1 Semi-discrete transport

City's inhabitants are located at $x \in X \subset R^2$ Assume that x are continuously distributed with density $n(x)$. X is a compact and convex set – the city location.

Let's normalize the total mass of inhabitants to one, that is

$$\int_X n(x)\,dx = 1.$$

A finite number of facilities (e.g. fountains) are located on the surface of the city. The coordinates of the facilities are $y \in Y = \{y_1, ..., y_J\}$. Assume that facility j has capacity m_j.

Let's assume that the total capacity of the fountains is equal to the total number of inhabitants, that is

$$\sum_j m_j = 1.$$

Assume that the cost of an inhabitant living at x getting water from fountain y is $|x-y|^2/2$.

Inhabitant at x picks fountain j such that

$$\min_j \left\{ |x-y_j|^2/2 \right\}.$$

(or equivalently, $\min_j \{|x-y_j|\}$).

Central planner's problem.

For every x and j, denote $\mu(x,j)$ the density of inhabitants that we send to fountain j.

What are the constraints on μ?

$\int \mu(x,j)\,dx = m_j$

$\sum_j \mu(x,j) = n(x)$

Take $\Phi(x,y) = -|x-y|^2/2$

The central planner's problem is thus

$$\begin{aligned} \max_{\mu(x,j)\geq 0} \quad & \sum_j \int \mu(x,j)\,\Phi(x,j)\,dx \\ s.t. \quad & \int \mu(x,j)\,dx = m_j \;\; [p_j] \\ & \sum_j \mu(x,j) = n(x) \quad [u(x)] \end{aligned}$$

$$\max_{\mu(x,j)\geq 0}\left\{\begin{array}{c}\sum_j \int \mu(x,j)\,\Phi(x,j)\,dx \\ +\min_{u(x)} \int n(x)\,u(x)\,dx - \sum_j \int \mu(x,j)\,u(x)\,dx \\ \min_{p_j} \sum_j m_j p_j - \sum_j \int \mu(x,j)\,p_j dx\end{array}\right\}$$

$$\begin{array}{ll}\min\limits_{u(x),p_j} & \displaystyle\int n(x)\,u(x)\,dx + \sum_j m_j p_j \\ s.t. & u(x) + p_j \geq \Phi(x,j) \;\; \forall x \forall j\end{array}$$

u, p satisfies the condition in the dual iff

$$u(x) \geq \Phi(x,j) - p_j$$

that is iff for all x,

$$u(x) \geq \max_j \{\Phi(x,j) - p_j\}$$

If (u,p) is optimal, then we have actually

$$u(x) = \max_j \{\Phi(x,j) - p_j\}$$

and the dual problem rewrites as

$$\min_{p_j} \int n(x) \max_j \{\Phi(x,j) - p_j\}\,dx + \sum_j m_j p_j$$

that is

$$\min_{p\in R^J} F(p)$$

where

$$F(p) = \int n(x) \max_j \{\Phi(x,j) - p_j\}\,dx + \sum_j m_j p_j.$$

F is a convex function. The optimality conditions are

$$\frac{\partial F(p)}{\partial p_j} = 0$$

thus

$$\int n(x) \frac{\partial}{\partial p_j} \max_{j'} \{\Phi(x,j') - p_{j'}\}\,dx + m_j = 0$$

but (envelope theorem)

$$\frac{\partial}{\partial p_j} \max_{j'} \{\Phi(x,j') - p_{j'}\} = -1\left\{j \in \arg\max_{j'} \{\Phi(x,j') - p_{j'}\}\right\},$$

so the optimality conditions become

$$-\int n(x)\,1\left\{j \in \arg\max_{j'}\left\{\Phi(x,j') - p_{j'}\right\}\right\} dx + m_j = 0$$

Define

$$D_j(p) = \int n(x)\,1\left\{j \in \arg\max_{j'}\left\{\Phi(x,j') - p_{j'}\right\}\right\} dx$$

which is the mass of the consumers who choose j, i.e. the aggregate demand for j. We have thus that the optimality conditions are

$$D_j(p) = m_j.$$

Monge-Kantorovich duality theorem.

Agorithm to minimize $F(p)$. Gradient descent

$$p^{t+1} = p^t - \epsilon \nabla F\left(p^t\right)$$

or in coordinates

$$p_j^{t+1} = p_j^t - \epsilon \frac{\partial F}{\partial p_j}\left(p^t\right)$$

but

$$\frac{\partial F}{\partial p_j}\left(p^t\right) = m_j - D_j(p)$$

is the excess supply of fountain j. Tatonnement algorithm.

Exercise. Consider a setting where there are n_x workers of type x and m_y firms of type y. If a worker of type $x \in \mathcal{X}$ is matched to a firm of type $y \in \mathcal{Y}$ with a gross wage w_{xy}, then the worker and the firm respectively get utility

$$\alpha_{xy} + (1-\tau)\,w_{xy} \text{ and } \gamma_{xy} - w_{xy}$$

where $\tau \in (0,1)$ is a linear tax rate, $\alpha_{xy} \leq 0$ is the job amenity, and $\gamma_{xy} \geq 0$ is the economic output productivity. Unmatched workers and firms get zero payoff.

(i) Show that the equilibrium matching μ_{xy}^τ is the optimal matching associated to surplus $\Phi_{xy}^\tau = \alpha_{xy} + (1-\tau)\,\gamma_{xy}$.

(ii) How is the Walrasian vector of wages w_{xy} determined based on the dual program?

(iii) Assume there are three types of workers and four types of firms, there is a mass one of each type of worker and of each type of firm, and

$$\alpha = \begin{pmatrix} -16 & -4 & -8 & -20 \\ -12 & -8 & -4 & -4 \\ -4 & -8 & -8 & -16 \end{pmatrix} \text{ and } \gamma = \begin{pmatrix} 4 & 8 & 4 & 2 \\ 4 & 3 & 6 & 6 \\ 9 & 4 & 8 & 2 \end{pmatrix}$$

Let

$$A = \sum_{xy} \mu^{\tau}_{xy} \alpha_{xy} \text{ and } \Gamma = \sum_{xy} \mu^{\tau}_{xy} \gamma_{xy}$$

Using Gurobi, fill out the following table

τ	.00	0.25	0.50	0.75
A				
Γ				

and comment.