Z-Transform

Z Transform

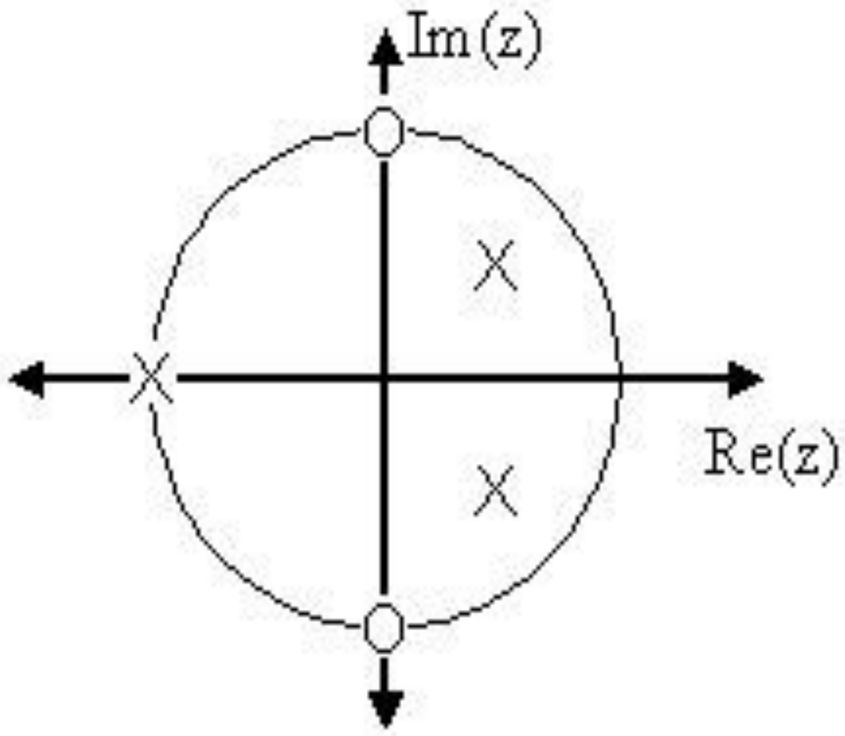
Z transform of discrete time signal x[n] is defined as the power series.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

where z is complex variable.

Z Transform

$$X(z) = \frac{P(z)}{Q(z)}$$



where $P(z)$ and $Q(z)$ are polynomials in z.

Zeros: The values of z's such that $X(z) = 0$

Poles: The values of z's such that $X(z) = \infty$

Z Transform

the z-transform of a discrete-time signal $x(n)$ may be expressed as

$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

where z is a complex variable.

The above expression is generally known as **two-sided z-transform.**

Z Transform

If discrete-time signal $x(n)$ is a causal signal *i.e.*, $x(n) = 0$ for $n < 0$, then the z-transform is called as **one sided z-transform** and is expressed as

$$Z\{x(n)\} = X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n} \qquad \text{...(3.7)}$$

In the above expression for causal signal, we obtain negative power of z in $X(z)$.

In fact, generally we assume that $x(n)$ is a causal discrete-time signal unless it is stated.

This means that generally, we analyze causal signal.

Z Transform

On the other hand, if $x(n)$ is a non-causal discrete-time signal *i.e.*, $x(n) = 0$ for $n \geq 0$, then its z-transform is expressed as

$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{-1} x(n)\, z^{-n}$$

The above expression contains positive powers of z in $X(z)$ and is also called a **one-sided z-transform.**

Z Transform

The inverse z-transform is used to get the original time-domain discrete signal $x(n)$ from its frequency-domain signal $X(z)$. Mathematically,

$$x(n) = Z^{-1}[X(z)]$$

$$x(n) \xleftrightarrow{Z} X(z)$$

Z Transform

3.3 REGION OF CONVERGENCE (ROC)

(U.P. Tech., Semester Exam., 2003-04)(05 marks)

We know that z-transform is expressed as

$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} \qquad \text{...(3.11)}$$

We may write, complex number $z = r\, e^{jw}$.

Putting $z = r\, e^{j\omega}$, equation (3.11) becomes

$$X(z)\big|_{\text{at } z = re^{jw}} = X(r\, e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, r^{-n}\, e^{-j\omega n} \qquad \text{...(3.12)}$$

The above expression is the discrete-time Fourier transform of the modified discrete-time signal $\{x(n)\, r^{-n}\}$.

Z Transform

Now if $r = 1$ then $|z| = 1$

In this case $X(z)$ (equation (3.12)) reduces to its Fourier transform.

Hence the expression in equation (3.12) will converge if $\{x(n)\ r^{-n}\}$ is absolutely summable.

Mathematically,

$$\sum_{n=-\infty}^{\infty} \left| x(n)\ r^{-n} \right| < \infty$$

Z Transform

Hence, for $x(n)$ to be finite, the magnitude of its z-transform, $X(z)$ must also be finite.

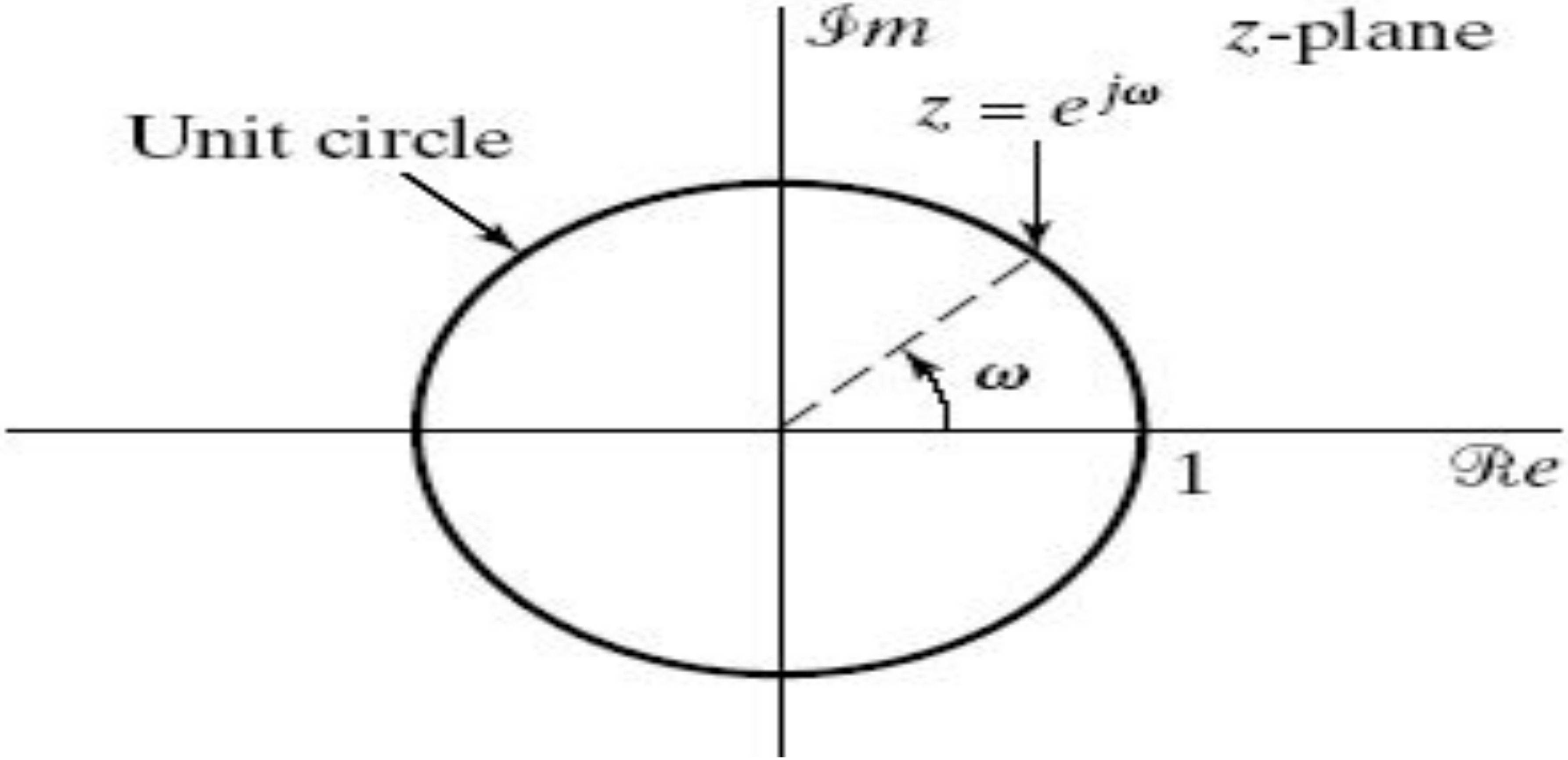
Therefore, the set of values of z in the z-plane for which the magnitude of $X(z)$ is finite, is called the **Region of Convergence (ROC).**

This means that convergence of

$\sum_{n=0}^{\infty} \left| x(n)\, r^{-n} \right|$ guarantees the convergence of the expression

$$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

Z Transform



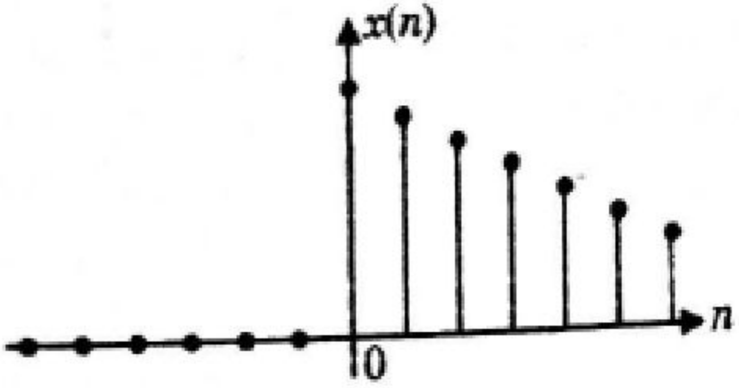
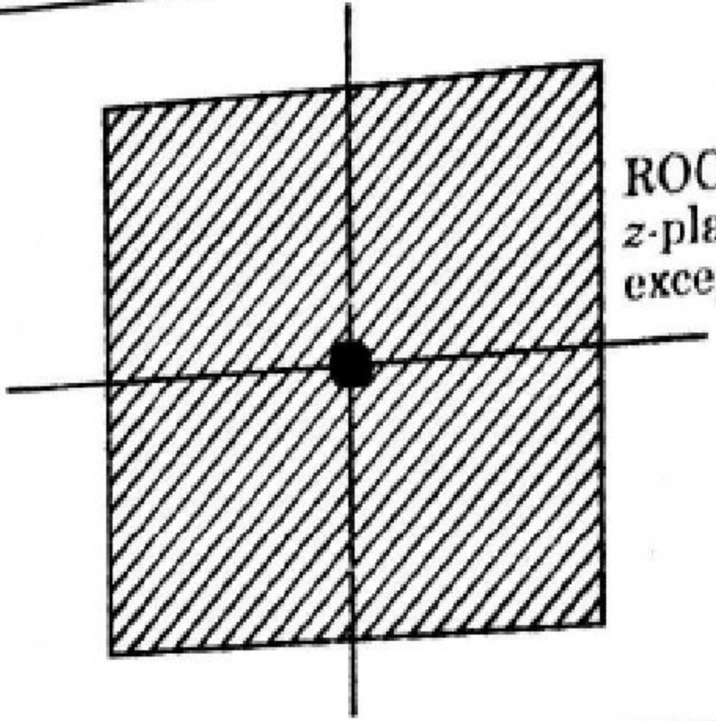
Z Transform(ROC)

- ROC defines the region of where the Z-transform exists.
- ROC of X(z) is the set of all values of z for which X(z) attains a finite value.
- X(z) exists if and only if the argument of z is inside the ROC

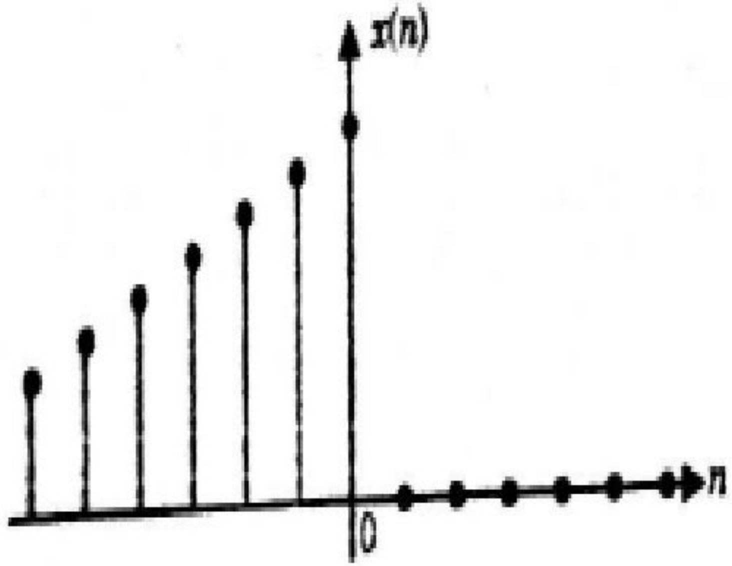
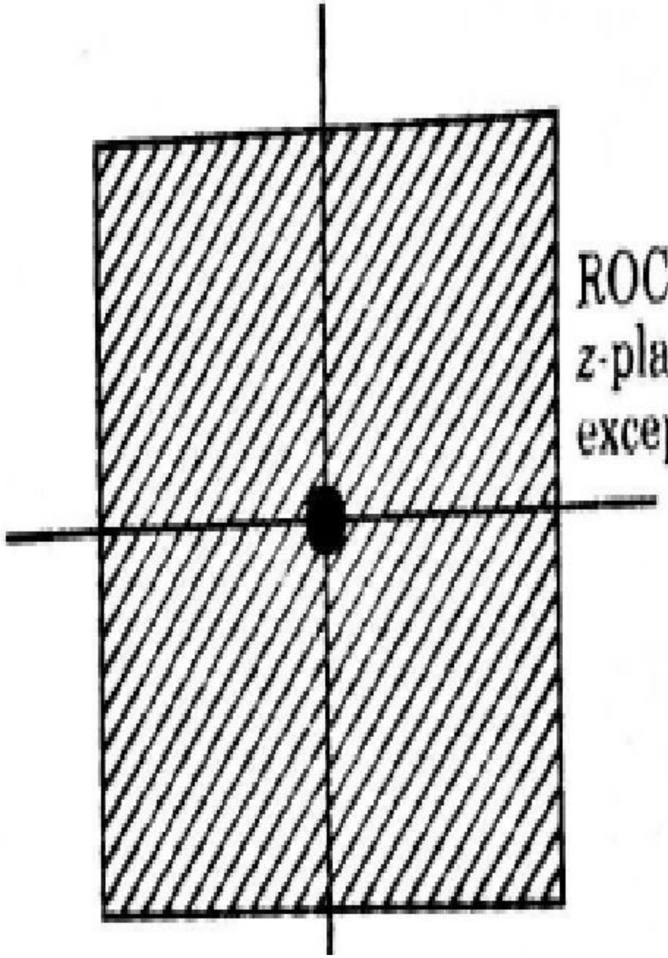
Z Transform

Table 3.1 shows finite-duration
ROCs.

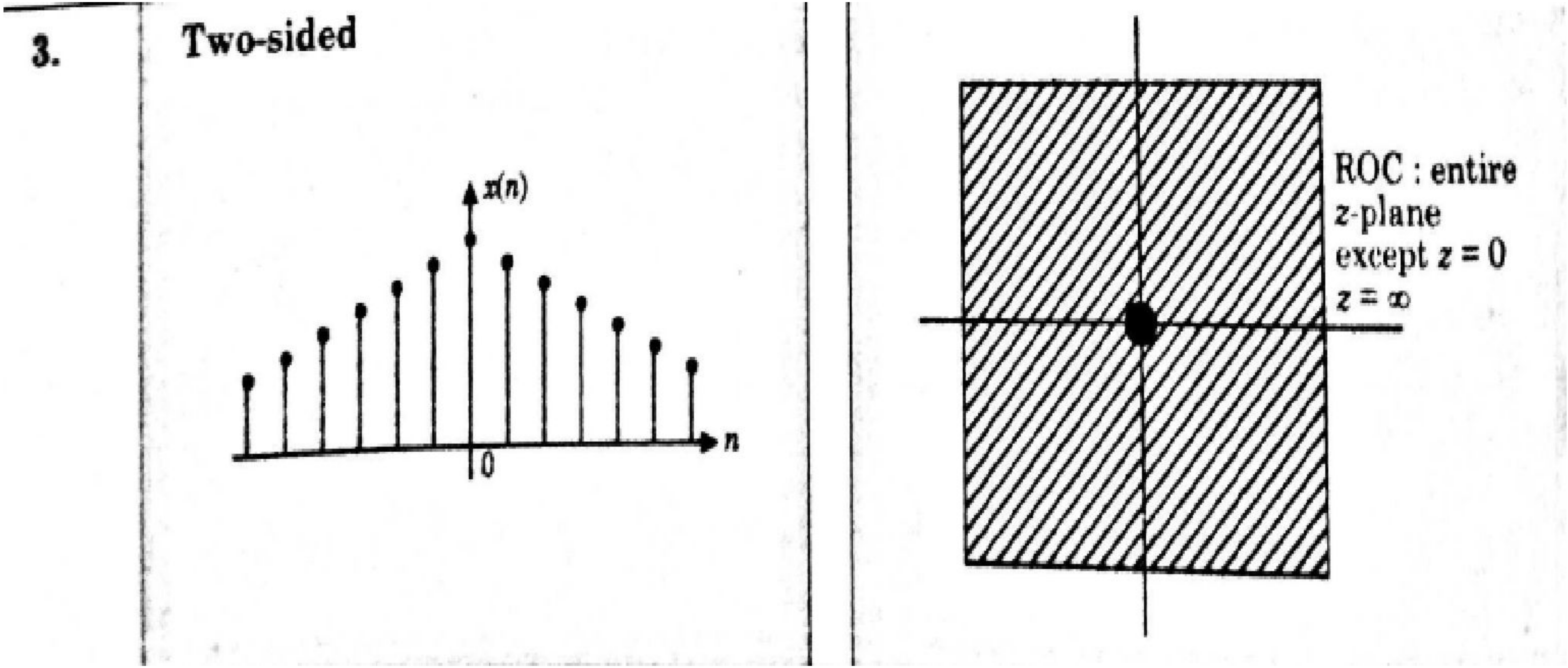
TABLE 3.1 *Finite-Duration Causal, Anitcausal and Two-Sided Signals With Their ROCs.*

S.No.	Signals	ROC's
1.	**Causal** 	 ROC : entire z-plane except $z = 0$

Z Transform

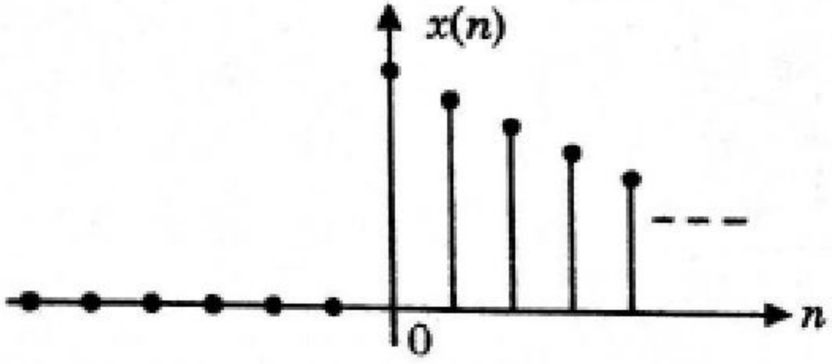
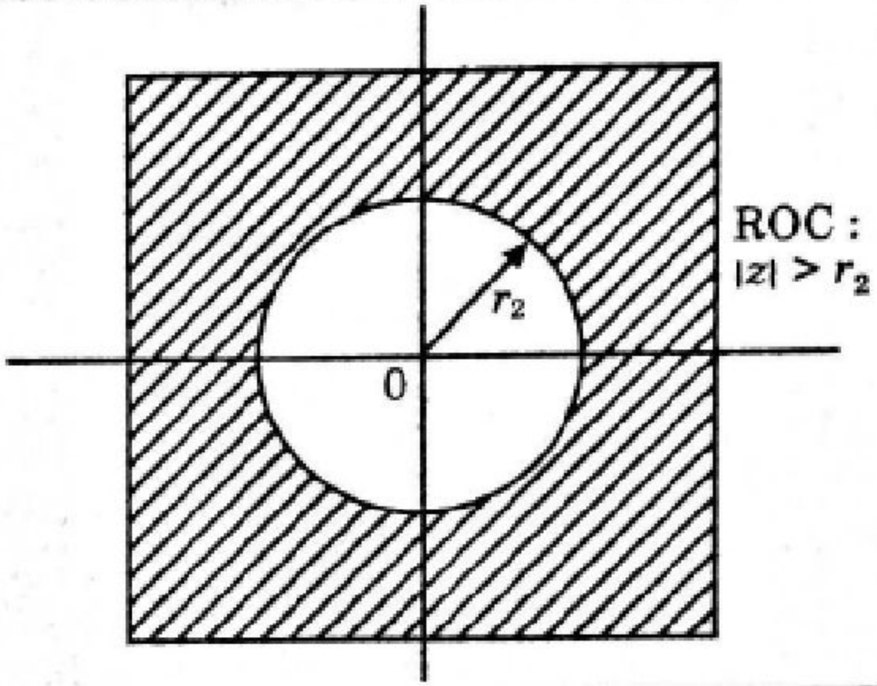
2.	Anti-causal 	 ROC : entire z-plane except $z = \infty$

Z Transform

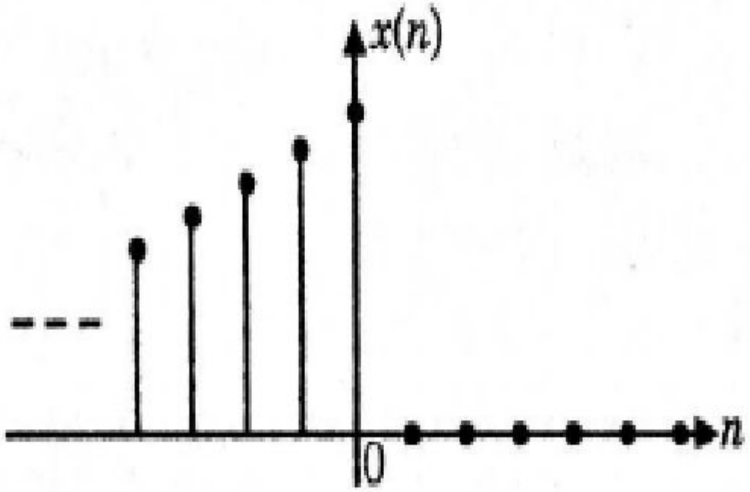
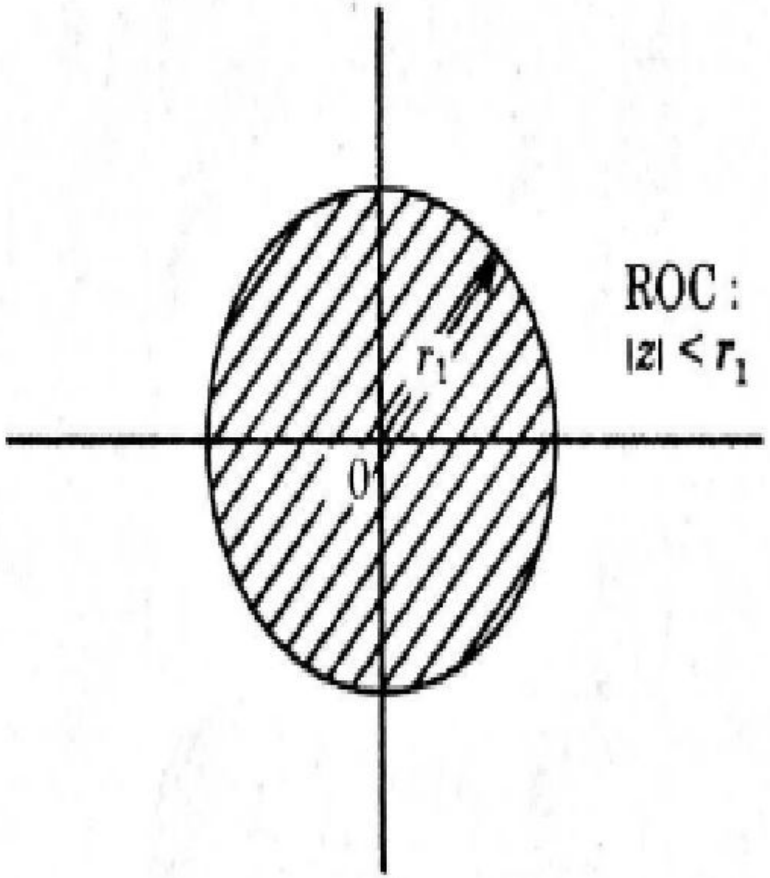


Z Transform

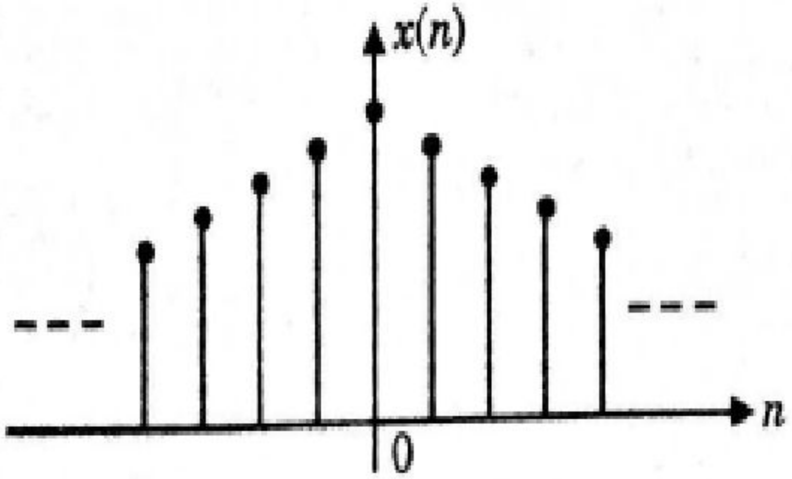
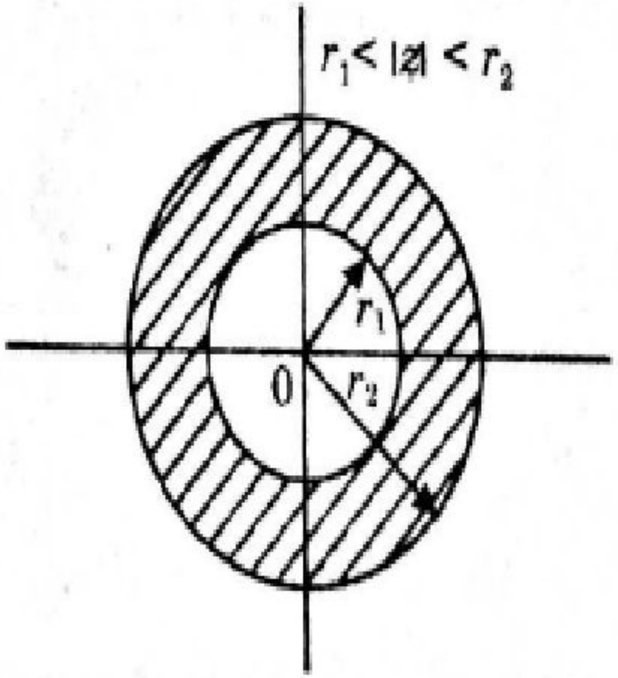
TABLE 3.2 *Infinite-Duration Causal, Anticausal and Two-Sided Signals with Their ROCs*

S.No.	Signals	ROC's
1.	Causal 	

Z Transform

2.	Anti-Causal 		

Z Transform

3.	Two-sided 	

Z Transform

- Z-transform of d[n]

Z Transform

- Z transform of d[n-k]

Z Transform

- Z transform of unit step signal.

Z Transform

Z-transform of $\boxed{x(n) = a^n u(n)}$

Example: A right sided Sequence (Causal Signal)

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z-a}$$

ROC $|z| > |a|$

Z Transform

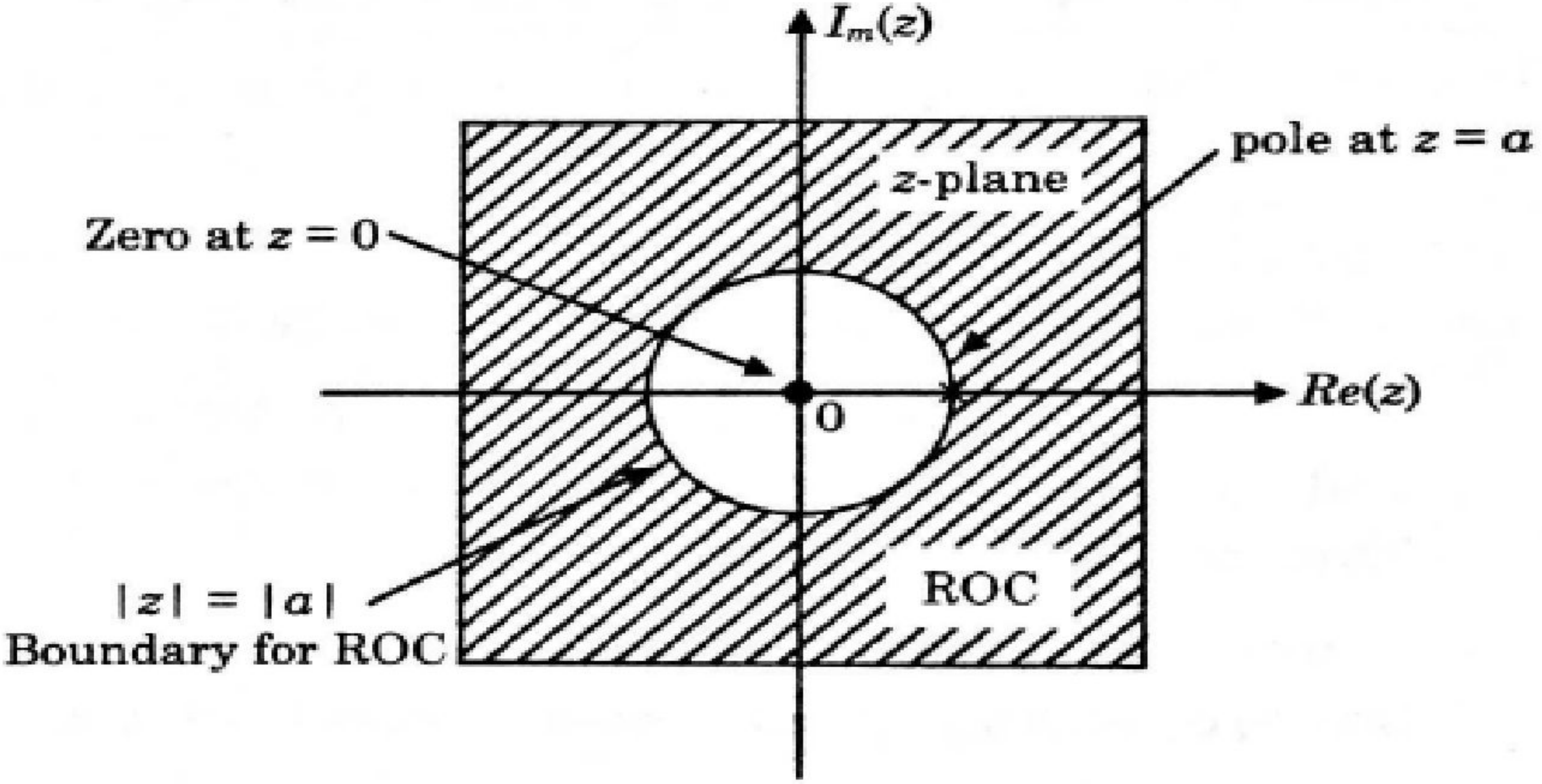


FIGURE 3.2

Z Transform

$$x(n) = -a^n u(-n-1)$$

Example: A left sided Sequence or anti causal signal

Z Transform

$$x(n) = -a^n u(-n-1)$$

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u(-n-1) z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n$$

$$= 1 - \sum_{n=0}^{\infty} a^{-n} z^n$$

$$X(z) = 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n = 1 - \frac{1}{1 - a^{-1} z}$$

$$= 1 - \frac{1}{1 - (z/a)}$$

$$= \frac{z}{z-a}$$

ROC $|z| < |a|$

Z Transform

Sequence	*Transform*	*ROC*
$\delta(n)$	1	$\forall\, z$
$u(n)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert > 1$
$-u(-n-1)$	$\frac{1}{1-z^{-1}}$	$\lvert z\rvert < 1$
$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-b^n u(-n-1)$	$\frac{1}{1-bz^{-1}}$	$\lvert z\rvert < \lvert b\rvert$

Z Transform

3.4 PROPERTIES OF z-TRANSFORM

In this section, we shall discuss various properties of z-transform.

3.4.1 Linearity Property

The z-transform is linear. This property states that the z-transform of a linear combination of discrete-time signal is equal to the same linear combination of their z-transform. Mathematically,

If $\quad x_1(n) \xleftrightarrow{Z} X_1(z)$

and $\quad x_2(n) \xleftrightarrow{Z} X_2(z)$

Then we have

$$x(n) = a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{Z} X(z) = a_1 X_1(z) + a_2 X_2(z) \qquad ...(3.17)$$

where a_1 and a_2 are any two arbitrary constants.

Z Transform

3.4.2 Time Reversal

Time Reversal property states that

If $x(n) \xleftrightarrow{Z} X(z)$ ROC : $r_1 < |z| < r_2$

Then $x(-n) \xleftrightarrow{Z} X(z^{-1})$ ROC $\frac{1}{r_2} < |z| < \frac{1}{r_1}$

3.4.3 Time Shifting Property

Time shifting property states that

If $x(n) \xleftrightarrow{z} X(z)$ ...(3.20)

Then $x(n-n_0) \xleftrightarrow{z} z^{-n_0} X(z)$ ...(3.21)

The region of convergence (ROC) of $z^{-n_0} X(z)$ will be the same as that of $X(z)$ except for $z = 0$ if $n_0 > 0$ and $z = \infty$ if $n_0 < 0$.

Z Transform

3.4.4 Scaling Property

The scaling property states that

If $\quad x(n) \xleftrightarrow{z} X(z) \quad$ ROC : $r_1 < |z| < r_2$

then $\quad a^n x(n) \xleftrightarrow{z} X(a^{-1} z) \quad$ ROC : $|a|\, r_1 < |z| < |a|\, r_2$

Here a is any constant which may be real or a complex quantity.

3.4.5 Differentiation Property

Differentiation property states that

If $\quad x(n) \xleftrightarrow{z} X(z)$

Then $\quad n\, x(n) \xleftrightarrow{z} -z \dfrac{dX(z)}{dz}$

or $\quad n\, x(n) \xleftrightarrow{z} z^{-1} \dfrac{dX(z)}{dz^{-1}}$

Z Transform

3.4.6 Convolution Property

We know that convolution of two discrete-time signals $x_1(n)$ and $x_2(n)$ is expressed as

$$x(n) = x_1(n) \otimes x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

Now, the convolution property of z-transform states that

If $\quad x_1(n) \xleftrightarrow{z} X_1(z)$

and $\quad x_2(n) \xleftrightarrow{z} X_2(z)$

Then $\quad x(n) = x_1(n) \otimes x_2(n) \xleftrightarrow{z} X(z) = X_1(z) \cdot X_2(z)$

Z Transform

$\ldots$ point $z = z_0$.

3.4.9 Initial Value Theorem

Initial value theorem states that if $x(n)$ is a causal discrete-time signal with z-transform $X(z)$, then the initial value may be determined by using the following expression

$$x(0) = \lim_{n \to 0} x(n) = \lim_{|z| \to \infty} X(z) \qquad \ldots(3.31)$$

Z Transform

the inverse z-transform of $X(z)$.

3.4.10 Final Value Theorem

The final value theorem states that for a discrete-time signal $x[n]$, if $X\{(z)\}$ and the poles of $X(z)$ are all inside the unit circle, then the final value of the discrete-time signal, $x(\infty)$, may be determined by using the following expression

$$x(\infty) = \lim_{n \to \infty} x(n) = \lim_{|z| \to 1} [(1 - z^{-1}) X(z)] \qquad \text{...(3.32)}$$

Z Transform

3.5 THE INVERSE Z-TRANSFORM

Mathematically, the inverse z-transform is expressed as

$$x(n) = Z^{-1}[X(z)] \quad \ldots(3.33)$$

To perform the inverse z-transform of the given expression, basically there are three methods as under :

(*i*) Long Division Method

(*ii*) Partial Fraction Expansion Method

(*iii*) Residue Method.

Now let us discuss all these methods one by one in the sections to follow :

Z Transform

EXAMPLE 3.11 **Using long division, determine the inverse *z*-transform of**

$$X(z) = \frac{1}{1-(3/2)z^{-1}+(1/2)z^{-2}}$$

when (*a*) ROC : $|z| > 1$

and (*b*) ROC : $|z| < \frac{1}{2}$

DO YOU KN

When the inverse z-

Z Transform

z

Solution : (*a*) Because the ROC : $|z| > 1$ is the exterior of a circle, therefore $x(n)$ is a causal signal. Hence, we see a power series expansion in negative powers of z. By dividing the numerator of $X(z)$ by its denominator, we get

Z Transform

$$
\begin{array}{r|l}
 & 1+\frac{3}{2}z^{-1}+\frac{7}{4}z^{-2}+\frac{15}{8}z^{-3}+\frac{31}{16}z^{-4}+\ldots\ldots \\
\hline
1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2} & 1 \\
 & 1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2} \\
\hline
 & \frac{3}{2}z^{-1}-\frac{1}{2}z^{-2} \\
 & \frac{3}{2}z^{-1}-\frac{9}{4}z^{-2}+\frac{3}{4}z^{-3} \\
\hline
 & \frac{7}{4}z^{-2}-\frac{3}{4}z^{-3} \\
 & \frac{7}{4}z^{-2}-\frac{21}{8}z^{-3}+\frac{7}{8}z^{-4} \\
\hline
 & \frac{15}{8}z^{-3}-\frac{7}{8}z^{-4} \\
 & \frac{15}{8}z^{-3}-\frac{15}{16}z^{-4}+\frac{15}{16}z^{-5} \\
\hline
 & \frac{31}{16}z^{-4}-\frac{15}{16}z^{-5} \\
\hline
\end{array}
$$

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Z Transform

Therefore, we have

$$X(z) = 1 + \frac{3}{2} z^{-1} + \frac{7}{4} z^{-2} + \frac{15}{8} z^{-3} + \frac{31}{16} z^{-4} + \ldots\ldots$$

Taking inverse z-transform, we get $x(n) = \left\{ \underset{\uparrow}{1}, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16}, \ldots \right\}$

Z Transform

(*b*) In this case the ROC : $|z| < 0.5$ is the interior of a circle. Consequently the signal $x(n)$ is anti-causal. To obtain a power series expansion in positive powers of z, we perform the long division in the following way.

Z Transform

$$
\begin{array}{r|l}
 & 2z^2 + 6z^3 + 14z^4 + 30z^5 + 62z^6 + \dots \\
\hline
\frac{1}{2}z^{-2} - \frac{3}{2}z^{-1} + 1 & 1 \\
 & 1 - 3z + 2z^2 \\
\hline
 & 3z - 2z^2 \\
 & 3z - 9z^2 + 6z^3 \\
\hline
 & 7z^2 - 6z^3 \\
 & 7z^2 - 21z^3 + 14z^4 \\
\hline
 & 15z^3 - 14z^4 \\
 & 15z^3 - 45z^4 + 30z^5 \\
\hline
 & 31z^4 - 30z^5
\end{array}
$$

Z Transform

Therefore, $X(z) = 2z^2 + 6z^3 + 14\,z^4 + 30\,z^5 + 62\,z^6 + \ldots$

Taking inverse z-transform, we get $x(n) = \left\{ \ldots, 62, 30, 14, 6, 2, \underset{\uparrow}{0}, \underset{\uparrow}{0}, \right\}$

Z Transform(Partial Fraction method)

EXAMPLE 3.12 Determine the inverse z-transform of

$$X(z) = \frac{z}{3z^2 - 4z + 1}$$

if the regions of convergence are (*i*) $|z| > 1$, (*ii*) $|z| < \frac{1}{3}$ and (*iii*) $\frac{1}{3} < |z| < 1$.

(Sem. Exam: WBTU. Kalkata. 2003-04)

Z Transform

(Sem, Exam; WBTU, Kalkata, 2003-0

Solution: The partial fraction expansion of $X(z)$ yields

$$F(z) = \frac{X(z)}{z} = \frac{1}{3z^2 - 4z + 1} = \frac{1}{3(z-1)\left(z-\frac{1}{2}\right)} = \frac{A_1}{(z-1)} + \frac{A_2}{\left(z-\frac{1}{3}\right)}$$

1/3

$$A_1 = F(z)\,(z-1)\big|_{z=1} = \left.\frac{1}{3\left(z-\frac{1}{3}\right)}\right|_{z=1} = \frac{1}{2}$$

$$A_2 = F(z)\left(z-\frac{1}{3}\right)\Bigg|_{z=\frac{1}{3}} = \left.\frac{1}{3(z-1)}\right|_{z=\frac{1}{3}} = -\frac{1}{2}$$

$$\frac{X(z)}{z} = \frac{\frac{1}{2}}{(z-1)} + \frac{-\frac{1}{2}}{\left(z-\frac{1}{3}\right)}$$

$$X(z) = \frac{\frac{1}{2}z}{(z-1)} - \frac{\frac{1}{2}z}{\left(z-\frac{1}{3}\right)}$$

Z Transform

$\left(\quad 3 \right)$

(i) When the ROC is $|z| > 1$, the signal $x(n)$ is causal and both terms are causal.

Therefore, $$x(n) = \frac{1}{2}(1)^n u(n) - \frac{1}{2}\left(\frac{1}{3}\right)^n u(n) = \frac{1}{2}\left[1 - \left(\frac{1}{3}\right)^n\right] u(n)$$

(ii) When the ROC is $|z| < \frac{1}{3}$, the signal $x(n)$ is anti-causal, *i.e.*, the inverse gives negative time sequences. Therefore,

$$x(n) = \left[-\frac{1}{2}(1)^n + \frac{1}{2}\left(\frac{1}{3}\right)^n\right] u(-n-1)$$

DO YOU KNOW?

Z Transform

(*iii*) Here the ROC $\frac{1}{3} < |z| < 1$ is a ring, which implies that the signal $x(n)$ is two-sided. Therefore one of the terms corresponds to a causal signal and the other to an anti-causal signal, obviously the given ROC is the overlapping of the regions $|z| > \frac{1}{3}$ and $|z| < 1$. The

$$x(n) = -\frac{1}{2}(1)^n u(-n-1) - \frac{1}{2}\left(\frac{1}{3}\right)^n u(n)$$