monday 4/14

The Inverse Function & its derivative

Recall $f: X \to Y$ is

injective (one to one) if for all $x_1, x_2 \in X$ $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

surjective (onto) if for each $y \in Y$, there is $x \in X$ st $f(x) = y$

bijective (a one to one correspondence) if it is both injective and surjective

if $f: X \to Y$ is bijective, then for each $y \in Y$ there is exactly one $x \in X$ st $f(x) = y$

so we can define the inverse function $f^{-1}: Y \to X$ by $f^{-1}(y) = x$ $\quad X \underset{f^{-1}}{\overset{f}{\rightleftarrows}} Y$

note: $(f^{-1} \circ f)(x) = x$ for each $x \in X$

$(f \circ f^{-1})(y) = y$ for each $y \in Y$

Suppose I is an interval and $f: I \to \mathbb{R}$.

if f is strictly monotone, then f is injective $(\not\Leftarrow)$

claim if $f: I \to \mathbb{R}$ is continuous and injective, then it is strictly monotonic

justification if f is not strictly monotone, then there is $a < b < c$ in I st

either $f(a) \leq f(b) \geq f(c)$ or $f(a) \geq f(b) \leq f(c)$

Then by using IVT, we can show that some value is attained at two different points

thm Let I be an open interval and let $f: I \to \mathbb{R}$ be continuous and strictly monotonic. Then

(i) $J = f(I)$ is an open interval, finite or infinite

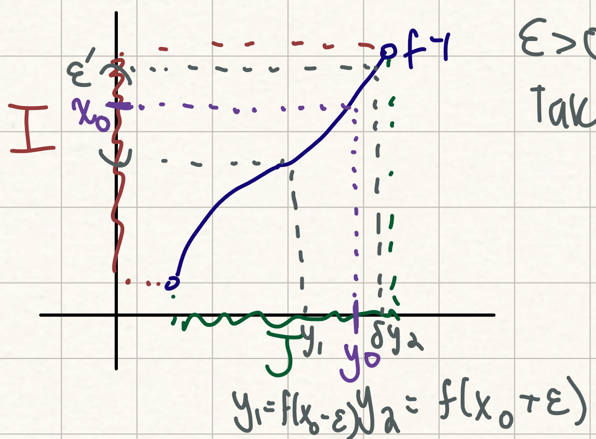
(ii) $f^{-1} J \to I$ is continuous & strictly monotone

PROOF let f be strictly increasing

(i) $J = (s_1, s_2)$ where $s_1 = \inf\{f(x) : x \in I\}$ and $s_2 = \sup\{f(x) : x \in I\}$

explain

(ii) Take $y_1 < y_2$ in J and let $x_1 = f^{-1}(y_1)$ and $x_2 = f^{-1}(y_2)$. Then $x_1 < x_2$ since f is strictly increasing. So f^{-1} is strictly increasing.



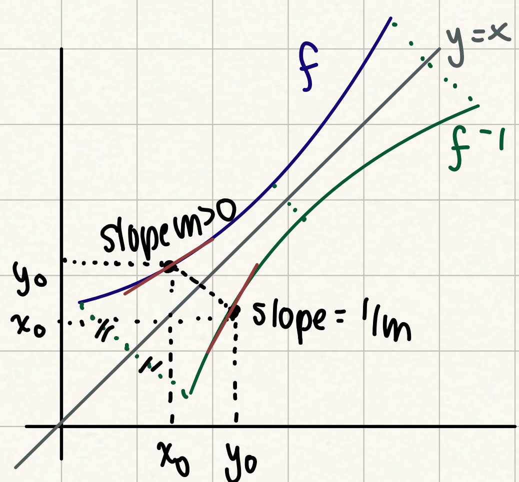
$\varepsilon>0$. Take $0<\varepsilon'<\varepsilon$.

Take $\delta=\min(f(x_0-\varepsilon), f(x_0+\varepsilon))$

Graph of f^{-1}

$f: x \to y$
$f^{-1}: y \mapsto x$ } reflect across $y=x$

Hence the graph of f^{-1} is the reflection of the graph of f across the line $y=x$



thm Let f be an injective continuous function on an open interval I. If f is differentiable at $x_0 \in I$ and $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}$$

PROOF Let $J = f(I)$. Then J is an open interval and $f^{-1}: J \to I$ is cont. & strictly monotone. Let (y_n) be a sequence in $J \setminus \{y_0\}$ converging to y_0. Let $x_n = f^{-1}(y_n)$. Then (x_n) is in $I \setminus \{x_0\}$ and $x_n \to x_0$.

Then $\lim_{n\to\infty} \frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0} = \lim_{n\to\infty} \frac{x_n - x_0}{f(x_n) - f(x_0)} = \lim_{n\to\infty} \frac{1}{\frac{f(x_n)-f(x_0)}{x_n - x_0}} = 1/f'(x_0)$

Hence $(f^{-1})'(y_0) = 1/f'(x_0)$

Friday 4/18

Linear Approximation & Taylor Polynomials

Let f be defined on an open interval I containing a.
Want to approx f by polynomials

Linear Approximation

Suppose f is differentiable at a. Then $0 = \lim_{x \to a} \left(\frac{f(x) - f(a)}{x-a} - f'(a) \right)$

$$= \lim_{x \to a} \frac{f(x) - (f(a) + f'(a) \cdot (x-a))}{x-a}$$

Let $T_1(x) = f(a) + f'(a)(x-a)$ and $R_1(x) = f(x) - T_1(x)$

Then $\lim_{x \to a} R_1(x)/(x-a) = 0$

So $R_1(x) = h(x)(x-a)$, where $h(x) \to 0$ as $x \to a$

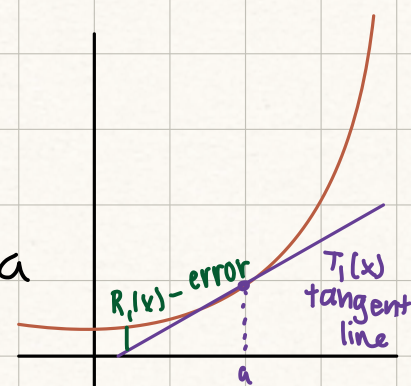
Thus if f is differentiable at a, then $f(x) = f(a) + f'(a)(x-a) + h(x)(x-a)$
where $h(x) \to 0$ as $x \to a$

So for x close to a,

$\Rightarrow f(x) \approx T_1(x) = f(a) + f'(a)(x-a)$

This is called the linear approximation of f at $x = a$

$T_1(a) = f(a)$, $T_1'(a) = f'(a)$



Taylor Polynomials

Suppose f is n times differentiable on I, ie $f'(x)$, $f''(x) = (f')'(x)$, $\dots$, $f^{(n)}(x)$ exist on I

Want: $T_n(x) = b_0 + b_1(x-a) + b_2(x-a)^2 + \dots + b_n(x-a)^n$ such that
$T_n(a) = f(a)$ and $T_n^{(k)}(a) = f^{(k)}(a)$ for $k = 1, \dots, n$

$T_n(a) = b_0$ Take $b_0 = f(a)$, $T_n^{(1)}(a) = b_1 + 2b_2(x-a) + \dots$

$T_n^{k}(a) = k!\, b_k$

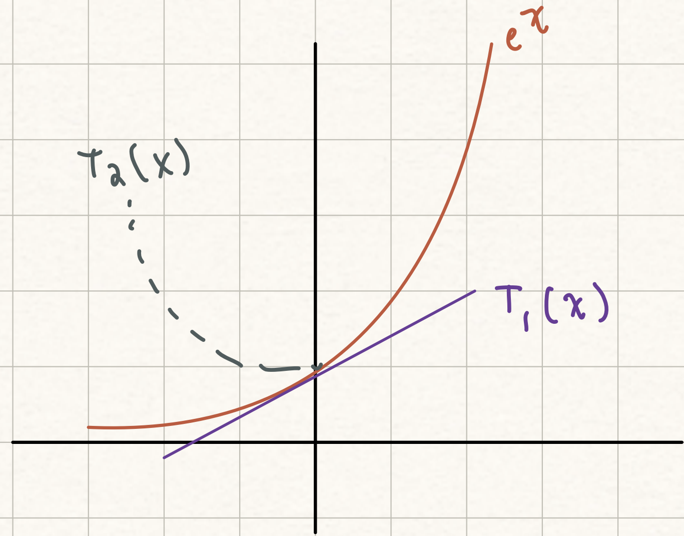
Take $b_k = f^{(k)}(a)/k!$

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f^2(a)}{2}(x-a)^2 + \dots + \frac{f^n(a)}{n!}(x-a)^n =$$

$$= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

is the n^{th} Taylor Polynomial of f at a.

ex| $f(x) = e^x$, $a=0$



$f^{(k)}(x) = e^x$ for all $k \in \mathbb{N}$

$f^{(k)}(0) = e^0 = 1$ for all $k \in \mathbb{N}$

$T_1(x) = 1 + x$

$T_2(x) = 1 + x + \frac{x^2}{2!}$

...

$T_n(x) = \sum_{k=0}^{n} \frac{1}{k!} x^k$

? if we approximate $f(x) \approx T_1(x)$, how to estimate the error
$R_n(x) = f(x) - T_n(x)$
Does $T_n(x) \to f(x)$ for every $x \in I$?

thm| Taylors Theorem - Suppose that f is $n+1$ differentiable on an open interval containing a. let T_n be the n^{th} Taylor polynomial of f at a. Let $R_n = f - T_n$
Then for each $x \neq a$ in I there exists a $y = y(x,n)$ between x and a st $R_n(x) = \frac{f^{n+1}(y)}{(n+1)!}(x-a)^{n+1}$

so $f(x) = T_n(x) + \frac{f^{(n+1)}(y)}{(n+1)!}(x-a)^{n+1}$

PROOF p. 251

ex Consider e^x on $(-M, M)$

for each $y \in (-M, M)$ and each $k \geq 0$,

$$|f^{(k)}(y)| = |e^y| \leq e^M$$

Hence for every $x \in (-M, M)$. $|R_n(x)| \leq \frac{e^M}{(n+1)!}|x|^{n+1} \to 0$ as $n \to \infty$

and hence $T_n(x) \to f(x)$

Since M was arbitrary, for each $x \in \mathbb{R}$, $T_n(x) \to e^x$

Corollary Suppose for every $k \in \mathbb{N}$, $f^{(k)}$ exists on I and there exists C such that $|f^{(k)}(y)| \leq C$ for all $y \in I$ and $k \in \mathbb{N}$

Then for every $x \in I$, $T_n(x) \to f(x)$ as $n \to \infty$

Taylor Series for f at a

$f(a) + f'(a) + f''(a)/2! + \dots =$

$$\sum_{k=0}^{\infty} \frac{f^k(a)}{k!}(x-a)^k$$