mathcal{E}p(\mathbf{c})$ thanks to viability Theorem 11.9, which also states that, whenever $(d,x,V(d,x)) \in \mathcal{E}p(V) \setminus \mathcal{E}p(\mathbf{c})$, i.e., whenever $V(d,x) < \mathbf{c}(d,x)$, there exists some $\upsilon \in \mathrm{Dom}(\ell)$ such that

$$(-\varphi(d), -\upsilon, -\ell(d,x;\upsilon)) \in T_{\mathcal{E}p(V)}(d,x,V(d,x)).$$

Recall that $N_{\mathcal{E}p(V)}(d,x,V(d,x)) := T^{\star}_{\mathcal{E}p(V)}(d,x,V(d,x))$ and that *the subdifferential* $\partial V(d,x)$ *is the set of* (p_d,p_x) *such that* $(p_d,p_x,-1) \in N_{\mathcal{E}p(V)}(d,x,V(d,x))$.

Therefore, we infer that

(55)
$$\forall\, \upsilon \in R_V(d,x),\ \forall\,(p_d,p_x) \in \partial V(d,x),\ \ 0 \le \langle p_d,\varphi(d)\rangle + \langle p_x,\upsilon\rangle - \ell(d,x;\upsilon)$$

and thus that, by taking the supremum over $F(d,x) := \mathrm{Dom}(\ell(d,x;\cdot))$,

$$\forall\,(p_d,p_x) \in \partial V(d,x),\ \ 0 \le \langle p_d,\varphi(d)\rangle + \ell^{\star}(d,x;p_x)\,. \tag{56}$$

2. Since $(d,x,\upsilon) \mapsto \ell(d,x;\upsilon)$ and the set-valued map $(d,x) \rightsquigarrow F(d,x)$ are Lipschitz, it is easy to observe that the set-valued map $(d,x) \rightsquigarrow \mathcal{E}p(\ell(d,x;\cdot))$ is Lipschitz. The viable-capture basin $\mathcal{E}p(V) := \mathrm{Capt}_{(13)}(\mathcal{E}p(\mathbf{k}),\mathcal{E}p(\mathbf{c}))$ is the smallest closed subset between $\mathcal{E}p(\mathbf{c})$ and $\mathcal{E}p(\mathbf{k})$ backward invariant relative to $\mathcal{E}p(\mathbf{k})$.
If we assume that $\mathcal{E}p(\mathbf{k})$ is itself backward invariant, then $\mathcal{E}p(V)$ is also backward invariant. It remains to translate properties of the invariance Theorem 11.10 in terms of subdifferentials.
Backward invariance of the epigraph $\mathcal{E}p(V)$ means that whenever $(d,x,V(d,x)) \in \mathcal{E}p(V)$, then

$$\forall\, \upsilon \in F(d,x),\ \forall\,(p_d,p_x) \in \partial V(d,x),$$
$$(\varphi(d),\upsilon,\ell(d,x;\upsilon)) \in T_{\mathcal{E}p(V)}(d,x,V(d,x)).$$

This amounts to saying that

$$\forall\, \upsilon \in F(d,x),\ \forall\,(p_d,p_x) \in \partial V(d,x),\ \ \langle p_d,\varphi(d)\rangle +\ \langle p_x,u\rangle - \ell(d,x;u) \le 0$$

and thus, that

$$\forall\,(p_d,p_x) \in \partial V(d,x),\ \ \langle p_d,\varphi(d)\rangle + \ell^{\star}(d,x;p_x) \le 0. \tag{57}$$

In the same way, the epigraph of $\mathbf{k}$ is backward invariant if

$$\forall\,(q_d,q_x) \in \partial\mathbf{k}(d,x),\ \ \langle q_d,\varphi(d)\rangle + \ell^{\star}(d,x;q_x) \le 0.$$

If $\mathbf{k}(d,x) < V(d,x) < \mathbf{c}(d,x)$, inequalities (56) and (57) imply that the viability solution satisfies

$$\forall\,(p_d,p_x) \in \partial V(d,x),\ \ \langle p_d,\varphi(d)\rangle + \ell^{\star}(d,x;p_x) = 0. \tag{58}$$

Taking into account that $\langle p_d,\varphi(d)\rangle = -\ell^{\star}(d,x;p_x)$ in inequality (55), we infer that

$$\forall\, \upsilon \in R_V(d,x),\ \forall\,(p_d,p_x) \in \partial V(d,x),\ \ 0 \le \langle p_x,\upsilon\rangle - \ell^{\star}(d,x;p_x) - \ell(d,x;\upsilon).$$

The Legendre property (9) of the Fenchel transform implies that this is equivalent to saying that $\upsilon \in \partial_p\ell^{\star}(d,x;p_x)$ or that $p_x \in \partial\ell_u(d,x;u)$. Consequently,

$$R_V(d,x) \subset \bigcap_{(p_d,p_x)\in\partial V(d,x)} \partial_p\ell^{\star}(d,x;p_x)$$

or, equivalently,

$$\bigcup_{(p_d,p_x)\in\partial V(d,x)} p_x \subset \bigcap_{u\in R_V(d,x)} \partial_u\ell(d,x;u).$$

Assuming that V is differentiable, we have proved formulas (52) and (53) and, consequently, that the viability solution is the unique solution to the structured Hamilton–Jacobi equation. □

Under the assumptions of Theorem 9.2, when the viability solution V is not differentiable, it is still the unique solution satisfying (58):

$$\forall\, (p_d, p_x) \in \partial V(d,x), \quad \langle p_d, \varphi(d)\rangle + \ell^\star(d,x;p_x) = 0.$$

THEOREM 9.3 (Barron–Jensen/Frankowska viscosity solution). *We posit the following assumptions:*

1. *The Lagrangian is Marchaud;*
2. *φ, $(d,x,\upsilon) \mapsto \ell(d,x;\upsilon)$, and the set-valued map $(d,x) \rightsquigarrow F(d,x)$ are Lipschitz;*
3. *the viability constraint function $\mathbf{k}$ satisfies*

$$\forall\, (q_d, q_x) \in \partial \mathbf{k}(d,x), \quad \langle q_d, \varphi(d)\rangle + \ell^\star(d,x;q_x) \le 0. \tag{59}$$

Then the viability solution is the unique function V satisfying conditions (12),

$$\mathbf{k}(d,x) \le V(d,x) \le \mathbf{c}(d,x),$$

and that is the solution to the Hamilton–Jacobi equation (11) *in the sense that*

$$\begin{cases} \text{(i)} & \textit{if } \mathbf{k}(d,x) < V(d,x) < \mathbf{c}(d,x), \textit{ then} \\ & \forall\, (p_d,p_x) \in \partial V(d,x), \quad \langle p_d, \varphi(d)\rangle + \ell^\star(d,x;p_x) = 0, \\ \text{(ii)} & \textit{if } \mathbf{k}(d,x) = V(d,x) \le \mathbf{c}(d,x), \textit{ then} \\ & \forall\, (p_d,p_x) \in \partial V(d,x), \quad \langle p_d, \varphi(d)\rangle + \ell^\star(d,x;p_x) \le 0, \end{cases} \tag{60}$$

which is the very definition of a Barron–Jensen/Frankowska viscosity solution for lower semicontinuous functions.

Furthermore, when $\mathbf{k}(d,x) < V(d,x)$, the regulation map and the subdifferential of the viability solution with respect to x are related by

$$\begin{cases} R_V(d,x) \subset \bigcap_{(p_d,p_x)\in\partial V(d,x)} \partial_p \ell^\star(d,x;p_x) \\ \textit{or, equivalently,} \\ \bigcup_{(p_d,p_x)\in\partial V(d,x)} \{p_x\} \subset \bigcap_{u\in R_V(d,x)} \partial_u \ell(d,x;u). \end{cases} \tag{61}$$

Remark. The theorem holds true by dropping assumption (59), but the conclusion translating the property that the epigraph of the viability solution is backward invariant relatively to the epigraph of $\mathbf{k}$ when $V(d,x) = \mathbf{k}(d,x)$ is quite technical and ugly:

(62)

$$\begin{cases} \text{If } \mathbf{k}(d,x) = V(d,x) \le \mathbf{c}(d,x), \text{ then} \\ \forall\, (p_d,p_x) \in \partial V(d,x), \ \inf_{(q_d,q_x)\in\partial\mathbf{k}(d,x)} (\langle p_d - q_d, \varphi(d)\rangle + \ell^\star(d,x;p_x - q_x)) \le 0. \end{cases}$$

Remark. Barron–Jensen/Frankowska solutions (see [18] and [26, 27]) are modifications of the concept of viscosity solutions for lower semicontinuous functions. The second viscosity inequality of an original viscosity solution in the sense of Crandall–Lions (see [21]) corresponds to normal conditions on the closure of the hypograph: to be equal to its hypograph, the solution must be continuous. Instead of using the forward invariance of the hypograph, Frankowska used the backward invariance of the epigraph, which requires only that the epigraph be closed. Barron–Jensen discovered independently the same property by partial differential equation techniques.

10. Notes on convex analysis. Let $\beta : u \mapsto \beta(u)$ be a lower semicontinuous convex function defined everywhere ($\mathrm{Dom}(\beta) = X$). We denote by ψ_F the *indicator* of F defined by $\psi_F(u) = 0$ if $u \in F$ and equal to $+\infty$ outside of F.

Remark (restriction of a function). The *restriction* $\ell(\cdot)$ of $\beta(\cdot)$ to F can be written in the form

$$\ell(u) := \beta(u) + \psi_F(u).$$

Denoting by $\sigma_F(p) := \sup_{u \in F} \langle p, u\rangle$ the support function of F, we can associate with any p at least one $\overline{q}$ such that

$$\ell^\star(p) := \inf_q (\beta^\star(q) + \sigma_F(p-q)) = \beta^\star(\overline{q}) + \sigma_F(p - \overline{q}).$$

- If $\beta(u) := 0$, then
$$\ell^\star(p) := \sigma_F(p).$$
- If $\beta(u) := \langle \pi, u\rangle + \gamma$, then $\beta^\star(\pi) = -\gamma$ and for $p \neq \pi$, $\beta^\star(\pi) = +\infty$. Therefore
$$\ell^\star(p) := \sigma_F(p - \pi) - \gamma.$$
- If $\beta(u) := t^\sharp \|u\| + \gamma$, then $\beta^\star(p) = \psi_{t^\sharp B_\star} - \gamma$. Therefore
$$\ell^\star(p) := \inf_{\|q\|_\star \le t^\sharp} \sigma_F(p-q) - \gamma. \qquad \square$$

Recall that two lower semicontinuous convex functions satisfy $\ell_1 \le \ell_2$ if and only if $\ell_2^\star \le \ell_1^\star$. We deduce from the above examples the following dual characterization of estimates of ℓ and $\ell^\star$.

LEMMA 10.1 (dual estimates). *Let ℓ be a lower semicontinuous convex function. We introduce two subsets $F \subset G$ and two finite lower semicontinuous convex functions $\alpha \le \beta$. The following conditions are equivalent:*

$$\forall\, p, \;\; \inf_q (\beta^\star(q) + \sigma_F(p-q)) \le \ell^\star(p) \le \inf_q (\alpha^\star(q) + \sigma_G(p-q)) \tag{63}$$

and

$$\begin{cases} \overline{co}(F) \subset \mathrm{Dom}(\ell) \subset \overline{co}(G), \\ \forall\, u \in \overline{co}(F), \;\; \ell(u) \le \beta(u), \\ \forall\, u \in \overline{co}(G), \;\; \alpha(u) \le \ell(u). \end{cases} \tag{64}$$

In particular, the two following conditions are equivalent:

$$\forall\, p, \;\; \sigma_{\mathrm{Dom}(\ell)}(p) - c \le \ell^\star(p) \le c\|p\|_\star + \alpha \tag{65}$$

and

$$\begin{cases} \mathrm{Dom}(\ell) \subset cB \textit{ and is closed}, \\ \forall\, u \in \mathrm{Dom}(\ell), \;\; -\alpha \le \ell(u) \le c. \end{cases} \tag{66}$$

Proof.

1. To say that for all $p, \inf_q (\beta^\star(q) + \sigma_F(p-q)) \le \ell^\star(p)$ amounts to saying that $\ell(u) \le \psi_{\overline{\mathrm{co}}(F)}(u) + \beta(u)$. This means that for all $u \in \overline{\mathrm{co}}(F)$, $\ell(u) \le \beta(u)$, and thus, since β is assumed to be finite, that $\overline{\mathrm{co}}(F) \subset \mathrm{Dom}(\ell)$.
2. To say that for all $p, \ell^\star(p) \le \inf_q (\alpha^\star(q) + \sigma_G(p-q))$ amounts to saying that $\psi_{\overline{\mathrm{co}}(G)}(u) + \alpha(u) \le \ell(u)$. This means that for all $u \in \mathrm{Dom}(\ell)$, $\alpha(u) \le \ell(u)$, and thus, since α is assumed to be finite, that $\mathrm{Dom}(\ell) \subset \overline{\mathrm{co}}(G)$.

The second statement is deduced by taking $F := \mathrm{Dom}(\ell)$, $G := cB$, $\beta(\cdot) \equiv c$, and $\alpha(\cdot) \equiv \alpha$. $\square$

Closed image theorem. Let us consider the *barrier cone* $K^\nabla := \{p \in X^\star \mid \sigma_K(p) < +\infty\} = \mathrm{Dom}(\sigma_K)$ and the *polar cone* $K^\star := \{p \in X^\star \mid \sigma_K(p) \le 0\}$ of a subset K.

THEOREM 10.2 (closed image theorem). *Let $A \in \mathcal{L}(X_1, X_2)$. If $K \subset X_1$ is closed, condition*

$$\mathrm{Im}(A^\star) + K^\nabla = X_1^\star \tag{67}$$

implies that $A(K)$ is closed and that the subsets $K \cap A^{-1}(x_2)$ are compact.

Proof. Assume that $Ax_n := y_n$ converges to y. Since any $p \in X_1^\star = A^\star q + r$, where $q \in X_2^\star$ and $r \in K^\nabla = \mathrm{Dom}(\sigma_K)$, we infer that

$$\begin{cases} \langle p, x_n\rangle \le \langle q, Ax_n\rangle + \langle r, x_n\rangle \\ \le \langle q, y_n\rangle + \sigma_K(r) < +\infty. \end{cases}$$

Hence x_n is bounded, and thus relatively compact, and converges to some x, so that $A(K)$ is closed. ▯

11. A viability survival kit.

Here we summarize material published in [9, 6, 7] and theorems to appear in [11] used in this study. This section presents a few selected statements that are most often used, but here are restricted to viable-capture basins. Three categories of statements are presented. The first provides characterizations of viable-capture basins as bilateral fixed points, which are simple and important and are valid without any assumption. The second provides characterizations in terms of local viability properties and backward invariance, involving topological assumptions on the evolutionary systems. The third characterizes viability kernels and viable-capture basins under differential inclusions in terms of tangential conditions, which furnishes the regulation map allowing us to pilot viable evolutions (and optimal evolutions in the case of optimal control problems).

We denote by $\mathcal{S}(x)$ the subset of solutions to differential inclusion $x'(t) \in F(x(t))$ starting at x. The set-valued map $x \rightsquigarrow \mathcal{S}(x)$ is called the *evolutionary system* generated by the differential inclusion.

DEFINITION 11.1 (Marchaud set-valued maps). *We say that F is a* Marchaud map *if*

$$\begin{cases} \text{(i)} & \text{the graph and the domain of } F \text{ are nonempty and closed;} \\ \text{(ii)} & \text{the values } F(x) \text{ of } F \text{ are convex;} \\ \text{(iii)} & \exists\, c > 0 \text{ such that } \forall x \in X,\ \|F(x)\| := \sup_{v \in F(x)} \|v\| \le c(\|x\| + 1). \end{cases} \tag{68}$$

11.1. Bilateral fixed point characterization.

We consider the maps $(K, C) \mapsto \mathrm{Capt}(K, C)$. The properties of these maps provide fixed point characterizations of viability kernels of the maps $K \mapsto \mathrm{Capt}(K, C)$ and $C \mapsto \mathrm{Capt}(K, C)$.

THEOREM 11.2 (the fundamental characterization of capture basins). *Let $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0, \infty; X)$ be an evolutionary system, $K \subset X$ be an environment, and $C \subset K$ be a nonempty target. The viable-capture basin $\mathrm{Capt}_{\mathcal{S}}(K, C)$ of C viable in K (see Definition 2.2) is the* unique *subset D between C and K that is both*

1. viable outside C *(and is the largest subset $D \subset K$ viable outside C), and*
2. *satisfies* $\mathrm{Capt}_{\mathcal{S}}(K, C) = \mathrm{Capt}_{\mathcal{S}}(K, \mathrm{Capt}_{\mathcal{S}}(K, C))$ *(and is the smallest subset $D \supset C$ to do so),*

i.e., the bilateral fixed point

$$\mathrm{Capt}_{\mathcal{S}}(\mathrm{Capt}_{\mathcal{S}}(K, C), C) = \mathrm{Capt}_{\mathcal{S}}(K, C) = \mathrm{Capt}_{\mathcal{S}}(K, \mathrm{Capt}_{\mathcal{S}}(K, C)). \tag{69}$$

11.2. Viability characterization. It happens that isolated subsets are, under adequate assumptions, backward invariant. Characterizing viability kernels and viable-capture basins in terms of forward viability and backward invariance allows us to use the results on viability and invariance.

DEFINITION 11.3 (local viability and backward relative invariance). *A subset K is said to be* locally viable *under $\mathcal{S}$ if from any initial state $x \in K$* there exists at least one evolution $x(\cdot) \in \mathcal{S}(x)$ *and a strictly positive $T_{x(\cdot)} > 0$ such that $x(\cdot)$ is viable in K on the nonempty interval $[0, T_{x(\cdot)}[$. It is a* repeller *under F if all solutions starting from K leave K in finite time.*

A subset D is locally backward invariant relatively to K *if all backward solutions starting from D viable in K are actually viable in K.*

If K is itself (backward) invariant, any subset (backward) invariant relative to K is (backward) invariant. If $C \subset K$ is (backward) invariant relative to K, then $C \cap \mathrm{Int}(K)$ is (backward) invariant.

PROPOSITION 11.4 (capture basins of relatively invariant targets). *Let $C \subset D \subset K$ be three subsets of X.*

1. *If D is backward invariant relative to K, then* $\mathrm{Capt}_{\mathcal{S}}(K, C) = \mathrm{Capt}_{\mathcal{S}}(D, C)$.
2. *If C is backward invariant relative to K, then* $\mathrm{Capt}_{\mathcal{S}}(K, C) = C$.

Using the concept of backward invariance, we provide a further characterization of viable-capture basins as follows.

THEOREM 11.5 (characterization of capture basins). *Let us assume that F is Marchaud, that the environment $K \subset X$ and that the target $C \subset K$ are closed subsets such that $K \setminus C$ is a repeller ($\mathrm{Viab}_{\mathcal{S}}(K \setminus C) = \emptyset$).*

Then the viable capture basin $\mathrm{Capt}_{\mathcal{S}}(K, C)$ *is the* unique *closed subset D satisfying $C \subset D \subset K$ and*

$$\left\{ \begin{array}{ll} \text{(i)} & D \backslash C \text{ is } \textit{locally viable} \text{ under } \mathcal{S}, \\ \text{(ii)} & D \textit{ is relatively backward invariant with respect to } K \text{ under } \mathcal{S}. \end{array} \right. \tag{70}$$

Under adequate assumptions, the viable-capture basin is given by the Lax–Hopf formula as follows.

THEOREM 11.6 (Lax–Hopf formula for capture basins). *Assume that the target C is contained in the environment $K \subset X$, that K is a closed convex subset, C is closed, and G is a constant set-valued map with a closed convex image G. Then the viable-capture basin enjoys the* Lax–Hopf formula

$$\mathrm{Capt}_G(K, C) = K \cap (C - \mathbb{R}_+ G). \tag{71}$$

11.3. Tangent and normal cones. Let $x \in K \subset X$. The *tangent cone* to K at x is defined by

$$T_K(x) := \left\{ v \in X \mid \liminf_{h \mapsto 0+} \frac{d(x + hv; K)}{h} = 0 \right\}.$$

Denoting by

$$P^\star := \{ p \in X^\star \text{ such that } \forall\, v \in P,\ \ \langle p, v \rangle \leq 0 \}$$

the polar of P,

- the normal cone $N_K(x) := T_K^\star(x)$,
- $T_K^{\star\star}(x)$ is the closed convex hull of $T_K(x)$.

We recall the characterization of the tangent cone to the image by a linear operator as follows.

THEOREM 11.7. *Let $A \in \mathcal{L}(X,Y)$ be a linear operator, $K \subset X$, and $x_0 \in K$. Then*

$$AT_K(x) \subset T_{A(K)}(Ax).$$

If we assume that for some $x_0 \in K$, there exist constants $c > 0$ and $\alpha \in [0,1[$ such that, for any $x \in K \cap B(x_0, \eta)$,

$$\forall\, v \in Y,\ \ \exists\, u \in T_K(x) \cap \|u\|B \text{ such that } \|Au - v\| \leq \alpha\|u\|, \tag{72}$$

then the equality

$$AT_K(x) = T_{A(K)}(Ax)$$

holds true.

Proof. The first inclusion is obvious, and for proving the converse, we take $v \in T_{A(K)}(Ax)$. Then, there exist $h_n > 0$ converging to 0 and v_n converging to v such that $Ax + h_n v_n \in A(K)$. Then constrained inverse function Theorem 3.4.5 on page 96 of [14] states that assumption (1) on page 6 implies that there exist constants $\gamma > 0$ and $\lambda > 0$ such that, for any $(x, Ax) \in K \cap B((x_0, Ax_0), \gamma)$,

$$\exists\, \xi \in K \text{ such that } A\xi = Ax + h_n v_n \text{ and } \|\xi - x\| \leq \lambda h_n \|v_n\|.$$

Setting $u_n := \frac{\xi - x}{h_n}$, we infer that $Au_n = v_n$ and that $\|u_n\| \leq \lambda\|v_n\| \leq \|v\| + 1$ for n large enough. Since the dimension of X is finite, this implies that there exists a subsequence (again denoted by) u_n converging to some u belonging to $T_K(x) \cap \lambda\|v\|B$ satisfying $Au = v$. ∎

Epiderivatives and subdifferentials of an extended function $\mathbf{v} : X \mapsto \mathbb{R} \cup \{+\infty\}$ are then defined as follows:

1. The *epiderivative* $D_{\uparrow}^{\star\star}\mathbf{v}(x)$ of $\mathbf{v}$ is defined as $\mathcal{E}p(D_{\uparrow}^{\star\star}\mathbf{v}(x)) := T^{\star\star}_{\mathcal{E}p(\mathbf{v})}(x, \mathbf{v}(x))$.
2. The *subdifferential* $\partial\mathbf{v}(x)$ of $\mathbf{v}$ at x is the set of p such that $(p, -1) \in N_{\mathcal{E}p(\mathbf{v})}(x, \mathbf{v}(x))$.

If $\mathbf{v}$ is differentiable, then

$$p = \frac{\partial \mathbf{v}(x)}{\partial x}.$$

11.4. The regulation map. These theorems, which are valid for any evolutionary systems, paved the way to go one step further when the evolutionary system is a differential inclusion.

We shall use the closed convex hull $T_K^{\star\star}(x)$ of the tangent cone.

Not only the viability theorem provides characterizations of viability kernels and viable-capture basins, but also the *regulation map* $R_D \subset F$ which governs viable evolutions.

DEFINITION 11.8 (regulation map). *Let us consider three subsets $C \subset D \subset K$ (where the target C may be empty) and a set-valued map $F : X \rightsquigarrow X$.*

The set-valued map $R_D : x \in D \rightsquigarrow F(x) \cap T_D^{\star\star}(x) \subset X$ is called the regulation map of F on $D \setminus C$ if

$$\forall\, x \in D \setminus C,\ \ R_D(x) := F(x) \cap T_D^{\star\star}(x) \neq \emptyset. \tag{73}$$

The viability theorem implies the following.

THEOREM 11.9 (tangential characterization of capture basins). *Let us assume that F is Marchaud and that the environment $K \subset X$ and the target $C \subset K$ are closed subsets such that $K \backslash C$ is a repeller ($\mathrm{Viab}_F(K\backslash C) = \emptyset$). Then the viable-capture basin* $\mathrm{Capt}_{\mathcal{S}}(K, C)$ *is the* largest *closed subset D satisfying $C \subset D \subset K$ and*

$$\forall x \in D\backslash C, \;\; F(x) \cap T_D^{\star\star}(x) \neq \emptyset.$$

Furthermore, for every $x \in D$, there exists at least one evolution $x(\cdot) \in \mathcal{S}(x)$ viable in D until it reaches the target C, and all evolutions $x(\cdot) \in \mathcal{S}(x)$ viable in D until they reach the target C are governed by the differential inclusion

$$x'(t) \in R_D(x(t)).$$

11.5. Frankowska characterizations of the regulation map. These fundamental theorems characterizing viability kernels and viable-capture basins justify further study of the regulation map and equivalent ways to characterize it. Actually, using the invariance theorem, we can go one step further and characterize viability kernels and viable-capture basins in terms of the *Frankowska property*, stated in two equivalent forms: the *tangential formulation*, expressed in terms of tangent cones, and its *dual version*, expressed in terms of normal cones.

11.5.1. Tangential Frankowska characterization of the regulation map. We begin with the case of tangential characterization as follows.

THEOREM 11.10 (tangential characterization of capture basins). *Let us assume that F is Marchaud and Lipschitz and that the environment $K \subset X$ and the target $C \subset K$ are closed subsets such that $K \setminus C$ is a repeller ($\mathrm{Viab}_F(K\backslash C) = \emptyset$). Then the viable-capture basin* $\mathrm{Capt}_{\mathcal{S}}(K, C)$ *is the* unique *closed subset D satisfying $C \subset D \subset K$ and the* Frankowska property*:*

$$(74) \qquad \begin{cases} \text{(i)} & \forall x \in D\backslash C, \;\; F(x) \cap T_D^{\star\star}(x) \neq \emptyset; \\ \text{(ii)} & \forall\, x \in \overset{\circ}{K} \cap D, \;\; -F(x) \subset T_D^{\star\star}(x); \\ \text{(iii)} & \forall x \in \partial K \cap D, \;\; -F(x) \cap T_K(x) \subset T_D^{\star\star}(x). \end{cases}$$

11.5.2. Dual Frankowska characterization of the regulation map. The dual formulation of the Frankowska property involves duality between the finite dimensional vector space X, its dual $X^\star := \mathcal{L}(X, \mathbb{R})$, and its duality pairing $\langle p, x\rangle := p(x)$ on $X^\star \times X$.

DEFINITION 11.11 (Hamiltonian of a differential inclusion). *We associate with the right-hand side F the* Hamiltonian $H : X \times X^\star \mapsto \mathbb{R} \cup \{+\infty\}$ *defined by*

$$(75) \qquad \forall\, x \in X, \; \forall\, p \in X^\star, \;\; H(x, p) = \inf_{v \in F(x)} \langle p, v\rangle\,.$$

The constrained Hamiltonian $H_K : \partial K \times X^\star \mapsto \mathbb{R} \cup \{+\infty\}$ *on K is defined by*

$$(76) \qquad \forall\, x \in K, \; \forall\, p \in X^\star, \;\; H_K(x, p) = \inf_{v \in F(x) \cap -T_K(x)} \langle p, v\rangle\,.$$

The function $p \mapsto H(x, p)$ is concave, positively homogeneous, and upper semicontinuous as the infimum of continuous affine functions.

The dual versions of the tangential conditions characterizing viability kernels and the viable-capture basin involve the Hamiltonian of F and "replace" tangent cones by "normal cones": the normal cone

$$N_K(x) := T_K(x)^\star := \{p \in X^\star \text{ such that } \forall\, v \in T_K(x), \langle p, v\rangle \leq 0\}$$

to K at x is defined as the polar cone to the tangent cone. Recall that the polar of the normal cone to K at x is equal to the closed convex hull $T_K^{\star\star}(x)$ thanks to the separation theorem.

THEOREM 11.12 (dual characterization of the regulation map). *Assume that the images $F(x)$ of a set-valued map F are compact, convex, and not empty on a subset D. Then*

$$\forall\, x \in D,\ \ R_D(x) = \{v \in F(x) \text{ such that } \forall\, p \in N_D(x),\ \ \langle p, v\rangle \leq 0\}.$$

If we assume furthermore that

$$\forall\, x \in D,\ \forall\, p \in N_D(x,p),\ \ H(x,p) \geq 0,$$

then

$$\forall\, x \in D,\ \ R_D(x) = \{v \in F(x) \text{ such that } \forall\, p \in N_D(x),\ \ \langle p, v\rangle = 0\}.$$

The "dual" version of the tangential characterization of viability kernels is stated in the following terms.

THEOREM 11.13 (dual characterization of capture basins). *Let us assume that F is Marchaud and Lipschitz, and that the environment $K \subset X$ and the target $C \subset K$ are closed subsets such that $K \setminus C$ is a repeller ($\mathrm{Viab}_{\mathcal{S}}(K \setminus C) = \emptyset$).*

Then the viable-capture basin $\mathrm{Capt}_{\mathcal{S}}(K, C)$ *is the* unique *closed subset satisfying $C \subset D \subset K$ and the* dual Frankowska property (77)*:*

$$\text{(77)}\qquad \left\{\begin{array}{ll} \text{(i)} & \forall x \in D \cap (\overset{\circ}{K} \setminus C),\ \forall\, p \in N_D(x),\ H(x,p) = 0;\\ \text{(ii)} & \forall x \in D \cap (\partial K \setminus C),\ \forall\, p \in N_D(x),\ H(x,p) \leq 0 \leq H_K(x,p);\\ \text{(iii)} & \forall x \in D \cap \partial K,\ \forall\, p \in N_D(x),\ 0 \leq H_K(x,p). \end{array}\right.$$

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