math436 week2

Week 2.1

thm 1.16 Steinitz Exchange Lemma

let V be an $\mathbb{F}$-vector space and $T = \{t_1, \dots, t_m\}$ a finite set of vectors that spans V. let $S = \{e_1, \dots, e_n\}$ be a linearly independent subset of V of size n. Then, there is a subset T' contained in T, also of size n such that $(T \setminus T') \cup S$ gives a spanning set

$\Rightarrow$ allows us to define dimension - size of base

PROOF Since T spans V, and $e_1 \in V$, we can write e_1 as a linear combo of elements of T

$\Rightarrow e_1 = \sum_{i=1}^{m} \lambda_i t_i$ where $\lambda_i \in \mathbb{F}$

let j be minimal st $\lambda_j \neq 0 \dots$ $e_1 = \lambda_j t_j + \sum_{i=j+1}^{m} \lambda_i t_i$

so we get $t_j = \frac{1}{\lambda_j}\left(e_1 - \sum_{i=j+1}^{n} \lambda_i t_i\right)$

... continue this w each e_i until T^n

corollary 1.7 let V be a vector space with a basis B of finite size n & a second basis B' of size m. Then m=n (proof by contradiction & use Steinitz Exchange lemma)

def 1.18 finite dimensional - an $\mathbb{F}$-vector space V is finite dimensional if it has a finite basis B

def 1.19 for a finite dimensional $\mathbb{F}$-vector space V, the dimension of V is the size of any $\mathbb{F}$-basis

remark 1.20 the field matters! $\mathbb{C}$ is dim 1 as a $\mathbb{C}$ vector space but dim 2 as an $\mathbb{R}$-vector space

2- Linear Maps

def 1.21 Given $\mathbb{F}$-vector spaces V and W, a function $\alpha: V \to W$ is a linear map if both hold:

L1) Scaling - for any $\lambda \in \mathbb{F}$ and any $v \in V$, $\alpha(\lambda \cdot v) = \lambda \cdot \alpha(v)$

L2) Additivity - for any $u, v \in V$, $\alpha(u+v) = \alpha(u) + \alpha(v)$

lemma 1.22 | lazy linear map def - for $\mathbb{F}$-vector spaces V and W, a function $\alpha: V \to W$ is a linear map iff for all $\lambda \in \mathbb{F}$ and all $u, v \in V$, we have

$$\alpha(\lambda \cdot u + v) = \lambda \alpha(u) + \alpha(v)$$

ex 1.23 |
- Scaling by any $\lambda \in \mathbb{F}$ gives $V \to V$. Scaling by $1_{\mathbb{F}}$ gives the identity map $Id_V: V \to V$
- the zero map $\alpha: V \to \{0\}$

lemma 1.24 | if $\alpha, \beta: V \to W$ are linear maps of vector spaces and $\{e_1, \dots, e_n\}$ is a basis for V, then $\alpha = \beta$ iff $\alpha(e_i) = \beta(e_i)$ for all i

lemma 1.25 | all the data of a linear map $\alpha: V \to W$, where V has basis $\{e_1, \dots, e_n\}$ and W has basis $\{f_1, \dots, f_m\}$ is captured by mn scalars $\lambda_{i,j}$ where $\alpha(e_i) = \sum_{j=1}^{m} \lambda_{ij} f_i$

Week 2 episode 2

def 2.6 | a linear map of $\mathbb{F}$-vector spaces $\alpha: V \to W$ has a Kernal

$$Ker(\alpha) = \{v \in V \mid \alpha(v) = 0\}$$ and image ...

$$Im(\alpha) = \{w \in W \mid \text{there is a } v \in V \text{ such that } \alpha(v) = w\}$$

we say α is surjective if $Im(\alpha) = W$

ex 2.7 |
- $Ker(Id_V) = \{0\}$ and $Im(Id_V) = V$
- for the zero map, the kernal is V & the image is $\{0\}$

lemma 2.8 | let $\alpha: V \to W$ be a linear map. Then $Ker(\alpha)$ is a subspace of V and $Im(\alpha)$ is a subspace of W

lemma 2.9 | a linear map $\alpha: V \to W$ is injective iff $Ker(\alpha) = \{0\}$

def 2.10 | the rank of a linear map α is $rk(\alpha) = dim(Im(\alpha))$ and the nullity is $n(\alpha) = dim(Ker(\alpha))$

thm 2.11 Rank-Nullity

let V be a finite dimensional vector spaces and let $\alpha: V \to W$ be a linear map. Then,

$$\dim(V) = \dim(\ker(\alpha)) + \dim(\mathrm{Im}(\alpha))$$

in particular, the image is finite dimensional

def 2.12 Suppose that U, W are subspaces of V. then V is a **direct sum** of U and W, written $V = U \oplus W$ if every $v \in V$ can be written as $u + w$ for some unique $u \in U$ and $w \in W$

def 2.14 for a vector space V and subspaces U, W, we say $V = U + W$ if every v can be written as $u + w$ for some (not necessarily unique) $u \in U$, $w \in W$