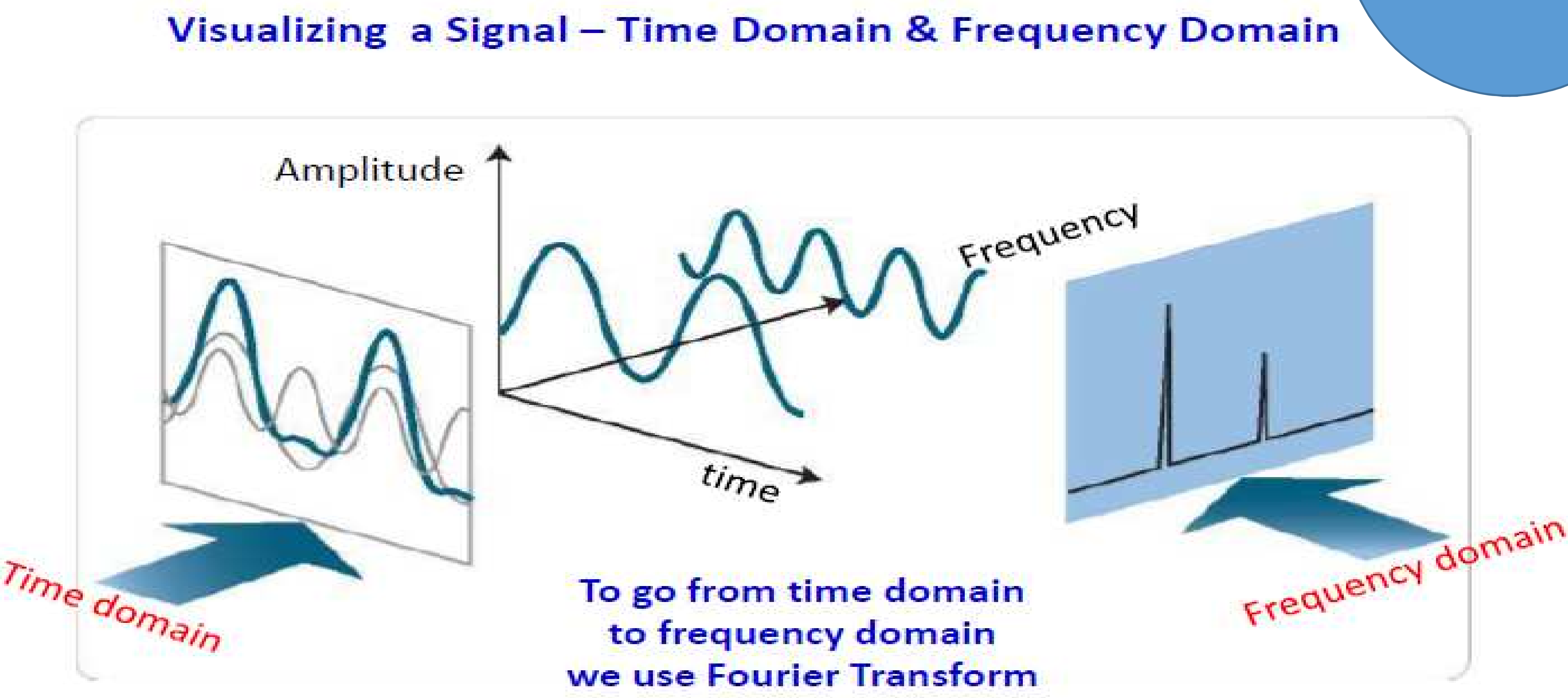
Fourier Transform

Fourier Transform

Frequency means how fast the signal is changing



Fourier Transform

The Fourier transform (*i.e.*, spectrum) of $f(t)$ is $F(\omega)$:

$$F(\omega) = \mathfrak{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt$$

$$f(t) = \mathfrak{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega$$

Therefore, $f(t) \Leftrightarrow F(\omega)$ is a Fourier Transform pair

Fourier Transform

Fourier Transform Produces a Continuous Spectrum

$\mathfrak{F}\{f(t)\}$ gives a spectra consisting of a continuous sum of exponentials with frequencies ranging from $-\infty$ to $+\infty$.

$$F(\omega) = \left|F(\omega)\right| \cdot e^{j\varphi(\omega)} \; ,$$

where $|F(\omega)|$ is the continuous amplitude spectrum of *f(t)*

and

$\varphi(\omega)$ is the continuous phase spectrum of *f(t)*.

Fourier Transform of special signals

- Impulse Function($\delta(t)$)

The Fourier transform (*i.e.*, spectrum) of $f(t)$ is $F(\omega)$:

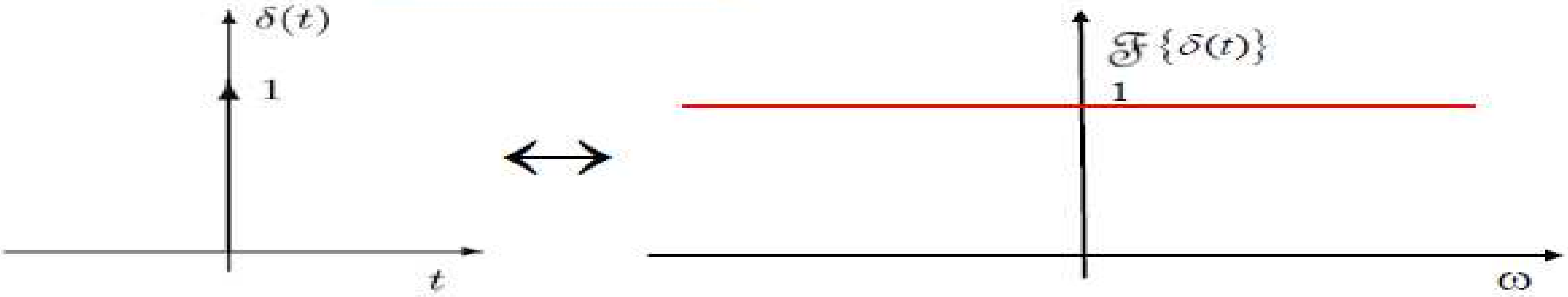
$$F(\omega) = \mathfrak{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

Fourier Transform of special signals

Example: Impulse Function δ(t)

$$F(\omega) = \mathfrak{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = e^{-j\omega t}\Big|_{t=0} = e^{j0} = 1$$

$$\boxed{\begin{array}{c} \delta(t) \Leftrightarrow 1 \\ 1 \Leftrightarrow 2\pi\delta(\omega) \end{array}}$$



$$\delta(t) = \begin{cases} \infty \text{ if } t = 0 \\ 0 \text{ if } t \neq 0 \end{cases}$$

Delta function has unity area.

Fourier Transform

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Therefore, $f(t) \Leftrightarrow F(\omega)$ is a Fourier Transform pair

Fourier Transform

- From earlier slides,Fourier Transform of impulse function = 1

Fourier Transform

- Find, Fourier transform of signal $x(t) = 1$

Fourier Transform

Fourier transform of signal $x(t) = 1$

Show that, the spectrum of constant signal $x(t)=1$ has impulse $2\pi\delta(\omega)$. [Special Case].

Solution:—

Let's start from $X(j\omega) = \delta(\omega)$.

$$\therefore x(t) = 1/2\pi \int_{-\infty}^{\infty} X(e^{j\omega}) e^{j\omega t} d\omega$$

$$= 1/2\pi \int_{-\infty}^{\infty} \delta(\omega) e^{j\omega t} d\omega.$$

$$= 1/2\pi \left[e^{j\omega t}\right]_{\omega=0} = 1/2\pi.$$

Fourier Transform

Fourier transform of signal $x(t) = 1$

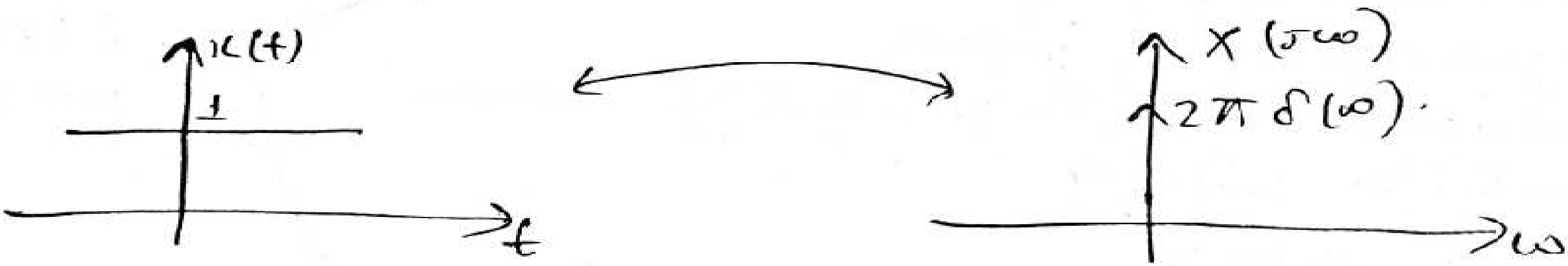
This gives, $F^{-1}[\delta(\omega)] = 1/2\pi$.

$$\therefore \delta[\omega] \Leftrightarrow F[1/2\pi]$$

So, $1/2\pi \longleftrightarrow \delta(\omega)$.

$$1 \longleftrightarrow 2\pi\delta(\omega). \;//$$

This shows, the spectrum of constant signal $x(t) = 1$ has impulse $2\pi\delta(\omega)$.



Fourier Transform of e^{jw_0t}

Find the inverse fourier transform of $X(j\omega) = \delta(\omega - \omega_0)$.

Solution:

$$x(t) = F^{-1}\left[\delta(\omega-\omega_0)\right].$$

$$= 1/2\pi \int_{-\infty}^{\infty} X(j\omega)\, e^{j\omega t}\, d\omega.$$

$$= 1/2\pi \int_{-\infty}^{\infty} \delta(\omega-\omega_0)\, e^{j\omega t}\, d\omega.$$

$$= 1/2\pi \left[e^{j\omega t}\right]_{\omega=\omega_0} = 1/2\pi\, e^{j\omega_0 t}.$$

Fourier Transform of e^{jw_0t}

This relation gives;

$$F^{-1}\left[\delta(\omega-\omega_0)\right] \longleftrightarrow 1/2\pi \; e^{j\omega_0 t}.$$

or $$\left[\delta(\omega-\omega_0)\right] \longleftrightarrow F\left[1/2\pi \; e^{j\omega_0 t}\right].$$

or $$(1/2\pi)\, e^{j\omega_0 t} \longleftrightarrow \delta(\omega-\omega_0)$$

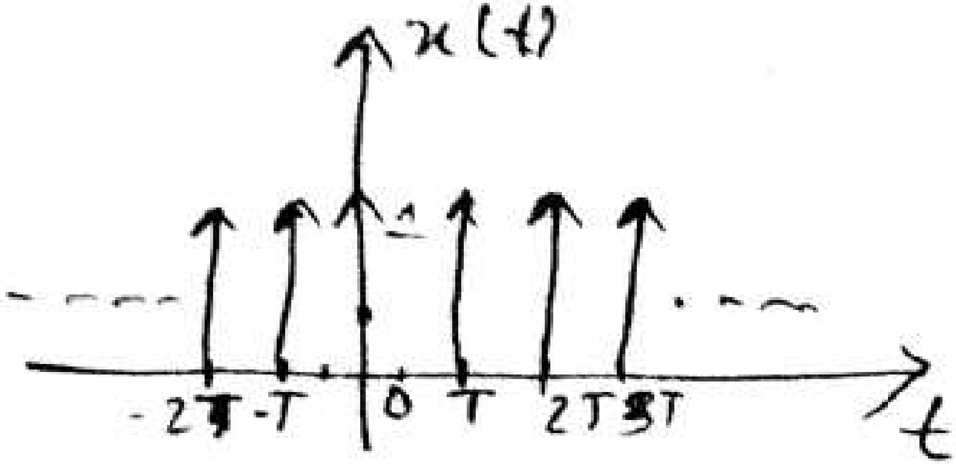
or $$e^{j\omega_0 t} \longleftrightarrow 2\pi\,\delta(\omega-\omega_0).$$

The above expression shows that the spectrum of an exponential signal $e^{j\omega_0 t}$ is $2\pi\,\delta(\omega-\omega_0)$.

Fourier Transform of Impulse Train

Find out the fourier transform of impulse train.

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT).$$



Fourier Transform of Impulse Train

Now, First we write $x(t) = \sum_{K=-\infty}^{\infty} a_K e^{jK\omega_0 t}$. [Fourier series]

where $\omega_0 = \frac{2\pi}{T}$.

$\therefore a_K = 1/T \int_T x(t) e^{-jk\omega_0 t} dt. = 1/T \int_T \left(\sum_{k=-\infty}^{\infty} \delta(t-kT) \right) e^{-jk\omega_0 t} dt$

$= 1/T \int_{-T/2}^{T/2} \left(\sum_{k=-\infty}^{\infty} \delta(t-kT) \right) e^{-jk\omega_0 t} dt.$ [Periodic Limit]

Fourier Transform of Impulse Train

In interval $[-T/2, T/2]$ $k=0$.

So, $a_k = 1/T \int_{-T/2}^{T/2} \delta(t)\, e^{-jk\omega_0 t}\, dt$

$= 1/T \left[e^{-j\omega_0 t}\right]_{t=0} = 1/T$,

Fourier Transform of Impulse Train

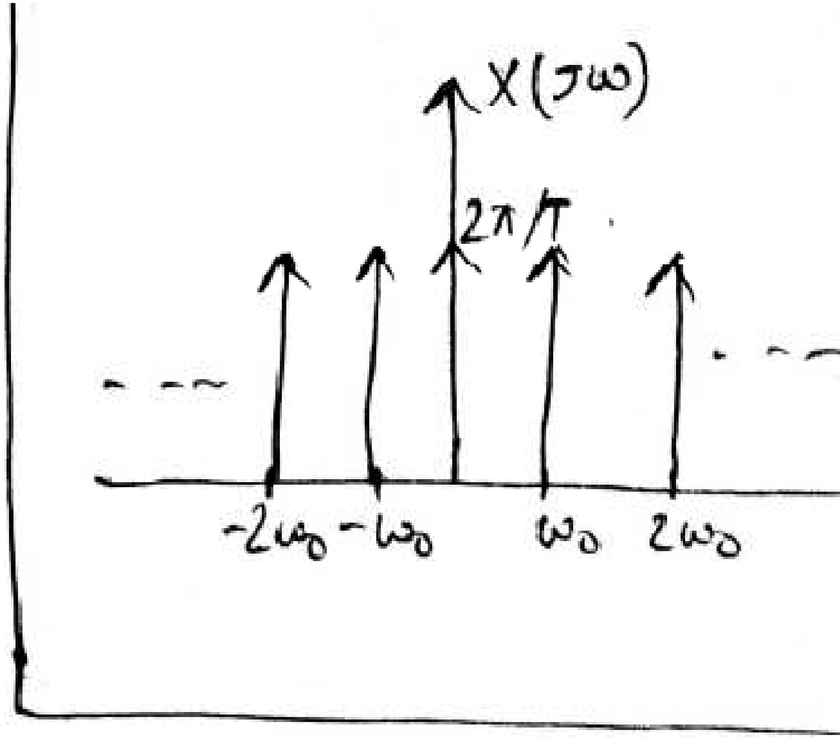
$$\therefore\ x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} 1/T\, e^{jk\omega_0 t}$$

$$= 1/T \sum_{k=-\infty}^{\infty} e^{jk\omega_0 t}$$

Now; $e^{jk\omega_0 t} \xleftrightarrow{F.T} 2\pi\,\delta(\omega - k\omega_0)$.

$$\therefore\ X(j\omega) = 1/T \sum_{k=-\infty}^{\infty} 2\pi\,\delta(\omega - k\omega_0) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0)$$

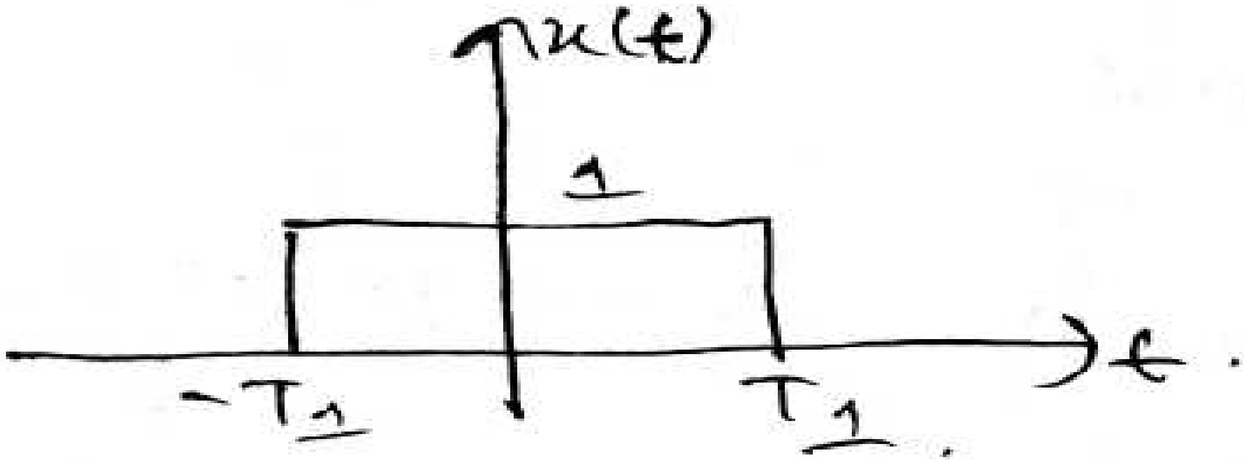


Fourier Transform of rectangle signal

Find the fourier transform of $x(t)$.

$$x(t) = \begin{cases} 1 & |t| \le T_1 \\ 0 & |t| > T_1 \end{cases}$$

Solution :—



Fourier Transform of rectangle signal

$$\therefore X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt.$$

$$= \int_{-T_1}^{T_1} x(t)\, e^{-j\omega t}\, dt$$

$$= -\frac{1}{j\omega}\left[e^{-j\omega t} \right]_{-T_1}^{T_1}$$

Fourier Transform of rectangle signal

$$= \frac{-1}{j\omega}\left[e^{-j\omega T_1} - e^{j\omega T_1}\right].$$

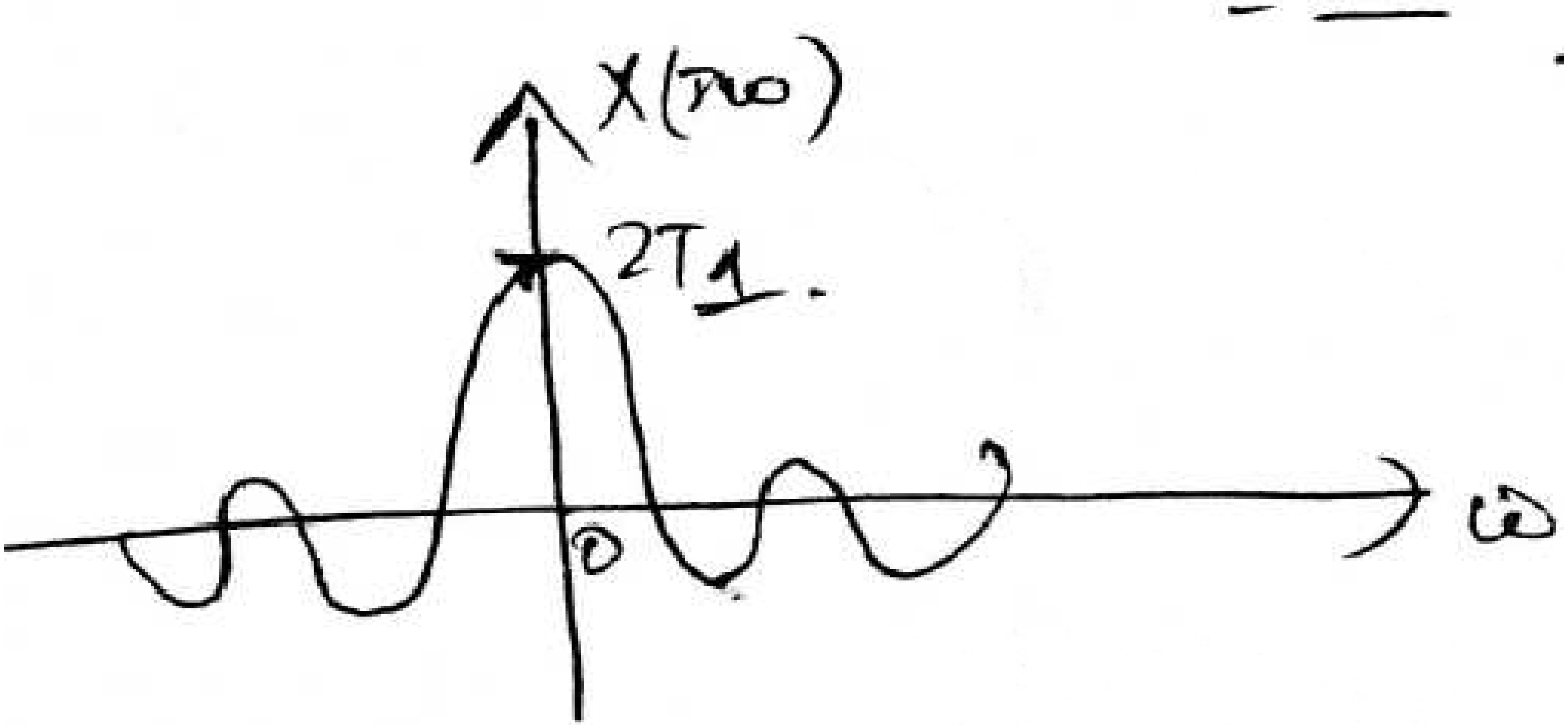
$$= \frac{1}{j\omega}\left[e^{j\omega T_1} - e^{-j\omega T_1}\right]$$

$$= \frac{1 \times 2}{\omega}\left[\frac{e^{j\omega T_1} - e^{-j\omega T_1}}{2j}\right]$$

$$= \frac{2}{\omega}\left(\sin \omega T_1\right).$$

$$= \frac{2T_1}{} \frac{\left(\sin \omega T_1\right)}{\omega T_1} = 2T_1 \operatorname{sinc}(\omega T_1).$$

Fourier Transform of rectangle signal



Relationship between exponentials and sinusoids

- Euler formula:

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$$e^{-j\omega t} = \cos(-\omega t) + j\sin(-\omega t)$$

$$= \cos(\omega t) - j\sin(\omega t)$$

$$\cos(\omega t) = \frac{e^{j\omega t} + e^{-j\omega t}}{2}$$

$$\sin(\omega t) = \frac{e^{j\omega t} - e^{-j\omega t}}{2j}$$

DTFT Properties

- In pdf(with some corrections)

$$X(w) = \sum_{n=-\infty}^{\infty} x[n] e^{-jwn}$$

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(w) e^{jwn} dw$$