math436 week 1

Fields

def 1.1 a **field** $\mathbb{F}$ is a set (of scalars) together with operations $\cdot, +$ such that if $a,b,c \in \mathbb{F}$

- $a+b = b+a \in \mathbb{F}$
- $(a+b)+c = a+(b+c)$

note: do not need to memorize

There is a 0 st $0+a=0$

$ab=ba$

$(ab)c = a(bc)$

$0 \cdot a = 0$

There is a 1 st $1 \cdot a = a$

+distribution

For every $a \neq 0$, there is an a^{-1} st $a \cdot a^{-1} = 1$

Slogan: a **field** is a set where multiplication and addition <u>are not</u> weird & where multiplication can be undone (where division works) " excluding 0

examples and nonexamples

$\mathbb{R}$ - this is a field

$\mathbb{C}$ - this is a field

$\mathbb{Z}$ - not a field

$\mathbb{Q} = \{\frac{a}{b} \mid a,b \in \mathbb{Z}, b \neq 0\}$ this is a field

$\mathbb{F}_p$ - for p a prime number is $\{0,1,\dots,p-1\}$ where addition is "addition mod p" and multiplication is mod p

ex) $\mathbb{F}_5 = \{0,1,2,3,4\}$

+	0	1	2	3	4
0	0	1	2	3	4
1		2	3	4	0
2			4	0	1
3				1	2
4					3

★ why will it not work for non prime & why does it work for prime

def 1.4 Characteristic of a field

if there is a number n st in a field $\mathbb{F}$,

$$\underbrace{1+1+\dots+1}_{n \text{ times}} = 0$$

then we say the field has a characteristic n

if there is no such number, the field has characteristic 0

1.2 - Vector Space

def 1.5 Vector space over a field

Given a field $\mathbb{F}$, an $\mathbb{F}$-vector space V is a set with an operation $+$ "vector addition" combining pairs of elements of V and $\cdot$ "scaling" combining elements of $\mathbb{F}$ with elements of V st:

(closure) for every $\lambda \in \mathbb{F}$ and every $v, w \in V$, $v+w$ is in V and $\lambda \cdot v$ is in V

(identity) there is a 0_V in V such that for every $v \in V$, $0_V + v = v$

(inverses) For every $v \in V$, there is a $w \in V$ st $v+w = 0_V$. This w is sometimes denoted by $-v$

(linearity) For every $\lambda, \mu \in \mathbb{F}$ and $v, w \in V$, $\lambda \cdot (v+w) = \lambda v + \lambda w$ and $(\lambda+\mu) \cdot v = \lambda v + \mu v$

(associative scaling) For $\lambda, \mu \in \mathbb{F}$ and any $v \in V$, $\lambda \cdot (\mu \cdot v) = (\lambda \cdot \mu) v$

(commutativity) For every $v, w \in V$, $v+w = w+v$

(associativity) For every $u, v, w \in V$, $(u+v)+w = u+(v+w)$

ex 1.6
- The plane $\mathbb{R}^2$ is an $\mathbb{R}$-vector space.
- The plane $\mathbb{R}^2$ is not a complete vector space with the natural definition of scaling.
- The complex numbers $\mathbb{C}$ are an $\mathbb{R}$-vector space. Prove this.
- The continuous functions from $\mathbb{R} \to \mathbb{R}$ form an $\mathbb{R}$-vector space $C(\mathbb{R})$.
  - $\to$ define $0(x) = 0 \ \forall x \in \mathbb{R}$
- Manhattan is not a vector space over any field of characteristic 0.

1.3 Linear Subspaces

def 1.7 Linear subspace

Let U be contained in an $\mathbb{F}$-vector space V (as a set). Then U is a subspace of V (or a linear subspace or vector subspace) if U is itself an $\mathbb{F}$-vector space, inheriting operations and identity from V.

note: quicker to check if something is a vector space when it already lives inside a known vector space since V4-V7 hold if a subspace is nonempty and V1 holds, V2 and V3 hold.

lemma 1.8 if U is a subset of an $\mathbb{F}$-vector space V, then U is a subspace of V if it is nonempty, and closed under addition and scaling

ex 1.9
- The plane $\mathbb{R}^2$ contains a copy of $\mathbb{R}$, as a subspace for example the x-axis. The y-axis is also a subspace. Any line thru the origin is a subspace
- For every vector space V, V is a subspace of V, and $\{0\}$ is a subspace of V
- The $\mathbb{R}$-vector space of functions $\mathbb{R} \to \mathbb{R}$ has a subspace of functions with finite support: these are the functions $f: \mathbb{R} \to \mathbb{R}$ that are zero everywhere except finitely many points

1.4 Bases

def 1.10 in an $\mathbb{F}$-vector space V, the span (or linear span or $\mathbb{F}$-span) of a finite subset $\{v_1, v_2, \dots v_n\} \in V$ is given by

$$\text{span}_{\mathbb{F}}(v_1, v_2, \dots, v_n) = \{\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n \mid \lambda_1, \lambda_2, \dots \lambda_n \in \mathbb{F}\}$$

remark 1.11 for an infinite set $S = \{v_i \mid i \in I\}$ the span is the set of linear combinations of any finite number of the v_i.

$$\text{span}_{\mathbb{F}}(S) = \{\lambda_1 v_{i_1} + \lambda v_{2 i_2} + \dots + \lambda_n v_{i_n} \mid \lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{F}, \ i_1 \dots i_n \in I\}.$$

def 1.12 linear independence - Let V be an $\mathbb{F}$-vector space and let $v_1, v_2, \dots, v_n \in V$. Then $\{v_1, v_2, \dots, v_n\}$ is linearly independent over $\mathbb{F}$ if, whenever there are scalars $\lambda_1, \lambda_2, \dots \lambda_n \in \mathbb{F}$ such that $\lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n = 0$, we must have $\lambda_1 = \lambda_2 = \dots = \lambda_n = 0$

A set is linearly dependent if it is not linearly independent

remark 1.13 an infinite set is linearly independent if every finite subset of it is linearly independent

def 1.14 Given an $\mathbb{F}$-vector space V, and a subset $B \subset V$, B is a $\mathbb{F}$-basis for V if B is linearly independent over $\mathbb{F}$ and $\operatorname{span}_{\mathbb{F}}(B) = V$