Discrete Filter Realization

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6.1 INTRODUCTION

Basically, the digital filters are discrete-time LTI systems. The discrete-time systems can be finite-impulse response (FIR) or infinite-impulse response (IIR) type. As discussed earlier, for the designing of digital filters, the system function $H(z)$ or the corresponding impulse response $h(n)$ should be specified. After that, we can implement or synthesize the digital filter structure in hardware or software form with the help of its difference equation which is obtained directly from the system function $H(z)$ or impulse response $h(n)$. Also, each difference equation or computational algorithm may be implemented simply by using a digital computer or a particular digital hardware or a particular programmable integrated circuit (IC).

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The various ways to implement difference equation are called realization structures of discrete time systems or digital filter structures. These structures are derived on the basis of computational complexity, ease of implementation, finite wordlength effects etc.

As a matter of fact, for implementing the specified difference equation of a system (*i.e.*, filter), the required basic operations are addition, delay and multiplication by a constant.

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6.2 BLOCK DIAGRAM REPRESENTATION

We know that the output of a finite order linear time invariant (LTI) system at time n may be expressed as a linear combination of the inputs and the outputs *i.e.*,

$$y(n) = -\sum_{k=1}^{N} a_k \cdot y(n-k) + \sum_{k=0}^{M} b_k \cdot x(n-k) \,. \qquad \text{...(6.1)}$$

not future

Here, a_k and b_k are constants with

Note that here the present output $y(n)$ is equal to the sum of past outputs, from $y(n-1)$ to $y(n-N)$, that are scaled by the delay-dependent feedback coefficients a_k, plus the sum of future, present and past inputs, which are scaled by the delay-dependent feed forward coefficient b_k.

Now, taking the z-transform of the output sequence, $y(n)$, expressed in equation (6.1), we obtain

$$Y(z) = \sum_{n=-\infty}^{\infty} y(n) \cdot z^{-n} = \sum_{n=-\infty}^{\infty} \left[-\sum_{k=1}^{N} a_k \cdot y(n-k) + \sum_{k=0}^{M} b_k \cdot x(n-k) \right] z^{-n} \qquad \text{...(6.2)}$$

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In equation (6.2), changing the order of summation, we get

$$Y(z) = \sum_{k=1}^{N} a_k \left[-\sum_{n=-\infty}^{\infty} y(n-k) z^{-n} \right] + \sum_{k=0}^{M} b_k \left[\sum_{n=-\infty}^{\infty} x(n-k) z^{-n} \right]$$

or $$Y(z) = -\sum_{k=1}^{N} a_k \cdot z^{-k} \cdot Y(z) + \sum_{k=0}^{M} b_k \cdot z^{-k} \cdot X(z)$$

These structures are derived on the basis of computation complexity, ease of implementation, finite wordlength effects etc.

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or $$Y(z)\left[1+\sum_{k=1}^{N} a_k \cdot z^{-k}\right] = \sum_{k=0}^{M} b_k \cdot z^{-k} \cdot X(z)$$

Thus, we write $$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k \cdot z^{-k}}{1+\sum_{k=1}^{N} a_k \cdot z^{-k}}$$ **Equation 6.3**

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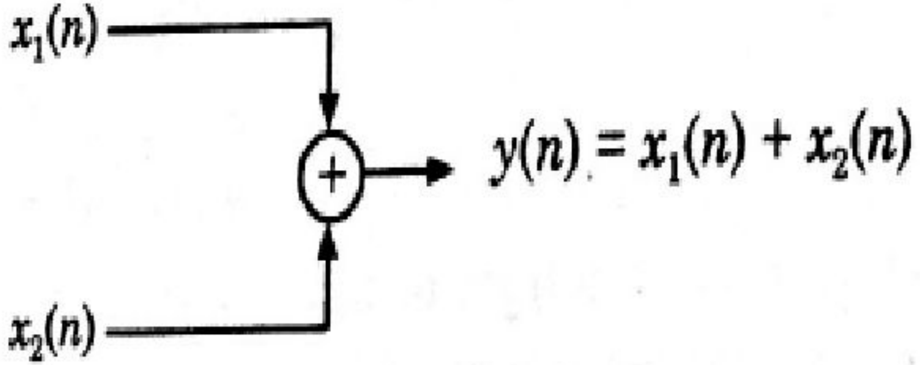
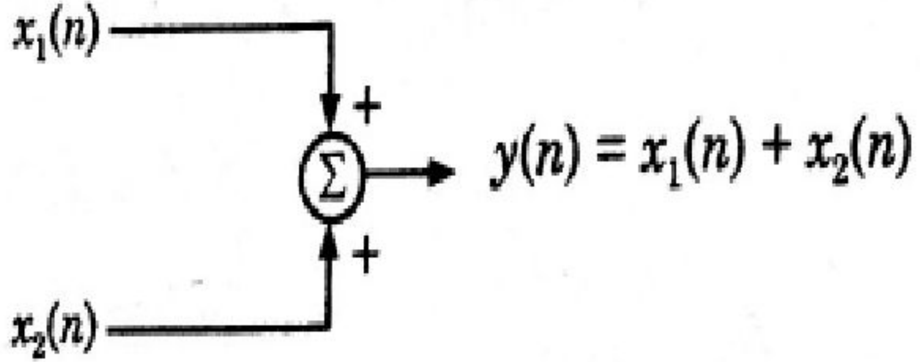
S. No.	Name of block	Symbol of block
1.	Adder	 

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2.	Constant multiplier	$x(n) \xrightarrow{a} y(n) = ax(n)$
3.	Signal multiplier	$x_1(n) \rightarrow \otimes \rightarrow y(n) = x_1(n) \,.\, x_2(n)$, with $x_2(n) \uparrow$ into $\otimes$
4.	Delay elements	$x(n) \rightarrow \boxed{z^{-1}} \rightarrow y(n) = x(n-1)$ or $x(n) \rightarrow \boxed{z^{-k}} \rightarrow y(n) = x(n-k)$
5.	Time advance elements	$x(n) \rightarrow \boxed{z} \rightarrow y(n) = x(n+1)$ $x(n) \rightarrow \boxed{z^{k}} \rightarrow y(n) = x(n+k)$

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Thus, equations (6.1) and (6.3) can be realized through various blocks as shown in above table. In fact, realization of the system means its actual implementation.

6.2.2 Advantages of Representing the Digital System (i.e., Filter) in Block Diagram Form

Advantages of representing the digital filter in the block diagram form can be listed as under:

(*i*) The computation algorithm can be easily written.

(*ii*) With the help of transfer function, a variety of equivalent block diagram representations can be easily developed.

(*iii*) The hardware requirements can be easily determined.

(*iv*) The relationship between the output and the input can be easily established.

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6.3 BASIC STRUCTURE FOR INFINITE IMPULSE RESPONSE IIR SYSTEMS (i.e., FILTERS)

As discussed earlier, the causal IIR systems (*i.e.* filters) are characterized* by the constant coefficient difference equation *i.e.*,

$$y(n) = -\sum_{k=1}^{N} a_k \cdot y(n-k) + \sum_{k=0}^{M} b_k \cdot x(n-k) \qquad ...(6.5)$$

where a_k and b_k are constants with $a_0 \neq 0$ and $M \leq N$.
or equivalently, by the real rational transfer function *i.e.*,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k \cdot z^{-k}}{1 + \sum_{k=1}^{N} a_k \cdot z^{-k}} \qquad ...(6.6)$$

Now, from these two equations (6.5) and (6.6), it may be observed that the realization of infinite duration impulse response (IIR) systems (*i.e.* filters) involves a recursive computational algorithm.

... filter structures namely direct forms I and II, cascade

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6.4 DIRECT FORM REALIZATION OF INFINITE IMPULSE RESPONSE (IIR) FILTERS

(Sem. Exam, MGU, Kerala, 2003–04)

We know that the standard form of the system transfer function for a discrete time LTI system is given by the following expression:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k \cdot z^{-k}}{1 + \sum_{k=1}^{N} a_k \cdot z^{-k}} \qquad \ldots(6.7)$$

By observation of this equation, we can draw the block diagram representation for the direct form realization. The multipliers in the feed forward paths are the numerator coefficients and

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the multipliers in the feedback paths are the negatives of the denominator coefficients. Because, the multiplier coefficients in the structures are exactly the coefficients of the transfer function, therefore, they are known as the **direct from structures**.

Now, let us $H_1(z)$ = Numerator term of equation (6.7)

$$H_1(z) = \sum_{k=0}^{M} b_k z^{-k} \qquad \text{...(6.8)}$$

and let

$$H_2(z) = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}} \qquad \text{...(6.9)}$$

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$k=1$

Substituting equations (6.8) and (6.9) in equation (6.7), we get

$$H(z) = H_1(z) \cdot H_2(z) \qquad \text{...(6.10)}$$

This equation shows that $H(z)$ can be represented as the cascade connection of $H_1(z)$ and $H_2(z)$ as shown in figure 6.3.

We know that $H_1(z)$ is the transfer function of the numerator term. Numerator always contain zeros of the system. So, $H_1(z)$ is called as **all zero system**. Now $H_2(z)$ is transfer function of denominator. Denominator always contain poles of the system. So, $H_2(z)$ is called as **all pole system**.

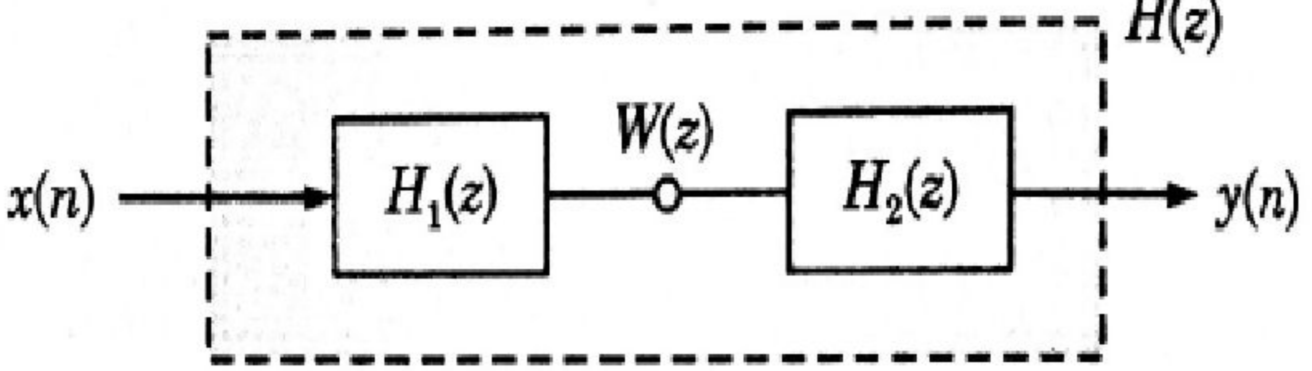

FIGURE 6.3 *Cascade connection of $H_1(z)$ and $H_2(z)$*

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6.5 DIRECT FORM-I STRUCTURE

The digital filter structure determined directly from either the equation

$$y(n) = -\sum_{k=1}^{N} a_k \cdot y(n-k) + \sum_{k=0}^{M} b_k \cdot x(n-k) \qquad \text{...(6.11)}$$

or the equation

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k \cdot z^{-k}}{1 + \sum_{k=1}^{N} a_k \cdot z^{-k}} \qquad \text{...(6.12)}$$

is known as the **direct form *I*.**

Basically, the direct form-I structure is obtained by cascading (connecting in series), the structure for $H_1(z)$ and $H_2(z)$. First, let us draw the direct form structure for $H_1(z)$.

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(i) Direct Form Structrue for $H_1(z)$

Recall the equation for $H_1(z)$ (equation (6.8)) *i.e.*,

$$H_1(z) = \sum_{k=0}^{M} b_k z^{-k} \qquad \ldots(6.13)$$

But we know that, $H(z) = \dfrac{\text{Output } [Y(z)]}{\text{Input } [X(z)]}$

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Now, let us write

$$H_1(z) = \frac{W(z)}{X(z)},$$

where $W(z)$ is output of first stage.

Using equation (6.13), we can write

$$\frac{W(z)}{X(z)} = \sum_{k=0}^{M} b_k z^{-k} \quad \text{...(6.14)}$$

or $$\frac{W(z)}{X(z)} = b_0 z^0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_M z^{-M}$$

But, $$z^0 = 1$$

Therefore,

$$W(z) = b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) + \dots + b_M z^{-M} X(z) \quad \text{..(6.15)}$$

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Taking inverse z-transform (IZT) of equation (6.15), we get

$$w(n) = b_0\, x(n) + b_1\, x(n-1) + b_2\, x(n-2) + \ldots + b_M\, x(n-M) \qquad \ldots(6.16)$$

Here, b_0, b_1, b_2 ... b_M are the coefficients. The direct form realization of equation(6.16) has been shown in figure (6.4).

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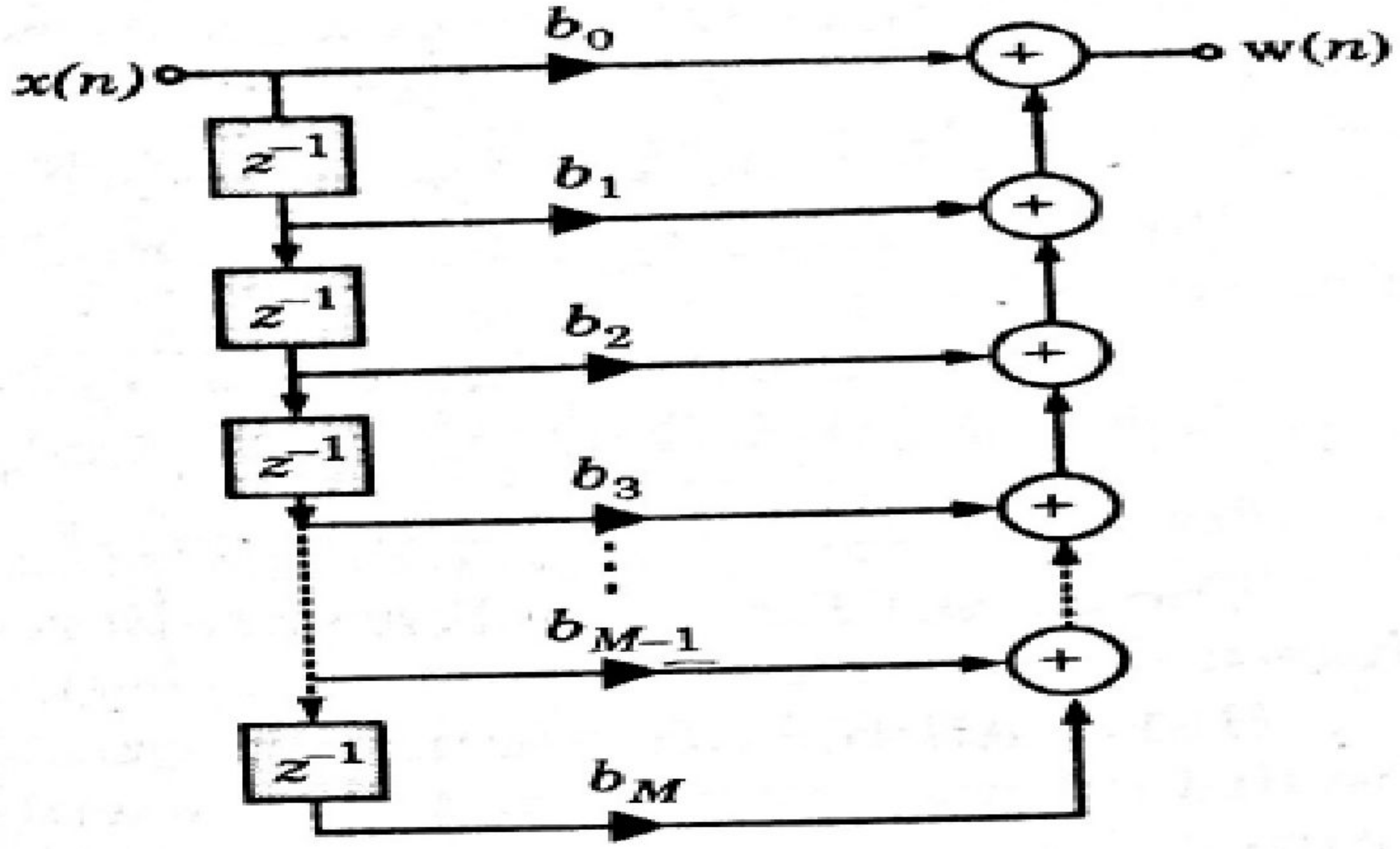

FIGURE 6.4 *Direct form realization of $H_1(z)$ (all zero system)*

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(ii) Direct Form Structure for $H_2(z)$

We know that expression for $H_2(z)$ is given by (equation (6.9))

$$H_2(z) = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}} \qquad \ldots(6.17)$$

We have,

$$H_2(z) = \frac{\text{Output } [Y(z)]}{\text{Input } [X(z)]} \qquad \ldots(6.18)$$

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Now $H_2(z)$ represents second stage of figure 6.3. Input of second stage is the output of first stage. Thus, input of $H_2(z)$ is $W(z)$ and output of $H_2(z)$ is the output of overall system which is $Y(z)$.

Thus, equation (6.18) becomes,

$$H_2(z) = \frac{Y(z)}{W(z)}$$

Substituting this value is equation (6.17), we get

$$\frac{Y(z)}{W(z)} = \frac{1}{1+\sum_{k=1}^{N} a_k z^{-k}}$$

or $$Y(z)\left[1+\sum_{k=1}^{N} a_k z^{-k}\right] = W(z)$$

or $$Y(z) + Y(z)\left[\sum_{k=1}^{N} a_k z^{-k}\right] = W(z)$$

or $$Y(z) = -\left[\sum_{k=1}^{N} a_k z^{-k}\right] Y(z) + W(z) \qquad \text{...(6.19)}$$

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Expanding the summation, we get

$$Y(z) = -[a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}]\ Y(z) + W(z)$$

or $Y(z) = -\,a_1 z^{-1}\, Y(z) - a_2 z^{-2}\, Y(z) .. - a_N z^{-N}\, Y(z) + W(z)$

Taking inverse z-transform (IZT) of above equation, we obtain

$$y(n) = -\,a_1\, y(n-1) - a_2\, y(n-2) \ldots - a_N\, y(n-N) + w(n) \qquad \ldots(6.20)$$

Here, $-\,a_1, -\,a_2 \ldots -\,a_N$ are the coefficients.

The direct form implementation of equation (6.20) has been shown in figure 6.5.

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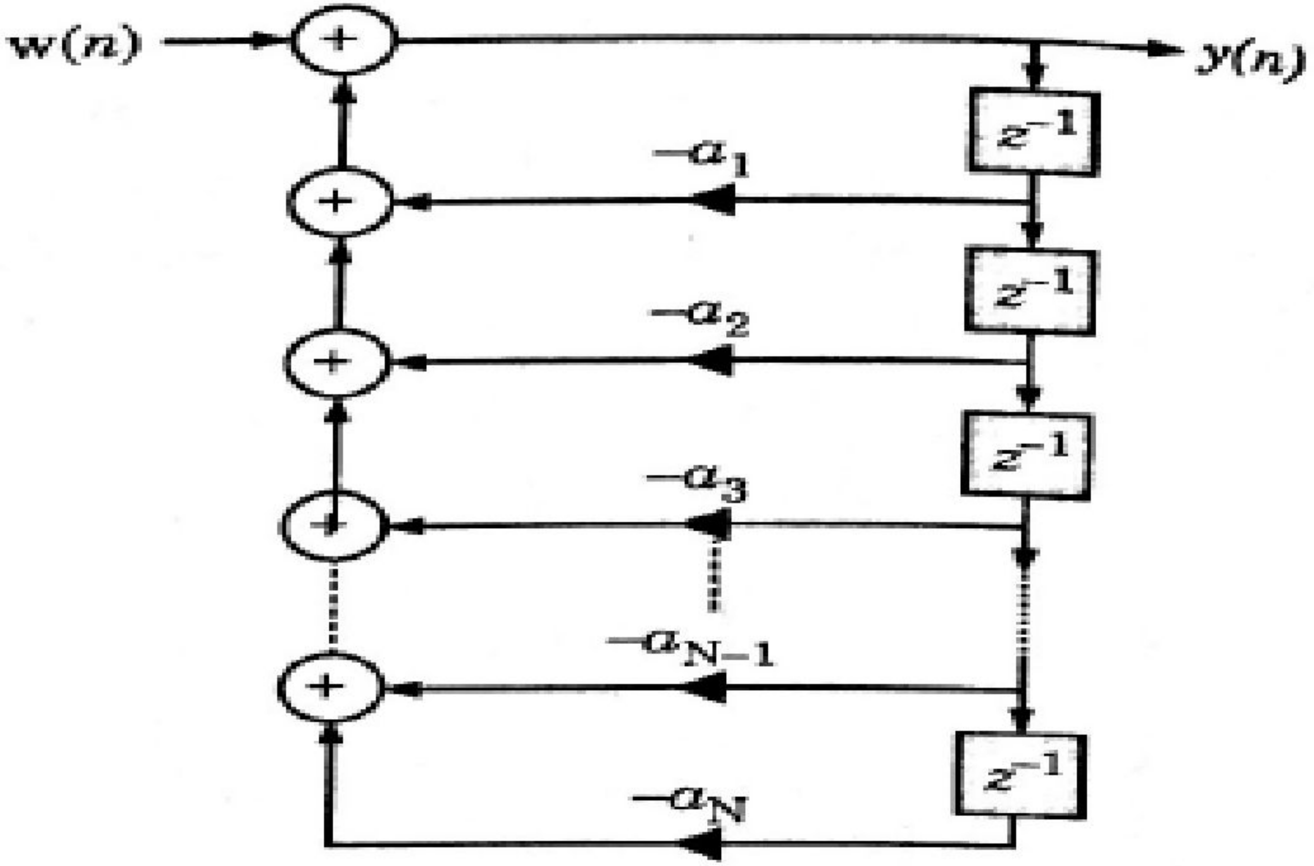

FIGURE 6.5 *Direct form realization of $H_2(z)$ (All pole system).*

Discrete Filter Realization

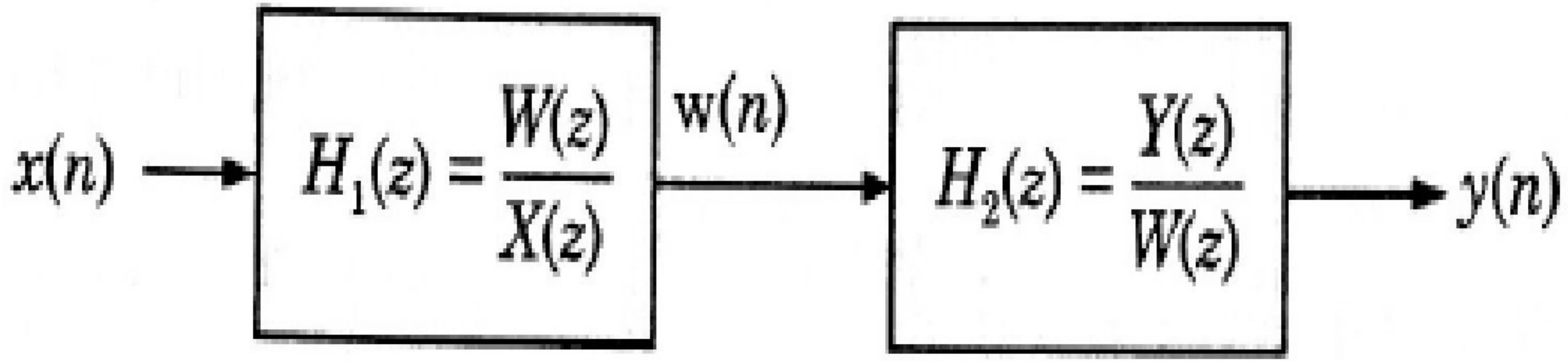

FIGURE 6.6 $H(z) = H_1(z) \cdot H_2(z)$ *represents cascading of two systems.*

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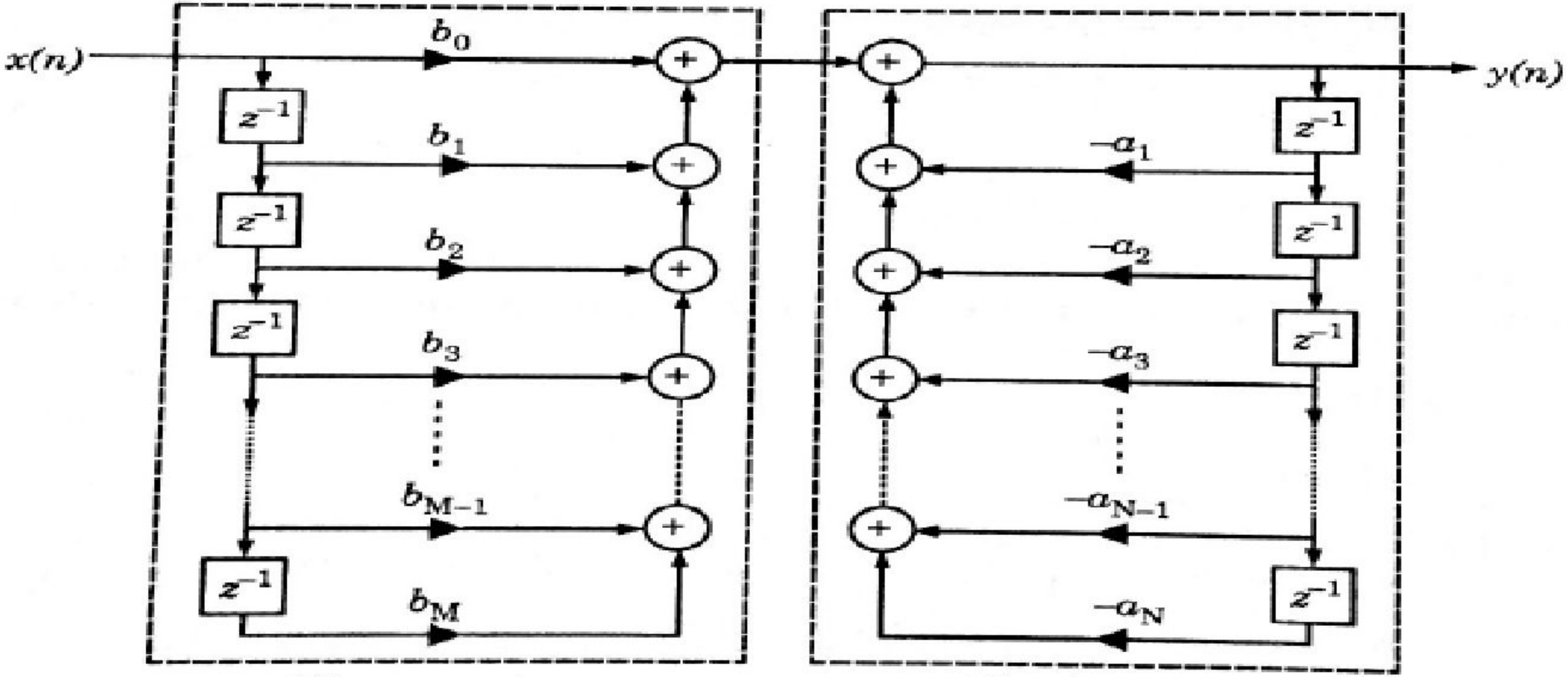

FIGURE 6.7 *Direct form-I realization of IIR system.*

Discrete Filter Realization

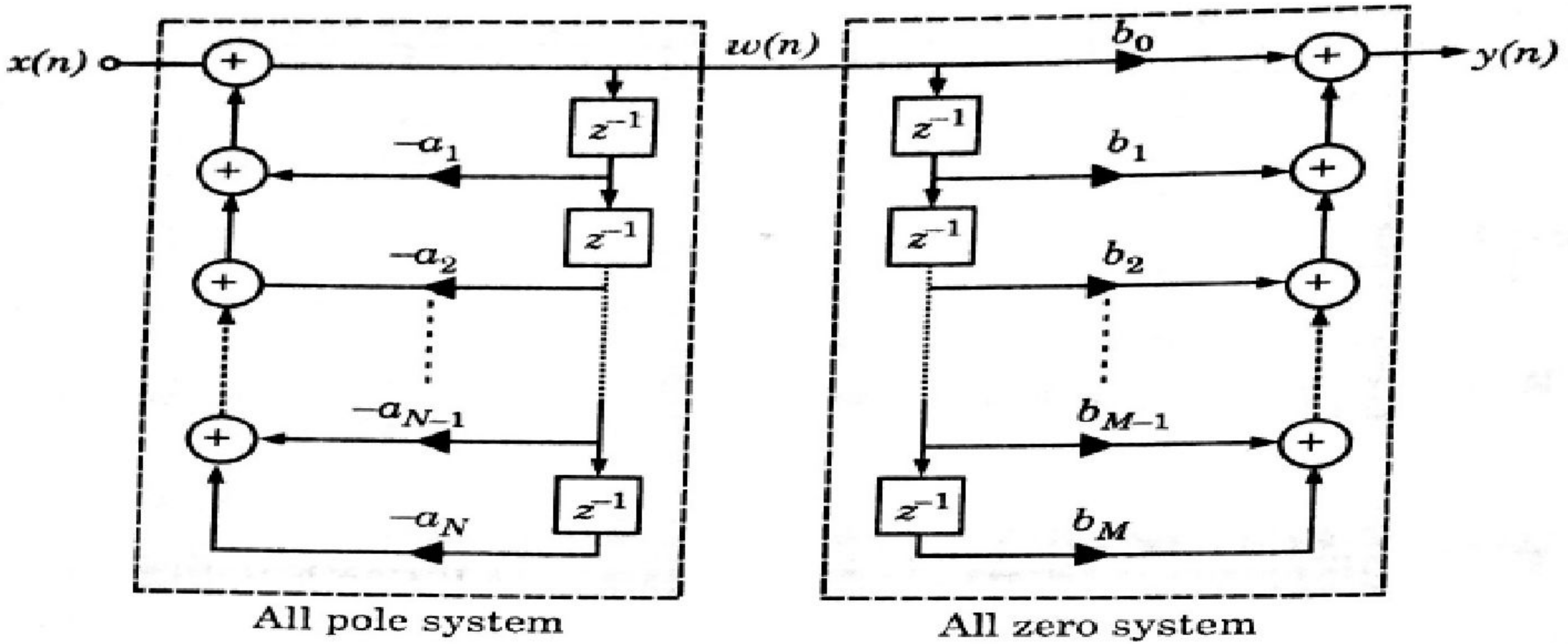

FIGURE 6.16 *Direct form-II realization of IIR system.*

Discrete Filter Realization

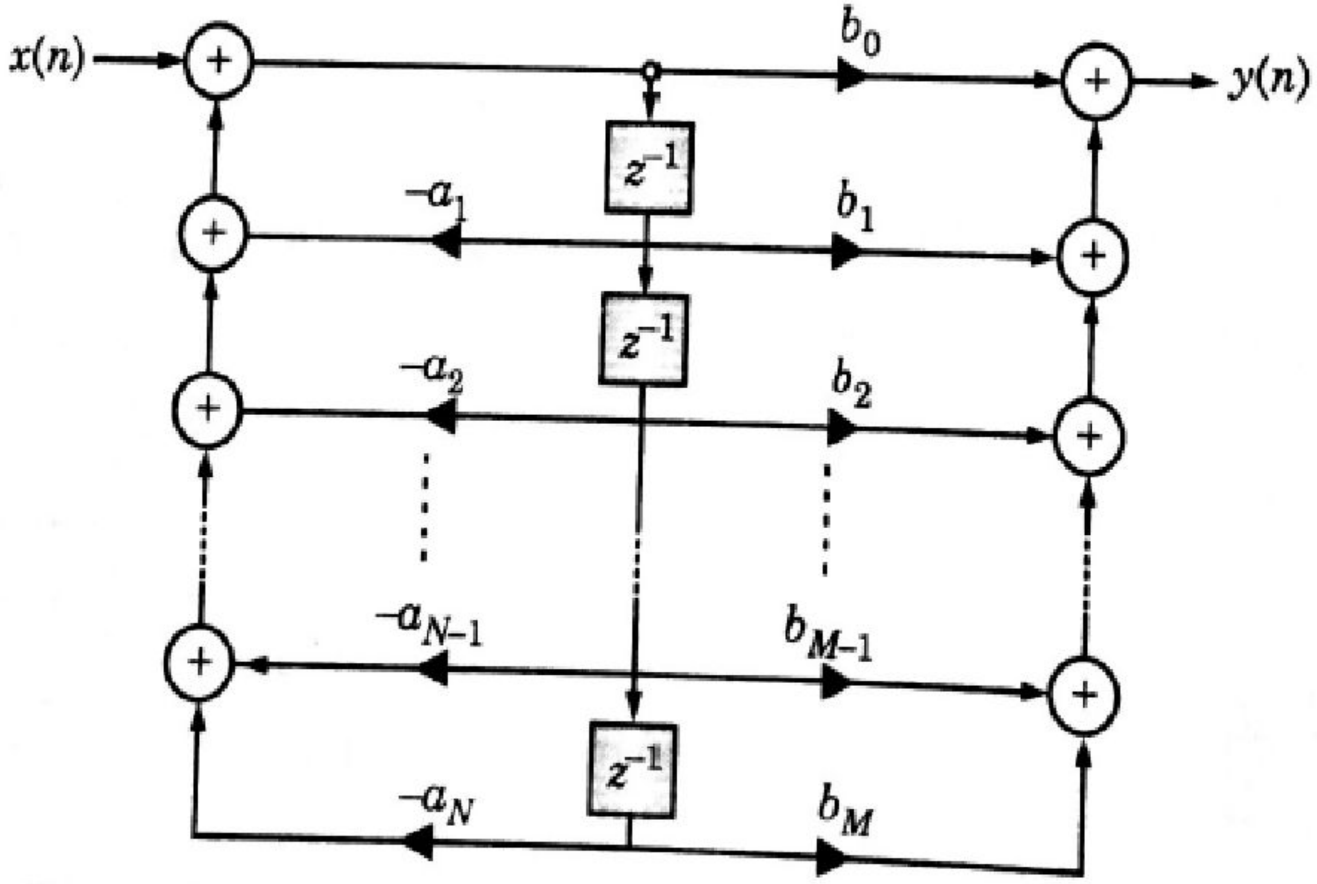

FIGURE 6.17 *Direct form-II realization using common delay elements. Here, $N = M$.*

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Direct I & Direct II (Conclusion)

- The Direct I and Direct II structures are obtained directly from the corresponding transfer functions without any rearrangements.
- The number of delay elements are reduced in Direct II form structure compared to Direct I form structure.

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6.17 BASIC FIR FILTER STRUCTURES

(Sem. Exam, JNTU, Hyderabad, 2004-05)

We know that FIR stands for finite impulse response. An FIR filter does not have feedback. The difference equation of an FIR filter is given by

$$y(n) = \sum_{k=0}^{M-1} b_k \, x(n-1) \qquad \ldots(6.82)$$

Above expression shows that the system has length M as the limits of summation are from 0 to $M-1$. These limits also indicate that the system is causal.

Now, taking z-transform of above expression, we obtain

$$Y(z) = \sum_{k=0}^{M-1} b_k \cdot z^{-k} X(z)^*$$

Hence, system transfer function $H(z) = \dfrac{Y(z)}{X(z)}$ becomes

$$H(z) = \sum_{k=0}^{M-1} b_k \cdot z^{-k}$$

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$k=0$

This is the system transfer function of FIR filter. Now, taking inverse z-transform of above expression, we get sample response of FIR filter *i.e.*,

$$h(n) = \begin{cases} b_n & \text{for} \quad 0 \le n \le M-1 \\ 0 & \quad \text{elsewhere} \end{cases}$$

FIR digital filters are generally preferred in several applications, because they provide an exact linear phase over the whole frequency range and they are always BIBO stable independent of the filter coefficients. Two simple realisation methods for FIR filters are direct form and cascade form have been discussed here.

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6.18 DIRECT FORM STRUCTURE OF FIR DIGITAL FILTERS

The direct form realization of FIR filter may be obtained by using the equation of linear convolution. It is

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) \qquad \ldots(6.83)$$

Now, if we consider that there are M samples, then equation (6.76) becomes

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$

Expanding above expression, we get

$$y(n) = h(0)\,x(n) + h(1)\,x(n-1) + h(2)\,x(n-2) + \ldots + h(M-1)\,x(n-M+1) \qquad \ldots(6.84)$$

To Draw Direct Form FIR Structure

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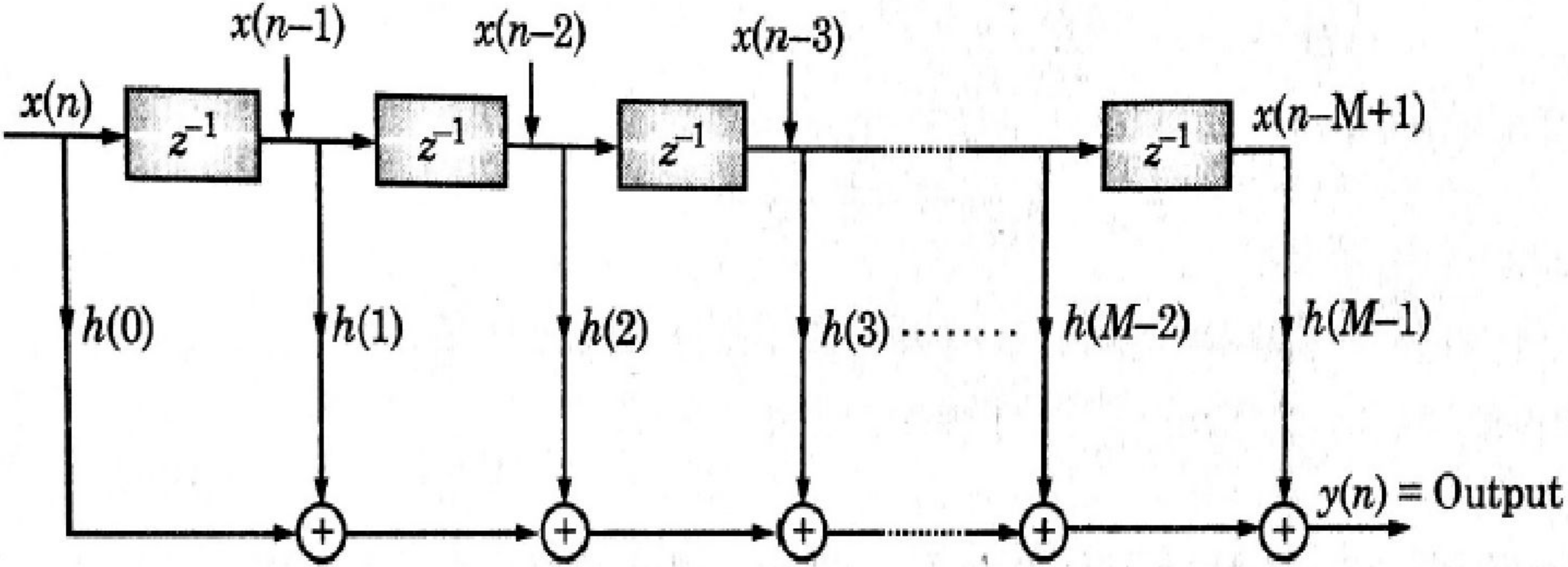

FIGURE 6.75 *Direct form realization of finite-impulse response (FIR) system.*

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- Lattice & Ladder Structure

$$k_m = a_m(m)$$

$$a_{m-1}(k) = \frac{a_m(k) - a_m(m)\, a_m(m-k)}{1 - a_m^2(m)}$$

$$c_m = b_m - \sum_{i=m+1}^{M} c_i a_i(i-m); \; m = M, M-1, \ldots 0$$

Ladder Structure

Lattice structure is generalize for all pole.

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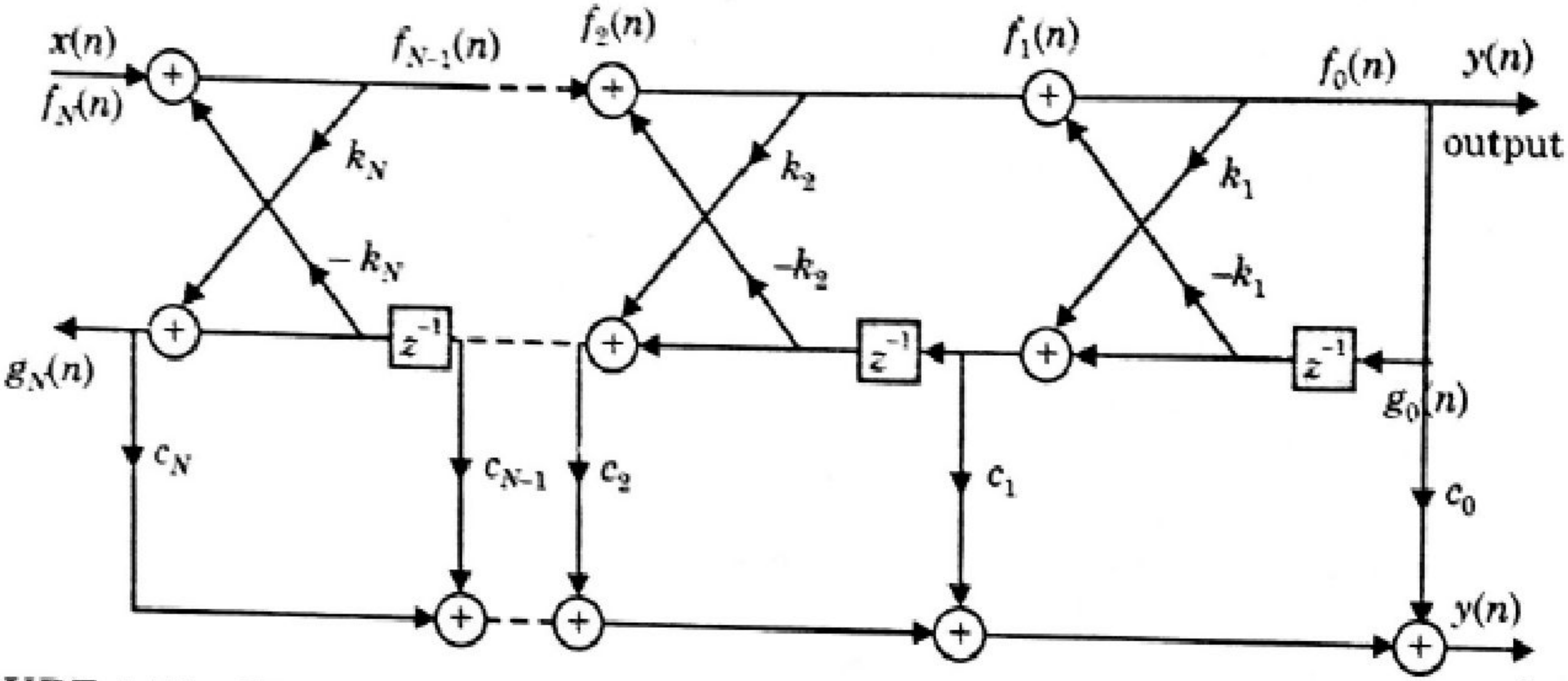

FIGURE 6.72 *Illustration of Lattice-ladder structure for realizing a pole-zero IIR system.*

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EXAMPLE 6.27 **Convert the following pole zero IIR filter into a Lattice ladder structure :**

$$H(z) = \frac{1 + 2z^{-1} + 2z^{-2} + z^{-3}}{1 + \frac{13}{24}z^{-1} + \frac{5}{8}z^{-2} + \frac{1}{3}z^{-3}}$$

Solution : Given that

$$b_M(z) = 1 + 2z^{-1} + 2z^{-2} + z^{-3}$$

and

$$A_N(z) = 1 + \frac{13}{24}z^{-1} + \frac{5}{8}z^{-2} + \frac{1}{3}z^{-3}$$

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Here, we have

$$a_3(0) = 1,\ a_3(1) = \frac{13}{24},\ a_3(2) = \frac{5}{8},\ a_3(3) = \frac{1}{3}$$

$$k_3 = a_3(3) = \frac{1}{3}$$

From equation (6.78), we get

$$a_{m-1}(k) = \frac{a_m(k) - a_m(m)\, a_m(m-k)}{1 - a_m^2(m)}$$

for $m = 3$ and $k = 1$, we have

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a3(3)

$$a_2(1) = \frac{a_3(1) - a_1(3)a_3(2)}{1 - a_3^2(3)} = \frac{\frac{13}{24} - \frac{1}{3}\left(\frac{5}{8}\right)}{1 - \left(\frac{2}{3}\right)^2} = \frac{3}{8}$$

for $m = 3$ and $k = 2$, we have

$$k_2 = a_2(2) = \frac{a_3(2) - a_3(3)a_3(1)}{1 - a_3^2(3)} = \frac{\frac{5}{8} - \frac{1}{3}\left(\frac{13}{24}\right)}{1 - \left(\frac{1}{3}\right)^2} = \frac{1}{2}$$

for $m = 2$ and $k = 1$, we have

$$k_1 = a_1(1) = \frac{a_2(1) - a_2(2)a_2(1)}{1 - a_2^2(2)} = \frac{\frac{3}{8} - \frac{1}{2}\left(\frac{3}{8}\right)}{1 - \left(\frac{1}{2}\right)^2} = \frac{1}{4}$$

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Therefore, for Lattice structure, we get

$$k_1 = \frac{1}{4}\, k_2 = \frac{1}{4}\, k_3 = \frac{1}{3}$$

For ladder structure, we have

$$c_m = b_m - \sum_{i=m+1}^{M} c_i a_i(i-m); \qquad m = M, M-1, \ldots 0$$

$$c_3 = b_3 = 1$$

$$c_2 = b_2 - c_3 a_3(1) = 2 - 1\left(\frac{13}{14}\right) = 1.4583$$

c2

$$c_1 = b_1 - \sum_{i=2}^{3} c_i\, a_i(i-m)$$

$$= b_1 - [c_2 a_2(1) + c_3 a_3(2)] = 2 - [1.4583)\ (3/8) + 5/8] = 0.8281$$

$$c_0 = b_0 - \sum_{i=1}^{3} c_i\, a_i(i-m) = b_0 - [c_1 a_1(1) + c_1 a_2(2) + c_3 a_3(3)]$$

or $$c_0 = 1 - \left[0.8281\left(\frac{1}{4}\right) + 1.4583\left(\frac{1}{2}\right) + \frac{1}{3}\right] = -\,0.2695$$

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The Lattice ladder structure for the given pole-zero filter has been shown in figure 6.73.

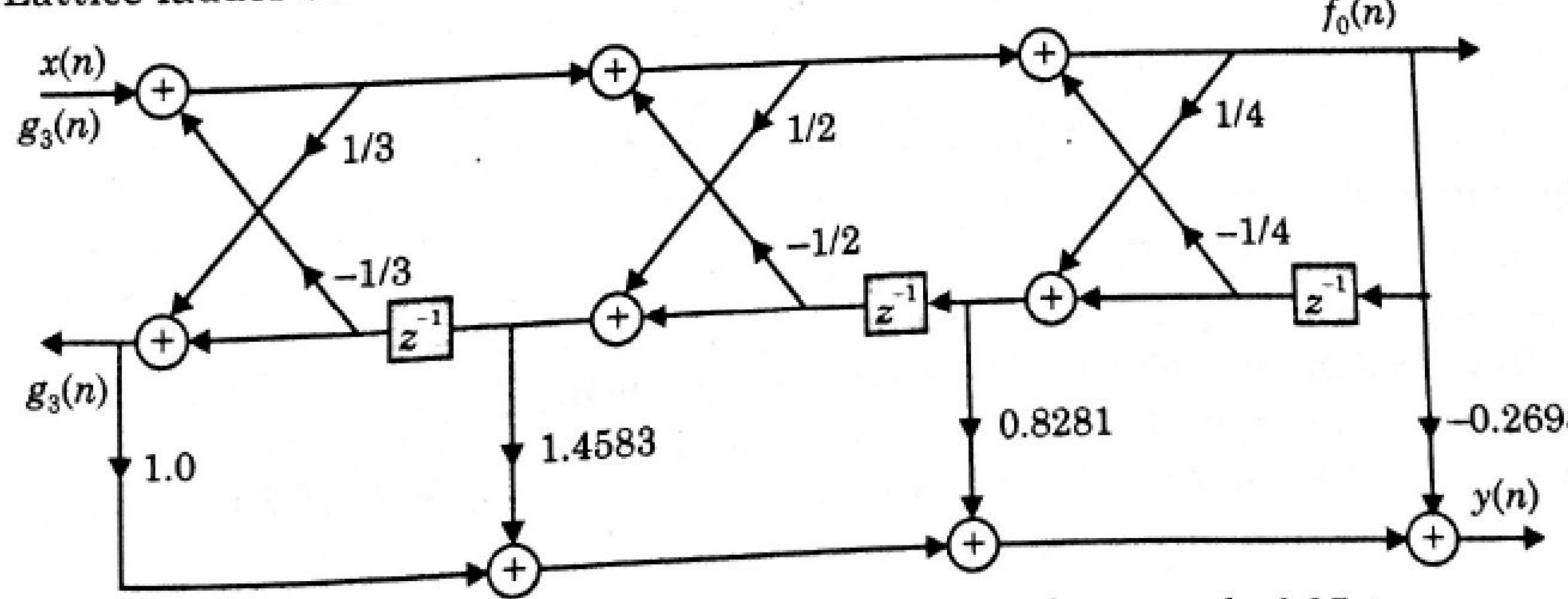

FIGURE 6.73 *Lattice-ladder form for the example 6.27.*