Figure 4.9 shows the graphs for $x(n)$ and $X(e^{j\omega})$ for the case for which $a = 0.8$.

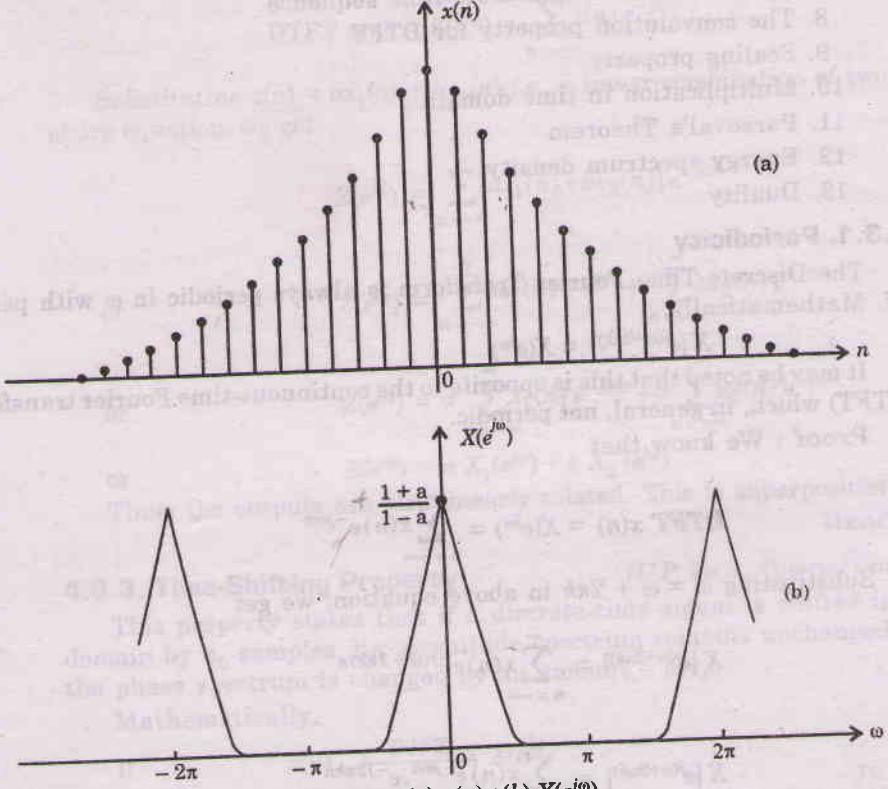

Fig. 4.9. (a) $x(n)$ (b) $X(e^{j\omega})$

4.3. Properties of Discrete-Time Fourier Transform (DTFT)

In this article, we shall discuss about the various properties of Fourier transform of discrete-time signals. Fourier transform of discrete-time signals is also referred to as discrete-time Fourier transform (DTFT). These properties are often useful in reducing the complexity in evaluation of the Discrete-Time Fourier Transform (DTFT) and inverse Discrete-time Fourier Transform (IDTFF). Also, we shall observe some of the similarities and differences between CTFT and DTFT. The derivation of DTFT properties is essentially identical to its continuous-time counterpart, *i.e.* CTFT.

Here, we shall use one notation similar to that used for CTFT to indicate the pairing of a signal and its Fourier transform, *i.e.*,

$$X(e^{j\omega}) = \text{DTFT}\,[x(n)]$$

or

$$x(n) = \text{Inverse DTFT}\,[X(e^{j\omega})]$$

or

$$x(n) \xleftrightarrow{\text{DTFT}} X(e^{j\omega})$$

In this section, we shall discuss following properties of the DTFT, in detail :

1. Periodicity of the DTFT
2. Linearity of the DTFT
3. Time shifting
4. Frequency shifting
5. Multiplication by n: Frequency Differentiation

6. Complex conjugation and conjugate symmetry
7. Time reversal of a discrete-time sequence
8. The convolution property for DTFT
9. Scaling property
10. Multiplication in time domain
11. Parseval's Theorem
12. Energy spectrum density
13. Duality

4.3.1. Periodicity

The Discrete-Time Fourier Transform is always periodic in ω with period 2π. Mathematically,

$$X[e^{j(\omega+2\pi k)}] = X(e^{j\omega})$$

It may be noted that this is opposite to the continuous-time Fourier transform (CTFT) which, in general, not periodic.

Proof : We know that

$$DTFT\ x(n) = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

Substituting $\omega = \omega + 2\pi k$ in above equation, we get

$$X[e^{j(\omega+2\pi k)}] = \sum_{n=-\infty}^{\infty} x(n)e^{-j(\omega+2\pi k)n}$$

or $$X[e^{j(\omega+2\pi k)}] = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}.e^{-j2\pi kn}$$

Now, $$e^{-j2\pi kn} = \cos(2\pi kn) - j\sin(2\pi kn) \quad ...(4.21)$$

It may be noted that in above equation both k and n are integers.

Thus $\cos(2\pi kn) = 1$ always and

$\sin(2\pi kn) = 0$ always

Hence, $e^{-j2\pi kn} = 1$

Therefore, equation (4.21) becomes

$$X[e^{j(\omega+2\pi k)}] = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

or $$X[e^{j(\omega+2\pi k)}] = X(e^{j\omega})$$

or $$X[e^{j(\omega+2\pi k)}] = \text{DTFT of } x(n)$$ **Hence Proved.**

4.3.2. Linearity

According to this property, the discrete-time Fourier transform (DTFT) is linear. Mathematically,

If $$x_1(n) \xleftrightarrow{DTFT} X_1(e^{j\omega})$$

and $$x_2(n) \xleftrightarrow{DTFT} X_2(e^{j\omega})$$

Then, according to this property, we have

$$z(n) = ax_1(n) + bx_2(n) \xleftrightarrow{DTFT} Z(e^{j\omega}) = aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

where a and b are constants.

Proof : We k

Substituting
above equation, we

or

or

or

Thus, the outpu

4.3.3. Time-Shiftin

This property st
domain by n_0 sampl
the phase spectrum
Mathematically,

If $x(n)$

Then $x(n-n_0)$

Proof :

We know that, $x(n$

or DT

Therefore,

$DTFT\,x($

Putting $n - n_0 = m$,

$DTFT\,x($

or $DTFT\,x(n$

or $DTFT \cdot x(n$

or $DTFT\,x(n$

Proof : We know that

$$\text{DTFT } z(n) = Z(e^{j\omega}) = \sum_{n=-\infty}^{\infty} z(n)\, e^{-j\omega n}$$

Substituting $z(n) = ax_1(n) + bx_2(n)$ *i.e.*, a linear combination of two inputs in above equation, we get

$$Z(e^{j\omega}) = \sum_{n=-\infty}^{\infty} [ax_1(n) + bx_2(n)]\, e^{-j\omega n}$$

or

$$Z(e^{j\omega}) = \sum_{n=-\infty}^{\infty} ax_1(n)\, e^{-j\omega n} + \sum_{n=-\infty}^{\infty} bx_2(n)\, e^{-j\omega n}$$

or

$$Z(e^{j\omega}) = a \sum_{n=-\infty}^{\infty} x_1(n)\, e^{-j\omega n} + b \sum_{n=-\infty}^{\infty} x_2(n)\, e^{-j\omega n}$$

or

$$Z(e^{j\omega}) = a\, X_1(e^{j\omega}) + b\, X_2(e^{j\omega})$$

Thus, the outputs are also linearly related. This is superposition principle.

Hence Proved.

4.3.3. Time-Shifting Property

(U.P. Tech., Tutorial Question Bank)

This property states that if a discrete-time signal is shifted in the time-domain by n_0 samples, its magnitude spectrum remains unchanged. However, the phase spectrum is changed by an amount $-\omega n_0$.

Mathematically,

If $\quad x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$

Then $x(n - n_0) \xleftrightarrow{DTFT} e^{-j\omega n_0} \cdot X(e^{j\omega})$; where n_0 is an integer.

Proof :

We know that, $x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$

or

$$DTFT\ x(n) = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

Therefore,

$$DTFT\ x(n - n_0) = \sum_{n=-\infty}^{\infty} x(n - n_0)\, e^{-j\omega n}$$

Putting $n - n_0 = m$, so that $n = m + n_0$, we get

$$DTFT\ x(n - n_0) = \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\omega(m+n_0)}$$

or

$$DTFT\ x(n - n_0) = \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\omega m} . e^{-j\omega n_0}$$

or

$$DTFT\ .\ x(n - n_0) = e^{-j\omega n_0} \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\omega m}$$

or

$$DTFT\ x(n - n_0) = e^{-j\omega n_0}\, X(e^{j\omega})$$

or $$x(n-n_0) \xleftrightarrow{DTFT} e^{-j\omega n_0}\, X(e^{j\omega})$$

Hence, according to above equation, time-shifting property states that delaying a discrete-time signal by n_0 units does not change its amplitude spectrum. The phase-spectrum, however, is changed by $-\omega n_0$.

This added phase is thus a linear function of ω with a slop $-n_0$.

This means that the time-delay in a discrete-time signal causes a linear phase shift in its spectrum.

4.3.4. Frequency-Shifting Property

This property states that multiplication of a sequence $x(n)$ by $e^{j\omega_0 n}$ is equivalent to a frequency translation of the spectrum $X(e^{j\omega})$ by ω_0. Since the spectrum $X(e^{j\omega})$ is periodic, the shift ω_0 applies to the spectrum of the signal in every period.

Mathematically, we have

If $$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$$

Then $$e^{j\omega_0 n}\, x(n)$$

Proof :

We know that

$$DTFT\, x(n) = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

then $$DTFT\; e^{j\omega_0 n}\, x(n) = \sum_{n=-\infty}^{\infty} x(n)\, e^{j\omega_0 n} \cdot e^{-j\omega n}$$

or $$DTFT\; e^{j\omega_0 n} \cdot x(n) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\omega-\omega_0)n}$$

or $$DTFT\; e^{j\omega_0 n} \cdot x(n) = X[e^{j(\omega-\omega_0)}]$$

or $$e^{j\omega_0 n} \cdot x(n) \xleftrightarrow{DTFT} X[e^{j(\omega-\omega_0)}]$$

Figure 4.10 illustrates the frequency shifting properly

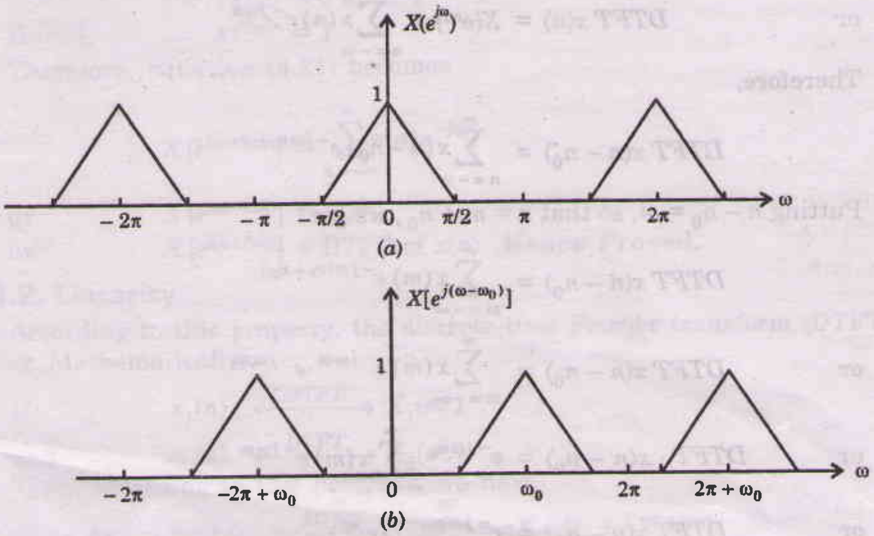

Fig. 4.10. *Illustration of frequency shifting property*

4.3.5. Multiplication by n : Frequency Differentiation

This property states that

If $$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$$

then $$n\,x(n) \xleftrightarrow{DTFT} j\,\frac{dX(e^{j\omega})}{d\omega}$$

Proof :

We know that

$$DTFT\,x(n) = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\,e^{-j\omega n}$$

Now $$\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{\infty} \frac{d}{d\omega}\left[x(n)\,e^{-j\omega n}\right]$$

or $$\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{\infty} -j\,n\,x(n)\,e^{-j\omega n}$$

or $$\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{\infty} -j\,n\,x(n)\,e^{-j\omega n} = DTFT\,[-jn\,x(n)]$$

or $$-j\,n\,x(n) \xleftrightarrow{DTFT} \frac{dX(e^{j\omega})}{d\omega}$$

or $$nx\,(n) \xleftrightarrow{DTFT} j\,\frac{dX(e^{j\omega})}{d\omega}$$ **Hence Proved.**

4.3.6. Complex Conjugation and Conjugate Symmetry

This property states that we can obtain complex conjugation of a complex discrete-time signal $x(n)$ by reversing the sign of the imaginary part of the complex signal, $x(n)$.

Mathematically,

If $$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$$

then $$x^*(n) \xleftrightarrow{DTFT} X^*(e^{-j\omega})$$

Now, if the discrete-time signal $x(n)$ is a real valued signal, then its DTFT $X(e^{jw})$ will be conjugate symmetric. This means that

$$X(e^{j\omega}) = X^*(e^{-j\omega}) \qquad \text{[if } x(n) \text{ is real]}$$

> **Note :** From above, it is clear that $Re[X(e^{j\omega})]$ is an even function of ω and $I_m\,[X(e^{j\omega})]$ is an odd function of ω. In the same way, the magnitude of $X(e^{j\omega})$ is an even function and the phase angle $\angle X(e^{j\omega})$ is an odd function.

4.3.7. Time Reversal

According to this property, if a discrete-time signal is folded about the origin in time, its magnitude spectrum remains unchanged, however, the phase spectrum undergoes a change in sign. Mathematically, we have

If $$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$$

then $$x(-n) \xleftrightarrow{DTFT} X(e^{-j\omega})$$

Proof :

Let there be a discrete-time signal $x(n)$ with *DTFT* $X(e^{j\omega})$, *i.e.,*

$$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$$

Let there be another discrete-time signal $y(n)$ with *DTFT* $Y(e^{j\omega})$, *i.e.,*

$$y(n) \xleftrightarrow{DTFT} Y(e^{j\omega}) \qquad \text{such that } y(n) = x(-n)$$

Now, $$Y(e^{j\omega}) = DTFT\, y(n) = \sum_{n=-\infty}^{\infty} y(n)\, e^{-j\omega n}$$

or $$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(-n)\, e^{-j\omega n}$$

Substituting $m = -n$ into above expression, we get

$$Y(e^{j\omega}) = \sum_{m=-\infty}^{\infty} x(m)\, e^{-j(-m)\omega}$$

or $$Y(e^{j\omega}) = \sum_{m=-\infty}^{\infty} x(m)\, e^{j\omega m}$$

or $$Y(e^{j\omega}) = X(e^{-j\omega}) = DTFT\,[x(-n)]$$

Therefore, $x(-n) \xleftrightarrow{DTFT} X(e^{-j\omega})$ **Hence Proved.**

4.3.8. Convolution Property

(U.P. Tech., Tutorial Question Bank)

In third chapter, we have discussed the importance of CTFT with respect to its effect on the convolution operation and its application in dealing with continuous-time LTI systems. In this section, we shall discuss the importance of DTFT with respect to its effect on the convolution operation and analysis of discrete-time LTI systems.

The convolution property states that

If $x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$

and $y(n) \xleftrightarrow{DTFT} Y(e^{j\omega})$

then $z(n) = x(n) \otimes y(n) \xleftrightarrow{DTFT} Z(e^{j\omega}) = X(e^{j\omega})\, Y(e^{j\omega})$

Proof :

We know that

$$DTFT\, z(n) = Z(e^{j\omega}) = \sum_{n=-\infty}^{\infty} z(n)\, e^{-j\omega n}$$

Substituting $$z(n) = x(n) \otimes y(n) = \sum_{k=-\infty}^{\infty} x(k)\, y(n-k)$$

in above equation, we get

$$Z(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x(k)\, y(n-k) \right]$$

Changing the o

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Thus convolutio
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Now, let us app
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figure 4.11.

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Using DTFT, th

Y

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respectively.

Now, combining

where $H(e^{j\omega})$ is
response $h(n)$ of the d
response of the di

Example 4.6. Deter
with impulse respo
system.

Solution : We know
equal to the DTFT
response is determi

Changing the order of summations we get

$$Z(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x(k) \sum_{n=-\infty}^{\infty} y(n-k)\, e^{-j\omega n}$$

Again, substituting $n - k = m$, the above equation becomes

$$Z(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x(k) \sum_{m=-\infty}^{\infty} y(m)\, e^{-j\omega(m+k)}$$

or
$$Z(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x(k) \sum_{m=-\infty}^{\infty} y(m)\, e^{-j\omega n}\, e^{-j\omega k}$$

or
$$Z(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x(k)\, e^{-j\omega k} \sum_{m=-\infty}^{\infty} y(m)\, e^{-j\omega n}$$

or
$$Z(e^{j\omega}) = X(e^{j\omega})\, Y(e^{j\omega})$$

Thus convolution of the two sequences is equivalent to multiplication of their spectrums.

Now, let us apply DTFT for representing and analyzing discrete-time LTI systems, particularly, when $x(n)$, $h(n)$ and $y(n)$ are the input, impulse response and the output, respectively of a discrete-time LTI system as shown in figure 4.11.

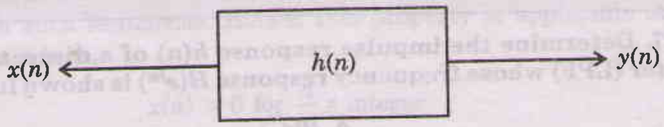

Fig. 4.11. *Discrete-time LTI system*

Now, here, the output $y(n)$ can be determined by convolving $x(n)$ and $h(n)$, *i.e.*,

$$y(n) = x(n) \otimes h(n) \qquad ...(4.22)$$

Using DTFT, the above equation takes the form

$$Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega}) \qquad ...(4.23)$$

where $X(e^{j\omega})$, $H(e^{j\omega})$ and $Y(e^{j\omega})$ are the DTFTs of $x(n)$, $h(n)$ and $y(n)$ respectively.

Now, combining equations (4.22) and (4.23), we have

$$y(n) = x(n) \otimes h(n) \xleftrightarrow{\text{DTFT}} Y(e^{j\omega}) = X(e^{j\omega})\, H(e^{j\omega}) \qquad ...(4.24)$$

where $H(e^{j\omega})$ is the discrete-time Fourier transform (DTFT) of the impulse response $h(n)$ of the discrete-time LTI system. **It is also known as the frequency response of the discrete-time LTI system.**

Example 4.6. Determine the frequency response of a discrete-time LTI system with impulse response $h(n) = \delta(n - n_0)$. Also determine the output for this system.

Solution : We know that the frequency response of a discrete-time LTI system is equal to the DTFT of the impulse response $h(n)$ of the system. The frequency response is determined as

$$H(e^{j\omega}) = \text{DTFT}\,[h(n)] = \sum_{n=-\infty}^{\infty} h(n)\, e^{-j\omega n}$$

or $$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta(n-n_0)\, e^{-j\omega n} = e^{-j\omega n_0}\,* \quad \ldots(i)$$

Now, according to the convolution property of DTFT, we have

$$y(n) = h(n) \otimes x(n) \xleftrightarrow{\text{DTFT}} Y(e^{j\omega}) = H(e^{j\omega})\, X(e^{j\omega})$$

or $$Y(e^{j\omega}) = H(e^{j\omega})\, X(e^{j\omega}) \quad \ldots(ii)$$

Substituting equation (*i*) in equation (*ii*), we have

$$Y(e^{j\omega}) = e^{-j\omega n_0}\, X(e^{j\omega}) \quad \ldots(iii)$$

The output $y(n)$ of above discrete-time LTI system is determined by taking the inverse DTFT of equation (*iii*).

Thus, taking inverse DTFT, we get

$$y(n) = \text{Inverse DTFT } [Y(e^{j\omega})]$$
$$= \text{Inverse DTFT } [e^{-j\omega n_0}\, X(e^{j\omega})]$$

or $$y(n) = x(n - n_0)$$

Hence, it may be noted that in this example, the output $y(n)$ is equal to the shifted version of the input $x(n)$ by a constant time n_0. The frequency response $H(e^{j\omega}) = e^{-j\omega n_0}$ is purely time-shifted and has unity magnitude at all the frequencies. Its phase characteristics are equal to $-\omega n_0$ *i.e.*, it is linear with frequency.

Example 4.7. Determine the impulse response $h(n)$ of a discrete-time ideal low-pass filter (LPF) whose frequency response $H(e^{j\omega})$ is shown in figure 4.12.

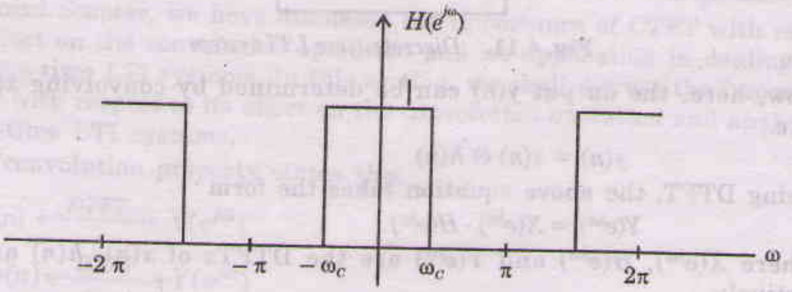

Fig. 4.12. *Frequency response of a discrete-time ideal low-pass filter.*

Solution: We know that the impulse response $h(n)$ of discrete-time ideal low pass filter is equal to the inverse DTFT of the frequency response $H(e^{j\omega})$ *i.e.*,

$$h(n) = \text{Inverse DTFT } [H(e^{j\omega})]$$

or $$h(n) = \frac{1}{2\pi} \int_{2\pi} H(e^{j\omega})\, e^{j\omega n}\, d\omega \quad \ldots(i)$$

In particular, using $-\pi \le \omega \le \pi$ as the interval of integration in equation (*i*), we have

$$h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega})\, e^{j\omega n} d\omega$$

or $$h(n) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} 1.e^{j\omega n}\, d\omega = \frac{1}{2\pi} \left[\frac{1}{jn} e^{j\omega n} \right]_{-\omega_c}^{\omega_c}$$

* Here, we have used shifting or sampling property of impulse function.

or $$h(n) = \frac{1}{n\pi}\left[\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j}\right] = \frac{1}{n\pi}.\sin\omega_c n = \frac{\sin\omega_c n}{\pi n}$$

This impulse response $h(n)$ has been shown in figure 4.13.

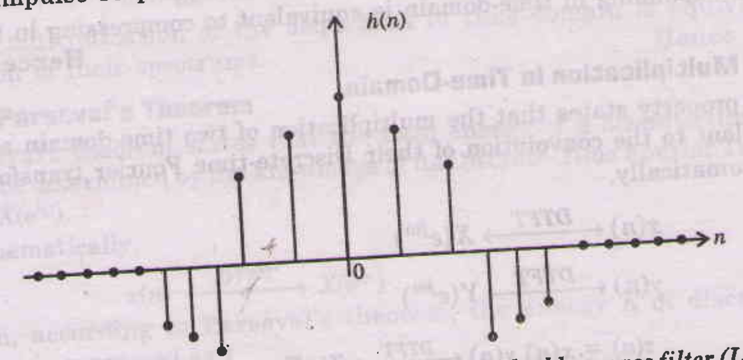

Fig. 4.13. ***Impulse response of a discrete-time ideal low-pass filter (LPF).***

4.3.9. Scaling

Let the discrete-time sequence be scaled as

$$y(n) = x(pn) \text{ for } p\text{-integer}$$

In this case, information in $x(n)$ is discarded. Then scaling property has no meaning with such sequences. Hence, this property is applicable only to those sequences for which,

$$x(n) = 0 \text{ for } \frac{n}{p} \neq \text{integer}$$

then $x(pn) \neq 0$ for all n values.

Then data will not be discarded. The discarded data due to scaling will be zeros.

Thus, the scaling property may be expressed as

If $$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$$

then $$y(n) = x(pn) \xleftrightarrow{DTFT} Y(e^{j\omega}) = X\left(\frac{\omega}{p}\right)$$

Proof : We know that

$$\text{DTFT } y(n) = Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y(n)\, e^{-j\omega n}$$

or $$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(pn)\, e^{-j\omega n} \quad ...(4.25)$$

Let us substitute $pn = m$

Now, since n has the range of $-\infty$ to ∞, m will also have the same range. then above equation (4.25) becomes

$$Y(e^{j\omega}) = \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\omega m/p}$$

or $$Y(e^{j\omega}) = \sum_{m=-\infty}^{\infty} x(m)\, e^{-j\left(\frac{\omega}{p}\right)m}$$

$$Y(e^{j\omega}) = X\left(e^{j\frac{\omega}{p}}\right)$$

Thus, expanding in time-domain is equivalent to compressing in frequency domain.

Hence Proved.

4.3.10. Multiplication in Time-Domain

This property states that the multiplication of two time-domain sequences is equivalent to the convolution of their Discrete-time Fourier transforms.

Mathematically,

If $\quad x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$

and $\quad y(n) \xleftrightarrow{DTFT} Y(e^{j\omega})$

then $\quad z(n) = x(n)\, y(n) \xleftrightarrow{DTFT} Z(e^{j\omega}) = \frac{1}{2\pi}[X(e^{j\omega}) \otimes Y(e^{j\omega})]$

Proof : We know that

$$\text{DTFT } z(n) = Z(e^{j\omega}) = \sum_{n=-\infty}^{\infty} z(n)\, e^{-j\omega n}$$

Substituting $z(n) = x(n)\, y(n)$ in above equation, we get

$$Z(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, y(n)\, e^{-j\omega n} \qquad \text{...(4.26)}$$

Also, inverse DTFT is expressed as

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\lambda})\, e^{j\lambda} d\lambda$$

Here, we have used separate frequency variable λ. Substituting the above expression of $x(n)$ in equation (4.26), we get

$$Z(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\lambda})\, e^{j\lambda n}\, d\lambda . y(n)\, e^{-j\omega n}$$

Now, interchanging the order of summation and integration, we get

$$Z(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\lambda}) \sum_{n=-\infty}^{\infty} y(n)\, e^{j\lambda n}\, e^{-j\omega n}\, d\lambda$$

or $$Z(e^{j\omega}) = \frac{1}{12}\int_{-\pi}^{\pi} X(e^{j\lambda}) \left[\sum_{n=-\infty}^{\infty} y(n)\, e^{-j(\omega-\lambda)n}\right] d\lambda$$

The term in brackets is equal to $Y[e^{j(\omega-\lambda)}]$.

Thus, above equation becomes

$$Z(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\lambda})\, Y[e^{j(\omega-\lambda)}]\, d\lambda$$

But, above equation represents convolution of $X(e^{j\omega})$ and $Y(e^{j\omega})$, *i.e.*,

$$Z(e^{j\omega}) = \frac{1}{2\pi}[X(e^{j\omega}) \otimes Y(e^{j\omega})]$$

Thus multiplication of the sequences in time-domain is equivalent to convolution of their spectrums. **Hence Proved**

4.3.11. Parseval's Theorem

Parseval's theorem states that the total energy of a discrete-time signal $x(n)$ may be determined by the knowledge of its Discrete-Time Fourier Transform (DTFT), $X(e^{j\omega})$.

Mathematically,

If $$x(n) \xleftrightarrow{DTFT} X(e^{j\omega})$$

Then, according to Parseval's theorem, the energy E of discrete-time signal $x(n)$ is expressed as

$$E = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|X(e^{j\omega})\right|^2 d\omega$$

Proof : We know that the energy of a discrete-time signal $x(n)$ is expressed as

$$E = \sum_{n=-\infty}^{\infty}|x(n)|^2$$

Since $$|x(n)| = x(n)\, x^*(n)$$

Therefore, $$E = \sum_{n=-\infty}^{\infty} x(n)\, x^*(n) \quad \text{...(4.27)}$$

We know that the inverse DTFT of $x^*(n)$ is expressed as

$$x^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\omega})e^{-j\omega n}d\omega$$

Thus, replacing $x^*(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\omega})\, e^{-j\omega n}\, d\omega$, in equation (4.27), we get

$$E = \sum_{n=-\infty}^{\infty} x(n)\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\omega})\, e^{-j\omega n}\, d\omega\right]$$

Integrating the order of integration and summation, we get

$$E = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\omega})\left[\sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}\right] d\omega$$

or $$E = \frac{1}{2\pi}\int_{-\pi}^{\pi} X^*(e^{j\omega})\, X(e^{j\omega})\, d\omega$$

But we know that $X^*(e^{j\omega})\, X(e^{j\omega}) = |X(e^{j\omega})|^2$

Therefore, we have

$$E = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|X(e^{j\omega})\right|^2 d\omega$$

Thus, energy 'E' of the discrete-time signal $x(n)$ is

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 \, d\omega \qquad ...(4.28)$$

This is the Parseval's theorem for discrete-time aperiodic signals with finite energy which states that energy of a discrete-time signal may also be obtained with the help of DTFT. **Hence Proved**

4.3.12. Energy Density Spectrum of Discrete-Time Aperiodic Signals

We know that energy of a discrete-time signal $x(n)$ is expressed as

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2 \qquad ...(4.29)$$

According to Parseval's theorem, this energy may also be expressed in terms of discrete-time Fourier transform as under :

$$E = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|X(e^{j\omega})\right|^2 d\omega$$

Substituting $|X(e^{j\omega})|^2 = \psi(e^{j\omega})$ in above expression, we get

$$E = \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi(e^{j\omega}) \, d\omega$$

Hence, it may be observed that the quantity

$$\psi(e^{j\omega}) = |X(e^{j\omega})|^2$$

represents the distribution of energy as a function of frequency and so it is known as the **energy density spectrum of discrete-time signal $x(n)$.**

It may also be noted that $\psi(e^{j\omega})$ does not contain any phase information.

Now, if $x(n)$ is real, then we have

$$X^*(e^{j\omega}) = X(e^{-j\omega})$$

or equivalently $\qquad |X(e^{-j\omega})| = |X(e^{j\omega})|$

This is called even symmetry.

Now, since $\qquad \psi(e^{j\omega}) = |X(e^{j\omega})|^2$

Therefore, it follows that

$$\psi(e^{-j\omega}) = \psi(e^{j\omega}) \qquad \text{(even symmetry)}$$ **Hence Proved.**

Example 4.8. A discrete-time signal is given as $x(n) = a^n \cdot u(n)$ for $-1 < a < 1$. Determine and sketch the energy density spectrum $\psi(e^{j\omega})$.

Solution : Given that

$$x(n) = a^n \cdot u(n) \text{ for } -1 < a < 1 \text{ or } |a| < 1$$

Since $|a| < 1$; the discrete-time signal $x(n)$ is absolutely summable. This can be verified by applying the geometric summation formula as

$$\sum_{n=-\infty}^{\infty} |x(n)| = \sum_{n=0}^{\infty} |a|^n$$

$$= \frac{1}{1-|a|} < \infty$$

Therefore, the Discrete-Time Fourier Transform (DTFT) of $x(n)$ exists and may be obtained as

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} x(n)e^{-j\omega n} = \sum_{n=0}^{\infty} a^n \cdot e^{-j\omega n}$$