Week 8

Chapter 4 - Inner Product Spaces

assume $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$

def 4.1 relative to the standard basis $\{e_1, \ldots, e_n\}$ on $\mathbb{R}^n$, the Euclidean norm of a vector $x = (x_1, \ldots x_n)$ is

$$\|x\| = \sqrt{x_1^2 + \ldots + x_n^2}$$

this norm is nonlinear

def 4.2 Relative to the standard basis of $\mathbb{R}^n$, the dot product of two vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ is

$$x \bullet y = x_1 y_1 + \ldots + x_n y_n$$

this is linear

def 4.3 the dot product on $\mathbb{C}^n$ relative to the standard basis $(e_1, \ldots, e_n)$ is defined as

$$z \bullet w = z_1 \bar{w}_1 + \ldots + z_n \bar{w}_n$$

def 4.4 an inner product is a map $\langle , \rangle : V \times V \to \mathbb{F}$ with the following properties

(positive definite) $\langle v, v \rangle \geq 0$, with equality if & only if $v = 0$

(linearity in the 1st argument) for all $x, y, z \in V$, and all $\lambda \in \mathbb{F}$, $\langle (x + \lambda y, z) \rangle = \langle x, z \rangle + \lambda \langle x, y \rangle$

(conjugate symmetry) for all $x, y, z \in V$, $\langle x, y \rangle = \overline{\langle y, x \rangle}$

ex 4.5 Let V be the vector space of continuous real-valued functions on $[-1, 1]$, Then

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x) dx$$

is an inner product on V

def 4.6 Orthogonal - two vectors u, v in an inner product space $(V, \langle , \rangle)$ are said to be orthogonal if $\langle u, v \rangle = 0$

Lemma 4.7 Pythagorean Thm - Let $u, v \in V$ where V has an inner product $\langle , \rangle$ with respect to which u, v are orthogonal. Then, $\|u + v\|^2 = \|u\|^2 + \|v\|^2$

thm 4.8 Cauchy-Schwarz - Let u, v be vectors in an inner product space V. Then

$$|\langle u, v\rangle| \leq \|u\| \|v\|$$

with equality if and only if one of the u, v is a scalar multiple of the other

thm 4.9 Triangle Inequality - Suppose $u, v \in V$ where V is an inner product space. Then,

$$\|u+v\| \leq \|u\| + \|v\|$$