FIR Filter

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7.1 INTRODUCTION

As a matter of fact, the digital filters are the discrete-time systems used mainly for filtering of arrays or sequences. The arrays or sequences are obtained by sampling the input analog signals. The digital filters perform the frequency related operations such as lowpass, highpass, band reject (notch), bandpass and allpass etc. Also, the design specifications include cut-off frequency, sampling frequency of input signal, passband variation, stop band attenuation, approximation, type of filter and realization form etc. Digital filters may be realized through hardware or software. Actually, the software digital filters need digital hardware for their operation. Now, we shall discuss few basic concepts about digital filters. We shall also study the difference between analog and digital filters and types of digital filters.

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7.1.1 Concept of Analog Low Pass Filter (LPF)

Let us consider an LC lowpass filter as shown in figure 7.1. This is a very standard and commonly used filter to remove ripple and noise in the input signal.

In figure 7.1, it may be observed that the input signal is noisy dc level of 5 volts. The noise is removed by the LC filter and the output signal is pure dc level of 5 volts. The noise is high frequency signal compared to dc level. The lowpass filter removes this noise and passes only dc level. This is how the simple LC lowpass filter works.

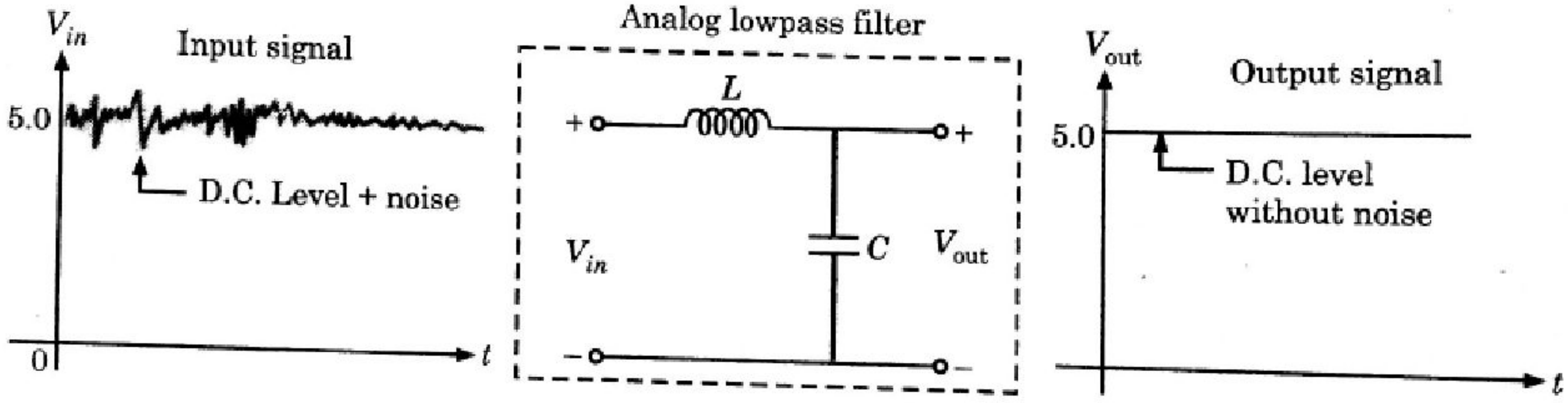

FIGURE 7.1 *Analog low pass filter (LPF)*

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7.1.2 Concept of a Digital Lowpass Filter

As a matter of fact, the digital filters do not use resistors, capacitors or inductors. We know that the discrete-time systems are represented by the difference equation. Let us consider a simple lowpass filter whose difference equation is given by

$$y(n) = \frac{1}{2}\,x(n) + \frac{1}{2}\,x(n-1) \quad ...(7.1)$$

Figure 7.2 shows the implementation of this digital lowpass filter. The noisy analog 5V signal is sampled and we get $x(n)$. It may be observed that the samples of $x(n)$ vary above and below 5V level. Infact, we have to remove this noise and all the samples at the output must be near 5V level.

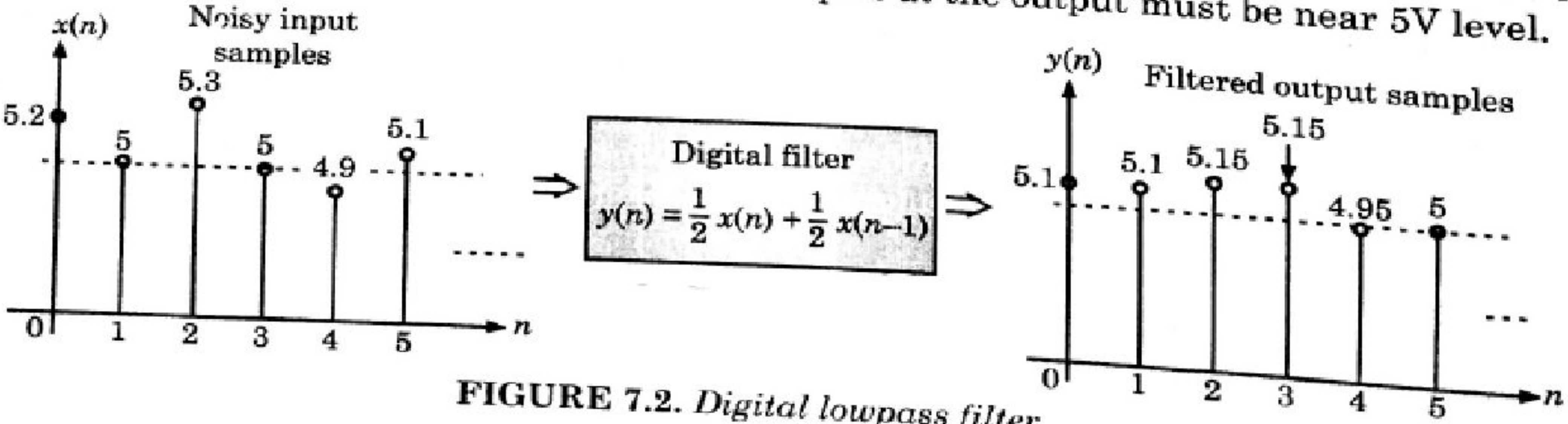

FIGURE 7.2. *Digital lowpass filter*

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This noisy $x(n)$ is applied to this digital lowpass filter as shown in figure 7.2. It may be observed that the digital lowpass filter is represented by the difference equation $y(n) = \frac{1}{2}x(n) + \frac{1}{2}x(n-1)$. The digital filter then computes various values of $y(n)$ according to this difference equation. Figure 7.2 shows the computation of first few samples of $y(n)$. For initial calculation, we need $x(-1)$. It is assumed to be 5.8. In figure 7.2, let us also note the values of $y(n)$. The noise in these samples is reduced compared to $x(n)$. This is how a digital filter works.

Now, let us assume $x(-1) = 5$

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Then, output samples $y(n)$ for various values of n may be obtained by using $y(n) = \frac{1}{2}\,x(n) + \frac{1}{2}\,x(n-1)$ as under :

$$y(0) = \frac{1}{2}\,x(0) + \frac{1}{2}\,x(-1) = \frac{1}{2}\,(5.2) + \frac{1}{2}\,(5) = 5.1$$

$$y(1) = \frac{1}{2}\,x(1) + \frac{1}{2}\,x(0) = \frac{1}{2}\,(5) + \frac{1}{2}\,(5.2) = 5.1$$

$$y(2) = \frac{1}{2}\,x(2) + \frac{1}{2}\,x(1) = \frac{1}{2}\,(5.3) + \frac{1}{2}\,(5) = 5.15$$

$$y(3) = \frac{1}{2}\,x(3) + \frac{1}{2}\,x(2) = \frac{1}{2}\,(5) + \frac{1}{2}\,(5.3) = 5.15$$

$$y(4) = \frac{1}{2}\,x(4) + \frac{1}{2}\,x(3) = \frac{1}{2}\,(4.9) + \frac{1}{2}\,(5) = 4.95$$

$$y(5) = \frac{1}{2}\,x(5) + \frac{1}{2}\,x(4) = \frac{1}{2}\,(5.1) + \frac{1}{2}\,(4.9) = 5$$

and so on.

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S.No.	Parameter of comparison	Analog Filters	Digital Filters
1.	Input/output signals	Analog or continuous-time	Digital or discrete-time sequences.
2.	Composition	Lumped elements such as R, L and C or analog ICs	Software + digital hardware.
3.	Filter representation	In terms of system components	By difference equation.
4.	Flexibility	Not flexible	Highly flexible.
5.	Portability	Not easily portable	Portable.
6.	Design objective and result	Specifications to values of R, L and C components	Specifications to difference equations.
7.	Environmental effects	Environment parameters affect the performance	Negligible effect of environmental parameters.
8.	Interference noise and other effects	Maximum effect	Minimum/negligible effect
9.	Storage/maintenance and failure.	Difficult storage and maintenance and higher failure rate	Easier storage and maintenance and reduced failure rate.

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7.1.5 Classification of Digital Filters

Upto now, we have studied the introductory part of digital filters. Now, let us study the types of digital filters. Basically, the digital filters are of following two types :

(*i*) Finite impulse response (FIR) filters, and

(*ii*) Infinite impulse response (IIR) filters.

Basically, digital filters are linear time invariant (LTI) systems which are characterized by unit sample response. The FIR system has finite duration unit sample response, *i.e.*,

$$h(n) = 0 \text{ for } n < 0 \qquad \text{...(7.2)}$$

This unit sample response exists only for the duration from 0 to M – 1. Therefore, this is a FIR system.

Also, the IIR system has infinite duration unit sample response, *i.e.*,

$$h(n) = 0 \text{ for } n < 0 \qquad \text{...(7.3)}$$

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This unit sample response exists for the duration from 0 to ∞. Therefore, this is an IIR system.

We know that FIR and IIR systems can be best described by difference equations. IIR systems can be easily described by recursive systems. FIR systems are nonrecursive. Hence, output of FIR filter depends only upon present and past inputs since it is nonrecursive, *i.e.*, it does not use feedback. The IIR filters are recursive *i.e.*, they use feedback. Therefore, output of IIR filter depends upon present input as well as past inputs and outputs.

DO YOU KNOW?

In case of FIR filters, since, no analog prototype is required in the synthesis procedure, digital filters can be designed that have no analog equivalent.

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We know that the difference equation of the LTI system is given by

$$y(n) = -\sum_{k=1}^{N} a_k\, y(n-k) + \sum_{k=0}^{M} b_k\, x(n-k) \quad \text{...(7.4)}$$

In above expression, the terms under first summation are past outputs, and terms under second summation are present and past inputs. Therefore, the expression in equation (7.4) represents IIR filter. For the FIR filter, the first summation will be absent and the difference equation becomes

$$y(n) = \sum_{k=0}^{M} b_k\, x(n-k) \quad \text{...(7.5)}$$

Further, we know that the output of the LTI system is given by convolution as under :

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

For the FIR filter, unit sample response $h(k)$ exists for the duration from 0 to $M-1$ as described in equation (7.2). Thus, output of FIR filter becomes

$$y(n) = \sum_{k=0}^{M-1} h(k)\, x(n-k) \quad \text{...(7.6)}$$

Here, it may be noted that equations (7.6) and (7.5) represent the output of FIR filter. Therefore, unit sample response of FIR filter is given by

$$h(k) = b_k \quad \text{...(7.7)}$$

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In short, FIR filters offer the following advantages over the IIR filters:

(i) FIR filters can have an exact linear phase, a characteristic which is very useful in speech processing.

(ii) FIR filters are al·vays stable because all the poles are at the origin.

(iii) For FIR filters, the design methods are generally linear.

(iv) FIR filters can be realized efficiently in hardware. This means that the question of physical realizability never arise, with a finite delay, it is always realizable.

(v) In case of FIR filter, the filter start-up transients have finite duration.

(vi) In case of FIR filters, errors arising from quantization is usually less, round-off noise can be made small by employing non-recursive technique of realization.

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7.3 THE MAGNITUDE RESPONSE AND PHASE RESPONSE OF DIGITAL FILTERS

(Sem. Exam, WBTU, Kolkata, 2001–02)

We know that the Discrete-Time Fourier Transform (DTFT) of a finite sequence impulse response $h(n)$ can be expressed as

$$H(e^{j\omega}) = DTFT\,[h(n)] = \sum_{n=0}^{M-1} h(n)e^{-j\omega nT}$$

Square Root

or

$$H(e^{j\omega}) = |H(e^{j\omega})|\,e^{j\phi(\omega)} \quad \text{...(7.12)}$$

Now, we can express the magnitude response and phase response as under:

Magnitude response

$$\begin{aligned} M(\omega) &= |H(e^{j\omega})| \\ &= \{Re[H(e^{j\omega})]^2 + Im[H(e^{j\omega})]^2\} \end{aligned} \quad \text{...(7.13)}$$

and Phase response

$$\theta(\omega) = \tan^{-1}\left\{\frac{Im\left[H\left(e^{j\omega}\right)\right]}{Re\left[H\left(e^{j\omega}\right)\right]}\right\} \quad \text{...(7.14)}$$

DO YOU KNOW?

Since, FIR filters have no feedback, they have no poles and are therefore always stable.

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As a matter of fact, filters can have a linear or non-linear phase depending upon the delay function, namely the phase delay and group delay. The phase and group delays of the filter can be given by

$$\tau_p = -\frac{\theta(\omega)}{\omega}$$

and $$\tau_g = -\frac{d\theta(\omega)}{d\omega}, \text{ respectively}$$

The group delay may be defined as the delayed response of the filter as a function of ω to a signal.

Linear phase filters are those filters in which the phase delay and group delay are constants *i.e.*, independent of frequency. Linear phase filters, some times, are also known as **constant time delay filters.**

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7.5 SYMMETRIC AND ANTISYMMETRIC FIR FILTERS

(Sem. Exam, MDU, Rohtak, 2005–06)

As a matter of fact, the symmetry and antisymmetry properties of FIR filters are related to their unit sample response $h(n)$.

Unit Sample Response of Symmetric FIR Filters

The unit sample response of FIR filters is symmetric if it satisfies following condition:

$$h(n) = h\,(M-1-n), \quad n = 0, 1, \ldots M-1 \qquad \ldots(7.22)$$

As an example, let us consider the unit sample response given in figure 7.3 for $M = 6$ samples.

In figure 7.3, $h(n)$ is given by

$$h\,(0) = 2,\ h(1) = 4,\ h(2) = 6$$

$$h(3) = 6,\ h(4) = 4,\ h(5) = 2$$

Using equation (7.22), we write

$$n = 0 \Rightarrow h(0) = h(6-1-0) = h(5) = 2$$

$$n = 1 \Rightarrow h(1) = h(6-1-1) = h(4) = 4$$

$$n = 2 \Rightarrow h(2) = h(6-1-2) = h(3) = 6$$

Hence, symmetry condition of equation (7.22) is satisfied and therefore, this is symmetric FIR filter.

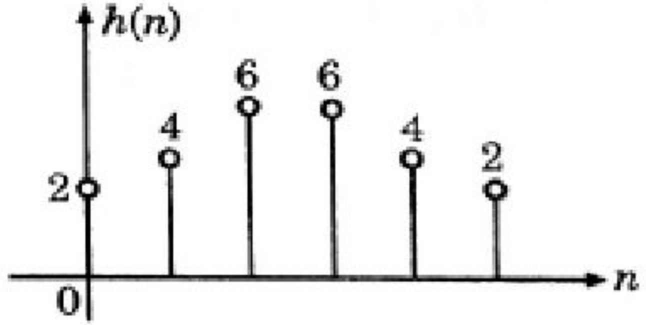

FIGURE 7.3 *Symmetric unit sample response of FIR filters. Here, $h(n) = h(M-1-n)$*

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Unit Sample Response of Antisymmetric FIR Filters

The unit sample response of FIR filters is antisymmetric if it satisfies following condition :

$$h(n) = -\,h(M-1-n), \quad n = 0, 1 \ldots M-1 \qquad \ldots(7.23)$$

As an example, let us consider the unit sample response shown in figure 7.4 for $M = 6$.

In the above figure, $h(n)$ is given as

$$h(0) = 2,\ h(1) = 4,\ h(2) = 6,$$

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$$h(3) = -6,\ h(4) = -4,\ h(5) = 2$$

Now, using equation (7.23), we write

$$n = 0 \Rightarrow h(0) = -h(5)$$

$$n = 1 \Rightarrow h(1) = -h(4)$$

$$n = 3 \Rightarrow h(2) = -h(3)$$

Therefore, antisymmetric condition is satisfied and hence this is called an *antisymmetric FIR filter*.

Thus, from above, we summarize that for linear phase FIR filter,

$$h(n) = \pm h(M - 1 - n) \qquad \text{...(7.24)}$$

The above condition should be satisfied.

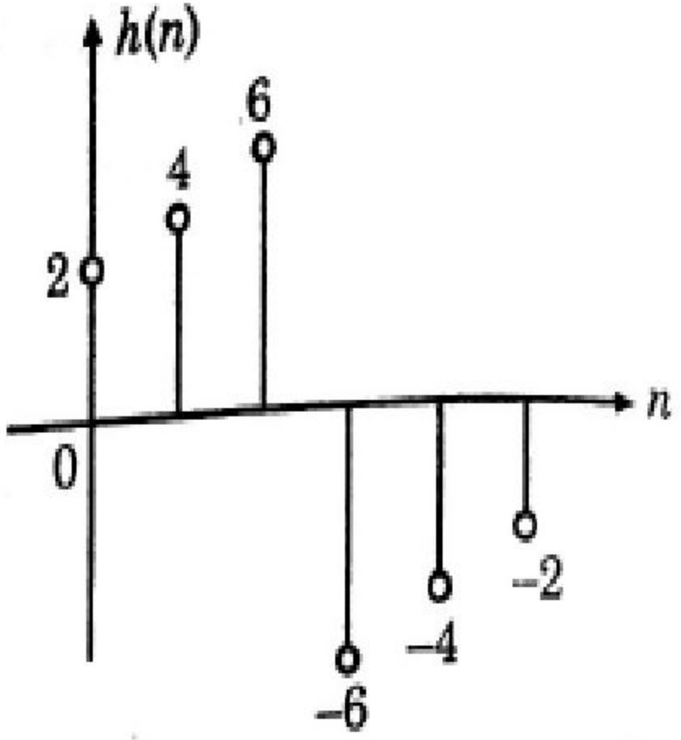

FIGURE 7.4 *Antisymmetric unit sample response of FIR filters. Here, $h(n) = -h(M - 1 - n)$*

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Note : *Linear phase is the most important feature of FIR filters. IIR filters cannot be designed with linear phase. The linear phase in FIR filters can be obtained if unit sample response satisfies equation (7.24). Several applications need linear phase filtering. As an example, the speech related applications require linear phase. Similarly, in data transmission applications, linear phase prevents pulse dispersion and the detection becomes more accurate. Therefore, whenever linear phase is desired, FIR filtering is used.*

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7.8 VARIOUS KINDS OF WINDOWS *(Sem. Exam, GGSIPU, Delhi, 2007–08)*

As a matter of fact, various window functions are used for filter design. First, let us consider the time and frequency domain representation of different types of windows.

Basically, following are the types of window functions :

(*i*) Rectangular window
(*ii*) Bartlett (*i.e.*, Triangular window)
(*iii*) Blackmann window
(*iv*) Hamming window
(*v*) Hanning window
(*vi*) Kaiser window

DO YOU KNOW?

The finite-duration impulse response (FIR) digital filter is based on a frequency-response specification, and the filter implementation is accomplished be taking the Fourier transform of the desired frequency-response specification.

7.8.1 Rectangular Window

As a matter of fact, different types of windows are used to design FIR filter. Now, let us discuss the design of FIR filter using rectangular window. Figure 7.6 shows the rectangular window.

$w_R(n)$

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window.

It is denoted by $w_R(n)$. Its magnitude is 1 for the range $n = 0$ to $M - 1$. Since, the shape of window function is rectangular; it is called as **rectangular window**.

The rectangular window of length 'M' is represented by $w_R(n)$ and it is expressed as under :

$$w_R(n) = \begin{cases} 1 & \text{for } n = 0, 1, 2, \ldots, M-1 \\ 0 & \text{elsewhere} \end{cases} \qquad \ldots(7.36)$$

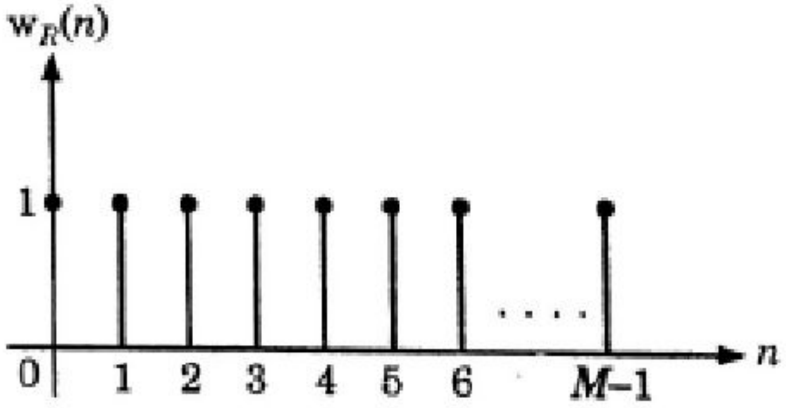

FIGURE 7.6 *Rectangular window.*

Now, let us consider the Fourier transform of rectangular window. It may be expressed as

$$W_R(\omega) = \sum_{n=0}^{M-1} w_R(n)\, e^{-j\omega n} \qquad \ldots(7.37)$$

Substituting for $w_R(n)$ from equation (7.36), equation (7.37) becomes,

$$W_R(\omega) = \sum_{n=0}^{M-1} e^{-j\omega n} \qquad \ldots(7.38)$$

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Here, let us use the standard series relation as under :

$$\sum_{k=N_1}^{N_2} a^k = \frac{a^{N_1} - a^{N_2+1}}{1-a} \qquad \ldots(7.39)$$

With $a = e^{-j\omega}$, equation (7.38) will take the following form:

$$W_R(\omega) = \frac{1-e^{-j\omega M}}{1-e^{-j\omega}} \qquad \ldots(7.40)$$

The last equation can be rearranged as under :

$$W_R(\omega) = \frac{e^{-j\omega\frac{M}{2}}.e^{j\omega\frac{M}{2}}.-e^{-j\omega\frac{M}{2}}.e^{-j\omega\frac{M}{2}}}{e^{-j\frac{\omega}{2}}.e^{j\frac{\omega}{2}}-e^{-j\frac{\omega}{2}}.e^{-j\frac{\omega}{2}}} = \frac{e^{-j\omega\frac{M}{2}}\left(e^{j\omega\frac{M}{2}}-e^{-j\omega\frac{M}{2}}\right)}{e^{-j\frac{\omega}{2}}\left(e^{j\frac{\omega}{2}}-e^{-j\frac{\omega}{2}}\right)}$$

We know that $e^{j\theta} - e^{-j\theta} = 2\,j \sin\theta$. Hence, above expression becomes

$$W_R(\omega) = \frac{e^{-j\omega\frac{M}{2}}.2\sin\left(\omega\frac{M}{2}\right)}{e^{-j\frac{\omega}{2}}\;.2\sin\left(\frac{\omega}{2}\right)}$$

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or $$W_R(\omega) = e^{-j\omega\left(\frac{M-1}{2}\right)} \cdot \frac{\sin\left(\frac{\omega M}{2}\right)}{\sin\left(\frac{\omega}{2}\right)} \quad ...(7.41)$$

The polar form of $W_R(\omega)$ is given by $|W_R(\omega)|\, e^{j\angle W_R(\omega)}$. Comparing with above equation, we get magnitude response of rectangular window as under :

$$|W_R(\omega)| = \frac{\left|\sin\left(\frac{\omega M}{2}\right)\right|}{\left|\sin\left(\frac{\omega}{2}\right)\right|} \quad ...(7.42)$$

Figure 7.7 illustrates the response of rectangular window given by above expression for $M = 50$.

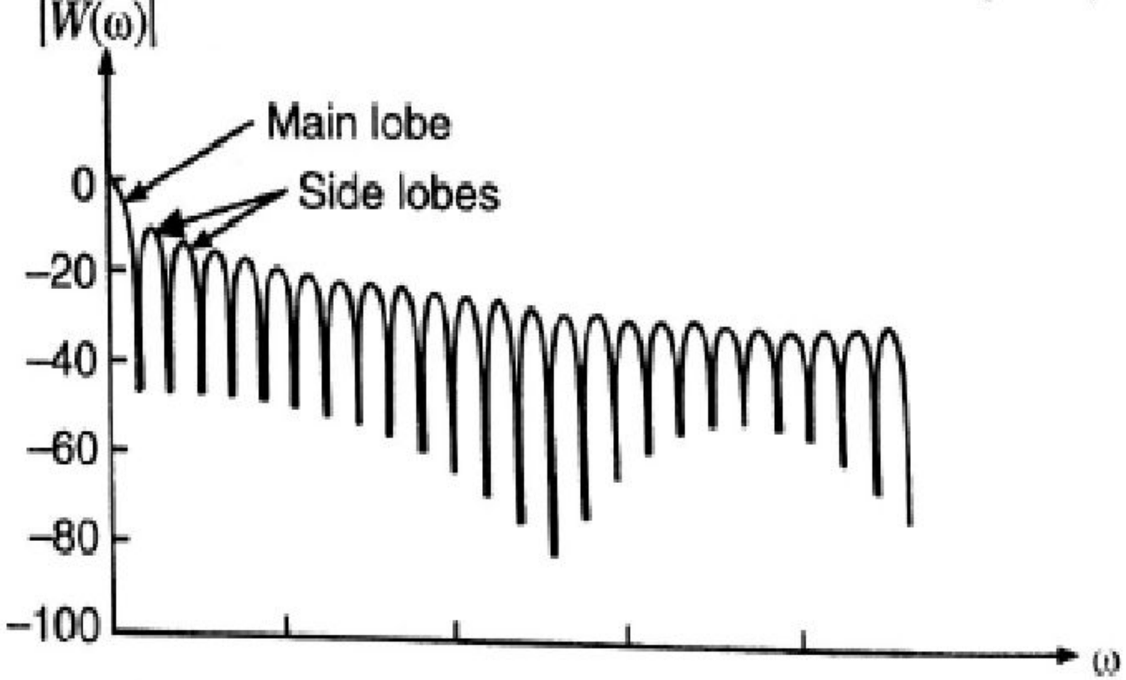

FIGURE 7.7 *Magnitude spectrum of rectangular window for M = 50.*

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7.8.2 Bartlett (i.e., Triangular) Window

Bartlett window is also called as triangular window. It is expressed mathematically as,

$$w_T(n) = \begin{cases} 1 - \dfrac{2\left|n - \dfrac{M-1}{2}\right|}{M-1} & \text{for } n = 0, 1, \ldots M-1 \\ 0 & \text{elsewhere} \end{cases} \qquad \ldots(7.43)$$

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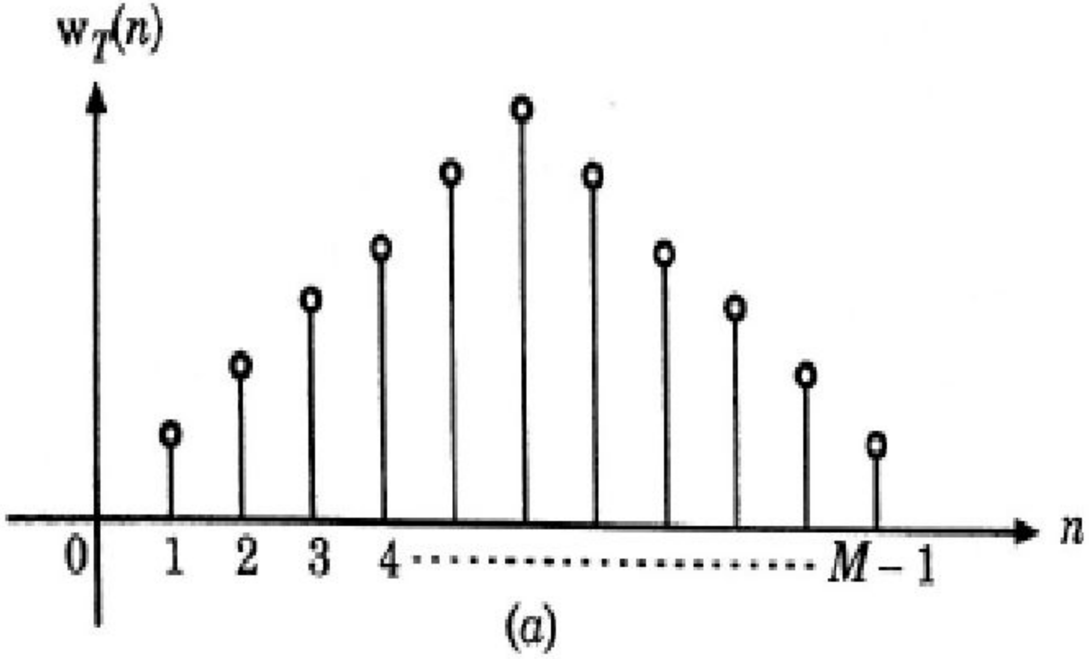
Fourier transform of this window can be obtained in a similar manner as discussed for rectangular window. Figure 7.8 (*a*) shows the sketch of this window. Figure 7.8 (*b*) shows the magnitude response of this window.



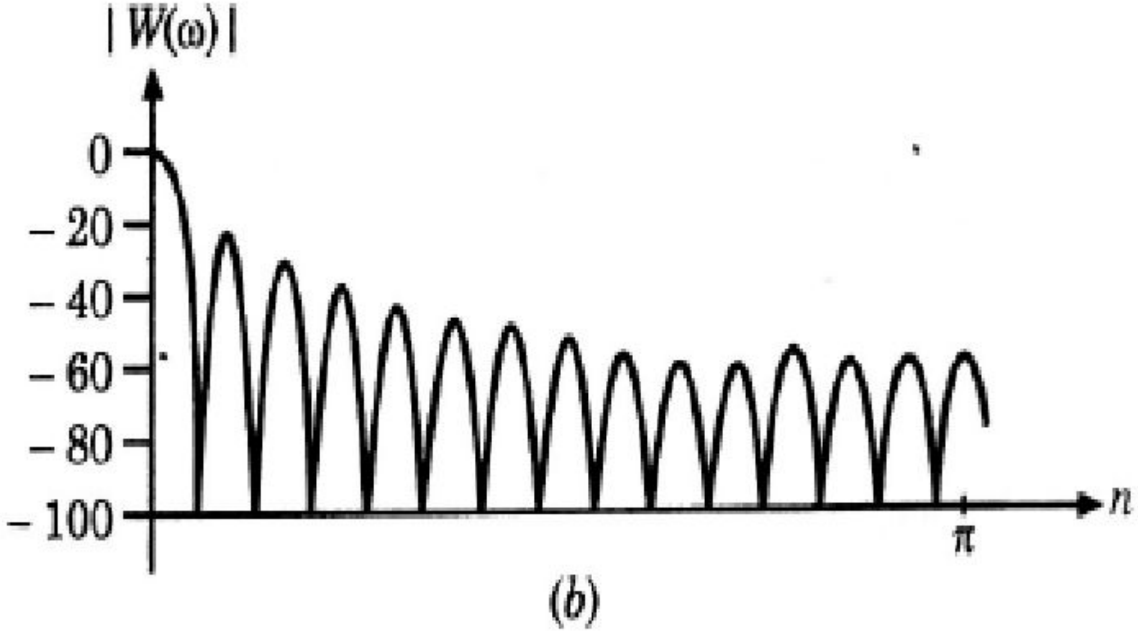


FIGURE 7.8 *Bartlett window (a) Time domain graph (b) Magnitude response.*

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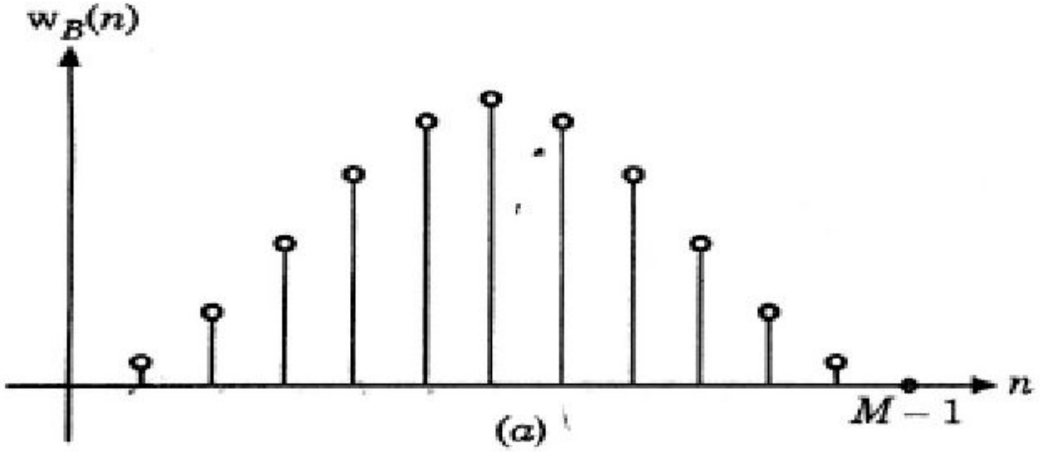
7.8.3 Blackmann Window

The Blackmann window has a bell like shape of its time-domain-samples. It is expressed mathematically as,

$$w_B(n) = \begin{cases} 0.42 - 0.5 \cos \dfrac{2\pi n}{M-1} + 0.8 \cos \dfrac{4\pi n}{M-1} & \text{for} \quad n = 0,\ 1,\ \ldots\ M-1 \\ 0 & \text{elsewhere} \end{cases} \qquad \ldots(7.44)$$

Figure 7.9 (*a*) shows the graph of Blackmann window. Figure 7.9 (*b*) shows its magnitude response. Here, it may be observed that the width of the main lobe is increased. However, it has very small sidelobes.



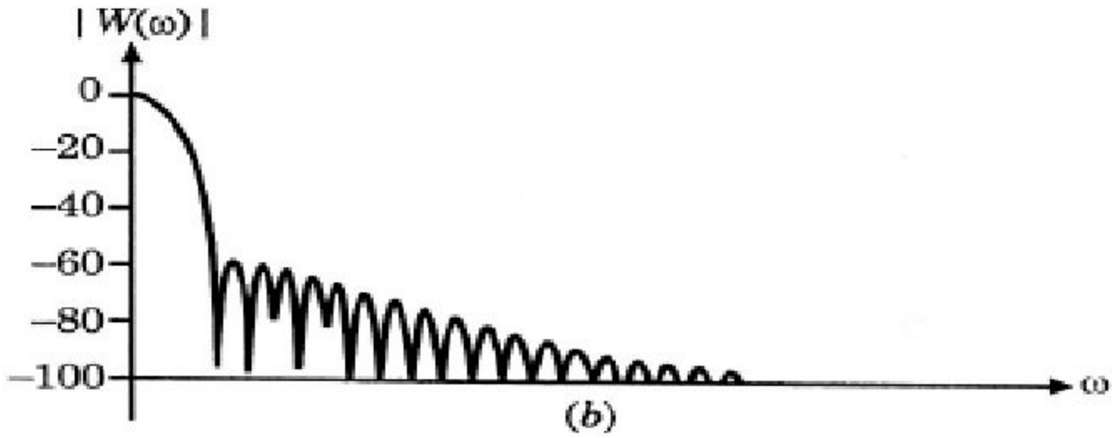


FIGURE 7.9 *Blackmann window (a) Time-domain graph (b) magnitude response.*

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7.8.4 Hamming Window

Hamming window is most commonly used window in speech processing. It is given as

$$w_H(n) = \begin{cases} 0.54 - 0.46 \cos \dfrac{2\pi n}{M-1} & \text{for} \quad n = 0, 1, \ldots M-1 \\ 0 & \text{elsewhere} \end{cases} \qquad \ldots(7.45)$$

This windows also has bell like shape. Its first and last samples are not zero. Figure 7.10 (*a*) shows the graph of Hamming window. Figure 7.10 (*b*) shows the magnitude response of this window. It has reduced sidelobes but slightly increased main lobe. The sidelobes are higher than Blackmann window.

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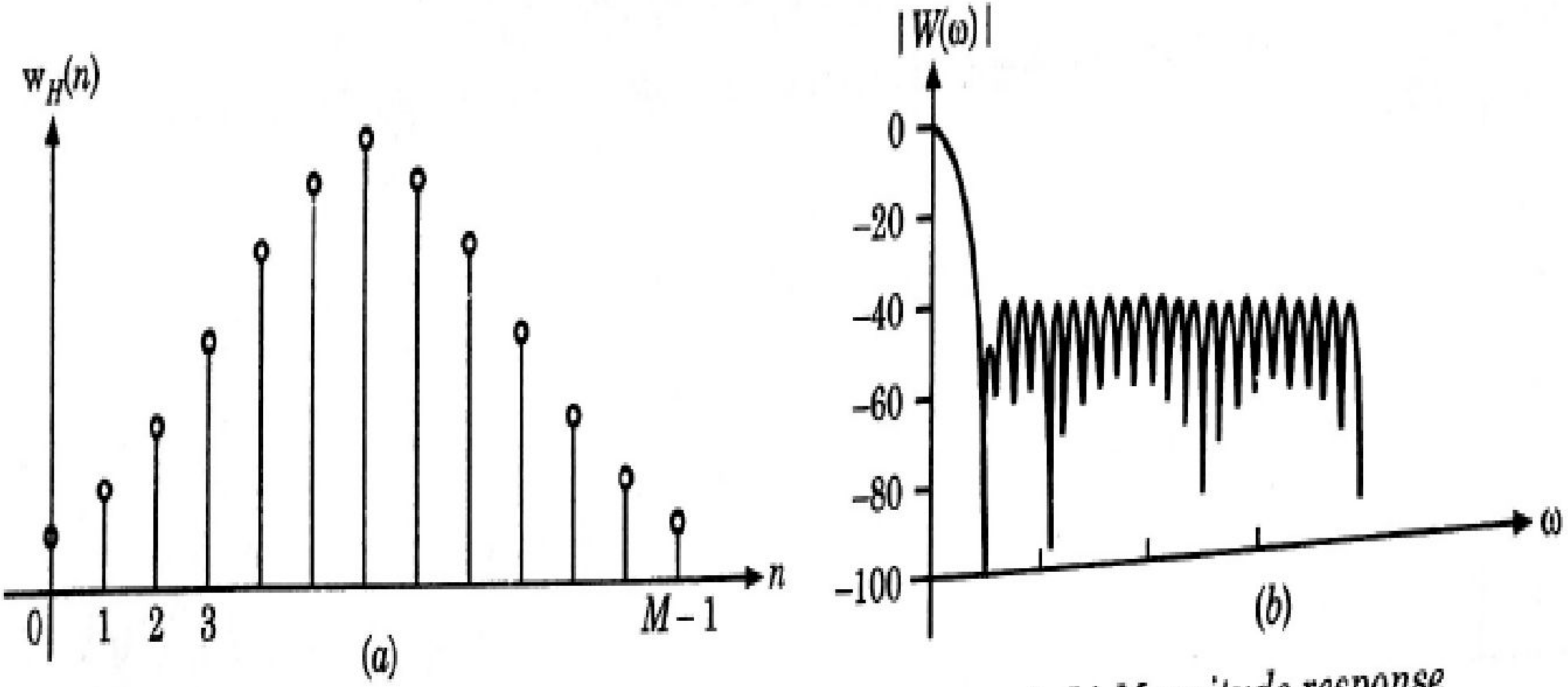

FIGURE 7.10 *Hamming window (a) Time-domain graph (b) Magnitude response*

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7.8.5 Hanning Window

Hanning window has shape similar to those of Blackmann and Hamming. Its first and last samples are zero. It is given as,

$$w_{HN}(n) = \begin{cases} \frac{1}{2}\left(1 - \cos\frac{2\pi n}{M-1}\right) & \text{for} \quad n = 0,\ 1,\ \ldots\ M-1 \\ 0 & \text{elsewhere} \end{cases} \qquad \ldots(7.46)$$

Figure 7.11 (*a*) shows the graph of this window. This window is commonly used for spectrum analysis, speech and music processing. Figure 7.11 (*b*) shows the magnitude response of Hanning window. Here, it may be observed that it has narrow main lobe, but first few sidelobes are significant. Then sidelobes reduce rapidly.

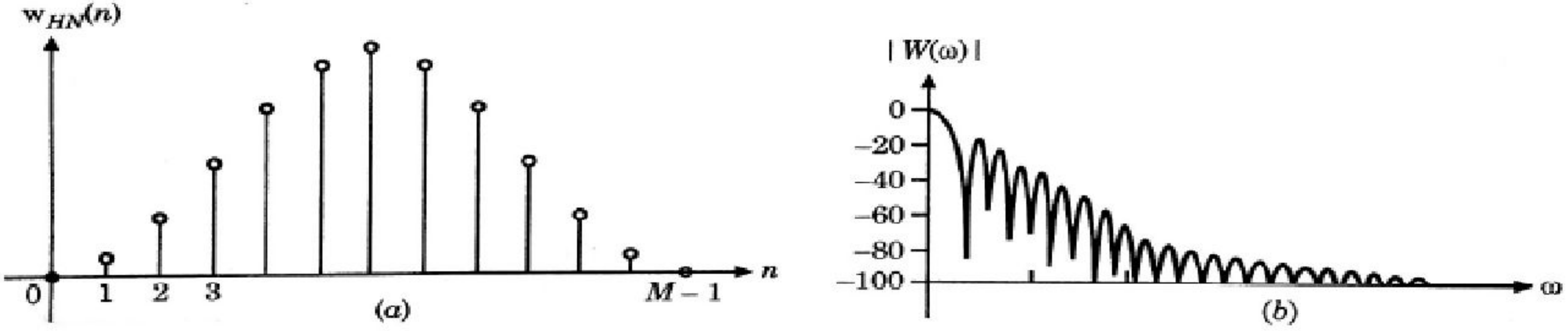

FIGURE 7.11 *Hanning window (a) Time-domain graph (b) Magnitude response.*

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7.8.6 Kaiser Window *(Sem. Exam., RTU, Jaipur, 2005–06)*

In previus subsections, we discussed four different types of windows. The values of samples of the windows depend only upon 'M', *i.e.*, length of the window. In other words, the width of the main lobe and attenuation, of sidelobes depends only upon length 'M' of the window. They cannot be controlled independently.

However, Kaiser window allows separate control of width of the main lobe and attenuation of sidelobes. The Kaiser window is defined as under:

$$w_k(n) = \begin{cases} \dfrac{I_0\left\{\beta\left[1-\left(\dfrac{n-\alpha}{\alpha}\right)^2\right]^{\frac{1}{2}}\right\}}{I_0(\beta)} & \text{for} \quad 0 \le n \le M \\ 0\,. & \text{elsewhere} \end{cases} \qquad \text{...(7.47)}$$

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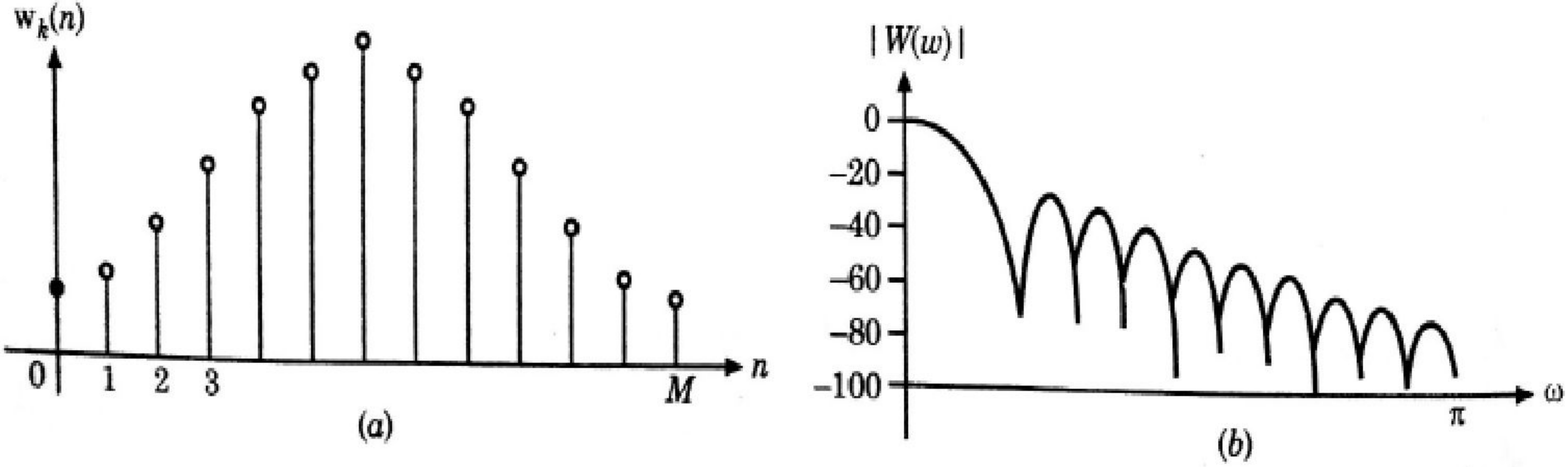

FIGURE 7.12 *Kaiser window (a) Time domain graph (b) Magnitude response.*

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7.9 DESIGN OF LINEAR PHASE FIR FILTERS USING WINDOWS

Let us consider that the digital filter which is to be designed have the frequency response $H_d(\omega)$. This is also called as desired frequency response. Let the corresponding unit sample response (desired) be $h_d(n)$. We know that $H_d(\omega)$ is Fourier transform of $h_d(n)$.

This means that
$$H_d(\omega) = \sum_{n=0}^{\infty} h_d(n)\, e^{-j\omega n} \quad \text{...(7.54)}$$

Also, $h_d(n)$ may be obtained by taking the inverse Fourier transform of $H_d(\omega)$.

This means that
$$h_d(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega \quad \text{...(7.55)}$$

Thus, the desired unit sample response can be obtained from desired frequency response by above expression.

Usually, the unit sample response obtained by equation (7.55) is infinite in duration. Since, we are designing a finite impulse response filter, the length of $h_d(n)$ must be made finite. If we want the unit sample response of length 'M', then $h_d(n)$ is truncated to length 'M'. This is equivalent to multiplying $h_d(n)$ by a window sequence w(n). This concept can be best explained by considering a particular type of window sequence. Here, let us consider rectangular window in the following subsection.

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7.9.1 Rectangular Window for FIR Filter Design

We know that the rectangular window is defined as,

$$w_R(n) = \begin{cases} 1 & \text{for} \quad n = 0, 1, \ldots, M-1 \\ 0 & \text{elsewhere} \end{cases} \qquad \ldots(7.56)$$

The unit sample response of the FIR filter is obtained by multiplying $h_d(n)$ by $w_R(n)$.

This means that $\quad h(n) = h_d(n)\, w_R(n) \qquad \ldots(7.57)$

From equation (7.56), substituting for $w_R(n)$ above expression will become,

$$h(n) = \begin{cases} h_d(n) & \text{for} \quad n = 0, 1, \ldots, M-1 \\ 0 & \text{elsewhere} \end{cases} \qquad \ldots(7.58)$$

FIR Filter

elsewhere

Hence, $h_d(n)$ is truncated to length 'M' to give $h(n)$. This truncation is obtained by multiplication of $h_d(n)$ and $w_R(n)$. This is also called as windowing since function $w_R(n)$ acts like a window for $h_d(n)$.

The Fourier transform of rectangular window is given by equation (7.41) and its magnitude response is given by equation (7.42), *i.e.*,

$$|W_R(\omega)| = \frac{\left|\sin\left(\frac{\omega M}{2}\right)\right|}{\left|\sin\left(\frac{\omega}{2}\right)\right|}$$

Figure 7.7 shows the magnitude frequency response given by above expression.

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Few Important Points

(*i*) From equation (7.57), we know that unit sample response of FIR filter is given as,

$$h(n) = h_d(n)\, \mathrm{w}(n)$$

Here, $\mathrm{w}(n)$ represents generalized window function.

(*ii*) The frequency response of FIR filter can be obtained by taking Fourier transform of above equation. This means that

$$H(\omega) = FT\{h_d(n) \,.\, \mathrm{w}(n)\} \qquad ...(7.59)$$

(*iii*) We know that Fourier transform of multiplication of two signals is equal to the convolution of their individual Fourier transforms. Then, above equation will become,

$$H(\omega) = H_d(\omega) \otimes W(\omega) \qquad ...(7.60)$$

This expression shows that the response of FIR filter is equal to the convolution of desired frequency response with that of window function. Because of convolution, $H(\omega)$ has the *smoothing effect*. The sidelobes of $W(\omega)$ create undesirable ringing effects in $H(\omega)$.

FIR Filter

7.9.2 Gibbs Phenomenon

(Sem. Exam., JNTU, Hyderabad, 2003-04,)

Let us consider the example of a lowpass filter having desired frequency response H_d (ω) as depicted of in figure 7.13 (a). This response has the cutoff frequency at ω_c.

Note: *In figure 7.13, oscillations or ringing takes place near band-edge (ω_c) of the filter. These oscillations or ringing is generated because of sidelobes in the frequency response W(ω) of the window function. This oscillatory behaviuor (i.e., ringing effect) near the band edge of the filter is known as Gibbs phenomenon. Thus, the ringing effect takes place because of sidelobes in W(ω). These sidelobes are generated because of abrupt discontinuity (in case of rectangular window) of the window function. In case of rectangular window, the sidelobes are larger in size because the discontinuity is abrupt. Therefore, ringing effect is maximum in rectangular window.*

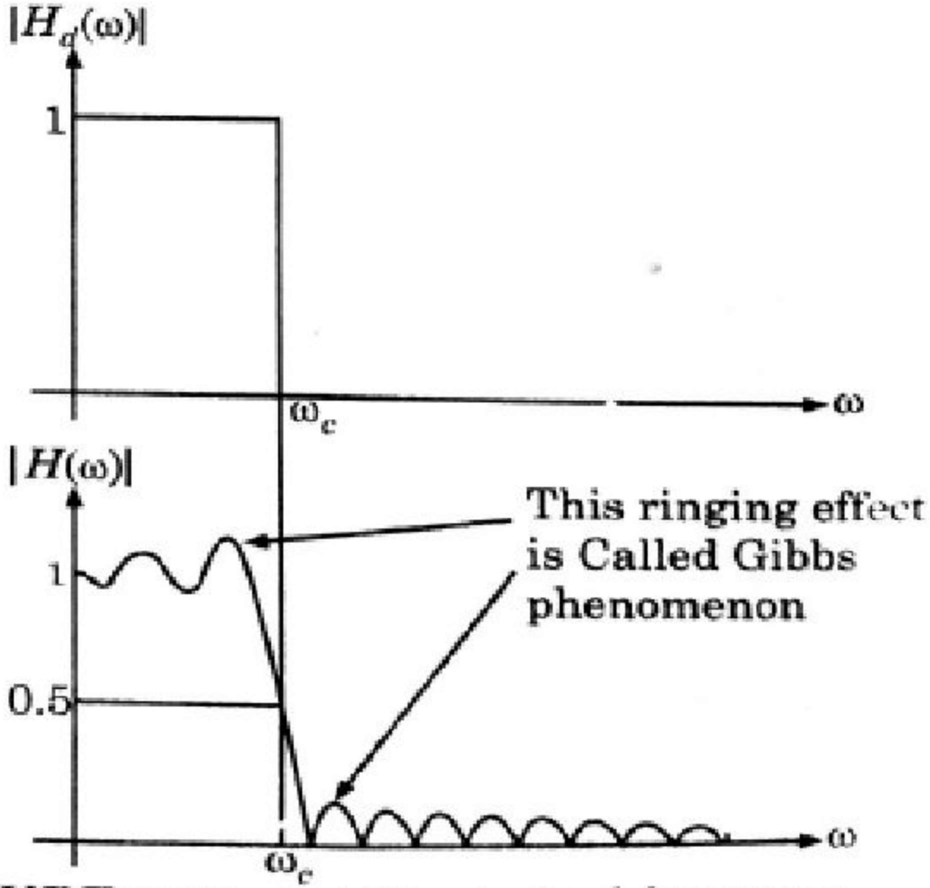

FIGURE 7.13 *(a) The desired frequency response $H_d(\omega)$ (b) The frequency response of FIR filter obtained by windowing. It has smoothing and ringing effect because of windowing.*

FIR Filter

EXAMPLE 7.9 **Design the symmetric FIR lowpass filter for which desired frequency response is expressed as**

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau} & \text{for } |\omega| \leq \omega_c \\ 0 & \text{elsewhere} \end{cases} \qquad \ldots(i)$$

The length of the filter should be 7 and $\omega_c = 1$ radians/sample. Make use of rectangular window.

Solution: Given that

Desired frequency response is $H_d(\omega)$

Length of the filter *i.e.*, $M = 7$

Cut-off frequency $\omega_c = 1$ radians/sample

FIR Filter

We know that the desired unit sample response is given by

$$h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega)\, e^{j\omega n}\, d\omega \qquad \ldots(ii)$$

According to equation (*i*), given frequency response $H_d(\omega)$ is

$$H_d(\omega) = \begin{cases} e^{-j\omega\tau} & \text{for} \\ 0 & \text{elsewhere} \end{cases} \quad -1 \le \omega \le 1 \;\; \text{since } \omega_c = 1$$

Therefore, equation (*ii*) will be given by

$$h_d(n) = \frac{1}{2\pi}\int_{-1}^{1} e^{-j\omega\tau}.e^{\,j\omega n}\, d\omega = \frac{1}{2\pi}\int_{-1}^{1} e^{j\omega(n-\tau)}\, d\omega \qquad \ldots(iii)$$

or

$$h_d(n) = \frac{1}{2\pi}\left[\frac{e^{j\omega(n-\tau)}}{j(n-\tau)}\right]_{-1}^{1} = \frac{\sin(n-\tau)}{\pi(n-\tau)} \quad \text{for } n \ne \tau$$

FIR Filter

When $n = \tau$, equation (iii) will become

$$h_d(n) = \frac{1}{2\pi}\int_{-1}^{1} d\omega = \frac{1}{\pi}$$

Therefore, we obtain $h_d(n)$ as under:

$$h_d(n) = \begin{cases} \dfrac{\sin(n-\tau)}{\pi(n-\tau)} & \text{for} \quad n \neq \tau \\ \dfrac{1}{\pi} & \text{for} \quad n = \tau \end{cases} \qquad \ldots(iv)$$

Now, let us determine the value of τ.

We know that the filter is symmetric, therefore, we have

$$h(n) = h(M-1-n)$$

Further, we know that $h(n) = h_d(n).\ w(n)$, thus, above condition becomes,

$$h_d(n)\,w(n) = h_d(M-1-n)\,w(n)$$

or

$$h_d(n) = h_d(M-1-n)$$

FIR Filter

Therefore, using equation (*iv*), we get

$$\frac{\sin(n-\tau)}{\pi(n-\tau)} = \frac{\sin(M-1-n-\tau)}{\pi(M-1-n-\tau)}$$

The above condition will be satisfied if, we have

$$-(n-\tau) = M-1-n-\tau$$

or

$$\tau = \frac{M-1}{2} \qquad \ldots(v)$$

Therefore, $h_d(n)$ in equation (*iv*) becomes,

$$h_d(n) = \begin{cases} \dfrac{\sin\left(n-\dfrac{M-1}{2}\right)}{\pi\left(n-\dfrac{M-1}{2}\right)} & \text{for} \quad n \neq \dfrac{M-1}{2} \\ \dfrac{1}{\pi} & \text{for} \quad n = \dfrac{M-1}{2} \end{cases} \qquad \ldots(vi)$$

FIR Filter

With $M = 7$, the last equation becomes,

$$h_d(n) = \begin{cases} \dfrac{\sin(n-3)}{\pi(n-3)} & \text{for} \quad n \neq 3 \\[2ex] \dfrac{1}{\pi} & \text{for} \quad n = 3 \end{cases} \qquad \text{...}(vii)$$

Table 7.4 presents the values of $h_d(n)$ calculated according to equation (*vii*).

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TABLE 7.4 *Calculation of $h_d(n)$*

S.No.	n	Value of coefficient $h_d(n)$ according to equation (*vii*)
1.	0	$h_d(0) = \frac{\sin(-3)}{-3\pi} = 0.01497$
2.	1	$h_d(1) = \frac{\sin(-2)}{-2\pi} = 0.14472$
3.	2	$h_d(2) = \frac{\sin(-1)}{-\pi} = 0.26785$
4.	3	$h_d(3) = \frac{1}{\pi} = 0.31831$
5.	4	$h_d(4) = \frac{\sin(1)}{\pi} = 0.26785$
6.	5	$h_d(5) = \frac{\sin(2)}{2\pi} = 0.14472$
7.	6	$h_d(6) = \frac{\sin(3)}{3\pi} = 0.01497$

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Now, let us obtain $h(n)$ by windowing operation.

Here, it may be noted that we have calculated only 7 values of $h_d(n)$. However, according to equation (*vii*), we can calculate infinite number of values of $h_d(n)$. Because we are using a window of '7' values, we will require only seven values of $h_d(n)$, *i.e.*,

$$h(n) = h_d(n) \cdot \mathrm{w}(n)$$

We know that for rectangular window, we have

$$\mathrm{w}(n) = \begin{cases} 1 & \text{for} \quad 0 \le n \le 6 \\ 0 & \text{elsewhere} \end{cases}$$

Therefore,

$$h(n) = \begin{cases} h_d(n) & \text{for} \quad 0 \le n \le 6 \\ 0 & \text{elsewhere} \end{cases}$$

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Hence, from Table 7.4, coefficients of FIR filter will become

$$h(0) = 0.01497$$

$$h(1) = 0.14472$$

$$h(2) = 0.26785$$

$$h(3) = 0.31831$$

$$h(4) = 0.26785$$

FIR Filter

$$h(5) = 0.14472$$

$$h(6) = 0.01497$$

This is the unit sample response of required FIR filter. Because, the filter is symmetric, above coefficients satisfy the condition of $h(n) = h(M - 1 - n)$ for $M = 7$, *i.e.*

$$h(n) = h(6 - n)$$

This means that

$$h(0) = h(6)$$

$$h(1) = h(5)$$

$$h(2) = h(4)$$

Thus , filter have linear phase.