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Mixture of q-Weibull Distribution

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Abstract

This study explores the use of q-Weibull distribution mixtures to model heterogeneous data. By integrating the flexibility of the q-Weibull distribution with mixture models, we develop a robust approach to capture complex data structures. The EM algorithm is employed for parameter estimation. Our results demonstrate significant improvements in modeling extreme events and non-Gaussian data, with potential applications in finance, healthcare, and reliability engineering.

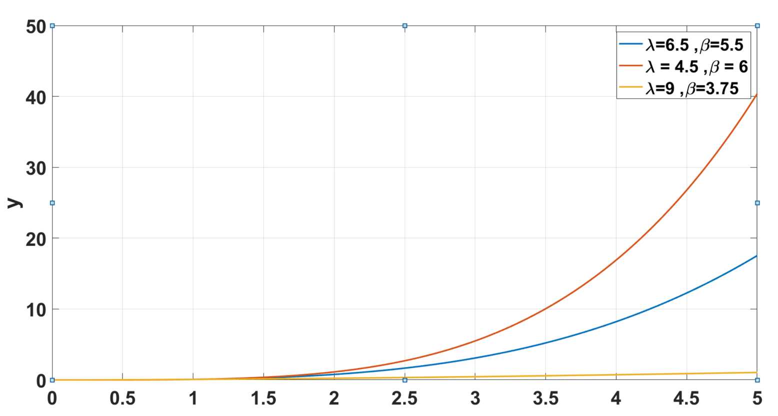
1) Definition

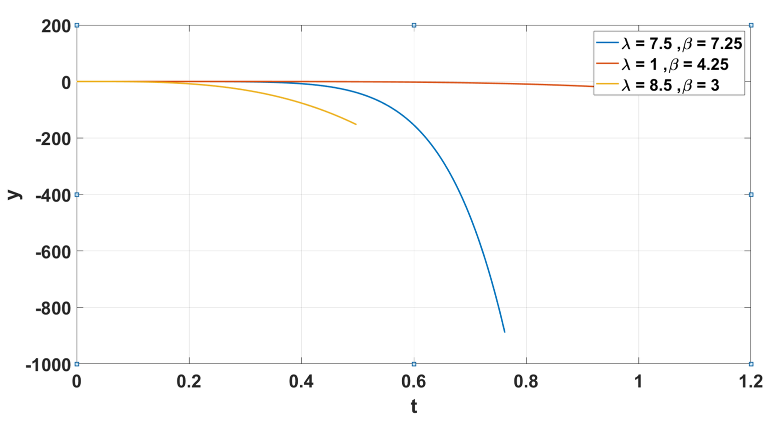
q-Weibull distribution: The pdf of the q-Weibull distribution is obtained from the classical Weibull model by the substitution of q-exponential:

$$f_q(t) = (2-q)\lambda\beta t^{\beta-1}\left[1-(1-q)\lambda t^{\beta}\right]^{1/1-q},$$

where

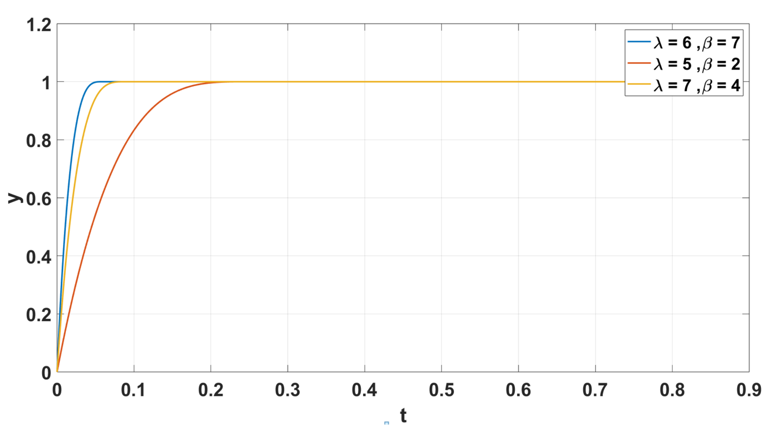
$$t \in \begin{cases} [0, +\infty[, & 1 < q < 2 \\ \left[0, \left[\lambda(1-q)\right]^{-1/\beta}\right], & q < 1 \end{cases}.$$

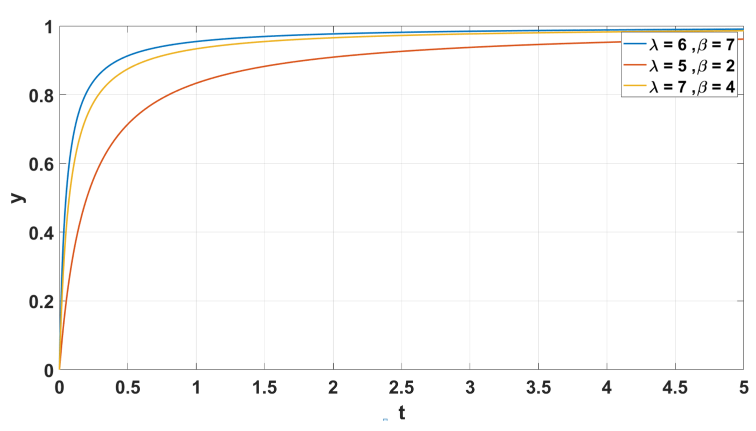




The cdf of the q-Weibull distribution is :

$$F_q(t) = 1 - \left[1-(1-q)\lambda t^{\beta}\right]^{\frac{2-q}{1-q}}.$$





2) Maximum likelihood estimation (MLE)

The MLE is a well known estimator. This estimator is important in statistics because of is asymptotic unbiasedness and minimal variance. Let $T = t_1, \ldots, t_n$ be a random sample from the q-Weibull distribution with set of parameters $\theta = (q, \beta, \lambda)$.The likelihood function $L = (X|\theta)$ is :

$$L(X|\theta) = L = [(2-q)\beta\lambda]^n \prod_{i=1}^{n}\left[t_i^{\beta-1}\left[1-(1-q)\lambda t^{\beta}\right]^{\frac{1}{1-q}}\right].$$

The first derivatives of the log-likelihood function with respect to q, β, λ are

$$\frac{\partial ln(L)}{\partial q} = \frac{n}{q-2} + \frac{\lambda}{1-q}\sum_{i=1}^{n}\frac{t_i^{\beta}}{1-(1-q)\lambda t_i^{\beta}} + \frac{1}{(1-q)^2}\sum_{i=1}^{n}\ln\left[1-(1-q)\lambda t_i^{\beta}\right],$$

$$\frac{\partial ln(L)}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\ln(t_i) - \sum_{i=1}^{n}\frac{\lambda t_i^{\beta}\ln(t_i)}{\left[1-(1-q)\lambda t_i^{\beta}\right]},$$

$$\frac{\partial ln(L)}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n}\frac{t_i^{\beta}}{\left[1-(1-q)\lambda t_i^{\beta}\right]}.$$

Bibliography

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3) Criteria for estimation

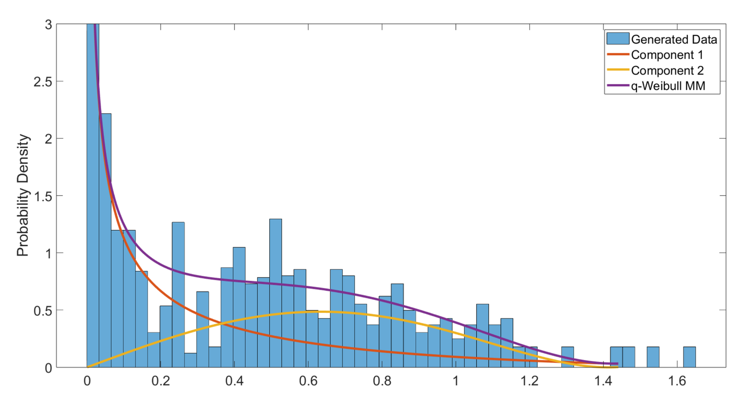
There are several selection criteria to determine which final model provides the best compromise between model fit and data complexity. The two most commonly used criteria are: AIC, BIC.

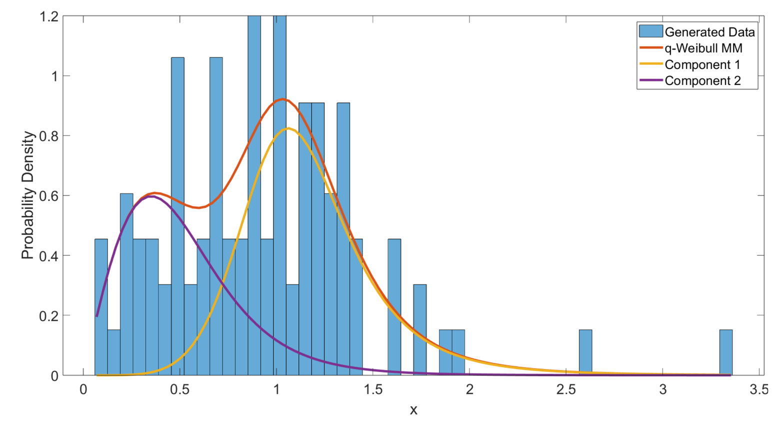
4) The mixture of q-Weibull distribution

The mixture of q-Weibull distribution is a method that combines several q-Weibull distribution to model complex data. Using the EM algorithm, the parameters of the mixture of q-Weibull distribution can be estimated from a set of observations.

$$f(x) = \sum_{i=1}^{N}\omega_i f_i(x, \theta_i),$$

where $f_i(x, \theta_i)$ is the probability density function of the q-Weibull distribution i with parameters θ_i, with $\theta_i = (q_i, \beta_i, \lambda_i)$ for $i \in \{1, \ldots, N\}$.





Algorithm 1 EM Algorithm for a mixture of q-Weibull distribution

Data q, β, λ, n
$X = qwbrlnd(q, \beta, \lambda, n)$ Estimation of parameters for the q-Weibull distribution
Maximum Likelihood
$\Theta = (\pi_1, \ldots, \pi_N, q_1, \ldots, q_N, \beta_1, \ldots, \beta_N, \lambda_1, \ldots, \lambda_N)$
Initialization Θ; **repeat**
Step E : For each x_t, calculate the conditional probabilities h_{tk} using the following formula :

$$h_{tk} = \frac{\pi_k f(x_t, q_k, \beta_k, \lambda_k)}{\sum_{j=1}^{N}\pi_j f(x_t, q_j, \beta_j, \lambda_j)}, \quad k = 1, \ldots, N,$$

where $f(x_t; q_k, \beta_k, \lambda_k)$ is the density of the q-Weibull distribution with parameters q_k, β_k, λ_k, evaluated at x_t.
Step M : Update the parameters as follows :

$$\pi_k = \frac{1}{n}\sum_{t=1}^{n}h_{tk}$$

$$q_k = \arg\max_{q_k} Q(\Theta, \Theta^{(l)}) = \arg\max_{q_k}\sum_{t=1}^{n}\sum_{k=1}^{N}h_{tk}\log f(x_t; q_k, \beta_k, \lambda_k)$$

$$\beta_k = \arg\max_{\beta_k} Q(\Theta, \Theta^{(l)}) = \arg\max_{\beta_k}\sum_{t=1}^{n}\sum_{k=1}^{N}h_{tk}\log f(x_t; q_k, \beta_k, \lambda_k)$$

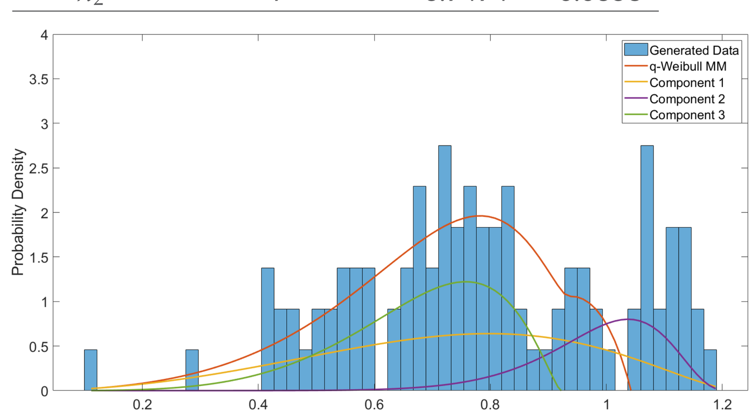
$$\lambda_k = \arg\max_{\lambda_k} Q(\Theta, \Theta^{(l)}) = \arg\max_{\lambda_k}\sum_{t=1}^{n}\sum_{k=1}^{N}h_{tk}\log f(x_t; q_k, \beta_k, \lambda_k)$$

until The convergence criterion is satisfied

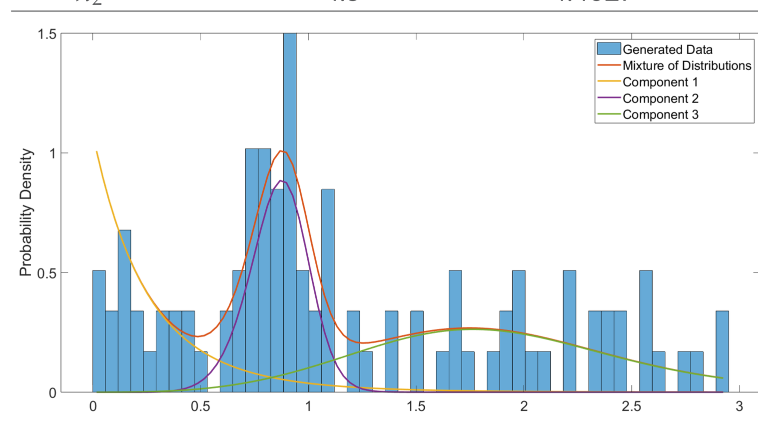
5) Result of simulation

To illustrate and highlight the performance of the EM algorithm described, some experimental results on simulated data, whose parameters are presented, are provided.

Parameter	True value	Estimated	std
q_1	0.4	0.3330	0.0474
β_1	2	1.9508	0.0348
λ_1	3	2.9470	0.0375
q_2	0.6	0.5396	0.0427
β_2	6	5.9702	0.0211
λ_2	7	6.9494	0.0358



Parameter	True value	Estimated	std
q_1	1.5	1.5041	0.0029
β_1	4	4.0162	0.0115
λ_1	3	2.9189	0.0573
q_2	1.3	1.3043	0.0030
β_2	1	1.0087	0.0062
λ_2	4.5	4.4027	0.0688



Conclusion

This research investigates advanced statistical distribution q-Weibull and their applications in reliability engineering and risk management. It discusses their versatility, parameter estimation methods(MLE), model selection criteria, and practical applications using (EM) algorithm, serving as a valuable resource for improving decision-making and system performance across various domains.