One-Shot Federated Learning: Theoretical Limits and Algorithms to Achieve Them [*]

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Editor:

Abstract

We consider distributed statistical optimization in one-shot setting, where there are m machines each observing n i.i.d. samples. Based on its observed samples, each machine sends a B-bit-long message to a server. The server then collects messages from all machines, and estimates a parameter that minimizes an expected convex loss function. We investigate the impact of communication constraint, B, on the expected error and derive a tight lower bound on the error achievable by any algorithm. We then propose an estimator, which we call *Multi-Resolution Estimator* (MRE), whose expected error (when $B \geq \log mn$) meet the aforementioned lower bound up to poly-logarithmic factors, and is thereby order optimal. We also address the problem of learning under tiny communication budget, and present lower and upper error bounds when B is a constant. The expected error of MRE , unlike existing algorithms, tends to zero as the number of machines (m) goes to infinity, even when the number of samples per machine (n) remains upper bounded by a constant. This property of the MRE algorithm makes it applicable in new machine learning paradigms where m is much larger than n.

Keywords: Federated learning, Distributed learning, Few shot learning, Communication efficiency, Statistical Optimization.

1. Introduction

In recent years, there has been a growing interest in various learning tasks over large scale data generated and collected via smart phones and mobile applications. In order to carry out a learning task over this data, a naive approach is to collect the data in a centralized server which might be infeasible or undesirable due to communication constraints or privacy reasons. For learning statistical models in a distributed fashion, several works have focused on designing communication-efficient algorithms for various machine learning applications (Duchi et al., 2012; Braverman et al., 2016; Chang et al., 2017; Diakonikolas et al., 2017; Lee et al., 2017).

∗. Parts of this work (including weaker versions of Theorems 3 and 6) are presented in Sharifnassab et al. (2019) at Neurips 2019.

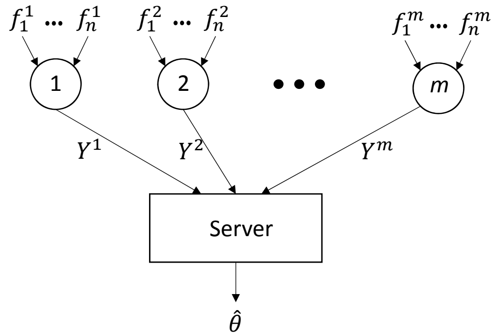


Figure 1: A distributed system of m machines, each having access to n independent sample functions from an unknown distribution P. Each machine sends a signal to a server based on its observations. The server receives all signals and output an estimate $\hat{\theta}$ for the optimization problem in (2).

In this paper, we consider the problem of statistical optimization in a distributed setting as follows. Consider an unknown distribution P over a collection, $\mathcal{F}$, of differentiable convex functions with Lipschitz first order derivatives, defined over a convex region in $\mathbb{R}^d$. There are m machines, each observing n i.i.d sample functions from P. Each machine processes its observed data, and transmits a signal of certain length to a server. The server then collects all the signals and outputs an estimate of the parameter θ^* that minimizes the expected loss, i.e., $\min_\theta \mathbb{E}_{f\sim P}\big[f(\theta)\big]$. See Fig. 1 for an illustration of the system model.

We focus on the distributed aspect of the problem considering arbitrarily large number of machines (m) and present tight lower bounds and matching upper bounds on the estimation error. In particular,

- Under general communication budget with $B \geq d\log mn$ bits per transmission, we present a tight lower bound and an order-optimal estimator that achieves this bounds up to poly-logarithmic factors. More specifically, we show that $\|\hat{\theta}-\theta^*\| = \tilde{\Theta}\left(\max\left(n^{-1/2}(mB)^{-1/d},\, (mn)^{-1/2}\right)\right)$.
- For the regime that the communication budget is very small with constant number of bits per transmission, we present upper and lower bounds on the estimation error and show that the error can be made arbitrarily small if m and n tend to infinity simultaneously.
- Compared to the previous works that consider function classes with Lipschitz continuous second or third order derivatives, our algorithms and bounds are designed and derived for a broader class of functions with Lipschitz continuous first order derivatives. This brings our model closer to real-world learning applications where the loss landscapes involved are highly non-smooth.

1.1 Background

The distributed setting considered here has recently employed in a new machine learning paradigm called *Federated Learning* (Konečný et al., 2015). In this framework, training data is kept in users' computing devices due to privacy concerns, and the users participate in the training process without revealing their data. As an example, Google has been working on this paradigm in their recent project, *Gboard* (McMahan and Ramage, 2017), the Google keyboard. Besides communication constraints, one of the main challenges in this paradigm is that each machine has a small amount of data. In other words, the system operates in a regime that m is much larger than n (Chen et al., 2017).

A large body of distributed statistical optimization/estimation literature considers “one-shot" setting, in which each machine communicates with the server merely once (Zhang et al., 2013). In these works, the main objective is to minimize the number of transmitted bits, while keeping the estimation error as low as the error of a centralized estimator, in which the entire data is co-located in the server.

If we impose no limit on the communication budget, then each machine can encode its entire data into a single message and sent it to the server. In this case, the sever acquires the entire data from all machines, and the distributed problem reduces to a centralized problem. We call the sum of observed functions at all machines as the centralized empirical loss, and refer to its minimizer as the centralized solution. It is part of the folklore that the centralized solution is order optimal and its expected error is $\Theta\big(1/\sqrt{mn}\big)$ (Lehmann and Casella, 2006; Zhang et al., 2013). Clearly, no algorithm can beat the performance of the best centralized estimator.

1.1.1 Upper bounds

Zhang et al. (2012) studied a simple averaging method where each machine obtains the empirical minimizer of its observed functions and sends this minimizer to the server through an $O(\log mn)$ bit message. Output of the server is then the average of all received empirical minimizers. Zhang et al. (2012) showed that the expected error of this algorithm is no larger than $O\big(1/\sqrt{mn}+1/n\big)$, provided that: 1- all functions are convex and twice differentiable with Lipschitz continuous second derivatives, and 2- the objective function $\mathbb{E}_{f\sim P}\big[f(\theta)\big]$ is strongly convex at θ^*. Under the extra assumption that the functions are three times differentiable with Lipschitz continuous third derivatives, Zhang et al. (2012) also present a bootstrap method whose expected error is $O\big(1/\sqrt{mn}\,+\,1/n^{1.5}\big)$. It is easy to see that, under the above assumptions, the averaging method and the bootstrap method achieve the performance of the centralized solution if $m\leq n$ and $m\leq n^2$, respectively. Recently, Jordan et al. (2018) proposed to optimize a surrogate loss function using Taylor series expansion. This expansion can be constructed at the server by communicating $O(m)$ number of d-dimensional vectors. Under similar assumption on the loss function as in (Zhang et al., 2012), they showed that the expected error of their method is no larger than $O\big(1/\sqrt{mn}+1/n^{9/4}\big)$. It, therefore, achieves the performance of the centralized solution for $m\leq n^{3.5}$. However, note that when n is fixed, all aforementioned bounds remain lower bounded by a positive constant, even when m goes to infinity.

In (Sharifnassab et al., 2019), we relaxed the second order differentiability assumption, and considered a model that allows for convex loss functions that have Lipschitz continuous first order derivatives. There we presented an algorithm (called MRE-C-log) with the communication budget of $\log mn$ bits per transmission, and proved the upper bound $\tilde{O}\big(m^{-1/\max(d,2)}n^{-1/2}\big)$ on its estimation error. In this work we extend this algorithm to general communication budget of B bits per transmission, for arbitrary values of $B\geq\log mn$. We also derive a lower bound on the estimation error of any algorithm. This lower bound meets the error-upper-bound of the MRE-C algorithm, showing that the MRE-C estimator has order optimal accuracy up to a poly-logarithmic factor.

1.1.2 Lower bounds

Shamir (2014) considered various communication constraints and showed that no distributed algorithm can achieve performance of the centralized solution with budget less than $\Omega(d^2)$ bits per machine. For the problem of sparse linear regression, Braverman et al. (2016) proved that any algorithm that achieves optimal minimax squared error, requires to communicate $\Omega(m \times \min(n, d))$ bits in total from machines to the server. Later, Lee et al. (2017) proposed an algorithm that achieves optimal mean squared error for the problem of sparse linear regression when $d < n$.

Zhang et al. (2013) derived an information theoretic lower bound on the minimax error of parameter estimation, in presence of communication constraints. They showed that, in order to acquire the same precision as the centralized solution for estimating the mean of a d-dimensional Gaussian distribution, the machines require to transmit a least total number of $\Omega\big(md/\log(m)\big)$ bits. Garg et al. (2014) improved this bound to $\Omega(dm)$ bits using direct-sum theorems (Chakrabarti et al., 2001).

1.1.3 One-shot vs. several-shot models

Besides the one-shot model, there is another communication model that allows for several transmissions back and forth between the machines and the server. Most existing works of this type (Bottou, 2010; Lian et al., 2015; Zhang et al., 2015; McMahan et al., 2017) involve variants of stochastic gradient descent, in which the server queries at each iteration the gradient of empirical loss at certain points from the machines. The gradient vectors are then aggregated in the server to update the model's parameters. The expected error of such algorithms typically scales as $\tilde{O}\big(1/\sqrt{k}\big)$, where k is the number of iterations.

The bidirectional communication in the several-shot model makes it convenient for the server to guide the search by sending queries to the machines (e.g., asking for gradients at specific points of interest). This powerful ability of the model typically leads to more efficient communication for the case of convex loss landscapes. However, the two-way communication require the users (or machines) be available during the time of training, so that they can respond to the server queries in real time. Moreover, in such iterative algorithms, the users should be willing to reveal parts of their information asked by the servers. In contrast to the several-shot model, in the one-shot setting, because of one-way communication, SGD-like iterative algorithms are not applicable. The one-shot setting calls for a totally different type of algorithms and lower bounds.

1.2 Our contributions

We study the problem of one-shot distributed learning under milder assumptions than previously available in the literature. We assume that loss functions, $f \in \mathcal{F}$, are differentiable with Lipschitz continuous first order derivatives. This is in contrast to the works of (Zhang et al., 2012) and (Jordan et al., 2018) that assume Lipschitz continuity of second or third derivatives. The assumption is indeed practically important since the loss landscapes involved in several learning applications are highly non-smooth. The reader should have in mind this model differences, when comparing our bounds with the existing results. See Table 1 for a summary of our results.

We consider a sitting where the loss landscape is convex, and derive a lower bound on the estimation error, under communication budget of B bits per transmission for all $B \geq d \log mn$. We also propose an algorithm (which we call Multi-Resolution Estimator for Convex setting (MRE-C)), and show that its estimation error meets the lower bound up to a poly-logarithmic

Communication Budget (B)	Assumptions	Result	Ref.
$B \geq d\log(mn)$	-	$\|\hat{\theta}-\theta^*\| = \tilde{\Omega}\left(\max\left(\frac{1}{\sqrt{n}\,(mB)^{1/d}},\, \frac{1}{\sqrt{mn}}\right)\right)$	Th. 1
		$\|\hat{\theta}-\theta^*\| = \tilde{O}\left(\max\left(\frac{1}{\sqrt{n}\,(mB)^{1/d}},\, \frac{1}{\sqrt{mn}}\right)\right)$	Th. 3
Constant B	$n=1$	$\|\hat{\theta}-\theta^*\| = \Omega\left(1\right)$	Th. 5
	$B=d$	$\|\hat{\theta}-\theta^*\| = O\left(\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{m}}\right)$	Th. 6

Table 1: Summary of our results.

factor. Therefore, MRE-C algorithm has order optimal accuracy. Combining these lower and upper bounds, we show that for any communication budget B no smaller than $d\log mn$, we have $\|\hat{\theta}-\theta^*\| = \tilde{\Theta}\left(\max\left(n^{-1/2}(mB)^{-1/d},\, (mn)^{-1/2}\right)\right)$. Moreover, computational complexity of the MRE-C algorithm is polynomial in m, n, and d. Our results also provide the minimum communication budget required for any estimator to achieve the performance of the centralized algorithm.

We also study a regime with tiny communication budget, where B is bounded by a constant. We show that when B is a constant and $n=1$, the error of any estimator is lower bounded by a constant, even when m tends to infinity. On the other hand, we propose an algorithm with the budget of $B=d$ bits per transmission and show that its estimation error is no larger than $O\left(n^{-1/2}+m^{-1/2}\right)$.

We evaluate the performance of MRE-C algorithm in two different machine learning tasks (with convex landscapes) and compare with the existing methods in (Zhang et al., 2012). We show via experiments, for the $n=1$ regime, that MRE algorithm outperforms these algorithms. The observations are also in line with the expected error bounds we give in this paper and those previously available. In particular, in the $n=1$ regime, the expected error of MRE algorithm goes to zero as the number of machines increases, while the expected errors of the previously available estimators remain lower bounded by a constant.

Unlike existing works, our results concern a regime where the number of machines m is large, and our bounds tend to zero as m goes to infinity, even if the number of per-machine observations (n) is bounded by a constant. This is contrary to the algorithms in (Zhang et al., 2012), whose errors tend to zero only when n goes to infinity. In fact, when $n=1$, a simple example[1] shows that the expected errors of the simple Averaging and Bootstrap algorithms in (Zhang et al., 2012) remain lower bounded by a constant, for all values of m. The algorithm in (Jordan et al., 2018) suffers from a similar problem and its expected error may not go to zero when $n=1$.

1.3 Outline

The paper is organized as follows. We begin with a detailed model and problem definition in Section 2. We then propose our lower bound on the estimation error in Section 3, under general communication constraints. In Section 4, we present the MRE-C algorithm and its error upper bound. Section 5 then

1. Consider two convex functions $f_0(\theta)=\theta^2+\theta^3/6$ and $f_1(\theta)=(\theta-1)^2+(\theta-1)^3/6$ over $[0,1]$. Consider a distribution P that associates probability $1/2$ to each function. Then, $\mathbb{E}_P[f(\theta)] = f_0(\theta)/2+f_1(\theta)/2$, and the optimal solution is $\theta^*=(\sqrt{15}-3)/2\approx 0.436$. On the other hand, in the averaging method proposed in (Zhang et al., 2012), assuming $n=1$, the empirical minimizer of each machine is either 0 if it observes f_0, or 1 if it observes f_1. Therefore, the server receives messages 0 and 1 with equal probability , and $\mathbb{E}\big[\hat{\theta}\big]=1/2$. Hence, $\mathbb{E}\big[|\hat{\theta}-\theta^*|\big] > 0.06$, for all values of m.

provides our results for the regime where communication budget is limited to constant number of bits per transmission. After that, we report our numerical experiments in Section 6. Finally, in Section 7 we conclude the paper and discuss several open problems and directions for future research. All proofs are relegated to the appendix for improved readability.

2. Problem Definition

Consider a positive integer d and a collection $\mathcal{F}$ of real-valued convex functions over $[-1,1]^d$. Let P be an unknown probability distribution over the functions in $\mathcal{F}$. Consider the expected loss function

$$F(\theta) = \mathbb{E}_{f\sim P}\big[f(\theta)\big], \qquad \theta \in [-1,1]^d. \tag{1}$$

Our goal is to learn a parameter θ^* that minimizes F:

$$\theta^* = \underset{\theta\in[-1,1]^d}{\operatorname{argmin}}\ F(\theta). \tag{2}$$

The expected loss is to be minimized in a distributed fashion, as follows. We consider a distributed system comprising m identical machines and a server. Each machine i has access to a set of n independently and identically distributed samples $\{f_1^i, \cdots, f_n^i\}$ drawn from the probability distribution P. Based on these observed functions, machine i then sends a signal Y^i to the server. We assume that the length of each signal is limited to b bits. The server then collects signals $Y^1, \ldots, Y^m$ and outputs an estimation of θ^*, which we denote by $\hat{\theta}$. See Fig. 1 for an illustration of the system model.[2]

We let the following assumptions be in effect throughout the paper:

Assumption 1 (Differentiability) *We assume:*

- *Each $f \in \mathcal{F}$ is once differentiable and its derivatives are bounded and Lipschitz continuous. More concretely, for any $f \in \mathcal{F}$ and any $\theta, \theta' \in [-1,1]^d$, we have $|f(\theta)| \le \sqrt{d}$, $\|\nabla f(\theta)\| \le 1$, and $\|\nabla f(\theta) - \nabla f(\theta')\| \le \|\theta - \theta'\|$.*

- *The minimizer of F lies in the interior of the cube $[-1,1]^d$. Equivalently, there exists $\theta^* \in (-1,1)^d$ such that $\nabla F(\theta^*) = \mathbf{0}$.*

In Assumption 1 we consider a class of functions with Lipschitz continuous first order derivatives, compared to previous works that consider function classes with Lipschitz continuous second or third order derivatives (Zhang et al., 2013; Jordan et al., 2018). This broadens the scope and applicability of our model to learning tasks where the loss landscape is far from being smooth (see Section 7 for further discussions).

Assumption 2 (Convexity) *We assume:*

- *Every $f \in \mathcal{F}$ is convex.*

- *Distribution P is such that F (defined in (1)) is strongly convex. More specifically, there is a constant $\lambda > 0$ such that for any $\theta_1, \theta_2 \in [-1,1]^d$, we have $F(\theta_2) \ge F(\theta_1) + \nabla F(\theta_1)^T(\theta_2 - \theta_1) + \lambda\|\theta_2 - \theta_1\|^2$.*

2. The considered model here is similar to the one in (Salehkaleybar et al., 2019).

The convexity assumption (Assumption 2) is common in the literature of distributed learning (Zhang et al., 2013; Jordan et al., 2018). When F is strongly convex, the objective is often designing estimators that minimize $\mathbb{E}\big[\|\hat{\theta}-\theta^*\|^2\big]$. Given the upper and lower bounds on the second derivative (in Assumptions 1 and 2), this is equivalent (up to multiplicative constants) with minimization of $\mathbb{E}\big[F(\hat{\theta})-F(\theta^*)\big]$. Note also that the assumption $\|\nabla F(x)\|\leq 1$ (in Assumption 2) implies that

$$\lambda \leq \frac{1}{\sqrt{d}}. \tag{3}$$

This is because if $\lambda > 1/\sqrt{d}$, then $\|\nabla F(x)\| > 1$, for some $x \in [-1,1]^d$.

3. Main Lower Bound

The following theorem shows that in a system with m machines, n samples per machine, and B bits per signal transmission, no estimator can achieve estimation error less than $\|\hat{\theta}-\theta^*\| = \tilde{\Omega}\left(\max\left(n^{-1/2}(mB)^{-1/d},\, (mn)^{-1/2}\right)\right)$. The proof is given in Appendix B.

Theorem 1 *Suppose that Assumption 2 is in effect for $\lambda \leq 1/\big(10\sqrt{d}\big)$. Then, for any estimator with output $\hat{\theta}$, there exists a probability distribution over $\mathcal{F}$ such that*

$$\Pr\left(\|\hat{\theta}-\theta^*\| = \tilde{\Omega}\left(\max\left(\frac{1}{\sqrt{n}\,(mB)^{1/d}},\, \frac{1}{\sqrt{mn}}\right)\right)\right) \geq \frac{1}{3}. \tag{4}$$

More specifically, for large enough values of mn, for any estimator there is a probability distribution over $\mathcal{F}$ such that with probability at least $1/3$,

$$\|\hat{\theta}-\theta^*\| \;\geq\; \max\left(\frac{1}{640\times 50^{1/d}\, d^{2.5}\, \log^{2+3/d}(mn)} \times \frac{1}{\sqrt{n}\,(mB)^{1/d}}\,,\; \frac{\sqrt{d}}{5\sqrt{mn}}\right) \tag{5}$$

In light of (3), the assumption $\lambda \leq 1/\big(10\sqrt{d}\big)$ in the statement of the theorem appears to be innocuous, and is merely aimed to facilitate the proofs. The proof is given in Section B. The key idea is to show that finding an $O\big(n^{-1/2}m^{-1/d}\big)$-accurate minimizer of F (i.e., $\|\hat{\theta}-\theta^*\| = O\big(n^{-1/2}m^{-1/d}\big)$) is as difficult as finding an $O\big(n^{-1/2}m^{-1/d}\big)$-accurate approximation of ∇F for all points in an $n^{-1/2}$-neighborhood of θ^*. This is quite counter-intuitive, because the latter problem looks way more difficult than the former. To see the unexpectedness more clearly, it suggests that in the special case where $n=1$, finding an $m^{-1/d}$-approximation of ∇F over the entire domain is no harder than finding an $m^{-1/d}$-approximation of ∇F at a single (albeit unknown) point θ^*. This provides a key insight beneficial for devising estimation algorithms:

Insight 1 *Finding an $\tilde{O}\big(n^{-1/2}m^{-1/d}\big)$-accurate minimizer of F is as difficult as finding an $O\big(n^{-1/2}m^{-1/d}\big)$-accurate approximation of ∇F over an $n^{-1/2}$-neighborhood of θ^*.*

This inspires estimators that first approximate ∇F over a neighborhood of θ^* and then choose $\hat{\theta}$ to be a point with minimum $\|\nabla F\|$. We follow a similar idea in Section 4 to design the MRE-C algorithm with order optimal error.

As an immediate corollary of Theorem 3, we obtain a lower bound on the moments of estimation error.

Corollary 2 *For any estimator $\hat{\theta}$, there exists a probability distribution over $\mathcal{F}$ such that for any $k \in \mathbb{N}$,*

$$\mathbb{E}\left[\|\hat{\theta} - \theta^*\|^k\right] = \tilde{\Omega}\left(\max\left(\frac{1}{\sqrt{n}\,(mB)^{1/d}}, \frac{1}{\sqrt{mn}}\right)^k\right). \tag{6}$$

In view of (6), no estimator can achieve performance of a centralized solution with the budget of $B = O(\log mn)$ when $d \geq 3$. As discussed earlier in the Introduction section, this is in contrast to the result in (Zhang et al., 2012) that a simple averaging algorithm achieves $O(1/\sqrt{nm})$ accuracy (similar to a centralized solution), in a regime that $n > m$. This apparent contradiction is resolved by the difference in the set of functions considered in the two works. The set of functions in (Zhang et al., 2012) are twice differentiable with Lipschitz continuous second derivatives, while we do not assume existence or Lipschitz continuity of second derivatives.

4. MRE-C Algorithm and its Error Upper Bound

Here, we propose an order optimal estimator under general communication budget B, for $B \geq d \log mn$. The high level idea, in view of Insight 1, is to acquire an approximation of derivatives of F over a neighborhood of θ^*, and then letting $\hat{\theta}$ be the minimizer of size of these approximate gradients. For efficient gradient approximation, transmitted signals are designed such that the server can construct a multi-resolution view of gradient of function $F(\theta)$ around a promising grid point. Thus, we call the proposed algorithm "Multi-Resolution Estimator for Convex loss (MRE-C)". The description of MRE-C is as follows:

Each machine i observes n functions and sends a signal Y^i comprising $\lceil B/(d \log mn)\rceil$ sub-signals of length $\lfloor d \log mn \rfloor$. Each sub-signal has three parts of the form (s, p, Δ). The three parts s, p, and Δ are as follows.

- Part s: Consider a grid G with resolution $\log(mn)/\sqrt{n}$ over the d-dimensional cube $[-1,1]^d$. Each machine i computes the minimizer of the average of its first $n/2$ observed functions,

$$\theta^i = \underset{\theta \in [-1,1]^d}{\operatorname{argmin}} \sum_{j=1}^{n/2} f_j^i(\theta). \tag{7}$$

  It then lets s be the closest grid point to θ^i. Note that all sub-signals of a machine have the same s-part.

- Part p: Let

$$\delta \triangleq 2d \log^3(mn) \max\left(\frac{1}{(mB)^{1/d}}, \frac{2^{d/2}}{m^{1/2}}\right). \tag{8}$$

  Let $t = \log(1/\delta)$. Without loss of generality we assume that t is a non-negative integer.[3] Let C_s be a d-dimensional cube with edge size $2\log(mn)/\sqrt{n}$ centered at s. Consider a sequence of $t+1$ grids on C_s as follows. For each $l = 0, \ldots, t$, we partition the cube C_s into 2^{ld} smaller equal sub-cubes with edge size $2^{-l+1}\log(mn)/\sqrt{n}$. The lth grid $\tilde{G}_s^l$ comprises the centers of

3. If $\delta > 1$, we reset the value of δ to $\delta = 1$. It is not difficult to check that the rest of the proof would not be upset in this spacial case.

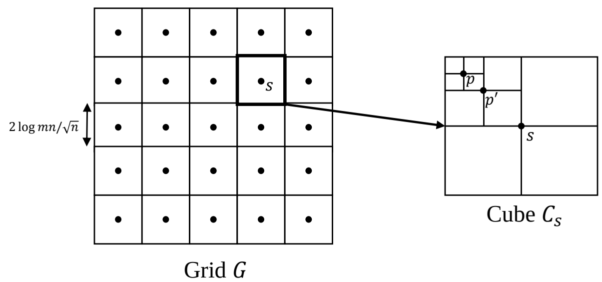


Figure 2: An illustration of grid G and cube C_s centered at point s for $d = 2$. The point p belongs to $\tilde{G}_s^2$ and p' is the parent of p.

these smaller cubes. Then, each $\tilde{G}_s^l$ has 2^{ld} grid points. For any point p' in $\tilde{G}_s^l$, we say that p' is the parent of all 2^d points in $\tilde{G}_s^{l+1}$ that are in the $\big(2^{-l} \times (2\log mn)/\sqrt{n}\big)$-cube centered at p' (see Fig. 2). Thus, each point $\tilde{G}_s^l$ $(l < t)$ has 2^d children.

In each sub-signal, to select p, we randomly choose an l from $0, \ldots, t$ with probability

$$\Pr(l) = \frac{2^{(d-2)l}}{\sum_{j=0}^{t} 2^{(d-2)j}} \tag{9}$$

We then let p be a uniformly chosen random grid point in $\tilde{G}_s^l$. The level l and point p chosen in different sub-signals of a machine are independent and have the same distribution. Note that $O(d\log(1/\delta)) = O(d\log(mn))$ bits suffice to identify p uniquely.

- Part Δ: We let

$$\hat{F}^i(\theta) \triangleq \frac{2}{n} \sum_{j=n/2+1}^{n} f_j^i(\theta), \tag{10}$$

and refer to it as the empirical function of the ith machine. For each sub-signal, if the selected p in the previous part is in $\tilde{G}_s^0$, i.e., $p = s$, then we set Δ to the gradient of $\hat{F}^i$ at $\theta = s$. Otherwise, if p is in $\tilde{G}_s^l$ for $l \geq 1$, we let

$$\Delta \triangleq \nabla \hat{F}^i(p) - \nabla \hat{F}^i(p'),$$

where $p' \in \tilde{G}_s^{l-1}$ is the parent of p. Note that Δ is a d-dimensional vector whose entries range over $\big(2^{-l}\sqrt{d}\log(mn)/\sqrt{n}\big) \times \big[-1, +1\big]$. This is due to the Lipschitz continuity of the derivative of the functions in $\mathcal{F}$ (see Assumption 1) and the fact that $\|p - p'\| = 2^{-l}\sqrt{d}\log(mn)/\sqrt{n}$. Hence, $O(d\log(mn))$ bits suffice to represent Δ within accuracy $2\delta\log(mn)/\sqrt{n}$.

At the server, we choose an $s^* \in G$ that has the largest number of occurrences in the received signals. Then, based on the signals corresponding to $\tilde{G}_{s^*}^0$, we approximate the gradients of F over C_{s^*} as follows. We first eliminate redundant sub-signals so that no two surviving sub-signals from a same machine have the same p-parts (consequently, for each machine, the surviving sub-signals are distinct). We call this process "redundancy elimination". We then let N_{s^*} be the total number of surviving sub-signals that contain s^* in their p part, and compute

$$\hat{\nabla}F(s^*) = \frac{1}{N_{s^*}} \sum_{\substack{\text{Subsignals of the form} \\ (s^*,s^*,\Delta) \\ \text{after redundancy elimination}}} \Delta,$$

Then, for any point $p \in \tilde{G}^l_{s^*}$ with $l \geq 1$, we let

$$\hat{\nabla}F(p) = \hat{\nabla}F(p') + \frac{1}{N_p} \sum_{\substack{\text{Subsignals of the form} \\ (s^*,p,\Delta) \\ \text{after redundancy elimination}}} \Delta, \tag{11}$$

where N_p is the number of signals having point p in their second argument, after redundancy elimination. Finally, the sever lets $\hat{\theta}$ be a grid point p in $\tilde{G}^t_{s^*}$ with the smallest $\|\hat{\nabla}F(p)\|$.

Theorem 3 *Let $\hat{\theta}$ be the output of the above algorithm. Then,*

$$\Pr\left(\|\hat{\theta}-\theta^*\| > \frac{4d^{1.5}\log^4(mn)}{\lambda}\,\max\left(\frac{1}{\sqrt{n}\,(mB)^{1/d}},\,\frac{2^{d/2}}{\sqrt{mn}}\right)\right) \,=\, \exp\Big(-\Omega\big(\log^2(mn)\big)\Big).$$

The proof is given in Appendix G. The proof goes by first showing that s^* is a closest grid point of G to θ^* with high probability. We then show that for any $l \leq t$ and any $p \in \tilde{G}^l_{s^*}$, the number of received signals corresponding to p is large enough so that the server obtains a good approximation of ∇F at p. Once we have a good approximation $\hat{\nabla}F$ of ∇F at all points of $\tilde{G}^t_{s^*}$, a point at which $\hat{\nabla}F$ has the minimum norm lies close to the minimizer of F.

Corollary 4 *Let $\hat{\theta}$ be the output of the above algorithm. There is a constant $\eta > 0$ such that for any $k \in \mathbb{N}$,*

$$\mathbb{E}\big[\|\hat{\theta}-\theta^*\|^k\big] \,<\, \eta\,\left(\frac{4d^{1.5}\log^4(mn)}{\lambda}\,\max\left(\frac{1}{\sqrt{n}\,(mB)^{1/d}},\,\frac{2^{d/2}}{\sqrt{mn}}\right)\right)^k.$$

Moreover, η can be chosen arbitrarily close to 1, for large enough values of mn.

The upper bound in Theorem 3 matches the lower bound in Theorem 1 up to a polylogarithmic factor. In this view, the MRE-C algorithm has order optimal error. Moreover, as we show in Appendix G, in the course of computations, the server obtains an approximation $\hat{F}$ of F such that for any θ in the cube C_{s^*}, we have $\|\nabla\hat{F}(\theta) - \nabla F(\theta)\| = \tilde{O}\big(m^{-1/d}n^{-1/2}\big)$. Therefore, the server not only finds the minimizer of F, but also obtains an approximation of F at all points inside C_{s^*}. This is in line with our previous observation in Insight 1.

5. Learning under Tiny Communication Budget

In this section, we consider the regime that communication budget per transmission is bounded by a constant, i.e., B is a constant independent of m an n. We present a lower bound on the estimation error and propose an estimator whose error vanishes as m and n tend to infinity.

We begin with a lower bound. The next theorem shows that when $n = 1$, the expected error is lower bounded by a constant, even if m goes to infinity.

Theorem 5 *Let $n = 1$ and suppose that the number of bits per signal, B, is limited to a constant. Then, there is a distribution P over $\mathcal{F}$ such that expected error, $\mathbb{E}_P\left[\|\hat{\theta} - \theta^*\|\right]$, of any randomized estimator $\hat{\theta}$ is lower bounded by a constant, for all $m \geq 1$. The constant lower bound holds even when $d = 1$.*

The proof is given in Appendix K. There, we construct a distribution P that associates non-zero probabilities to $2^b + 2$ polynomials of order at most $2^b + 2$. Theorem 5 shows that the expected error is bounded from below by a constant regardless of m, when $n = 1$ and B is a constant.

We now show that the expected error can be made arbitrarily small as m and n go to infinity simultaneously.

Theorem 6 *Under the communication budget of $B = d$ bits per transmission, there exists a randomized estimator $\hat{\theta}$ such that*

$$\mathbb{E}\left[\|\hat{\theta} - \theta^*\|^2\right]^{1/2} = O\left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{m}}\right).$$

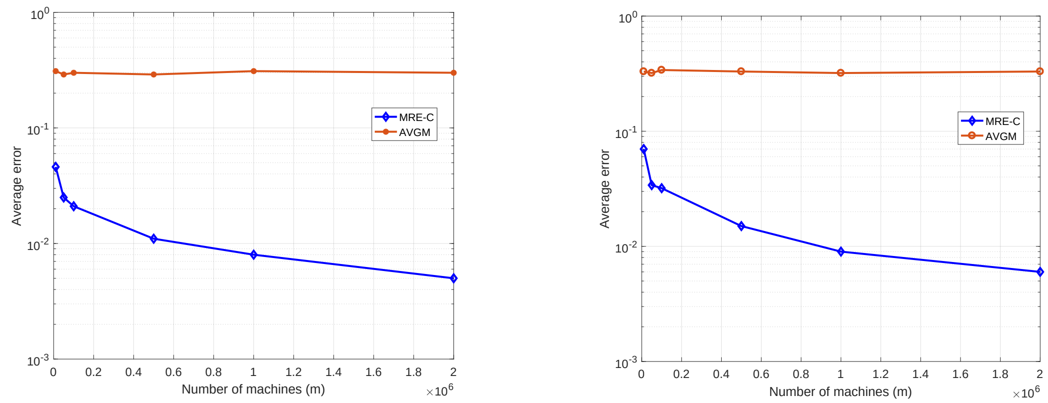
The proof is given in Appendix L. There, we propose a simple randomized algorithm in which each machine i first computes an $O(1/\sqrt{n})$-accurate estimation θ^i based on its observed functions. It then generates as its output signal a random binary sequence of length d whose jth entry is 1 with probability $(1 + \theta^i_j)/2$, where θ^i_j is the jth entry of θ^i. The server then computes $\hat{\theta}$ based on the average of received signals.

6. Experiments

We evaluated the performance of MRE-C on two learning tasks and compared with the averaging method (AVGM) in (Zhang et al., 2012). Recall that in AVGM, each machine sends the empirical risk minimizer of its own data to the server and the average of received parameters at the server is returned in the output.

The first experiment concerns the problem of ridge regression. Here, each sample (X, Y) is generated based on a linear model $Y = X^T\theta^* + E$, where X, E, and θ^* are sampled from $N(\mathbf{0}, I_{d\times d})$, $N(0, 0.01)$, and uniform distribution over $[0,1]^d$, respectively. We consider square loss function with l_2 norm regularization: $f(\theta) = (\theta^T X - Y)^2 + 0.1\|\theta\|_2^2$. In the second experiment, we perform a logistic regression task, considering sample vector X generated according to $N(\mathbf{0}, I_{d\times d})$ and labels Y randomly drawn from $\{-1, 1\}$ with probability $\Pr(Y = 1|X, \theta^*) = 1/(1 + \exp(-X^T\theta^*))$. In both experiments, we consider a two dimensional domain ($d = 2$) and assumed that each machine has access to one sample ($n = 1$).

In Fig. 3, the average of $\|\hat{\theta} - \theta^*\|_2$ is computed over 100 instances for the different number of machines in the range $[10^4, 10^6]$. Both experiments suggest that the average error of MRE-C keep decreasing as the number of machines increases. This is consistent with the result in Theorem 3, according to which the expected error of MRE-C is upper bounded by $\tilde{O}(1/\sqrt{mn})$. It is evident from the error curves that MRE-C outperforms the AVGM algorithm in both tasks. This is because where m is much larger than n, the expected error of the AVGM algorithm typically scales as $O(1/n)$, independent of m.



(a) Ridge regression

(b) Logistic regression

Figure 3: The average of MRE-C and AVGM algorithms versus the number of machines in two different learning tasks.

7. Discussion

We studied the problem of statistical optimization of convex loss landscapes in a distributed system with one-shot communications. We present matching upper and lower bounds on estimation error under general communication constraints. We showed that the expected error of any estimator is lower bounded by $\|\hat{\theta} - \theta^*\| = \tilde{\Omega}\left(\max\left(n^{-1/2}(mB)^{-1/d},\, (mn)^{-1/2}\right)\right)$. We proposed an algorithm called MRE-C , whose estimation errors meet the above lower bound up to a poly-logarithmic factor. More specifically, the MRE-C algorithm has error no large than $\|\hat{\theta} - \theta^*\| = \tilde{O}\left(\max\left(n^{-1/2}(mB)^{-1/d},\, (mn)^{-1/2}\right)\right)$. Aside from being order optimal, the MRE-C algorithm has the advantage over the existing estimators that its error tends to zero as the number of machines goes to infinity, even when the number of samples per machine is upper bounded by a constant and communication budget is limited to $d \log mn$ bits per transmission. This property is in line with the out-performance of the MRE-C algorithm in the $m \gg n$ regime, as verified in our experimental results.

The key insight behind the proof of the lower bound and the design of our algorithm is an observation that emerges from our proofs: in the one-shot model, finding an $O\big(n^{-1/2}m^{-1/d}\big)$-accurate minimizer of F is as difficult as finding an $O\big(n^{-1/2}m^{-1/d}\big)$-accurate approximation of ∇F for all points in an $n^{-1/2}$-neighborhood of θ^*. Capitalizing on this observation, the MRE-C algorithm computes, in an efficient way, an approximation of the gradient of the expected loss over a neighborhood of θ^*. It then output a minimizer of approximate gradient norms as its estimate of the loss minimizer. It is quite counter intuitive that while MRE-C algorithm carries out an intricate and seemingly redundant task of approximating the loss function for all points in a region, it is still very efficient, and indeed order optimal in terms of estimation error and sample complexity. This remarkable observation is in line with the above insight that finding an approximate minimizer is as hard as finding an approximation of the function over a relatively large neighborhood of the minimizer.

We also addressed the problem of distributed learning under tiny (constant) communication budget. We showed that when budget B is a constant and for $n = 1$, the expected error of any estimator is lower bounded by a constant, even when m goes to infinity. We then proposed an estimator with the budget of $B = d$ bits pert transmission and showed that its expected error is no larger than $O\left(n^{-1/2} + m^{-1/2}\right)$.

Our algorithms and bounds are designed and derived for a broader class of functions with Lipschitz continuous first order derivatives, compared to the previous works that consider function classes with Lipschitz continuous second or third order derivatives. The assumption is indeed both practically important and technically challenging. For example, it is well-known that the loss landscapes involved in learning applications and neural networks are highly non-smooth. Therefore, relaxing assumptions on higher order derivatives is actually a practically important improvement over the previous works. On the other hand, considering Lipschitzness only for the first order derivative renders the problem way more difficult. To see this, note that when $n > m$, the existing upper bound $O\left((mn)^{-1/2} + n^{-1}\right)$ for the case of Lipschitz second derivatives goes below the $O(m^{-1/d}n^{-1/2})$ lower bound in the case of Lipschitz first derivatives.

A drawback of the MRE algorithms is that each machine requires to know m in order to set the number of levels for the grids. This however can be resolved by considering infinite number of levels, and letting the probability that p is chosen from level l decrease exponentially with l. The constant lower bound in Theorem 5 decreases exponentially with B. This we expect because when $B = d\log mn$, error of the MRE-C algorithm is proportional to an inverse polynomial of m and n (see Theorem 3, and therefore decays exponentially with B.

There are several open problems and directions for future research. The first group of problems involve the constant bit regime. It would be interesting if one could verify whether or not the bound in Theorem 6 is order optimal. We conjecture that this bound is tight, and no estimator has expected error smaller than $o\left(n^{-1/2} + m^{-1/2}\right)$, when the communication budget is bounded by a constant. This would essentially be an extension of Theorem 5 for $n > 1$. Another interesting problem involves the regime $B < d$, and best accuracy achievable with $B < d$ bits per transmission?

As for the MRE-C estimator, the estimation error of these algorithms are optimal up to poly-logarithmic factors in m and n. However, the bounds in Theorem 3 have an extra exponential dependency on d. Removing this exponential dependency is an important problem to address in future works.

More importantly, an interesting problem involves the relaxation of the convexity assumption (Assumption 2) and finding tight lower bounds and order-optimal estimators for general non-convex loss landscapes, in the one-shot setting. This we address in an upcoming publication (see Sharifnassab et al. (2020) for first drafts).

Another important group of problems concerns a more restricted class of functions with Lipschitz continuous second order derivatives. Despite several attempts in the literature, the optimal scaling of expected error for this class of functions in the $m \gg n$ regime is still an open problem.

Acknowledgments

This research was supported by INSF under contract No. 97012846. The first author thanks Nitin Vaidya for giving invaluable insights about the problem of distributed learning. The second author thanks John N. Tsitsiklis and Philippe Rigollet for fruitful discussions on Lemma 10.

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Appendices

Appendix A. Preliminaries

In this appendix, we review some preliminaries that will be used in the proofs of our main results.

A.1 Concentration inequalities

We collect two well-known concentration inequalities in the following lemma.

Lemma 7 *(Concentration inequalities)*

(a) (Hoeffding's inequality) Let $X_1, \cdots, X_n$ be independent random variables ranging over the interval $[a, a+\gamma]$. Let $\bar{X} = \sum_{i=1}^n X_i/n$ and $\mu = \mathbb{E}[\bar{X}]$. Then, for any $\alpha > 0$,

$$\Pr\left(|\bar{X} - \mu| > \alpha\right) \leq 2\exp\left(\frac{-2n\alpha^2}{\gamma^2}\right).$$

(b) (Theorem 4.2 in Motwani and Raghavan (1995)) Let $X_1, \cdots, X_n$ be independent Bernoulli random variables, $X = \sum_{i=1}^n X_i$, and $\mu = \mathbb{E}[X]$. Then, for any $\alpha \in (0,1]$,

$$\Pr\left(X < (1-\alpha)\mu\right) \leq \exp\left(-\frac{\mu\alpha^2}{2}\right).$$

A.2 Binary hypothesis testing

Let P_1 and P_2 be probability distributions over $\{0,1\}$, such that $P_1(0) = 1/2 - 1/5\sqrt{mn}$ and $P_2(0) = 1/2 + 1/5\sqrt{mn}$. The following lemma provides a lower bound on the error probability of binary hypothesis tests between P_1 and P_2. By the error probability of a test, we mean the maximum between the probabilities of type one and type two errors.

Lemma 8 *Consider probability distributions P_1 and P_2 as above. Then, for any binary hypothesis test $\mathcal{T}$ between P_1 and P_2 with mn samples, the error probability $\mathcal{T}$ is at least* $1/3$.

Proof Let P_1^{mn} and P_2^{mn} be the product-distributions of mn samples. Then, $\mathbb{E}\big[P_1^{mn}\big] = mn\left(1/2 - 1/5\sqrt{mn}\right)$, $\mathbb{E}\big[P_2^{mn}\big] = mn\left(1/2 + 1/5\sqrt{mn}\right)$, and $\mathrm{var}\left(P_1^{mn}\right) = \mathrm{var}\left(P_2^{mn}\right) = mn/4 - 1/25$. For large values of mn, the central limit theorem implies that the CDFs of P_1^{mn} and P_2^{mn} converge to the CDFs of $N_1 \triangleq N\big(mn(1/2 - 1/\sqrt{mn}),\, mn/4 - 1/25\big)$ and $N_2 \triangleq N\big(mn(1/2 + 1/\sqrt{mn}),\, mn/4 - 1/25\big)$, respectively. Therefore, for large enough values of mn,

$$\begin{aligned} d_{TV}\big(P_1^{mn}, P_2^{mn}\big) \;&\leq\; d_{TV}\big(N_1, N_2\big) \;+\; 0.02 \\ &=\; 1 - 2Q\left(\frac{1/5\sqrt{mn}}{\sqrt{1/4mn - 1/\big(25(mn)^2\big)}}\right) \;+\; 0.02 \\ &>\; 1 - 2Q\,(0.4) \;+\; 0.02. \end{aligned} \tag{12}$$

where $d_{TV}\big(P_1^{mn}, P_2^{mn}\big)$ is the total variation distance between P_1^{mn} and P_2^{mn}, $Q(\cdot)$ is the Q-function of normal distributions, and the equality is because the variances of N_1 and N_2 are the same and the two distributions have the same values at $1/5\sqrt{mn}$. On the other hand, it is well-known that the maximum of the two type of errors in any binary hypothesis test is lower bounded by $1/2(1-\delta)$, where δ is the total variation distance between the underlying distributions corresponding to the hypotheses. Therefore, in our case, for large enough values of mn, the error probability of $\mathcal{T}$ is at least

$$\frac{1-d_{TV}\big(P_1^{mn},P_2^{mn}\big)}{2} > \frac{2Q\,(0.4)\,-\,0.02}{2} \;>\; 0.3445-0.01 \;>\frac{1}{3}. \tag{13}$$

This completes the proof of Lemma 8. ■

A.3 Fano's inequality

In the rest of this appendix, we review a well-known inequality in information theory: Fano's inequality (Cover and Thomas, 2012). Consider a pair of random variables X and Y with certain joint probability distribution. Fano's inequality asserts that given an observation of Y no estimator $\hat{x}$ can recover x with probability of error less than $\big(H(X|Y)-1\big)/\log(|X|)$, i.e.,

$$\Pr(e) \,\triangleq\, \Pr\big(\hat{x}\neq x\big) \,\geq\, \frac{H(X|Y)-1}{\log(|X|)},$$

where $H(X|Y)$ is the conditional entropy and $|X|$ is the size of probability space of X. In the special case that X has uniform marginal distribution, the above inequality further simplifies as follows:

$$\begin{aligned}
H(X|Y) &= H(X,Y)-H(Y) \geq^a H(X)-H(Y)\\
&=^b \log(|X|)-H(Y) \geq \log(|X|)-\log(|Y|)\\
&\implies \Pr(e) \geq \frac{H(X|Y)-1}{\log(|X|)} \geq \frac{\log(|X|)-\log(|Y|)-1}{\log(|X|)} = 1-\frac{\log(|Y|)+1}{\log(|X|)},
\end{aligned} \tag{14}$$

(a) Since $H(X,Y)\geq H(X)$.
(b) X has uniform distribution.

Appendix B. Proof of Theorem 1

Here, we present the proof of Theorem 1. The high level idea is that if there is an algorithm that finds a minimizer of F with high probability, then there is an algorithm that finds a fine approximation of ∇F over $O(1/\sqrt{n})$-neighborhood of θ^*. The key steps of the proof are as follows.

We first consider a sub-collection $\mathcal{S}$ of $\mathcal{F}$ such that for any pair f,g of functions in $\mathcal{S}$, there is a point θ in the $O(1/\sqrt{n})$-neighborhood of θ^* such that $\|\nabla f(\theta)-\nabla g(\theta)\|\geq \epsilon/\sqrt{n}$. We develop a metric-entropy based framework to show that such collection exists and can have as many as $\Omega(1/\epsilon^d)$ functions. Consider a constant $\epsilon>0$ and suppose that there exists an estimator $\hat{\theta}$ that finds an $O(\epsilon/\sqrt{n})$-approximation of θ^* with high probability, for all distributions. We generate a distribution P that associates probability $1/2$ to an arbitrary function $f\in\mathcal{S}$, while distributes the remaining $1/2$ probability unevenly over $2d$ linear functions. The priory unknown probability distribution of these linear functions can displace the minimum of F in an $O(1/\sqrt{n})$-neighborhood. Capitalizing on this observation, we show that the server needs to obtain an $\big(\epsilon/\sqrt{n}\big)$-approximation of ∇f all over this $O(1/\sqrt{n})$-neighborhood; because otherwise the server could mistake f for another function $g\in\mathcal{S}$, which leads to $\Omega\big(\epsilon/\sqrt{n}\big)$-error in $\hat{\theta}$ for specific choices of probability distribution over the linear functions. Therefore, the server needs to distinguish which function f out of $1/\epsilon^d$ functions in $\mathcal{S}$ has positive probability in P. Using information theoretic tools (Fano's inequality (Cover and Thomas, 2012)) we conclude that the total number of bits delivered to the server (i.e., mb bits) must exceed the size of $\mathcal{S}$ (i.e., $\Omega(1/\epsilon^d)$). This implies that $\epsilon \geq (mb)^{1/d}$, and no estimator has error less than $O\big((mb)^{-1/d}n^{-1/2}\big)$.

B.1 Preliminaries

Before going through the details of the proof, in this subsection we present some definitions and auxiliary lemmas. Here, we will only consider a sub-collection of functions in $\mathcal{F}$ whose derivatives vanish at zero, i.e. $\nabla f(\mathbf{0}) = \mathbf{0}$, where $\mathbf{0}$ is the all-zeros vector. Throughout the proof, we fix constant

$$c \triangleq 4d\log(mn). \tag{15}$$

Let $\mathcal{F}_\lambda$ be a sub-collection of functions in $\mathcal{F}$ that are λ-strongly convex, that is for any $f \in \mathcal{F}_\lambda$ and any $\theta_1, \theta_2 \in [0,1]^d$, we have $f(\mathbf{0}) = 0$, $\nabla f(\mathbf{0}) = \mathbf{0}$ and $f(\theta_2) \geq f(\theta_1) + (\theta_2 - \theta_1)^T \nabla f(\theta_1) + \lambda\|\theta_2 - \theta_1\|^2$.

Definition 9 ((ϵ,δ)-packing and metric entropy) *Given $\epsilon, \delta > 0$, a subset $S \subseteq \mathcal{F}_\lambda$ is said to be an (ϵ,δ)-packing if for any $f \in S$, $f(\mathbf{0}) = 0$ and $\nabla f(\mathbf{0}) = \mathbf{0}$; and for any $f, g \in S$, there exists $x \in [-\delta,\delta]^d$ such that $\|\nabla f(x) - \nabla g(x)\| \geq \epsilon$. We denote an (ϵ,δ)-packing with maximum size by $S^*_{\epsilon,\delta}$, and refer to $K_{\epsilon,\delta} \triangleq \log|S^*_{\epsilon,\delta}|$ as the (ϵ,δ)-metric entropy.*

Lemma 10 *For any $\epsilon, \delta \in (0,1)$ with $10\sqrt{d}\epsilon \leq \delta$, we have $K_{\epsilon,\delta} \geq \Big(\delta/20\epsilon\sqrt{d}\Big)^d$.*

The proof is given in Appendix C. The proof is constructive. There we devise a set of functions in $\mathcal{F}_\lambda$ as convolutions of a collection of impulse trains by a suitable kernel, and show that they form a packing.

Choose an arbitrary ϵ such that

$$\epsilon < \frac{1}{80d^{1.5}\log^{1+3/d}(mn)}\left(\frac{1}{50mB}\right)^{1/d}, \tag{16}$$

and consider an $\big(\epsilon/\sqrt{n}, 1/(c\sqrt{n})\big)$-packing with maximum size and denote it by S^* (see Definition 9), where c is the constant in (15). We fix this ϵ and S^* for the rest of the proof. Then,

$$K_{\epsilon/\sqrt{n},1/(c\sqrt{n})} \;\geq\; \left(\frac{1/\left(c\sqrt{n}\right)}{20\sqrt{d}\;\times\left(\epsilon/\sqrt{n}\right)}\right)^d > 50mB\,\log^3(mn), \tag{17}$$

where the first inequality follows from Lemma 10 and the second inequality is by substituting the values of c and ϵ from (15) and (16).

We now define a collection $\mathcal{C}$ of probability distributions over $\mathcal{F}_\lambda$.

Definition 11 (Collection $\mathcal{C}$ of probability distributions) *Let $\mathbf{e}_i$ be a vector whose i-th entry equals 1 and all other entries equal zero. Consider a pair of linear functions $g_i^+(x) = d\mathbf{e}_i^T x$ and*

$g_i^-(x) = -d\mathbf{e}_i^T x$. Then, the collection $\mathcal{C}$ consists of probability distributions P of the following form:

$$P : \begin{cases} P(f) = \frac{1}{2}, & \text{for some } f \in S^* \\ P(g_i^+) \in \left[\frac{1}{4d} - \frac{1}{2c\sqrt{n}}, \frac{1}{4d} + \frac{1}{2c\sqrt{n}}\right], & i = 1, \dots, d, \\ P(g_i^-) = \frac{1}{2d} - P(g_i^+), & i = 1, \dots, d, \end{cases}$$

where c is the constant defined in (15)*. For each $f \in S^*$, we refer to any such P as a corresponding distribution of f.*

Note each $f \in S^*$, corresponds to infinite number of distributions in $\mathcal{C}$. In order to simplify the presentation, we use the shorthand notations $P_0 = P(f) = 1/2$, $P_i^+ = P(g_i^+)$, and $P_i^- = P(g_i^-)$, for $i = 1, \dots, d$. The following lemma provides a bound on the distance of distinct distributions in $\mathcal{C}$, in a certain sense.

Lemma 12 *Consider a function $f \in S^*$ and two corresponding distributions $P, P' \in \mathcal{C}$. We draw n i.i.d. samples from P. Let $n_0, n_{1+}, \dots, n_{d+}, n_{1-}, \dots, n_{d-}$ be the number of samples that equal $f, g_1^+, \dots, g_d^+, g_1^-, \dots, g_d^-$, respectively. Let $\underline{n} = [n_0, n_{1+}, n_{1-}, \dots, n_{d+}, n_{d-}]$. Then,*

$$\Pr_{\underline{n} \sim P}\left(\frac{1}{2} \le \frac{P(\underline{n})}{P'(\underline{n})} \le 2\right) = 1 - \exp\Big(-\Omega\big(\log^2(mn)\big)\Big).$$

The proof is based on the Hoeffding's inequality (Lemma 7 (a)) and is given in Appendix D.

B.2 Proof of $\Omega\big((mB)^{-1/d} n^{-1/2}\big)$ bound

Recall the definition of ϵ and c in (16) and (15), respectively. Here, we prove the difficult part of the theorem and show that for large enough values of mn, for any estimator there is a probability distribution under which with probability at least $1/3$ we have

$$\|\hat{\theta} - \theta^*\| \ge \epsilon/(2c\sqrt{n}) = \frac{1}{640 \times 50^{1/d}\, d^{2.5}\, \log^{2+2/d}(mn)} \times \frac{1}{\sqrt{n}\,(mB)^{1/d}}. \tag{18}$$

In order to draw a contradiction, suppose that there exists an estimator $\hat{E}_1$ such that in a system of m machines and n samples per machine, $\hat{E}_1$ has error less than $\epsilon/(2c\sqrt{n})$ with probability at least $2/3$, for all distributions P satisfying Assumptions 1 and 2. Note that since $\hat{E}_1$ cannot beat the estimation error $1/\sqrt{mn}$ of the centralized solution, it follows that

$$\epsilon \ge \frac{2c}{\sqrt{m}} \ge m^{-1/2}. \tag{19}$$

We will show that $\epsilon = \tilde{\Omega}(m^{-1/d})$.

We first improve the confidence of $\hat{E}_1$ via repetitions to obtain an estimator $\hat{E}_2$, as in the following lemma.

Lemma 13 *There exists an estimator $\hat{E}_2$ such that in a system of $m\log^2(mn)$ machines and n samples per machine, $\hat{E}_2$ has estimation error less than $\epsilon/(c\sqrt{n})$ with probability at least $1 - \exp\left(-\Omega(\log^2 mn)\right)$, for all distributions P satisfying Assumptions 1 and 2.*

The proof is fairly standard, and is given in Appendix E.

For any $f \in S^*$, consider a probability distribution $P \in \mathcal{C}$ such that: $P(f) = 1/2$, $P_{i^+} = P_{i^-} = 1/(4d)$, $i = 1, \ldots, d$. Suppose that each machine observes n samples from this distribution. Let $n_0, n_{1^+}, \ldots, n_{d^+}, n_{1^-}, \ldots, n_{d^-}$ be the number of that equal from $f, g_1^+, \ldots, g_d^+, g_1^-, \ldots, g_d^-$, respectively. We refer to $\underline{n} = [n_0, n_{1^+}, n_{1^-}, \ldots, n_{d^+}, n_{d^-}]$ as the observed frequency vector of this particular machine. We denote by $Y(f, \underline{n}^j)$ the signal generated by estimator $\hat{E}_2$ (equivalently by $\hat{E}_1$) at machine j, corresponding to the distribution P and the observed frequency vector $\underline{n}^j$.

Definition 14 *Consider a system of* $8m\log^2(mn)$ *machines. For any* $f \in S^*$*, we define* $\mathcal{W}_f$ *as the collection of pairs* $\big(\underline{n}^1, Y(f, \underline{n}^1)\big), \ldots, \big(\underline{n}^{8m\log^2(mn)}, Y(f, \underline{n}^{8m\log^2(mn)})\big)$ *that are generated via the above procedure.*

We now present the main technical lemma of this proof. It shows that employing $\hat{E}_2$, given $\mathcal{W}_f$, we can uniquely recover f out of all functions in S^*, with high probability.

Lemma 15 *There exists an algorithm* $\hat{E}_3$ *that for any* $f \in S^*$*, given* $\mathcal{W}_f$*, it outputs an* $h \in S^*$ *such that* $h = f$ *with probability* $1 - \exp(-\Omega(\log^2(mn)))$.

Proof Consider the collection $\mathcal{C}$ of probability distributions defined in Definition 11. The high level idea is as follows. We first show that for any distribution $P' \in \mathcal{C}$ corresponding to f, there is a sub-sampling of $\mathcal{W}_f$ such that the sub-sampled pairs are i.i.d. and have distribution P'. As a result, employing estimator $\hat{E}_2$, we can find the minimizer of $\mathbb{E}_{g\sim P'}\big[g(\cdot)\big]$. On the other hand, for any $x \in [-1/(c\sqrt{n}), 1/(c\sqrt{n})]^d$, we can choose a $P' \in \mathcal{C}$ such that x be the minimizer of $\mathbb{E}_{g\sim P'}\big[g(\cdot)\big]$, or equivalently $\nabla\mathbb{E}_{g\sim P'}\big[g(x)\big] = \nabla f(x)/2 + \nabla h(x) \simeq 0$, for some known linear function h (see Definition 11). Therefore, the fact that $\hat{E}_2$ we can find the minimizer of $\mathbb{E}_{g\sim P'}\big[g(\cdot)\big]$ for all $P' \in \mathcal{C}$ implies that $\hat{E}_2$ has sufficient information to obtain a good approximation of $\nabla f(x)$, for all $x \in [-1/(c\sqrt{n}), 1/(c\sqrt{n})]^d$, with high probability. This enables us to recover ∇f from $\mathcal{W}_f$ with high probability via some algorithm that we then call $\hat{E}_3$. We now go over the details of the proof.

Consider a cube $A = \big[-d/(c\sqrt{n}), d/(c\sqrt{n})\big]^d$ and suppose that A^* is a minimum $\epsilon\lambda/(4\sqrt{n})$-covering[4] of A. A regular grid yields a simple bound on the size of A^*:

$$|A^*| \leq (8\sqrt{d}/(c\lambda\epsilon))^d. \tag{20}$$

Let $P \in \mathcal{C}$ be the probability distribution with $P(f) = 1/2$ and $P_{i^+} = P_{i^-} = 1/(4d)$, for $i = 1, \ldots, d$. Moreover, for any $v \in A^*$ consider the probability distributions $P^v \in \mathcal{C}$: $P^v(f) = 1/2$, $P^v_{i^+} = 1/(4d) + v_i/4$, and $P^v_{i^-} = 1/(4d) - v_i/4$, for $i = 1, \ldots, d$ (note that $P^v_{i^+}, P^v_{i^+} \geq 0$, because $v \in A$ and $c > d$).

It follows from Lemma 12 that for any observed frequency vector $\underline{n}$ in $\mathcal{W}_f$, we have with probability at least $1 - \exp(-\Omega(\log^2(mn)))$,

$$\frac{1}{4} \leq \frac{P^v(\underline{n})}{2P(\underline{n})} \leq 1. \tag{21}$$

We sub-sample $\mathcal{W}_f$, and discard from $\mathcal{W}_f$ any pair $\big(\underline{n}, Y(f, \underline{n})\big)$ whose $\underline{n}$ does not satisfy (21). Otherwise, if $\underline{n}$ satisfies (21), we then keep the pair $\big(\underline{n}, Y(f, \underline{n})\big)$ with probability $P^v(\underline{n})/(2P(\underline{n}))$. We denote the set of surviving samples by $\mathcal{W}_f^v$.

4. By a covering, we mean a set A^* such that for any $x \in A$, there is a point $p \in A^*$ such that $\|x - p\| \leq \epsilon\lambda/(4\sqrt{n})$.

Claim 1 *With probability* $1 - \exp\big(\Omega(-\log^2(mn))\big)$, *at least* $m\log^2(mn)$ *pairs* $\big(\underline{n}, Y(f, \underline{n})\big)$ *survive the above sub-sampling procedure; these pairs are i.i.d and the corresponding* $\underline{n}$*'s have distribution* P^v.

The proof is given in Appendix F.1.

Let $\hat{x}_v$ be the output of the server of estimator $\hat{E}_2$ to the input $\mathcal{W}_f^v$. It follows from Lemma 13 and Claim 1 that with probability at least $1 - \exp\big(\Omega(-\log^2(mn))\big)$,

$$\|\hat{x}_v - x^*_{v,f}\| \le \frac{\epsilon}{c\sqrt{n}}, \tag{22}$$

where $x^*_{v,f}$ is the minimizer of function $\mathbb{E}_{g\sim P^v}[g(x)] = \frac{1}{2}(f(x) + v^T x)$. By repeating this process for different v's, we compute $\hat{x}_v$ for all v in A^*. We define event $\mathcal{E}$ as follows:

$$\mathcal{E}:\ \|\hat{x}_v - x^*_{v,f}\| \le \frac{\epsilon}{c\sqrt{n}},\ \forall v \in A^*.$$

Then,

$$\begin{aligned}
\Pr\big(\mathcal{E}\big) &\ge 1 - |A^*|\exp(-\Omega(\log^2(mn)))\\
&\ge 1 - \left(\frac{8\sqrt{d}}{\lambda c\epsilon}\right)^d \exp\big(-\Omega(\log^2(mn))\big)\\
&= 1 - \Big(\frac{8\sqrt{d}}{4\lambda d}\Big)^d \Big(\frac{1}{\epsilon\log(mn)}\Big)^d \exp\big(-\Omega(\log^2(mn))\big)\\
&\ge 1 - \Big(\frac{2}{\lambda\sqrt{d}}\Big)^d \Big(\frac{\sqrt{m}}{\log(mn)}\Big)^d \exp\big(-\Omega(\log^2(mn))\big)\\
&= 1 - \exp\big(-\Omega(\log^2(mn))\big),
\end{aligned}$$

where the first four relations follow from the union bound, (20), the definition of c in (15), and (19), respectively.

The algorithm $\hat{E}_3$ then returns, as its final estimation of f, an $\hat{f} \in S^*$ of the form

$$\hat{f} \in \operatorname*{argmin}_{g\in S^*} \max_{v\in A^*} \|\hat{x}_v - x^*_{v,g}\| \tag{23}$$

We now bound the error probability of $\hat{E}_3$, and show that $\hat{f} = f$ with probability at least $1 - \exp(-\Omega(\log^2(mn)))$.

Claim 2 *For any* $g \in S^*$ *with* $g \neq f$, *there is a* $v \in A^*$ *such that* $\|x^*_{v,g} - x^*_{v,f}\| \ge \epsilon/(2\sqrt{n})$.

The proof is given in Appendix F.2.

Suppose that event $\mathcal{E}$ has occurred and consider a $g \in S^*$ with $g \neq f$. Then, it follows from Claim 2 that there is a $v \in A^*$ such that

$$\begin{aligned}
\|\hat{x}_v - x^*_{v,g}\| &\ge \|x^*_{v,g} - x^*_{v,f}\| - \|x^*_{v,f} - \hat{x}_v\|\\
&\ge^a \frac{\epsilon}{2\sqrt{n}} - \|x^*_{v,f} - \hat{x}_v\|\\
&\ge^b \frac{\epsilon}{2\sqrt{n}} - \frac{\epsilon}{c\sqrt{n}}\\
&>^c \frac{\epsilon}{c\sqrt{n}}.
\end{aligned}$$

(a) Due to Claim 2.
(b) According to the definition of event $\mathcal{E}$.
(c) Based on the definition of $c = 4d\log(mn)$ in (15), we have $c > 4$.
Therefore, with probability at least $\Pr(\mathcal{E}) = 1 - \exp\big(-\Omega(\log^2(mn))\big)$, for any $g \in S^*$ with $g \neq f$,

$$\max_{v\in A^*} \|\hat{x}_v - x^*_{v,g}\| \;>\; \frac{\epsilon}{c\sqrt{n}} \;\geq\; \max_{v\in A^*} \|\hat{x}_v - x^*_{v,f}\|.$$

It then follows from (23) that $\hat{f} = f$ with probability $1 - \exp\big(-\Omega(\log^2(mn))\big)$. This shows that the error probability of $\hat{E}_3$ is $\exp\big(-\Omega(\log^2(mn))\big)$, and completes the proof of Lemma 15. ■

Back to the proof of Theorem 1, consider a random variable X that has uniform distribution over the set of functions S^* and a random variable W over the domain $\{\mathcal{W}_f\}_{f\in S^*}$ with the following distribution:

$$\Pr(W|X) = \Pr\big(\mathcal{W}_f = W \mid f = X\big).$$

Note that each $\mathcal{W}_f$ consists of at most $8m\log^2(mn)$ pairs, each containing a signal Y of length B bits and a vector $\underline{n}$ of $2d+1$ entries ranging over $[0,n]$. Therefore, each $\mathcal{W}_f$ can be expressed by a string of length at most $8m\log^2(mn)\big(B + (2d+1)\log(n+1)\big)$ bits. As a result, we have the following upper bound on the size $|W|$ of the state space of random variable X:

$$\log(|W|) \;\leq\; 8m\log^2(mn)\Big(B + (2d+1)\log(n+1)\Big). \tag{24}$$

Based on Lemma 15, there exists an estimator $\hat{E}_3$ which observes W and returns the correct X with probability at least $1 - \exp\big(-\Omega(\log^2(mn))\big)$. Let $\Pr(e)$ be the probability of error of this estimator. Then,

$$\Pr(e) \;\leq\; \exp\big(-\Omega(\log^2(mn))\big) \;<\; \frac{1}{2}, \tag{25}$$

for large enough mn. On the other hand, it follows from the Fano's inequality in (14) that

$$\begin{aligned}
\Pr(e) \;&\geq\; 1 - \frac{\log(|W|)+1}{\log(|X|)}\\
&\geq\; 1 - \frac{8m\log^2(mn)\Big(B + (2d+1)\log(n+1)\Big)}{\log(|X|)}\\
&\geq\; 1 - \frac{25mB\log^3(mn)}{\log(|X|)}\\
&=\; 1 - \frac{25mB\log^3(mn)}{K_{\epsilon/\sqrt{n},1/(c\sqrt{n})}}\\
&>\; 1 - \frac{25mB\log^3(mn)}{50mB\log^3(mn)}\\
&=\; \frac{1}{2},
\end{aligned} \tag{26}$$

where the first equality is by Definition 9, the second inequality follows from (24), and the last inequality is due to (17). Eq. (26) contradicts (25). Hence, our initial assumption that there is an estimator $\hat{E}_1$ with accuracy $\epsilon/(2c\sqrt{n})$ and confidence $2/3$ is incorrect. This implies (18) and completes the proof of Theorem 1.

B.3 Proof of $\Omega\big(1/\sqrt{mn}\big)$ bound

We now go over the easier part of the theorem and show the $1/\sqrt{mn}$ lower bound on the estimation error. The $\Omega\left(1/\sqrt{mn}\right)$ barrier is actually well-known to hold in several centralized scenarios. Here we adopt the bound for our setting. Without loss of generality, suppose that the communication budget is infinite and the system is essentially centralized. Consider functions $f_1, f_2 \in \mathcal{F}$, such that $f_1(\theta) = \|\theta - \mathbf{1}\|^2/4\sqrt{d}$ and $f_1(\theta) = \|\theta + \mathbf{1}\|^2/4\sqrt{d}$, for all $\theta \in [-1,1]^d$. We define two probability distributions P_1 and P_2 as follows

$$P_1 : \begin{cases} \Pr(f_1) = 1/2 - 1/5\sqrt{mn}, \\ \Pr(f_2) = 1/2 + 1/5\sqrt{mn}, \end{cases}$$

$$P_2 : \begin{cases} \Pr(f_1) = 1/2 + 1/5\sqrt{mn}, \\ \Pr(f_2) = 1/2 - 1/5\sqrt{mn}. \end{cases}$$

Then, the minimizers of $\mathbb{E}_{f\sim P_1}\big(f(\cdot)\big)$ and $\mathbb{E}_{f\sim P_2}\big(f(\cdot)\big)$ are $\theta_1 \triangleq -\mathbf{1}/5\sqrt{mn}$ and $\theta_2 \triangleq \mathbf{1}/5\sqrt{mn}$, respectively. Therefore, $\|\theta_1 - \theta_2\| = 2\sqrt{d}/5\sqrt{mn}$.

We now show that no estimator has estimation error less than $\sqrt{d}/5\sqrt{mn}$ with probability at least $2/3$. In order to draw a contradiction, suppose that there is an estimator $\hat{E}$ for which $\|\hat{\theta} - \theta^*\| < \sqrt{d}/5\sqrt{mn}$ with probability at least $2/3$. Based on $\hat{E}$ we devise a binary hypothesis test $\mathcal{T}$ as follows. This $\mathcal{T}$ tests between hypothesis $\mathcal{H}_1$ that sample are drown from distribution P_1 and hypothesis $\mathcal{H}_2$ that sample are drown from distribution P_2. It works as follows: for the output $\hat{\theta}$ of $\hat{E}$, $\mathcal{T}$ chooses $\mathcal{H}_1$ if $\|\hat{\theta} - \theta^1\| < \|\hat{\theta} - \theta^2\|$, and $\mathcal{H}_2$ otherwise. Since $\|\hat{\theta} - \theta^*\| < \sqrt{d}/5\sqrt{mn}$ with probability at least $2/3$, it follows that the error success of $\mathcal{T}$ is at least $2/3$. This contradicts Lemma 8, according to which no binary hypothesis test can distinguish between P_1 and P_2 with probability at least $2/3$. Therefor, there is no estimator for which $\|\hat{\theta} - \theta^*\| < \sqrt{d}/5\sqrt{mn}$ with probability at least $2/3$. Combined with (18), this completes the proof of Theorem 1.

Appendix C. Proof of Lemma 10

We assume

$$10\sqrt{d}\epsilon \leq \delta \leq 1, \tag{27}$$

and show that

$$K_{\epsilon,\delta} \geq \Big(\frac{1}{20\sqrt{d}}\Big)^d \Big(\frac{\delta}{\epsilon}\Big)^d.$$

We begin by a claim on existence of a kernel function with certain properties.

Claim 3 *There exists a continuously twice differentiable function $h : \mathbb{R}^d \to \mathbb{R}$ with the following properties:*

$$
\begin{aligned}
h(x) &= 0, && \textit{for} \quad x \notin (-1,1)^d, && (28)\\
|h(x)| &\le 1, && \textit{for} \quad x \in \mathbb{R}^d && (29)\\
\|\nabla h(0)\| &> \frac{1}{2}, && && (30)\\
\|\nabla h(x)\| &\le 3, && \textit{for} \quad x \in \mathbb{R}^d, && (31)\\
-4I_{d\times d} \preceq \nabla^2 h(x) &\preceq 4I_{d\times d}, && \textit{for} \quad x \in \mathbb{R}^d. && (32)
\end{aligned}
$$

Proof We show that the following function satisfies (28)–(32):

$$
h(x) = \begin{cases} \frac{8}{27}\Big(1 - \frac{9}{4}\,\|x + \frac{1}{3}e_1\|^2\Big)^3, & \text{if } \|x + \frac{1}{3}e_1\|_2 \le \frac{2}{3}, \\ 0, & \text{otherwise,} \end{cases} \tag{33}
$$

where e_1 is a vector whose first entry equals one and all other entries equal zero. Note that if $\|x + e_1/3\| \ge 2/3$, then we have $h(x) = 0$, $\nabla h(x) = \mathbf{0}_{n\times 1}$, and $\nabla^2 h(x) = \mathbf{0}_{n\times n}$. Therefore, the function value and its the first and second derivatives are continuous. Hence, h is continuously twice differentiable. The gradient and Hessian of function h are as follows:

$$
\nabla h(x) = -4\Big(1 - \frac{9}{4}\|x + \frac{1}{3}e_1\|^2\Big)^2\Big(x + \frac{1}{3}e_1\Big) \tag{34}
$$

$$
\nabla^2 h(x) = 36\Big(1 - \frac{9}{4}\|x + \frac{1}{3}e_1\|^2\Big)\big(x + e_1/3\big)\big(x + e_1/3\big)^T - 4\Big(1 - \frac{9}{4}\|x + \frac{1}{3}e_1\|^2\Big)^2 I. \tag{35}
$$

We now examine properties (28)–(32). For (28), note that if $x \notin (-1,1)^d$, then $\|x+e_1/3\| \ge 2/3$, and as a result, $h(x) = 0$. Eq. (29) is immediate from the definition of h in (33). Property (30) follows because $\|\nabla h(0)\| = 3/4 > 1/2$. For (31), consider any x such that $\nabla h(x) \neq 0$. Then, $\|x + e_1/3\| \le 2/3$, and (34) implies that

$$
\|\nabla h(x)\| \;=\; 4\left(1 - \frac{9}{4}\|x + \frac{1}{3}e_1\|^2\right)^2 \|x + e_1/3\| \;\le\; 4 \times \frac{2}{3} \;<\; 3.
$$

Based on (35), at any point x where $\|x + e_1/3\| < 2/3$, the largest and smallest eigenvalues of the Hessian matrix are

$$
\begin{aligned}
\lambda_{min} &= -4\Big(1 - \frac{9}{4}\|x\|^2\Big)^2 \ge -4,\\
\lambda_{max} &= 36\Big(1 - \frac{9}{4}\|x + e_1/3\|^2\Big)\|x + e_1/3\|^2 - 4\Big(1 - \frac{9}{4}\|x + e_1/3\|^2\Big)^2.
\end{aligned} \tag{36}
$$

Letting $\alpha = \big(3\|x + e_1/3\|/2\big)^2$, we have

$$
\lambda_{max} \;\le\; \sup_{\alpha\in[0,1]} 16(1-\alpha)\alpha - 4(1-\alpha)^2 \;\le\; \sup_{\alpha\in[0,1]} 4 - 4(1-\alpha)^2 \;\le\; 4. \tag{37}
$$

Property (32) follows from (36) and (37). This completes the proof of the claim. ■

Consider a function h as in Claim (3), and let

$$k(x) = 10\sqrt{d}\epsilon^2\, h\left(\frac{x}{10\sqrt{d}\epsilon}\right),$$

for all $x \in \mathbb{R}^d$. Also let $\epsilon' = 10\sqrt{d}\epsilon$. Then, (27) implies that $\delta \geq \epsilon'$. It follows from Claim 3 that $k(\cdot)$ is continuously twice differentiable, and

$$\begin{aligned}
k(x) &= 0, && \text{for} \quad x \notin (-\epsilon', \epsilon')^d, && (38)\\
|k(x)| &\leq 10\sqrt{d}\epsilon^2, && \text{for} \quad x \in \mathbb{R}^d && (39)\\
\|\nabla k(0)\| &> \frac{\epsilon}{2}, && && (40)\\
\|\nabla k(x)\| &\leq 3\epsilon, && \text{for} \quad x \in \mathbb{R}^d, && (41)\\
-\frac{4}{10\sqrt{d}}\, I_{d\times d} \preceq \nabla^2 k(x) &\preceq \frac{4}{10\sqrt{d}}\, I_{d\times d}, && \text{for} \quad x \in \mathbb{R}^d. && (42)
\end{aligned}$$

Consider a regular $2\epsilon'$-grid G inside $[-\delta, \delta]^d$, such that the coordinates of points in G are odd multiples of ϵ', e.g., $[\epsilon', \epsilon', \ldots, \epsilon'] \in G$. Let M be the collection of all functions $s : G \to \{-1, +1\}$ that assign ± 1 values to each grid point in G. Therefore , $\mathcal{M}$ has size

$$|\mathcal{M}| \;=\; 2^{|G|} \;=\; 2^{\left(2\left\lfloor\frac{\delta+\epsilon'}{2\epsilon'}\right\rfloor\right)^d} \;\geq\; 2^{\left(\frac{\delta}{2\epsilon'}\right)^d}, \tag{43}$$

where in the last inequality is because $\delta \geq \epsilon'$.

For any function $s \in \mathcal{M}$, we define a function $f_s : \mathbb{R}^d \to \mathbb{R}$ of the following form

$$f_s(x) \;=\; \left(\sum_{p\in G} s(p) k(x-p)\right) + \frac{1}{4\sqrt{d}}\|x\|^2. \tag{44}$$

There is an equivalent representation for f_s as follows. For any $x \in \mathbb{R}^d$, let $\pi(x)$ be the closest point to x in G. Then, it follows from (38) that for any $x \in \mathbb{R}^d$,

$$f_s(x) \;=\; s\big(\pi(x)\big)\, k\big(x - \pi(x)\big) \;+\; \frac{1}{4\sqrt{d}}\|x\|^2. \tag{45}$$

Claim 4 *For any $s \in \mathcal{M}$, we have $f_s \in \mathcal{F}_\lambda$, $f_s(\mathbf{0}) = 0$, and $\nabla f_s(\mathbf{0}) = \mathbf{0}$.*

Proof First note that $\pi(0) = [\epsilon', \ldots, \epsilon'] = \epsilon'\mathbf{1}$, and it follows from (45) that

$$\begin{aligned}
f_s(\mathbf{0}) \;&=\; k(\epsilon'\mathbf{1}) = 0,\\
\nabla f(\mathbf{0}) \;&=\; \nabla k(\epsilon'\mathbf{1}) \;=\; \mathbf{0},
\end{aligned}$$

where the second and last equalities are due to (38). Moreover, since $k(\cdot)$ is continuously twice differentiable, so is f_s. We now show that $f_s \in \mathcal{F}_\lambda$, i.e.,

$$\begin{aligned}
|f_s(x)| \;&\leq\; \sqrt{d}, && (46)\\
\|\nabla f_s(x)\| \;&\leq\; 1, && (47)\\
\lambda I_{d\times d} \;\preceq\; \nabla^2 f_s(x) \;&\preceq\; \frac{1}{\sqrt{d}} I_{d\times d}, && (48)
\end{aligned}$$

for all $x \in [-1,1]^d$

From (45) and (39) , we have for $x \in [-1,1]^d$,

$$|f_s(x)| \;\le\; |k(x)| + \frac{1}{4\sqrt{d}}\|x\|^2 \;\le\; 10\sqrt{d}\epsilon^2 + \frac{\sqrt{d}}{4} \;\le^a\; \epsilon + \frac{\sqrt{d}}{4} \;\le^b\; \sqrt{d},$$

and

$$\begin{aligned}\|\nabla f_s(x)\| \;&=\; \left\|\nabla k(x-\pi(x)) + \frac{x}{2\sqrt{d}}\right\| \\ &\le\; \|\nabla k(x-\pi(x))\| + \frac{\|x\|}{2\sqrt{d}} \\ &\le^c 3\epsilon + \frac{\sqrt{d}}{2\sqrt{d}} \\ &\le^d 1,\end{aligned}$$

where (a), (b), and (d) are due to the assumption $10\sqrt{d}\epsilon \le 1$ in (27), and (c) follows from (41). For (48), we have from (45),

$$\nabla^2 f(x) \;=\; s\big(\pi(x)\big)\nabla^2 k\big(x-\pi(x)\big) + \frac{1}{2\sqrt{d}}I.$$

It then follows from (42) that for any $x \in [-1,1]^d$,

$$\lambda I \;\preceq\; \frac{1}{10\sqrt{d}}I \;=\; \frac{-4}{10\sqrt{d}}I + \frac{1}{2\sqrt{d}} \;\preceq\; \nabla^2 f(x) \;\preceq\; \frac{4}{10\sqrt{d}}I + \frac{1}{2\sqrt{d}}I \;\prec\; \frac{1}{\sqrt{d}}I.$$

where the first inequality is from the assumption $\lambda \le 1/\big(10\sqrt{d}\big)$ in the statement of Theorem 1. Therefore $f_s \in \mathcal{F}_\lambda$ and the claim follows. ■

Let $\mathcal{S}_{\epsilon,\delta}$ be the collection of functions f_s defined in (44), for all $s \in \mathcal{M}$; i.e.,

$$\mathcal{S}_{\epsilon,\delta} \;=\; \big\{f_s \;\mid\; s \in \mathcal{M}\big\}. \tag{49}$$

We show that $\mathcal{S}_{\epsilon,\delta}$ is an (ϵ,δ)-packing. Consider a pair of distinct functions $s_1, s_2 \in M$. Then, there exists a point $p \in G$ such that $s_1(p) \ne s_2(p)$; equivalently, $|s_1(p) - s_2(p)| = 2$. Therefore,

$$\begin{aligned}\|\nabla f_{s_1}(p) - \nabla f_{s_2}(p)\| \;&=\; \|s_1(p)\nabla k(0) - s_2(p)\nabla k(0)\| \\ &=\; |s_1(p) - s_2(p)| \times \|\nabla k(0)\| \\ &>\; 2 \times \frac{\epsilon}{2} \\ &=\; \epsilon,\end{aligned}$$

where the first equality is due to (45), and the inequality follows from (41). This shows that for any pair of distinct $f, g \in S_{\epsilon,\delta}$ functions, $\|\nabla f(p) - \nabla g(p)\| \ge \epsilon$, for some $p \in [-\delta,\delta]^d$. Therefore, $S_{\epsilon,\delta}$ is an (ϵ,δ)-packing.

Finally, it follows from (43) that

$$K_{\epsilon,\delta} \;\ge\; \log\big(|S_{\epsilon,\delta}|\big) \;=\; \log\big(|M|\big) \;\ge\; \left(\frac{\delta}{2\epsilon'}\right)^d \;=\; \left(\frac{1}{20\sqrt{d}}\right)^d \left(\frac{\delta}{\epsilon}\right)^d,$$

and Lemma 10 follows.

Appendix D. Proof of Lemma 12

We define an event $\mathcal{E}_0$ as follows:

$$\left|n_{i^+} - \frac{n}{4d}\right| \le \frac{\sqrt{n}\log(mn)}{16d}, \quad \text{and} \quad \left|n_{i^-} - \frac{n}{4d}\right| \le \frac{\sqrt{n}\log(mn)}{16d}, \quad i = 1, \cdots, d.$$

We first show that $\mathcal{E}_0$ occurs with high probability. Let n_i be the i-th entry of $\underline{n}$ for $i \ge 1$. Then,

$$\begin{aligned}
\Pr_{\underline{n}\sim P}\left(\left|n_i - \frac{n}{4d}\right| \ge \frac{\sqrt{n}\log(mn)}{16d}\right) &\le \Pr_{\underline{n}\sim P}\left(\Big|n_i - \mathbb{E}_P[n_i]\Big| + \left|\mathbb{E}_P[n_i] - \frac{n}{4d}\right| \ge \frac{\sqrt{n}\log(mn)}{16d}\right) \\
&\le^a \Pr_{\underline{n}\sim P}\left(\Big|n_i - \mathbb{E}_P[n_i]\Big| \ge \frac{\sqrt{n}}{16d}\Big(\log(mn) - \frac{2}{\log(mn)}\Big)\right) \\
&\le^b \Pr_{\underline{n}\sim P}\left(\Big|n_i - \mathbb{E}_P[n_i]\Big| \ge \frac{\sqrt{n}\log(mn)}{32d}\right) \\
&= \Pr_{\underline{n}\sim P}\left(\frac{1}{n}\Big|n_i - \mathbb{E}_P[n_i]\Big| \ge \frac{\log(mn)}{32d\sqrt{n}}\right) \\
&\le^c \exp\left(-2n\left(\frac{\log(mn)}{32\sqrt{n}d}\right)^2\right) \\
&= \exp\Big(-\Omega\big(\log^2(mn)\big)\Big),
\end{aligned}$$

where (a) is due to that fact that $\big|\mathbb{E}[n_i] - n/(2d)\big| = \big|P_i n - n/(4d)\big| \le n/(2c\sqrt{n}) = \sqrt{n}/\big(8d\log(mn)\big)$, (b) is valid for $mn \ge 4$, and (c) follows from the Hoeffding's inequality. It then follows from the union bound that

$$\Pr(\mathcal{E}_0) \;\ge\; 1 - d\exp\Big(-\Omega\big(\log^2(mn)\big)\Big) \;=\; 1 - \exp\Big(-\Omega\big(\log^2(mn)\big)\Big). \tag{50}$$

Consider a pair $\underline{n}$ and $\underline{n}'$ of vectors that satisfy $\mathcal{E}_0$. Then,

$$
\begin{aligned}
\frac{P(\underline{n})/P'(\underline{n})}{P(\underline{n}')/P'(\underline{n}')} &= \frac{\prod_{i=0}^{2d}\left(P_i/P_i'\right)^{n_i}}{\prod_{i=0}^{2d}\left(P_i/P_i'\right)^{n_i'}} \\
&= \prod_{i=0}^{2d}\left(\frac{P_i}{P_i'}\right)^{n_i-n_i'} \\
&=^a \prod_{i=1}^{2d}\left(\frac{P_i}{P_i'}\right)^{n_i-n_i'} \\
&\leq^b \prod_{i=1}^{2d}\left(\frac{1/(4d)+\left(2c\sqrt{n}\right)}{1/(4d)-1/\left(2c\sqrt{n}\right)}\right)^{2\sqrt{n}\log(mn)/(16d)} \\
&= \prod_{i=1}^{2d}\left(\frac{1+2d/\left(c\sqrt{n}\right)}{1-2d/\left(c\sqrt{n}\right)}\right)^{\sqrt{n}\log(mn)/(8d)} \\
&\leq^c \prod_{i=1}^{2d}\left(1+4\times\frac{2d}{c\sqrt{n}}\right)^{\sqrt{n}\log(mn)/(8d)} \\
&\leq^d \exp\left(2d\frac{8d}{c\sqrt{n}}\times\frac{\sqrt{n}\log(mn)}{8d}\right) \\
&= \exp\left(\frac{16d^2\log(mn)}{32d^2\log(mn)}\right) \\
&= \sqrt{e},
\end{aligned}
$$

(a) Due to the fact that: $P_0 = P_0' = 1/2$.
(b) Since $\underline{n}$ and $\underline{n}'$ satisfy event $\mathcal{E}_0$ and $P, P' \in \mathcal{C}$.
(c) Because $2d/c \leq 1/2$.
(d) Due to the fact that $1 + x \leq \exp(x)$.
Therefore,

$$
\begin{aligned}
\sup_{\underline{n}\in\mathcal{E}_0}\frac{P(\underline{n})}{P'(\underline{n})} &\leq \sqrt{e}\ \inf_{\underline{n}\in\mathcal{E}_0}\frac{P(\underline{n})}{P'(\underline{n})} \\
&\leq \sqrt{e}\ \frac{\sum_{\underline{n}\in\mathcal{E}_0}P(\underline{n})}{\sum_{\underline{n}\in\mathcal{E}_0}P'(\underline{n})} \\
&= \sqrt{e}\ \frac{P(\mathcal{E}_0)}{P'(\mathcal{E}_0)} \\
&\leq \sqrt{e}\Big(1-\exp\big(-\Omega(\log^2(mn))\big)\Big) \\
&\leq 2,
\end{aligned}
$$

where the inequality before the last one is due to (50) and the last inquality is valid for sufficiently large values of mn. Interchanging the roles of P and P', it follows that $\inf \underline{n} \in \mathcal{E}_0 P(\underline{n})/P'(\underline{n}) \geq 1/2$. This completes the proof of Lemma 12.

Appendix E. Proof of Lemma 13

The server subdivides the $m\log^2(mn)$ machines into $\log^2(mn)$ groups of m machines, and employs $\hat{E}_1$ to obtain an estimate $\hat{\theta}_i$ for each group $i = 1, \dots, \log^2(mn)$. Let $k = \log(mn)^2$ and without loss of generality suppose that k is an even integer. Consider a d-dimensional ball B of smallest radius that encloses at least $k/2+1$ points from $\hat{\theta}_1, \dots, \hat{\theta}_k$. Let $\hat{\theta}$ be the center of B. The estimator $\hat{E}_2$ then outputs $\hat{\theta}$ as an estimation of θ^*.

We now show that $\|\hat{\theta} - \theta^*\| \leq \epsilon/(c\sqrt{n})$ with high probability. Let B' be the ball of radius $\epsilon/(2c\sqrt{n})$ centered at θ^*, and let q be the number of point from $\hat{\theta}_1, \dots, \hat{\theta}_k$ that lie in B'. Since the error probability of $\hat{E}_1$ is less than $1/3$, we have $\mathbb{E}[q] \geq 2k/3$. Then,

$$\begin{aligned}
\Pr\big(q \leq k/2\big) \;&\leq\; \Pr\big(q - \mathbb{E}[q] \leq -k/6\big)\\
&\leq\; \exp\left(\frac{-2k}{36}\right)\\
&=\; \exp\left(\frac{-\log^2(mn)}{18}\right)\\
&=\; \exp\Big(-\Omega\big(\log^2(mn)\big)\Big),
\end{aligned}$$

where the second inequality follows from the Hoeffding's inequality.

Therefore, with probability $1 - \exp\big(-\Omega(\log^2 mn)\big)$, B' encloses at least $k/2+1$ points from $\hat{\theta}_1, \dots, \hat{\theta}_k$. In this case, by definition, the radius r of B would be no larger that the radius fo B', i.e., $r \leq \epsilon/(2c\sqrt{n})$. Moreover, since B and B' each encapsulate at least $k/2+1$ points out of k points, they intersect (say at point p). Then, with probability at least $1 - \exp\big(-\Omega(\log^2 mn)\big)$,

$$\|\hat{\theta} - \theta^*\| \;\leq\; \|\hat{\theta} - p\| + \|p - \theta^*\| \;\leq\; r + \frac{\epsilon}{2c\sqrt{n}} \;\leq\; \frac{\epsilon}{c\sqrt{n}},$$

and the lemma follows.

Appendix F. Missing Parts of the Proof of Lemma 15

F.1 Proof of Claim 1

Since $\mathcal{W}_f$ consists of $8m\log^2(mn)$ samples, the union bound implies that all received $\underline{n}$'s satisfy (21) with probability at least $1 - 8m\log^2(mn)\exp\big(-\Omega(\log^2(mn))\big) = 1 - \exp\big(-\Omega(\log^2(mn))\big)$. Thus, with probability at least $1 - \exp\big(-\Omega(\log^2(mn))\big)$, the survived sub-sampled pairs are i.i.d. with distribution P^v. Moreover, if we denote the number of sub-sampled pairs by t, then,

$$\mathbb{E}[t] \;\geq\; \left(\frac{1}{4} - \exp\Big(-\Omega(-\log^2(mn))\Big)\right) \times 8m\log^2(mn) \;\geq\; \frac{3}{2}m\log^2(mn),$$

for large enough values of mn. Then,

$$\begin{aligned}
\Pr\big(t < m\log^2(mn)\big) \;&\leq\; \Pr\left(t - \mathbb{E}[t] \leq -\frac{1}{2}m\log^2(mn)\right) \\
&\leq\; \Pr\left(\frac{t-\mathbb{E}[t]}{8m\log^2(mn)} \leq -\frac{1}{16}\right) \\
&\leq\; \exp\left(-16m\log^2(mn)\left(\frac{1}{16}\right)^2\right) \\
&=\; \exp\Big(-\Omega\big(\log^2(mn)\big)\Big),
\end{aligned} \tag{51}$$

where the last inequality is due to Hoeffding's inequality. This completes the proof of the claim.

F.2 Proof of Claim 2

According to the definition of S^* and the fact that $g, f \in S^*$ and $f \neq g$, there exists $x \in [-1/(c\sqrt{n}), 1/(c\sqrt{n})]^d$ such that:

$$\|\nabla f(x) - \nabla g(x)\| \geq \epsilon/\sqrt{n}. \tag{52}$$

It follows from the assumption $\nabla f(0) = 0$ that

$$\|\nabla f(x)\| \;=\; \|\nabla f(x) - \nabla f(0)\| \leq^a \|x\| \;\leq\; \frac{\sqrt{d}}{c\sqrt{n}},$$

where the first inequality is due to the Lipschitz continuity of derivative of f (cf. Assumption 1). Then,

$$-\nabla f(x) \in A. \tag{53}$$

Let $v \in A^*$ be the closest point of A^* to $-\nabla f(x)$. Since A^* is an $\big(\epsilon\lambda/(4\sqrt{n})\big)$-covering of A, it follows from (53) that

$$\|\nabla f(x) + v\| \;\leq\; \frac{\epsilon\lambda}{4\sqrt{n}}. \tag{54}$$

According to Assumption 1, f is λ-strongly convex. Then,

$$\|x - x^*_{v,f}\| \;\leq\; \frac{\|\nabla f(x) - \nabla f(x^*_{v,f})\|}{\lambda} \;=\; \frac{\|\nabla f(x) + v\|}{\lambda} \;\leq\; \frac{\epsilon\lambda/(4\sqrt{n})}{\lambda} \;=\; \frac{\epsilon}{4\sqrt{n}}, \tag{55}$$

where the first equality is because $x^*_{v,f}$ is a minimizer of $f(x) + v^T x$. On the other hand, it follows from (52) and (54) that

$$\begin{aligned}
\|\nabla g(x) + v\| \;&\geq\; \|\nabla g(x) - \nabla f(x)\| - \|\nabla f(x) + v\| \\
&\geq\; \frac{\epsilon}{\sqrt{n}} - \frac{\epsilon\lambda}{4\sqrt{n}} \\
&\geq\; \frac{\epsilon}{\sqrt{n}} - \frac{\epsilon}{4\sqrt{n}} \\
&=\; \frac{3\epsilon}{4\sqrt{n}},
\end{aligned}$$

where the last inequality is because ∇f is Lipschtz continuous with constant 1 and as a result, $\lambda \leq 1$. Then,

$$\|x_{v,g}^* - x\| \geq \|\nabla g(x) + v\| \geq \frac{3\epsilon}{4\lambda\sqrt{n}} \geq \frac{3\epsilon}{4\sqrt{n}}. \tag{56}$$

Combining (55) and (56), we obtain

$$\|x_{v,g}^* - x_{v,f}^*\| \geq \|x_{v,g}^* - x\| - \|x_{v,f}^* - x\| \geq \frac{3\epsilon}{4\sqrt{n}} - \frac{\epsilon}{4\sqrt{n}} = \frac{\epsilon}{2\sqrt{n}}.$$

This completes the proof of Claim 2.

Appendix G. Proof of Theorem 3

We first show that s^* is a closest grid point of G to θ^* with high probability. We then show that for any $l \leq t$ and any $p \in \tilde{G}_{s^*}^l$, the number of sub-signals corresponding to p after redundancy elimination is large enough so that the server obtains a good approximation of ∇F at p. Once we have a good approximation of ∇F at all points of $\tilde{G}_{s^*}^t$, a point with the minimum norm for this approximation lies close to the minimizer of F.

Recall the definition of s^* as the grid point of G that appears for the most number of times in the s component of the received signals. Let m^* be the number of machines that select $s = s^*$. We let $\mathcal{E}'$ be the event that $m^* \geq m/2^d$ and θ^* lies in the $\big(2\log(mn)/\sqrt{n}\big)$-cube C_{s^*} centered at s^*, i.e.,

$$\|s^* - \theta^*\|_\infty \leq \frac{\log(mn)}{\sqrt{n}}. \tag{57}$$

Then,

Lemma 16

$$\Pr\left(\mathcal{E}'\right) = 1 - \exp\big(-\Omega(\log^2(mn))\big). \tag{58}$$

The proof relies on concentration inequalities, and is given in Appendix H.

We now turn our focus to the inside of cube C_{s^*}. Let

$$\epsilon \triangleq \frac{2\sqrt{d}\,\log(mn)}{\sqrt{n}} \times \delta = 4d^{1.5}\log^4(mn)\,\max\left(\frac{1}{\sqrt{n}\,(mB)^{1/d}}, \frac{2^{d/2}}{\sqrt{mn}}\right). \tag{59}$$

For any $p \in \bigcup_{l\leq t} \tilde{G}_{s^*}^l$, let N_p be the number of machines that select point p in at least one of their sub-signals. Equivalently, N_p is the number of sub-signals after redundancy elimination that have point p as their second argument. Let $\mathcal{E}''$ be the event that for any $l \leq t$ and any $p \in \tilde{G}_{s^*}^l$, we have

$$N_p \geq \frac{4d^2 2^{-2l}\log^6(mn)}{n\epsilon^2}. \tag{60}$$

Then,

Lemma 17 $\Pr\left(\mathcal{E}''\right) = 1 - \exp\big(-\Omega(\log^2(mn))\big)$.

The proof is based on the concentration inequality in Lemma 7 (b), and is given in Appendix I.

Capitalizing on Lemma 17, we now obtain a bound on the estimation error of gradient of F at the grid points in $\tilde{G}^l_{s^*}$. Let $\mathcal{E}'''$ be the event that for any $l \leq t$ and any grid point $p \in \tilde{G}^l_{s^*}$, we have

$$\big\|\hat{\nabla} F(p) - \nabla F(p)\big\| \,<\, \frac{\epsilon}{4}.$$

Lemma 18 $\Pr\big(\mathcal{E}'''\big) \,=\, 1 - \exp\big(-\Omega(\log^2(mn))\big)$.

The proof is given in Appendix J and relies on Hoeffding's inequality and the lower bound on the number of received signals for each grid point, driven in Lemma 17.

In the remainder of the proof, we assume that (57) and $\mathcal{E}'''$ hold. Let p^* be the closest grid point in $\tilde{G}^t_{s^*}$ to θ^*. Therefore,

$$\|p^* - \theta^*\| \,\leq\, \sqrt{d}\, 2^{-t}\frac{\log(mn)}{\sqrt{n}} \,=\, \delta \times \frac{\sqrt{d}\,\log(mn)}{\sqrt{n}} \,=\, \epsilon/2. \tag{61}$$

Then, it follows from $\mathcal{E}'''$ that

$$\begin{aligned}
\|\hat{\nabla} F(p^*)\| &\leq \big\|\hat{\nabla} F(p^*) - \nabla F(p^*)\big\| \,+\, \|\nabla F(p^*)\| \\
&\leq\, \epsilon/4 + \|\nabla F(p^*)\| \\
&=\, \epsilon/4 + \big\|\nabla F(p^*) - \nabla F(\theta^*)\big\| \\
&\leq\, \epsilon/4 + \big\|p^* - \theta^*\big\| \\
&\leq\, \epsilon/4 + \epsilon/2 \\
&=\, 3\epsilon/4,
\end{aligned} \tag{62}$$

where the second inequality is due to $\mathcal{E}'''$, the third inequality follows from the Lipschitz continuity of ∇F, and the last inequality is from (61). Therefore,

$$\begin{aligned}
\|\hat{\theta} - \theta^*\| \,&\leq\, \frac{1}{\lambda}\,\big\|\nabla F(\hat{\theta}) - \nabla F(\theta^*)\big\| \\
&=\, \frac{1}{\lambda}\,\|\nabla F(\hat{\theta})\| \\
&\leq\, \frac{1}{\lambda}\,\|\hat{\nabla} F(\hat{\theta})\| + \frac{1}{\lambda}\,\big\|\hat{\nabla} F(\hat{\theta}) - \nabla F(\hat{\theta})\big\| \\
&\leq^a\, \frac{1}{\lambda}\,\|\hat{\nabla} F(\hat{\theta})\| + \frac{\epsilon}{4\lambda} \\
&\leq^b\, \frac{1}{\lambda}\,\|\hat{\nabla} F(p^*)\| + \frac{\epsilon}{4\lambda} \\
&\leq^c\, \frac{3\epsilon}{4\lambda} + \frac{\epsilon}{4\lambda} \\
&=\, \frac{\epsilon}{\lambda},
\end{aligned}$$

(a) Due to event $\mathcal{E}'''$.
(b) Because the output of the server, $\hat{\theta}$, is a grid point p in $\tilde{G}^t_{s^*}$ with smallest $\|\hat{\nabla} F(p)\|$.
(c) According to (62).

Finally, it follows from (57) and Lemma 18 that the above inequality holds with probability $1 - \exp\big(-\Omega(\log^2(mn))\big)$. Equivalently,

$$\Pr\left(\|\hat{\theta} - \theta^*\| \geq \frac{\epsilon}{\lambda}\right) \;=\; \exp\Big(-\Omega\big(\log^2(mn)\big)\Big),$$

and Theorem 3 follows.

Appendix H. Proof of Lemma 16

Suppose that machine i observes functions $f_1^i, \dots, f_n^i$. Recall the definition of θ^i in (7). The following proposition provides a bound on $\theta^i - \theta^*$, which improves upon the bound in Lemma 8 of (Zhang et al., 2013).

Claim 5 *For any $i \leq m$,*

$$\Pr\left(\|\theta^i - \theta^*\| \geq \frac{\alpha}{\sqrt{n}}\right) \;\leq\; d\exp\left(\frac{-\alpha^2\lambda^2}{d}\right),$$

where λ is the lower bound on the curvature of F (cf. Assumption 1).

Proof [Proof of Claim] Let $F^i(\theta) = \sum_{j=1}^{n/2} f_j^i(\theta)$, for all $\theta \in [-1,1]^d$. From the lower bound λ on the second derivative of F, we have

$$\|\nabla F(\theta^i) - \nabla F^i(\theta^i)\| \;=\; \|\nabla F(\theta^i)\| \;=\; \|\nabla F(\theta^i) - \nabla F(\theta^*)\| \;\geq\; \lambda\|\theta^i - \theta^*\|,$$

where the two equalities are because θ^i and θ^* are the the minimizers of F^i and F, respectively. Then,

$$\begin{aligned}
\Pr\left(\|\theta^i - \theta^*\| \geq \frac{\alpha}{\sqrt{n}}\right) \;&\leq\; \Pr\left(\|\nabla F(\theta^i) - \nabla F^i(\theta^i)\| \geq \frac{\lambda\alpha}{\sqrt{n}}\right)\\
&\leq^a \; \sum_{j=1}^{d} \Pr\left(\Big|\frac{\partial F^i(\theta^i)}{\partial\theta_j} - \frac{\partial F(\theta^i)}{\partial\theta_j}\Big| > \frac{\alpha\lambda}{\sqrt{d}\sqrt{n}}\right)\\
&=\; d\,\Pr\left(\Big|\frac{2}{n}\sum_{l=1}^{n/2}\frac{\partial}{\partial\theta_j}f_l^i(\theta^i) \;-\; \mathbb{E}_{f\sim P}\big[\frac{\partial}{\partial\theta_j}f(\theta^i)\big]\Big| \geq \frac{\alpha\lambda}{\sqrt{d}\sqrt{n}}\right)\\
&=^b \; d\exp\left(-\frac{\alpha^2\lambda^2}{d}\right),
\end{aligned} \tag{63}$$

(a) Follows from the union bound and the fact that for any d-dimensional vector v, there exists an entry v_i such that $\|v\| \leq |v_i|/\sqrt{d}$.
(b) Due to Hoeffding's inequality (cf. Lemma 7 (a)).
This completes the proof of Claim 5. ■

Based on Claim 5, we can write

$$
\begin{aligned}
\Pr\left(\|\theta^i-\theta^*\| \le \frac{\log(mn)}{2\sqrt{n}},\ \text{for } i=1,\dots,m\right)& \\
&\ge\ 1-m\Pr\left(\|\theta^1-\theta^*\|\ge\frac{\log(mn)}{2\sqrt{n}}\right)\\
&\ge 1-md\exp\left(\frac{-\lambda^2\log^2(mn)}{4d}\right)\\
&=\ 1-\exp\Big(-\Omega\big(\log^2(mn)\big)\Big),
\end{aligned}
\tag{64}
$$

where the first inequality is due to the union bound and the fact that the distributions of $\theta^1,\dots,\theta^m$ are identical, and the second inequality follows from Lemma 5. Thus, with probability at least $1-\exp\big(-\Omega(\log^2(mn))\big)$, every θ^i is in the distance $\log(mn)/2\sqrt{n}$ from θ^*. Recall that for each machine i, all sub-signals of machine i have the same s-component. Hereafter, by the s-component of a machine we mean the s-component of the sub-signals generated at that machine. For each i, let s^i be the s-component of machine i. Therefore, with probability at least $1-\exp\big(-\Omega(\log^2(mn))\big)$, for any machine i,

$$
\begin{aligned}
\Pr\left(\|s^i-\theta^*\|_\infty>\frac{\log(mn)}{\sqrt{n}}\right)\ &\le\ \Pr\left(\|s^i-\theta^i\|_\infty+\|\theta^i-\theta^*\|_\infty>\frac{\log(mn)}{\sqrt{n}}\right)\\
&\le\ \Pr\left(\|s^i-\theta^i\|_\infty>\frac{\log(mn)}{2\sqrt{n}}\right)\\
&\qquad+\Pr\left(\|\theta^i-\theta^*\|_\infty>\frac{\log(mn)}{2\sqrt{n}}\right)\\
&=\ 0+\Pr\left(\|\theta^i-\theta^*\|_\infty>\frac{\log(mn)}{2\sqrt{n}}\right)\\
&=\ \exp\Big(-\Omega\big(\log^2(mn)\big)\Big),
\end{aligned}
$$

where the first equality is due to the choice of s^i as the nearest grid point to θ^i, and the last equality follows from (64). Recall that s^* is the grid point with the largest number of occurrences in the received signals. Therefore, (57) is valid with probability at least $1-\exp\big(-\Omega(\log^2(mn))\big)$. Equivalently, θ^* lies in the $\big(2\log(mn)/\sqrt{n}\big)$-cube C_{s^*} centered at s^*.

Since grid G has block size $\log(mn)/\sqrt{n}$, there are at most 2^d points s of the grid that satisfy $\|s-\theta^*\|_\infty\le\log(mn)/\sqrt{n}$. It then follows from (57) that $\Pr\big(\mathcal{E}'\big)=1-\exp\big(-\Omega(\log^2(mn))\big)$. This completes the proof of Lemma 16.

Appendix I. Proof of Lemma 17

Suppose that the s-component of machine i is $s=s^*$ and assume that $\mathcal{E}'$ is valid. We begin with a simple inequality: for any $x\in[0,1]$ and any $k>0$,

$$
1-(1-x)^k\ \ge\ 1-e^{kx}\ \ge\ \frac{1}{2}\min\big(kx,1\big). \tag{65}
$$

Let Q_p be the probability that p appears in the p-component of at least one of the sub-signals of machine i. Then, for $p \in \tilde{G}^l_{s^*}$,

$$
\begin{aligned}
Q_p \;&=\; 1 - \left(1 - 2^{-dl} \times \frac{2^{(d-2)l}}{\sum_{j=0}^{t} 2^{(d-2)j}}\right)^{\lfloor B/(d\log mn)\rfloor} \\
&\geq\; \frac{1}{2}\min\left(\frac{2^{-2l}\,\lfloor B/(d\log mn)\rfloor}{\sum_{j=0}^{t} 2^{(d-2)j}},\, 1\right) \\
&\geq\; \frac{1}{2}\min\left(\frac{2^{-2l}B}{d\log(mn)\,\sum_{j=0}^{t} 2^{(d-2)j}},\, 1\right)
\end{aligned}
$$

where the equality is due to the probability of a point p in $\tilde{G}^l_{s^*}$ (see (9)) and the number $\lfloor B/(d\log mn)\rfloor$ of sub-signals per machine, and the first inequality is due to (65). Assuming $\mathcal{E}'$, we then have

$$
\mathbb{E}\big[N_p\big] \;=\; Q_p m^* \;\geq\; \min\left(\frac{2^{-2l}mB}{2^{d+1}d\log(mn)\,\sum_{j=0}^{t} 2^{(d-2)j}},\, \frac{m}{2^{d+1}}\right). \tag{66}
$$

We now bound the two terms on the right hand side of (66). For the second term on the right hand side of (66), we have

$$
\begin{aligned}
\frac{m}{2^{d+1}} \;&=\; \frac{m\epsilon^2}{2^{d+1}\epsilon^2} \\
&\geq\; \frac{16\,md^3\log^8(mn)2^d}{2^{d+1}mn\epsilon^2} \\
&>\; \frac{8d^2\log^6(mn)}{n\epsilon^2},
\end{aligned} \tag{67}
$$

where the first inequality is from the definition of ϵ in (59). For the first term at the right hand side of (66), if $d \leq 2$, then

$$
\left(\frac{1}{\delta}\right)^{\max(d,2)} \;=\; \left(\frac{1}{\delta}\right)^{2} \;\leq\; \frac{m}{4d^2\log^6(mn)\,2^d},\; < \; \frac{mB}{4d^2\log^6(mn)\,2^d}. \tag{68}
$$

On the other hand, if $d \geq 3$, then

$$
\left(\frac{1}{\delta}\right)^{\max(d,2)} \;=\; \left(\frac{1}{\delta}\right)^{d} \;\leq\; \frac{mB}{2^d d^d\log^{3d}(mn)} \;<\; \frac{mB}{4d^2\log^6(mn)\,2^d}. \tag{69}
$$

Moreover,

$$
1/\delta \;\leq\; \frac{1}{2d\log^3 mn} \times \frac{\sqrt{m}}{2^{d/2}} \;<\; m. \tag{70}
$$

It follows that for any $d \geq 1$,

$$\begin{aligned}
\sum_{j=0}^{t} 2^{(d-2)j} &\leq t 2^{t \max(d-2,0)} \\
&\leq \log(mn)\, 2^{t \max(d-2,0)} \\
&= \log(mn) \left(\frac{1}{\delta}\right)^{\max(d-2,0)} \\
&= \log(mn)\, \delta^2 \left(\frac{1}{\delta}\right)^{\max(d,2)} \\
&\leq \log(mn)\, \delta^2 \frac{mB}{4d^2 \log^6(mn)\, 2^d} \\
&= \log(mn) \times \frac{n\epsilon^2}{4d \log^2(mn)} \times \frac{mB}{4d^2 \log^6(mn)\, 2^d} \\
&= \frac{nmB\epsilon^2}{16d^3 \log^7(mn)\, 2^d},
\end{aligned}$$

where the second inequality is due to (70) and $t = \log(1/\delta)$. third inequality follows from (68) and (69), the third equality is from the definition of ϵ in (59). Then,

$$\begin{aligned}
\frac{2^{-2l} mB}{2^{d+1} d \log(mn) \sum_{j=0}^{t} 2^{(d-2)j}} &\geq \frac{2^{-2l} mB}{2^{d+1} d \log(mn)} \times \frac{16d^3 \log^7(mn)\, 2^d}{nmB\epsilon^2} \\
&= \frac{8d^2 \log^6(mn)\, 2^{-2l}}{n\epsilon^2}.
\end{aligned} \tag{71}$$

Plugging (67) and (71) into (66), it follows that for $l = 0, \ldots, t$ and for any $p \in \tilde{G}_{s^*}^{l}$,

$$\mathbb{E}\big[N_p\big] \geq \frac{8d^2 \log^6(mn)\, 2^{-2l}}{n\epsilon^2}. \tag{72}$$

Given the bound in (72), for $l = 0, \ldots, t$, we have

$$\begin{aligned}
\frac{1}{8}\mathbb{E}\big[N_p\big] &\geq \frac{d^2 \log^6(mn)\, 2^{-2l}}{n\epsilon^2} \\
&\geq \frac{d^2 \log^6(mn)\, 2^{-2t}}{n\epsilon^2} \\
&= \frac{d^2 \log^6(mn)\, \delta^2}{n^2\epsilon^2} \\
&= \frac{d^2 \log^6(mn)\, \delta^2}{4d \log^2(mn)\, \delta^2} \\
&= \frac{d \log^4(mn)}{4},
\end{aligned} \tag{73}$$

where the first equality is from definition of t and the second equality is by the definition of ϵ in (59). Then, for any $l \in 0, \dots, t$ and any $p \in \tilde{G}_{s^*}^l$,

$$\begin{aligned} \Pr\left(N_p \le \frac{4d^2 \log^6(mn)\, 2^{-2l}}{n\epsilon^2}\right) &\le \Pr\left(N_p \le \frac{\mathbb{E}[N_p]}{2}\right) \\ &\le 2^{-(1/2)^2 \mathbb{E}[N_p]/2} \\ &\le 2^{-d\log^4(mn)/4} \\ &< \exp\big(-d\log^4(mn)/6\big), \end{aligned} \tag{74}$$

where the first inequalities are due to (72), Lemma 7 (b), (73), and the fact that $\ln(2)/4 > 1/6$ respectively. Then,

$$\begin{aligned} \Pr\big(\mathcal{E}'' \mid \mathcal{E}'\big) &= \Pr\left(N_p \ge \frac{4d^2 \log^6(mn)\, 2^{-2l}}{n\epsilon^2}, \quad \forall p \in \tilde{G}_{s^*}^l \text{ and for } l = 0, \dots, t\right) \\ &\ge 1 - \sum_{l=0}^{t} \sum_{p \in \tilde{G}_{s^*}^l} \Pr\left(N_p \le \frac{4d^2 \log^6(mn)\, 2^{-2l}}{n\epsilon^2}\right) \\ &\ge 1 - t 2^{dt} \exp\big(-d\log^4(mn)/6\big) \\ &= 1 - \log(1/\delta)\left(\frac{1}{\delta}\right)^d \exp\big(-d\log^4(mn)/6\big) \\ &\ge 1 - \log(m)\, m^d \exp\big(-d\log^4(mn)/6\big) \\ &= 1 - \exp\Big(-\Omega\big(\log^4(mn)\big)\Big), \end{aligned}$$

where the first equality is by the definition of $\mathcal{E}''$, the first inequality is from union bound, the second inequality due to (74), and the third equality follows from (70). It then follows from Lemma 16 that $\Pr\big(\mathcal{E}''\big) = 1 - \exp\big(-\Omega(\log^2(mn))\big)$ and Lemma 17 follows.

Appendix J. Proof of Lemma 18

For any $l \le t$ and any $p \in \tilde{G}_{s^*}^0$, let

$$\hat{\Delta}(p) = \frac{1}{N_p} \sum_{\substack{\text{Subsignals of the form} \\ (s^*, p, \Delta) \\ \text{after redundancy elimination}}} \Delta,$$

and let $\Delta^*(p) = \mathbb{E}[\hat{\Delta}(p)]$.

We first consider the case of $l = 0$. Note that $\tilde{G}_{s^*}^0$ consists of a single point $p = s^*$. Moreover, the component Δ in each signal is the average over the gradient of $n/2$ independent functions. Then, $\hat{\Delta}(p)$ is the average over the gradient of $N_p \times n/2$ independent functions. Given event $\mathcal{E}''$, for any entry j of the gradient, it follows from Hoeffding's inequality (Lemma 7 (a)) that

$$\begin{aligned}
&\Pr\left(\big|\hat{\Delta}_j(s^*) - \Delta_j^*(s^*)\big| \geq \frac{\epsilon}{4\sqrt{d}\log(mn)}\right)\\
&\qquad\leq\ \exp\left(-N_{s^*}n \times \left(\frac{\epsilon}{4\sqrt{d}\log(mn)}\right)^2\ /\,2^2\right)\\
&\qquad\leq \exp\left(-n\frac{4d^2\log^6(mn)}{n\epsilon^2} \times \frac{\epsilon^2}{16\,d\log^2(mn)}\right)\\
&\qquad= \exp\left(\frac{-d\log^4(mn)}{4}\right)\\
&\qquad=\ \exp\Big(-\Omega\big(\log^4(mn)\big)\Big).
\end{aligned} \tag{75}$$

For $l \geq 1$, consider a grid point $p \in \tilde{G}_{s^*}^l$ and let p' be the parent of p. Then, $\|p-p'\| = \sqrt{d}\,2^{-l}\log(mn)/\sqrt{n}$. Furthermore, for any function $f \in \mathcal{F}$, we have $\|\nabla f(p) - \nabla f(p')\| \leq \|p-p'\|$. Hence, for any $j \leq n$,

$$\Big|\frac{\partial f(p)}{\partial x_j} - \frac{\partial f(p')}{\partial x_j}\Big|\ \leq\ \|p-p'\|\ =\ \frac{\sqrt{d}\log(mn)2^{-l}}{\sqrt{n}}.$$

Therefore, $\hat{\Delta}_j(p)$ is the average of $N_p \times n/2$ independent variables with absolute values no larger than $\gamma \triangleq \sqrt{d}\log(mn)2^{-l}/\sqrt{n}$. Given event $\mathcal{E}''$, it then follows from the Hoeffding's inequality that

$$\begin{aligned}
&\Pr\left(\big|\hat{\Delta}_j(p) - \Delta_j^*(p)\big| \geq\ \frac{\epsilon}{4\sqrt{d}\log(mn)}\right)\\
&\leq\ \exp\left(-nN_p \times \frac{1}{(2\gamma)^2} \times \left(\frac{\epsilon}{4\sqrt{d}\log(mn)}\right)^2\right)\\
&\leq\ \exp\left(-n\frac{4d^2 2^{-2l}\log^6(mn)}{n\epsilon^2} \times \frac{n}{4d2^{-2l}\log^2(mn)} \times \frac{\epsilon^2}{16d\log^2(mn)}\right)\\
&=\ \exp\big(-n\log^2(mn)/16\big)\\
&=\ \exp\Big(-\Omega\big(\log^2(mn)\big)\Big),
\end{aligned}$$

where the second inequality is by substituting N_p from (60). Employing union bound, we obtain

$$\begin{aligned}
&\Pr\left(\big\|\hat{\Delta}(p) - \Delta^*(p)\big\| \geq \frac{\epsilon}{4\log(mn)}\right)\\
&\qquad\leq\ \sum_{j=1}^{d}\Pr\left(\big|\hat{\Delta}_j(p) - \Delta_j^*(p)\big| \geq \frac{\epsilon}{4\sqrt{d}\log(mn)}\right)\\
&\qquad=\ d\,\exp\Big(-\Omega\big(\log^2(mn)\big)\Big)\\
&\qquad= \exp\Big(-\Omega\big(\log^2(mn)\big)\Big).
\end{aligned}$$

Recall from (11) that for any non-zero $l \le t$ and any $p \in \tilde{G}^l_{s^*}$ with parent p',

$$\hat{\nabla} F(p) - \nabla F(p) \;=\; \hat{\nabla} F(p') - \nabla F(p') + \hat{\Delta}(p) - \Delta^*(p).$$

Then,

$$\begin{aligned}
&\Pr\left(\left\|\hat{\nabla} F(p) - \nabla F(p)\right\| > \frac{(l+1)\epsilon}{4\log(mn)}\right) \\
&\quad\le\; \Pr\left(\left\|\hat{\nabla} F(p') - \nabla F(p')\right\| > \frac{l\epsilon}{4\log(mn)}\right) \\
&\qquad\quad + \Pr\left(\left\|\hat{\Delta}(p) - \Delta^*(p)\right\| > \frac{\epsilon}{4\log(mn)}\right) \\
&\quad\le\; \Pr\left(\left\|\hat{\nabla} F(p') - \nabla F(p')\right\| > \frac{l\epsilon}{4\log(mn)}\right) + \exp\Big(-\Omega\big(\log^2(mn)\big)\Big).
\end{aligned}$$

Employing an induction on l, we obtain for any $l \le t$,

$$\Pr\left(\left\|\hat{\nabla} F(p) - \nabla F(p)\right\| > \frac{(l+1)\epsilon}{4\log(mn)}\right) \;\le\; \exp\Big(-\Omega\big(\log^2(mn)\big)\Big).$$

Therefore, for any grid point p,

$$\begin{aligned}
\Pr\left(\left\|\hat{\nabla} F(p) - \nabla F(p)\right\| > \frac{\epsilon}{4}\right) \;&\le\; \Pr\left(\left\|\hat{\nabla} F(p) - \nabla F(p)\right\| > \frac{(t+1)\epsilon}{4\log(mn)}\right) \\
&= \exp\Big(-\Omega\big(\log^2(mn)\big)\Big),
\end{aligned}$$

where the inequality is because $t+1 = \log(1/\delta) + 1 \le \log(mn)$. It then follows from the union bound that

$$\begin{aligned}
\Pr\left(\mathcal{E}''' \mid \mathcal{E}''\right) \;&\ge\; 1 - \sum_{l=0}^{t} \sum_{p \in \tilde{G}^l_{s^*}} \Pr\left(\left\|\hat{\nabla} F(p) - \nabla F(p)\right\| > \frac{\epsilon}{4}\right) \\
&\ge\; 1 - t2^{dt} \exp\Big(-\Omega\big(\log^2(mn)\big)\Big) \\
&=\; 1 - \log(1/\delta)\left(\frac{1}{\delta}\right)^d \exp\Big(-\Omega\big(\log^2(mn)\big)\Big) \\
&\ge\; 1 - m\log(m) \exp\Big(-\Omega\big(\log^2(mn)\big)\Big) \\
&=\; 1 - \exp\Big(-\Omega\big(\log^2(mn)\big)\Big).
\end{aligned} \tag{76}$$

On the other hand, we have from Lemma 17 that $\Pr\left(\mathcal{E}''\right) = 1 - \exp\big(-\Omega(\log^2(mn))\big)$. Then, $\Pr\left(\mathcal{E}'''\right) = 1 - \exp\big(-\Omega(\log^2(mn))\big)$ and Lemma 18 follows.

Appendix K. Proof of Theorem 5

Let $\mathcal{F}_\lambda$ be a sub-collection of functions in $\mathcal{F}$ that are λ-strongly convex. Consider 2^B+2 convex functions in $\mathcal{F}_\lambda$:

$$f(\theta,i) \triangleq \theta^2 + \frac{\theta^i}{i!}, \qquad \text{for} \quad \theta\in[-1,1] \quad \text{and} \quad i=1,\dots,2^B+2.$$

Consider a probability distribution P over these functions that, for each i, associates probability p_i to function $f(\cdot,i)$. With an abuse of the notation, we use P also for a vector with entries p_i. Since $n=1$, each machine observes only one of f_i's and it can send a B-bit length signal out of 2^B possible messages of length B bits. As a general randomized strategy, suppose that each machine sends j-th message with probability a_{ij} when it observes function $f(\cdot,i)$. Let A be a $(2^B+2)\times 2^B$ matrix with entries a_{ij}. Then, each machine sends j-th message with probability $\sum_i p_i a_{ij}$.

At the server side, we only observe the number (or frequency) of occurrences of each message. In view of the law of large number, as m goes to infinity, the frequency of j-th message tends to $\sum_i p_i a_{ij}$, for all $j\le 2^B$. Thus, in the case of infinite number of machines, the entire information of all transmitted signals is captured in the vector A^TP.

Let $\hat{G}$ denote the estimator located in the server, that takes the vector A^TP and outputs an estimate $\hat\theta=\hat{G}(A^TP)$ of the minimizer of $F(\theta)=\mathbb{E}_{x\sim P}\big[f(\theta,x)\big]$. We also let $\theta^*=G(P)$ denote the optimal solution (i.e., the minimizer of F). In the following, we will show that the expected error $\mathbb{E}\big[|\hat\theta-\theta^*|\big]=\mathbb{E}\big[|\hat{G}(A^TP)-G(P)|\big]$ is lower bounded by a universal constant, for all matrices A and all estimators $\hat{G}$.

We say that vector P is central if

$$\sum_{i=1}^{2^B+1} p_i = 1, \qquad \text{and} \qquad p_i \ge \frac{1}{2^B+2}, \quad \text{for} \quad i=1,\cdots,2^B+1. \tag{77}$$

Let $\mathcal{P}_c$ be the collection of central vectors P. We define two constants

$$\begin{aligned}\theta_1 &\triangleq \inf_{P\in\mathcal{P}_c} \operatorname*{argmin}_\theta \sum_{i=1}^{2^B+2} p_i f(\theta,i),\\ \theta_2 &\triangleq \sup_{P\in\mathcal{P}_c} \operatorname*{argmin}_\theta \sum_{i=1}^{2^B+2} p_i f(\theta,i).\end{aligned}$$

For any central P, the minimizer of $\mathbb{E}_{x\sim P}[f(\theta,x)]$ lies in the interval $[\theta_1,\theta_2]$. Furthermore, since functions $f(\cdot,1)$ and $f(\cdot,2)$ have different minimizers, we have $\theta_1\neq\theta_2$. Let

$$\epsilon \triangleq \inf_{\substack{v\in\mathbb{R}^{2^b+2}\\ \|v\|=1}} \ \sup_{\theta\in[\theta_1,\theta_2]} \left| \sum_{i=1}^{2^B+2} v_i\, f'(\theta,i)\right|, \tag{78}$$

where $f'(\theta,i)=d/d\theta\, f(\theta,i)$. We now show that $\epsilon>0$. In order to draw a contradiction, suppose that $\epsilon=0$. In this case, there exists nonzero vector v such that the polynomial $\sum_{i=1}^{2^B+2} v_i f'(\theta,i)$ is equal to zero for all $\theta\in[\theta_1,\theta_2]$. On the other hand, it follows from the definition of $f(\cdot,i)$ that for any nonzero vector v, $\sum_{i=1}^{2^B+2} v_i f'(\theta, i)$ is a nonzero polynomial of degree no larger than $2^B + 1$. As a result, the fundamental theorem of algebra (Krantz, 2012) implies that this polynomial has at most $2^B + 1$ roots and it cannot be zero over the entire interval $[\theta_1, \theta_2]$. This contradict with the earlier statement that the polynomial of interest equals zero throughout the interval $\theta \in [\theta_1, \theta_2]$. Therefore, $\epsilon > 0$.

Let v be a vector of length $2^B + 2$ such that $A^T v = 0$, $\|v\| = 1$, and $\sum_i v_i = 0$. Note that such v exists and lies in the null-space of matrix $[A|\mathbf{1}]^T$, where $\mathbf{1}$ is the vector of all ones. Let θ' be the solution of the following optimization problem

$$\theta' = \operatorname*{argmax}_{\theta \in [\theta_1, \theta_2]} \left| \sum_{i=1}^{2^B+2} v_i f'(\theta, i) \right|,$$

and assume that P is a central vector such that $G(P) = \theta'$. Then, it follows from (78) that

$$\left| \sum_{i=1}^{2^B+2} v_i f'(\theta', i) \right| \geq \epsilon. \tag{79}$$

Let $Q = P + 2^{-(B+2)} v$. Then, from the conditions in (77) and $\|v\| = 1$, we can conclude that Q is a probability vector. Furthermore, based on the definition of v,

$$A^T Q = A^T P + A^T v = A^T P. \tag{80}$$

It then follows from (79) that

$$\left| \frac{d}{d\theta} \mathbb{E}_{i \sim Q}[f(\theta, i)] \Big|_{\theta = \theta'} \right| = \left| \sum_{i=1}^{2^B+2} \left(p_i + \frac{v_i}{2^{B+2}} \right) f'(\theta', i) \right| = \frac{1}{2^{B+2}} \left| \sum_{i=1}^{2^B+2} v_i f'(\theta', i) \right| \geq \frac{\epsilon}{2^{B+2}}, \tag{81}$$

where the last equality is due to the fact that θ' minimizes $\mathbb{E}_{i \sim P}[f(\theta, i)]$.

Let $\theta'' = G(Q)$ be the minimizer of $\mathbb{E}_{i \sim Q}[f(\theta, i)]$. Then,

$$\frac{d}{dt} \mathbb{E}_{i \sim Q}[f(\theta, i)] \Big|_{\theta = \theta''} = 0. \tag{82}$$

Furthermore, for any $i \leq 2^B + 2$ and any $\theta \in [-1, 1]$, its easy to see that $|f''(\theta, i)| \leq 4$. Consequently, $\left| d^2/d\theta^2 \, \mathbb{E}_{i \sim Q}[f(\theta, i)] \right| \leq 4$, for all $\theta \in [-1, 1]$. It follows that

$$\begin{aligned} |G(Q) - G(P)| &= |\theta'' - \theta'| \\ &\geq \frac{1}{4} \left| \frac{d}{d\theta} \mathbb{E}_{i \sim Q}[f(\theta, i)] \Big|_{\theta = \theta''} - \frac{d}{d\theta} \mathbb{E}_{i \sim Q}[f(\theta, i)] \Big|_{\theta = \theta'} \right| \\ &= \frac{1}{4} \left| \frac{d}{d\theta} \mathbb{E}_{i \sim Q}[f(\theta, i)] \Big|_{\theta = \theta'} \right| \\ &\geq \frac{\epsilon}{2^{B+4}}, \end{aligned}$$

where the last two relations are due to (82) and (81), respectively. Then,

$$\begin{aligned} \left| \hat{G}(A^T P) - G(P) \right| + \left| \hat{G}(A^T Q) - G(Q) \right| &\geq \left| G(Q) - G(P) + \hat{G}(A^T P) - \hat{G}(A^T Q) \right| \\ &= \left| G(Q) - G(P) \right| \\ &\geq \frac{\epsilon}{2^{B+4}}, \end{aligned}$$

where the equality follows from (80). Therefore, the estimation error exceeds $\epsilon/2^{B+5}$ for at least one of the probability vectors P or Q. This completes the proof of Theorem 5.

Appendix L. Proof of Theorem 6

For simplicity, in this proof we will be working with the $[0,1]^d$ cube as the domain. Consider the following randomized algorithm:

- Suppose that each machine i observes n function $f_1^i, \cdots, f_n^i$ and finds the minimizer of $\sum_{j=1}^n f_j^i(\theta)$, which we denote by θ^i. Machine i then lets its signal Y^i be a randomized binary string of length d of the following form: for $j = 1, \ldots, d$,

$$Y_j^i = \begin{cases} 0, & \text{with probability } \theta_j^i, \\ 1, & \text{with probability } 1 - \theta_j^i, \end{cases}$$

  where Y_j^i is the j-th bit of Y^i, and θ_j^i is the j-th entry of θ^i.

- The server receives signals from all machines and outputs $\hat{\theta} = 1/m \sum_{i=1}^m Y^i$.

For the above algorithm, we have for $j = 1, \ldots, d$,

$$\operatorname{var}\left(\hat{\theta}_j\right) = \operatorname{var}\left(\frac{1}{m}\sum_{i=1}^m Y_j^i\right) = \frac{1}{m}\operatorname{var}\left(Y_j^1\right) = O\left(\frac{1}{m}\right), \tag{83}$$

where the last equality is because Y_j^1 is a binary random variable. Then,

$$\begin{aligned}
\mathbb{E}\Big[\|\hat{\theta} - \theta^*\|^2\Big] &= \sum_{j=1}^d \mathbb{E}\Big[\big(\hat{\theta}_j - \theta_j^*\big)^2\Big] \\
&= \sum_{j=1}^d \mathbb{E}\Big[\big(\hat{\theta}_j - \mathbb{E}[\hat{\theta}_j] + \mathbb{E}[\hat{\theta}_j] - \theta_j^*\big)^2\Big] \\
&= \sum_{j=1}^d \mathbb{E}\Big[\big(\hat{\theta}_j - \mathbb{E}[\hat{\theta}_j]\big)^2\Big] + \sum_{j=1}^d \mathbb{E}\Big[\big(\mathbb{E}[\hat{\theta}_j] - \theta_j^*\big)^2\Big] \\
&= \sum_{j=1}^d \operatorname{var}\left(\hat{\theta}_j\right) + \sum_{j=1}^d \Big(\mathbb{E}\big[\hat{\theta}_j - \theta_j^*\big]\Big)^2 \\
&= O\left(\frac{d}{m}\right) + O\left(\frac{d}{n}\right),
\end{aligned}$$

where the last equality is due to (83) and Claim 5. This completes the proof of Theorem 6.