IIR(cont..)

IIR

EXAMPLE 8.1 Determine $H(z)$ using impulse invariance method at 5 Hz sampling frequency from $H_a(s)$ as given below :

$$H_a(s) = \frac{2}{(s+1)(s+2)}$$

IIR

Solution. Given analog transfer function is,

$$H_a(s) = \frac{2}{(s+1)(s+2)}$$

Let us expand $H_a(s)$ using partial fraction expansion, *i.e.*,

$$H_a(s) = \frac{A_1}{(s+1)} + \frac{A_2}{(s+2)}$$

IIR

$$(s+1) \qquad (s+2)$$

Thus, we find that poles are at $p_1 = -1$ and $p_2 = -2$.
Now, values of A_1 and A_2 can be calculated as under:

$$A_1 = (s - p_1)\,H(s)\big|_{s=p_1} = (s+1)\,.\,\frac{2}{(s+1)(s+2)}\bigg|_{s=-1}$$

Therefore, $$A_1 = \frac{2}{-1+2} = 2$$

Also, $$A_2 = (s - P_2)\,H(s)\big|_{s=p2} = (s+2)\,.\,\frac{2}{(s+1)(s+2)}\bigg|_{s=-2}$$

Therefore, $$A_2 = \frac{2}{-2+1} = -2$$

IIR

Substituting values of A_1 and A_2 in equation (*ii*), we obtain

$$H_a(s) = \frac{2}{(s+1)} - \frac{2}{(s+2)}$$

Now, let us obtain the *z*-transform using impulse invariance transformation equation. It is given by

$$\frac{1}{s-p_i} \rightarrow \frac{1}{1-e^{p_i T}.z^{-1}} \qquad \text{...(iv)}$$

IIR

Here, T = Sampling time. Now, given sampling frequency is $f_s = 5$ Hz.

Also $$T = \frac{1}{f_s} = \frac{1}{5} = 0.2 \text{ sec.}$$

We have poles at $p_1 = -1$ and $p_2 = -2$

IIR

Hence, using equation (*iv*), we obtain

$$\frac{1}{s+1} \rightarrow \frac{1}{1-e^{-1(0.2)}.z^{-1}} = \frac{1}{1-e^{-0.2}.z^{-1}}$$

and

$$\frac{1}{s+2} \rightarrow \frac{1}{1-e^{-2(0.2)}.z^{-1}} = \frac{1}{1-e^{-0.4}.z^{-1}}$$

IIR

The transfer function of digital filter is given by

$$H(z) = \sum_{i=1}^{N} \frac{A_i}{1 - e^{p_i \cdot T} \cdot z^{-1}}$$

IIR

In this case, we obtain

$$H(z) = \frac{A_1}{1 - e^{p_1 . T} . z^{-1}} + \frac{A_2}{1 - e^{p_2 . T} . z^{-1}}$$

Using equations (v) and (vi), we obtain

$$H(z) = \frac{2}{1 - e^{-0.2} . z^{-1}} - \frac{2}{1 - e^{-0.4} . z^{-1}} = \frac{2}{1 - 0.818\, z^{-1}} - \frac{2}{1 - 0.67\, z^{-1}}$$

IIR

To convert each term into positive powers of z, let us multiply each term by z to get

$$H(z) = \frac{2z}{z-0.818} - \frac{2z}{z-0.67} = \frac{2z(z-0.67)-2z(z-0.818)}{(z-0.818)(z-0.67)}$$

or

$$H(z) = \frac{2z^2-1.34z-2z^2+1.636z}{z^2-0.67z-0.818z+0.54} = \frac{0.29z}{z^2-1.488z+0.54}$$

This is the required transfer function for digital IIR filter. **Ans.**

IIR

Design of IIR filter by Bilinear transformation method

This limitation

Due to spectrum aliasing, Impulse invariant method technique are not suitable for high pass or band reject filter.

techniques are not suitable for high-pass or band-reject filters. This limitation has been removed in the mapping technique which is popularly known as the **bilinear transformation**. This transformation is a one-to-one mapping from the s-domain to the z-domain. That means that the bilinear transformation is a conformal mapping which transforms the $j\Omega$-axis into the unit circle in the z-plane only once, and hence avoiding aliasing in frequency components. Further, the transformation of a stable analog filter results in a stable digital filter since all the poles in the left half of the s-plane are mapped onto points inside the unit circle of the z-domain. We can obtain the bilinear transformation by using the trapezoidal formula for numerical integration.

Let us consider that the

IIR

...ation by using the trapezoidal formula for numerical int

Let us consider that the system function of the analog filter is expressed as under :

$$H_a(s) = \frac{b}{s+a}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b}{s+a}$$

IIR

or $$sY(s) + aY(s) = bX(s)$$

Taking inverse Laplace transform of both sides, we get

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

Integrating the above equation between the limits $(nT - T)$ and nT, *i.e.*,

$$\int_{nT-T}^{nT} \frac{dy(t)}{dt} dt + a \int_{nT-T}^{nT} y(t)dt = b \int_{nT-T}^{nT} x(t)dt \qquad \ldots(8.44)$$

IIR

The trapezoidal rule for numeric integration is expressed as

$$\int_{nT-T}^{nT} a(t)dt = \frac{T}{2}[a(nT) + a(nT-T)] \qquad ..(8.45)$$

IIR

Using equation (8.45) in equation (8.44), we have

$$y(nT) - y(nT - T) + \frac{aT}{2} y(nT) + \frac{aT}{2} y(nT - T) = \frac{bT}{2} x(nT) + \frac{bT}{2} x(nT - T)$$

IIR

Taking z-transform, we get:—

$$Y(z) - Y(z)z^{-1} + \frac{aT}{2}Y(z) + \frac{aT}{2}Y(z)z^{-1} = \frac{bT}{2}X(z) + \frac{bT}{2}X(z)z^{-1}$$

or $Y(z)\left[1+\frac{aT}{2}\right] - \left[1-\frac{aT}{2}\right]z^{-1}Y(z) = \frac{bT}{2}\left[X(z)+X(z)z^{-1}\right].$

or $Y(z)\left[1+\frac{aT}{2} - \left(1-\frac{aT}{2}\right)z^{-1}\right] = X(z)\left[bT/2\left(1+z^{-1}\right)\right].$

Now, Transfer function of the equivalent digital filter is:—

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\left[bT/2\left(1+z^{-1}\right)\right]}{\left[1+\frac{aT}{2} - \left(1-\frac{aT}{2}\right)z^{-1}\right]}.$$

IIR

$$\therefore H(z) = \frac{\frac{bT}{2}\left(1+z^{-1}\right)}{1+\frac{aT}{2}-z^{-1}-\frac{aT}{2}z^{-1}} = \frac{\frac{bT}{2}\left(1+z^{-1}\right)}{\left(1-z^{-1}\right)+\frac{aT}{2}\left(1+z^{-1}\right)} = \frac{bT/2}{\frac{\left(1-z^{-1}\right)}{\left(1+z^{-1}\right)}+\frac{aT}{2}\cdot\frac{\left(1+z^{-1}\right)}{\left(1+z^{-1}\right)}}$$

$$= \frac{\frac{bT}{2}}{\frac{\left(1-z^{-1}\right)}{\left(1+z^{-1}\right)}+\frac{aT}{2}} = \frac{b}{a+\frac{2}{T}\frac{\left(1-z^{-1}\right)}{\left(1+z^{-1}\right)}}.$$

IIR

Taking z-transform, the system function of the digital filter will be

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+a}$$

IIR

The general characteristic of the mapping $z = e^{sT}$ may be obtained by putting $s = \sigma + j\,\Omega$ and expressing the complex variable z in the polar form as $z = re^{j\omega}$ in equation (8.47).

Thus, $$s = \frac{2}{T}\left(\frac{z-1}{z+1}\right) = \frac{2}{T}\left(\frac{re^{j\omega}-1}{re^{j\omega}+1}\right)$$

IIR

Frequency Relation

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \;\; \& \;\; s = \sigma + j\Omega$$

IIR

If $r=1$ in $z = re^{j\omega}$

$$s = \frac{2}{T}\left[\frac{1-e^{-j\omega}}{1+e^{-j\omega}}\right]$$

$$= \frac{2}{T}\left[\frac{e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)}{e^{-j\omega/2}\left(e^{j\omega/2} + e^{-j\omega/2}\right)}\right]$$

IIR

$$S = \frac{2}{T}\left[\frac{2J \sin\omega/2}{2 \cos\omega/2}\right]$$

$$= \frac{2}{T} J \tan\omega/2 .$$

IIR

Comparing with slide 20, we get

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2} \quad \ldots(8.51)$$

$$\omega = 2\tan^{-1}\frac{\Omega T}{2} \quad \ldots(8.52)$$

IIR

Note : *Equation (8.51) provides the relationship between the frequencies in the two domains and this has been shown in figure 8.13. It can be noted that the entire range in Ω is mapped only once into the range $-\pi \leq \omega \leq \pi$. But, as seen in figure 8.13, the mapping is non-linear and the lower frequencies in analog domain are expanded in the digital domain, whereas the higher frequencies are compressed. This is due to the non-linearity of the arc tangent function and usually known as* ***frequency warping****.*

IIR

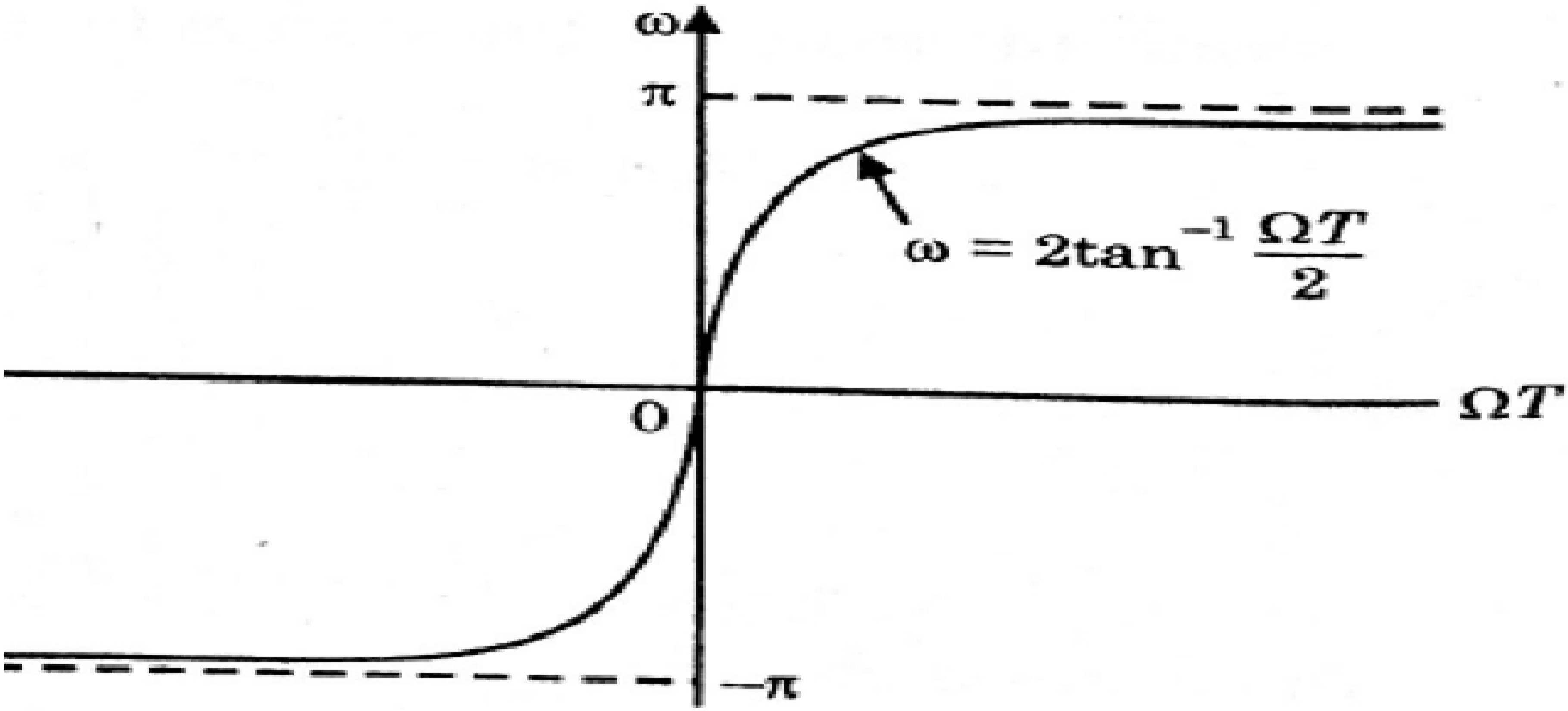

FIGURE 8.13 *Mapping between 'Ω' and ω in bilinear transformation.*

IIR

The main difference between impulse invariance and bilinear transformation is that there is no aliasing effect in bilinear transformation. Infact, this is the major advantage of bilinear transformation. Observe that the complete jΩ axis is mapped on the unit circle only once. But in impulse invariance the segments $\frac{(2k-1)\pi}{T} \leq \Omega \leq \frac{(2k+1)\pi}{T}$ of $j\Omega$ axis are all mapped on unit circle repeatedly. Thus the transformation is many to one. Hence problem of aliasing takes place in impulse invariance method. The problem with bilinear transformation is that the frequency relationship is nonlinear.

IIR

8.9.1 Advangates of Bilinear Transformation Method

(*i*) There is one to one transformation from the *s*-domain to the z-domain.

(*ii*) The mapping is one to one.

(*iii*) There is no aliasing effect.

(*v*) Stable analog filter is transformed into the stable digital filter.

IIR

8.9.2 Disadvantages of Bilinear Transformation Method

(*i*) The mapping is non-linear and because of this, the frequency warping effect takes place.

IIR

8.9.3 Comparison between Impulse Invariance Method and Bilinear Transformation Method

S. No.	Impulse invariance method	Bilinear transformation method
1.	Poles are transferred by using the expression: $\frac{1}{s-p_i} \rightarrow \frac{1}{1-e^{p_i T}\cdot z^{-1}}$	1. Poles are transferred by using the expression; $s = \frac{2}{T}\left[\frac{z-1}{z+1}\right]$
2.	Mapping is many to one.	2. Mapping is one to one.
3.	Aliasing effect is present.	3. Aliasing effect is not present.
4.	It is not suitable to design high-pass filter and band reject filter.	4. High pass filter and bandreject filter can be designed.
5.	Only poles of the system can be mapped.	5. Poles as well as zeros can be mapped.
6.	No frequency warping effect.	6. Frequency warping effect is present.

IIR

EXAMPLE 8.12 The system function of the analog filter is given as,

$$H_a(s) = \frac{s+0.1}{(s+0.1)^2+16}$$

Obtain the system function of the digital filter using bilinear transformation which is resonant at $\omega_r = \frac{\pi}{2}$.

IIR

Solution : From the denominator of $H_a(s)$, we can write the poles of analog filter as under:

$$(s+0.1)^2+16=(s+0.1-j4)(s+0.1+j4)$$

or

$$s=-0.1+j4 \text{ and } s=-0.1-j4$$

There are the two complex conjugate poles. We know that $s=\sigma+j\Omega$. Hence, the values of 'σ' and 'Ω' for these two poles will be

$$\sigma=-0.1 \text{ and } \Omega=\pm 4$$

A function is said to be resonant at its poles. Therefore, $H_a(s)$ will be resonant at,

$$s=-0.1\pm j4$$

IIR

In other words, we can state that $H_a(s)$ will be resonant at $\Omega = 4$. It is needed that the digital filter must be resonant at $\omega_r = \frac{\pi}{2}$. This means that the bilinear transformation must map $\Omega = 4$ into $\omega_r = \frac{\pi}{2}$. We know that the relationship between 'Ω' and 'ω' is given by

IIR

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

or

$$T = \frac{2}{\Omega}\tan\frac{\omega}{2}$$

Substituting for $\Omega = 4$ and $\omega_r = \omega = \frac{\pi}{2}$, we get

$$T = \frac{2}{4}\tan\frac{\pi}{4} = \frac{1}{2}$$

IIR

This means that if we select $T = \frac{1}{2}$, then the resonant frequency $\Omega = 4$ of analog filter will map into $\omega_r = \frac{\pi}{2}$ of digital filter in bilinear transformation. The bilinear transformation is given by

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

IIR

At $\quad T = \frac{1}{2}, \quad$ we have, $\quad s = 4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$

Substituting for this value of s in $H_a(s)$, we get $H(z)$ *i.e.*,

$$H(z) = H_a(s)\Big|_{s=4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)} = \frac{4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.1}{\left[4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+0.1\right]^2+16}$$

On simplifying the above expression, we get

$$H(z) = \frac{0.128+0.006z^{-1}-0.122z^{-2}}{1+0.0006z^{-1}+0.975z^{-2}} = \frac{0.128z^2+0.006z-0.122}{z^2+0.0006z+0.975}$$

IIR

8.10 BASIC ANALOG FILTER APPROXIMATIONS

As discussed earlier that the digital IIR filters are designed from the analog filters. Many times, it is necessary to approximate the characteristics of analog filter. This approximation is required because the practical characteristic of a filter is not identical to the ideal characteristics. There are three different types of approximation techniques as under :

(*i*) Butterworth filter approximation.

(*ii*) Chebyshev filter approximation.

(*iii*) Elliptic filter approximation.

IIR

8.11 BUTTERWORTH FILTER APPROXIMATION

A typical characteristic of a butterworth low-pass filter (LPF) is as shown in figure 8.14.

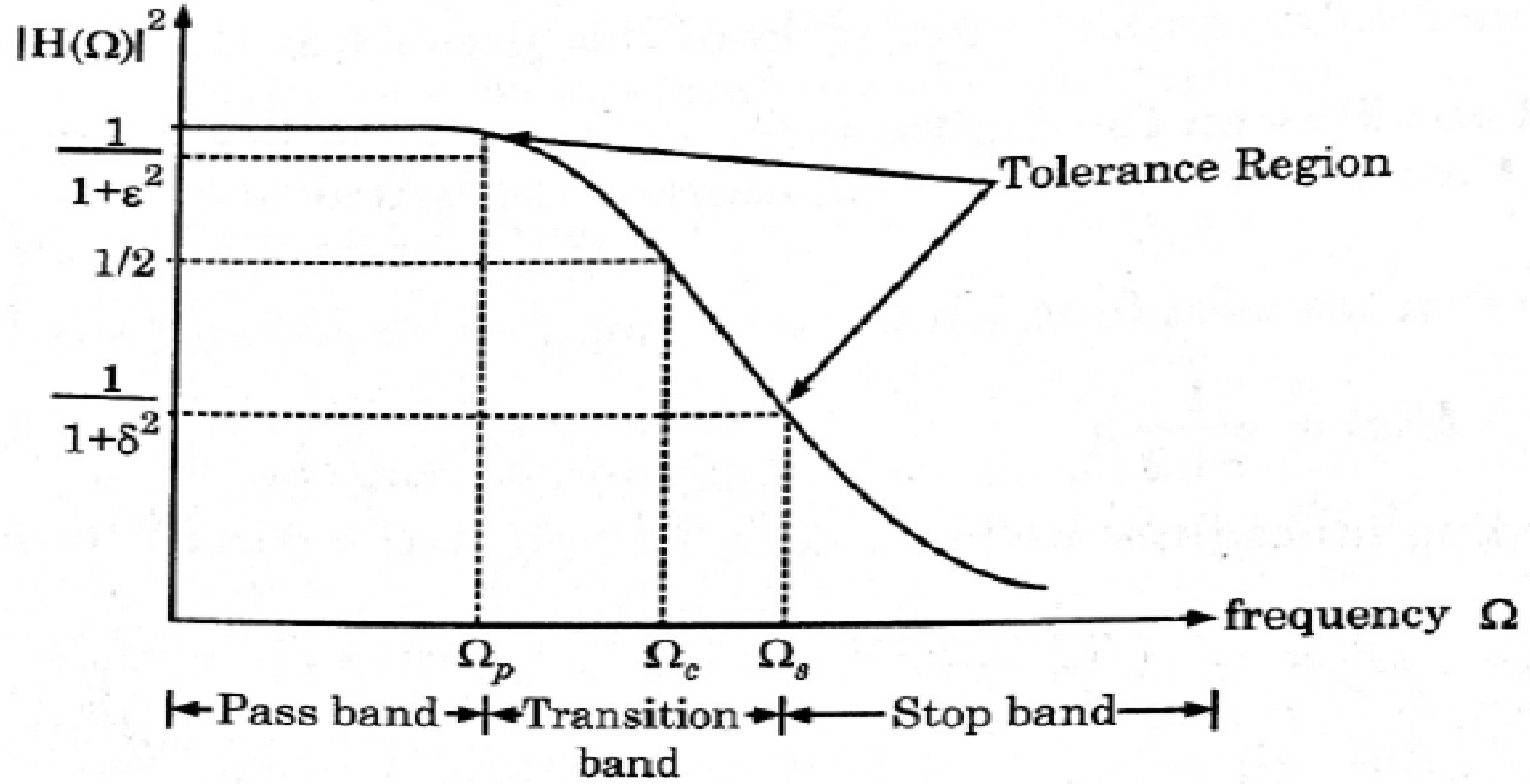

FIGURE 8.14 *Typical characteristics of analog low-pass filter (LPF).*

IIR

This type of response is called as Butterworth response because its main characteristic is that the passband is maximally flat. This means that there are no variations (ripples) in the passband.

Now, the magnitude squared response of low pass Butterworth filter is given by

$$|H(\Omega)|^2 = \frac{1}{1+\left(\frac{\Omega}{\Omega_c}\right)^{2N}} \qquad \text{...(8.53)}$$

This equation may also be expressed as,

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2\left(\frac{\Omega}{\Omega_p}\right)^{2N}} \qquad \text{...(8.54)}$$

IIR

Here, $|H(\Omega)|$ = Magnitude of analog low pass filter (LPF).

Ω_c = Cut-off frequency (– 3 dB frequency).

Ω_p = Passband edge frequency.

$1 + \epsilon^2$ = Passband edge value.

$1 + \delta^2$ = Stopband edge value.

ϵ = Parameter related to ripples in passband.

δ = Parameter related to ripples in stopband.

N = Order of the filter.

IIR

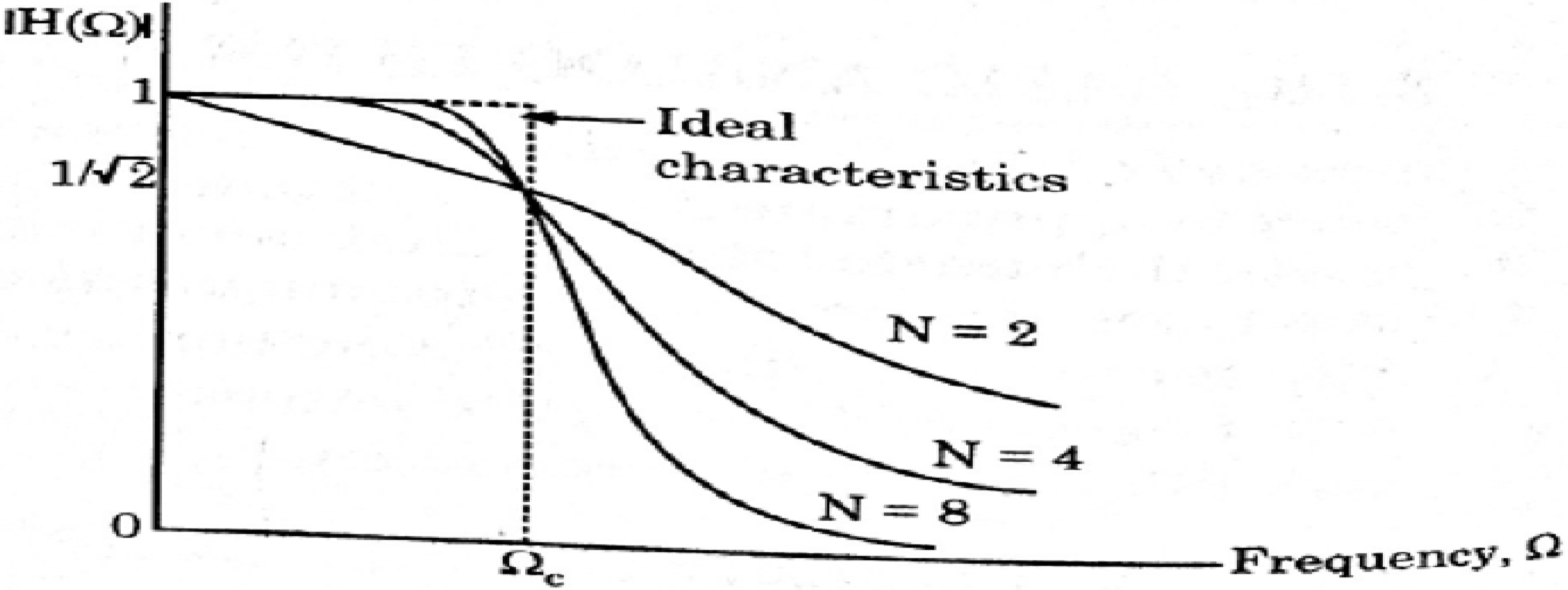

FIGURE 8.15 *Effect of N on frequency response characteristics.*

IIR

We know that, in case of low pass filter (LPF), the frequencies will pass upto the value of cut-off frequency (Ω_c). This is called as passband. After that the frequencies are attenuated. This is called as stop band. Ideal characteristic is shown by dotted line in figure 8.15. Ideally, at the value of cut-off frequency (Ω_c) the frequencies should be stopped. However, in practical cases this is not happening.

IIR

Now the order of filter is denoted by N. Roughly we can say order of filter means, the number of stages used in the design of analog filter. As the order of filter N increases, the response of filter is more close to the ideal response as shown in figure 8.15.

IIR

8.11.1 Salient Features of Low Pass Butterworth Filter

(*i*) The magnitude response is nearly constant (equal to 1) at lower frequencies. This means that the passband is maximally flat.

(*ii*) There are no ripples in the pass band and stopband.

(*iii*) The maximum gain occurs at $\Omega = 0$ and it is $|H(0)| = 1$.

(*iv*) The magnitude response is monotonically decreasing.

IIR

8.11.2 Designing Expressions and Designing Steps

Let A_p = Attenuation in passband.

A_s = Attenuation in stopband.

Ω_p = Pass band edge frequency.

Ω_c = Cut-off frequency

Ω_s = Stopband edge frequency.

IIR

In numerical problems, the specifications of required digital filter is usually given and it is asked to design a particular discrete time Butterworth filter. Then the following steps must be used :

1. From the given specifications of digital filter, we obtain equivalent analog filter as under :

(a) For *impulse invariance method*, we have

$$\Omega = \frac{\omega}{T}$$

(b) For *bilinear transformation method*, we have

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

Here, Ω = Frequency of analog filter

ω = Frequency of digital filter

T = Sampling time

IIR

2. We evaluate the order N of filter using the following exprression:

$$N = \frac{1}{2} \times \frac{\log\left[\dfrac{\left(\dfrac{1}{A_s^2}-1\right)}{\left(\dfrac{1}{A_p^2}-1\right)}\right]}{\log\left(\dfrac{\Omega_s}{\Omega_p}\right)} = \frac{\log\left[\left(1/\delta_s^2\right)-1\right]}{2\log\left(\Omega_s/\Omega_c\right)}$$

Here, δ_s = Attenuation in stop band.

IIR

If the specifications are given in decibels (dB) then, we make use of the following expression:

$$N = \frac{1}{2}\frac{\log\left[\dfrac{10^{0.1 A_s(dB)}-1}{10^{0.1 A_p(dB)}-1}\right]}{\log\left(\dfrac{\Omega_s}{\Omega_p}\right)}$$

IIR

3. After that we determine cut-off frequency (Ω_c).

The cut-off frequency (Ω_c) of analog filter is calculated as under :

(a) For *impulse invariance method*, we have

$$\Omega_c = \frac{\omega_c}{T}$$

(b) For *bilinear transformation method*, we have

$$\Omega_c = \frac{2}{T} \tan\frac{\omega_c}{2}$$

IIR

In case, ω_c is not given then we the following expression:

(i) $$\Omega_c = \frac{\Omega_p}{\left(\frac{1}{A_p^2}-1\right)^{1/2N}}$$ and if specifications are in dB then

(ii) $$\Omega_c = \frac{\Omega_p}{\left[10^{0.1A_p}-1\right]^{1/2N}}$$

IIR

4. Next, we determine the poles using the following expression :

$$p_i = \pm \Omega_c \, e^{j(N+2i+1)\pi/2N}, \quad i = 0, 1, 2, \ldots N-1.$$

If the poles are complex conjugate then organize the poles (p_i) as complex conjugate pairs that means, s_1 and s_2^*, s_2 and s_2^* etc.

IIR

5. Next, we calculate the system transfer function of analog filter using following expression:

$$H_a(s) = \frac{\Omega_c^N}{(s-p_1)(s-p_2)\ldots}$$

and if poles are complex conjugate then, we have

$$H_a(s) = \frac{\Omega_c^N}{(s-s_1)(s-s_1^*)(s-s_2)(s-s_2^*)}$$

6. Lastly, we design the digital filter using impulse invariance method or bilinear transformation method.

IIR

8.12.1 Type-1 Chebyshev Filter

These filters are all pole filters. In the passband, these filters show equiripple behaviour and they have monotonic characteristics in the stopband.

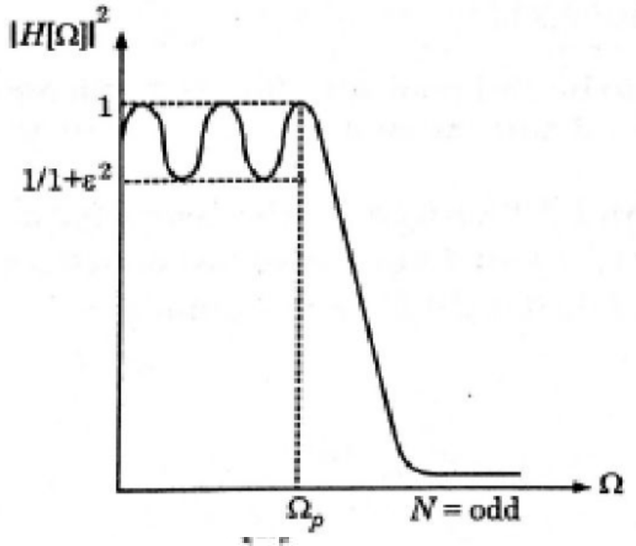

FIGURE 8.20 *Type-1 Chebyshev filter characteristics.*

IIR

The magnitude squared frequency response is given by

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2 C_N^2(\Omega/\Omega_p)}$$

Here, ϵ = Ripple parameter in the passband

Ω_p = Passband frequency

$C_N(x)$ = Chebyshev polynomial of order N.

IIR

Type II Chebyshev Filter

The filter contains zeroes as well as poles. They have monotonic in the passband and equiripple in a stopband.

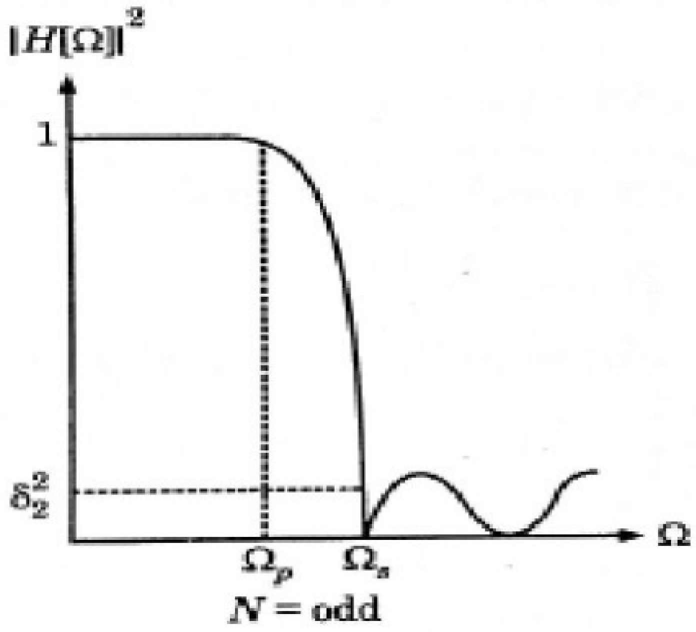

FIGURE 8.21 *Type-II Chebyshev filter characteristic.*

IIR

The magnitude squared response is given by

$$|H(\Omega)|^2 = \frac{1}{1+\epsilon^2\left[C_N^2\left(\frac{\Omega_s}{\Omega_p}\right)\Big/C_N^2\left(\frac{\Omega_s}{\Omega}\right)\right]}$$

Here $C_N(x) = N^{\text{th}}$ order polynomial

$\Omega_s =$ Stopband frequency

$\Omega_p =$ Passband frequency