Homework 6

5.1.8

Using Taylor Expansions around x:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + O(h^4)$$

$$f(x+2h) = f(x) + (2h)f'(x) + \frac{(2h)^2}{2!}f''(x) + \frac{(2h)^3}{3!}f'''(x) + O(h^4)$$
$$= f(x) + 2h f'(x) + 2h^2 f''(x) + \frac{4h^3}{3} f'''(x) + O(h^4)$$

$-f(x+2h) + 4f(x+h) - 3f(x)$

Subbing in:

$$= -\left[f(x) + 2hf'(x) + 2h^2f''(x) + \frac{4h^3}{3}f'''(x) + O(h^4)\right] + 4\left[f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + O(h^4)\right] - 3f(x)$$

$$= -f(x) + 4f(x) - 3f(x) - 2hf'(x) + 4hf'(x) - 2h^2f''(x) + 2h^2f''(x) - \frac{4}{3}h^3f'''(x) + \frac{4}{6}h^3f'''(x)$$

$\rightarrow (-1+4-3)f(x) = 0f(x) = 0$

$\rightarrow (-2+4)hf'(x) = 2hf'(x)$

$\rightarrow (-2+2)h^2f''(x) = 0h^2f''(x) = 0$

$\rightarrow \left(\frac{-4}{3} + \frac{2}{3}\right)h^3f'''(x) = -\frac{2}{3}h^3f'''(x)$

$$\therefore = \frac{2hf'(x) - \frac{2}{3}h^3f'''(x) + O(h^4)}{2h}$$

$$= f'(x) - \frac{1}{3}h^2f'''(x) + O(h^3)$$

Since $-\frac{1}{3}h^2f'''(x)$ is of the order $O(h^2)$ and $O(h^3)$ is also of the order $O(h^2)$ (as h^3 goes to zero faster than h^2), we can write the error term as $O(h^2)$

$$\therefore f'(x) = \frac{-f(x+2h) + 4f(x+h) - 3f(x)}{2h} + O(h^2)$$

5.1.10

$$f'(x) = 4f(x+h) - 3f(x) - f(x-2h) / 6h$$

Using Taylor expansions

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{4}(x) + O(h^5)$$

$$f(x-2a) = f(x) + (-2a) f'(x) + \frac{(-2a)^2}{2!} f''(x) + \frac{(-2a)^3}{3!} f'''(x) + \frac{(2a)^4}{4!} f^4(x) + O(a^5)$$

$$= f(x) - 2a f'(x) + \frac{4a^2}{2} f''(x) - \frac{8a^3}{6} f'''(x) + \frac{16a^4}{24} f^4(x) + O(a^5)$$

$$= f(x) - 2a f'(x) + 2a^2 f''(x) - \frac{4}{3} a^3 f'''(x) + \frac{2}{3} a^4 f^4(x) + O(a^5)$$

plugging into $4f(x+a) - 3f(x) - f(x-2a)$

$$= 4\left[f(x) + a f'(x) + \frac{a^2}{2} f''(x) + \frac{a^3}{6} f'''(x) + \frac{a^4}{24} f^4(x) + O(a^5)\right] - 3f(x) - \left[f(x) - 2a f'(x) + 2a^2 f''(x) - \frac{4a^3}{3} f'''(x) + \frac{2a^4}{3} f^4 + O(a^5)\right]$$

$$= 4f(x) + 4a f'(x) + 2a^2 f''(x) + \frac{4}{6} a^3 f'''(x) + \frac{4}{24} a^4 f^4(x) + O(a^5) - 3f(x) - f(x) + 2a f'(x) - 2a^2 f''(x) + \frac{4}{3} a^3 f'''(x) - \frac{2}{3} a^4 f^4(x) + O(a^5)$$

$f(x)$: $(4-3-1) f(x) = 0$, $af'(x)$: $(4+2) af'(x) = 6 a f'(x)$

$a^2 f''(x)$: $(2-2) a^2 f''(x) = 0 a^2 f''(x) = 0$

$a^3 f'''(x)$: $\left(\frac{4}{6} + \frac{4}{3}\right) a^3 f'''(x) = \frac{6}{3} a^3 f'''(x) = 2a^3 f'''(x)$

$a^4 f^4(x)$: $\left(\frac{4}{24} - \frac{2}{3}\right) a^4 f^4(x) = \left(\frac{1}{6} - \frac{4}{6}\right) a^4 f^4(x) = -\frac{3}{6} a^4 f^4 x = -\frac{1}{2} a^4 f^4(x)$

$\therefore N = 6a f'(x) + 2a^3 f'''(x) - \frac{1}{2} a^4 f^4 x + O(a^5)$

Numerator$/6$

$$= f'(x) + \frac{1}{3} a^2 f'''(x) - \frac{1}{12} a^3 f^4(x) + O(a^4)$$

$$f'(x) = \frac{4f(x+a) - 3f(x) - f(x+2a)}{6a} - \frac{1}{3} a^2 f'''(x) + \frac{1}{12} a^3 f^4(x) + O(a^4)$$

$\therefore$ error $= \underbrace{-\frac{1}{3} a^2 f'''(x)}_{\text{leading}} + \frac{1}{12} a^3 f^4 x + O(a^4)$

$\therefore$ order of approximation $= \boxed{O(a^2)}$

5.1.18 in the end

5.8.20

$$f'''(x) = \frac{f(x-3a) - 6f(x-2a) + 12f(x-a) - 10f(x) + 3f(x+a)}{2a^3} + O(a^2)$$

Using Taylor expansions

$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f^4(x) + \frac{h^5}{120}f^5(x) + O(h^6)$

$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f^4(x) - \frac{h^5}{120}f^5(x) + O(h^6)$

$f(x-2h) = f(x) - 2hf'(x) + \frac{(2h)^2}{2}f''(x) - \frac{(2h)^3}{6}f'''(x) + \frac{(2h)^4}{24}f^4(x) - \frac{(2h)^5}{120}f^5(x)$
$+ O(h^6)$

$= f(x) - 2hf'(x) + 2h^2f''(x) - \frac{4}{3}h^3f'''(x) + \frac{2}{3}h^4f^4(x) -$
$\frac{4}{15}h^5f^5(x) + O(h^6)$

$f(x-3h) = f(x) - 3hf'(x) + \frac{(3h)^2}{2}f''(x) - \frac{(3h)^3}{6}f'''(x) + \frac{(3h)^4}{24}f^4(x) -$
$\frac{(3h)^5}{120}f^5(x) + O(h^6)$

$= f(x) - 3hf'(x) + \frac{9}{2}h^2f''(x) - \frac{9}{2}h^3f'''(x) + \frac{27}{8}h^4f^4(x)$
$- \frac{81}{40}h^5f^{(5)}(x) + O(h^6)$

$N = f(x-3h) - 6f(x-2h) + 12f(x-h) - 10f(x) + 3f(x+h)$

$f(x): 1 - 6(1) + 12(1) - 10 + 3(1) = 0$

$hf'(x): -3 - 6(-2) + 12(-1) - 10(0) + 3(1) = 0$

$h^2f''(x): \frac{9}{2} - 6(2) + 12(\frac{1}{2}) - 10(0) + 3(\frac{1}{2}) = 0$

$h^3f'''(x): -\frac{9}{2} - 6(-\frac{4}{3}) + 12(-\frac{1}{6}) - 10(0) + 3(\frac{1}{6}) = 2$

$h^4f^4(x) = \frac{27}{8} - 6(\frac{2}{3}) + 12(\frac{1}{24}) - 10(0) + 3(\frac{1}{24}) = 0$

$h^5f^5(x): -\frac{81}{40} - 6(-\frac{4}{15}) + 12(-\frac{1}{120}) - 10(0) + 3(\frac{1}{120})$

$= \frac{-81 + 64 - 4 + 1}{40} = -\frac{1}{2}$

Summing each: $2h^3f'''(x) - \frac{1}{2}h^5f^5(x) + O(h^6)$

dividing by $2h^3$

$= f'''(x) - \frac{1}{4}h^2f^5(x) + O(h^3)$

formula is thus: $\frac{f(x-3h) - 6f(x-2h) + 12f(x-h) - 10f(x) + 3f(x+h)}{2h^3} + \frac{1}{4}h^2f^5x + O(h^3)$

Since $\frac{1}{4}h^2f^5(x)$ is an $O(h^2)$ term, and $O(h^3)$ is also contained within $O(h^2)$, we can write the error term as $O(h^2)$

$\therefore \boxed{f'''(x) = \frac{f(x-3h) - 6f(x-2h) + 12f(x-h) - 10f(x) + 3f(x+h)}{2h^3} + O(h^2)}$

5.24

b) $\int_0^1 \frac{dx}{1+x^2}\,dx$

$\tan^{-1}x \Big|_0^1$

$\tan^{-1}x(1) - \tan^{-1}x(0)$

$= \frac{\pi}{4} - 0 = \frac{\pi}{4} = 0.7853981634$

one panel: m=1:

2 subintervals, so N=2. Step size is $h = \frac{b-a}{N} = \frac{1-0}{2} = \frac{1}{2}$

$x_0 = 0, x_1 = \frac{1}{2}, x_2 = 1$

formula thus is: $\int_a^b f(x)dx \approx \frac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right)$ [1 panel, 2 intervals]

$f(x_0) = f(0) = \frac{1}{1+0^2} = 1$

$f(x_1) = f(\tfrac{1}{2}) = \frac{1}{1+(\frac{1}{2})^2} = \frac{4}{5} = 0.8$

$f(x_2) = f(1) = \frac{1}{1+1^2} = 0.5$

$S_{m=1} = \frac{1/2}{3}\left(1 + 4(0.8) + 0.5\right) = \frac{1}{6}(1+3.2+0.5) = \frac{1}{6}(4.7) = 0.78333$

two panels: m=2

4 subintervals, N=4, step size is $h = \frac{b-a}{N} = \frac{1-0}{4} = \frac{1}{4}$

$x_0 = 0, x_1 = \frac{1}{4}, x_2 = \frac{1}{2}, x_3 = \frac{3}{4}$ and $x_4 = 1$

for m=2 ~~pa~~ (panels, 4 intervals) $\approx \frac{h}{3}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right)$

$f(x_0) = f(0) = 1$

$f(x_1) = f(\tfrac{1}{4}) = \frac{1}{\frac{17}{16}} \approx 0.9411764706$

$f(x_2) = f(\tfrac{1}{2}) = 0.8$ (from above)

$f(x_3) = f(\tfrac{3}{4}) = \frac{16}{25} = 0.64$

$f(x_4) = f(1) = 0.5$ (from above)

$S_{m=2} = \frac{1/4}{3}\left(1 + 4(0.9411764706) + 2(0.8) + 4(0.64) + 0.5\right) \approx 0.7853921569$

1) Four panels, $m=4$

For $m=4$, there are 8 subintervals, so $N=8$

$$h = \frac{b-a}{N} = \frac{1-0}{8} = \frac{1}{8}$$

$$x_0 = 0,\ x_1 = \frac{1}{8},\ x_2 = \frac{2}{8} = \frac{1}{4},\ x_3 = \frac{3}{8},\ x_4 = \frac{4}{8} = \frac{1}{2}$$

$$x_5 = \frac{5}{8},\ x_6 = \frac{6}{8} = \frac{3}{4},\ x_7 = \frac{7}{8},\ x_8 = 1$$

for $m=4$ (four panels, 8 intervals)

$$\int_a^b f(x)\,dx \approx \frac{h}{3}\big(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + 4f(x_5) + 2f(x_6) + 4f(x_7) + f(x_8)\big)$$

$f(0) = 1$

$$f(1/8) = \frac{1}{1+\left(\frac{1}{8}\right)^2} = \frac{1}{1+1/64} = \frac{64}{65} \approx 0.9846153846$$

$f(1/4) = 0.9411764706$

$f(3/8) = 0.8767123288$

$f(1/2) = 0.8$

$f(5/8) = 0.7191011236$

$f(3/4) = 0.64$

$f(7/8) = 0.5663716814$

$f(1) = 0.5$

$$S_{m=4} = \frac{1/8}{3}\big(1 + 4(0.9846153846) + 2(0.9411764706) + 4(0.8767123288) + 2(0.8) + 4(0.7191011236) + 2(0.64) + 4(0.5663716814) + 0.5\big)$$

≈ 0.7853981212

It gets closer and closer to 0.7853981634 as no. of panels increases

Error analysis:

exact value: 0.7853981634 ($\frac{\pi}{4}$)

for $m=1$ ($h=1/2$):

error $= |0.7853981634 - 0.7833333333| \approx 2.06 \times 10^{-3}$

for $m=2$ ($h=1/4$):

error $= |0.7853981634 - 0.7853921569| \approx 6.01 \times 10^{-6}$

for $m=4$ ($h=1/8$):

error $= |0.7853981634 - 0.7853981212| \approx 4.22 \times 10^{-8}$

5.2.6b

· declare the function

```
import numpy as np
f = lambda x: (np.exp(x) -1) / x
```

· getting the approximate "trusted" value using scipy:

```
integral1 = scipy.integrate.quadrature (f, 0,1) [0]
print (f" Trusted value: {integral1}")
```

Result:

integral1 = 0.7182818284590453

· one panel, $m=1$, $x=0.5$, $h=1$

integral2 = $1 \times f(0.5)$

Result:

integral 2 = 0.6487212707001282

error = 6.96 e-02

· $m=2$, $x=0.25, 0.75$, $h=0.5$:

integral 3 = $0.5 * (f(0.25) + f(0.75))$

Result:

integral3 = 0.701839845

error = 1.64e-02

· $m=4$, $x=0.125, 0.375, 0.625, 0.875$, $h=0.25$

integral4 = $0.25 \times (f(0.125) + f(0.375) + f(0.625) + f(0.875))$

Result:

integral4 = 0.714607

error = 3.67e-03

We see that as the no of panels increases, the approximation get better and better. (get closer to 0.7182818284590453)

5.3.26

$$\int_0^1 \frac{dx}{1+x^2}\, dx$$

This is the same integral as 5.2.46, we recall that the exact value of the integral we are trying to compute is $\frac{\pi}{4} = 0.7853981634$

code:

```
f = lamda x : 1/(1+x**2)
h1 = 1.0
h2 = 1.0/2.0
h3 = 1.0/4.0

R11 = (h1/2 * (f(0)+f(1))
R21 = (1/2) * R11 + h2 * f(1/2))
R31 = (1/2) * R21 + h3 * (f(1/4) + f(3/4))
R22 = (4* R21 - R11) / 3
R32 = (4 * R31 - R21) / 15
R33 = (16 * R32 - R22) / 15
```

R33 = 0.78552941176 5

R11 = 0.750000000000

R21 = 0.775000000000   R31 = 0.782794117647

R22 = 0.783333333333   R32 = 0.7853

R31 = 0.782794117647   R32 = 0.785392156863   R33 = 0.785529411765

errors

3.54×10^{-2}

1.04×10^{-2} 2.06×10^{-3}

2.60×10^{-3} 6.01×10^{-6} 1.31×10^{-4}

We note that R_{22} is exactly Simpson with n=1 (2 subintervals) as computed in 5.2.45, and R_{32} is exactly Simpsons rule with n=2 (4 subintervals as computed in 5.2.45. This was expected

5.1.18

Goal: To find the upper bound for the machine approximation error, $E(h)$, of the two-point forward-difference formula and the optimal step size h that minimizes that error.

Total error function

- Truncation error: bounded by $\frac{hM}{2}$, where M is the maximum absolute value of the second derivative of $|f''(x)|$
- Rounding error: Bound by $\frac{2\varepsilon}{h}$

$$\therefore E(h) = \frac{hM}{2} + \frac{2\varepsilon}{h}$$

minimizing error,

$$E'(h) = \frac{d}{dh}\left(\frac{hM}{2} + \frac{2\varepsilon}{h}\right) = \frac{M}{2} - \frac{2\varepsilon}{h^2}$$

$$E'(0) = \frac{M}{2} - \frac{2\varepsilon}{h^2} \rightarrow h^2 = \frac{4\varepsilon}{M}$$

$$h_{min} = \sqrt{\frac{4\varepsilon}{M}} = 2\sqrt{\frac{\varepsilon}{M}}$$

Upper bound for $E(h)$

$$E(h_{min}) = \frac{M}{2}\left(2\sqrt{\frac{\varepsilon}{M}}\right) + \frac{2\varepsilon}{2\sqrt{\frac{\varepsilon}{M}}}$$

$$= \sqrt{M\varepsilon} + \sqrt{M\varepsilon}$$

$$\therefore \boxed{E(h) = 2\sqrt{M\varepsilon}}$$