IIR Filter Design

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8.1 INTRODUCTION

As discussed earlier, digital filters are of two types depending on the number of sample points used to determine the unit sample (*i.e.*, impulse) response of a LTI discrete-time system. Also, if infinite number of sample points are used to determine the unit-sample response then these digital filters are known Infinite-duration Impulse Response (IIR) digital filters.

It may be noted that Infinite-duration Impulse Response (IIR) digital filter design procedures are extensions of those originally developed for analog filters. In fact, IIR digital filters are commonly used to replace existing analog filters.

... available for the design of digital

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As a matter of fact, there are various techniques which are available for the design of digital filters having an infinite duration unit impulse response. The analog filter design is well-developed and the techniques discussed in this chapter are all based on taking an analog filter and converting it into a digital filter. This means that the design of an IIR filter involves design of a digital filter in the analog domain and transforming the design into the digital domain.

We can write the system function describing an analog filter as under :

$$H_a(s) = \frac{B(s)}{A(s)} = \frac{\sum_{k=0}^{M} b_k . s^k}{\sum_{k=0}^{N} a_k . s^k} \qquad \text{...(8.1)}$$

Here, $\{a_k\}$ and $\{b_k\}$ are called the filter coefficients.

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The transform impulse response of these filter coefficients is related to $H_a(s)$ by the Laplace transform, *i.e.*,

$$H_a(s) = \int_{-\infty}^{\infty} h(t)e^{-st}.dt \qquad ...(8.2)$$

Thus, the system function $H_a(s)$ can also be obtained from impulse response $h(t)$.

Furthermore, the analog filter having the rational system $H(s)$ expressed in equation (8.1) can also be defined by the linear constant-coefficient differential equations as under :

$$\sum_{k=0}^{N} a_k \cdot \frac{d^k.y(t)}{dt^k} = \sum_{k=0}^{M} b_k \cdot \frac{d^k.x(t)}{dt} \qquad ...(8.3)$$

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Here, $x(t)$ is the input signal of the filter and $y(t)$ is the output signal of the filter.

It may be noted that the above three equivalent characterization (*i.e.*, equations (8.1) (8.2) and (8.3) of an analog filter leads to three alternative methods to transform an analog filter into the digital domain. Also, the designing

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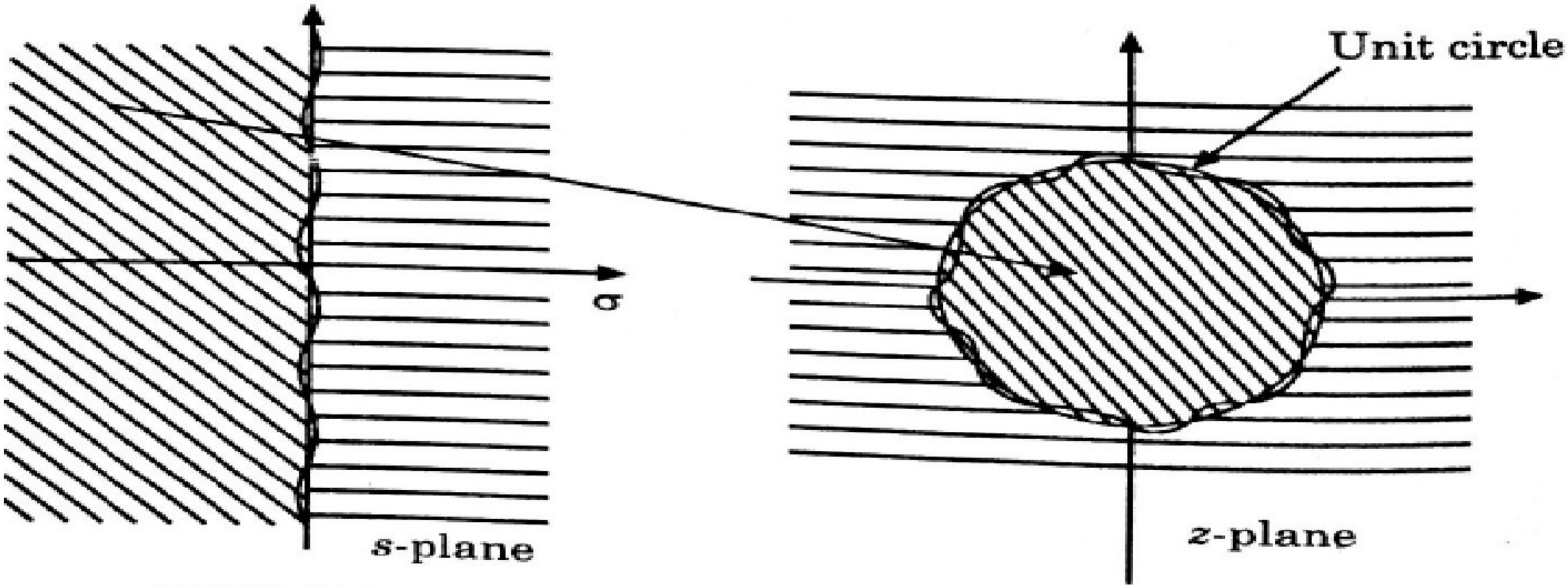

FIGURE 8.1 *Illustration of mapping from s-plane to z-plane.*

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the digital domain. Also, the designing techniques for Infinite Impulse Response (IIR) filters are presented with the restriction that the filters be realizable and stable.

We know that an analog filter having system function $H(s)$ would be stable if all its poles lie in the left-half of the s-plane. Now, as a result of this, if the conversion techniques are to be effective the technique must posses the properties as listed ahead :

(*i*) The $j\Omega$ axis in the s-plane must map onto the unit circle in the

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z-plane. This aspect provides a direct relationship between the two frequency variables in the two domains.

(*ii*) The left-half plane of the s-plane must map into the inside of the unit circle in the z-plane to convert a stable analog filter into a stable digital filter.

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8.2 BASIC FUNCTION OF DIGITAL FILTER

As a matter of fact the basic function of a digital filter is to eliminate the noise and to extract the signal of interest from other signals. A digital filter is a basic device used in digital signal processing (DSP).

There are several techniques available to design the digital filters. But, generally while designing a digital filter, first an analog filter is designed and then it is converted into the corresponding digital filter.

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8.3 REVIEW OF ANALOG FILTERS

Before discussing digital filters, in detail, let us discuss some fundamental aspects related to analog filters. This is necessary because, generally digital filters are designed using analog filters.

8.3.1 Few Parameters Related to Analog Filters

First, we shall go through few parameters which are related to analog filters as under:

(*i*) Passband :

It passes certain range of frequencies. In the passband, attenuation is zero.

(*ii*) Stopband :

It suppresses certain range of frequencies. In the stopband, attenuation is infinity.

(*iii*) Cut-off frequency :

This is the frequency which separates passband and stopband.

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8.3.2 Classification of Analog Filters

The different types of analog filters are as under :

(*i*) Low Pass Filter (LPF)

(*ii*) High Pass Filter (HPF)

(*iii*) Band Pass Filter (BPF)

(*iv*) Band Reject Filter (BRF)

(*v*) All Pass Filter (APF)

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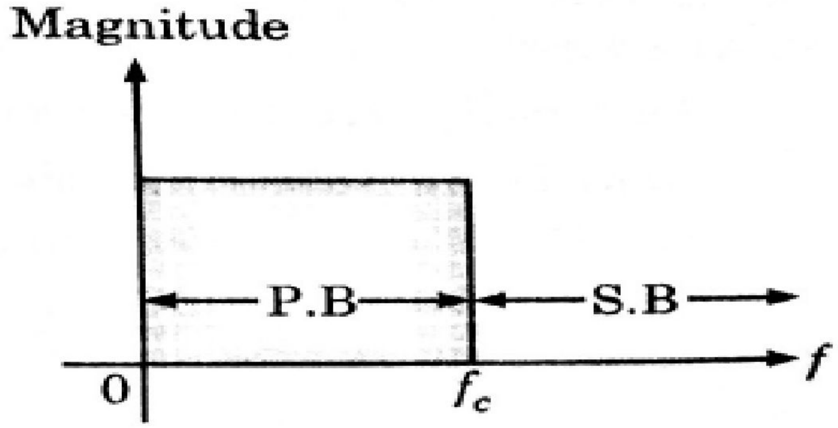

FIGURE 8.2 *Ideal low pass filter (LPF) characteristic.*

(*i*) Low Pass Filter (LPF)

It passes the frequency from zero upto some designated frequency, called as cut-off frequency. After cut-off frequency, it will not allow any signal to pass through it. An ideal low pass filter characteristic is shown in figure 8.2.

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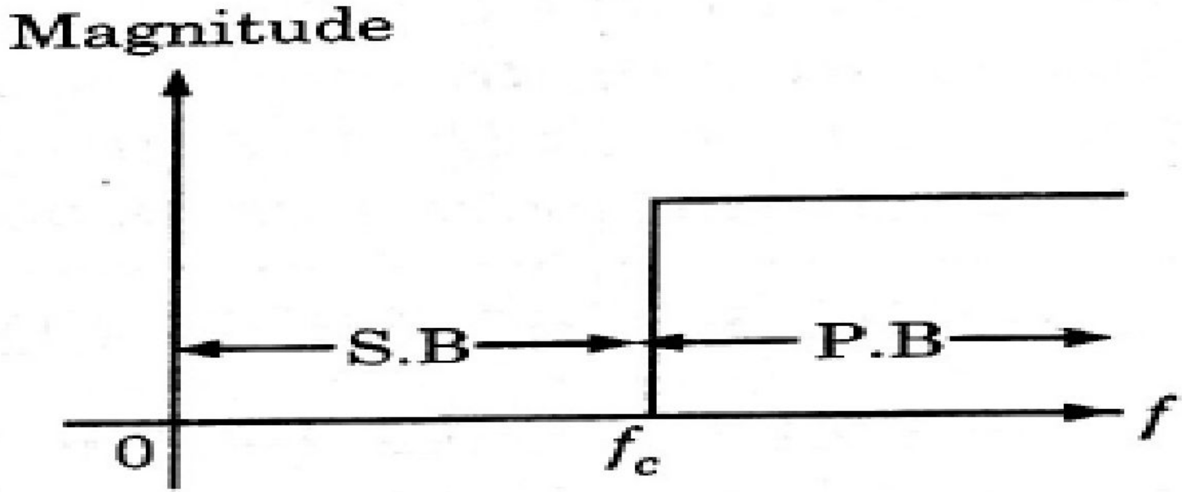

FIGURE 8.4 *Ideal ~~BPF~~ characteristic.*

(*ii*) High Pass Filter (HPF)

It passes the frequency above some designated frequency called as cut-off frequency. If input signal frequency is less than the cut-off frequency; then this signal is not allowed to pass through it. An ideal HPF characteristic is shown in figure 8.4.

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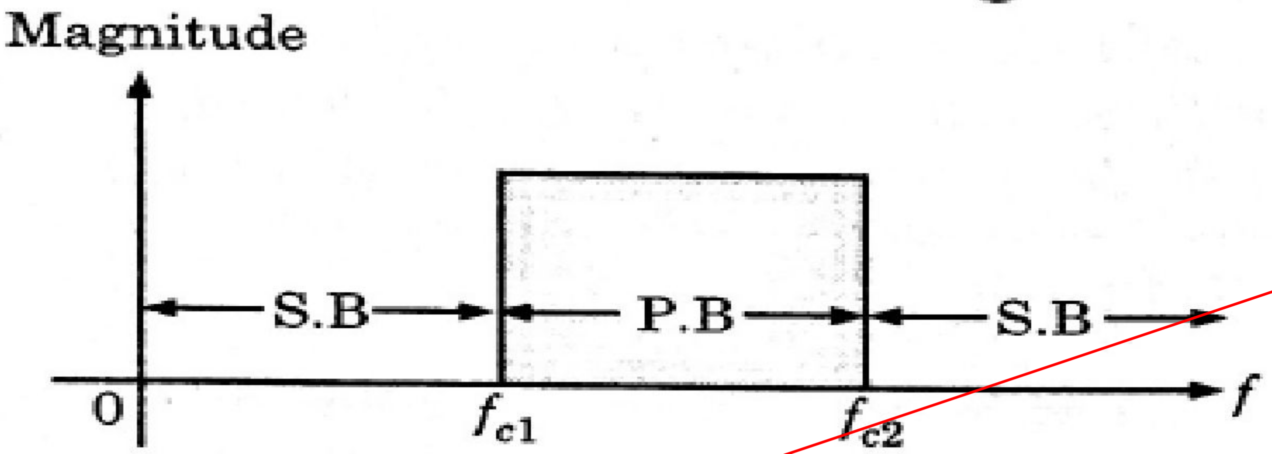

Band Pass Filter

FIGURE 8.3 *Ideal high pass filter (~~HPF~~) characteristic.*

(*iii*) Band Pass Filter (BPF)

It allows the frequencies between two designated cut-off frequencies (say f_{c1} and f_{c2}). An ideal BPF characteristic is shown in figure 8.4.

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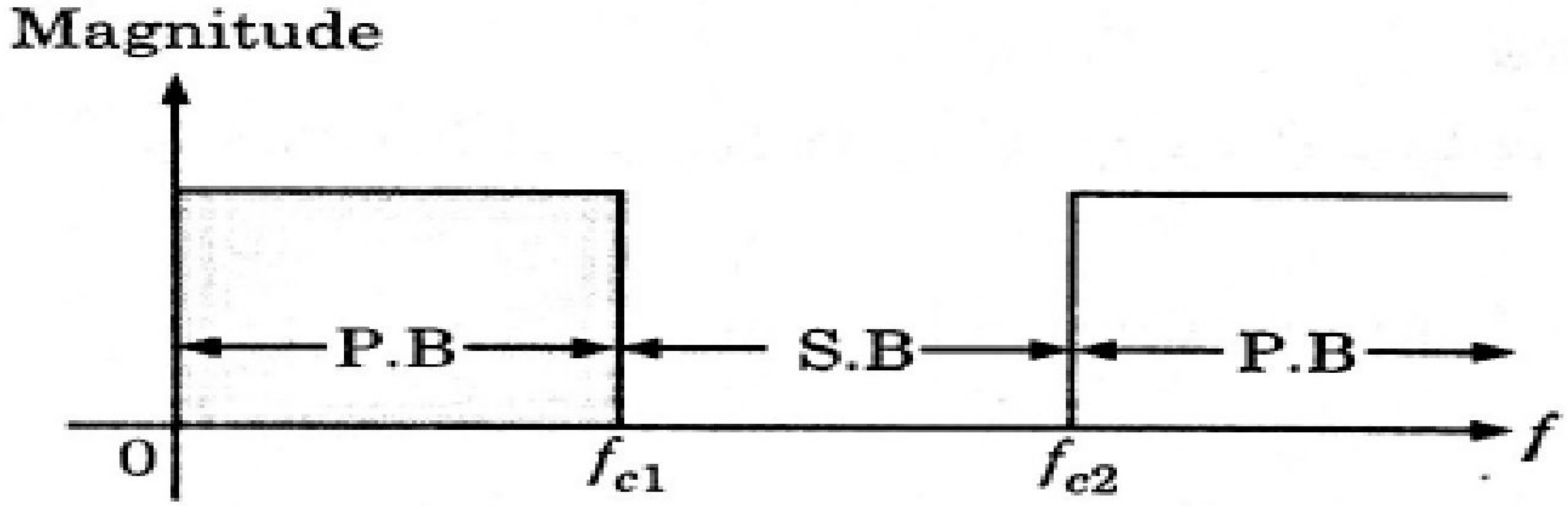

FIGURE 8.5 *Ideal band reject filter (BRF) characteristic.*

(*iv*) Band Reject Filter (BRF)

It attenuates all frequencies between two designated cut-off frequencies. At the sametime, it passes all other frequencies. An ideal band reject filter (BRF) characteristic is shown in figure 8.5.

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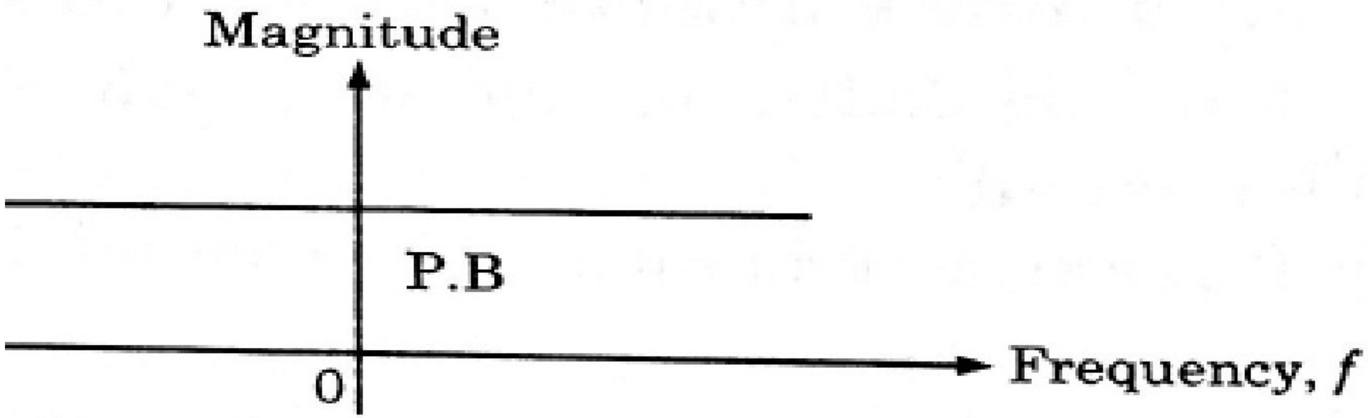

FIGURE 8.6 *All pass filter characteristic.*

(v) All Pass Filter :

It passes all the frequencies. However, by using this filter the phase of input signal can be modified.

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8.4 FILTERING CONCEPT

(Important)

Analog filters are designed using analog components like resistors (R), inductors (L) and capacitors (C). On the other hand, digital filters are implemented using difference equation.

The digital filters described by differential equations can be implemented using software like C or assembly language. Since, we can easily change the algorithm so, we can easily change the filter characteristics according to our requirement.

Basically, there are two types of filters as under:

(*i*) FIR (Finite impulse response) filter.

(*ii*) IIR (Infinite impulse response) filter.

Let us compare analog and digital filters by studying advantages and disadvantages of digital filters.

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8.4.1 Advantages of Digital Filters

1. Many input signals can be filtered by one digital filter without replacing the hardware.
2. Digital filters have characteristic like linear phase response. Such characteristic is not possible to obtain in case of analog filters.
3. The performance of digital filters does not vary with environmental parameters. However, the environmental parameters like temperature, humidity etc., change the values of components in case of analog filters. Therefore, it is required to calibrate analog filters periodically.
4. In case of digital filters, since the filtering is done with the help of digital computer, both filtered and unfiltered data can be saved for further use.

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filtered and unfiltered data can be saved for further use.

5. Unlike analog filters, the digital filters are portable.
7. The digital filters are highly flexible.
8. Using VLSI technology, the hardware of digital filters can be reduced. Similarly the power consumption can be reduced.
9. Digital filters can be used at very low frequencies, for example, in Biomedical applications.
10. In case of analog filters, maintenance is frequently required. However, for digital filters, it is not required.

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8.4.2 Disadvantages of Digital Filters

(*i*) Speed limitation :

In case of digital filters, ADC and DAC are used. So, the speed of digital filter depends on the conversion time of ADC and the settling time of DAC. Similarly, the speed of operation of digital filter depends on the speed of digital processor. Thus, the bandwidth of input signal processed is limited by ADC and DAC. In real time applications, the bandwidth of digital filter is much lower than analog filters.

(*ii*) Finite wordlength effect :

The accuracy of digital filter depends on the wordlength used to encode them in binary from. Wordlength should be long enough to obtain the required accuracy.

The digital filters are also affected by the ADC noise, resulting from the quantization of continuous signals. Similarly, the accuracy of digital filters is also affected by the roundoff noise occurred during computation.

(*iii*) Long design and development time :

An initial design and development time for digital hardware is more than analog filters.

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8.5 IDEAL FILTERS AND APPROXIMATIONS

As a matter of fact, Ideal frequency selective filters are practically not realizable because of the following two important characteristics :

(*i*) Ideal filters have constant gain in the passband and zero gain in the stop- band.

(*ii*) Ideal filter has linear phase response.

Also, in order to design a digital filter, an important condition is that the response of filter must be causal.

Let us consider that the impulse response of an LTI system is denoted by $h(n)$. Now, system is causal when $h(n)$ has some value for positive values of n and $h(n)$ is zero for negative values of n. Therefore, the condition of causality can be expressed as under:

$$h(n) = 0 \text{ for } n < 0 \qquad \text{...(8.4)}$$

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As an example, let us consider the magnitude response of ideal low pass filter (LPF) as shown in figure 8.7.

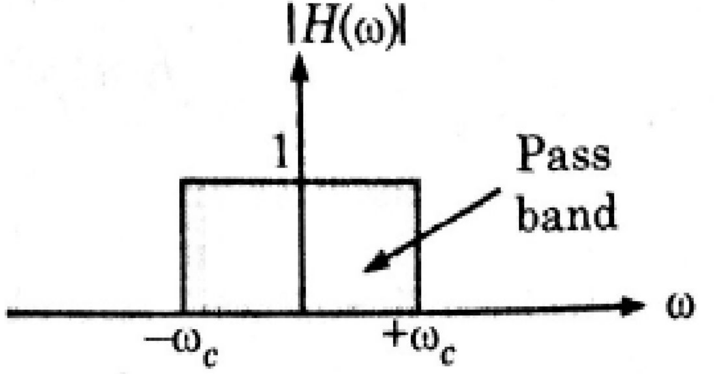

FIGURE 8.7 *Magnitude response of ideal LPF.*

In figure (8.7), ω_c = Cut-off frequency

and $|H(\omega)|$ = Magnitude of filter

According to figure (8.7), the magnitude is unity in the frequency range $-\omega_c$ to $+\omega_c$.

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Thus, we have

$$H(\omega) = \begin{cases} 1 & \text{for} \quad -\omega_c \le \omega \le \omega_c \\ 0 & \text{elsewhere} \end{cases} \qquad \text{...(8.5)}$$

We can obtain the value of $h(n)$ simply by taking inverse Fourier transform of $H(\omega)$.
Hence, we have

$$h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(\omega)\, e^{j\omega n}\, d\omega \qquad \text{...(8.6)}$$

Making use of equation (8.5), we get

$$h(n) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} 1.\, e^{j\omega n}\, d\omega \qquad \text{...(8.7)}$$

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$-\omega_c$

Now, let us consider two conditions as under :

Condition (*i*) : When *n* = 0 :

Substituting $n = 0$ in equation (8.7), we get

$$h(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1.e^0\, d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} d\omega \qquad (\because e^0 = 1)$$

or
$$h(n) = \frac{1}{2\pi}[\omega]_{-\omega_c}^{\omega_c}$$

or
$$h(n) = \frac{\omega_c}{\pi} \qquad \text{for } n = 0$$

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Condition (*ii*) : When $n \neq 0$:

Taking integration of equation (8.7), we get

$$h(n) = \frac{1}{2\pi}\left[\frac{e^{j\omega n}}{jn}\right]_{-\omega_c}^{\omega_c} = \frac{1}{2\pi jn}\left[e^{j\omega_c n} - e^{-j\omega_c n}\right]$$

or

$$h(n) = \frac{1}{\pi n}\left[\frac{e^{j\omega_c n} - e^{-j\omega_c n}}{2j}\right]$$

But according to Euler's identity, we know that

$$\frac{e^{j\theta} - e^{-j\theta}}{2j} = \sin\theta$$

Thus equation (8.9) becomes,

$$h(n) = \frac{\sin \omega_c n}{\pi n} \qquad \text{...for } n \neq 0$$

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Combining two conditions, we can write

$$h(n) = \begin{cases} \dfrac{\sin \omega_c n}{\pi n} & \text{for} \quad n \neq 0 \\ \dfrac{\omega_c}{\pi} & \text{for} \quad n = 0 \end{cases}$$

In figure 8.8, we have sketched $h(n)$ for different values of n by taking $\omega_c = \dfrac{\pi}{4}$.

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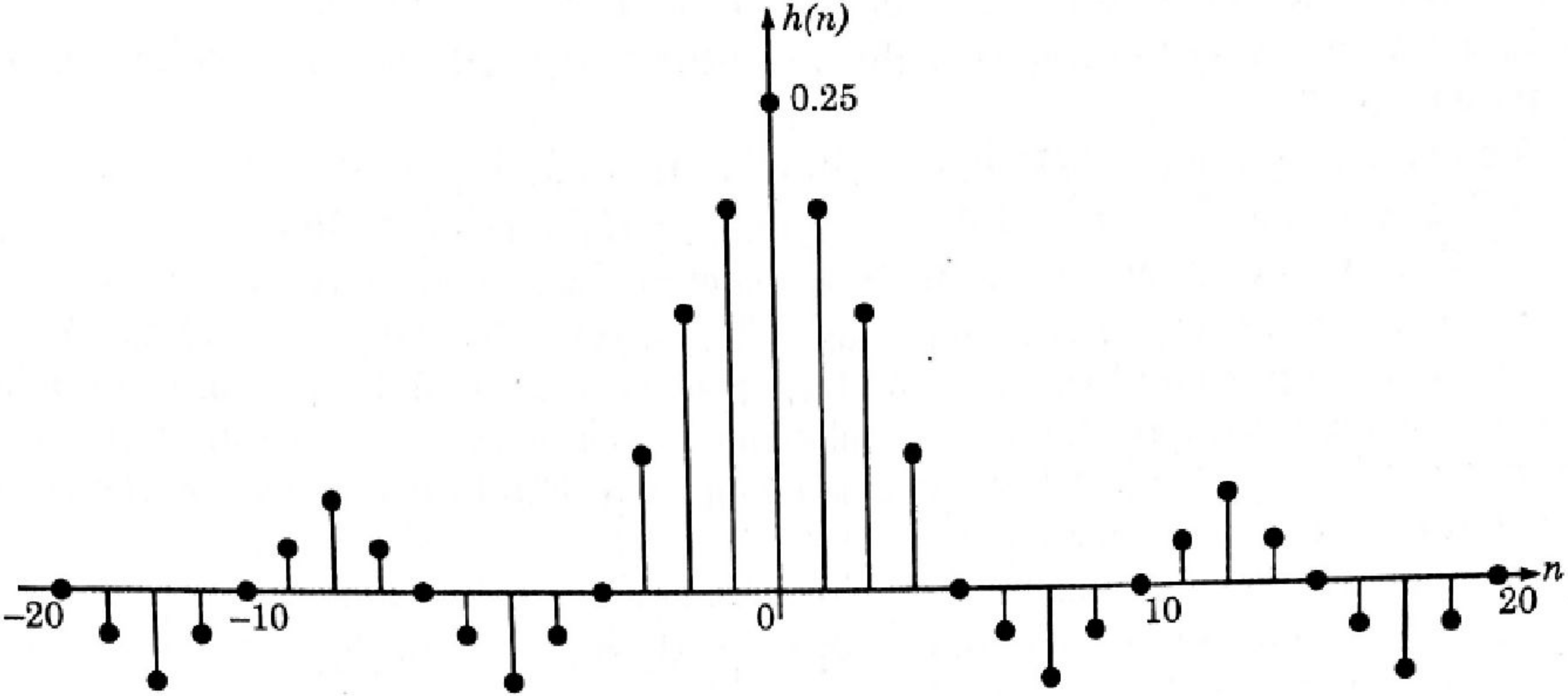

FIGURE 8.8 *Response h(n) of ideal low pass filter (LPF).*

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Note : *From figure 8.8, we can easily conclude that h(n) is present for negative values of n. This means that it is noncausal. However, we know that if the frequency response of filter is causal then only it can be realized. Hence, ideal filters are practically not realizable.*

Therefore, in practical cases, it is not possible to realize an ideal filter. An important reason for this is that ideal filters are non-causal.

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8.5.1 Magnitude Characteristics of Physically Realizable Filter

The frequency response characteristic of a filter is denoted by $H(\omega)$. An ideal filter cannot be realized because the requirements of ideal filter cannot be fulfilled. But if these requirements are relaxed then it is possible to realize causal filters having characteristics which are approximately similar to ideal filters. The magnitude characteristics of a physically realizable filter is shown in figure 8.9.

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The different notations used in figure 8.9 are as under :

δ_1 = Passband ripple
δ_2 = Stopband ripple
ω_p = Passband edge frequency
ω_s = Stopband edge frequency

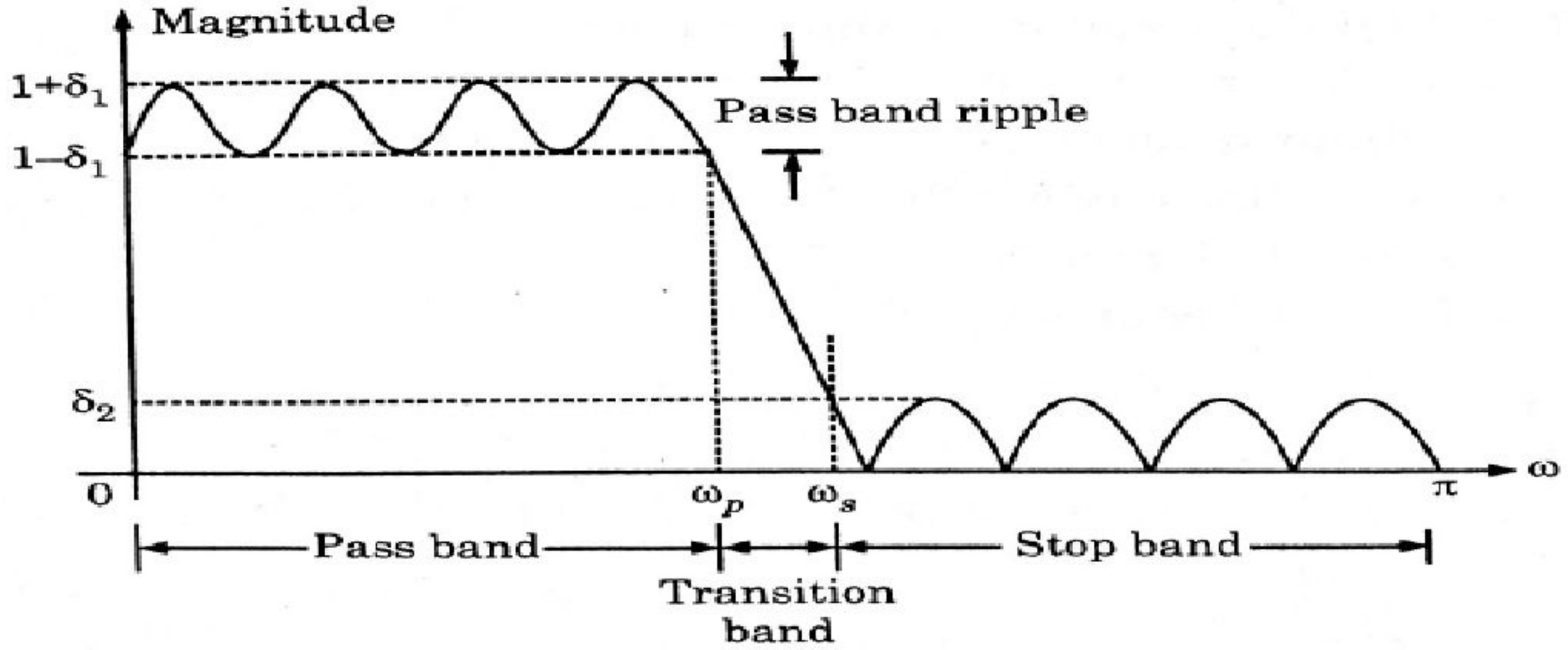

FIGURE 8.9 *Magnitude characteristics of a physically realizable filter.*

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The different conditions that can be relaxed to design such filters are as under:

(i) It is not necessary to insist that the magnitude $|H(\omega)|$ is constant in the entire range of passband.

(ii) A small amount of ripple in the passband is allowed.

(iii) It is not necessary that the filter response in the stopband should be perfectly zero. A small amount of ripple in the stopband is tolerable.

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small amount of ripple in the stopband is tolerable.

As shown in figure 8.9, the frequency response from zero to ω_p is called as passband. The frequency ω_p denotes the edge of passband whereas the frequency ω_s denotes the beginning of stopband. The transition between passband to stopband is called as transition band. Thus the frequency range of transition band is $\omega_p - \omega_s$. The width of passband is called as bandwidth of a filter. Hence, as shown in figure 8.9, the bandwidth of filter is ω_p.

A ripple in the passband is denoted by δ_1 and ripple in stop and is denoted by δ_2. The magnitude in the passband varies between the limits $1 \pm \delta_1$.

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Specifications Required

In any filter design, following specifications are required :

(*i*) The maximum tolerable passband ripple (δ_1)

(*ii*) The maximum tolerable stopband ripple (δ_2)

(*iii*) Passband edge frequency (ω_p)

(*iv*) Stopband edge frequency (ω_s)

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8.6 IIR FILTER DESIGN FROM CONTINUOUS-TIME FILTERS

As discussed earlier, in order to design the digital IIR filter, first analog IIR filter is designed. Then analog filter is converted into the digital filter because of following two reasons :

(i) The procedure to design analog filter is readily available and it is highly advanced.

(ii) When we design digital filter using analog filter then the implementation becomes simple.

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8.7 IMPULSE INVARIANT METHOD

In this method, the design starts from the specifications of analog filter. Here, we have to replace analog filter by digital filter. This is achieved if impulse response of digital filter resembles the sampled version of impulse response of analog filter. If impulse response of both, analog and digital filter matches then, both filters perform in a similar manner.

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In this method, we shall use the following different notations :

$h_a(t) =$ Impulse response in time domain

$H_a(s) =$ Transfer function of analog filter, here 's' is Laplace operator

$h_a(nT) =$ Sampled version of $h_a(t)$, obtained by replacing t by nT

$H(z) =$ z-transform of $h(nT)$. This is response of digital filter.

$\Omega =$ Analog frequency

$\omega =$ Digital frequency

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8.7.1 Transformation of Analog System Function $H_a(s)$ to Digital System Function H(z)

Now, let the system transfer function of analog filter be $H_a(s)$. We can express $H_a(s)$ in terms of partial fraction expansion. This means that

$$H_a(s) = \frac{A_1}{s-p_1} + \frac{A_2}{s-p_2} + \frac{A_3}{s-p_3} \text{----------}$$

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Therefore, $$H_a(s) = \sum_{i=1}^{N} \frac{A_i}{s - p_i} \quad \text{...(8.12)}$$

Here, $A_i = A_1, A_2 \ldots A_N$ are the coefficients of partial fraction expansion.

and $p_i = p_1, p_2 \ldots p_N$ are the poles.

Here, 's' is the Laplace operator. Hence, we can obtain impulse response of analog filter, $h_a(t)$ from $H_a(s)$ by taking inverse Laplace transform of $H_a(s)$. Therefore, using standard relation of inverse Laplace transform, we obtain

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$$h_a(t) = \sum_{i=1}^{N} A_i\, e^{p_i t} \qquad \text{...(8.13)}$$

Now, unit impulse response for discrete structure is obtained by sampling $h_a(t)$. This means that $h(n)$ can be obtained from $h_a(t)$ by replacing t by nT in equation (8.13).

Thus, $$h(n) = \sum_{i=1}^{N} A_i\, e^{p_i nT} \qquad \text{...(8.14)}$$

Here, T is the sampling time.

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The system transfer function of digital filter is denoted by $H(z)$. It is obtained by taking Z-transform of $h(n)$. According to the definition of Z-transform for causal system, we have

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} \quad \text{...(8.15)}$$

Substituting equation (8.14) in equation (8.15), we get

$$H(z) = \sum_{n=0}^{\infty} \left[\sum_{i=1}^{N} A_i e^{p_i n T} \right] . z^{-n} = \sum_{i=1}^{N} A_i \sum_{n=0}^{\infty} e^{p_i nT} . z^{-n}$$

or

$$H(z) = \sum_{i=1}^{N} A_i \sum_{n=0}^{\infty} (e^{p_i T} . z^{-1})^n \quad \text{...(8.16)}$$

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Let us use the following standard summation formula:

$$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$$

With the help of above standard expression, equation (8.16) becomes,

$$H(z) = \sum_{i=1}^{N} A_i \cdot \frac{1}{1-e^{p_i T} z^{-1}} \quad \text{...(8.17)}$$

This is the required transfer function of digital filter.

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8.7.2 Mapping of Poles

Thus, comparing equations (8.12) and (8.17), we can say that the transfer function of digital filter is obtained from the transfer function of analog filter by doing the following transformation:

$$\frac{1}{s-p_i} \rightarrow \frac{1}{1-e^{p_i T}.z^{-1}} \qquad \text{...(8.18)}$$

Equation (8.18) shows how the poles from analog domain are transferred into the digital domain. This transformation of poles is called as **mapping of poles.**

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8.7.3 Relationship of s-plane to z-plane

We know that the poles of analog filters are located at $s = p_i$. Now, from equation (8.18), we can say that the poles of digital filter, $H(z)$ are located at,

$$z = e^{p_i T} \quad \text{...(8.19)}$$

This equation indicates that the poles of analog filter at $s = p_i$ are transformed into the poles of digital filter at $z = e^{p_i T}$. Thus, the relationship between Laplace (s domain) and z-domain is given by,

$$z = e^{sT} \quad \text{...(8.20)}$$

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Here, $s = p_i$ and T is the sampling time.

Now, s is the Laplace operator and it is expressed as,

$$s = \sigma + j\Omega \qquad \text{...(8.21)}$$

Here, σ = attenuation factor

and Ω = analog frequency

We know that z can be expressed in polar form as under:

$$z = re^{j\omega} \qquad \text{...(8.22)}$$

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$$z = re^{j\omega} \quad \text{...(8.22)}$$

Here, r is magnitude and 'ω' is the digital frequency.

Substituting equations (8.21) and (8.22) in equation (8.20), we obtain

$$re^{j\omega} = e^{(\sigma + j\Omega)T} = e^{\sigma T} \cdot e^{j\Omega T} \quad \text{...(8.23)}$$

Separating real and imaginary parts of equation (8.23), we obtain

$$r = e^{\sigma T} \quad \text{...(8.24)}$$

and $\quad e^{j\omega} = e^{j\Omega T}$

Thus, we have $\quad \omega = \Omega T \quad$ ...(8.25)

Now, we will find the relationship between s-plane and z-plane...

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...(8.25)

Now, we will find the relationship between s plane and z plane. Basically, plot in 's'-domain means that σ is plotted on X-axis and $j\Omega$ is plotted on Y-axis. Also, z-domain representation means that real z is plotted on X-axis and imaginary Z is plotted on Y-axis.

Let us consider equation (8.24), *i.e.*,

$$r = e^{\sigma T}$$

we shall discuss the following conditions :

(*i*) If $\sigma < 0$, then r is equal to reciprocal of e raise to some constant. Hence, range of r will be 0 to 1 *i.e.*,

$$\sigma < 0 \Rightarrow 0 < r < 1 \qquad ...(8.22)$$

Now, $\sigma < 0$ means the negative values of ...

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...(8.22)

Now, $\sigma < 0$ means the negative values of σ. That is L.H.S. of s plane. We know that r is the radius of circle in z plane.

Therefore, $0 < r < 1$ indicates interior part of unit circle. Thus we can conclude that,

L.H.S. of s plane is mapped inside the unit circle.

(*ii*) If $\sigma = 0$ then $r = e^0 = 1$ *i.e.*,

$$\sigma = 0 \Rightarrow r = 1$$

Now, $\sigma = 0$ indicates $j\Omega$ axis and $r = 1$ indicates unit circle. Thus,

$j\Omega$ axis in s plane is mapped on the unit circle.

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(*iii*) If $\sigma > 0$ then, r is equal to e raise to some constant. That means $r > 1$ *i.e.*,

$$\sigma > 0 \Rightarrow r > 1$$

Now, $\sigma > 0$ indicates R.H.S. of s plane and $r > 1$ indicates exterior part of unit circle. Thus,

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R.H.S. of s-plane is mapped outside the unit circle.

Combining all the above conditions, this mapping is shown in figure 8.10.

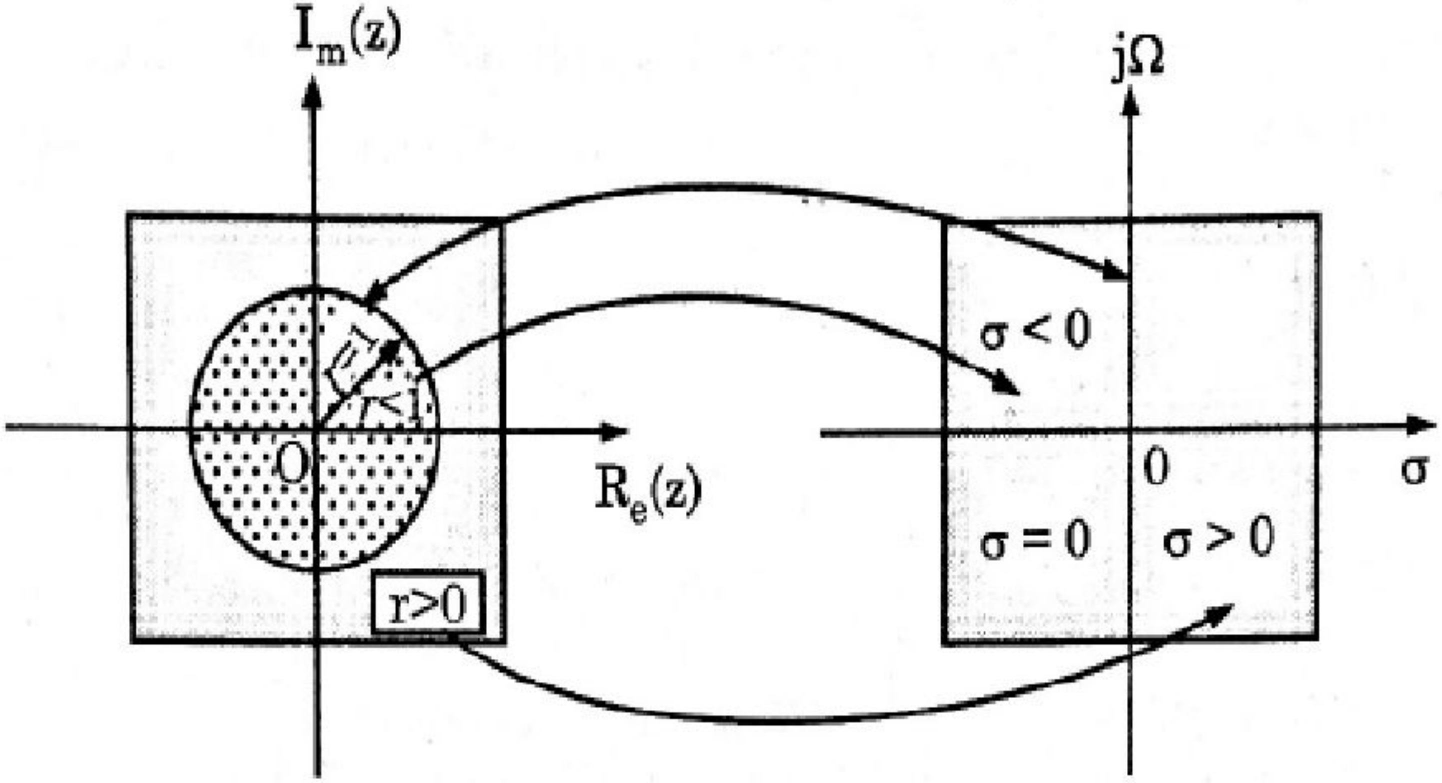

FIGURE 8.10 *Illustration of the mapping $z = e^{sT}$*

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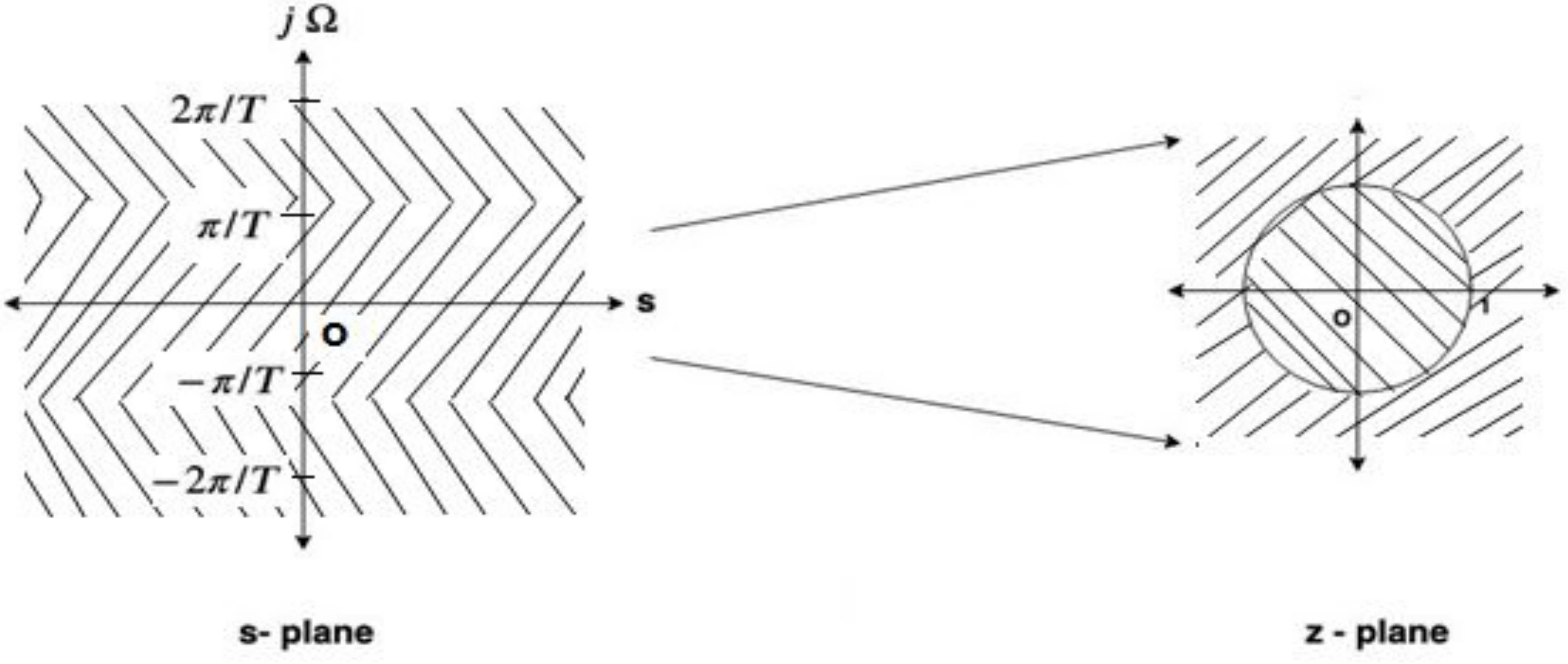

Mapping of a single strip in s-plane onto z plane.

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8.7.4 Drawbacks of Impulse Invariance Method

(i) We know that Ω is analog frequency and its range is form $\frac{\pi}{T}$ to $-\frac{\pi}{T}$. While the digital frequency ω varies from $-\pi$ to π. This means that from $\frac{\pi}{T}$ to $-\frac{\pi}{T}$, ω maps from $-\pi$ to π. Let i be any integer. Then, we can write the general range of Ω as $(i-1)\frac{\pi}{T}$ to $(i+1)\frac{\pi}{T}$. However, for this range also, ω maps from $-\pi$ to π. Hence, mapping from analog frequency Ω to digital frequency ω is termed as **many to one**. This mapping is not one to one.

(ii) Analog filters are not band limited so there will be aliasing due to the sampling process. Because of this aliasing, the frequency response of resulting digital filter will not be identical to the original frequency response of analog filter.

(iii) The change in the value of sampling time (T) has no effect on the amount of aliasing.

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8.7.5 Standard Expressions

Some standard formulae for transformation in impulse invariance method are as under :

(1) $$\frac{1}{s-p_i} \rightarrow \frac{1}{1-e^{p_i T}.z^{-1}}$$

(2) $$\frac{s+a}{(s+a)^2+b^2} \rightarrow \frac{1-e^{-aT}[\cos bT]z^{-1}}{1-2e^{-aT}[\cos bT]z^{-1}+e^{-2aT}.z^{-2}}$$

(3) $$\frac{b^{2}}{(s+a)^2+b^2} \rightarrow \frac{e^{-aT}[\sin bT]z^{-1}}{1-2e^{-aT}[\cos bT]z^{-1}+e^{-2aT}.z^{-2}}$$

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8.7.6 Design Steps for Impulse Invariance Method

(*i*) In numerical problems, analog frequency transfer function $H_a(s)$ is usually be given. If it is not given, then, we obtain expression of $H_a(s)$ from the given specifications.

(*ii*) If required, we expand $H_a(s)$ by using partial fraction expansion (PFE).

(*iii*) Then, we obtain z-transform of each PFE term using impulse invariance transformation equation.

(*iv*) We obtain $H(z)$, this is required digital IIR filter.