monday 2.24

Series (Sec 14)

Notations:

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \ldots + a_n = \sum_{i=m}^{n} a_i$$

$$\sum_{k=m}^{\infty} a_k = a_m + a_{m+1} + a_{m+2} + \ldots$$

usually, $m=0$ or 1

Meaning of $\sum_{k=m}^{\infty} a_k$

$\Rightarrow$ we consider partial sums of the series, $S_n = \sum_{k=m}^{n} a_k$, $n \geq m$

if the sequence of partial sums (S_n) converges, we say that the series $\sum_{k=m}^{\infty}$ converges

in this case, we define $\sum_{k=m}^{\infty} a_k = \lim_{n\to\infty} S_n$

Thus $\sum_{k=m}^{\infty} a_k = S$ means that $\lim_{n\to\infty} \left(\sum_{k=m}^{n} a_k \right) = S$

if the sequence of partial sums (S_n) diverges, we say that the series diverges

if $\lim_{n\to\infty} S_n = \pm\infty$, we say that the series diverges to $\pm\infty$ &

we write $\sum_{k=m}^{\infty} a_k = \pm\infty$

note: if $a_k \geq 0$ for all k then $\sum a_k$ either converges or diverges to $+\infty$

ex (1) $\sum_{k=1}^{\infty} 1$ ($a_k = 1$ for each k)

diverges to $+\infty$, since for each n, $S_n = 1 + \ldots + 1 = n$ & so $\lim S_n = +\infty$

(2) $\sum_{k=1}^{\infty} (-1)^n = -1 + 1 - 1 + 1 - \ldots$

diverges since $S_n = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$, and so (S_n) diverges

(3) (a telescoping series)

$$\sum_{k=1}^{\infty} \left(\frac{1}{\sqrt{k+1}} - 1/\sqrt{k} \right) = \left(\frac{1}{\sqrt{2}} - 1,\ -1 + \cancel{\frac{1}{\sqrt{2}}} + \frac{1}{\sqrt{3}} - \cancel{\frac{1}{\sqrt{2}}},\ -1 + \cancel{\frac{1}{\sqrt{2}}} + \cancel{\frac{1}{\sqrt{3}}} - \cancel{\frac{1}{\sqrt{2}}} + \frac{1}{2} - \cancel{\frac{1}{\sqrt{3}}},\ \ldots = -1 + \frac{1}{\sqrt{n+1}} \to -1$$

$\therefore$ this series converges & its sum is -1

(4) (p- series)

$\sum_{n=1}^{\infty} 1/n^p$ converges for $p > 1$
diverges for $p \leq 1$

in particular, the harmonic series $\sum_{n=1}^{\infty} 1/n$ diverges (hw)

(5) (geometric series)

$\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \ldots$ where $a \neq 0$

let $a=1$ $\sum_{n=0}^{\infty} r^n$

Then $S_n = 1 + r + r^2 + \ldots + r^n$

$-rS_n = r + r^2 + \ldots + r^n + r^{n-1}$

$(1-r) S_n = 1 - r^{n+1}$

Hence for $r \neq 1$, $S_n = \frac{1-r^{n+1}}{1-r}$

if $|r|<1$ then $r^{n+1} \to 0$ & hence $S_n \to \frac{1}{1-r}$ as $n \to \infty$

thus if $|r|<1$ $\sum_{n=0}^{\infty} r^n$ converges & its sum is $\frac{1}{1-r}$,

also, $\sum_{n=0}^{\infty} ar^n$ converges & its sum is $a/1-r$

for $|r| \geq 1$, the series diverges

Cauchy Criterion for Series

$\sum a_k$ conv $\Leftrightarrow$ the seq (s_n) of partial sums converges $\Leftrightarrow$ (s_n) is Cauchy $\Leftrightarrow$

for each $\varepsilon > 0$, there is N' st $n > m > N'$ implies $|s_n - s_m| < \varepsilon$ $\Leftrightarrow$

for each $\varepsilon > 0$, there is N st $n \geq m > N$ implies $|s_n - s_{m-1}| < \varepsilon$

$|a_n + a_{n-1} + \ldots + a_m|$

thm Cauchy Criterion for Series - a series $\sum a_k$ converges if & only if it satisfies the Cauchy Criterion for series,

for every $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $n \geq m > N$ implies $\left| \sum_{k=m}^{n} a_k \right| < \varepsilon$

corollary if $\sum a_n$ converges then $\lim a_n = 0$

PROOF Suppose $\sum a_n$ converges, then it satisfies the Cauchy criterion, taking $m=n$, we see that for each $\varepsilon>0$, there is N st $n>N$ implies $|a_n|<\varepsilon$

this means $a_n \to 0$ $\square$

Corollary if $a_n \not\to 0$ then $\sum a_n$ diverges

ex $\sum ar^n$, $a \neq 0$

if $|r| \geq 1$, then $a_n = ar^n \not\to 0$ and so the series diverges

T/F if $a_n \to 0$ then $\sum a_n$ converges False $\sum 1/n$

! $a_n \to 0$ tells us <u>nothing</u> about con/div of $\sum a_n$

wednesday 2.26

Absolute Convergence (see 14)

def We say that $\sum a_n$ **converges absolutely** if $\sum |a_n|$ converges

thm if $\sum |a_n|$ converges, then $\sum a_n$ converges

i.e. absolute convergence implies convergence

PROOF Suppose $\sum |a_n|$ converges

let $\varepsilon>0$. Since $\sum |a_n|$ converges, it satisfies the Cauchy Criterion, and so there is $N \in \mathbb{N}$ st for all $n \geq m > N$, $\sum_{k=m}^{n} |a_k| < \varepsilon$

Since $\left|\sum_{k=m}^{n} a_k\right| \leq \sum_{k=m}^{n} |a_k|$,

for all $n \geq m > N$, $\left|\sum_{k=m}^{n} a_k\right| < \varepsilon$

thus $\sum a_n$ satisfies the Cauchy Crit & hence it converges note: convergence $\not\Rightarrow$ absolute convergence

$\sum \frac{(-1)^n}{n}$ converges but $\sum \left| \frac{(-1)^n}{n} \right| = \sum \frac{1}{n}$ diverges

(or a suitable telescoping series)

Comparison test

let $a_n \geq 0$ for all n

(i) if $\sum a_n$ converges & $|b_n| \leq a_n$ for all n, then $\sum b_n$ converges (absolutely)

(ii) if $\sum a_n$ diverges, i.e. $\sum a_n = +\infty$, and $b_n \geq a_n$ for all n, then $\sum b_n = +\infty$

PROOF

(i) Since $|b_n| \leq a_n$ for all n, for any $n \geq m$,

(*) $\left| \sum_{k=m}^{n} b_k \right| \leq \sum_{k=m}^{n} |b_k| \leq \sum_{k=m}^{n} a_k = \left| \sum_{k=m}^{n} a_k \right|$

Since $\sum a_k$ converges, it satisfies the Cauchy Criterion, & it follows by * (explain) that $\sum b_k$ and $\sum |b_k|$ also satisfy the CC, & hence they converge

(ii) let s_n and t_n be partial sums for $\sum a_n$ and $\sum b_n$ respectivly

Since $\sum a_n = +\infty$, $s_n \to \infty$

Since $t_n \geq s_n$ for each n, it follows (explain) that $t_n \to +\infty$ and so $\sum b_n = +\infty$

note: it suffices to assume that $|b_n| \leq a_n$ or $b_n \geq a_n$ for all $n \geq n_0$

root test

let $\sum a_n$ be a series and let $\alpha = \limsup |a_n|^{1/n}$

(i) if $\alpha < 1$, then $\sum a_n$ converges absolutely

(ii) if $\alpha > 1$, then $\sum a_n$ diverges

(iii) if $\alpha = 1$, then the test is inconclusive, the series may converge or diverge

PROOF (iii) $\sum 1/n$ $\alpha = \limsup 1/n^{1/n} = 1$
diverges
$\sum 1/n^2$ converges $\alpha = 1$

(i) idea: comparison w a geometric series
let $\alpha < 1$. We take β st $\alpha < \beta < 1$
Since $\alpha = \limsup |a_n|^{1/n}$ and $\beta > \alpha$
there is N st for all $n > N$, $|a_n|^{1/n} < \beta$ and so $|a_n| < \beta^n$
Since $0 < \beta < 1$ and so $|\beta| < 1$, the geometric series $\sum \beta^n$ converges
and hence by the Comparison test, $\sum a_n$ converges absolutely

(ii) (IDEA: $a_n \not\to 0$)
Suppose $\alpha = \limsup |a_n|^{1/n} > 1$
then there are infinitely many n st $|a_n|^{1/n} > 1$ and hence $|a_n| > 1$
it follows that $a_n \not\to 0$ and hence $\sum a_n$ diverges

ex) $\sum a_n$, where $a_n = \left(\frac{-3}{5 + (-1)^n}\right)^n$

$$a_n = \begin{cases} (-3/4)^n, & n \text{ is odd} \\ (-1/2)^n, & n \text{ is even} \end{cases}$$

$\alpha = \limsup |a_n|^{1/n} = 3/4 < 1$ so $\sum a_n$ conv. by root test
OR for each n, $|a_n| \leq (3/4)^n$ and $|3/4| < 1$, the geometric series
$\sum (3/4)^n$ conv. & hence $\sum a_n$ conv by comparison test

ratio test
let $\sum a_n$ be a series of non-zero terms
(i) if $\limsup \left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum a_n$ converges absolutely
(ii) if $\liminf \left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\sum a_n$ diverges
(iii) otherwise, $\liminf |a_{n+1}/a_n| \leq 1 \leq \limsup |a_{n+1}/a_n|$, the test is inconclusive

PROOF (HW) same ideas as root test

Corollary let $\sum a_n$ be a series of non-zero terms
Suppose that $L = \lim \left|\frac{a_{n+1}}{a_n}\right|$ exists in $\mathbb{R} \cup \{+\infty\}$
(i) if $L<1$, then $\sum a_n$ conv abs
(ii) if $L>1$, then $\sum a_n$ diverges
(iii) if $L=1$, the test gives no info

ex $\sum \frac{100^n}{n!}$ $\quad \left|\frac{a_{n+1}}{n!}\right| = \frac{100}{n+1} \to 0$
so the series conv by the ratio test

ex 2 $\sum \frac{n^5}{2^n}$
ratio: $\frac{(n+1)^5}{n^5 \cdot 2} = (1+1/n)^5 \cdot 1/2 \to 1^5 \cdot 1/2 = 1/2 < 1$ $\therefore$ conv. by ratio test
root: $|a_n|^{1/n} = \frac{(n^5)^{1/n}}{2} = \frac{(n^{1/n})^5}{2} \to 1^5/2 = 1/2 < 1$ $\therefore$ conv by root test

friday 2.28

note: comparing ratio & root tests
- root does not require $a_n \neq 0$
- One can show (thm 12.2) that if $a_n \neq 0$ for all n, then

$$\liminf \left|\frac{a_{n+1}}{a_n}\right| \le \liminf |a_n|^{1/n} \le \limsup |a_n|^{1/n} \le \limsup \left|\frac{a_{n+1}}{a_n}\right|$$

if root test gives no information then ratio test gives us no info
but, there are cases where ratio test gives us no info but root test does
↳ example 8 page 103

thm Alternating Series Thm (Model: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - 1/2 + 1/3 - \dots$) (sec. 15)

if sequence (a_n) is decreasing, then $a_n \geq 0$ for all n & $\lim a_n = 0$

then the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \dots$ converges

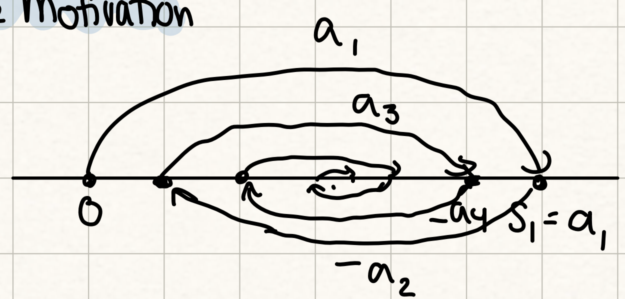
Moreover, for each $n \in \mathbb{N}$:

$$\left| \sum_{k=n+1}^{\infty} (-1)^{k+1} a_k \right| \leq a_n,$$

ie $|s - s_n| \leq a_n$

Note: s_n is the error when we use s_n to estimate the sum S

Picture Motivation



"jumping" back & forth

PROOF

The sequence (s_{2k}) is increasing since for each k, $s_{2k+2} - s_{2k} = a_{2k+1} - a_{2k+2} \geq 0$ since (a_n) is decreasing

also, (s_{2k}) is bounded above by a_1, since for each k,

$$s_{2k} = a_1 + (-a_2 + a_3) + \dots + (-a_{2k} + a_{2k+1}) - a_{2k} \leq a_1$$

Since (s_{2k}) is increasing & bdd above, it converges to some $s \in \mathbb{R}$

Since $s_{2k+1} = \underset{\text{conv. to } s}{s_{2k}} + \underset{\text{conv. to } 0}{a_{2k+1}}$ $\therefore$ $s_{2k+1} \to s$

Since $s_{2k} \to s$ and $s_{2k+1} \to s$, $s_n \to s$ (explain)

Moreover part:

(s_{2k+1}) is decreasing (explain) so for each k, $s_{2k} \leq s \leq s_{2k+1}$ then

$|s - s_{2k+1}| \leq |s_{2k+1} - s_{2k}| = a_{2k+1}$

$|s - s_{2k+1}| \leq |s_{2k+1} - s_{2k}| \leq a_{2k}$

So $|s - s_n| \leq a_n$ for each n $\square$

ex $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges

and for each n, $|s - s_n| \le a_n = 1/n$

Power Series (sec 23)

Power Series: $\displaystyle\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$ where $a_n \in \mathbb{R}$

more generally: $\displaystyle\sum_{n=0}^{\infty} a_n \underbrace{(x-c)}_{y}^n$ reduces to previous case if $y = x - c$

→ for each fixed $x \in \mathbb{R}$, $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ is a series of real numbers.

? For which x does it converge?

always converges for $x = 0$

We can fix x & apply the Root Test

$$\alpha_x = \limsup |a_n x^n|^{1/n} = \limsup \left(|x| \, |a_n|^{1/n} \right) = |x| \cdot \beta \text{ where } \beta = \limsup |a_n|^{1/n}$$

≥ 0

if $\beta = 0$, then $\alpha_x = 0 < 1$ for every $x \in \mathbb{R}$ & so the series $\displaystyle\sum_{n=0}^{\infty} a_n x^n$ conv. abs for all x

if $\beta = +\infty$, then $\alpha_x = +\infty > 1$ for all $x \neq 0$, & so the series diverges for all $x \neq 0$

if $0 < \beta < +\infty$, then $\alpha_x < 1$ for $|x| < 1/\beta$ and

$\alpha_x > 1$ for $|x| > 1/\beta$

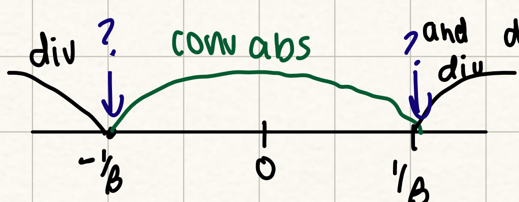
thm For a power series $\displaystyle\sum_{n=0}^{\infty} a_n x^n$, let $\beta = \limsup |a_n|^{1/n}$

(i) if $\beta = 0$, then the series conv. abs. for all $x \in \mathbb{R}$

(ii) if $\beta = +\infty$, then the series conv. for $x = 0$ only

(iii) if $0 < \beta < 1$, then the series conv. abs. for $|x| < 1/\beta$

and diverges for $|x| > 1/\beta$



$R = 1/\beta$ is called the radius of convergence of the series note: If $\lim\left|\frac{a_{n+1}}{a_n}\right|$ exists, then $\beta = \lim\left|\frac{a_{n+1}}{a_n}\right|$

(show using ratio test - exercise)

ex | (1) $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\quad a_n = 1/n!$ $\quad \lim\left(\frac{a_{n+1}}{a_n}\right) = \lim\left(\frac{1}{n+1}\right) = 0$

so $\beta = 0$ and hence the series conv abs for all $x \in \mathbb{R}$ (to e^x)

(2) $\sum_{n=1}^{\infty} \frac{n^n}{2^n} x^n$ $\quad a_n = \frac{n^n}{2^n}$ $\quad \beta = \lim|a_n|^{1/n} = \lim \frac{n}{2} = +\infty$

conv for only $x = 0$

(3) $\sum_{n=1}^{\infty} \left(-\frac{x^n}{n}\right)$ $\quad a_n = -1/n$ $\quad \lim\left(\frac{n+1}{n}\right) = 1$ $\quad \beta = 1$

conv. abs for $x \in (-1, 1)$
(to $\ln(1-x)$)
div for $|x| > 1$
div for $x = 1$
ex $x = -1$