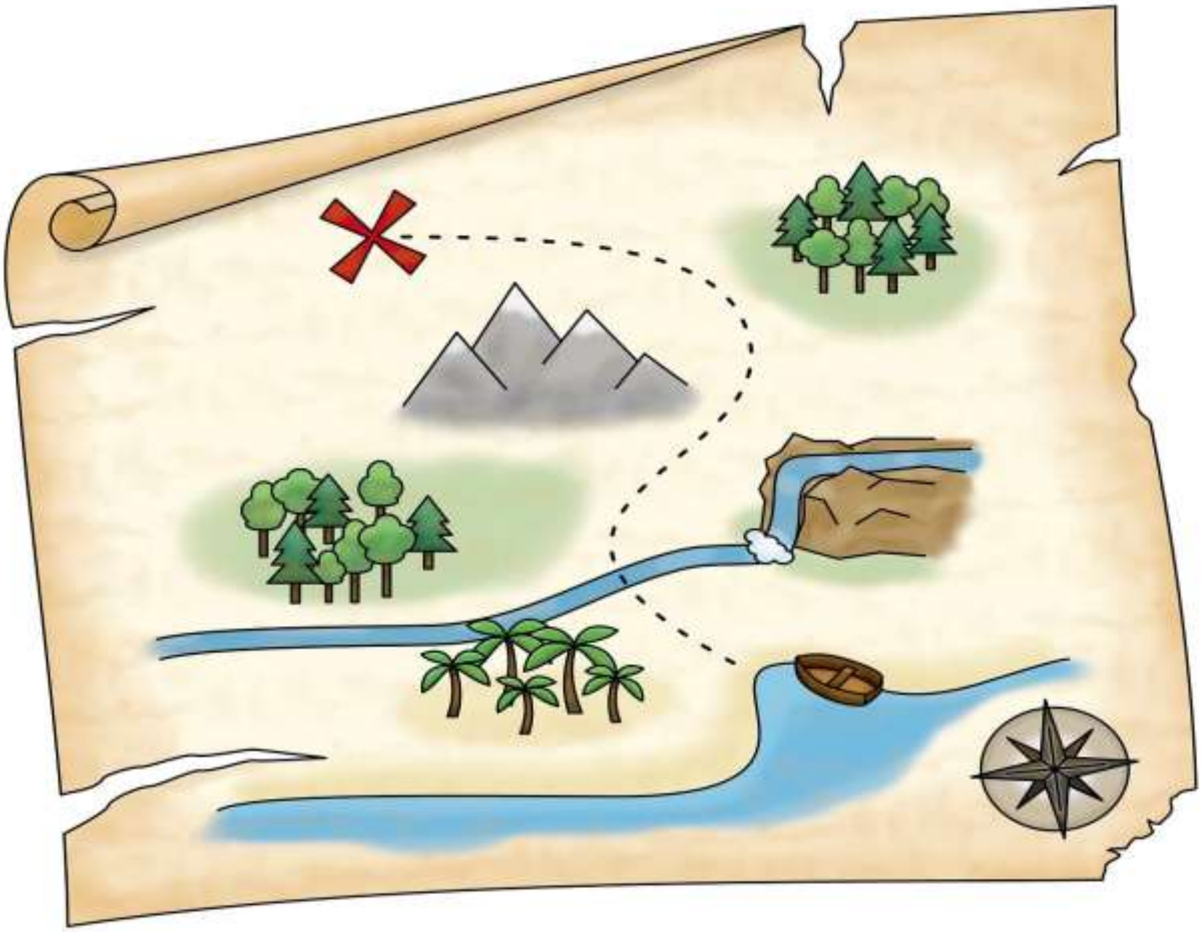
FIXED POINT ITERATION



Imagine you're trying to find a specific spot on a map, like a treasure hidden in a forest. Instead of wandering randomly, you decide to follow a set of clear instructions: "From your current position, move to the spot marked on your map." You keep repeating this process—moving to the new spot you found—until you finally get to the treasure.

In math, the **Fixed Point Iteration Method** works similarly. You start with an initial guess for the solution to an equation, and then you repeatedly plug that guess into a function to get a new guess. The idea is that if you keep doing this, your guesses will eventually settle down to a specific value—like finding the treasure.

A *fixed point* for a function is a number at which the value of the function does not change when the function is applied.

Definition: The number p is a **fixed point** for a function g if $g(p) = p$.

Example: Determine any fixed points of

$$g(x) = x^2 - 2.$$

Solution: A fixed point p for g has property

$$x = g(x) = x^2 - 2.$$

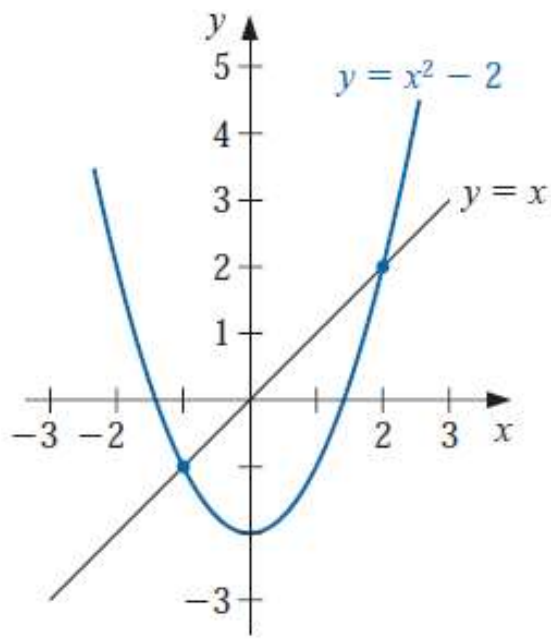
A fixed point for g occurs

when graph of $y = g(x)$ intersects $y = x$.

So g has two fixed points,

$p = -1$ and $p = 2$ as

$-1 = g(-1)$ and $2 = g(2)$.



Why we are finding fixed point here?

An important application of a fixed point is that if we write an equation $f(x) = 0$ in the fixed-point form $x = g(x)$ using simple algebraic manipulation, same x will also be the solution of the equation $f(x) = 0 = x - g(x)$. It means above calculated fixed points are also the solutions of equation $x^2 - x - 2 = 0$.

The following theorem gives sufficient conditions for the existence and uniqueness of a fixed point.

Theorem: **(i)** If $g \in C[a, b]$ and $g(x) \in [a, b]$ for all $x \in [a, b]$, then g has at least one fixed point in $[a, b]$.

(ii) If, in addition, $g'(x)$ exists on (a, b) and a positive constant $k < 1$ exists with $|g'(x)| \leq k$, for all $x \in (a, b)$, then there is exactly one fixed point in $[a, b]$.

To approximate the fixed point of a function g, we choose an initial approximation p_0 and generate the sequence $\{p_n\}_{n=0}^{\infty}$ by letting $p_n = g(p_{n-1})$, for each $n \geq 1$.

Such sequence converges to the unique fixed point p in $[a, b]$.

Illustration: The equation $x^3 + 4x^2 - 10 = 0$ has a unique root in [1, 2]. There are many ways to change the equation to the fixed-point form $x = g(x)$ using simple algebraic manipulation.

Here, to obtain the function g, we can manipulate the equation $x^3 + 4x^2 - 10 = 0$ in the following four ways:

$$x^3 = 10 - 4x^2 \;\Rightarrow \qquad (a)\;\; x = \sqrt{\frac{10}{x} - 4x}$$

$$4x^2 = 10 - x^3 \;\Rightarrow \qquad (b)\;\; x = \frac{1}{2}\sqrt{10 - x^3}$$

$$x^2(x + 4) = 10 \;\Rightarrow \qquad (c)\;\; x = \sqrt{\frac{10}{4+x}}$$

and by some other means $\quad (d)\;\; x = x - \dfrac{x^3+4x^2-10}{3x^2+8x}$

To start the sequence for each function, let us take $p_0 = 1.5$ from [1, 2].

We have summarized these four sequences in the form of a table as:

n	(a) $x = \sqrt{\frac{10}{x} - 4x}$	(b) $\boldsymbol{x} = \frac{\mathbf{1}}{2}\sqrt{10 - x^3}$	(c) $\boldsymbol{x} = \sqrt{\frac{\mathbf{10}}{4+\boldsymbol{x}}}$	(d) $\boldsymbol{x} = x - \frac{\boldsymbol{x^3+4x^2-10}}{3x^2+8x}$
0	1.5	1.5	1.5	1.5
1	0.8165	1.28695	1.34840	1.37333
2	2.9969	1.40254	1.36738	1.36526
3	$\sqrt{-8.65}$	1.34546	1.36496	1.36523
4		1.37517	1.365265	1.36523
5		1.36009	1.365226	
6		1.36785	1.36523	
7		1.36389		
8		1.36592		
9		1.36488		
10		1.36541		
11		1.36514		
12		1.36528		

We found that sequence obtained by function in (d) rapidly converges to a solution.

Now, the **Newton-Raphson Method** can be thought of as a more refined version of the fixed-point iteration process. While fixed-point iteration uses a single function $g(x)$, Newton-Raphson uses the derivative of the function you're trying to solve, making it faster and more efficient.