3/25

Reality Check #2

1) Prove $f''''(x) = \dfrac{f(x-2h) - 4f(x-h) + 6f(x) - 4f(x+h) + f(x+2h)}{h^4} + O(h^2)$

Taylor Expansion

$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f^{4}(x) + \frac{h^5}{120}(x) + \frac{h^6}{720}f^{6}(x) + \dots$

$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f'''(x) + \frac{h^4}{24}f(x) - \frac{h^5}{120}f^{5}(x) + \frac{h^6}{720}f^{6}(x) - \dots$

$f(x+2h) = f(x) + 2hf'(x) + \frac{4h^2}{2}f''(x) + \frac{8h^3}{6}f'''(x) + \frac{16h^4}{24}f^{4}(x) + \frac{32h^5}{120}f^{5}(x)$
$+ \frac{64h^6}{720}f^{6}x + \dots$

$f(x-2h) = f(x) - 2hf'(x) + \frac{4h^2}{2}f''(x) - \frac{8h^3}{6}f'''(x) + \frac{16h^4}{24}f^{4}(x) - \frac{32h^5}{120}f^{5}(x)$
$+ \frac{64h^6}{720}f^{6}(x) - \dots$

Assembling numerator

$f(x)$ terms: $(1-4+6-4+1)\,f(x) = 0$

$f'(x)$ terms: $(-2h - 4(-h) - 4(h) + 2h)\,f'(x) = (-2+4-4+2)hf'(x) = 0$

$f''(x)$ terms: $\left(\frac{4h^2}{2} - 4\frac{h^2}{2} - 4\frac{h^2}{2} + \frac{4h^2}{2}\right) f''(x) = (2-2-2+2)h^2 f''(x) = 0$

$f'''(x)$ terms: $\left(\frac{-8h^3}{6} - 4\left(\frac{-h^3}{6}\right) - 4\left(\frac{h^3}{6}\right) + \frac{8h^3}{6}\right) f'''(x)$

$= \left(\frac{-4}{3} + \frac{2}{3} - \frac{2}{3} + \frac{4}{3}\right) h^3 f'''(x) = 0$

$f^{4}(x)$ terms: $\left(\frac{16h^4}{24} - 4\frac{h^4}{24} + 6(0) - 4\frac{h^4}{24} + \frac{16h^4}{24}\right) f^{4}(x) =$

$= \left(\frac{2}{3} - \frac{1}{6} - \frac{1}{6} + \frac{2}{3}\right) h^4 f^{4}(x) = \left(\frac{16-4-4+16}{24}\right) h^4 f^{4}(x)$

$= h^4 f^{4}(x)$

$f^{5}(x)$ terms: $\left(\frac{-32h^5}{120} - 4\left(\frac{-h^5}{120}\right) - 4\left(\frac{h^5}{120}\right) + \frac{32h^5}{120}\right) f^{5}(x)$

$= (-32+4-4+32)\frac{h^5}{120} f^{5}(x) = 0$

$f^{6}(x)$ terms: $\left(\frac{64h^6}{720} - \frac{4h^6}{720} - \frac{4h^6}{720} + \frac{64h^6}{720}\right) f^{6}(x) = \left(\frac{64-4-4+64}{720}\right) h^6 f^{6}(x)$

$= \frac{120}{720} h^6 f^{6}(x) = \frac{1}{6} h^6 f^{6}(x)$

Combining these terms gives:

$$\rightarrow h^4 f''''(x) + \frac{1}{6} h^6 f^6(x) + \cdots$$

Dividing by h^4:

$$= f(x-2h) - 4f(x-h) + 6f(x) - 4f(x+h) + f(x+2h) / h^4$$

$$= f''''(x) + \frac{1}{6} h^2 f^6(x) + \cdots$$

Since the error term is dominated by the lowest power of h, which is h^2, we write:

$$\boxed{f''''(x) = \frac{f(x-2h) - 4f(x-h) + 6f(x) - 4f(x+h) + f(x+2h)}{h^4} + O(h^2)}$$

4) Proof of Sinusoidal Load Solution

The total force is $f(x) = f_{const} + S(x)$, where $f_{const} = -480 wdg$ and $S(x) = -pg \sin\left(\frac{\pi}{L} x\right)$.

proposed solution is: $y(x) = \underbrace{\frac{f_{const}}{24EI} x^2 (x^2 - 4Lx + 6L^2)}_{y_c(x)} \underbrace{- \frac{pgL}{EI\pi}\left(\frac{L^3}{\pi^3}\sin\left(\frac{\pi}{L}x\right) - \frac{x^3}{6} + \frac{L}{2}x^2 - \frac{L^2}{\pi^2}x\right)}_{y_s(x)}$

Part 1: Verify it satisfies the diff eq. $EIy''''(x) = f(x)$

from constant load case, we know $EIy_c''''(x) = f_{const}$.

$y_s^4(x)$:

$$y_s'(x) = -\frac{pgL}{EI\pi}\left[\frac{L^2}{\pi^2}\cos\left(\frac{\pi}{L}x\right) - \frac{x^2}{2} + Lx - \frac{L^2}{\pi^2}\right]$$

$$y_s''(x) = -\frac{pgL}{EI\pi}\left[-\frac{L}{\pi}\sin\left(\frac{\pi}{L}x\right) - x + L\right]$$

$$y_s'''(x) = -\frac{pgL}{EI\pi}\left[-\cos\left(\frac{\pi}{L}x\right) - 1\right]$$

$$y_s''''(x) = -\frac{pgL}{EI\pi}\left[\frac{\pi}{L}\sin\left(\frac{\pi}{L}x\right)\right] = -\frac{pg}{EI}\sin\left(\frac{\pi}{L}x\right)$$

$$\therefore EIy_s^4(x) = -pg\sin\left(\frac{\pi}{L}x\right) = S(x).$$

Combining the parts:

$$EIy^{4}(x) = E(y_c + y_s)^{4} = EI y_c^{4} + EI y_s^{4}$$
$$= f_{const} + s(x) = f(x).$$

$\therefore$ the solution satisfies the differential equation.

Part 2: verify the clamped-free boundary conditions

$y(0)=0, \quad y'(0)=0, \quad y''(L)=0, \quad y'''(L)=0$

$y(0)$

$y_c(0)=0$

$y_s(0) = \frac{-\rho g L}{EI\pi}\left[\frac{L^3}{\pi^3}\sin(0) - 0 + 0 - 0\right] = 0$

$\therefore y(0) = 0$

$y'(0)$:

$y_c'(x) = \frac{f_{const}}{24EI}(4x^3 - 12Lx^2 + 12L^2x)$, so $y_c'(0) = 0$

$y_s'(0) = \frac{-\rho g L}{EI\pi}\left[\frac{L^3}{\pi^2}\cos(0) - 0 + 0 - \frac{L^2}{\pi^2}\right] = \frac{-\rho g L}{EI\pi}\left[\frac{L^2}{\pi^2} - \frac{L^2}{\pi^2}\right] = 0$

$\therefore y'(0) = 0$

$y''(L)$:

$y_c''(x) = \frac{f_{const}}{24EI}(12x^2 - 24Lx + 12L^2)$, so $y_c''(L) = \frac{f_{const}}{24EI}(12L^2 - 24L^2 + 12L^2)$

$= 0$

$y_s''(L) = \frac{-\rho g L}{EI\pi}\left[-\frac{L}{\pi}\sin(\pi) - L + L\right] = \frac{-\rho g L}{EI\pi}\left[0 - L + L\right] = 0$

$\therefore y''(L) = 0$

$y'''(L)$:

$y_c'''(x) = \frac{f_{const}}{24EI}(24x - 24L)$, so $y_c'''(L) = \frac{f_{const}}{24EI}(24L - 24L) = 0$

$$y_s'''(L) = -\frac{\rho g L}{EI\pi}\left[-\cos(\pi) - 1\right] = -\frac{\rho g L}{EI\pi}\left[-(-1) - 1\right] = -\frac{\rho g L}{EI\pi}\left[1 - 1\right] = 0$$

$\therefore\ y'''(L) = 0$

$\therefore$ all boundary conditions are satisfied.