1 The diet problem

Average person:

needs d_i units of nutrient i, $i \in \{1, ..., 13\}$

One unit of food $j \in \{1, ..., 77\}$ contains N_{ij} in nutrient i.

We are looking for a diet that is a combination of foods such that we meet all the nutrient requirements.

Look for $q_j \geq 0$ such that $\sum_j q_j N_{ij} \geq d_i$ for all i that is in vector notation

$$Nq \geq d$$

Let c_j be the cost of one unit of food j, we are looking for

$$\begin{aligned} V_P = \min_{(q_j) \geq 0} \quad & \sum_j q_j c_j \\ s.t. \quad & \sum_j q_j N_{ij} \geq d_i \;\; [y_i \geq 0] \end{aligned}$$

or in vector notations

$$\begin{aligned} V_P = \min_{(q_j) \geq 0} \quad & q^\top c \\ s.t. \quad & Nq \geq d. \end{aligned}$$

This is the optimal diet problem.

Duality: we rewrite the problem as

$$V_P = \min_{(q_j) \geq 0} q^\top c + \sum_i L\left(d_i - \sum_j q_j N_{ij}\right)$$

where $L(z) = 0$ if $z \leq 0$ and $+\infty$ otherwise. Claim: we can write L as

$$L(z) = \max_{y \geq 0} zy.$$

The program becomes

$$\begin{aligned} & \min_{(q_j) \geq 0} q^\top c + \sum_i \max_{y_i \geq 0} y_i \left(d_i - \sum_j q_j N_{ij}\right) \\ = \quad & \min_{(q_j) \geq 0} \max_{(y_i) \geq 0} \left\{ q^\top c + y^\top (d - Nq) \right\} = V_P \end{aligned}$$

The weak duality inequality always holds, that is

$$\min_q \max_y f(q, y) \geq \max_y \min_q f(q, y)$$

Exercise: prove it.

We have

$$V_P \geq V_D = \max_{(y_i)\geq 0} \min_{(q_j)\geq 0} \left\{ q^\top c + y^\top (d - Nq) \right\}$$

Let's study V_D. We have

$$\begin{aligned} V_D &= \max_{(y_i)\geq 0} y^\top d + \min_{(q_j)\geq 0} \left\{ q^\top c - y^\top N q \right\} \\ &= \max_{(y_i)\geq 0} y^\top d + \min_{(q_j)\geq 0} \left\{ q^\top c - q^\top N^\top y \right\} \\ &= \max_{(y_i)\geq 0} y^\top d + \min_{(q_j)\geq 0} \left\{ q^\top \left(c - N^\top y \right) \right\} \\ &= \max_{(y_i)\geq 0} y^\top d + \sum_j \min_{q_j \geq 0} \left\{ q_j \left(c_j - \left(N^\top y \right)_j \right) \right\} \end{aligned}$$

We need to evaluate

$$\begin{aligned} \min_{q\geq 0} qw &= -\infty \text{ if } w < 0 \\ &= 0 \text{ if } w \geq 0 \end{aligned}$$

We have thus

$$\begin{aligned} V_D = & \quad \max_{(y_i)\geq 0} y^\top d \\ s.t. & \quad c_j - \left(N^\top y \right)_j \geq 0 \end{aligned}$$

that is

$$\begin{aligned} V_D &= \max_{(y_i)\geq 0} y^\top d \\ s.t. & \quad \left(N^\top y \right)_j \leq c_j \end{aligned}$$

which we can write as

$$\begin{aligned} V_D &= \max_{(y_i)\geq 0} \sum_i y_i d_i \\ s.t. & \quad \sum_i N_{ij} y_i \leq c_j \;\; [q_j \geq 0] \end{aligned}$$

Interpretation of y_i.

We say that q is a primal feasible solution if it satisfies the constraints of the primal problem, ie $q \geq 0$ and $Nq \geq d$.

When q is a solution to the primal problem, we say that q is a primal optimal solution.

Same notions for the dual.

Theorem (duality). (i) If the primal or the dual is feasible (that is, if there are primal or dual feasible solutions) then strong duality holds, i.e. $V_P = V_D$.

(ii) If BOTH the primal and the dual are feasible, then $V_P = V_D$ are both finite.

Remark: if $V_P \neq V_D$ then $V_P = +\infty$ and $V_D = -\infty$. This happens if and only if both the primal and the dual are not feasible.

Exercise: give a simple example of such a situation.

Theorem (complementary slackness). Let q be a primal feasible solution and y a dual feasible solution. The following two statements are equivalent:

(i) q is optimal for the primal and y is optimal for the dual;

and

(ii) complementary slackness relations hold, that is: $q_j > 0 \implies c_j = \sum_i N_{ij} y_i$ and $y_i > 0 \implies d_i = \sum_j N_{ij} q_j$.

Back to the diet problem.

We have $N_{ij} \geq 0$, so the dual is feasible as $y = 0$ does the trick.

Thus we have $V_P = V_D$, and these values are either a finite number, or $+\infty$.

The primal is feasible if $\max_j N_{ij} > 0$ for every i.

y_i is the shadow cost of nutrient i..

2 The optimal assignment problem

Assume we are trying to match workers and jobs (or machines)

There are $n_x > 0$ workers of type $x \in X$

and $m_y > 0$ jobs of type $y \in Y$.

A matching is the specification of how many workers of type x we match with jobs of type y. We shall call the corresponding number μ_{xy}, and the constraint on μ is

$$\begin{aligned} \sum_y \mu_{xy} &= n_x \\ \sum_x \mu_{xy} &= m_y \end{aligned}$$

The constraints on μ is thus

$$\begin{aligned} \mu_{xy} &\geq 0 \\ \sum_y \mu_{xy} &= n_x \\ \sum_x \mu_{xy} &= m_y \end{aligned}$$

Question: when is there a feasible solution? yes provided $\sum_x n_x = \sum_y m_y = N$. => is clear; conversely, if $\sum_x n_x = \sum_y m_y = N$, then define μ as

$$\mu_{xy} = \frac{n_x m_y}{N}$$

(independent matching) which satisfies the constraints. Assume that if x works for job y, then x produces output

$$\Phi_{xy}$$

then the total output is

$$\sum_{xy} \mu_{xy} \Phi_{xy}$$

Therefore the optimal assignment problem is

$$\begin{aligned}
\max_{\mu \geq 0} \quad & \sum_{xy} \mu_{xy} \Phi_{xy} \\
s.t. \quad & \sum_{y} \mu_{xy} = n_x \ \ [u_x] \\
& \sum_{x} \mu_{xy} = m_y \ \ [v_y]
\end{aligned}$$

Write this as a max-min problem. We have

$$\max_{\mu \geq 0} \left\{ \begin{array}{c} \sum_{xy} \mu_{xy} \Phi_{xy} \\ +\sum_x L\left(n_x - \sum_y \mu_{xy}\right) \\ +\sum_y L\left(m_y - \sum_x \mu_{xy}\right) \end{array} \right\}$$

where $L(z) = 0$ if $z = 0$, $L(z) = -\infty$ otherwise.

1) What is $L(z) = \min_{w \in R} wz$

2) formulate the max min and the the min max problem

$$\max_{\mu \geq 0} \left\{ \begin{array}{c} \sum_{xy} \mu_{xy} \Phi_{xy} \\ +\sum_x \min_{u_x} u_x \left(n_x - \sum_y \mu_{xy}\right) \\ +\sum_y \min_{v_y} v_y \left(m_y - \sum_x \mu_{xy}\right) \end{array} \right\}$$

$$\begin{aligned}
& \leq \min_{(u_x, v_y)} \max_{(\mu_{xy}) \geq 0} \left\{ \begin{array}{c} \sum_{xy} \mu_{xy} \Phi_{xy} \\ +\sum_x u_x \left(n_x - \sum_y \mu_{xy}\right) \\ +\sum_y v_y \left(m_y - \sum_x \mu_{xy}\right) \end{array} \right\} \\
& = \min_{(u_x, v_y)} \sum_x u_x n_x + \sum_y v_y m_y + \max_{(\mu_{xy}) \geq 0} \left\{ \begin{array}{c} \sum_{xy} \mu_{xy} \Phi_{xy} \\ -\sum_{xy} u_x \mu_{xy} \\ -\sum_{xy} v_y \mu_{xy} \end{array} \right\} \\
& = \min_{(u_x, v_y)} \sum_x u_x n_x + \sum_y v_y m_y + \max_{(\mu_{xy}) \geq 0} \left\{ \ \sum_{xy} \mu_{xy} \left(\Phi_{xy} - u_x - v_y\right) \ \right\}
\end{aligned}$$

3) write down the dual

This yields

$$\begin{aligned}
\min_{(u_x, v_y)} \quad & \sum_x u_x n_x + \sum_y v_y m_y \\
s.t. \quad & u_x + v_y \geq \Phi_{xy} \ \ \left[\mu_{xy} \geq 0\right]
\end{aligned}$$

The primal is feasible. The dual too. Hence strong duality holds, and by complementary slackness, if μ is feasible for the primal and (u, v) for the dual, then the following two conditions are equivalent:

(i) μ is optimal for the primal and (u, v) for the dual

and

(ii) $\mu_{xy} > 0$ implies $u_x + v_y = \Phi_{xy}$.

4) intepret the dual as an equilibrium in the labor market where u_x is worker x's wage, and v_y is firm y's profit.

Walrasian equilibrium picture:

u_x=wage of worker x

Supply for workers x=n_x.

What is the demand for worker x?

Aggregate demand for x is the sum of the demand for x from y for each y. Thus this is

$$\sum_y \mu_{xy}$$

where μ_{xy} is the demand by firm y of worker x. We should have

$$\mu_{xy} > 0 \implies x \in \arg\max_{x'} \{\Phi_{x'y} - u_{x'}\}$$

We have

$$\sum_x \mu_{xy} = m_y$$

To summarize we have:

$\sum_y \mu_{xy} = n_x$

$\sum_x \mu_{xy} = m_y$ (primal feasibility)

$\mu_{xy} > 0 \implies x \in \arg\max_{x'} \{\Phi_{x'y} - u_{x'}\}$

Let's define v_y as

$v_y := \max_x \{\Phi_{xy} - u_x\}$

By definition we have

$v_y \geq \Phi_{xy} - u_x$ for all x for all y

thus

$u_x + v_y \geq \Phi_{xy}$ for all x for all y – dual feasibility.

$\mu_{xy} > 0 \implies \max_{x'} \{\Phi_{x'y} - u_{x'}\} = \Phi_{xy} - u_x$

but that means

$\mu_{xy} > 0 \implies v_y = \Phi_{xy} - u_x$

hence

$\mu_{xy} > 0 \implies u_x + v_y = \Phi_{xy}$ which is complementary slackness.

Exercise: show the converse is true.

Computation.

$$\begin{aligned}
\max_{\mu \geq 0} \quad & \sum_{xy} \mu_{xy} \Phi_{xy} \\
s.t. \quad & \sum_{y} \mu_{xy} = n_x \ \ [u_x] \\
& \sum_{x} \mu_{xy} = m_y \ \ [v_y]
\end{aligned}$$

If we view μ_{xy} as a matrix, say $\tilde{\mu}$, we can rewrite the first set of constraints as

$$\begin{aligned}
\tilde{\mu} 1_Y &= n \\
\tilde{\mu}^\top 1_X &= m
\end{aligned}$$

How do I convert a matrix into a vector? vectorize by stacking rows (row-major order; C; python/numpy) or columns (column-major order; matlab; R; Fortran). For instance

$$\tilde{\mu} = \begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{21} & \mu_{22} \end{pmatrix}$$

$vec_R(\tilde{\mu}) = (\mu_{11}, \mu_{12}, \mu_{21}, \mu_{22})^\top$.

Kronecker product

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B & ... & a_{1m}B \\ a_{21}B & & & \\ ... & & & \\ a_{n1}B & & & a_{nm}B \end{pmatrix}$$

We have the vectorization identity

$$vec_R(AXB) = \left(A \otimes B^\top\right) vec_R(X).$$

Here we have

$$\begin{aligned}
I_X \tilde{\mu} 1_Y &= n \\
1_X^\top \tilde{\mu} I_Y &= m^\top
\end{aligned}$$

which we vectorize into

$$\begin{aligned}
vec_R(I_X \tilde{\mu} 1_Y) &= vec_R(n) = n \\
vec_R\left(1_X^\top \tilde{\mu} I_Y\right) &= vec_R\left(m^\top\right) = m
\end{aligned}$$

and applying the identity gets us

$$\begin{aligned}
\left(I_X \otimes 1_Y^\top\right) \mu &= n \\
\left(1_X^\top \otimes I_Y\right) \mu &= m
\end{aligned}$$

Therefore the problem becomes

$$\begin{aligned} \max_{\mu \geq 0} \quad & \mu^\top \Phi \\ s.t. \quad & \left(I_X \otimes 1_Y^\top\right) \mu = n \\ & \left(1_X^\top \otimes I_Y\right) \mu = m \end{aligned}$$

The dual

$$\begin{aligned} \min_{u,v} \quad & u^\top n + v^\top m \\ s.t. \quad & \left(I_X \otimes 1_Y^\top\right)^\top u + \left(1_X^\top \otimes I_Y\right)^\top v \geq \Phi \end{aligned}$$

but we will use the fact $(A \otimes B)^\top = A^\top \otimes B^\top$, thus the dual is

$$\begin{aligned} \min_{u,v} \quad & u^\top n + v^\top m \\ s.t. \quad & \left(I_X \otimes 1_Y\right) u + \left(1_X \otimes I_Y\right) v \geq \Phi. \end{aligned}$$

3 Installing Docker and the Gurobi license

Install docker

https://www.docker.com/

Get a WLS Gurobi license

https://www.gurobi.com/features/academic-wls-license/

Run the m+e+c docker image by

docker run -it –rm -p 8888:8888 -v //c/Users/alfre/Desktop/docker-tmp:/src/notebooks/my-work -v //c/Users/alfre/Dropbox/AGResearch/code/gurobi/gurobi.lic:/opt/gurobi/gurobi.lic:ro –cgroupns=host alfredgalichon/mec_optim:2022-01

//c/Users/alfre/Desktop/docker-tmp is an empty folder on your computer

//c/Users/alfre/Dropbox/AGResearch/code/gurobi/gurobi.lic is the address of the file where you Gurobi license file is stored