Homework 5

A1 ~~###~~.1 4.1.1

a) $\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$

$A^TA = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1^2+0^2+2^2 & 1\cdot2+0\cdot1+2\cdot1 \\ 2\cdot1+1\cdot0+1\cdot2 & 2^2+1^2+1^2 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 4 & 6 \end{bmatrix}$

$A^Tb = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1\cdot3+0\cdot1+2\cdot1 \\ 2\cdot3+1\cdot1+1\cdot1 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$

$\begin{bmatrix} 5 & 4 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$

$5x_1 + 4x_2 = 5 \quad \times 3$

$4x_1 + 6x_2 = 8 \quad \times 2$

$\begin{cases} 15x_1 + 12x_2 = 15 \\ 8x_1 + 12x_2 = 16 \end{cases}$

$7x_1 = -1$

$x = -1/7$

$5x_1 + 4x_2 = 5$

$5\left(-\frac{1}{7}\right) + 4x_2 = 5 \Rightarrow -\frac{5}{7} + 4x_2 = 5 \Rightarrow x_2 = \frac{10}{7}$

$\therefore x = \begin{bmatrix} -1/7 \\ 10/7 \end{bmatrix}$

residual $r = b - Ax$

$$Ax = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1/7 \\ 10/7 \end{bmatrix} = \begin{bmatrix} 1\cdot(-1/7) + 2\left(\frac{10}{7}\right) \\ 0 + 1\cdot\frac{10}{7} \\ 2\cdot\left(\frac{-1}{7}\right) + 1\cdot\left(\frac{10}{7}\right) \end{bmatrix} = \begin{bmatrix} \frac{19}{7} \\ 10/7 \\ 8/7 \end{bmatrix}$$

$$r = b - Ax = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 19/7 \\ 10/7 \\ 8/7 \end{bmatrix} = \begin{bmatrix} 2/7 \\ -3/7 \\ -1/7 \end{bmatrix}$$

2-norm error

$$\|r\|_2 = \sqrt{\left(\frac{2}{7}\right)^2 + \left(\frac{-3}{7}\right)^2 + \left(\frac{-1}{7}\right)^2} = \frac{1}{7}\sqrt{4+9+1} = \frac{\sqrt{14}}{7}$$

$$\boxed{\therefore x = \begin{bmatrix} -1/7 \\ 10/7 \end{bmatrix}, \quad \|r\|_2 = \frac{\sqrt{14}}{7}}$$

4.11

c)

$$\begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 2 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1^2+1^2+2^2+2^2 & 1\cdot2+1\cdot1+2\cdot1+2\cdot2 \\ 2^2+1^2+1^2+2^2 & \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 9 \\ 9 & 10 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 2 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1\cdot3+1\cdot3+2\cdot3+2\cdot2 \\ 2\cdot3+1\cdot3+1\cdot3+2\cdot2 \end{bmatrix} = \begin{bmatrix} 16 \\ 16 \end{bmatrix}$$

$$\begin{bmatrix} 10 & 9 \\ 9 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 16 \\ 16 \end{bmatrix}$$

$10x_1 + 9x_2 = 16$

$9x_1 + 10x_2 = 16$

$$\begin{cases} 100x_1 + 90x_2 = 160 \\ 81x_1 + 90x_2 = 144 \end{cases}$$

$(100-81)x_1 = 16$

$x_1 = \frac{16}{19}$

plug into $10x_1 + 9x_2 = 16$

$10 \cdot \left(\frac{16}{19}\right) + 9x_2 = 16$

$x_2 = \frac{16}{19}$ $\rightarrow$ $x = \begin{bmatrix} 16/19 \\ 16/19 \end{bmatrix}$

residual $r = b - Ax$

$$Ax = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 16/19 \\ 16/19 \end{bmatrix} = \begin{bmatrix} 48/19 \\ 32/19 \\ 48/19 \\ 64/19 \end{bmatrix}$$

$$r = b - Ax = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 2 \end{bmatrix} - \begin{bmatrix} 48/19 \\ 32/19 \\ 48/19 \\ 64/19 \end{bmatrix} = \begin{bmatrix} 9/19 \\ 25/19 \\ 9/19 \\ -26/19 \end{bmatrix}$$

$$\|r\|_2 = \sqrt{\left(\frac{9}{19}\right)^2 + \left(\frac{25}{19}\right)^2 + \left(\frac{9}{19}\right)^2 + \left(\frac{-26}{19}\right)^2} = \frac{\sqrt{1463}}{19}$$

$$\boxed{\therefore x = \begin{bmatrix} 16/19 \\ 16/19 \end{bmatrix}, \quad \|r\|_2 = \frac{\sqrt{1463}}{19}}$$

4.1.2

a) $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

$A^TA = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 3 & 3 & 2 \\ 3 & 6 & 3 \\ 2 & 3 & 3 \end{bmatrix}$

$A^Tb = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \\ 4 \end{bmatrix}$

$= \begin{bmatrix} 9 \\ 12 \\ 9 \end{bmatrix}$

$$\begin{bmatrix} 3 & 3 & 2 \\ 3 & 6 & 3 \\ 2 & 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \\ 9 \end{bmatrix}$$

$x_1 \rightarrow \quad 3x_1 + 3x_2 + 2x_3 = 9 \rightarrow x_1 + x_2 + \frac{2x_3}{3} = 9$

$x_2 \rightarrow \quad 3x_1 + 6x_2 + 3x_3 = 12 \Rightarrow x_1 + 2x_2 + x_3 = 4$

$x_1 - x_2 = (x_1 + 2x_2 + x_3) - (x_1 + x_2 + \frac{2}{3}x_3) = 4 - 3$

$x_2 + \frac{1}{3}x_3 = 1$

$x_2 = 1 - \frac{1}{3}x_3$

$2x_1 + 3x_2 + 3x_3 = 9$

$2x_1 + 3(1 - \frac{1}{3}x_3) + 3x_3 = 9$

$2x_1 + 3 - x_3 + 3x_3 = 9$

$2x_1 + 3 + 2x_3 = 9 \rightarrow 2x_1 = 6 - 2x_3 \Rightarrow x_1 = 3 - x_3$

$\therefore x_2 = 1 - \frac{1}{3}x_3$

$$x = \begin{bmatrix} 3 - x_3 \\ 1 - \frac{1}{3}x_3 \\ x_3 \end{bmatrix}$$

let's pick $x_3 = 0$ for minimum norm solution

residual $r = b - Ax$

$$Ax = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3+1 \\ 1+0 \\ 3+2 \\ 3+0 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 5 \\ 3 \end{bmatrix}$$

$$r = b - Ax = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 4 \\ 1 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

$$RMSE = \sqrt{\frac{(-2)^2 + (1)^2 + (-2)^2 + 1^2}{4}} = \sqrt{\frac{4+1+4+1}{4}} = \sqrt{\frac{10}{4}} = \frac{\sqrt{10}}{2}$$

$$\therefore x = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \quad \boxed{RMSE = \frac{\sqrt{10}}{2}}$$

4.1.8b

b) $(1,2), (3,2), (4,1), (6,3)$

$y = \alpha + \beta x$

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 4 \\ 1 & 6 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 3 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 4 & 6 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 1 & 4 \\ 1 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 14 \\ 14 & 56 \end{bmatrix}$$

$(1,1)$: $1+1+1+1=4$

$(1,2)$: $1\cdot 1+1\cdot 3+1\cdot 4+1\cdot 6=14$

$(2,2)$: $1^2+3^2+4^2+6^2=62$

$$\therefore A^TA=\begin{bmatrix}4 & 14\\ 14 & 62\end{bmatrix}$$

$$A^TB=\begin{bmatrix}1&1&1&1\\1&3&4&6\end{bmatrix}\begin{bmatrix}2\\2\\1\\3\end{bmatrix}=\begin{bmatrix}8\\30\end{bmatrix}$$

$2+2+1+3=8$

$1(2)+3(2)+4(1)+6(3)=2+6+4+18=30$

$$\therefore A^Tb=\begin{bmatrix}8\\30\end{bmatrix}$$

$$\begin{bmatrix}4 & 14\\ 14 & 62\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix}=\begin{bmatrix}8\\30\end{bmatrix}$$

$$\left.\begin{array}{l}4\alpha+14\beta=8 \quad \cdot 14\\ 14\alpha+62\beta=30 \quad \cdot 4\end{array}\right\}\Rightarrow \begin{array}{l}56\alpha+196\beta=112\\ 56\alpha+248\beta=120\end{array}$$

$(56\alpha+248\beta)-(56\alpha+196\beta)=120-112$

$52\beta=8$

$\beta=\frac{2}{13}$

$4\alpha=8-\frac{28}{13}=\frac{104-28}{13}=\frac{76}{13}$

$\alpha=\frac{19}{13}$

$\therefore \alpha=\frac{19}{13},\ \beta=\frac{2}{13}$

$$\therefore \boxed{y=\frac{19}{13}+\frac{2}{13}x}$$

residuals

for $x=1$: $\hat{y}=\frac{19}{13}+\frac{2}{13}(1)=\frac{21}{13}$; $x=3$: $\hat{y}=\frac{19}{13}+\frac{2}{13}(3)=\frac{25}{13}$

residual: $2-\frac{21}{13}=5/13$; residual: $1-\frac{27}{13}=\frac{-14}{13}$

$x=4$: $\hat{y} = \frac{19}{13} + \frac{2}{13}(4) = \frac{27}{13}$ $\quad$ $x=6$: $\hat{y} = \frac{19}{13} + \frac{2}{13}(6) = \frac{31}{13}$

residual $1 - \frac{27}{13} = \frac{-14}{13}$ $\quad$ residual $3 - \frac{31}{13} = \frac{8}{13}$

$$r = \begin{bmatrix} 5/13 \\ 1/13 \\ -14/13 \\ 8/13 \end{bmatrix}$$

RMSE

$$\|r\|_2^2 = \left(\frac{5}{13}\right)^2 + \left(\frac{1}{13}\right)^2 + \left(\frac{-14}{13}\right)^2 + \left(\frac{8}{13}\right)^2$$

$$= \sqrt{\frac{286}{169}} = \frac{\sqrt{286}}{13} \approx 0.649$$

$\therefore$ $\boxed{y = \frac{19}{13} + \frac{2}{13}x, \quad RMSE = \frac{\sqrt{236}}{26} \approx 0.649}$

4.19.b

b) $(1,2), (3,2), (4,1), (6,3)$

$y = ax^2 + bx + c$

$$A = \begin{bmatrix} 1^2 & 1 & 1 \\ 3^2 & 3 & 1 \\ 4^2 & 4 & 1 \\ 6^2 & 6 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \\ 36 & 6 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 3 \end{bmatrix}$$

Using python we get:

$$x = \begin{bmatrix} -0.121 \\ 1.030 \\ 0.758 \end{bmatrix}$$

$\therefore$ $\boxed{y = -0.121x^2 + 1.030x + 0.758}$

Using python,

$RMSE = \sqrt{\frac{1}{4}\left[(2-\hat{y}_1)^2 + (2-\hat{y}_2)^2 + (1-\hat{y}_3)^2 + (3-\hat{y}_4)^2\right]} = \boxed{0.399}$

From 4.18.b, the best line fit had $RMSE = 0.694$ $\therefore$ $\boxed{\text{parabola is better}}$ $\quad$ $0.694 > 0.399$

CP 4.2.6
CP 4.2.7 } In Collab Notebook.
CP 4.2.8
CP 4.2.9

Levenberg-Marquardt.

4.3.2

b) Classical Gram-Schmidt Orthogonalization

$$\begin{bmatrix} -4 & -4 \\ -2 & 7 \\ 4 & -5 \end{bmatrix}$$

$$a_0 = \begin{bmatrix} -4 \\ -2 \\ 4 \end{bmatrix} \qquad a_1 = \begin{bmatrix} -4 \\ 7 \\ -5 \end{bmatrix}$$

Let $w_0 = a_0$

$$r_{00} = \|w_0\|_2 = \sqrt{(-4)^2 + (-2)^2 + 4^2}$$
$$= \sqrt{16+4+16}$$
$$= 6$$

$$q_0 = \frac{w_0}{r_{00}} = \frac{1}{6}\begin{bmatrix} -4 \\ -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix}$$

$$r_{01} = q_0^T a_1 = \begin{bmatrix} -2/3 & -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} -4 \\ 7 \\ -5 \end{bmatrix}$$

$$= \left(\frac{-2}{3}\right)(-4) + \left(\frac{-1}{3}\right)(7) + \left(\frac{2}{3}\right)(-5)$$

$$= \frac{-9}{3} = -3$$

$$w_1 = a_1 - q_0 r_{01}$$

$$q_0 r_{01} = -3 \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

$$w_1 = a_1 - q_0 r_{01} = (-4 \;\; 7 \;\; -5) - (2 \;\; 1 \;\; -2) = (-6 \;\; 6 \;\; 3)$$

$r_{11} = ||w_1||_2 = \sqrt{(-6)^2 + 6^2 + (-3)^2}$
$= \sqrt{81} = 9$

$q_1 = \frac{w_1}{r_{11}} = \frac{1}{9}\begin{bmatrix} -6 \\ 6 \\ -3 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 2/3 \\ -1/3 \end{bmatrix}$

$\therefore \boxed{Q = \begin{bmatrix} -2/3 & -2/3 \\ -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix} \quad R = \begin{bmatrix} 6 & -3 \\ 0 & 9 \end{bmatrix}}$

$\therefore A = QR$, $Q^TQ = I$, R is upper triangular.

4.3.7

b). $\begin{bmatrix} -4 & -4 \\ -2 & 7 \\ 4 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ 0 \end{bmatrix}$

from 4.3.2b) found:

$Q = \begin{bmatrix} -2/3 & -2/3 \\ -1/3 & 2/3 \\ 2/3 & -1/3 \end{bmatrix}, \quad R = \begin{bmatrix} 6 & -3 \\ 0 & 9 \end{bmatrix}$

$Q^Tb = \begin{bmatrix} -2/3 & -1/3 & 2/3 \\ -2/3 & 2/3 & -1/3 \end{bmatrix}\begin{bmatrix} 3 \\ 9 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} \frac{-2}{3}(3) + \frac{-1}{3}(9) + \frac{2}{3}(0) \\ \frac{-2}{3}(3) + \frac{2}{3}(9) + \frac{-1}{3}(0) \end{bmatrix} = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$

$Rx = Q^Tb$

$\begin{bmatrix} 6 & -3 \\ 0 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -5 \\ 4 \end{bmatrix}$

$9x_2 = 4$

$x_2 = 4/9$

$6x_1 - 3x_2 = -5$

$6x_1 - \cancel{3}^{1}\left(\frac{4}{\cancel{9}_3}\right) = -5 \rightarrow 3(6x_1) - 3\left(\frac{4}{3}\right) = {}^{3}(-5) \rightarrow 18x_1 - 4 = -15$

$18x_1 = -11$

$x_1 = \frac{-11}{18}$

$$\boxed{\therefore x = \begin{bmatrix} -11/18 \\ 4/9 \end{bmatrix} \quad \text{or } x \approx \begin{bmatrix} -0.6111 \\ 0.4444 \end{bmatrix}}$$

Residual Orthogonality Check

$$A = \begin{bmatrix} -4 & -4 \\ -2 & 7 \\ 4 & -5 \end{bmatrix}, \quad x = \begin{bmatrix} -11/18 \\ 4/9 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 9 \\ 0 \end{bmatrix}$$

check whether: $A^T(b - Ax) = 0$

Computing Ax

$$(-4)\left(\frac{-11}{18}\right) + -4\left(\frac{4}{9}\right) = \frac{44}{18} - \frac{16}{9} = \frac{44}{18} - \frac{16}{9} = \frac{2}{3}$$

$$-2\left(\frac{-11}{18}\right) + 7\left(\frac{4}{9}\right) = \frac{22}{18} + \frac{28}{9} = \frac{78}{18} = \frac{13}{3}$$

$$4\left(\frac{-11}{18}\right) + -5\left(\frac{4}{9}\right) = \frac{-44}{18} - \frac{40}{18} = \frac{-84}{18} = \frac{-14}{3}$$

Hence $Ax = \begin{bmatrix} 2/3 \\ 13/3 \\ -14/3 \end{bmatrix}$

$$A^T r = \begin{bmatrix} -4 & -2 & 4 \\ -4 & 7 & -5 \end{bmatrix} \begin{bmatrix} 7/3 \\ 14/3 \\ 14/3 \end{bmatrix}$$

$$-4\left(\frac{2}{3}\right) + -2\left(\frac{14}{3}\right) + 4\left(\frac{14}{3}\right) = 0$$

$$-4\left(\frac{7}{3}\right) + 7\left(\frac{14}{3}\right) - 5\left(\frac{14}{3}\right) = 0$$

$\therefore A^T r = 0$

$\therefore A^T(b - Ax) = 0$

So, the residual is orthogonal to the columns of A

4.3.8

b) $\begin{bmatrix} 2 & 4 \\ 0 & -1 \\ 2 & -1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 2 \\ 1 \end{bmatrix}$

$a_0 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad a_1 = \begin{bmatrix} 4 \\ -1 \\ -1 \\ 3 \end{bmatrix}$

$\|a_0\|_2 = \sqrt{2^2+0^2+2^2+1^2} = \sqrt{9} = 3$

$q_0 = \frac{a_0}{3} = \begin{bmatrix} 2/3 \\ 0 \\ 2/3 \\ 1/3 \end{bmatrix}$

$r_{01} = q_0^T a_1$

$= \left(\frac{2}{3}\right)(4) + (0)(-1) + \left(\frac{2}{3}\right)(-1) + \left(\frac{1}{3}\right)(3)$

$= \frac{8}{3} + 0 - \frac{2}{3} + 1 = \frac{8-2+3}{3} = 3$

$w_1 = a_1 - q_0 r_{01}$ $\qquad w_1 = a_1 - q_0 \cdot 3$

$q_0 \cdot 3 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}$ $\qquad = \begin{bmatrix} 4 \\ -1 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -3 \\ 2 \end{bmatrix}$

$r_{11} = \|w_1\|_2 = \sqrt{4+1+9+4}$

$= \sqrt{18} = 3\sqrt{2}$

$q_1 = \frac{w_1}{r_{11}} = \frac{1}{3\sqrt{2}} \begin{bmatrix} 2 \\ -1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2/3\sqrt{2} \\ -1/3\sqrt{2} \\ -1/\sqrt{2} \\ 2/3\sqrt{2} \end{bmatrix}$

$$\therefore\ Q = \begin{bmatrix} 2/3 & 2/3\sqrt{2} \\ 0 & -1/3\sqrt{2} \\ 2/3 & -1/\sqrt{2} \\ 1/3 & 2/3\sqrt{2} \end{bmatrix} \qquad R = \begin{bmatrix} 3 & 3 \\ 0 & 3\sqrt{2} \end{bmatrix}$$

Hence $A = QR$

Least Squares $Ax = b$

$Q^T b$

$$q_0^T b = \frac{2}{3}(-1) + 0(3) + \frac{2}{3}(2) + \frac{1}{3}(1) = 1$$

$$q_1^T b = \frac{2}{3\sqrt{2}}(-1) + \left(\frac{-1}{3\sqrt{2}}\right)(3) + \left(\frac{-1}{\sqrt{2}}\right)(2) + \frac{2}{3\sqrt{2}}(1) = \frac{-9}{3\sqrt{2}} = \frac{-3}{\sqrt{2}}$$

$$Q^T b = \begin{bmatrix} 1 \\ \frac{-3}{\sqrt{2}} \end{bmatrix}$$

$Rx = Q^T b$

$$\begin{bmatrix} 3 & 3 \\ 0 & 3\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{-3}{\sqrt{2}} \end{bmatrix}$$

$$3\sqrt{2}\,x_2 = -3/\sqrt{2}$$
$$x_2 = -3/\sqrt{2} \cdot 1/3\sqrt{2}$$
$$= -1/2$$

$$3x_1 + 3x_2 = 1$$
$x_2 = \frac{-1}{2}$ $$3x_1 + 3\left(\frac{-1}{2}\right) = 1$$
$$3x_1 = \frac{5}{2}$$
$$x_1 = \frac{5}{6}$$

$$\boxed{\therefore\ x = \begin{bmatrix} 5/6 \\ -1/2 \end{bmatrix} \qquad x \approx \begin{bmatrix} 0.8333 \\ -0.5 \end{bmatrix}}$$