LAGRANGE INTERPOLATION

WHICH SOFT WARES/TOOLS USE THIS METHOD?

AUTODESK® MAYA®

ROS Robot Operating System

GAZEBO

Autodesk Maya

- In Autodesk Maya, Lagrange interpolation is used in the animation process to create smooth transitions between keyframes. This method is often implemented in the underlying algorithms for motion paths, allowing animators to define movements that interpolate accurately between specified positions over time.

Gazebo

- Gazebo, a robotics simulation software, utilizes Lagrange interpolation to model sensor data and simulate smooth movement paths for robotic entities. While developing simulation scenarios, this interpolation is incorporated in the code to generate realistic trajectories for robots as they navigate their environments.

ROS (Robot Operating System)

- ROS employs Lagrange interpolation for various applications, including path planning and trajectory generation for robots. This method is often implemented in the code to ensure that robots follow smooth, continuous paths between waypoints, enhancing their navigation and movement precision in dynamic environments.

MATHEMATICAL REPRESENTATION

Lagrange Interpolating Polynomials

A polynomial of degree one passing through points (x_0, y_0) and (x_1, y_1) is same as approximating a function f with $y_0 = f(x_0)$ and $y_1 = f(x_1)$.

Lagrange polynomial for it is

$$P_1(x) = L_0(x)f(x_0) + L_1(x)f(x_1) \quad \text{with}$$

$$L_0(x) = \frac{x-x_1}{x_0-x_1} \quad \text{and} \quad L_1(x) = \frac{x-x_0}{x_1-x_0}\,.$$

Similarly for three points $(x_0, y_0), (x_1, y_1)$ and (x_2, y_2), we have to calculate three coefficient polynomials $L_0(x), L_1(x), L_2(x)$ and Lagrange polynomial for it is

$$P_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) \quad \text{with}$$

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)},\ L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \quad \text{and} \quad L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

Next, can you guess Lagrange polynomial for four points?

Yes…

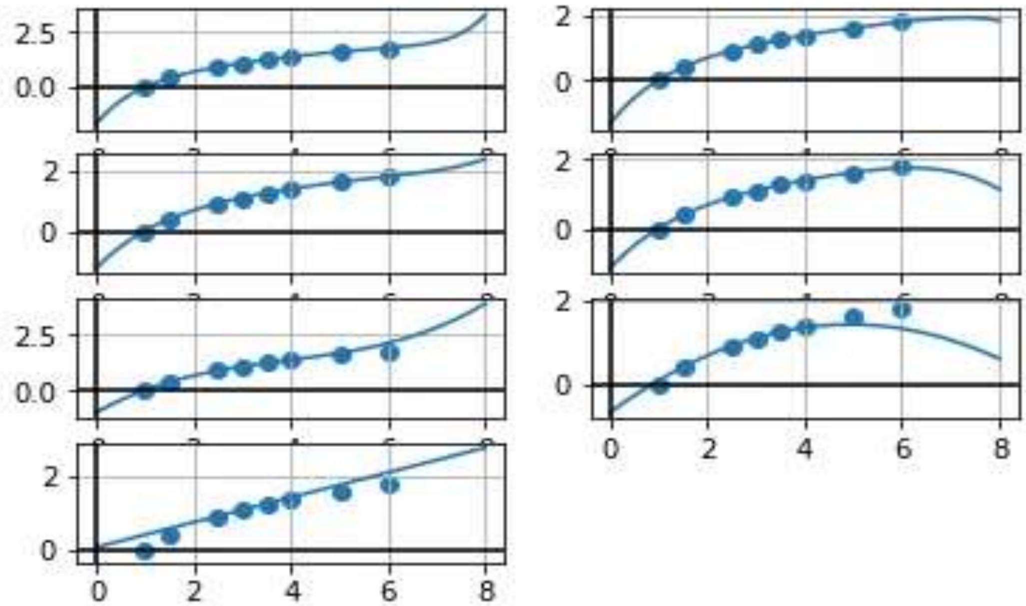
For $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ and (x_3, y_3), Lagrange polynomial will be

$$P_3(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) + L_3(x)f(x_3) \quad \text{with}$$

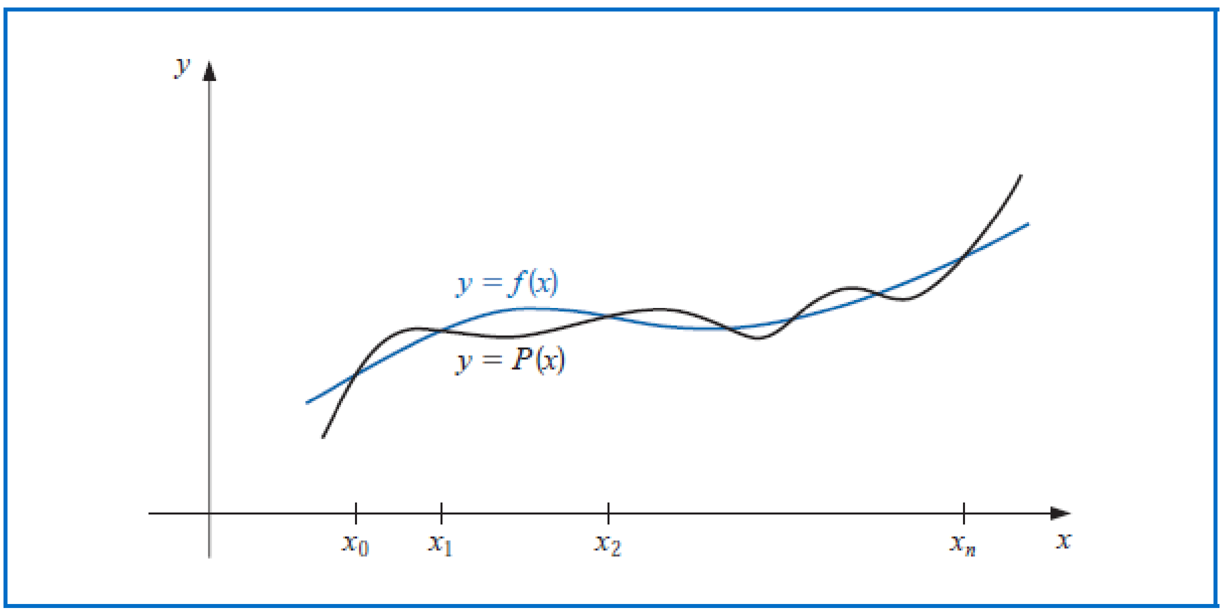
$$L_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}, \qquad L_1(x) = \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)},$$

$$L_2(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_1)(x_2-x_1)(x_2-x_3)}, \qquad L_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}.$$

GRAPHICAL REPRESENTATION



Graph of different orders of interpolation using Lagrange



Example

Determine the linear Lagrange interpolating polynomial that passes through the points $(2,\ 4)$ and $(5,\ 1)$.

Solution. Lagrange polynomial for $(x_0, y_0) = (2,\ 4)$ and $(x_1, y_1) = (5,\ 1)$ is

$$P_1(x) = L_0(x)f(x_0) + L_1(x)f(x_1)$$

with

$$L_0(x) = \frac{x-x_1}{x_0-x_1} = \frac{x-5}{-3} \quad \text{and} \quad L_1(x) = \frac{x-x_0}{x_1-x_0} = \frac{x-2}{3}.$$

Hence,

$$P_1(x) = -\frac{1}{3}(x-5)4 + \frac{1}{3}(x-2)1 = -x + 6\,.$$

*You can see that it is the same linear equation, you have derived in calculus from the point slope form of the equation using two points. The advantage is that the Lagrange polynomial is not only for polynomials of degree one but can be extended to polynomials of degree n using n+1 points of any given data.

Example

Use $x_0 = 2,\ \ x_1 = 2.75$ and $x_2 = 4$ to find Lagrange polynomial for $f(x) = \frac{1}{x}$. Use this polynomial to approximate $f(3)$.

Solution. First determine coefficient polynomials $L_0(x), L_1(x)$, and $L_2(x)$ as

$$L_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-2.75)(x-4)}{(2-2.75)(2-4)} = \frac{2}{3}(x-2.75)(x-4),$$

$$L_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-2)(x-4)}{(2.75-2)(2.75-4)} = -\frac{16}{15}(x-2)(x-4),$$

and $$L_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-2)(x-2.75)}{(4-2)(4-2.75)} = \frac{2}{5}(x-2)(x-2.75).$$

Therefore, polynomial for it is

$$P_2(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$$

$$= \frac{2}{3}(x-2.75)(x-4)\frac{1}{2} - \frac{16}{15}(x-2)(x-4)\frac{1}{2.75} + \frac{2}{5}(x-2)(x-2.75)\frac{1}{4}$$

$$\Longrightarrow P_2(x) = \frac{1}{22}x^2 - \frac{35}{88}x + \frac{49}{44}$$

So, $P_2(3) = 0.32955$ while $f(3) = \frac{1}{3} = 0.33333.$

PRACTICE

Problem 1

The following data of the velocity of a body is given as a function of time.

Time (s)	10	15	18	22	24
Velocity (m/s)	22	24	37	25	123

A quadratic Lagrange interpolating polynomial is formed using three data points, $t = 15, 18$, & 22. Use this information to evaluate that at what times (in seconds) is the velocity of the body $26\ m/s$ during the time interval of $t = 15$ to $t = 22$ seconds.

Problem 2

The following table shows the population of US from 1960 to 2010

Year	1960	1970	1980	1990	2000	2010
Population (in thousands)	179,323	203,302	226,542	249,633	281,422	308,746

Use Lagrange Interpolation to approximate the population in the years 1965, 1975, 2014, and 2005.

Problem 3

A software development team is working on an application that models the temperature change in a specific area over time. They collected temperature data at certain points: at time $t = 0$ hours, the temperature was 0°C ; at $t = 0.5$ hours, the temperature was y℃ ; at $t = 1$ hour, the temperature reached 3℃ ; and at $t = 2$ hours, it cooled down to 2℃ .

To create a smooth temperature model, they decide to use the Lagrange interpolating polynomial $P_3(x)$ based on the collected data points. The team also learns that the coefficient of x^3 in this polynomial must be 6.

a. Determine the value of y that satisfies this condition.
b. Calculate the complete polynomial $P_3(x)$ that models the temperature changes over the observed time intervals.