DFT(cont..)

DFT

4.12 PROPERTIES OF DISCRETE FOURIER TRANSFORM (DFT)

The properties of Discrete Fourier Transform (DFT) are quite useful in the practical techniques for processing signals. At this point, it may be noted that most of the properties of the DFT and z-transform have some similarity since they have some relationship between them.

The properties of Discrete Fourier Transform (DFT) can be listed as under :

1. Periodicity
2. Linearity
3. Shifting property
4. Circular convolution

Now, let us discuss these properties one by one in the subsequent sub-sections.

DFT

4.12.1 Periodicity

(i) Statement

This property states that if a discrete-time signal is periodic then its DFT will also be periodic. Also, if a signal or sequence repeats its waveform after N number of samples then it is called a periodic signal or sequence and N is called the period of signal. Mathematically,

If $X(k)$ is an N-point DFT of $x(n)$ *i.e.*,

If $x(n) \xleftrightarrow[N]{DFT} X(k)$, then we have

$$x(n+N) = x(n) \text{ for all values of } n \qquad \text{...(4.34)}$$

$$X(k+N) = X(k) \text{ for all values of } k \qquad \text{...(4.35)}$$

DFT

(ii) Proof :

According to the definition of DFT, we have

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$$

Replacing k by $k + N$, we obtain

$$X(k+N) = \sum_{n=0}^{N-1} x(n)\, W_N^{(k+N)n} = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}\, W_N^{Nn}$$

We know that W_N is a twiddle factor and it is expressed as

$$W_N = e^{-j\frac{2\pi}{N}}$$

Also,

$$W_N^{Nn} = \left(e^{-j\frac{2\pi}{N}}\right)^{Nn} = e^{-j\frac{2\pi}{N}.Nn} = e^{-j2\pi n}$$

or

$$W_N^{Nn} = \cos 2\pi n - j \sin 2\pi n$$

Since, n is an integer, therefore, we have

$$\cos 2\pi n = 1 \quad \text{and} \quad \sin 2\pi n = 0$$

Hence,

$$W_N^{Nn} = 1$$

DFT

Substituting this value in equation (4.37), we have

$$X(k+N) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

Comparing equations (4.36) and (4.39), we get

$$X(k+N) = X(k)$$ **Hence proved.**

DFT

4.12.2 Linearity

combination

Linearity properties states that DFT of linear combination of two or more signals is equal to the sum of linear combinatin of DFT of individual signals. Let us consider that $X_1(k)$ and $X_2(k)$ are the N-points DFTs of $x_1(n)$ and $x_2(n)$ respectively, and a and b are arbitrary constants either real or complex-valued, then mathematically

If $x_1(n) \xleftrightarrow[N]{DFT} X_1(k)$ and $x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$ then,

$$a\,x_1(n) + b\,x_2(n) \xleftrightarrow[N]{DFT} a\,X_1(k) + b\,X_2(k)$$

Here, a and b are some constants

DFT

Proof: According to the definition of DFT, we have

$$X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \quad \text{...(4.41)}$$

Here, $$x(n) = a\, x_1(n) + b\, x_2(n)$$

Therefore, $$X(k) = \sum_{n=0}^{N-1} [a\, x_1(n) + b\, x_2(n)]\, W_N^{kn} = \sum_{n=0}^{N-1} a\, x_1(n)\, W_N^{kn} + \sum_{n=0}^{N-1} b\, x_2(n)\, W_N^{kn}$$

Since, a and b are constants, therefore, we can take them out of the summation sign.

Hence, $$X(k) = a\sum_{n=0}^{N-1} x_1(n)\, W_N^{kn} + b\sum_{n=0}^{N-1} x_2(n)\, W_N^{kn} \quad \text{...(4.42)}$$

DFT

Comparing equation (6.57) with the definition of DFT, we obtain

$$X(k) = a\,X_1(k) + b\,x_2(k)$$

Hence Proved

DFT

4.12.3 Circular Symmetries of a Sequence *(Sem. Exam, GGSIPU, Delhi, 2006-07)*

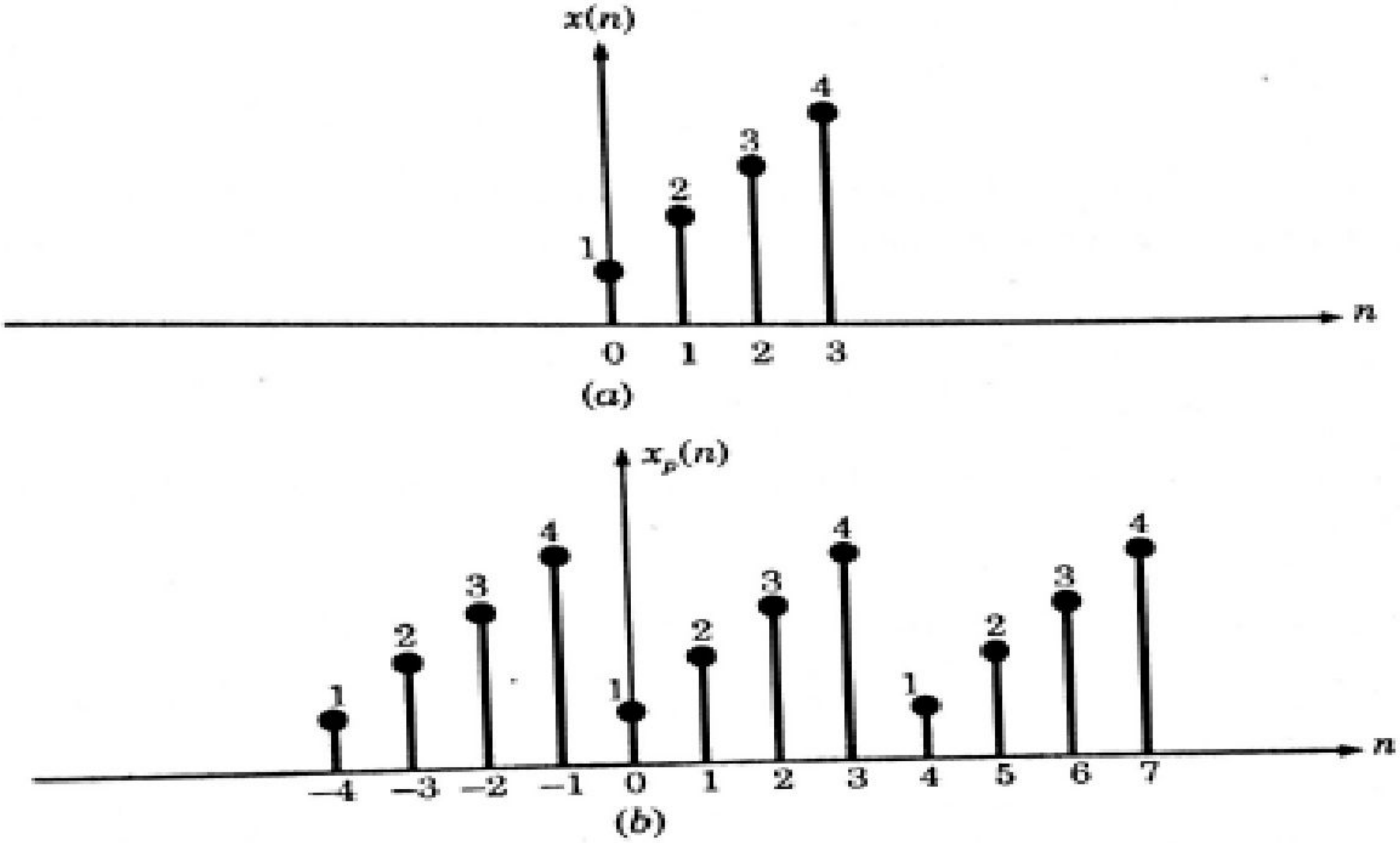
We have discussed the periodicity property of discrete Fourier transform (DFT). Let us consider that input discrete time sequence is $x(n)$ then, the periodic sequence is denoted by $x_p(n)$. The period of $x_p(n)$ is N which means that after N the sequence $x(n)$ repeats itself.

Now, we can write the sequence $x_p(n)$ as under:

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN) \qquad \text{...(4.43)}$$

Now, let us consider one example.

DFT



DFT

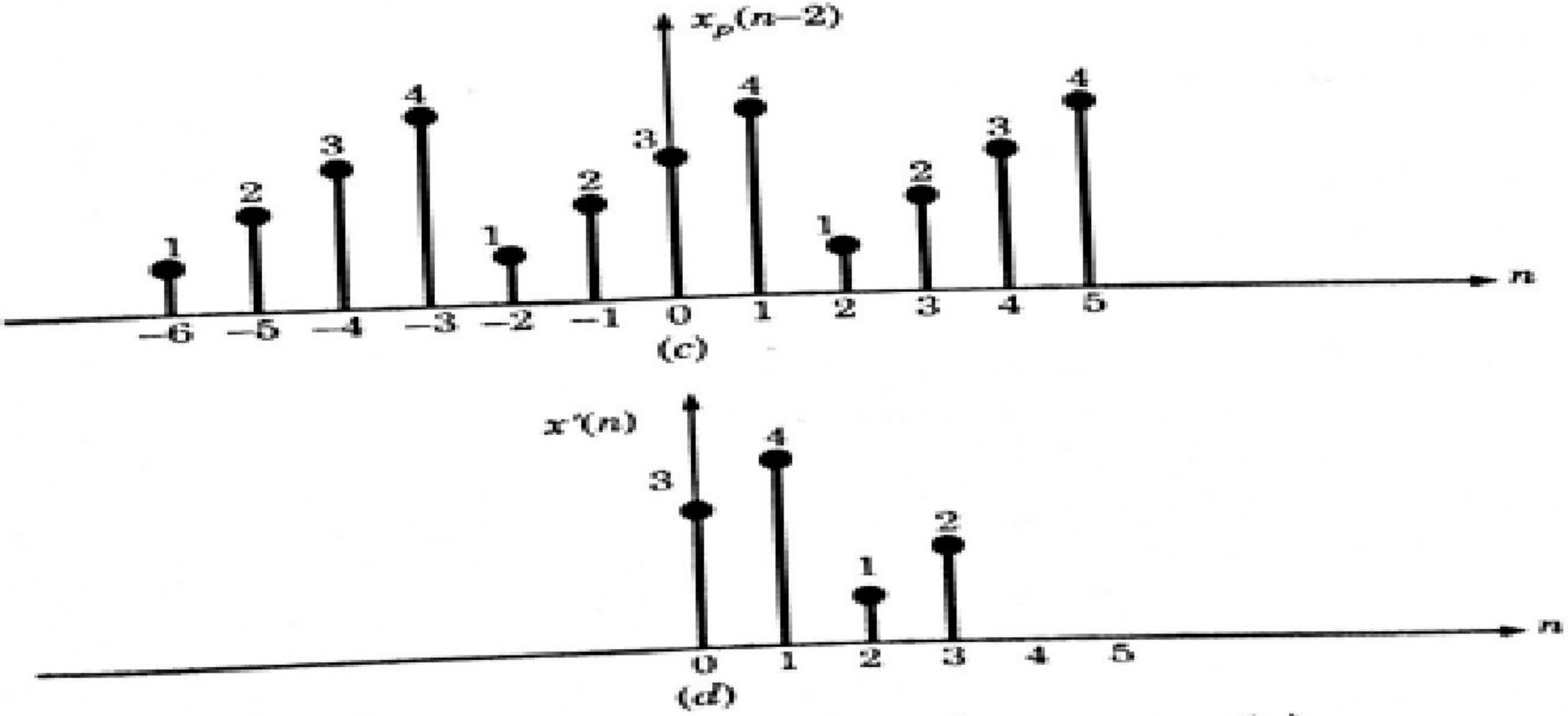


FIGURE 4.10 *Shifting of sequence x(n).*

DFT

Let $x(n) = \{1, 2, 3, 4\}$. This sequence is shown in figure 4.10(a). The periodic sequence $x_p(n)$ is shown in figure 4.10(b). We shall delay the periodic sequence $x_p(n)$ by two samples as shown in figure 4.10(c). This sequence is denoted by $x_p(n - 2)$. Now, the original signal is present in the range $n = 0$ to $n = 3$. In the same range, we shall write the shifted signal as shown in figure 4.10(d). This signal is denoted by $x'(n)$.

Now, from figure 4.10, we can write every sequence as under:

$$x(n) = \{\underset{\uparrow}{1}, 2, 3, 4\} \quad ...(4.44)$$

$$x_p(n) = \{... 1, 2, 3, 4, \underset{\uparrow}{1}, 2, 3, 4, 1, 2, 3, 4, ...\} \quad ...(4.45)$$

$$x_p(n - 2) = \{... 1, 2, 3, 4, 1, 2, \underset{\uparrow}{3}, 4, 1, 2, 3, 4, ...\} \quad ...(4.46)$$

and

$$x'(n) = \{\underset{\uparrow}{3}, 4, 1, 2\} \quad ...(4.47)$$

DFT

↑

Now, from equation (4.44) and (4.47), we can say that the sequence $x'(n)$ is obtained by **circularly shifting** sequence $x(n)$, by two samples. This means that **$x'(n)$ is related to $x(n)$ by circular shift.**

Notation

This relation of circular shift is denoted by,

$$x'(n) = x(n - k, \text{ modulo } N) \quad \text{...(4.48)}$$

It means that divide $(n - k)$ by N and retain the remainder only. We can also use the short hand notation as under :

$$x'(n) = x((n - k))_N \quad \text{...(4.49)}$$

Here, k indicates the number of samples by which $x(n)$ is delayed and N indicates N-point DFT. In the present example, the sequence $x(n)$ is delayed by two samples; thus $k = 2$. Because, there are four samples in $x(n)$, this is 4-point DFT. Hence, $N = 4$.

Now, for this example, equation (4.48) becomes,

$$x'(n) = x((n - 2))_4 \quad \text{...(4.50)}$$

Graphical Representation

The circular shifting of a sequence can be plotted graphically as under:

(i) Circular plot of sequence $x(n)$

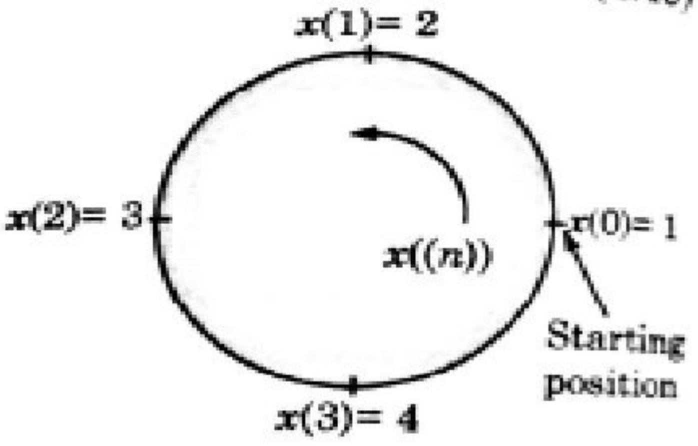
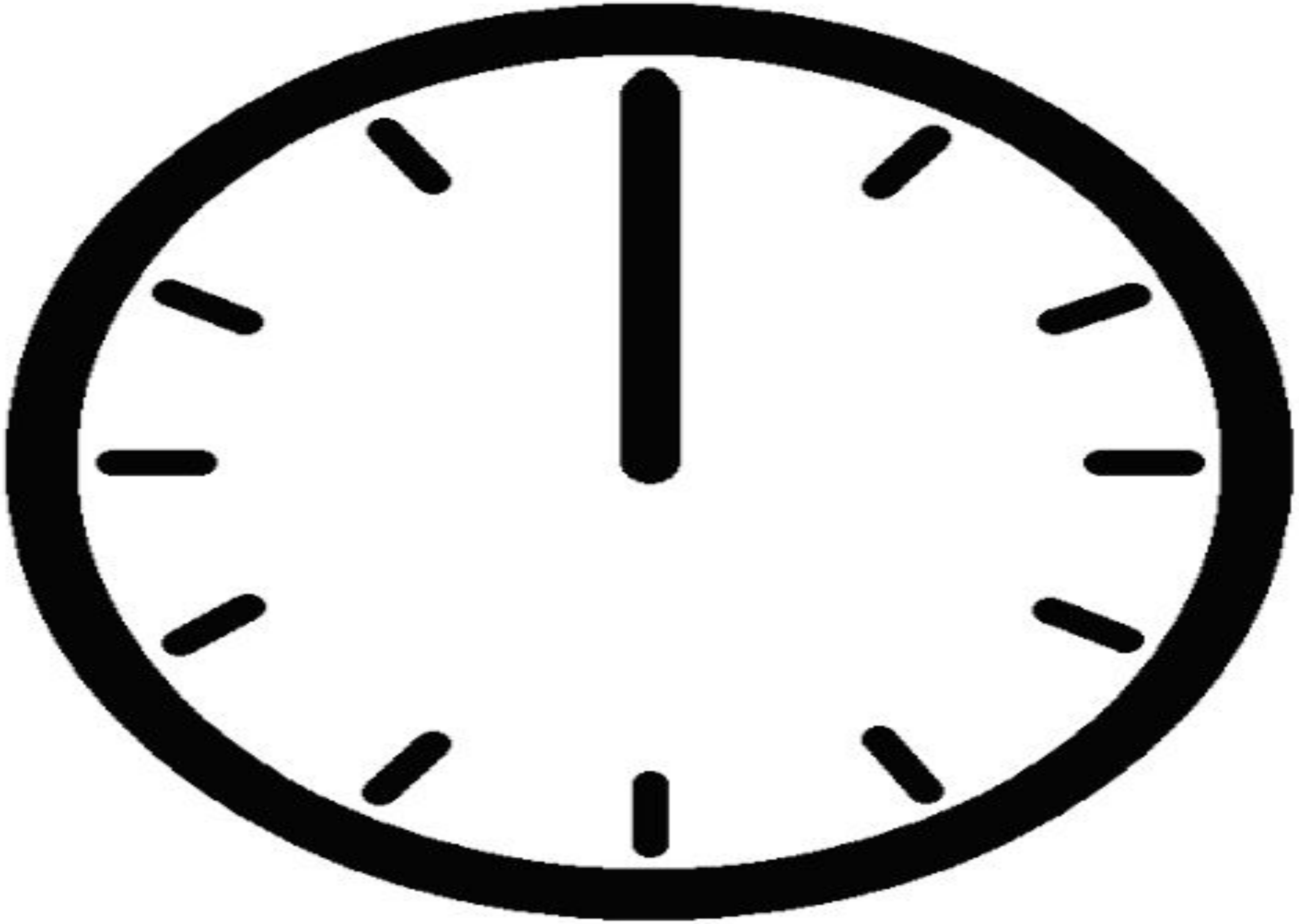


FIGURE 4.11 $x((n))_4$ – *The samples of $x(n)$ are plotted circularly anticlockwise.*

DFT



DFT

(i) Circular plot of sequence $x(n)$

Here, we have considered,

$$x(n) = \{\underset{\uparrow}{1}, 2, 3, 4\}$$

Circular plot of $x(n)$ is denoted by $x((n))_4$. This plot is obtained by writing the samples of $x(n)$ circularly anticlockwise. It is shown in figure 4.11.

(ii) Circular delay by one sample

To delay sequence $x(n)$ circularly by one sample, shift every sample circularly in anticlockwise direction by 1. This is shown in figure 4.12. This operation is denoted by $x((n-1))$.

It may be noted that **delay by k samples means shift the sequence circularly in anticlockwise direction by k.**

samples of $x(n)$ are plotted circularly anticlockwise.

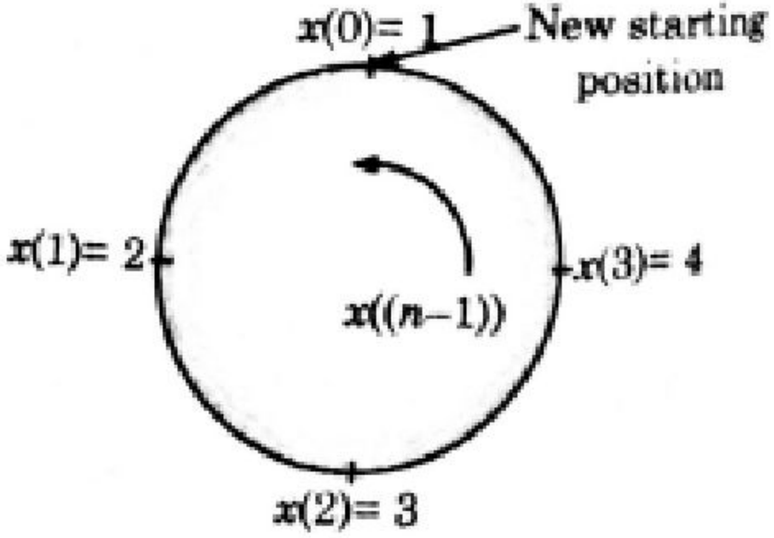


FIGURE 4.12 *$x((n-1))$ shift every sample by 1 in anticlockwise direction.*

DFT

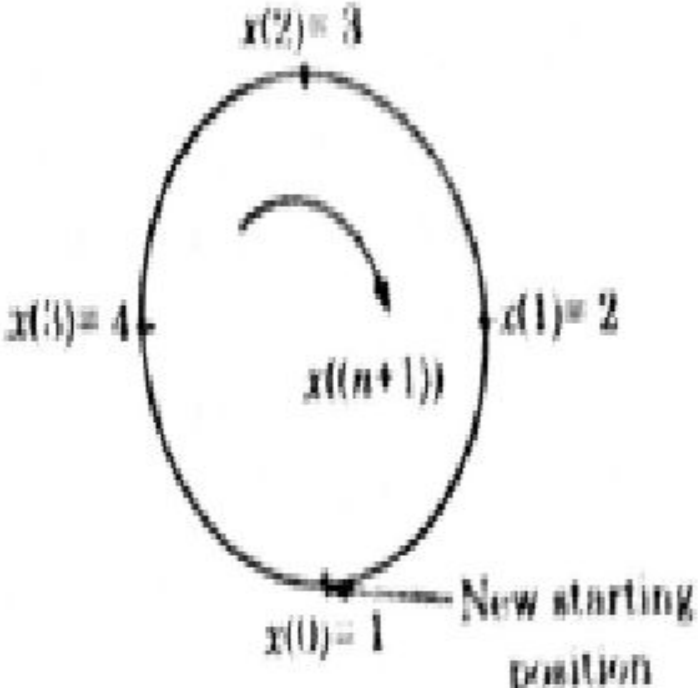


FIGURE 4.13 *x((n + 1)) shift every sample by one in clockwise direction.*

x(3) = 4

(iii) Circular advance by one sample

To advance sequence $x(n)$ circularly by one sample shift every sample circularly in clockwise direction by 1 sample. This sequence is denoted by $x((n + 1))$. It is shown in figure 4.13.

It may be noted that **advance by *k* samples means shift the sequence circularly in clockwise direction by *k*.**

DFT

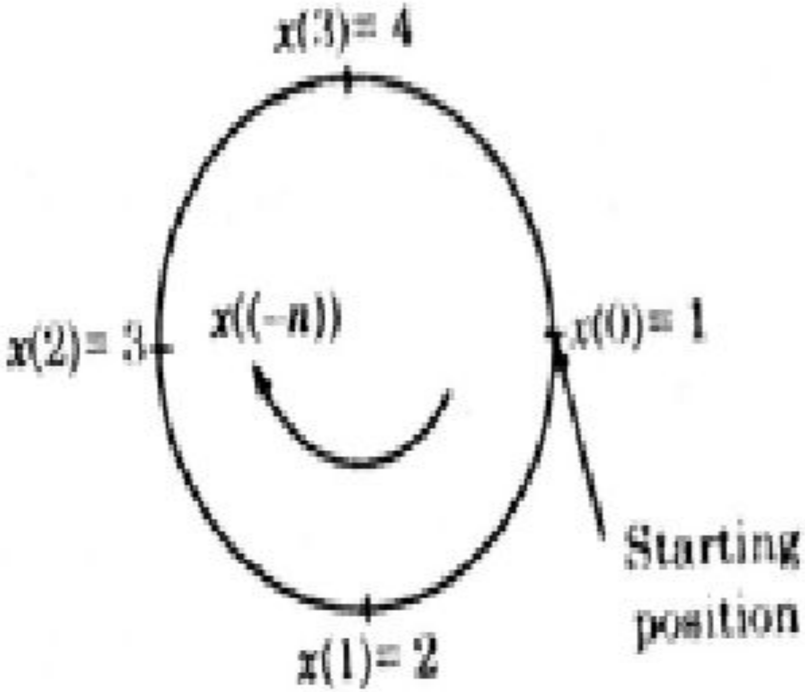


FIGURE 4.14 *$x((-n))$ samples are plotted circularly clockwise.*

by ...

(iv) Circularly folded sequence

A circularly folded sequence is denoted by $x((-n))$. We have plotted sequence $x((n))$ in anticlockwise direction. So, folded sequence $x((-n))$ is plotted in clockwise direction. It is shown in figure 4.14.

It may be noted that **circular folding means plot the samples in clockwise direction.**

Now recall equation (4.50) it is,

$$x'(n) = x((n-2))_4$$

It indicates delay of sequence $x(n)$ by two samples. It is obtained by rotating samples of figure 4.16 in anticlockwise direction by two samples. This sequence is shown in figure 4.15.

DFT

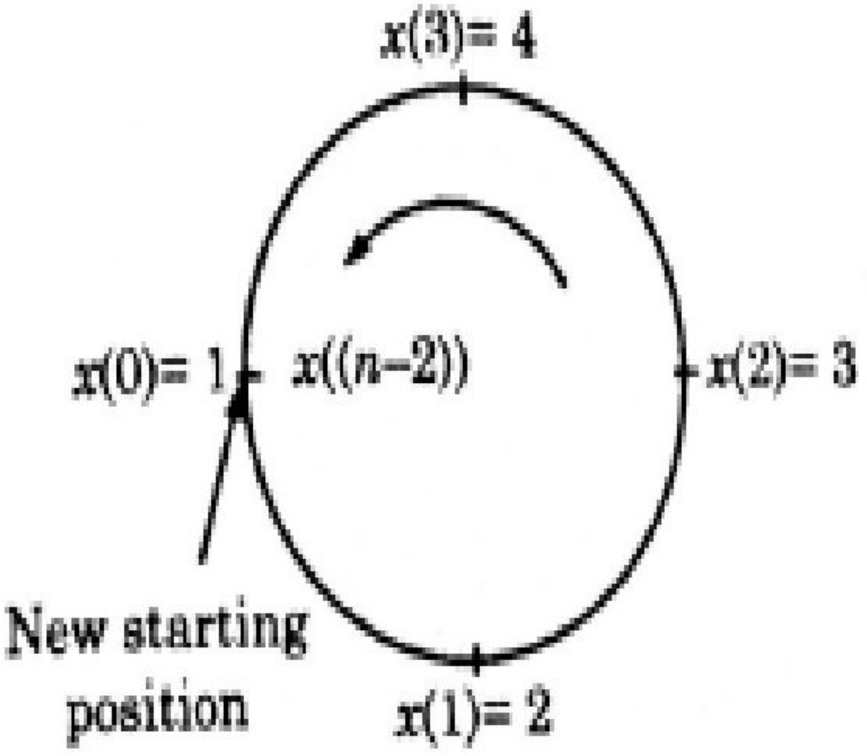


FIGURE 4.15 *Plot of* $x((n-2))_4$

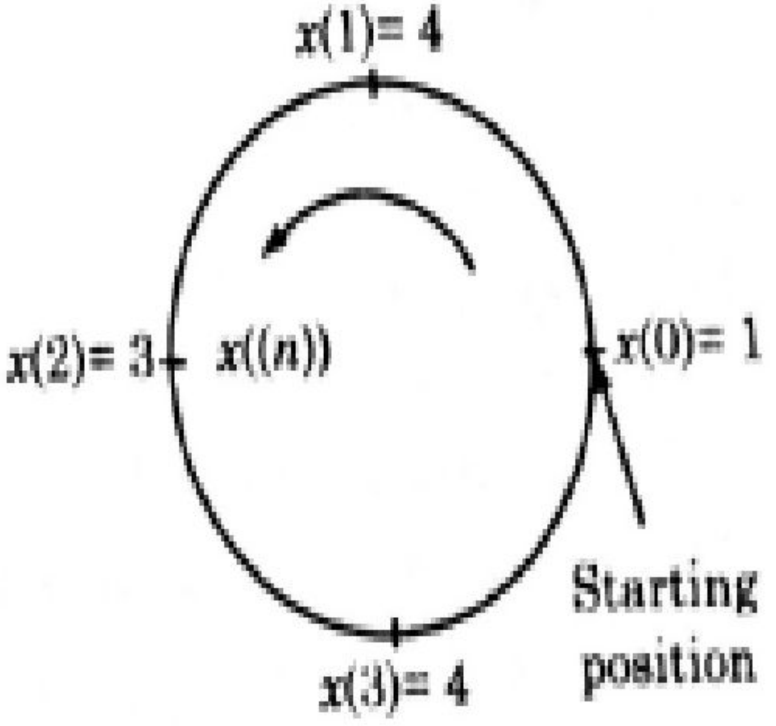


FIGURE 4.16 $x(n) = \{1, 4, 3, 4\}$

DFT

(v) Circularly even sequence

The N-point discrete time sequence is circularly even if it is symmetric about the point zero on the circle.

This means that

$$x(N-n) = x(n),\ 1 \le n \le N-1$$

Now, let us consider the sequence

$$x(n) = \{1, 4, 3, 4\}.$$

It has been plotted as shown in figure 4.16.

It may be noted that this sequence is symmetric about point zero on the circle. So, it is circularly even sequence. We can also verify it using mathematical equation,

The sequence is $x(n) = \{1, 4, 3, 4\}$

$\therefore\ x(0) = 1,\quad x(1) = 4,\quad x(2) = 3$ and $x(3) = 4$

We have the following condition for circularly even sequence:

DO YOU KNOW?

The use of window functions to minimize spectral leakage requires windows having transforms with narrow mainlobs and low side-lobs. Useful window functions having transforms with sidelobes lower than those of rectangular windows include triangular, Hanning, Hamming, and Kaiser-Bessel windows.

DFT

$$x(N - n) = x(n) \qquad \text{...(4.51)}$$

Here, $N = 4$.

Let us check this condition by substituting different values of n as under :

For $n = 1$, we have

$$x(4 - 1) = x(1) \text{ this means } x(3) = x(1) = 4$$

For $n = 2$, we have

$$x(4 - 2) = x(2) \text{ this means } x(2) = x(2) = 3$$

For $n = 3$, we have

$$x(4 - 3) = x(3) \text{ this means } x(1) = x(3) = 4.$$

Since for all values of n, equation (4.51) is satisfied, the given sequence is circularly ever

tric about point

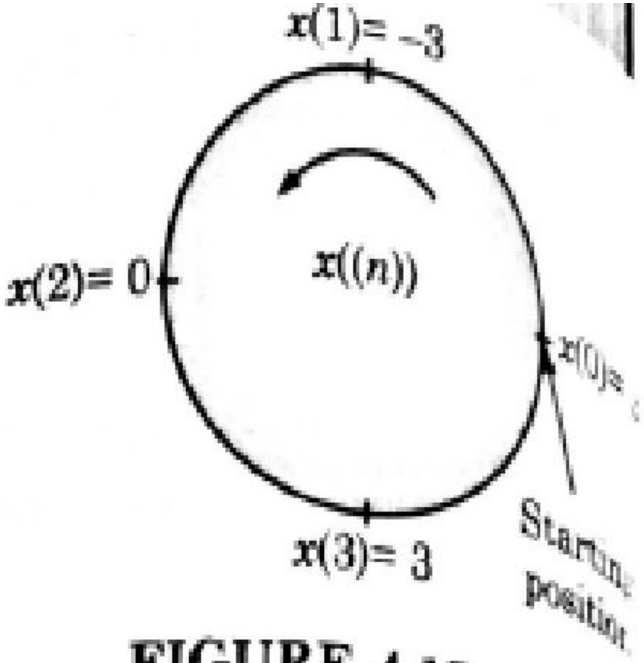


FIGURE 4.17 Plot of $x(n) = \{2, -3, 0, 3\}$.

DFT

Since for all values of n, equation ...

(vi) Circularly odd sequence

A N-point sequence is called circularly odd if it is antisymmetric about point zero on the circle.

This means that

$$x(N-n) = -x(n), \quad 1 \le n \le N-1$$

Let us consider the sequence,

$$x(n) = \{2, -3, 0, 3\}$$

This sequence has been plotted as shown in figure 4.17.

Here, $x(0) = 2$, $x(1) = -3$, $x(2) = 0$ and $x(3) = 3$.

We have the following condition for circularly odd sequence:

$$x(N-n) = -x(n), \qquad \text{for} \quad 1 \le n \le N-1 \quad \ldots(4.64)$$

For $n = 1$, we have

$$x(4-1) = -x(1) \text{ this means } x(3) = -x(1)$$

For $n = 2$, we have

$$x(4-2) = -x(2) \text{ this means } x(2) = -x(2)$$

For $n = 3$, we have

$$x(4-3) = -x(3) \text{ this means } x(1) = -x(3)$$

Thus, for all values of n, equation (4.64) is satisfied. Hence, the sequence is circularly odd

DFT

Summary of Circular Properties

Table 4.2 shows summary of circular property.

TABLE 4.2

S.No.	Sequence	Expression	Explanation
1.	Input sequence	$x((n))$	Plot the samples of $x(n)$ in anti-clockwise direction. Anticlockwise means positive direction.
2.	Circular delay	$x((n-k))$	Shift sequence $x(n)$ in anticlock-wise direction by k samples.
3.	Circular advance	$x((n+k))$	Shift sequence $x(n)$ in clockwise direction by k samples.
4.	Circular folding	$x((-n))$	Plot the samples of $x(n)$ in clockwise direction. Clockwise means negative direction.
5.	Circularly even	$x(N-n) = x(n)$	Sequence is symmetric about the point zero on the circle.
6.	Circularly odd	$x(N-n) = -x(n)$	Sequence is antisymmetric about the point zero on the circle.

DFT

Therefore, $x(n)$ ... N

4.12.6 Multiplication of Two DFTs and Circular Convolution

(Sem. Exam, WBTU, Kolkata, 2002-03)

This property states that the multiplication of two DFTs is equivalent to the circular convolution of their sequences in time domain.

Mathematically, we have

If $x_1(n) \xleftrightarrow[N]{DFT} X_1(k)$

and $x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$ then,

$$x_1(n) \circledN x_2(n) \xleftrightarrow[N]{DFT} X_1(k) \cdot X_2(k) \quad \text{...(4.71)}$$

Here, $\circledN$ indicates circular convolution.

Let the result of circular convolution of $x_1(n)$ and $x_2(n)$ be $y(m)$ then the circular convolution can also be expressed as,

$$y(m) = \sum_{n=0}^{N-1} x_1(n)x_2((m-n))_N, \quad m = 0, 1, \dots N-1 \quad \text{...(4.72)}$$

Here, the term $x_2((m-n))_N$ indicates the circular convolution.

DFT

EXAMPLE 4.12 **Given the two sequences of length 4 as under:**

$$x(n) = \{0, 1, 2, 3\}$$
$$h(n) = \{2, 1, 1, 2\}$$

Compute the circular convolution.

Solution : According to the definition of circular convolution, we have

$$y(m) = \sum_{n=0}^{N-1} x_1(n) \, . \, x_2((m-n))_N \qquad ...(i)$$

Here, given sequences are $x(n)$ and $h(n)$. The length of sequence is 4 that means $N = 4$. Thus equation (i) becomes,

$$y(m) = \sum_{n=0}^{3} x(n)h((m-n))_4 \qquad ...(ii)$$

DFT

(*i*) **We draw $x(n)$ and $h(n)$ as shown in figure 4.19(a) and (b).**

It may be noted that $x(n)$ and $h(n)$ are plotted in anticlockwise direction.

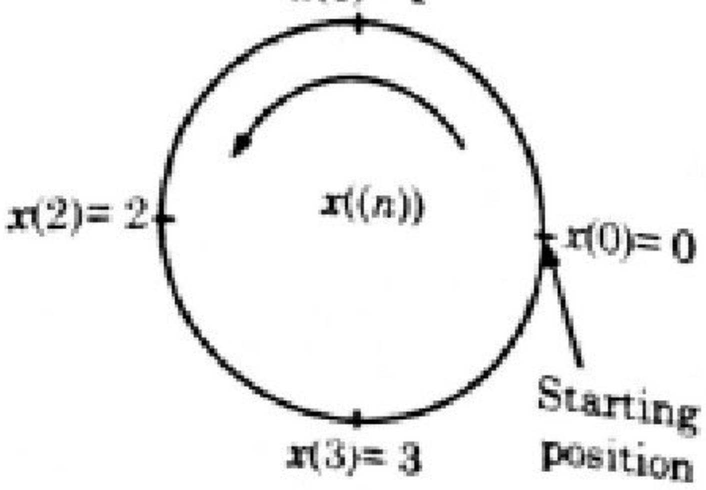


FIGURE 4.19 (a) $x(n) = \{0, 1, 2, 3\}$.

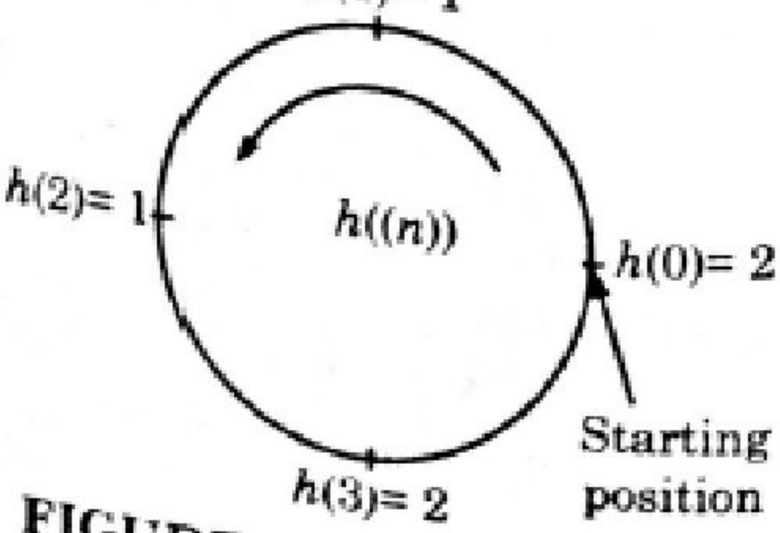


FIGURE 4.19 (b) $h(n) = \{2, 1, 1, 2\}$.

DFT

Now, let us calculate different values of $y(m)$ by putting $m = 0$ to $m = 3$ in equation (*ii*).

(*ii*) **Calculation of $y(0)$**

Substituting $m = 0$ in equation (*ii*), we get

$$y(0) = \sum_{n=0}^{3} x(n)h((-n))_4 \quad \text{...(iii)}$$

Equation (*iii*) shows that we have to obtain the product of $x(n)$ and $h((-n))$, and then we have to take the summation of product elements. Using graphical method, this calculation is done as follows.

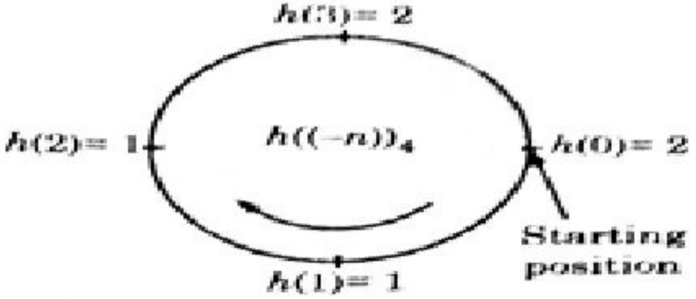


FIGURE 4.19 *(c) h(n) is plotted in clockwise direction.*

The sequence $h((-n))_4$ indicates circular folding of $h(n)$. This sequence is obtained by plotting $h(n)$ in a clockwise direction as shown in figure 4.19(*c*).

To do the calculations, we plot $x(n)$ and $h((-n))$ on two concentric circles as shown in figure 4.19(*d*). Also, $x(n)$ is plotted on the inner circle and $h((-n))$ is plotted on the outer circle.

Now, according to equation (*ii*), individual values of product $x(n)$ and $h((-n))$ are obtained by multiplying two sequences point by point. Then, $y(0)$ is obtained by adding all product terms.

Therefore, we have

$$y(0) = (0 \times 2) + (1 \times 2) + (1 \times 2) + (3 \times 1) = 0 + 2 + 2 + 3$$

or $$y(0) = 7$$

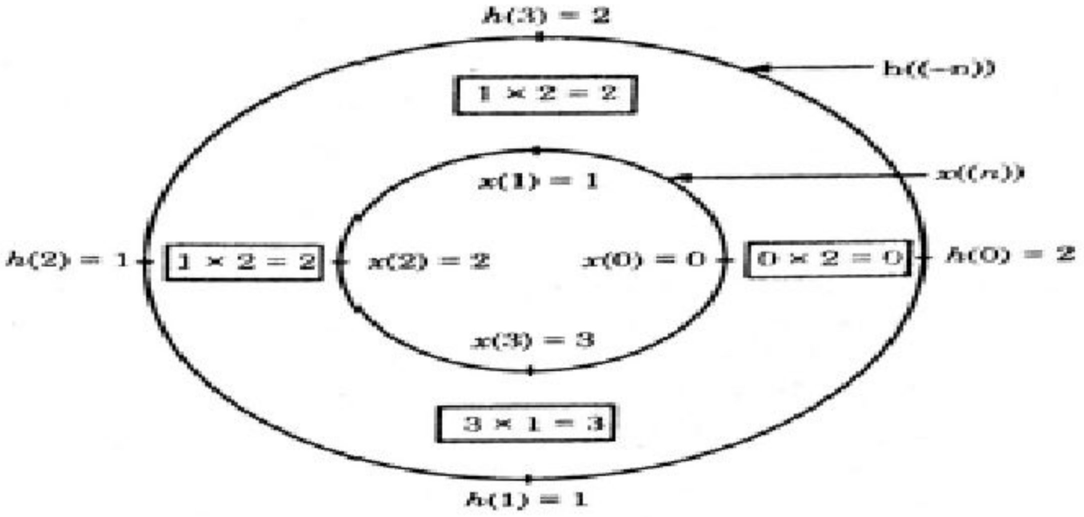


FIGURE 4.19 *(d)* $\sum_{n=0}^{3} x(n)h((-n))_4$

DFT

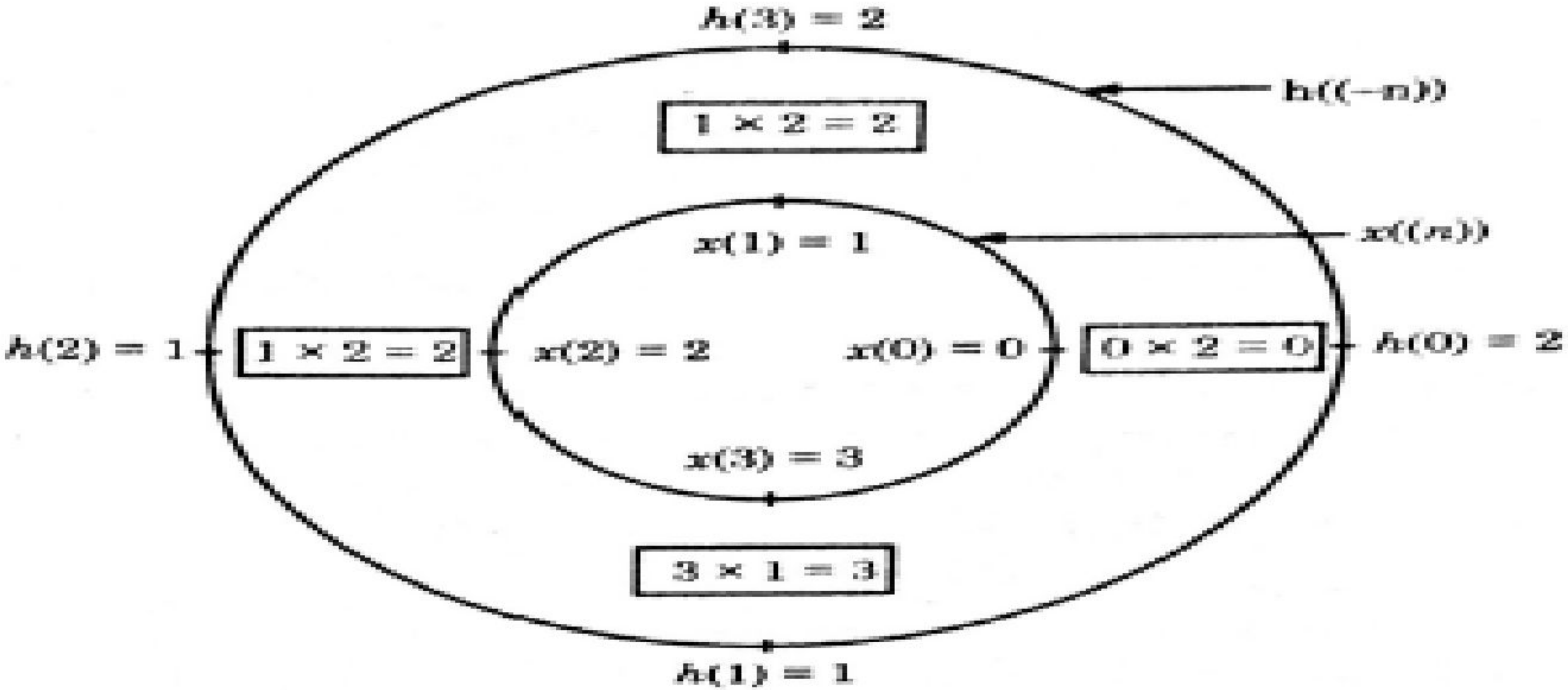


FIGURE 4.19 *(d)* $\sum_{n=0}^{3} x(n)h((-n))_4$

DFT

(iii) Calculation of $y(1)$

Substituting $m = 1$ in equation (ii), we have

$$y(1) = \sum_{n=0}^{3} x(n)h((1-n))_4 \qquad ...(iv)$$

Here, $h((1-n))_4$ is same as $h((-n+1))_4$. This indicates delay of $h((-n))$ by 1 sample. This is obtained by shifting $h((-n))$ in anticlockwise direction by 1 sample, as shown in figure 4.19(e).

We have already drawn the sequence $x(n)$ as shown in figure 4.19(a). To do the calculations, according to equation (iv), two sequences $x(n)$ and $h((1-n))_4$ are plotted on two concentric circles as shown in figure 4.19(f). Also, $y(1)$ is obtained by adding the product of individual terms.

Therefore, we write

$$y(1) = (0 \times 1) + (3 \times 1) + (2 \times 2) + (1 \times 2)$$
$$= 0 + 3 + 4 + 2$$

or $\quad y(1) = 9$

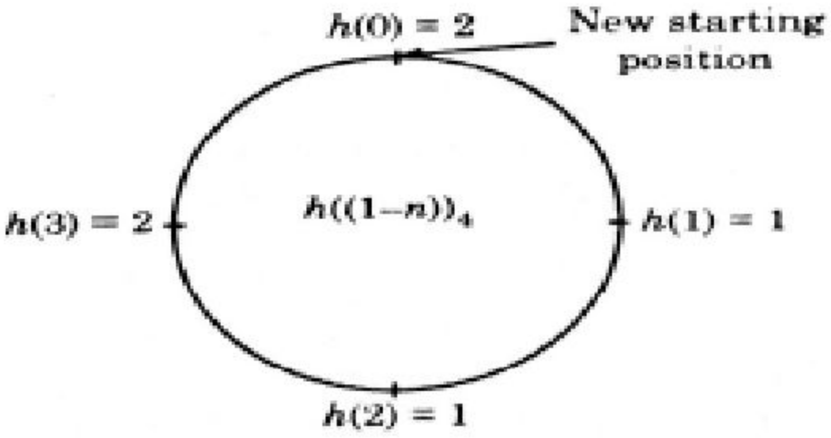


FIGURE 4.19 (e). $h((-n+1))_4$.

DFT

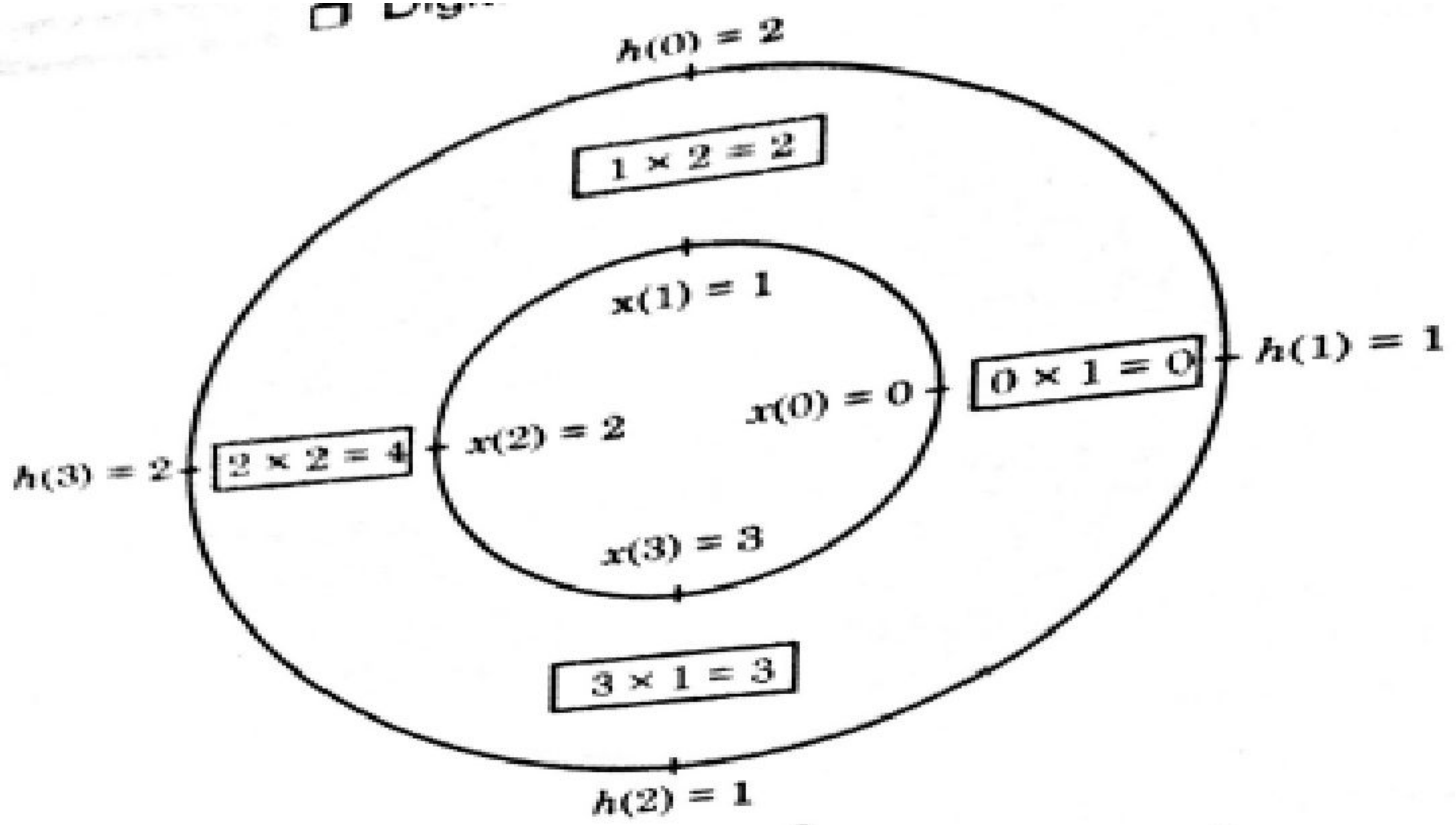


FIGURE 4.19 (f) $y(1) = \sum_{n=0}^{3} x(n)h((1-n))_4$.

DFT

(iv) Calculation of $y(2)$

Substituting $m = 2$ in equation (*ii*), we obtain,

$$y(2) = \sum_{n=0}^{3} x(n)h((2-n))_4 \qquad ...(v)$$

Here, $h((2-n))_4$ is same as $h((-n+2))_4$. It indicates delay of $h((-n))_4$ by 2 samples. It is obtained by shifting $h((-n))_4$ by two samples in anticlockwise direction as shown in figure 4.19(*g*).

According to equation (*iv*), the value of $y(2)$ is obtained adding individual product terms as shown in figure 4.19(*h*).

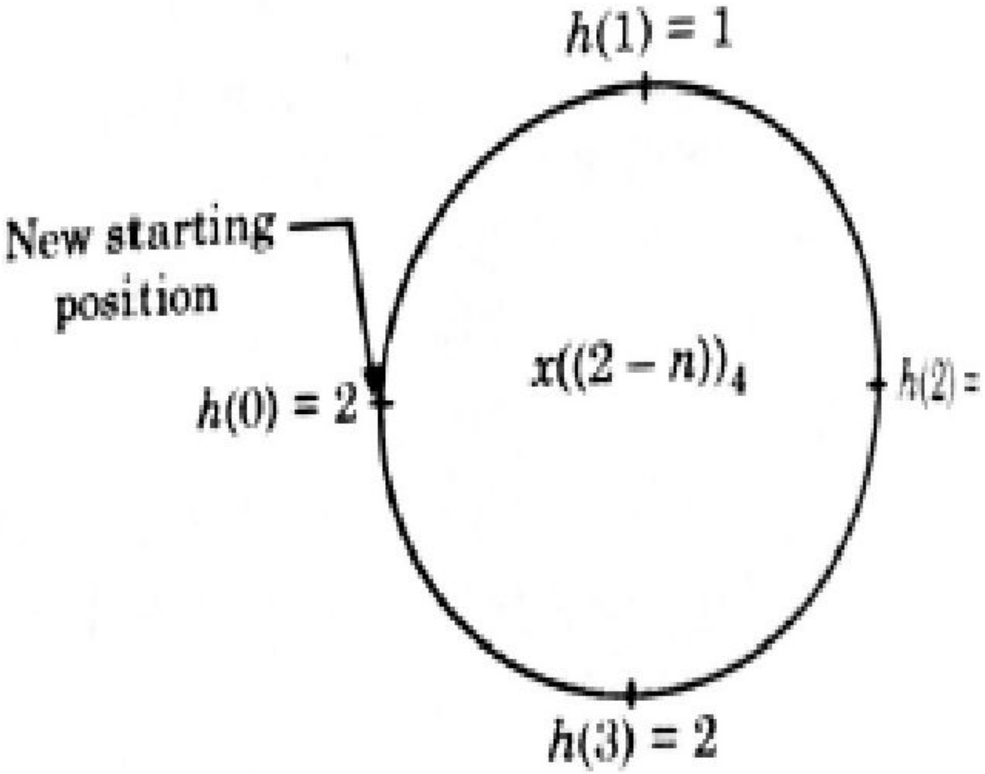


FIGURE 4.19 (*g*) $h((-n+2))_4$

DFT

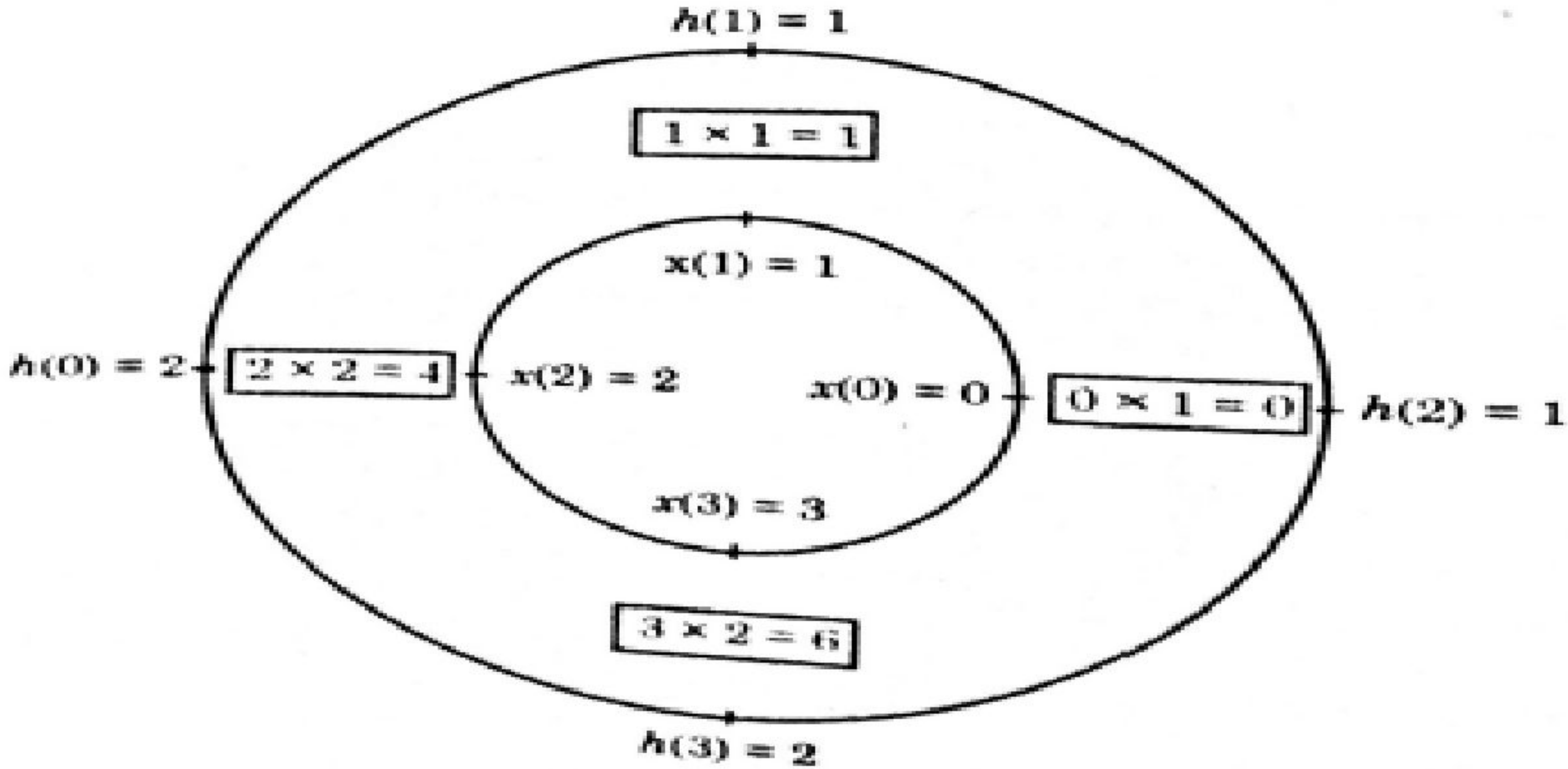


FIGURE 4.19 (h) $y(2) = \sum_{n=0}^{3} x(n)\, h((2-n))_4$

DFT

Therefore, we have

$$y(2) = (0 \times 1) + (3 \times 2) + (2 \times 2) + (1 \times 1)$$

or

$$y(2) = (0 + 6 + 4 + 1)$$

or

$$y(2) = 11$$

DFT

Step V : Calculation of $y(3)$

Substituting $m = 3$ in equation (*ii*), we get

$$y(3) = \sum_{n=0}^{3} x(n)h((3-n))_4 \qquad \ldots(iv)$$

Here, $h((3-n))_4$ is same as $h((-n+3))_4$. It indicates delay of $h((-n))_4$ by 3 samples. It is obtained by shifting $h((-n))_4$ by 3 samples in anticlockwise direction as shown in figure 4.19(*i*).

According to equation (*v*), $y(3)$ is obtained by adding individual product terms as shown in figure 4.19(*j*) *i.e.*,

$$y(3) = (0 \times 2) + (3 \times 2) + (2 \times 1) + (1 \times 1)$$

or $$y(3) = 0 + 6 + 2 + 1$$

or $$y(3) = 9$$

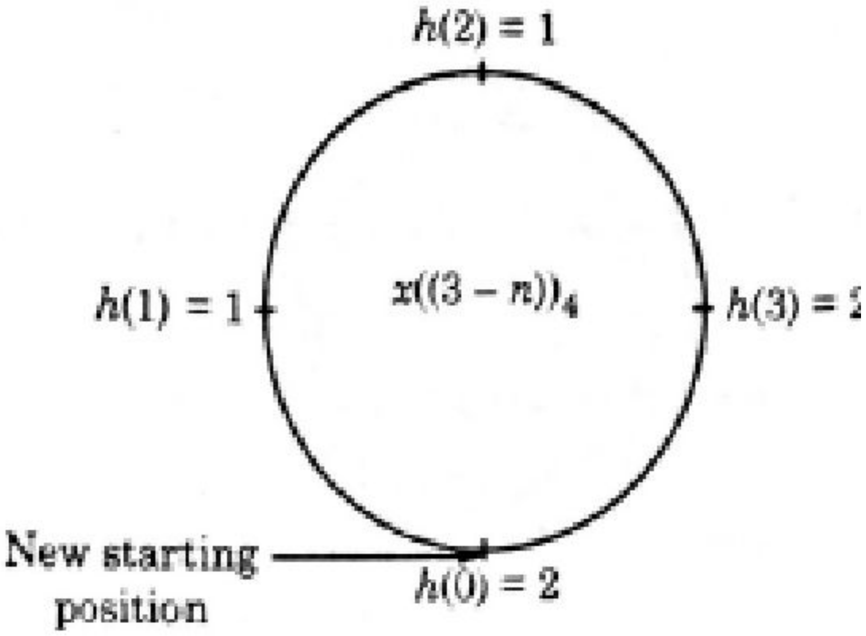


FIGURE 6.19 (*i*). $h((-n+3))_4$

DFT

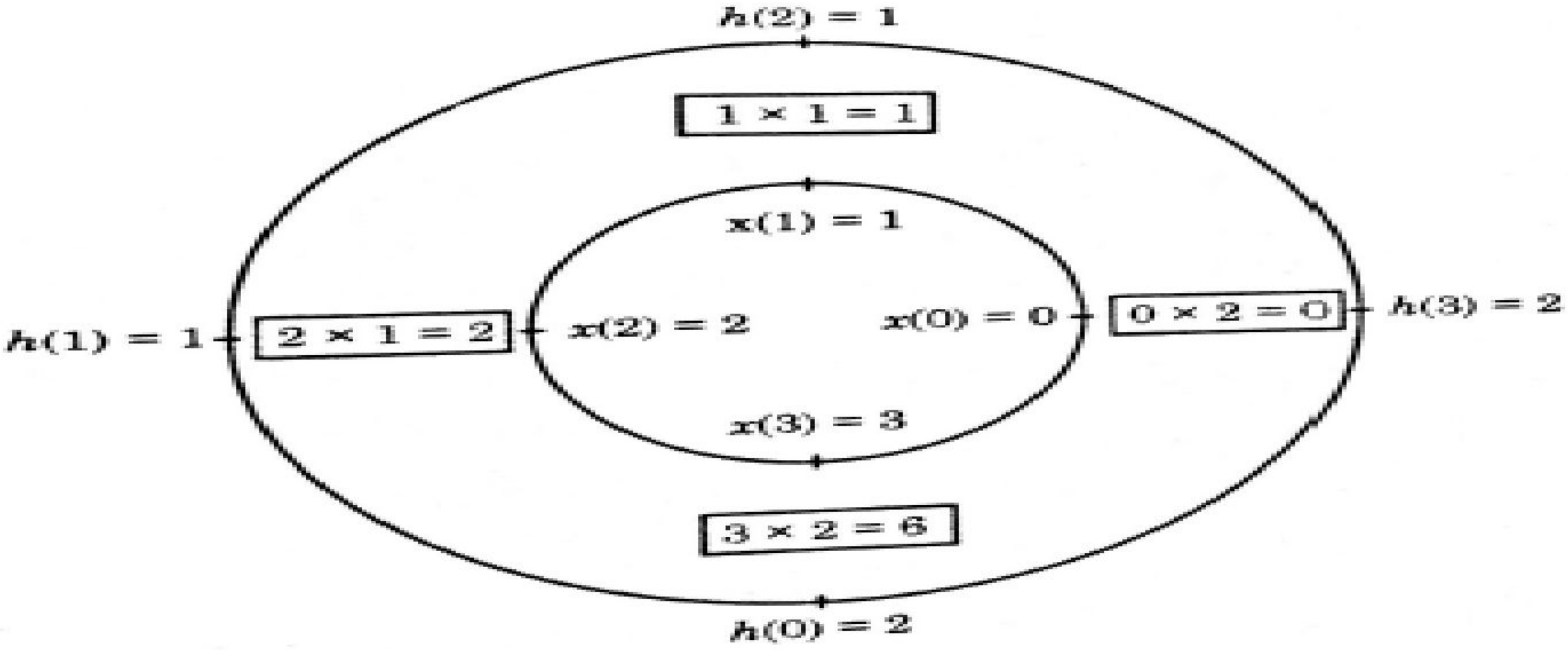


FIGURE 4.19 (j). $y(3) = \sum_{n=0}^{3} x(n)h((3-n))_4$

DFT

Now, the resultant sequence $y(m)$ can be written as under:

$$y(m) = \{y(0), y(1), y(2), y(3)\}$$

or

$$y(m) = \{7, 9, 11, 9\}. \quad \textbf{Ans.}$$

Matrix Method

DFT

- Linear Convolution Vs Circular Convolution

- Linear Convolution:

L = Number of samples in $x(n)$

M = Number of samples in $h(n)$

N = Number of samples in the result of linear convolution.

Hence, for the linear convolution, we can write the equation,

$$N = L + M - 1$$

Circular Convolution:

The output contains the same number of samples as that of x[n] and h[n].