Problem Set 1 Submission

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Problem 1. Prove that $log_4 6$ is irrational.

Proof. By contradiction, let us suppose that $log_4 6$ is rational.

$$\begin{aligned}
log_4 6 &= \frac{log_3 6}{log_3 4} && \text{(base change rule)} \\
&= \frac{log_3 3 + log_3 2}{log_3 2 + log_3 2} && \text{(product rule)} \\
&= \frac{1 + log_3 2}{2 log_3 2} \\
&= \frac{1}{2 log_3 2} + \frac{log_3 2}{2 log_3 2} \\
&= \frac{1}{2} \times \frac{1}{log_3 2} + \frac{1}{2} \\
&= \frac{1}{2} \times log_2 3 + \frac{1}{2}. && \text{(base switch rule)}
\end{aligned}$$

Since $\frac{1}{2}$ is rational, the last expression requires that $log_2 3$ be rational for the entire expression to be rational.

$$\begin{aligned}
log_2 3 &= \frac{m}{n} && \text{(m and n are integers, n} \neq 0\text{)} \\
3 &= 2^{\frac{m}{n}} && \text{(2 raised to power of each side)} \\
3^n &= 2^m. && \text{(each side raised to power of n)}
\end{aligned}$$

However, because 3 and 2 are prime with respect to each other (no common factors), the finally equality cannot hold, leading to a contradiction. $log_2 3$ is therefor irrational, and because it is a component of $log_4 6$, $log_4 6$ is also irrational.

□

Problem 2. Prove that $n \leq 3^{n/3}$ for all nonnegative integers.

Proof. **Lemma 1.** P is true for {0, 1, 2, 3, 4}

$$\begin{aligned}
n \leq 3^{n/3} &= n^3 \leq 3^n && \text{(cube both sides)} \\
0^3 \leq 3^0 &= 0 \leq 1 && \text{(n = 0, T)} \\
1^3 \leq 3^1 &= 1 \leq 3 && \text{(n = 1, T)} \\
2^3 \leq 3^2 &= 8 \leq 9 && \text{(n = 2, T)} \\
3^3 \leq 3^3 &= 27 \leq 27 && \text{(n = 3, T)} \\
4^3 \leq 3^4 &= 64 \leq 81 && \text{(n = 4, T)}
\end{aligned}$$

Theorem . By contradiction. Assume that P is false, i.e. that $n^3 > 3^n$ for some $n > 4$. Counterexamples to P are collected in nonempty set C:

$C ::= \{n \in \mathbb{N} | n^3 > 3^n \wedge n > 4\}$

Because C is a nonempty set of nonnegative integers, by the Well Ordering Principle there is a minimum element that we'll call r.

By assumption, $(r-1)^3 \leq 3^{r-1}$ is true, because $r - 1 < r$ and r is the minimum element in the set C.

$$\begin{aligned}
(r-1)^3 \leq 3^{r-1} &\equiv 3(r-1)^3 \leq 3^r && \text{(multiply both sides by 3)} \\
&\equiv 3^r - 3(r-1)^3 \geq 0 && \text{(move terms to left side)} \\
&\equiv 3^r - 3r^3 - 9r^2 + 9r - 3 \geq 0 && \text{(expand terms)} \\
&\equiv r^3 - 3r^3 - 9r^2 + 9r - 3 \geq 0 && \text{(by assumption that } r^3 > 3^r) \\
&\equiv -2r^3 - 9r^2 \geq -9r + 3 \\
&\equiv 2r^3 + 9r^2 \leq 9r - 3
\end{aligned}$$

Because r is a positive integer ≥ 5, the cubic term is *at least* $\frac{50}{9}\times$ larger than the 9r term in the right hand side, and the quadratic term in the left hand side is at least $5\times$ larger. This means the inequality cannot hold, and therefore the assumption that $n^3 > 3^n$ for $n > 4$ leads to a contradiction.

□

Problem 3.

(a) Verify by truth table that (P IMPLIES Q) OR (Q IMPLIES P)

P	Q	P $\implies$ Q	$\vee$	Q $\implies$ P
T	T	T	***T***	T
T	F	F	***T***	T
F	T	T	***T***	F
F	F	T	***T***	T

(b) Let P and Q be propositional formulas. Describe a single formula, R, using only ANDs, ORs, NOTs, and copies of P and Q, such that R is valid iff P and Q are equivalent.

$R = (P \wedge Q) \vee (\neg P \wedge \neg Q)$

(c) Explain why P is valid iff NOT(P) is not satisfiable.

In order for NOT(P) to be satisfiable, there must be some assignment of truth values to P that results in P being false, which in direct conflict with the meaning of "valid", (i.e.) a propositional formula is valid iff it *always* evaluates to true.

(d) A set of propositional formulas P1,. . . ,Pk is consistent iff there is an environment in which they are all true. Write a formula, S, so that the set P1,. . . ,Pk is not consistent iff S is valid.

$P \vee \neg P$

Problem 4.

(a) A 1-bit add1 module just has input a_0. Write propositional formulas for its outputs c and p_0.

$$c = a_0 \wedge 1$$
$$p_0 = a_0 \oplus 1$$

(b) Explain how to build an $(n + 1)$-bit parallel half-adder from an $(n + 1)$-bit add1 module by writing a propositional formula for the half-adder output, o_i, using only the variables a_i, p_i, and b.

(c) Write a formula for the carry, c, in terms of c_1 and c_2.

(d) Complete the specification of the double-size module by writing propositional formulas for the remaining outputs, p_i, for . The formula for pi should only involve the variables a_i, $r_{i-(n+1)}$, and c_i.

(e) Parallel half-adders are exponentially faster than ripple-carry half-adders. Confirm this by determining the largest number of propositional operations required to compute any one output bit of an n-bit add module. (You may assume n is a power of 2.)