monday 3/2

Decimal expansions of $\mathbb{R}$ numbers

T/F every real number has exactly one decimal expansion?

$1 = 1.00\ldots = .999\ldots$

similarly any $n \in \mathbb{N}$ has 2 decimal expansions

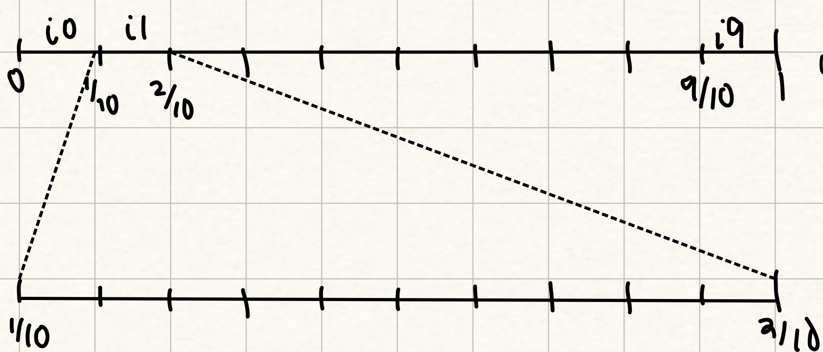
let $x \in (m, m+1)$, $m \in \mathbb{N}$

$x = m.a_1a_2\ldots \iff x - m = 0.a_1a_2\ldots$

decimal exp of $x>0$ is $-$(decimal exp of $|x|$)

$\therefore$ we focus on $[0,1]$

Geometric Approach



closed intervals $[\frac{i}{10}, \frac{i+1}{10}]$ $i=0,\ldots,9$

$x = 0.a_1a_2\ldots$ means x is in interval a_1

more specifically, in its subinterval a_1a_2

———————— '' ———————— $a_1a_2a_3\ldots$ & so on

ex) $x = .2417$

x is in $[2/10, 3/10]$

x is in $[24/100, 25/100]$

x is in $[241/1000, 242/1000]$

$\ldots$

every $x \in [0,1]$ has at least 1 decimal expansion

endpoints of the intervals ie $\frac{m}{10^k}$ where $0 < m < 10^k$ have 2 decimal exp.
every other $x \in (0,1)$ has exactly 1

$0.a_1 a_2 a_3 \dots$

for any sequence $(a_1, a_2, \dots)$ where $a_n \in \{0, \dots, 9\}$ there is exactly one $x \in [0,1]$ with decimal expansion $0.a_1 a_2 \dots$

follows from:

Lemma: let $I_1 \supset I_2 \supset \dots$ be nonempty bounded closed intervals with $|I_n| \to 0$
then $\bigcap_{n=1}^{\infty} I_n$ consists of a single point

Outline of Proof

$\Rightarrow$ existence of a point $c \in \bigcap_{n=1}^{\infty} I_n$

for each n, choose $c_n \in I_n$

Then (c_n) is Cauchy $(\dots)$

Then $c_n \to c \in \mathbb{R}$

$c \in I_n$ for all n $(\dots)$

Since $|I_n| \to 0$, there cannot exist 2 points w/ dist 0 $\therefore$ it is unique c

algebraic approach

$a_n \in \{0, 1, \dots 9\}$

$x = 0.a_1 a_2 \dots$ means $x = \sum_{n=1}^{\infty} \frac{a_n}{10^n}$ converges

every $x \in [0,1]$ has a decimal expansion

Pick largest a_1 st $\frac{a_1}{10} \le x$ | Then $|x - 0.a_1 a_2 \dots a_n| \le 1/10^n$

Pick largest a_2 st $\frac{a_2}{100} < x$ | for each $n \in \mathbb{N}$

....

$\sum_{n=1}^{\infty} \frac{9}{10^n} = 1$ $\qquad$ $\sum_{n=k+1}^{\infty} \frac{9}{10^n} = \frac{1}{10^k}$

$0.a_1 \dots \underset{\neq 9}{a_k} 999 \dots = 0.a_1 \dots a_{k-1}(a_k + 1)$

? decimal expansion of rationals?

friday 3.7

countable & uncountable sets

def a function $f: X \to Y$ is bijective (a bijection) if it is both one-to-one and onto

$$\begin{array}{ccc} X & & Y \\ \circ & \to & \circ \\ \circ & \to & \circ \\ \vdots & & \end{array}$$

def a set X is finite if either $X = \emptyset$ or for some $n \in \mathbb{N}$, there is a bijection from X to $\{1, 2, \dots, n\}$. i.e. $X = \{x_1, x_2, \dots, x_n\}$

def X is countable infinite if there is a bijection from X to $\mathbb{N}$ ie we can list the ele. of X:

$X = \{x_1, x_2, x_3, \dots\}$

def X is countable if it is either finite or countably infinite.

def X is uncountable if it is not countable ie too large for its elements to be listed

ex
- $\mathbb{N}$ is countably infinite
- $\mathbb{Z}$ is countable infinite

  $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \dots\}$

  exercise: write a formula to map

- $\mathbb{Q}$ is countably infinite (refer to table)

$$\begin{array}{cccccccc} 0 \to & 1 \to & -1 & 2 & -2 & 3 & -3 & \dots \\ 1/2 & -1/2 & \cancel{2/2} & \cancel{-2/2} & 3/2 & -3/2 & & \dots \\ 1/3 & -1/3 & 2/3 & -2/3 & 3/3 & -3/3 & & \dots \\ \dots & & & & & & & \end{array}$$

X-skip b/c in list

- $\mathbb{R}$ - look @ $(0,1) = \{x \in \mathbb{R} : 0 < x < 1\}$ is uncountable

PROOF We show that for any list of numbers from $(0,1)$, there is a number from $(0,1)$ that is not on this list

$x_1 = 0.a_{11}a_{12}a_{13}\dots$

$x_2 = 0.a_{21}a_{22}a_{23}\dots$

$x_3 = 0.a_{31}a_{32}a_{33}\dots$

$\dots$

let $b = 0.b_1b_2b_3\dots$

where $b_i = \begin{cases} 1 & \text{if } a_{ii} \neq 1 \\ 2 & \text{if } a_{ii} = 1 \end{cases}$

- if X is countable, then any subset of X is countable
- $\mathbb{R}$ is uncountable

Open and Closed Sets

Basic examples: open and closed intervals (a,b) and $[a,b]$ in $\mathbb{R}$

for any $x \in (a,b)$ there is $r>0$ st $(x-r, x+r) \subseteq (a,b)$

Can take $r = \min\{x-a, b-x\}$

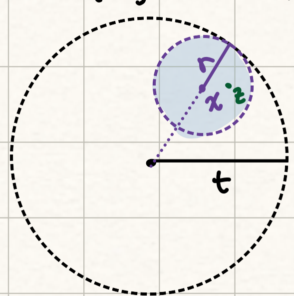
def A set U in a metric space (M,d) is open if for every point $x \in U$ there is $r>0$ such that:

$$B_r(x) \subseteq U$$

note: $\emptyset$ and M are open, trivially

prop: any open ball $\in (M,d)$ is open

PROOF: $B_t(y)$ in (M,d)



let $x \in B_t(y)$

let $r = t - d(x,y)$

Show for any $z \in B_r(x)$, $d(z,y) < t$ (triangle inequality) and so $z \in B_t(y)$

For a closed interval $[a,b]$ in $\mathbb{R}$,

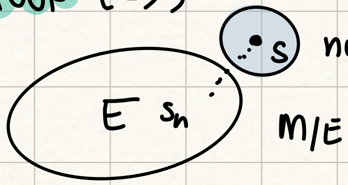
- its complement $(-\infty, a) \cup (b, \infty)$ is open
- for any sequence (s_n) in $[a,b]$ that converges in $\mathbb{R}$, $\lim s_n \in [a,b]$

def A set E in (M,d) is closed if its complement $M \setminus E$ is open

note M and $\emptyset$ are closed (and open!)

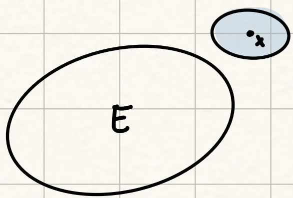
thm a set E in (M,d) is closed iff (*) for any sequence $(s_n) \in E$ that converges in M, $\lim s_n \in E$

PROOF $(\Rightarrow)$ NOTE: Check notes for 3/7 for written proof



no elements of E in S

$(\Leftarrow)$



$r_n = 1/n$

$s_n \in B_{1/n}(x)$

$s_n \to x$

ex in $\mathbb{R}$

- $[a, \infty)$ closed
- $\{\frac{1}{n} : n \in \mathbb{N}\}$ neither
- $\{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ closed
- $\mathbb{Q}$ - neither