DFT

Discrete Fourier Transform

DFT

When a Fourier transform is calculated only at discrete points then it is called as discrete fourier transform (DFT).

Now, if we have aperiodic time domain signal then discrete time Fourier transform (DTFT) is obtained. But, DTFT is continous in nature and its range is from $-\infty$ to $+\infty$. Then a finite range sequence is obtained by extracting a particular protion from such infinite sequence.

portion

DFT

Since, $X(\omega)$ is a continuous-time signal, a discrete-time signal is obtained by sampling $X(\omega)$. A particular sequence which is extracted from infinite sequence is called is **windowed sequence.** A windowed signal is considered as periodic signal. We can obtain periodic extension of this signal. This periodic extension in frequency domain is called is **Discrete Fourier Transform** (DFT). From this original sequence. $x(n)$ is obtained by performing inverse process which is known as **Inverse Discrete Fourier Transform (IDFT).**

DFT

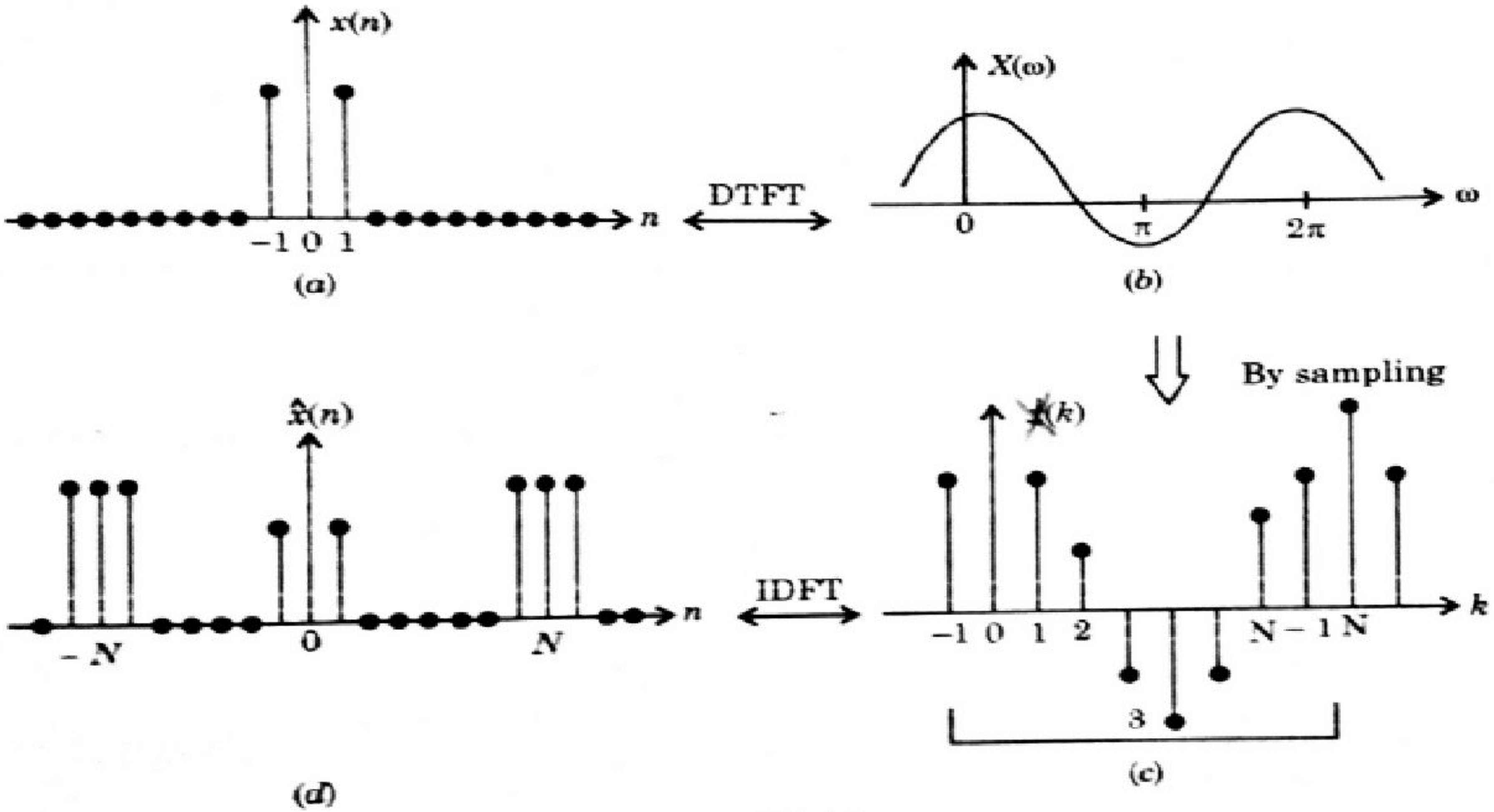


FIGURE 4.1.

DFT

This process is explained graphically as shown in figure 4.1. Figure 4.1(a) shows discrete time signal $x(n)$. By taking DTFT of $x(n)$, $X(\omega)$ is obtained as shown in figure 4.1(b). The sampled version of $X(\omega)$ is denoted by $X(k)$ which is called as DFT. It is shown in figure 4.1(c). By performing IDFT, original signal is obtained. It is denoted by $\tilde{x}(n)$. It is shown in figure 4.1(d). It is periodic extension of sequence $x(n)$.

Here, N denotes the number of samples of input sequence and the number of frequency points in the DFT output.

DFT

(i) Definition of DFT

It is a finite duration discrete frequency sequence which is obtained by sampling one period of Fourier transform. Sampling is done at 'N' equally spaced points over the period points over the period extending from $\omega = 0$ to $\omega = 2\pi$.

(ii) Mathematical Expressions

The DFT of discrete sequence $x(n)$ is denoted by $X(k)$. It is given by,

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j2\pi kn/N} \qquad \ldots(4.13)$$

Here, $k = 0, 1, 2 \ldots N-1$

Since, this summation is taken for N points; it is called as **N-point DFT.**

We can obtain discrete sequence $x(n)$ from its DFT. It is called as inverse discrete fourier transform (IDFT). It is given by,

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi kn/N} \qquad \ldots(4.14)$$

Here, $n = 0, 1, 2, \ldots N-1$

This is called as **N-point IDFT.**

DFT

(*iii*) Twiddle Factor and its Importance

Now, we will define the new term W as,

$$W_N = e^{-j2\pi/N} \quad \text{...(4.15)}$$

This is called as **twiddle factor.** Twiddle factor makes the computation of DFT a bit easy and fast.

Using twiddle factor, we can write equations of DFT and IDFT as under :

$$X(k) = \sum_{n=0}^{N-1} X(n)\, W_N^{kn}$$

Here, $n = 0, 1, 2 \ldots N-1$ ...(4.16)

and $$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k)\, W_N^{-kn}$$

Here, $n = 0, 1, 2, \ldots N-1$ ...(4.17)

DFT

(iv) Linear Transformation

Let us view the DFT and IDFT as linear transformations on sequences $\{x(n)\}$ and $\{X(k)\}$, respectively. Let us define an N-point vector x_N of frequency samples, and an $N \times N$ matrix W_N as

$$x_N = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}, \quad X_N = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

$$\mathbf{W}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & W_N^{1(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

With these definitions, the N-point DFT may be expressed in matrix form as

$$X_N = \mathbf{W}_N x_N$$

DFT

$$X_N = W_N x_N$$

where W_N is the matrix of the linear transformation. We observe that W_N is a symmetric matrix. If we assume that the inverse of W_N exists, then the last expression can be inverted by premultiplying both sides by W_n^{-1}. Thus we obtain

$$x_N = W_N^{-1} X_N \qquad \text{...(4.18)}$$

But this is just an expression for the IDFT.

In fact, the IDFT can be expressed in matrix form as under

$$x_N = \frac{1}{N} W_N^{*} X_N \qquad \text{...(4.19)}$$

DFT

4.4 COMPARISON OF DTFT AND DFT

We know that the DTFT is discrete time Fourier transform and is given by,

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \qquad ...(4.22)$$

DFT

The range of ω is from $-\pi$ to π or 0 to 2π.

Now, we know that discrete Fourier transform (DFT) is obtained by sampling one cycle of Fourier transform. Also, DFT of $x(n)$ is given by,

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \quad \text{...(4.2}$$

Comparing equations (4.22) and (4.23), we can say that DFT is obtained from DTFT by substituting $\omega = \dfrac{2\pi k}{N}$.

Hence, $\qquad X(k) = X(\omega)\Big|_{\omega=\frac{2\pi k}{N}}$

DFT

Few Important Points

By comparing DFT with DTFT, we can write

(i) The continuous frequency spectrum $X(\omega)$ is replaced by discrete Fourier spectrum $X(k)$.

(ii) Infinite summation in DTFT is replaced by finite summation in DFT.

(iii) The continuous frequency variable is replaced by finite number of frequencies located at $\frac{2\pi k}{NT_s}$, where T_s is called as *sampling time.*

DFT

4.5 DISCRETE FOURIER TRANSFORM (DFT) OF SOME STANDARD SIGNALS

In this article, let us obtain DFT of few standard signals in the form of solved examples as follows :

EXAMPLE 4.1 Obtain DFT of unit impulse $\delta(n)$.

Solution : Here $x(n) = \delta(n)$ ...(i)

According to the definition of DFT, we have,

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \quad ...(ii)$$

But $\delta(n) = 1$ only at $n = 0$.

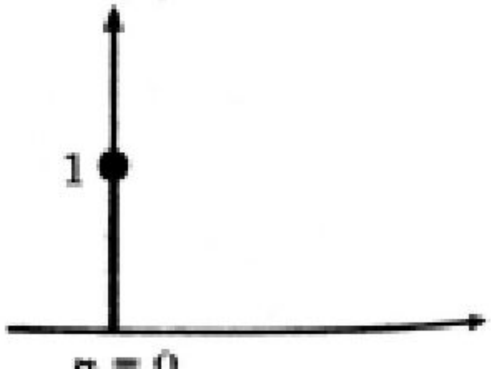
Thus, equation (ii) becomes,

$$X(k) = \delta(0)e^{0} = 1$$

Therefore, we can write

$$\delta(n) \xleftrightarrow{DFT} 1$$

This is the standard DFT pair.



DFT

EXAMPLE 4.2 Obtain DFT of delayed unit impulse $\delta(n - n_0)$.

FIGURE 4.4.

Solution : We know that $\delta(n - n_0)$ indicates unit impulse delayed by 'n_0' samples.

Here, $$x(n) = \delta(n - n_0) \quad \text{...(i)}$$

Now, we have, $$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \quad \text{...(ii)}$$

But, $\delta(n - n_0) = 1$ only at $n = n_0$.

Thus, equation (ii) becomes,

$$X(k) = 1 \,.\, e^{-j2\pi k n_0/N}$$

Hence $\delta(n - n_0) \xleftrightarrow{DFT} e^{-j2\pi k n_0/N}$

Similarly, we can write,

$$\delta(n + n_0) \xleftrightarrow{DFT} e^{j2\pi k n_0/N}$$

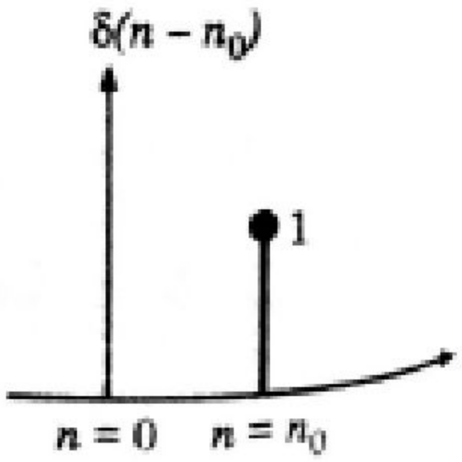


FIGURE 4.5.

DFT

EXAMPLE 4.3 **Compute N-point DFT of the following exponential sequence :**

$$x(n) = a^n u(n) \quad \text{for } 0 \le n \le N-1$$

(Sem. Exam, GGSIPU, Delhi, 2005–06)

Solution : According to the definition of DFT, we have

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \qquad \ldots(i)$$

Here, $\quad x(n) = a^n u(n)$

The multiplication of a^n with $u(n)$ indicates that sequence is positive. Substituting $x(n) = a^n$ in equation (*i*), we obtain

$$X(k) = \sum_{n=0}^{N-1} a^n e^{-j2\pi kn/N}$$

or

$$X(k) = \sum_{n=0}^{N-1} \left(ae^{-j2\pi k/N}\right)^n \qquad \ldots(ii)$$

Now, let us use the following standard summation expression:

$$\sum_{k=N_1}^{N_2} A^k = \frac{A^{N_1} - A^{N_2+1}}{1-A}$$

Here, $N_1 = 0$, $N_2 = N - 1$ and $A = ae^{-j2\pi k/N}$

DFT

Therefore, $$X(k) = \frac{\left(ae^{-j2\pi k/N}\right)^0 - \left(ae^{-j2\pi k/N}\right)^{N-1+1}}{1-ae^{-j2\pi k/N}} = \frac{1-a^N e^{-j2\pi k}}{1-ae^{-j2\pi k/N}} \qquad ...(iii)$$

Making use of Euler's identity to the numerator term, we shall have

$$e^{-j2\pi k} = \cos 2\pi k - j \sin 2\pi k$$

But, k is an integer

Therefore, $\cos 2\pi k = 1$ and $\sin 2\pi k = 0$

or $$e^{-j2\pi k} = 1 - j0 = 1$$

or $$X(k) = \frac{1-a^N}{1-ae^{-j2\pi k/N}}$$

Hence, $$a^n u(n) \xleftrightarrow{DFT} \frac{1-a^N}{1-ae^{-j2\pi k/N}}$$

DFT

4.6 DETAILED EXPLANATION OF CYCLIC PROPERTY OF TWIDDLE FACTOR

(Important)

The twiddle factor is denoted by W_N and is given by,

$$W_N = e^{-j2\pi/N} \quad \text{...(4.24)}$$

Now, the discrete time sequence $x(n)$ can be denoted by x_N. Here, N stands for N point DFT. While in case of N point DFT, the range of n is from 0 to $N-1$.

Now, the sequence x_N can be represented in the matrix from as under:

$$x_N = \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \cdot \\ \cdot \\ x(N-1) \end{bmatrix}_{N\times 1} \quad \text{...(4.25)}$$

DFT

This is a $N \times 1$ matrix and n varies from 0 to $N-1$. We have denoted $x(n)$ by x_N. Similarly, we can denote $X(k)$ by X_N K In the matrix form, X_k can be represented as under:

$$X_k = \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ \cdot \\ \cdot \\ X(N-1) \end{bmatrix}_{N\times 1} \qquad \text{...(4.26)}$$

This is also $N \times 1$ matrix and k varies from 0 to $N-1$. Recall the definition of DFT *i.e.*,

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad (4.27)$$

DFT

We can also represent W_N^{kn} in the matrix form. Further, since k varies from 0 to $N-1$ and n also varies from 0 to $N-1$, therefore, we have

$$W_N{}^{kn} = \begin{array}{c} \\ k=0 \\ k=1 \\ k=2 \\ \vdots \\ k=N-1 \end{array} \begin{array}{c} \begin{array}{ccccc} n=0 & n=1 & n=2 & \cdots & n=N-1 \end{array} \\ \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ W_N^0 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ W_N^0 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix}_{N \times N} \end{array} \quad \text{...(4.28)}$$

Note that each value is obtained by taking multiplication of k and n.

As an example, if $k = 2$, $n = 2$, then we get $W_N^{kn} = W_N^4$.

Thus, DFT can be represented in the matrix form as,

$$X_N = [W_N]x_N \quad \text{...(4.29)}$$

DFT

Similarly, IDFT can be represented in the matrix forms as,

$$x_N = \frac{1}{N}\left[W_N^*\right]X_N \qquad \text{...(4.30)}$$

Here, W_N^* is complex conjugate of W_N.

Now, let us show that W_N possess the periodicity property. This means that after some period, the value of W_N repeats. Let us consider 8-point DFT, *i.e.*, $N = 8$.

We have, $\quad W_N = e^{-\frac{j2\pi}{N}}$

Therefore, $\quad W_N^{kn} = e^{-\frac{j2\pi}{N}\times kn}$

But, $N = 8$

Hence, $\quad W_8^{kn} = e^{-\frac{j2\pi}{8}\times kn} = e^{-\frac{j\pi}{4}\times kn} \qquad \text{...(4.31)}$

Now, let us obtain W_8^{kn} by substituting different values of kn. This has been shown in Table 4.1.

DFT

Here, it may be noted that all calculations have been done by making use of Euler's identity. For example, when kn = 1, equation (4.47) becomes,

$$W_8^1 = e^{-j\frac{\pi}{4}} = \cos\frac{\pi}{4} - j\sin\frac{\pi}{4} = 0.707 - j\,0.707$$

DFT

W_8

TABLE 4.1.

S.No.	Value of kn	$W_8^{kn} = e^{-j\frac{\pi}{4}\times kn}$
1	0	$W_8^0 = e^0$
2	1	$W_8^1 = e^{-j\frac{\pi}{4}\times 1} = e^{-j\frac{\pi}{4}}$
3	2	$W_8^2 = e^{-j\frac{\pi}{4}\times 2} = e^{-j\frac{\pi}{2}}$
4	3	$W_8^3 = e^{-j\frac{\pi}{4}\times 3} = e^{-j\frac{3\pi}{4}}$
5	4	$W_8^4 = e^{-j\frac{\pi}{4}\times 4} = e^{-j\pi}$
6	5	$W_8^5 = e^{-j\frac{\pi}{4}\times 5} = e^{-j\frac{5\pi}{4}}$
7	6	$W_8^6 = e^{-j\frac{\pi}{4}\times 6} = e^{-j\frac{3\pi}{2}}$
8	7	$W_8^7 = e^{-j\frac{\pi}{4}\times 7} = e^{-j\frac{7\pi}{4}}$
9	8	$W_8^8 = e^{-j\frac{\pi}{4}\times 8} = e^{-j2\pi}$
10	9	$W_8^9 = e^{-j\frac{\pi}{4}\times 9} = e^{-j\frac{9\pi}{4}}$
11	10	$W_8^{10} = e^{-j\frac{\pi}{4}\times 10} = e^{-j\frac{5\pi}{2}}$
12	11	$W_8^{11} = e^{-j\frac{\pi}{4}\times 11} = e^{-j\frac{11\pi}{4}}$

From table 4.1, it may be observed ...

DFT

TABLE 4.1.

S.No.	Value of kn	$W_8^{kn} = e^{-j\frac{\pi}{4}\times kn}$	Value of the phasor
1	0	$W_8^0 = e^0$	1
2	1	$W_8^1 = e^{-j\frac{\pi}{4}\times 1} = e^{-j\frac{\pi}{4}}$	$0.707 - j\,0.707$
3	2	$W_8^2 = e^{-j\frac{\pi}{4}\times 2} = e^{-j\frac{\pi}{2}}$	$0 - j\,1$
4	3	$W_8^3 = e^{-j\frac{\pi}{4}\times 3} = e^{-j\frac{3\pi}{4}}$	$-0.707 - j\,0.707$
5	4	$W_8^4 = e^{-j\frac{\pi}{4}\times 4} = e^{-j\pi}$	-1
6	5	$W_8^5 = e^{-j\frac{\pi}{4}\times 5} = e^{-j\frac{5\pi}{4}}$	$-0.707 + j\,0.707$
7	6	$W_8^6 = e^{-j\frac{\pi}{4}\times 6} = e^{-j\frac{3\pi}{2}}$	$0 + j\,1$
8	7	$W_8^7 = e^{-j\frac{\pi}{4}\times 7} = e^{-j\frac{7\pi}{4}}$	$0.707 + j\,0.707$
9	8	$W_8^8 = e^{-j\frac{\pi}{4}\times 8} = e^{-j2\pi}$	1
10	9	$W_8^9 = e^{-j\frac{\pi}{4}\times 9} = e^{-j\frac{9\pi}{4}}$	$0.707 - j\,0.707$
11	10	$W_8^{10} = e^{-j\frac{\pi}{4}\times 10} = e^{-j\frac{5\pi}{2}}$	$0 - j\,1$
12	11	$W_8^{11} = e^{-j\frac{\pi}{4}\times 11} = e^{-j\frac{11\pi}{4}}$	$-0.707 - j\,0.707$

From table 4.1, it may be observed

DFT

$- 0.707 - j\,0.707$

From table 4.1, it may be observed that the value of W_8^0 is same as W_8^8. Similarly, W_8^1 [cut off] same as W_8^9 and W_8^2 is same as W_8^{10}. Since, this is 8-point DFT (N = 8), after 8 points, the val[cut off] repeats.

This means that

$$W_8^0 = W_8^8 = W_8^{16} \;\ldots$$

$$W_8^1 = W_8^9 = W_8^{17} \;\ldots$$

$$W_8^2 = W_8^{10} = W_8^{18} \;\ldots$$

$$\vdots$$

$$W_8^7 = W_8^{15} = W_8^{23} \;\ldots$$

DFT

This property of twiddle factor is called as ***periodicity property*** *or* ***cyclic property.***

Few Important Points

(i) In Table 4.1, every value of W_N^{kn} can be represented in terms of magnitude and angle. For example, we have $W_8^0 = e^0$. We know that magnitude and angle can be expressed as, **magnitude** $e^{j\text{ angle}}$. Thus, for $W_8^0 = 1.e^0$. Here, magnitude is 1 and angle is zero.

(ii) Similarly, we have $W_8^1 = 1 . e^{-j\frac{\pi}{4}}$, so magnitude is 1 and angle is $- . \frac{\pi}{4}$.

(iii) Likewise, we can write the

DFT

, so magnitude is 1 and angle is $-\frac{\pi}{4}$.

(iii) Likewise, we can write the magnitude and angle of each value. The cyclic property of twiddle factor has been illustrated in figure 4.6.

(iv) In figure 4.6, we have drawn unit circle that means a circle having radius equal to 1. Every point is in the clockwise direction because we have negative angles. Here, we have considered 8-point DFT. Therefore, the circle is divided into 8 points. This spacing of DFT or the resolution of DFT is also called as the bin spacing of DFT output.

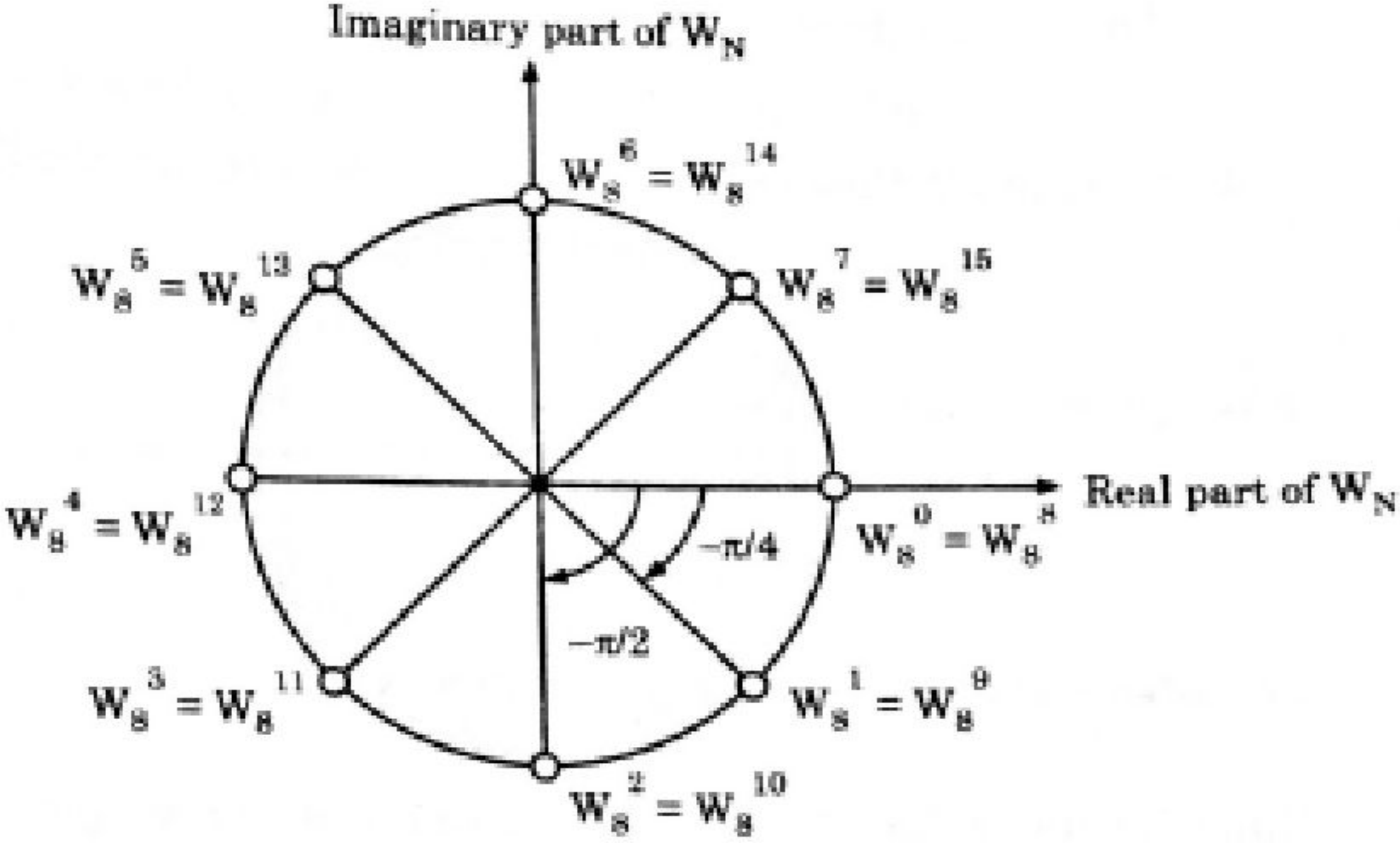


FIGURE 4.6 *Cyclic property of twiddle factor.*

DFT

EXAMPLE 4.5 **Compute 2-point and 4-point DFT of the following sequence:**

$$x(n) = u(n) - u(n-2)$$

Sketch the magnitude of DFT in both the cases.

(Sem. Exam, JNTU, Hyderabad, 2006–07)

Solution : First, let us obtain the sequence $x(n)$. It has been represented as shown in figure 4.7.

Thus, from figure (4.7), we get

$$x(n) = \{1, 1\} \quad \text{...}(i)$$

(i) Determination of 2-point DFT

For 2-point DFT, $N = 2$

We have, $W_N = e^{-j\frac{2\pi}{N}}$

so that $W_2 = e^{-j\frac{2\pi}{2}} = e^{-j\pi}$

Hence, $W_2^{kn} = e^{-j\pi kn} \quad \text{...}(ii)$

We know that n is from 0 to $N-1$. In this case, n is from 0 to 1. Similarly, k is from 0 to $N-1$. In this case, k is from 0 to 1.

Now, the matrix $W_N = W_2^{kn} = e^{-j\pi kn}$ can be written as under :

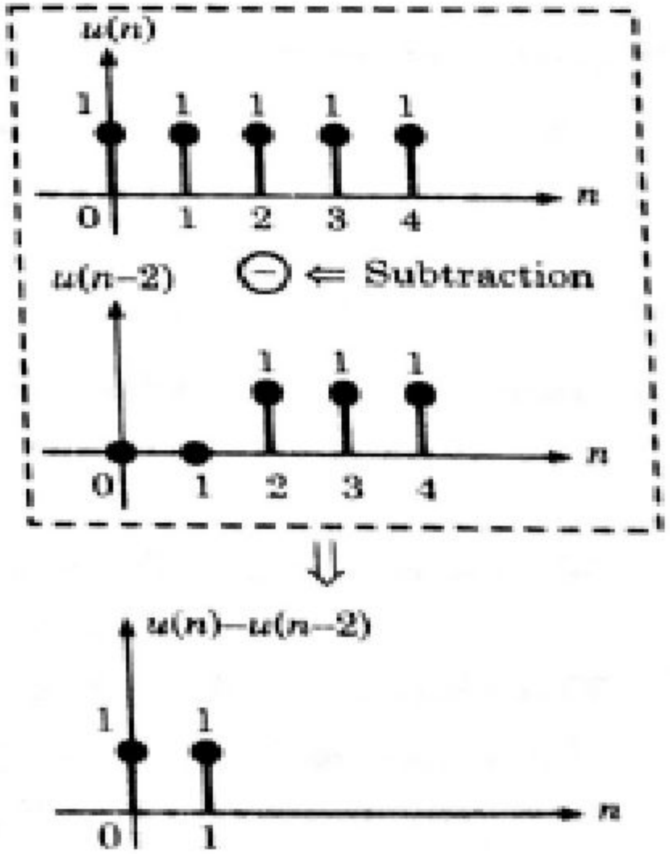


FIGURE 4.7 $x(n) = u(n) - u(n-2)$.

DFT

$$W_2^{kn} = \begin{array}{c} \\ k=0 \\ k=1 \end{array} \begin{array}{c} \begin{array}{cc} n=0 & n=1 \end{array} \\ \begin{bmatrix} W_2^0 & W_2^0 \\ W_2^0 & W_2^1 \end{bmatrix} \end{array}$$

According to equation (*ii*), we have

$$W_2^{kn} = e^{-j\pi kn}$$

For $kn = 0$, we have

$$W_2^0 = e^{-j\pi \times 0} = e^0 = 1$$

For $kn = 1$, we have

$$W_2^1 = e^{-j\pi \times 1} = e^{-j\pi} = \cos \pi - j \sin \pi = -1$$

Substituting all these values in equation (*iii*), we shall get

$$W_2^{kn} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

DFT

Also, given sequence is $x(n) = \{1, 1\}$.

In the matrix form, this sequence can be written as,

$$x_N = x(n) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

For above sequence, the DFT matrix is given by,

$$X_N = [W_N]x_N$$

Substituting values from equation (*iv*) and (*v*), we get

$$X_N = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} (1\times1)+(1\times1) \\ (1\times1)+(1\times-1) \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

Therefore, the 2-point DFT will be

$$X(k) = \{2, 0\}$$

DFT

Magnitude plot

We know that magnitude = $\sqrt{(\text{Real part})^2 + (\text{Imaginary part})^2}$

In equation (*vi*), the imaginary part is zero.

Hence, the magnitude at $k = 0$ is 2 and magnitude at $k = 1$ is 0.

This magnitude plot has been shown in figure 4.8.

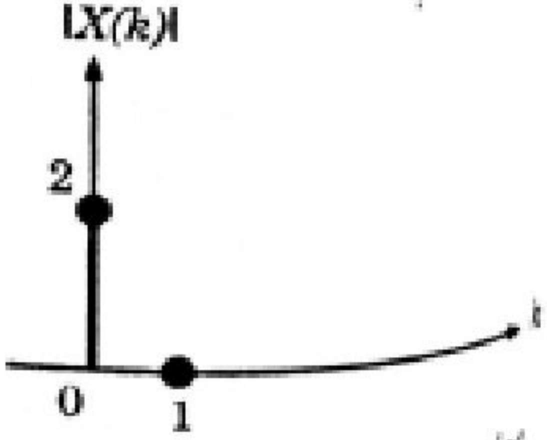


FIGURE 4.8 *Magnitude plot.*

...(*vii*)

(*ii*) Determination of 4-point DFT

For 4-point DFT, $N = 4$

We have, $W_N = W_4 = e^{-j\frac{2\pi}{4}} = e^{-j\frac{\pi}{2}}$

Therefore, $W_N^{kn} = e^{-j\frac{\pi}{2}kn}$

The range of k and n is from 0 to $N - 1$, *i.e.*, 0 to 3.

Now, the matrix $W_N = W_4 = W_4^{kn}$ can be written as

DFT

Discrete Fourier Transform (DFT)

$$[W_4] = W_4^{kn} = \begin{array}{c} \\ k=0 \\ k=1 \\ k=2 \\ k=3 \end{array} \begin{array}{c} \begin{array}{cccc} n=0 & n=1 & n=2 & n=3 \end{array} \\ \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \end{array}$$

Making use of equation (vii), we obtain

$$W_4^0 = e^{-j\frac{\pi}{2}\times 0} = e^0 = 1$$

$$W_4^1 = e^{-j\frac{\pi}{2}\times 1} = e^{-j\frac{\pi}{2}} = \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} = -j$$

$$W_4^2 = e^{-j\frac{\pi}{2}\times 2} = e^{-j\pi} = \cos\pi - j\sin\pi = -1$$

$$W_4^3 = e^{-j\frac{\pi}{2}\times 3} = e^{-j\frac{3\pi}{2}} = \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} = +j$$

DFT

$$= \cos\frac{\quad}{2} - j\sin\frac{\quad}{2} = +j$$

According to cyclic property of DFT, we know that

$$W_4^0 = W_4^4 = 1$$

$$W_4^1 = W_4^5 = W_4^9 = -j$$

$$W_4^2 = W_4^6 = W_4^{10} = -1$$

and $$W_4^3 = W_4^7 = W_4^{11} = +j$$

Substituting all these values in equation (*viii*), we shall get the matrix $[W_4]$ *i.e.*,

$$[W_4] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

DFT

The given sequence is $x(n) = \{1, 1\}$. Since, the desirable length of this sequence is equal to 4, it is obtained by adding zeros at the end of sequence. This is called as **zero padding.**

Thus, with the help of zero padding, we get

$$x(n) = \{1, 1, 0, 0\}$$

Hence, $$x_N = x_4 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \ldots(x)$$

Now, the discrete Fourier Transform (DFT) is given by

$$X_N = [W_N]x_N$$

or $$X_N = X_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+1+0+0 \\ 1-j+0+0 \\ 1-1+0+0 \\ 1+j+0+0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1-j \\ 0 \\ 1+j \end{bmatrix}$$

or $$X_4 = \{2, \ 1-j, \ 0, \ 1+j\}$$

DFT

The above DFT sequence can also be written as under:

$$X_4 = \{2 + j\,0,\ 1 - j,\ 0 + j\,0,\ 1 + j\}$$
$$\uparrow$$
$$k = 0$$

Magnitude Plot

The magnitude at different values can be obtained as under:

For $k = 0$, we have

$$|X(k)| = \sqrt{(2)^2 + (0)^2} = 2$$

For $k = 1$, we have

$$|X(k)| = \sqrt{(1)^2 + (-1)^2} = \sqrt{2} = 1.414$$

For $k = 2$, we have

$$|X(k)| = \sqrt{0 + 0} = 0$$

For $k = 3$, we have

$$|X(k)| = \sqrt{(1)^2 + (1)^2} = \sqrt{2} = 1.414$$

This magnitude plot has been shown in figure 4.9.

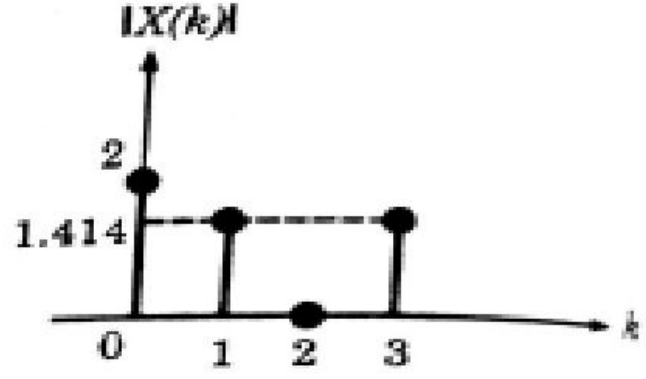


FIGURE 4.9 *Magnitude plot.*

DFT

EXAMPLE 4.7 **Compute the length-4 sequence from its DFT which is given by**

$$X(k) = \{4, 1 - j, - 2, 1 + j\}$$

Solution: We know that the IDFT in matrix form is expressed as

$$\text{IDFT} = x(n) = x_N = \frac{1}{N}\left[W_N^*\right] . X_N \qquad ...(i)$$

Here, X_N is the given DFT matrix. Also, '*' indicates complex conjugate. To obtain the complex conjugate, we have to change the sign of j term. For example, complex conjugate of $1 - j1$ is $1 + j1$.

Now, we have already obtained the matrix W_4 in previous examples. It is reproduced here i.e.,

DFT

Discrete Fourier Transform (DFT)

$$[W_4] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

Therefore, $$\left[W_4^*\right] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

Given matrix of DFT is

$$X_N = X_4 = \begin{bmatrix} 4 \\ 1-j \\ -2 \\ 1+j \end{bmatrix}$$

DFT

Substituting equations (*iii*) and (*iv*), and substituting $N = 4$ in equation (*i*), we shall have

$$x_N = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}\begin{bmatrix} 4 \\ 1-j \\ -2 \\ 1+j \end{bmatrix}$$

or $$x_N = \frac{1}{4}\begin{bmatrix} 4+1-j-2+1+j \\ 4+j-j^2+2-j-j^2 \\ 4-1+j-2-1-j \\ 4-j+j^2+2+j+j^2 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ 4+2+1+1 \\ 4-4 \\ 4+2-2 \end{bmatrix} \qquad (\because j^2 = -1)$$

Simplifying, we get

or $$x_N = \frac{1}{4}\begin{bmatrix} 4 \\ 8 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

or $x(n) = \{1, 2, 0, 1\}$ **Ans.**