H_∞ Control for Discrete-Time Nonlinear Stochastic Systems

Nadav Berman and Uri Shaked

***Abstract*—In this note, we develop an H_∞-type theory for a large class of discrete-time nonlinear stochastic systems. In particular, we establish a bounded real lemma (BRL) for this case. We introduce the notion of stochastic dissipative systems, analogously to the familiar notion of dissipation associated with deterministic systems, and utilize it in the derivation of the BRL. In particular, this BRL establishes a necessary and sufficient condition, in terms of a certain Hamilton Jacobi inequality (HJI), for a discrete-time nonlinear stochastic system to have l_2 − gain ≤ γ. The time-invariant case is also considered as a special case. In this case, the BRL guarantees necessary and sufficient conditions for the system to have l_2 − gain ≤ γ, by means of a solution to a certain algebraic HJI. An application of this theory to a special class of systems which is a characteristic of numerous physical systems, yields a more tractable HJI which serves as a sufficient condition for the underlying system to possess l_2 − gain ≤ γ. Stability in both the mean square sense and in probability, is also discussed. Systems that possess a special structure (norm-bounded) of uncertainties in their model are considered. Application of the BRL to this class of systems yields a linear state-feedback stabilizing controller which achieves l_2 − gain ≤ γ, by means of certain linear matrix inequalities (LMIs).**

***Index Terms*—Discrete-time H_∞ control, dissipative systems, nonlinear stochastic systems, stochastic BRL.**

I. Introduction

In recent years, there has been a growing interest, as is reflected in the various published research works, in the extension of H_∞ control and estimation theory to accommodate stochastic systems (see e.g., [2]–[5] and [7]–[9]). Most of the research that has been done this area deals with linear systems. The works of [12] and [13] investigate the problem of H_∞ control for continuous-time nonlinear stochastic systems. While [12] has taken the differential games approach to achieve an H_∞ output control, [13] and [17] present an H_∞ control theory for continuous-time systems based on the concept of stochastic dissipation. The work of Ugrinovskii [14] is also related to our work as it deals with filtering of uncertain linear stochastic systems where the uncertainty is characterized in terms of probability distribution.

The main thrust for these efforts stems from the attempt to model system uncertainties as a stochastic process, in particular, as a white noise, or formally as a Wiener process. This has led to the development of an H_∞ control and estimation theory for stochastic linear systems with multiplicative noise.

H_∞ control for discrete-time deterministic nonlinear systems has been considered by numerous researchers, see e.g., [16], which utilize, in part, the dissipativity concept. In order to develop an analogous theory for the stochastic counterpart, we setup, in the present note, some theory of stochastic dissipativity. A related topic is the risk sensitive control problems (see, e.g., [15] and references therein) which, in general, deal with optimization of stochastic systems where the cost function involves an exponential.

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N. Berman is with the Department of Mechanical Engineering, Ben-Gurion University, Beer-Sheva 84105, Israel (e-mail: nadav@menix.bgu.ac.il).

U. Shaked is with the School of Electrical Engineering, Tel Aviv University, Tel Aviv 69978, Israel (e-mail: shaked@eng.tau.ac.il).

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This note may be viewed as an effort to extend the work presented in [13], in order to include discrete-time nonlinear stochastic systems. The contribution of this note is in that it provides means [the bounded real lemma (BRL)] for synthesizing a state-feedback H_∞ controller for a large class of nonlinear stochastic systems. In addition, the BRL facilitates, in a natural way, the utilization of the linear matrix inequality (LMI) techniques to achieve $l_2 -$ gain $\leq \gamma$ for a large class of uncertain nonlinear systems with norm-bounded uncertainties. Another contribution of this note is the introduction of what we call stochastic dissipation, which serves as a basis for the H_∞ control theory developed in the sequel.

In this note, we consider the following stochastic system:

$$x_{k+1} = f_k(x_k, v_k, u_k, \omega_k) \tag{1}$$

where $\{x_k\}_{k\geq 0}$ is a solution to (1), with an initial condition x_0, an exogenous disturbances $\{v_k\}_{k\geq 0}$, a control signal $\{u_k\}_{k\geq 0}$, and a white-noise sequence $\omega = \{\omega_k\}_{k\geq 0}$ defined on a probability space (Ω, F, P). In the sequel, $\omega = \{\omega_k\}_{k\geq 0}$ describes both exogenous random inputs and parameter uncertainty of the system. The following will be assumed to hold throughout this note.

1) Let $(\Omega, F, \{F_k\}_{k\geq 0}, P)$ be a filtered probability space where $\{F_k\}_{k\geq 0}$ is the family of sub σ-algebras of F generated by $\{\omega_k\}_{k\geq 0}$, where ω_k are assumed to be R^l-valued. In fact, each F_k is assumed to be the minimal σ-algebra generated by $\{\omega_i\}_{0\leq i\leq k-1}$ while F_0 is assumed to be some given sub σ-algebra of F, independent of F_i for all $i > 0$.
2) $\{u_k\}_{k\geq 0}$ and $\{v_k\}_{k\geq 0}$ are nonanticipative (that is u_k, v_k are independent of $\{F_i, i > k\}$), R^m and R^{m_1} valued, respectively, stochastic processes defined on $(\Omega, F, \{F_k\}_{k\geq 0}, P)$. The vectors x_k are R^n valued, and for each k, $f_k : R^n \times R^{m_1} \times R^m \times R^l \to R^n$ is assumed to be continuous on $R^n \times R^{m_1} \times R^m \times R^l$.
3) $\{v_k\}_{k\geq 0}$ satisfies $E\{\sum_{k=0}^N \|v_k\|^2\} < \infty \ \forall N \geq 0$, where E stands for the expectation operation.
4) x_0 is assumed to be F_0-measurable and to satisfy $E\{\|x_0\|^2\} < \infty$.
5) The following notation will be used in the sequel. Let X, Y be R^{n_1} and R^{n_2}-valued random variables defined on (Ω, F, P) and let $V : R^{n_1} \times R^{n_2} \to R$. Let P_y be the probability distribution of Y. Then the random variable $E_y\{V(X, Y)\}$ is defined by $E_y\{V(X, Y)\} \doteq \int_{R^{n_2}} V(X, y) dP_y(y)$.

In this note, we investigate the problem of stochastic H_∞ state-feedback control which is formulated as follows. Given a controlled output

$$z_k(x, u) = \begin{bmatrix} h_k(x) \\ u \end{bmatrix} \tag{2}$$

where $h_k : R^n \to R^r$, synthesize a controller $u_k = K_k(x_k)$ such that, for a given $\gamma > 0$, the following H_∞ criterion is satisfied:

$$E\left\{\sum_{i=j}^{k-1} \|z_i(x_i, u_i)\|^2\right\} \leq E\{\beta_j(x_j)\} + \sum_{i=j}^{k-1} \gamma^2 E\left\{\|v_i\|^2\right\} \tag{3}$$

for all $0 \leq j < k$, for all $\{v_k\}_{k\geq 0}$ such that $\sum_{i=0}^k E\{\|v_i\|^2\} < \infty$, where $\beta_j : R^n \to R^+$ are positive functions satisfying $E\{\beta_j(x)\} < \infty$ whenever x is an F-measurable random variable satisfying $E\{\|x\|^2\} < \infty$. A system that satisfies this property is said to be $l_2 -$ gain $\leq \gamma$. We also denote the family of all exogenous disturbances v which satisfy $\sum_{i=0}^N E\{\|v_i\|^2\} < \infty$, and such that

(1) possesses a solution $\{x_k\}_{k\geq 0}$ with $E\{\|x_k\|^2 < \infty\}$ for some $\{u_k\}_{k\geq 0}$, by $\mathcal{A}_u$.

In Section II, we introduce the notion of dissipative stochastic systems, which is a natural extension of the dissipation notion regarding deterministic systems, as it was introduced first by Willems [11]. Dissipation is then utilized in establishing the BRL for general nonlinear time-varying stochastic systems. We also consider the time-invariant case where this yields a necessary and sufficient condition for the underlying system to possess $l_2 -$ gain $\leq \gamma$ in terms of a certain HJI. Stability in the mean square sense and the probability sense is established under certain conditions.

In Section III, we consider a special case (a characteristic of a large class of mechanical systems, in particular of robotic systems) where we establish sufficient conditions, in terms of a HJI with quadratic storage function, for a system to possess $l_2 -$ gain $\leq \gamma$.

In Section IV, we use the theory in the design of a linear controller for a stochastic nonlinear system with norm-bounded uncertainties. Exploiting the underlying system's norm-bounded uncertainties feature, we establish sufficient conditions under which the satisfaction of the HJI is guaranteed by the existence of certain LMIs. In fact, as the l_2-gain property is implied by the satisfaction of HJI (see (16) in the sequel), we establish a bound for this HJI by means of some LMIs which guarantee the l_2-gain property. Utilizing these LMIs and the BRL that have been introduced earlier, we synthesize a linear controller which renders $l_2 -$ gain $\leq \gamma$. An example of a single degree of freedom inverted pendulum with multiplicative white noise is given in Section V.

II. Dissipative Systems and the BRL for Discrete-Time Nonlinear Stochastic Systems

In this section, we establish what we call the BRL for discrete-time stochastic nonlinear systems, on the grounds of a theory of dissipative stochastic systems, which we develop in the sequel. Thus, we first define, in analogy to the continuous-time case, the notion of stochastic dissipative system.

Definition 1: Consider the discrete-time stochastic system (1) defined on the discrete time interval $[0, \infty)$, with the associated controlled output (2) and let $S : R^{r+m} \times R^{m_1} \to R$ be a Borel measurable function on $R^{r+m} \times R^{m_1}$. Then, the system (1) is said to be dissipative with respect to S if, for some admissible control sequence $\{u_k\}_k$, there exists a family of functions $V_k : (R^n \times N^+) \to R$ with $V_k(x) \geq 0\ \forall x \in R^n, k \in N^+$ (N^+ denotes the positive integers), so that $V_k(0) = 0\ \forall k \in N^+$, $E\{V_0(x_0)\} < \infty$, and

$$E\{V_k(x_k)\} \leq E\{V_j(x_j)\} + E\left\{\sum_{i=j}^{k-1} S(z_i, v_i)\right\} \quad (4)$$

for all $k : k \geq j \geq 0$, with the convention: $\sum_{i=j}^{l-1} S(z_i, v_i) = 0$ whenever $l = j$, and for all nonanticipative $v = \{v_i\}_{i\geq 0}$ with $E\{\sum_{i=0}^{\infty} \|v_i\|^2\} < \infty$. We call S supply rate (see e.g., [11] for the deterministic case), and the family $V = \{V_k\}_k$ is said to be a storage function of the system.

We have now the following trivial result which connects the l_2-gain property to the notion of stochastic dissipation.

Lemma 1: Consider the system (1) and let the supply rate be $S(z, v) = \gamma^2\|v\|^2 - \|z\|^2$. Assume the system possesses a storage function V so that the system is dissipative with respect to this S. Then, the system has $l_2 -$ gain $\leq \gamma$.

In analogy to the continuous-time case (see [13] and [17]), we establish now conditions under which the system (1) possesses a storage function (that is, the system is dissipative). We introduce below a particular positive function, which is shown to be a storage function, provided its expected value is finite.

Definition 2: Consider the system (1). Given an integer $j \geq 0$ and let x be an F_j-measurable, R^n valued random variable defined on (Ω, F, P) with $E\{\|x\|^2\} < \infty$. Let $S(z_i, v_i)$ be as in Definition 1. Define

$$V_j^a(x) = \sup_{\substack{l\geq j\\ v\in\mathcal{A}}} \left[-E\left\{\left[\sum_{i=j}^{l-1} S(z_i, v_i)\right] |x\right\}\right] \quad (5)$$

where $\{x_i\}_{i\geq j}$ is the solution to (1) with the initial condition $x_j = x$.

Remark 1: We note that in the deterministic setup V_j^a is called an available storage. Theorem 1, as stated below, characterizes the smallest storage function of a system. For the continuous time counterpart, see, e.g., [17].

Theorem 1: Given a Borel function $S(z, v)$, the function $V^a = \{V_i\}_{0\leq i\leq N}$, as defined by (5), is a storage function for the system (1) (or equivalently, the system (1) is dissipative with respect to the supply rate S) iff $E\{V_i^a(x)\}$ is finite for all $i \geq 0$ and for all F_i-measurable, R^n valued random variable x with $E\{\|x\|^2\} < \infty$. Moreover, if V_j is any other storage function for the system (1) which is associated with the same supply rate, then $V_j^a(x) \leq V_j(x)$ for all $j \geq 0$ and for all $x \in R^n$.

Proof: Assume first that $E\{V_j^a(x)\}$ is finite for all $j \geq 0$, and for all x which satisfies the hypothesis of the theorem. It is obvious that $V_j^a(x) \geq 0\ P - a.e.$ as $l = j$ yields the smallest candidate: $V_j^a(x) = 0$ P-a.e. Let now k be arbitrarily fixed as long as it satisfies $0 \leq j \leq k$. Then

$$\begin{aligned} E\left\{V_j^a(x_j)\right\} &= E\left\{\sup_{l\geq j, v\in\mathcal{A}}\left[-E\left\{\left[\sum_{j}^{l-1} S(z_i, v_i)\ |x_j\right\}\right]\right]\right\} \\ &\geq E\left\{-E\left\{\left[\sum_{i=j}^{l} S(z_i, v_i)\right] |x_j\right\}\right\} \end{aligned}$$

for all $l : l \geq j$ and for all $v \in \mathcal{A}$, which implies: $E\{V_j^a(x_j)\} \geq E\{-E\{[\sum_j^{k-1} S(z_i, v_i)]|x_j\}\} + E\{-E\{[\sum_k^{l-1} S(z_i, v_i)]|x_j\}\}\ \forall l > k,\ \forall v \in \mathcal{A}$.

Using now elementary properties of the conditional expectation, and the fact that the last inequality holds for all $v \in \mathcal{A}$, we obtain the following:

$$\begin{aligned} E\left\{V_j^a(x_j)\right\} \geq &- E\left\{\sum_{i=j}^{k-1} S(z_i, v_i)\right\} \\ &+ E\left[\sup_{\substack{l\geq k\\ v\in\mathcal{A}}} -E\left\{\left[\sum_{i=k}^{l-1} S(z_i, v_i)\right] |x_k\right\}\right] \\ = &- E\left\{\sum_{i=j}^{k-1} S(z_i, v_i)\right\} + E\left\{V_k^a(x_k)\right\} \\ &\forall k \geq j, \qquad v \in \mathcal{A}. \end{aligned}$$

This establishes the system's dissipativity.

Suppose now that the system is dissipative with respect to the supply rate S, that is, it possesses a storage function V which satisfies $0 \leq$

$E\{V_k(x_k)\} \leq E\{V_j(x_j)\} + E\{\sum_{i=j}^{k-1} S(z_i, v_i)\}$ for all $k > j \geq 0$. This implies

$$-E\left\{\sum_{i=j}^{k-1} S(z_i, v_i)\right\} \leq E\left\{V_j(x_j)\right\} < \infty \qquad \forall v \in \mathcal{A}. \quad (6)$$

Since $-E\{\sum_{i=j}^{k-1} S(z_i, v_i)\} = E\{-E\{[\sum_{i=j}^{k-1} S(z_i, v_i)]|x_j\}\}$ it follows that:

$$E\left\{V_j^a(x_j)\right\} = E\left\{\sup_{k \geq j, v \in \mathcal{A}} \left[-E\left\{\left[\sum_{i=j}^{k-1} S(z_i, v_i)\right] |x_j\right\}\right]\right\} \quad (7)$$

is finite for all $j : j \geq 0$. In view of (5) and by (6), it follows that for an arbitrary $x \in R^n$

$$V_j^a(x) \leq V_j(x) \qquad \forall j \geq 0.$$

This completes the proof. □

Having proved Theorem 1, we now state and prove what we call the BRL for stochastic nonlinear systems.

Theorem 2: Consider the system (1) with the controlled output (2) and with $u_k = 0, \forall k \geq 0$.

a) Let $\gamma > 0$, and let $\{V_k\}_{k \geq 0}$ be a family of positive real valued functions: $V_k : R^n \to R^+$ satisfying the following HJI:

$$V_k(x) \geq \sup_{v \in R^{m_1}} \left\{\|z_k\|^2 - \gamma^2 \|v\|^2 + E_{\omega_k}\left\{V_{k+1}\left[f_k(x, v, \omega_k)\right]\right\}\right\} \quad (8)$$

for all $x \in R^n$ and for all $v \in R^{m_1}$.
Assume also that $E\{V_i(x)\} < \infty$, $\forall i \geq 0$ and for all F-measurable x with $E\{\|x\|^2\} < \infty$. Then, the system (1) has $l_2 -$ gain $\leq \gamma$.

b) Assume the system (1) has $l_2 -$ gain $\leq \gamma$ for some $\gamma > 0$. Then, there is a family $\{V_k\}_{k \geq 0}$ of positive functions which satisfy the HJI of (8).

Proof:

a) Let $\{v_k\}_{k \geq 0}$ be any sequence of random variables in $\mathcal{A}$, and let x_0 satisfy the above hypothesis. Let $\{x_k\}_{k \geq 0}$ be the solution to (1). Note that ω_k is independent of $\{x_k, v_k,\}$ as ω_k is a white sequence, and $\{x_k, v_k,\}$ are F_k-measurable. This implies

$$E_{\omega_k}\{V_{k+1}[f_k(x_k, v_k, \omega_k)]\} = E\{V_{k+1}[f_k(x_k, v_k, \omega_k)] \,|x_k, v_k\}.$$

Therefore, the HJI yields

$$V_k(x_k) \geq \|z_k(x_k)\|^2 - \gamma^2 \|v_k\|^2 + E\left\{V_{k+1}\left[f_k(x_k, v_k, \omega_k)\right] |x_k, v_k\right\}$$

P-a.e. Applying now the expectation operation to both sides of the last inequality, we obtain

$$E\left\{V_k(x_k)\right\} \geq E\left\{\|z_k(x_k)\|^2\right\} - \gamma^2 E\left\{\|v_k\|^2\right\} + E\left\{V_{k+1}(x_{k+1})\right\}$$

which clearly implies dissipativity. Taking now $\beta_k = V_k$, this yields $l_2 -$ gain $\leq \gamma$.

b) Note that since the system has $l_2 -$ gain $\leq \gamma$, it follows:

$$\sum_{i=j}^{k-1} E\left\{\|z_i\|^2\right\} - \gamma^2 \sum_{i=j}^{k-1} E\left\{\|v_i\|^2\right\} < \infty$$

for all $0 \leq j \leq k$ and for all $\{v_i\}_{i \geq 0} \in \mathcal{A}$. Therefore, by Theorem 1, there is a family of positive functions $\{V_k\}_{k \geq 0}$ so that the system (1) is dissipative with respect to the supply rate $S(z, v) = \gamma^2 \|v\|^2 - \|z\|^2$. Fix $0 \leq k$, let $x \in R^n$, $v \in R^{m_1}$, and take $x_k = x$, $v_k = v$. Then, the dissipation implies

$$E_\omega\left\{V_{k+1}\left(f_k(x, v, \omega_k)\right)\right\} \leq E\left\{V_k(x)\right\} + \gamma^2 E\left\{\|v\|^2\right\} - E\left\{\|z_k\|^2\right\} \quad (9)$$

for all $x \in R^n$, and for all $v \in R^{m_1}$. This obviously implies the inequality (8). □

Remark 2: No boundary conditions have been imposed on $\{V_k\}_k$. In order to obtain a particular solution to (8) such boundary conditions should be specified. In the sequel, we focus on the stationary case where such conditions are irrelevant.

The question of stability is now in order. The particular definition of the l_2-gain property in this work is closely related to the system stability in various probability senses.

We first note that whenever the system (1) has $l_2 -$ gain $\leq \gamma$, and in the absence of exogenous disturbances, the following holds:

$$E\left\{\sum_{i=0}^{\infty} \|z_k\|^2\right\} \leq \gamma^2 \left[E\left\{\|x_0\|^2\right\}\right] \quad (10)$$

which implies $\lim_{k \to \infty} E\{\|z_k\|^2\} = 0$, from which it also follows that $\lim_{k \to \infty} E\{\|h_k(x_k)\|^2\} = 0$. This means that $h_i(x_i)$ converges to zero in the mean square sense and, hence, also in probability. Assume now that $h_i(x) = 0$ implies $x = 0$ for all $i \in N^+$ and for all $x \in R^n$. Since as $h_i(x_i)$ converges to zero in probability it follows that the sequence $\{x_i\}_{i \geq 0}$ converges to zero in probability, that is, the system is stable in the probability sense.

Obviously, in the case for which $\|h_i(x)\| \geq \alpha_i \|x\|$, the system is also stable in the mean square sense.

As an application of the BRL (Theorem 2), we consider now the time-invariant case

$$x_{k+1} = f(x_k, v_k, u_k, \omega_k) \quad (11)$$

with the controlled output (2), and with a time invariant h. As the underlying system is now time invariant, the l_2-gain property, as defined previously, is now equivalent to

$$E\left\{\sum_{i=0}^{k} \|z_i\|^2\right\} \leq \gamma^2 E\left\{\|x_0\|^2\right\} + E\left\{\sum_{i=0}^{k} \left[\gamma^2 \|v_i\|^2 - \|z_i\|^2\right]\right\} \qquad \forall k \geq 0.$$

We will assume in the sequel that $\sum_{i=1}^{\infty} E\{\|v_i\|^2\} < \infty$. For the time invariant case, Theorem 1 is valid when the available storage function is taken to be time invariant as follows:

$$V^a(x) = \sup_{k\geq 0, v\in\mathcal{A}} \left[-E\left\{\left[\sum_{i=0}^{k-1} S(z_i, v_i)\right] |x\right\}\right].$$

In this case, one easily establishes, analogously to the time varying case, a BRL which is a trivial corollary of Theorem 2. It yields a time-invariant algebraic HJI, the counterpart of the well-known linear stochastic case. This corollary is stated as follows.

Corollary 1: Consider the system (11) with the controlled output (2), and let $u_k = 0\ \forall k \geq 0$.

a) Let $\gamma > 0$, and let V be a positive real valued function: $V : R^n \to R^+$ satisfying the following Hamilton–Jacobi algebraic inequality (HJAI)

$$V(x) \geq \sup_{v\in R^n} \left\{\|z(x)\|^2 - \gamma^2\|v\|^2 + E_{\omega_k}\left\{V\left[f(x, v, \omega_k)\right]\right\}\right\} \quad (12)$$

for all $x \in R^n$. Assume also: $E\{V(x_0)\} \leq \gamma^2 E\{\|x_0\|^2\}$ for all F_0-measurable random variables x_0 with $E\{\|x_0\|^2\} < \infty$. Then, the system of (11) has $l_2 -$ gain $\leq \gamma$.

b) Assume there is a control signal $\{u_k\}_{k\geq 0}$ which makes the l_2-gain of the system (11) less than or equal to γ for some $\gamma > 0$ and for all F_0-measurable x_0 with $E\{\|x_0\|^2\} < \infty$. Then, there is a positive function V which satisfies the HJI (12).

III. Sufficient Conditions for $l_2 -$ Gain $\leq \gamma$: A Special Case

In this section, we consider a special case of (1) which is characteristic of a large class of mechanical systems. For this class of systems, we establish a more tractable sufficient conditions in terms of a HJI. Thus, we consider the following system:

$$x_{k+1} = f_k(x_k, v_k, \omega_k) \quad z_k = h_k(x_k)$$

In what follows, we assume a special case of the above nonlinear system:

$$x_{k+1} = f_k(x_k) + g_k^1(x_k)v_k + g_k^2(x_k)v_k\omega_k^2 + G_k(x_k)\omega_k^1 \quad (13)$$

where $\{\omega_k\} \triangleq \{[(\omega_k^1)^T\ \ \omega_k^2]^T\}$ is an R^{l+1}-valued white-noise sequence with uncorrelated components defined on the probability space (Ω, F, P), with the covariance $E\{\omega_k\omega_k^T\} = \bar{R} = \mathrm{diag}\{r_1, r_2, \ldots, r_l, r\} = \mathrm{diag}\{R, r\}$. Obviously, $R = E\{\omega_k^1\omega_k^{1T}\}$ and $r = E\{(\omega_k^2)^2\}$. The exogenous disturbance $\{v_k\}_{k\geq 0}$ is assumed to be $\{F_k\}_{k\geq 0}$-adapted, that is v_k is F_{k-1}-measurable for all k, where F_k is the minimal σ-algebra generated by $\{\omega_i : i \leq k-1\}$. Moreover, it is assumed that $\sum_{k=1}^{\infty} \|v_k\|^2 < \infty$. The functions $f_k : R^n \to R^n$, $g_k^1 : R^n \to R^{n\times m_1}$, $g_k^2 : R^n \to R^{n\times m_1}$, and $G_k : R^n \to R^{n\times l}$ are all assumed to be continuous. It is further assumed that the aforementioned functions are of the following structure. $G_k(x)$ is taken to be an $n \times l$ matrix, so that

$$G_k(x) = \left[G_k^1(x)x \quad G_k^2(x)x \cdots G_k^l(x)x\right] \quad (14)$$

where $G_k^i(x)$ are $n \times n$ matrices. It follows that $G_k(x)\omega_k = \sum_{i=1}^{l} G_k^i(x)x\omega_k^i$.

It is also assumed that $f_k(x) = F_k(x)x$ and $h_k(x) = H_k(x)x$. Application of the BRL yields the following.

Lemma 2: Consider the system (13), with $f_k, g_k^1, g_k^2, G_k, h_k$ as aforementioned

a) Let $\{Q_k(x)\}_{k\geq 0}$ be a family of symmetric positive matrices for all $x \in R^n$ which satisfies

i)

$$\phi_k(x) \triangleq \left[\gamma^2 I - g_k^{1T}(x)Q_{k+1}(x)g_k^1(x) - rg_k^{2T}(x)Q_{k+1}(x)g_k^2(x)\right] > 0 \quad (15)$$

for all x in R^n;

ii)

$$\begin{aligned} x^T Q_k(x)x \geq x^T &\left\{F^T(x)Q_{k+1}(x)F(x) + \sum_{i=1}^{l} r_i \left(G_k^i(x)\right)^T Q_{k+1}(x)G_k^i(x)\right\} x \\ &+ F^T(x)Q_{k+1}(x)g_k^1(x)\phi_k^{-1}(x)g_k^{1T}(x) \\ &\times Q_{k+1}(x)F(x) \end{aligned} \quad (16)$$

for all x in R^n, where $r_i = E(\omega_k^i\omega_k^i)$;

iii)

$$Q_0(x) \leq \gamma^2 I \quad \forall x \in R^n. \quad (17)$$

Then, the system (13) has $l_2 -$ gain $\leq \gamma$.

b) Assume the system (13) is dissipative with respect to the supply rate $\gamma^2\|v\|^2 - \|z\|^2$ for some $\gamma > 0$, with a storage function $V_i(x) = x^T Q_i(x)x$, where $\{Q_i(x)\}_{i\geq 0}$ are symmetric with $Q_i(x) \geq 0$ for all $x \in R^n$. Then, the inequality (16) holds.

Proof:

a) Define $V_k(x)$ as $V_k(x) = x^T Q_k(x)x$, for all $x \in R^n$. Utilizing (16), it is straightforward to show that $\{V_k(x)\}_k$ satisfies (8), which together with iii) imply, in view of the BRL, that the system (13) has $l_2 -$ gain $\leq \gamma$. In fact, in view of the particular structure of $V_k(x)$, we have

$$\begin{aligned} &\sup_v \left\{|z_k|^2 - \gamma^2|v|^2 + E_{\omega_k}\left[V_{k+1}\left\{f_k(x_k, v, \omega_k)\right\}\right]\right\} \\ &= \sup_v \left\{x^T H_k^T(x)H_k(x)x - \gamma^2|v|^2 \right. \\ &\quad + \left[F_k(x)x + g_k^1(x)v\right]^T Q_{k+1}(x)\left[F_k(x)x + g_k^1(x)v\right] \\ &\quad + \mathrm{trace}\left[R^{\frac{1}{2}}G_k^T(x)Q_{k+1}G_k(x)R^{\frac{1}{2}}\right]\Big\} \\ &\quad + rv^T g_k^{2T}Q_{k+1}g_k^2 v \end{aligned} \quad (18)$$

where $R = \mathrm{diag}\{[r_i]_{i=1}^l\}$.

From (15) it follows that there is a unique maximizing $v_k = v_k(x)$ given by:

$$v_k = \left[\gamma^2 I - g_k^{1T}(x)Q_{k+1}(x)g_k^1(x) - rg_k^{2T}(x)Q_{k+1}(x)g_k^2(x)\right]^{-1} g_k^T(x)Q_{k+1}(x)F_k(x)x. \quad (19)$$

Substituting (19) in (18) and using the following identity:

$$\text{trace}\left[R^{\frac{1}{2}}G_k^T(x)Q_{k+1}G_k(x)R^{\frac{1}{2}}\right] = x^T\left\{\sum_i r_i\left[G_k^i(x)\right]^T Q_{k+1}(x)G_k^i(x)\right\}x \quad (20)$$

in (18) we conclude, in view of (16), that the inequality (8) holds with $\{V_k(x)\}_k$ as aforementioned. Therefore, by the BRL the system has $l_2 -$ gain $\leq \gamma$.

b) Since the sequence $\{x^TQ_k(x)x\}$ is a storage function for system (13), it satisfies the HJI (8). Evaluating now the maximizing v, inequality (16) is satisfied. □

IV. Norm-Bounded Uncertainty

We consider the infinite-time horizon case where the underlying stochastic system is time-invariant. The system considered is

$$\begin{aligned} x_{k+1} &= f(x_k)x_k + G(x_k)w_k + g_1(x_k)v_k + \bar{g}(x_k)u_k \\ z_k &= \text{col}\left\{h(x_k),\ \bar{D}u_k\right\} \end{aligned} \quad (21)$$

where $f(x) = (A + HF(x)E_1)$, $G(x) = (A_1 + HF(x)E_3)x$, $g_1(x) = B_1$, $\bar{g}(x) = B_2 + HF(x)E_2$, and $h(x) = \bar{C}_1x$ and where $\bar{D}$ is a constant matrix of appropriate dimensions. The sequence $\{w_k\}$ is a standard white-noise scalar sequence and u_k is the control input $G \in \mathcal{R}^{n\times n}$, $g \in \mathcal{R}^{n\times r}$, and $\bar{g} \in \mathcal{R}^{n\times m}$. The nonlinear part $F(x)$, which is in $\mathcal{R}^{q_1\times q_2}$, is assumed to be bounded. Namely

$$F(x)^TF(x) \leq I. \quad (22)$$

In the sequel, we view $F(x)$ as the norm-bounded uncertainty and we consider the case where $z_k(x) = C_1x_k + Du_k$, where C_1 and D are constant matrices and $D^TD > 0$, namely we take $C_1 = \left[\begin{smallmatrix}\bar{C}_1\\0\end{smallmatrix}\right]$ and $D = \left[\begin{smallmatrix}0\\\bar{D}\end{smallmatrix}\right]$, and we seek a state-feedback control $u_k = Kx_k$, where $K \in \mathcal{R}^{m\times n}$ is a constant gain matrix, that achieves a prescribed bound γ on the system's ℓ_2-gain.

The latter problem can be readily solved in the deterministic case by applying a linear approach. The extension of this approach to the stochastic case is, however, not immediate and one has to base his solution on the theory of the above sections in order to obtain a correct answer. Choosing $V_k(x_k) = x_k^TQx_k$, where $0 < Q \in \mathcal{R}^{n\times n}$, it follows from Lemma 4 that the inequality which guarantees the required bound on the system's gain is given by:

$$\begin{aligned} &-Q + A_F^TQA_F + A_F^TQB_1\left(\gamma^2I - B_1^TQB_1\right)^{-1}B_1^TQA_F \\ &+ (A_1 + HF(x)E_3)^TQ(A_1 + HF(x)E_3) \\ &+ C_1^TC_1 + K^TD^TDK < 0 \end{aligned} \quad (23)$$

for all $x \in \mathcal{R}^n$, where $A_F = A + B_2K + HF(x)(E_1 + E_2K)$. The latter can be put in an equivalent matrix inequality applying Schur's complement formula. The resulting inequality is

$$\begin{bmatrix} -Q+C_1^TC_1 & A_F^TQ & 0 & (A_1+HF(x)E_3)^TQ & K^T \\ * & -Q & QB_1 & 0 & 0 \\ * & * & -\gamma^2I & 0 & 0 \\ * & * & * & -Q & 0 \\ * & * & * & * & -I \end{bmatrix} \leq 0. \quad (24)$$

Multiplying the latter, from both sides, by $\text{diag}\{\bar{Q}, \bar{Q}, I, \bar{Q}, I\}$ where $\bar{Q} \triangleq Q^{-1}$ and defining $Y = K\bar{Q}$ the following LMI is obtained:

$$\begin{bmatrix} -\bar{Q} & \bar{Q}A_F^T & 0 & \bar{Q}(A_1+HF(x)E_3)^T & Y^TD^T & \bar{Q}C_1^T \\ * & -\bar{Q} & B_1 & 0 & 0 & 0 \\ * & * & -\gamma^2I & 0 & 0 & 0 \\ * & * & * & -\bar{Q} & 0 & 0 \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -I \end{bmatrix} \leq 0. \quad (25)$$

The latter inequality can also be written as

$$\Gamma + \Phi_1F(x)\Phi_2^T + \Phi_2F^T(x)\Phi_1^T + \Phi_3F(x)\Phi_4^T + \Phi_4F^T(x)\Phi_3^T \leq 0 \quad (26)$$

where

$$\begin{aligned} \Phi_1 &= [0\ H^T\ 0\ 0\ 0\ 0]^T & \Phi_2 &= [E_1\bar{Q} + E_2Y\ 0\ 0\ 0\ 0\ 0]^T \\ \Phi_3 &= [0\ 0\ 0\ H^T\ 0\ 0]^T & \Phi_4 &= [E_3\bar{Q}\ 0\ 0\ 0\ 0\ 0]^T \end{aligned} \quad (27)$$

and where

$$\Gamma = \begin{bmatrix} -\bar{Q} & \bar{Q}A^T + Y^TB_2^T & 0 & \bar{Q}A_1^T & Y^TD^T & \bar{Q}C_1^T \\ * & -\bar{Q} & B_1 & 0 & 0 & 0 \\ * & * & -\gamma^2I & 0 & 0 & 0 \\ * & * & * & -\bar{Q} & 0 & 0 \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -I \end{bmatrix}. \quad (28)$$

Using the fact that for any two matrices α and β of compatible dimensions and for any positive definite matrix $\bar{R}$ of compatible dimensions the following holds: $\alpha\beta^T + \beta\alpha^T \leq \alpha\bar{R}\alpha^T + \beta\bar{R}^{-1}\beta^T$, together with the bound of (22), we obtain the following:

$$\Gamma + \Phi_1\bar{R}_1\Phi_1^T + \Phi_2\bar{R}_1^{-1}\Phi_2^T + \Phi_3\bar{R}_2\Phi_3^T + \Phi_4\bar{R}_2^{-1}\Phi_4^T \leq 0 \quad (29)$$

for some positive definite matrices $\bar{R}_1$ and $\bar{R}_2$. Choosing the latter matrices to be scalar matrices we obtain the following.

Theorem 3: Consider the system of (21). Given the scalar $0 < \gamma$, there exists a constant state-feedback controller $u_k = Kx_k$ that globally stabilizes the closed-loop exponentially, in the mean-square sense, and achieves $l_2 -$ gain $\leq \gamma$ if there exist $0 < \bar{Q} \in R^{n\times n}$, $Y \in R^{p\times n}$, and scalars ρ_1 and ρ_2, so that the following LMI is satisfied, as shown in

$$\begin{bmatrix} -\bar{Q} & \bar{Q}A^T + Y^T B_2^T & 0 & \bar{Q}A_1^T & Y^T D^T & \bar{Q}C_1^T & 0 & 0 & \bar{Q}E_1^T + Y^T E_2^T & \bar{Q}E_3^T \\ * & -\bar{Q} & B_1 & 0 & 0 & 0 & H\rho_1 & 0 & 0 & 0 \\ * & * & -\gamma^2 I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\bar{Q} & 0 & 0 & 0 & H\rho_2 & 0 & 0 \\ * & * & * & * & -I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -I & 0 & 0 & 0 & 0 \\ * & * & * & * & * & * & -\rho_1 I & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -\rho_2 I & 0 & 0 \\ * & * & * & * & * & * & * & * & -\rho_1 I & 0 \\ * & * & * & * & * & * & * & * & * & -\rho_2 I \end{bmatrix} < 0. \tag{30}$$

(30) at the top of the page. If a solution to this LMI exists, the state-feedback gain is given by $K = Y\bar{Q}^{-1}$.

V. Example

We consider the following model for a single degree of freedom inverted pendulum with the multiplicative white noise ω:

$$ml^2\ddot{\theta} - mgl\sin(\theta) + (\varsigma + \omega)\dot{\theta} + \kappa\theta = u + 2v \tag{31}$$

where κ is the spring coefficient and ς is a damping coefficient. The signal v is a deterministic disturbance acting on the control input u and ω is a stochastic uncertainty in the damping. In this model, θ is the inclination angle of the pendulum, l and m are its length and mass, respectively, and g is the gravitation coefficient.

Denoting $x_1 = \theta$ and $x_2 = ml^2\dot{\theta}$, we discretize the model of (31) and obtain the following discrete-time state-space representation of the system:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_{k+1} = \begin{bmatrix} 1 & \frac{T}{ml^2} \\ Tmgl\frac{\sin(x_{1,k})}{x_{1,k}} - \kappa T & 1 - \frac{T\varsigma}{ml^2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}_k + \begin{bmatrix} 0 \\ T \end{bmatrix} u_k + \begin{bmatrix} 0 \\ 2T \end{bmatrix} v_k + \begin{bmatrix} 0 \\ -\sqrt{T}\frac{x_{2,k}}{ml^2} \end{bmatrix} \omega_k$$

where T is the sampling period. Regarding now the notation associated with (21) we have now

$$A = \begin{bmatrix} 1 & \frac{T}{ml^2} \\ -\kappa T & 1 - \frac{\varsigma T}{ml^2} \end{bmatrix} \quad H = \begin{bmatrix} 0 \\ Tmgl \end{bmatrix}$$
$$E_1 = [1 \quad 0] \quad F(x) = \frac{\sin(x)}{x}$$
$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{1}{ml^2} \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ 2T \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ T \end{bmatrix}.$$

For $m = 0.5$ kg, $l = 0.7$ m, $\kappa = 0.5$ N/m, and $\varsigma = 0.25$, we choose $T = 0.01$ s

$$C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}$$

and we require $\gamma = 0.23$.

Applying the LMI of Theorem 3 (for $\rho_2 \to 0$), we obtain a feasible solution with

$$\bar{Q} = 10^3 \begin{bmatrix} 3.4 & -1.8 \\ -1.8 & 9.3 \end{bmatrix} \quad Y = [0.0033 \quad -0.8124] \quad \text{and}$$
$$\rho = 0.3067.$$

The latter leads to $K = -[51.1802 \quad 97.5356]$. The corresponding function $V = x^T\bar{Q}^{-1}x$ then satisfies the HJI of (23).

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