1 Solution to the exercise

Exercise. Consider a setting where there are n_x workers of type x and m_y firms of type y. If a worker of type $x \in \mathcal{X}$ is matched to a firm of type $y \in \mathcal{Y}$ with a gross wage w_{xy}, then the worker and the firm respectively get utility

$$\alpha_{xy} + (1-\tau)\, w_{xy} \text{ and } \gamma_{xy} - w_{xy}$$

where $\tau \in (0,1)$ is a linear tax rate, $\alpha_{xy} \leq 0$ is the job amenity, and $\gamma_{xy} \geq 0$ is the economic output productivity. Unmatched workers and firms get zero payoff.

(i) Show that the equilibrium matching μ^{τ}_{xy} is the optimal matching associated to surplus $\Phi^{\tau}_{xy} = \alpha_{xy} + (1-\tau)\,\gamma_{xy}$.

(ii) How is the Walrasian vector of wages w_{xy} determined based on the dual program?

(iii) Assume there are three types of workers and four types of firms, there is a mass one of each type of worker and of each type of firm, and

$$\alpha = \begin{pmatrix} -16 & -4 & -8 & -20 \\ -12 & -8 & -4 & -4 \\ -4 & -8 & -8 & -16 \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} 4 & 8 & 4 & 2 \\ 4 & 3 & 6 & 6 \\ 9 & 4 & 8 & 2 \end{pmatrix}$$

Let

$$A = \sum_{xy} \mu^{\tau}_{xy}\alpha_{xy} \text{ and } \Gamma = \sum_{xy} \mu^{\tau}_{xy}\gamma_{xy}$$

Using Gurobi, fill out the following table

τ	.00	0.25	0.50	0.75
A				
Γ				

and comment.

1.1 Solution

Question (i)

Consider an equilibrium matching (μ_{xy}, w_{xy})

(1) Population constraints.

$\sum_y \mu_{xy} \leq n_x$

$\sum_x \mu_{xy} \leq m_y$

(2)

for each x,

$$\mu_{xy} > 0 \implies \begin{cases} y \in \arg\max_{y'} \{\alpha_{xy'} + (1-\tau)\, w_{xy'}, 0\} \\ x \in \arg\max_{x'} \{\gamma_{x'y} - w_{x'y}, 0\} \end{cases}$$

Introduce

$$\begin{aligned} u_x &= \max_{y'} \{\alpha_{xy'} + (1-\tau)\, w_{xy'}, 0\} \\ v_y &= \max_{x'} \left\{\gamma_{x'y} - w_{x'y}, 0\right\} \end{aligned}$$

and rewrite the conditions as

$$\mu_{xy} > 0 \implies \left\{ \begin{array}{l} \alpha_{xy} + (1-\tau)\, w_{xy} = u_x \\ \gamma_{xy} - w_{xy} = v_y \end{array} \right.$$

which implies

$$\mu_{xy} > 0 \implies \left\{ \begin{array}{l} \alpha_{xy} + (1-\tau)\, w_{xy} = u_x \\ \gamma_{xy}\,(1-\tau) - w_{xy}\,(1-\tau) = v_y\,(1-\tau) \end{array} \right.$$

that is

$$\mu_{xy} > 0 \implies \alpha_{xy} + (1-\tau)\,\gamma_{xy} = u_x + (1-\tau)\, v_y$$

However in general

$$\begin{aligned} u_x + (1-\tau)\, v_y \quad &\geq \quad \alpha_{xy} + (1-\tau)\, w_{xy} \\ &\quad\quad + (1-\tau)\,\gamma_{xy} - (1-\tau)\, w_{xy} \end{aligned}$$

thus

$$u_x + (1-\tau)\, v_y \geq \alpha_{xy} + (1-\tau)\,\gamma_{xy}$$

Let's recap:
Define $\tilde{v}_y = (1-\tau)\, v_y$
We have:
(1)
$\sum_y \mu_{xy} \leq n_x$
$\sum_x \mu_{xy} \leq m_y$
(2) $u_x + \tilde{v}_y \geq \Phi^{\tau}_{xy}$
$u_x \geq 0$
$\tilde{v}_y \geq 0$
(3)
if $\mu_{xy} > 0$ then $u_x + \tilde{v}_y = \Phi^{\tau}_{xy}$
if $n_x > \sum_y \mu_{xy}$ then $u_x = 0$
if $m_y > \sum_x \mu_{xy}$ then $\tilde{v}_y = 0$

These are the CS conditions associated with

$$\begin{aligned} \max_{\mu \geq 0} \quad & \sum_{xy} \mu_{xy} \Phi^{\tau}_{xy} \\ s.t. \quad & \sum_{y} \mu_{xy} \leq n_x \\ & \sum_{x} \mu_{xy} \leq m_y \end{aligned}$$

which has dual

$$\begin{aligned} \min_{u_x \geq 0, \tilde{v}_y \geq 0} \quad & \sum_x n_x u_x + \sum_y m_y \tilde{v}_y \\ s.t. \quad & u_x + \tilde{v}_y \geq \Phi^{\tau}_{xy} \end{aligned}$$

(ii) Assume $(u, \tilde{v})$ is a solution to the dual problem above. How can we recover w_{xy} based on u and $\tilde{v}$?

We have

$$\begin{aligned} u_x &= \max_{y'} \{\alpha_{xy'} + (1-\tau) w_{xy'}, 0\} \\ \tilde{v}_y &= (1-\tau) v_y = \max_{x'} \left\{(1-\tau)\gamma_{x'y} - (1-\tau) w_{x'y}, 0\right\} \end{aligned}$$

thus

$$\begin{aligned} u_x &\geq \alpha_{xy} + (1-\tau) w_{xy} \\ \tilde{v}_y &\geq (1-\tau)\gamma_{xy} - (1-\tau) w_{xy} \end{aligned}$$

that is $(1-\tau) w_{xy} \geq (1-\tau)\gamma_{xy} - \tilde{v}_y$, hence

$$\frac{u_x - \alpha_{xy}}{1-\tau} \geq w_{xy} \geq \gamma_{xy} - \frac{\tilde{v}_y}{(1-\tau)}$$

Claim: if w_{xy} satisfies these inequalities, then it is a vector of equilibrium wages.

Indeed, if $\mu_{xy} > 0$ then $u_x + \tilde{v}_y = \alpha_{xy} + (1-\tau)\gamma_{xy}$, which implies

$$\frac{u_x - \alpha_{xy}}{1-\tau} = \gamma_{xy} - \frac{\tilde{v}_y}{(1-\tau)}$$

2 Continuous optimal transport

Recall discrete case

$$\begin{aligned} \max_{\mu_{xy} \geq 0} \quad & \sum_{xy} \mu_{xy} \Phi_{xy} \\ s.t. \quad & \sum_y \mu_{xy} = n_x \\ & \sum_x \mu_{xy} = m_y \end{aligned}$$

which has dual

$$\begin{aligned} \min_{u,v} \quad & \sum_x n_x u_x + \sum_y m_y v_y \\ s.t. \quad & u_x + v_y \geq \Phi_{xy} \end{aligned}$$

Let's assume $\sum_x n_x = \sum_y m_y = 1$. In that case μ_{xy} satisfies

$$\sum_{xy} \mu_{xy} = 1$$

Thus μ_{xy} can be interpreted as a probability on $\mathcal{X} \times \mathcal{Y}$. If (X, Y) is a random pair with probability μ, then the constraints on μ imply that

$$\begin{aligned} \Pr(X = x) &= \sum_{x'y'} \mu_{x'y'} 1\{x' = x\} = \sum_{y'} \mu_{xy'} = n_x \\ \Pr(Y = y) &= m_y \end{aligned}$$

Thererefore we can rewrite the problem as

$$\begin{aligned} &\max_{(X,Y)\sim\mu} E[\Phi_{XY}] \\ &X \sim n \\ &Y \sim m \end{aligned}$$

and the dual problem as

$$\begin{aligned} \min_{u,v} \quad & E_n[u_X] + E_m[v_Y] \\ s.t. \quad & u_x + v_y \geq \Phi_{xy} \end{aligned}$$

In the sequel, $\mathcal{X}$ and $\mathcal{Y}$ will be R^n and $\Phi_{xy} = x^\top y$.

Let's assume $u(x)$ and $v(y)$ are solutions to the dual. We have

$$v(y) = u^*(x) \text{ and } u(x) = v^*(y),$$

where

$$\begin{aligned} u^*(y) &= \max_x \left\{x^\top y - u(x)\right\} \\ v^*(x) &= \max_y \left\{x^\top y - v(y)\right\} \end{aligned}$$

We have that u^* and v^* are convex.

Do we always have $u^{**}(x) = u(x)$. NO if u is not convex.
YES if u is convex.

In the finite case, the support of μ is the set of xy such that $\mu_{xy} > 0$.
Complementary slackness
$(x, y) \in Supp(\mu) \implies x \in \arg\max_{x'} \left\{x'^\top y - u(x')\right\}$
this implies optimality conditions
$y = \nabla u(x)$

This means that if $(X, Y) \sim \mu$ is an optimal solution and if (u, v) is an optimal dual solution, then

$$Y = \nabla u(X).$$

This leads us the the Brenier theorem.

If n has a density wrt Lebesgue and finite second moments then for any m with finite second moments there is a convex function u, such that there is a $X \sim n$ and $Y \sim m$ and

$$Y = \nabla u(X).$$

(Extended by Gangbo and McCann)

In dimension 1, what does this mean?

In dimension 1, gradient of convex function = derivative of a convex function = nondecreasing function.

Assume $n = U([0, 1])$, and assume m is a distribution on the real line with cdf F_m. My statement means that for $X \sim U([0, 1])$ there is a $Y \sim F_m$ and

$$Y = T(X)$$

where T is nondecreasing. In this case, $T(x) = F_m^{-1}(x)$.

In deed, $\Pr\left(F_m^{-1}(X) \leq y\right) = \Pr(X \leq F_m(y)) = F_m(y)$.

Yesterday, we saw the case when n is continuous and m is discrete. In that case, u is convex and piecewise affine, that is

$$u(x) = \max_j \left\{x^\top y_j - p_j\right\}$$

We have that

$$\nabla u(x) = y_j \text{ where } \{j\} = \arg\max_j \left\{x^\top y_j - p_j\right\}.$$

3 Network flow problems

Consider a directed network.

Nodes (=cities) $z \in Z$

A city can be a producer, a consumer or a transit point.

For each z, q_z =net consumption of silk at node z.

(=consumption - production)

$q_z > 0$ means z is a consumer

$q_z < 0$ means z is a producer

$q_z = 0$ means z is a transit point.

$\sum_{z:q_z<0} (-q_z) = \sum_{z:q_z>0} q_z$

$\sum_{z \in Z} q_z = 0$ means that overall production=overall consumption.

Arcs. The set of arcs is a subset A of $Z \times Z$.

if $xy \in A$, then x is the origin of xy and y is the destination.

Paths. A path from x to y is a series of nodes $x_0 = x, x_1, ..., x_T = y$ where $x_t x_{t+1} \in A$.

Assumption (connectedness) – there is a path from any producer to any consumer.

Define a flow as a vector $\left(\mu_{xy}\right) \in \mathbb{R}_+^A$.
Conditions on μ to be a feasible flow.
$\mu_{xy} \geq 0$ for all $xy \in A$

Consider a node $z \in Z$
$\sum_{x:xz\in A} \mu_{xz} - \sum_{y:zy\in A} \mu_{zy} = q_z$

A flow μ that satisfies such equation is called a feasible flow.

Central planner's problem.
Let's assume c_a is the cost of shipping one unit of the commodity through arc a. The central planner's problem is

$$\begin{aligned} \min_{\mu_{xy}\geq 0} \quad & \sum_{xy\in A} \mu_{xy} c_{xy} \\ s.t. \quad & \sum_{x:xz\in A} \mu_{xz} - \sum_{y:zy\in A} \mu_{zy} = q_z \;\; [p_z]\,. \end{aligned}$$

$$\min_{\mu_{xy}\geq 0} \sum_{xy\in A} \mu_{xy} c_{xy} + \sum_z p_z \left(q_z - \sum_{x:xz\in A} \mu_{xz} + \sum_{y:zy\in A} \mu_{zy} \right)$$

$$\min_{\mu_{xy}\geq 0} \sum_{xy\in A} \mu_{xy} c_{xy} + \sum_z p_z q_z - \sum_{xz\in A} \mu_{xz} p_z + \sum_{zy\in A} \mu_{zy} p_z$$

$$\min_{\mu_{xy}\geq 0} \max_{p_z} \sum_{xy\in A} \mu_{xy} c_{xy} + \sum_z p_z q_z - \sum_{xy\in A} \mu_{xy} p_y + \sum_{xy\in A} \mu_{xy} p_x$$

$$\max_{p_z} \sum_z p_z q_z + \min_{\mu_{xy}\geq 0} \sum_{xy\in A} \mu_{xy} \left(c_{xy} - p_y + p_x\right)$$

that is

$$\begin{aligned} \max_{p_z} \quad & \sum_z p_z q_z \\ s.t. \quad & p_y - p_x \leq c_{xy} \end{aligned}$$

$\mu_{xy} > 0 \implies p_y - p_x = c_{xy}$
Interpetation of p_z: price of one unit of the commodity at z.
$p_y - p_x - c_{xy} \leq 0$ is a no-arbitrage condition.
$\mu_{xy} > 0 \implies p_y - p_x = c_{xy}$ means that traders trade only if there is nonnegative profit.

Cheapest path. Recall that a path from x to y is a series of nodes $x_0 = x, x_1, ..., x_T = y$ where $x_t x_{t+1} \in A$.

The cost of a path $x_0 x_1 ... x_T$ is $c_{x_0 x_1} + c_{x_1 x_2} + ... + c_{x_{T-1} x_T}$.

The cheapest path from x to y is the path that achieves minimimum cost.

Exercise. Consider the problem of the cheapest path from any point x to a destination y^*.

Let P_x^t be the cost of going from x to y^* in at most t steps if it is possible, and $P_x^t = +\infty$ otherwise.

1) What is P_x^1

2) Show that P_x^{t+1} and P_x^t are related by a Bellman equation.

3) Deduce an algorithm to determine the cheapest path from any point x to a destimation y^*.