UNIT-3

GRAPH THEORY

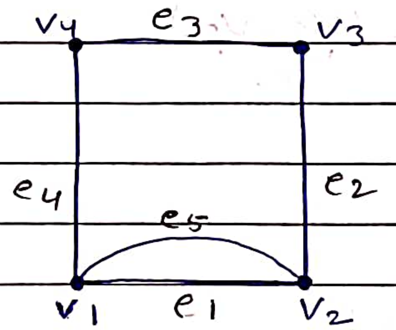
Graph:

A Graph is a pair of (V, E) where $V = \{v_1, v_2, v_3 \ldots\}$ is called the set of vertex or nodes and $E = \{e_1, e_2, e_3 \ldots\}$ is a set such that each element e_k of E is an unordered pair of (v_i, v_j) of vertices.

$E = \{e_1, e_2, e_3 \ldots\}$ is called set of edges

Parallel Edges in the graph-

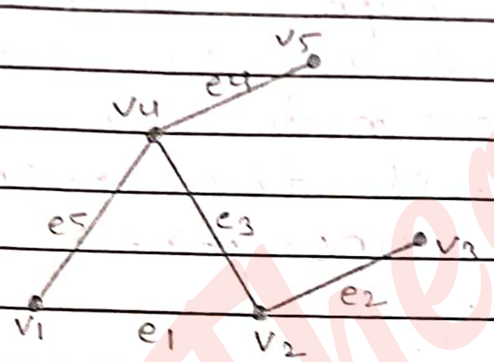
An Edges having the same pair of end vertex are called parallel edges.



e_1 & e_5 are ~~partrallel~~ parallel edges

Self-loop in the Graph:

A Graph is called simple graph if graph is free from parallel edges and self-loop.



Finite Graph:

A graph is said to be finite if the set v and E are finite.

$V = \{V_1, V_2, V_3\}$

$E = \{\underbrace{(V_1, V_2)}_{e_1}, \underbrace{(V_3, V_2)}_{e_2}\}$

$\therefore E = \{e_1, e_2\}$

Infinite Graph:

A graph is said to be infinite if the set v and E are infinite.

$V = \{V_1, V_2, V_3 \text{------}\}$

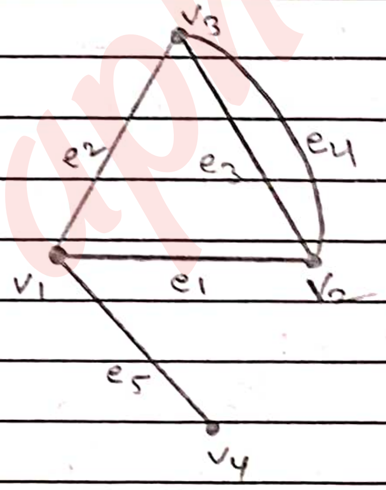
$E = \{e_1, e_2, e_3 \text{------}\}$

Incident and degree:

Let e_k be any edge joining two vertices v_i and v_j in the graph G. Then the edge e_k is said to be incident on v_i and v_j.

Adjacent vertex:

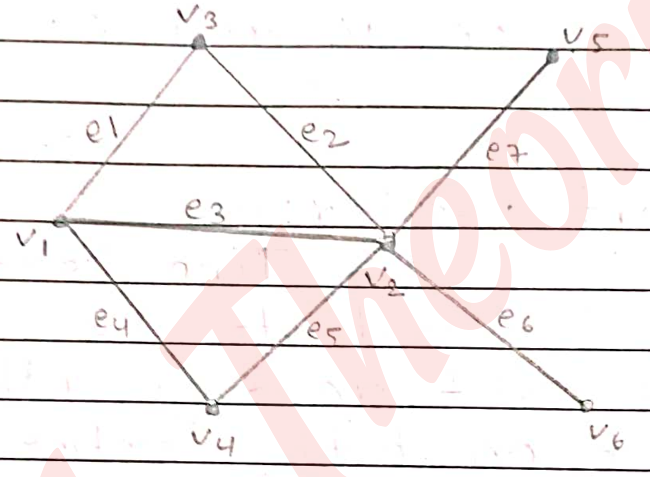
Two vertices in the graph is said to be adjacent if there exist an edge which are incident on both vertices.



$\{(v_1, v_4), (v_1, v_3), (v_1, v_2), (v_2, v_3)\}$ are adjacent.
but,
$\{(v_2, v_4), (v_3, v_4)\}$ are not adjacent.

Adjacent Edges:

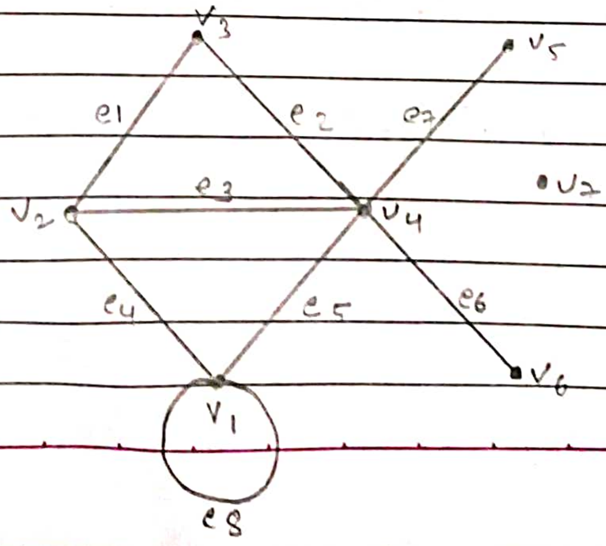
Two non parallel edges which incident on same vertex is called adjacent edges.



$(e_1, e_2), (e_3, e_1), (e_3, e_2)$ .... are adjacent edges but (e_2, e_4) are not adjacent.

* Degree of vertex:

The degree of vertex v is equal to the number of edges which are incident on v



- Denoted by $d(v)$
- Degree of self-loop is counted as two

$d(v_1) = 4$
$d(v_2) = 3$
$d(v_3) = 2$
$d(v_4) = 5$
$d(v_5) = 1$
$d(v_6) = 1$
$d(v_7) = 0$

Isolated vertex:

A vertex which has degree zero is called isolated vertex

Pendant vertex:

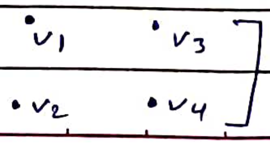
A vertex of degree one(1) is called pendent vertex.

Null Graph:

A graph is said to be null graph if the set of vertex is non-empty but the set of edges are empty is called null graph.

Eg:-

$V = \{v_1, v_2, v_3, v_4\}$



Null Graph

* Sum of degrees of all vertices will be twice the no. of edges

Proof

Let G be any graph and $v_1, v_2, v_3 \ldots v_n$ n vertices since each edge incident on two vertex, so each edge contribute 2 times when we find the degree of edges.
Thus the form of all vertex
= 2 x Number of degree.

Handshaking lemma Mathematically

$$\sum_{i=1}^{n} d(v_i) = 2e$$

$$d(v_1) + d(v_2) + d(v_3) + d(v_4) = 2e$$

Even vertex:

A vertex v is said to be even if degree of vertex v is even.

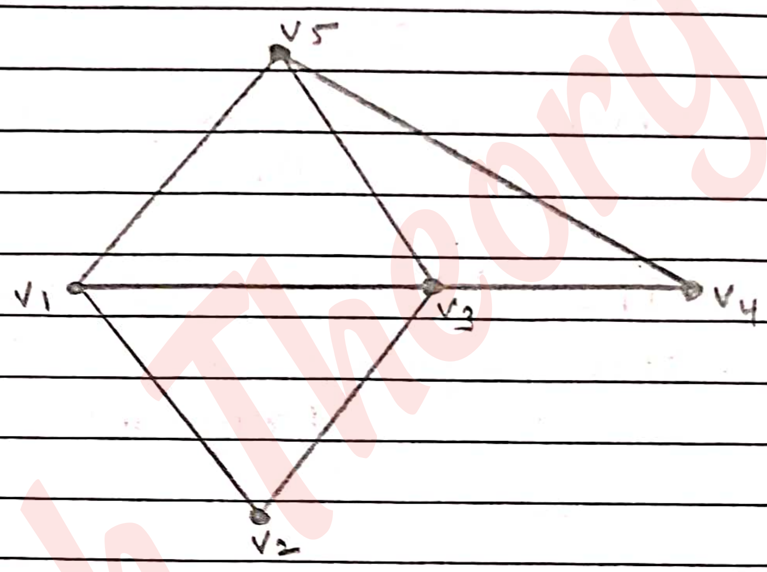
Odd vertex:

A vertex v is said to be odd if degree of vertex v is odd.

VVI

Th-2

The number of odd vertex in a graph is always even



Let G be any graph.

Let $v_1, v_2, v_3, \ldots\ldots v_n$ are n vertex
Let k vertices is of odd degree. Then number of even vertex is equal to n-k.

Now,

$$\underbrace{\sum_{i=1}^{n} d(v_i)}_{\text{Even No.}} = \underbrace{\sum_{i=1}^{n} d(v_i)}_{\text{I even no.}} + \underbrace{\sum_{i=n-k}^{n} d(v_i)}_{\text{II even no.}}$$

since the sum of L.H.S term is an even number and the sum of R.H.S 2nd term is also even

$\Rightarrow$ The sum $\sum_{i=1}^{k} d(v_i)$ is also even

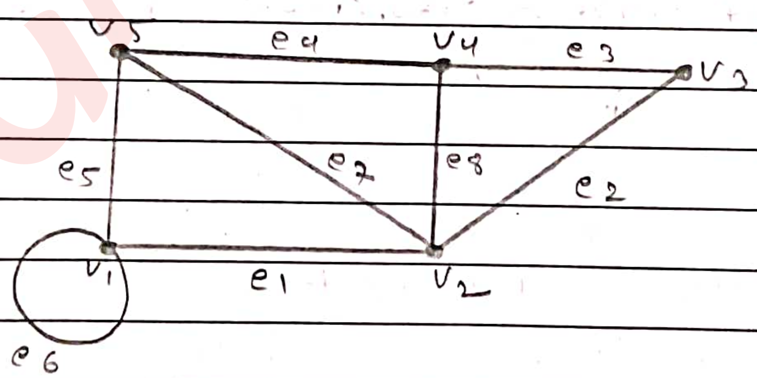
since sum $\sum_{i=1}^{n} d(v_i)$ is even

$\Rightarrow$ The number of odd vertices is also even

Proved

walk:

A walk is an alternating sequence of vertex and edges in which repetition of edges and vertex are allowed



$v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_8 v_2 e_1 v_1$ is walk

Find walk start and end at v_5 and cover all edges and vertex at least one time

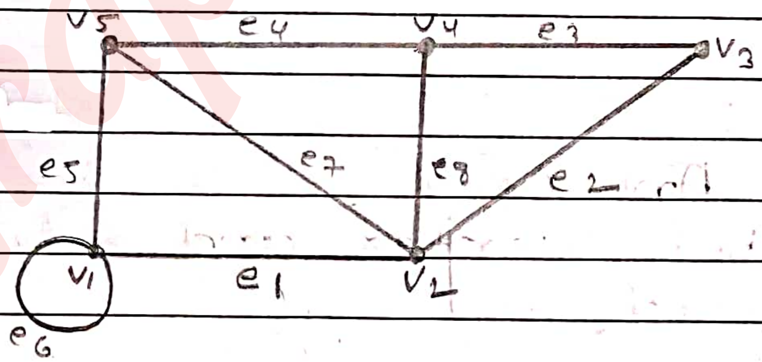
$v_5 e_5 v_1 e_6 v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_8 v_2 e_7 v_5 e_4 v_4 e_8 v_2$ $e_7 v_5$

Open walk:

A walk in which end point are distinct is called open walk.

Note:

In open walk vertices and edges may be respected. except end vertices.

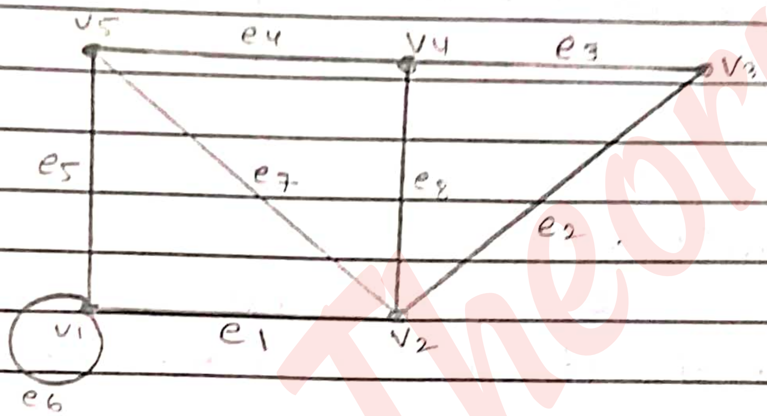


(i) $v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_5$ is an open walk.

(ii) $v_5 e_5 v_1 e_1 v_2 e_2 v_3$ is an open walk.

Closed walk:

A walk is said to be closed if the end vertices of the walk are same.



(i) $v_1 e_6 v_1$ is a closed walk

(ii) $v_5 e_5 v_1 e_1 v_2 e_8 v_4 e_4 v_5$ is a closed walk

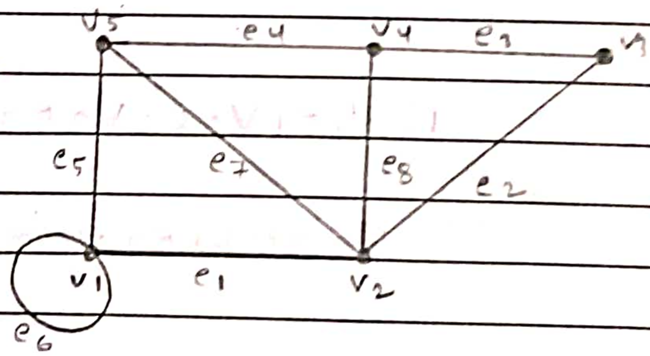
Path:

An open walk in which number vertex and edges are repeated is called path.

(i) $v_1 e_1 v_2 e_7 v_5$ is path

(ii) $v_1 e_6 v_1 e_5 v_5$ is not path

(iii) $v_4 e_4 v_5 e_5 v_1$ is a path.

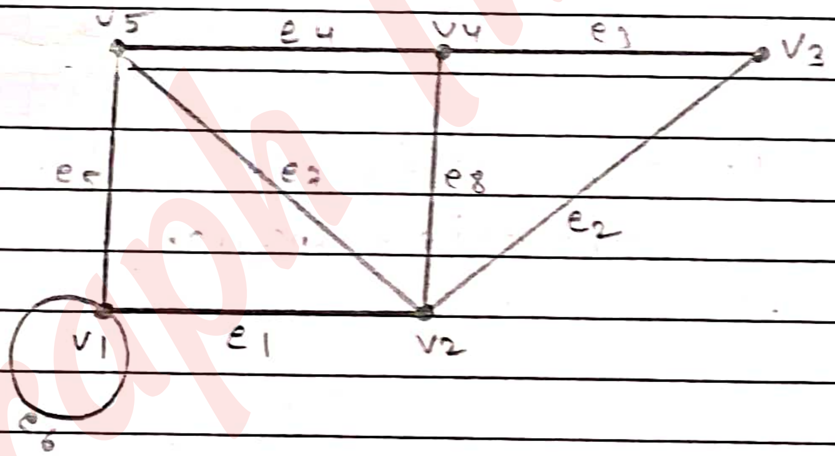


Length of the Path:

The number of edges in the path is called length of the path.

circuit (cycle):

A closed walk in which number vertex and edges are repeated except and vertices is called circuit.



v1 e6 v1 is a circuit.

v1 e1 v2 e8 v4 e4 v5 e5 v1 is a circuit

Note:

In circuit the degree of each vertex is equal to 2.

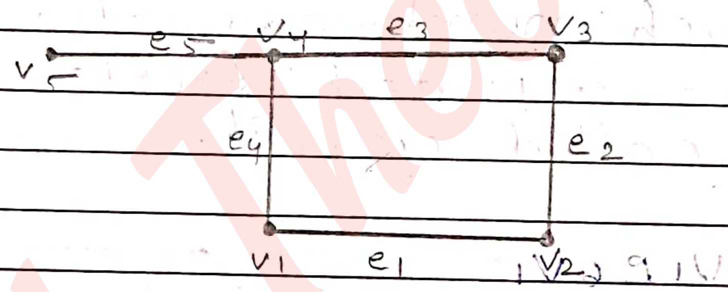
Note:

Every self-loop is a circuit but every circuit is not a self-loop

Connected & Disconnected Graph:

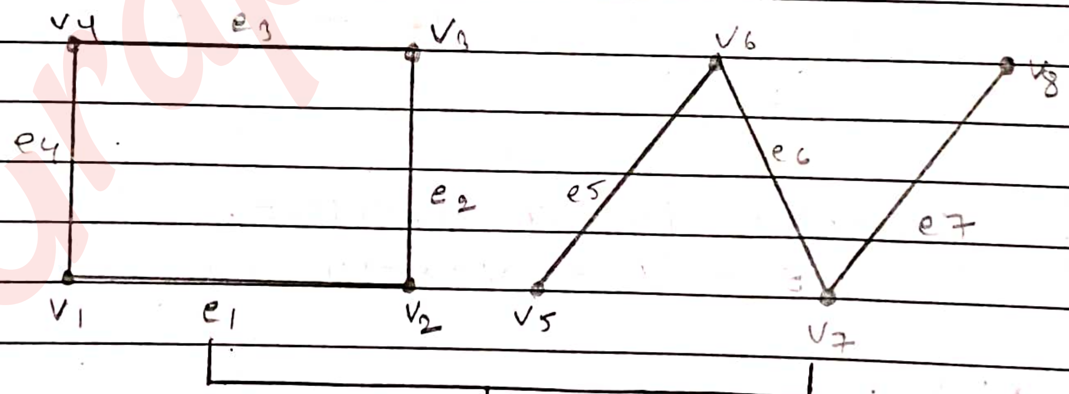
A graph G is said to be connected if there exist a path between each pair of vertices. other graph is said to be disconnected.

Eg:-



$(v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_5 v_5)$

Connected Graph



Disconnected Graph

* In disconnected graph, Every component of disconnected graph is connected.

Component:

A component of a graph G is maximal connected graph G.

V.V.I.

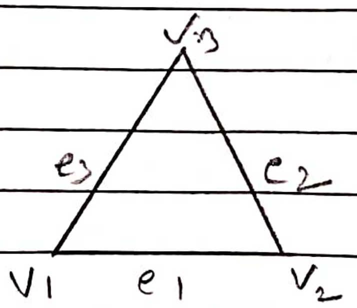
Complete Graph:

A simple graph is said to be complete if each pair of vertices are adjacent in graph.

- If there are n vertices in a complete graph their it is denoted by K_n
- The number of edge in complete graph is $\frac{n(n-1)}{2}$
- The degree of each vertex is equal to $(n-1)$

Eg:-

K_3



* no. of edge $= \frac{3(3-1)}{2}$
$= 3$

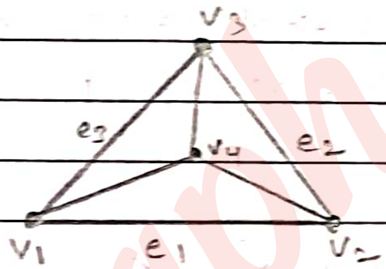
* degree of vertex $= K_3$
$= (3-1)$
$= 2$

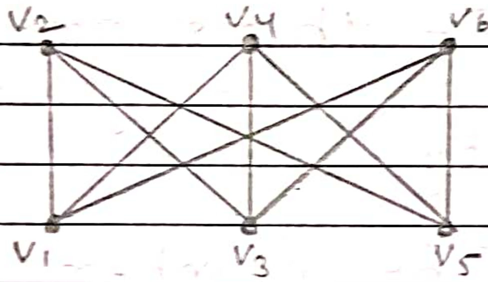
Regular Graph:

A graph G is said to be regular if the degree of each vertex is equal.

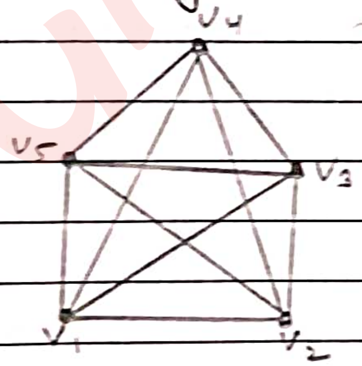
Let the degree of each vertex in a regular graph is equal to r then graph is called r- regular graph.

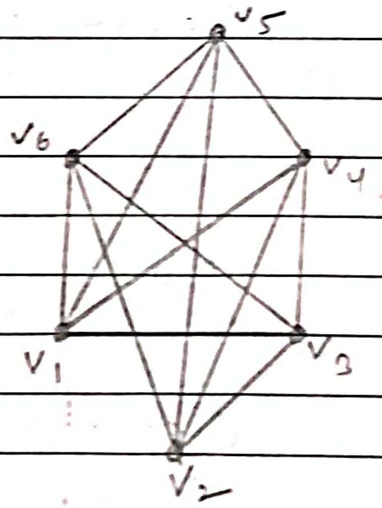
3 - regular graph





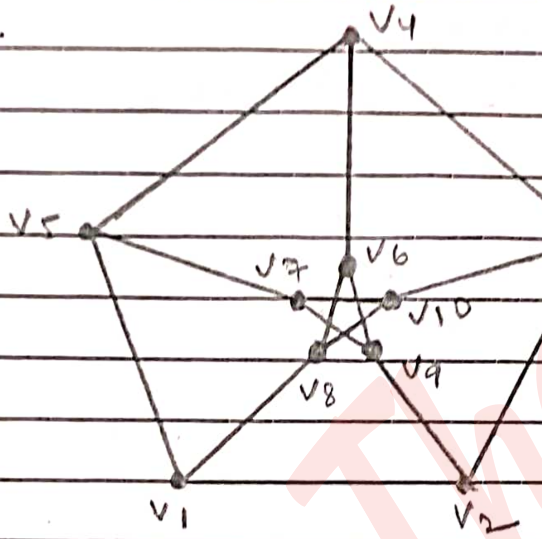
4- regular graph





Peterson Graph:

A 3-regular graph which is as follows is called peterson graph.



subgraph:

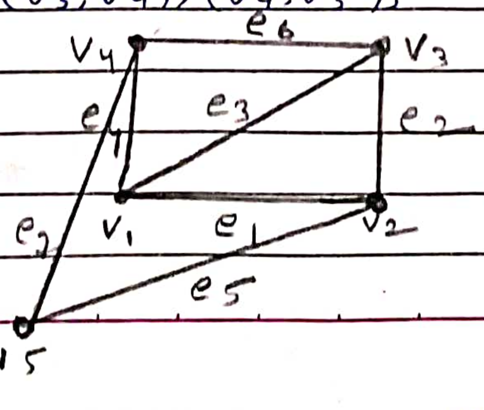
Let $G=(V,E)$ be a graph then $H=(V',E')$ is said to be subgraph if V' is a subset of V and E' is a subset of E.

Graph

$G=(V,E)$

$V=\{v_1, v_2, v_3, v_4, v_5\}$

$E=\{(v_1,v_2),(v_2,v_3),(v_3,v_1),(v_4,v_1),(v_5,v_2),$
$(v_3,v_4),(v_4,v_5)\}$

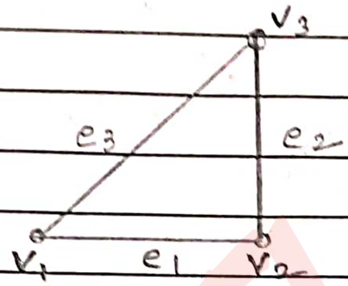


Subgraph

$H = (V', E')$

$V' = \{V_1, V_2, V_3\}$

$E' = \{(V_1, V_2), (V_2, V_3), (V_3, V_1)\}$



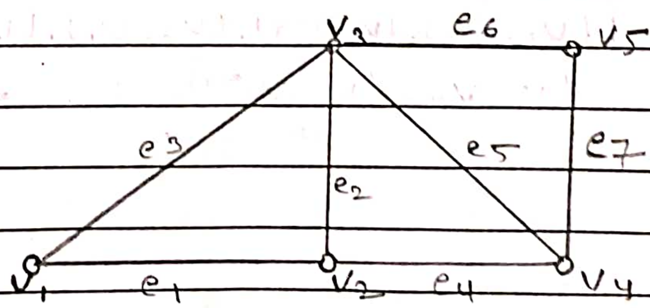
Edge-Disjoint Subgraph:

Two subgraphs H_1 and H_2 of Graph G is said to be edge-disjoint subgraph if number edge is common in both subgraph.

$G = (V, E)$

$V = \{V_1, V_2, V_3, V_4, V_5\}$

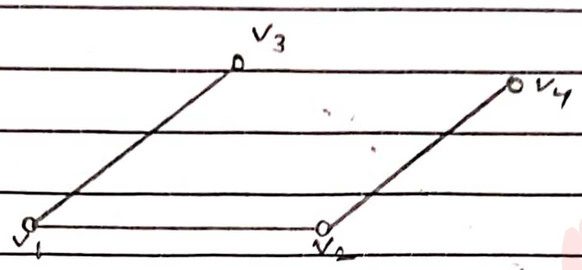
$E = \{(V_1, V_2), (V_2, V_3), (V_3, V_1), (V_2, V_4), (V_4, V_3), (V_3, V_5), (V_5, V_4)\}$



$H_1 = (V_1, E_1)$

$V_1 = \{v_1, v_2, v_3, v_4\}$

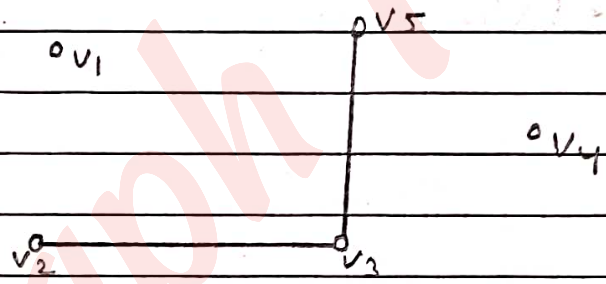
$E_1 = \{(v_1, v_2), (v_3, v_1), (v_2, v_4)\}$



$H_2 = (V_2, E_2)$

$V_2 = \{v_1, v_2, v_3, v_4, v_5\}$

$E_2 = \{(v_2, v_3), (v_3, v_5)\}$



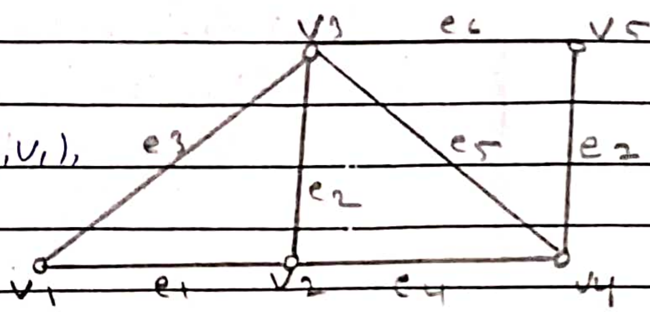
Vertex-Disjoint subgraph:

Two subgraph H_1 and H_2 of graph G is said to be vertex-Disjoint subgraph if number vertex is common in both subgraph.

$G = (V, E)$

$V = \{v_1, v_2, v_3, v_4, v_5\}$

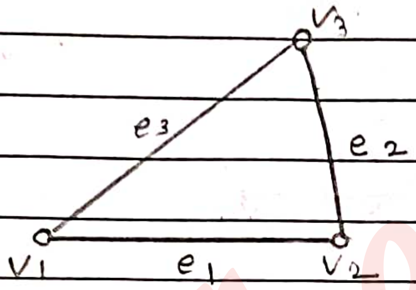
$E = \{(v_1, v_2), (v_2, v_3), (v_3, v_1), (v_2, v_4), (v_4, v_3), (v_3, v_5), (v_5, v_4)\}$



$H_1 = (V_1, E_1)$

$V_1 = \{V_1, V_2, V_3\}$

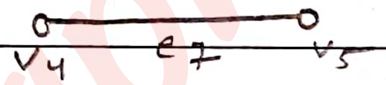
$E_1 = \{(V_1, V_2), (V_2, V_3), (V_3\ V_1)\}$



$H_2 = (V_2, E_2)$

$V_2 = \{V_4, V_5\}$

$E_2 = \{(V_4, V_5)\}$



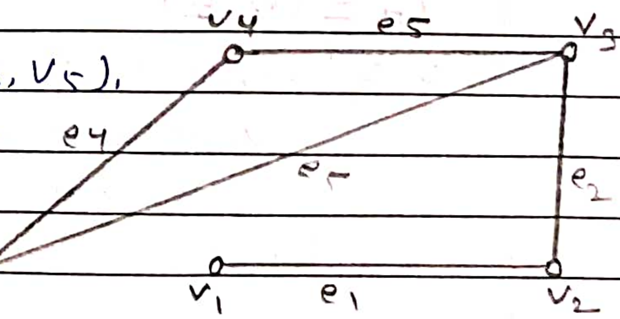
Spanning Subgraph:

A subgraph H of graph G is called spanning subgraph if all the vertices are present in subgraph.

$H = (V_1, E_1)$

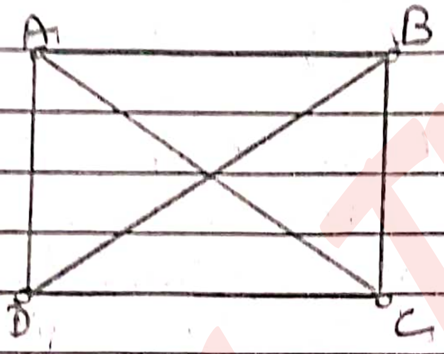
$V1 = \{V_1, V_2, V_3, V_4, V_5\}$

$E_1 = \{(V_1, V_2), (V_2, V_3), (V_3, V_5),$
$(V_5, V_4), (V_4, V_3)\}$

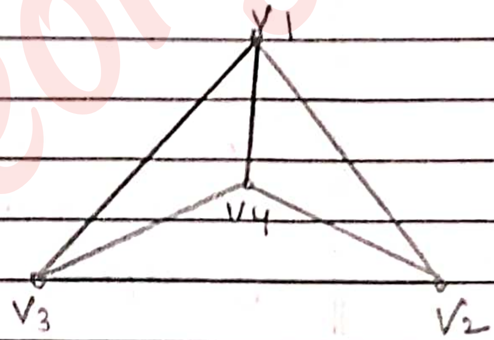


Isomorphic Graph:

Two graphs G_1 and G_2 are said to be isomorphic if there exist one to one correspondence between their vertices and their edges which are incident on vertices.



G_1

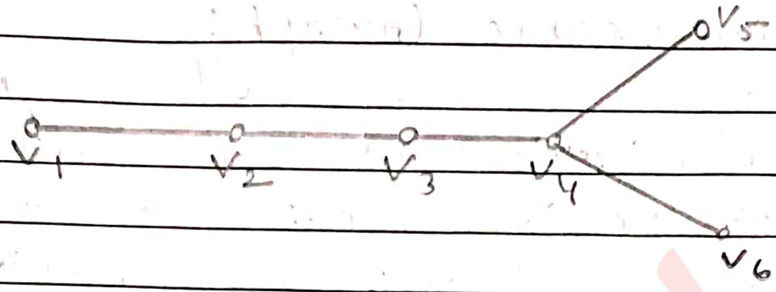


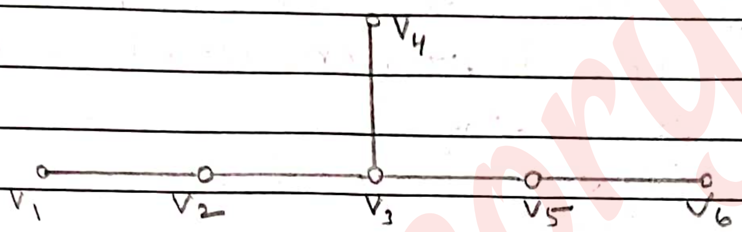
G_2

Number of vertex = 4   Number of vertex = 4
Number of Edge = 6   Number of Edge = 6

Vertex correspondence	Edge correspondence
$A \longleftrightarrow V_1$	$(A, B) \longleftrightarrow (V_1, V_2)$
$B \longleftrightarrow V_2$	$(B, C) \longleftrightarrow (V_2, V_3)$
$C \longleftrightarrow V_3$	$(C, D) \longleftrightarrow (V_3, V_1)$
$D \longleftrightarrow V_4$	$(D, A) \longleftrightarrow (V_1, V_4)$
	$(A, C) \longleftrightarrow (V_2, V_4)$
	$(B, D) \longleftrightarrow (V_3, V_4)$

Both graph are isomorphic

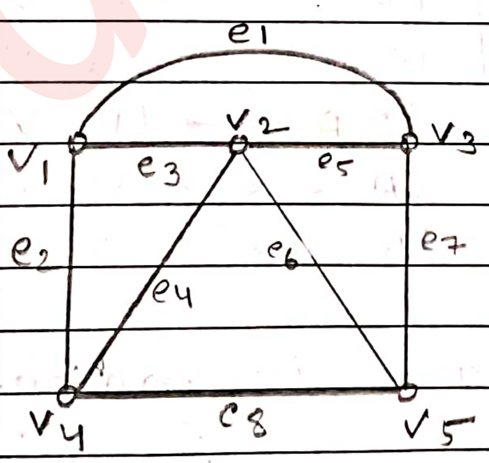




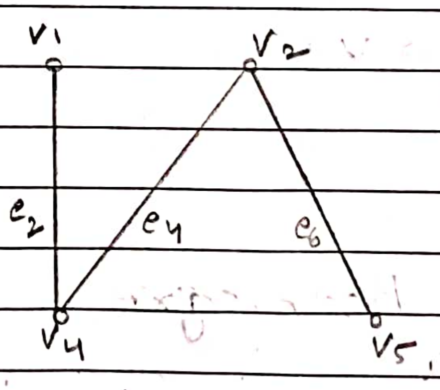
There does not exist one to one correspondence between edges so graphs are not isomorphic.

Complement of a subgraph:

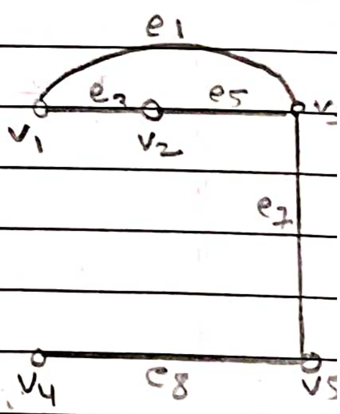
Let $H = (V', E')$ be a subgraph of a graph $G = (V, E)$.
Then complement of a subgraph H is $\bar{H} = (V, E - E')$.



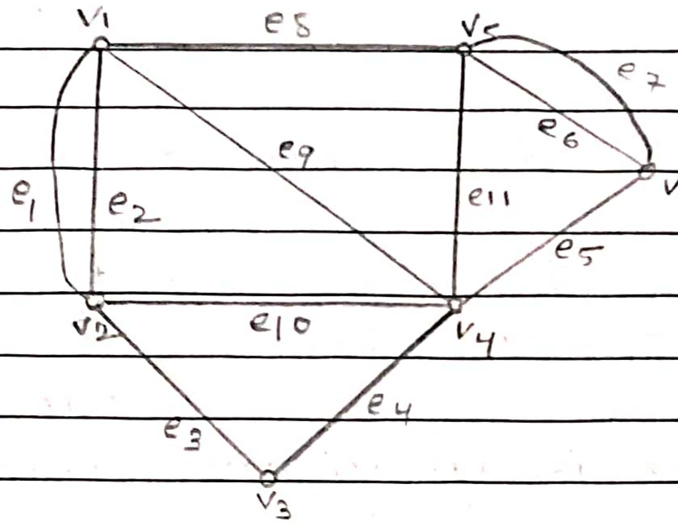
Graph-G



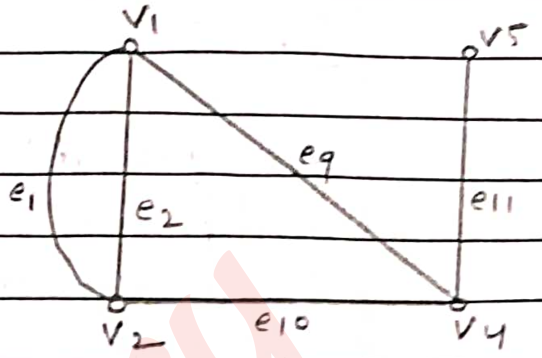
subgraph-H



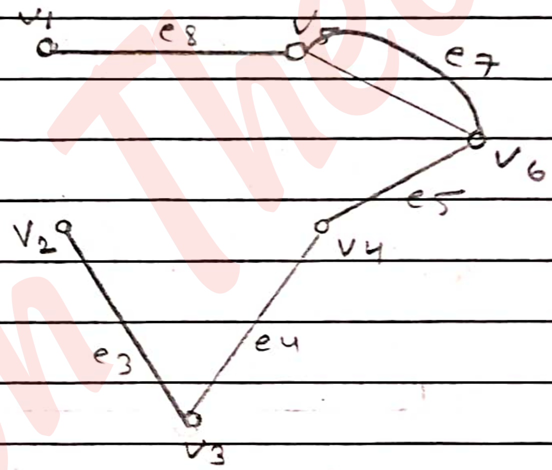
component of subgraph



Graph - G



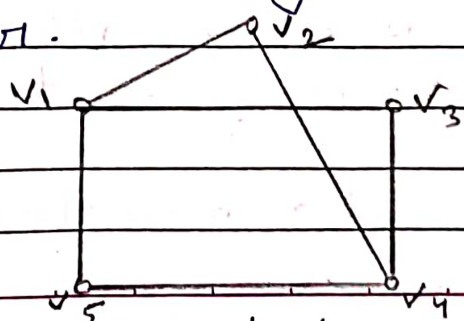
subgraph - H



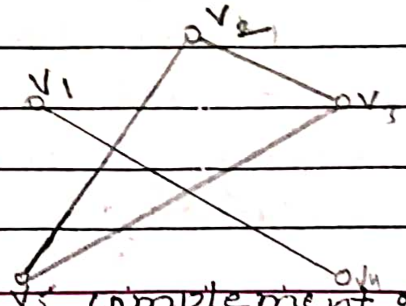
complement of subgraph

complement of simple graph:

Let $G = (V, E)$ be a simple Graph. Then complement of simple graph G is equal to the graph in which number Edge is present which is in G.



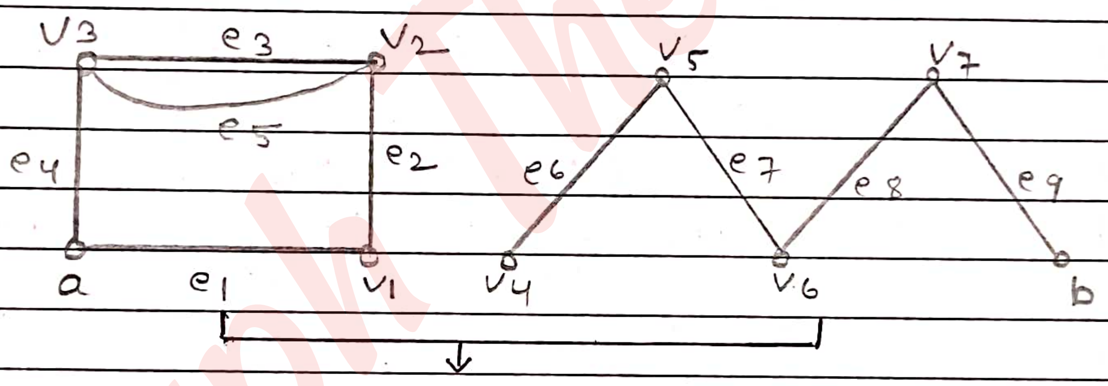
Graph - G



complement of simple Graph

Th-1

A graph G is disconnected if and only if the vertex V of G can be partitioned into two non empty disjoint subsets V_1 and V_2 such that there exist number edge in G whose one end vertex is in V_1 and other in V_2



Disconnected

$V = \{a, v_1, v_2, v_3, v_4, v_5, v_6, v_7, b\}$
$V_1 = \{a, v_1, v_2, v_3\}$
$V_2 = \{v_4, v_5, v_6, v_7, b\}$

Proof

Let G be a disconnected graph
Let a be any vertex in G.
Let V_1 be the set of all those vertices which are join to "a" by any path.
since graph is disconnected.

$\Rightarrow V_1$ does not contains all the vertices of V Let V_2 be the set of all those vertices which are not in V_1

$\Rightarrow V_1$ and V_2 are required partition

Coversely,

Let V_1 and V_2 be two partition of V which are disjoint Let $a \in V_1$ and $b \in V_2$

Since there does not exist any path from a to b

$\Rightarrow G$ is disconnected.

V.V.I.

Note-1

The maximum number of edge in simple Graph with n vertices is equal to $\frac{n(n-1)}{2}$

Th-1

A simple graph with n vertices and k components have at most $\frac{(n-k)(n-k+1)}{2}$ edges

Proof

Let G be a simple graph with n vertices and k components. Let $n_1, n_2, n_3 \ldots n_k$ be the number of vertices of k components.

$$\Rightarrow n_1 + n_2 + n_3 + \ldots\ldots + n_k = n \quad \text{—— (1)}$$

Now,

$$\sum_{i=1}^{k} (n_i - 1) = \sum_{i=1}^{k} n_i - \sum_{i=1}^{k} 1$$

$$\sum_{i=1}^{k} (n_i - 1) = (n_1 + n_2 + n_3 + \ldots + n_k) - (1+1+1 \ldots + k)$$

$$\sum_{i=1}^{k} (n_i - 1) = n - k \quad \text{—— (2)}$$

squaring both side of eqn (2)

$$\left[\sum_{i=1}^{k} (n_i - 1)\right]^2 = (n-k)^2$$

$$\sum_{i=1}^{k} (n_i - 1)^2 + 2(n_i - 1)(n_j - 1) = n^2 + k^2 - 2nk$$

$$\sum_{i=1}^{k} (n_i - 1)^2 \le n^2 + k^2 - 2nk$$

$$\sum_{i=1}^{k} (n_i^2 + 1 - 2i) \le n^2 + k^2 - 2nk$$

$$\sum_{i=1}^{k} n_i^2 + \sum_{i=1}^{k} 1 - 2 \sum_{i=1}^{k} \le n^2 + k^2 - 2nk$$

$$\sum_{i=1}^{k} n_i^2 + (1+1+1+\cdots\cdots+k) - 2(n_1+n_2+n_3+\cdots\cdots+n_k)$$
$$\le n^2 + k^2 - 2nk$$

$$\sum_{i=1}^{k} n_i^2 + k - 2n \le n^2 + k^2 - 2nk$$

$$\sum_{i=1}^{k} n_i^2 \le n^2 + k^2 - 2nk - k + 2n$$

$$\sum_{i=1}^{k} n_i^2 \le n^2 + k(k-2n) - 1(k-2n)$$

$$\sum_{i=1}^{k} n_i^2 \le n^2 + (k-2n)(k-1) \longleftarrow ③$$

Now,

Maximum number of edge in simple graph with n vertex
$\frac{n(n-1)}{2}$

Now,

Let maximum of edge in i^{th} component is $\frac{n_i(n_i-1)}{2}$

Now,

Again we have total k-component

$$\sum_{i=1}^{k} n_i(n_i-1) = \sum_{i=1}^{k} \frac{(n_i^2 - n_i)}{2}$$

$$= \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i\right]$$

$$= \frac{1}{2}\left[n^2 + (k-2n)(k-1) - (n_1+n_2+n_3+\ldots+n_k)\right]$$

$$= \frac{1}{2}\left[n^2 + (k-2n)(k-1) - n\right]$$

$$= \frac{1}{2}\left[n^2 + k^2 - k - 2nk + 2n - n\right]$$

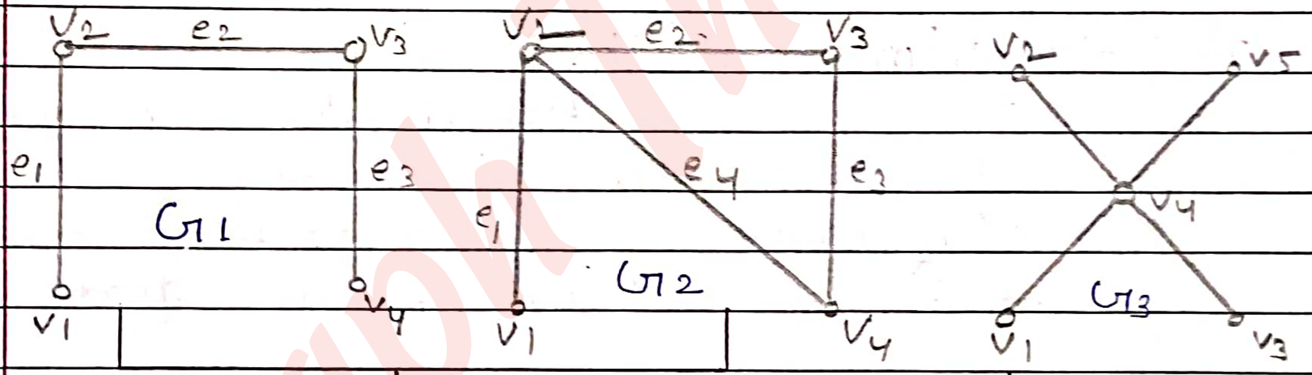
$$= \frac{1}{2}\left[n^2 + k^2 - k - 2nk + n\right]$$

$$= \frac{1}{2}\left[(n-k)(n-k+1)\right]$$

Proved

Euler Path:

An open walk in a graph which contains all edges of the graph called Euler path.



Euler Path

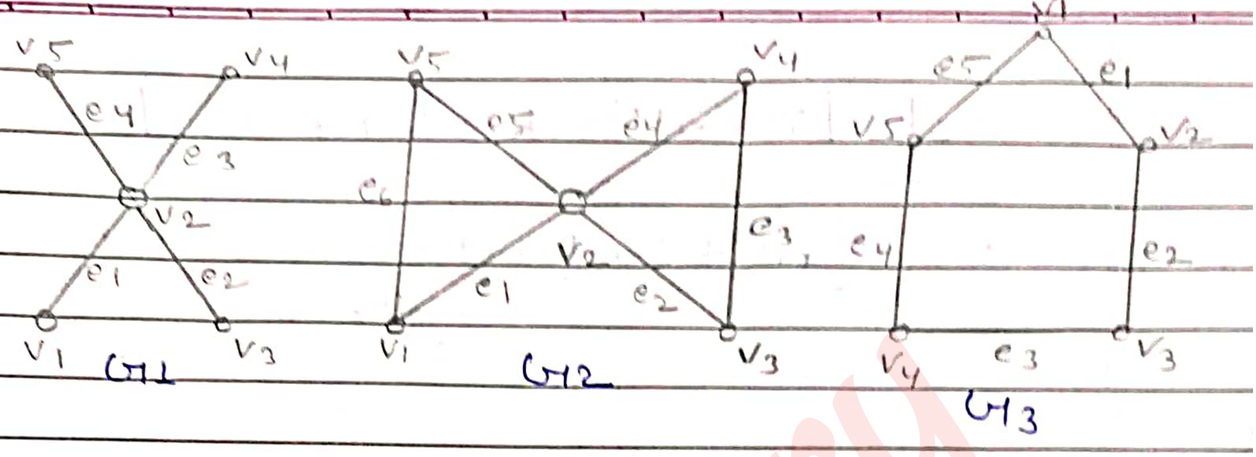
Not contain Euler Path

$G_1 = V_1 e_1 V_2 e_2 V_3 e_3 V_4$

$G_2 = V_1 e_1 V_2 e_2 V_3 e_3 V_4 e_4 V_2$

Euler Graph:

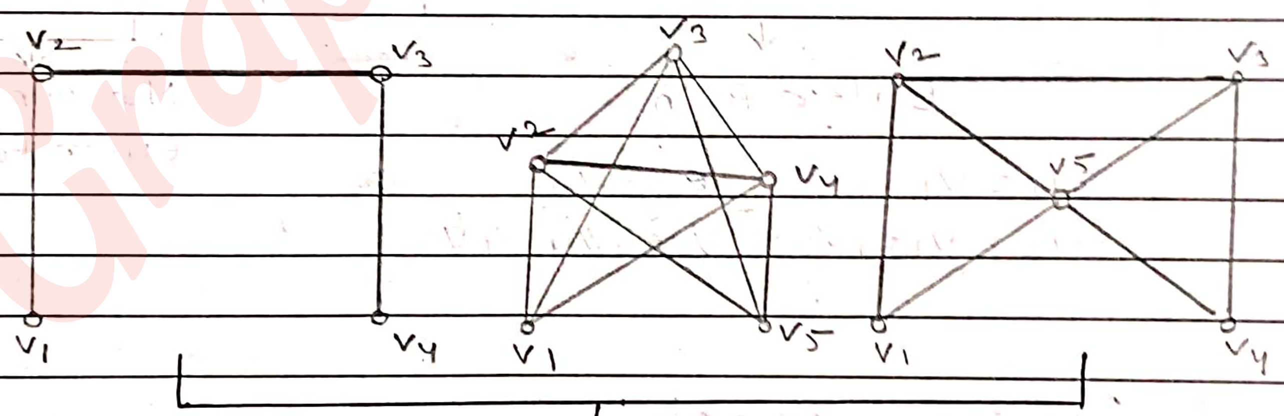
A closed walk in a graph which contains all edges of the graph called Euler graph or Euler circuit or Euler line.



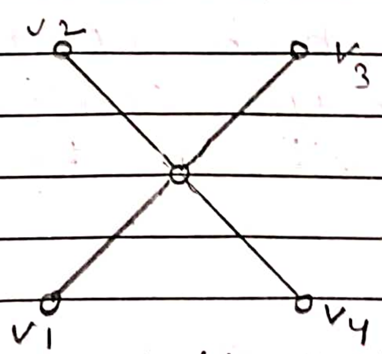
G1 = Not an Euler graph
G2 = v1 e1 v2 e2 v3 e3 v4 e4 v2 e5 v5 e6 v1] → Euler graph
G3 = v1 e1 v2 e2 v3 e3 v4 e4 v5 e5 v1

Hamiltonian Path:

An walk in a graph which covers all the vertices exactly once without repeating end vertex called Hamiltonian path.



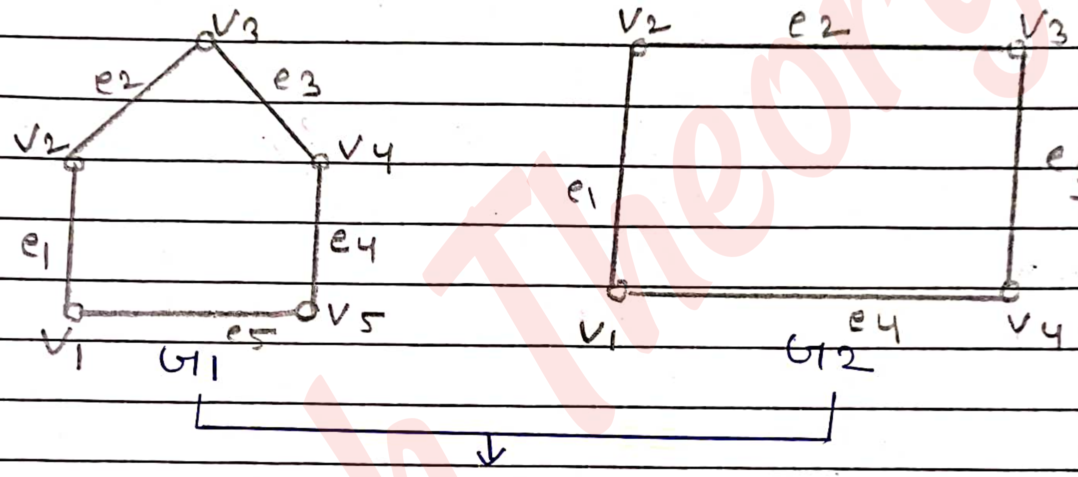
All are Hamiltonian Path



Not Hamiltonian Path

Hamiltonian Graph:

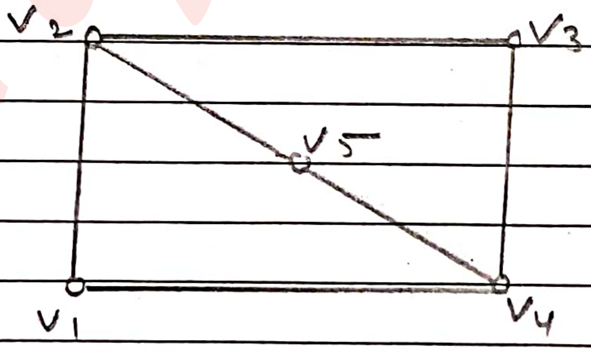
A closed walk in a graph which covers all the vertices exactly once except end points is called Hamiltonian graph or Hamiltonian circuit.



Hamiltonian Graph

$G1 = v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_5 e_5 v_1$

$G2 = v_1 e_1 v_2 e_2 v_3 e_3 v_4 e_4 v_1$



Not a Hamiltonian Graph