exam 2

DEFINITIONS

- let (s_n) be a sequence in $\mathbb{R}$ or in (M,d), a **subsequence** of (s_n) is a sequence $(s_{n_k})_{k\in\mathbb{N}}$ where $n_k \in \mathbb{N}$ and $n_1 < n_2 < n_3 < \dots$

- if the sequence of partial sums (s_n) converges, we say that the series **converges**
  if the sequence of partial sums (s_n) diverges, we say that the series **diverges**

- We say that $\sum a_n$ **converges absolutely** if $\sum |a_n|$ converges

- **Cauchy Criterion for series** - for each $\varepsilon > 0$, there exists a number N such that

$$n \geq m > N \text{ implies } \left|\sum_{k=m}^{n} a_k\right| < \varepsilon$$

- a set U in a metric space (M,d) is **open** if for every point $x \in U$, there is $r>0$ such that $B_r(x) \subseteq U$

- a set E in (M,d) is closed if its complement $M \setminus E$ is open

- a set X is **finite** if either $X = \emptyset$ or for some $n \in \mathbb{N}$, there is a bijection from X to $\{1, 2, \dots, n\}$

Moreover, for each $n \in \mathbb{N}$:

$$\left|\sum_{k=n+1}^{\infty} (-1)^{k+1} a_k\right| \leq a_n$$

ie $|s - s_n| \leq a_n$

- X is **countably infinite** if there is a bijection from X to $\mathbb{N}$ ie we can list the elements of X

- X is **countable** if it is either finite or countably infinite

- X is **uncountable** if it is not countable ie too large for its elements to be listed a function f is continuous at $x_0 \in dom(f)$ if for every sequence (x_n) in $dom(f)$ converging to x_0, $\lim(f(x_n)) = f(x_0)$ ie $f(x_n) \to f(x_0)$

a function f is continuous at $x_0 \in dom(f)$ if for every $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in dom(f)$ and $|x - x_0| < \delta$ implies that $|f(x) - f(x_0)| < \varepsilon$

a function f is continuous if f is continuous at each $x_0 \in dom(f)$

THEOREMS AND COROLLARIES

if the sequence (s_n) converges, then every subsequence converges to the same limit

Bolzono-Weierstrass Theorem - every bounded sequence in $\mathbb{R}$ has a convergent subsequence

let (s_n) be any sequence. There exists a monotonic subsequence whose limit is $\limsup s_n$ and there exists a monotonic subsequence whose limit is $\liminf s_n$

a series converges if and only if it satisfies the Cauchy criterion

if a series $\sum a_n$ converges, then $\lim a_n = 0$

Comparison Test - let $\sum a_n$ be a series where $a_n \geq 0$ for all n

(i) if $\sum a_n$ converges and $|b_n| \leq a_n$ for all n, then $\sum b_n$ converges

(ii) if $\sum a_n = +\infty$ and $b_n \geq a_n$ for all n, then $\sum b_n = +\infty$

absolutely convergent series are convergent

Ratio Test - a series $\sum a_n$ of nonzero terms

(i) converges absolutely if $\limsup\left|\frac{a_{n+1}}{a_n}\right|<1$

(ii) diverges if $\liminf\left|\frac{a_{n+1}}{a_n}\right|>1$

(iii) if $\liminf\left|\frac{a_{n+1}}{a_n}\right|\leq 1\leq \limsup\left|\frac{a_{n+1}}{a_n}\right|$ then the test gives no information

Root Test - let $\sum a_n$ be a series and let $\alpha=\limsup|a_n|^{1/n}$. The series $\sum a_n$

(i) converges absolutely if $\alpha<1$

(ii) diverges if $\alpha>1$

(iii) if $\alpha=1$ we get no information

p-series - $\sum_{n=1}^{\infty}\frac{1}{n^p}$

(i) converges for $p>1$

(ii) diverges for $p\leq 1$

geometric series - $\sum_{n=0}^{\infty} ar^n$ where $a\neq 0$

(i) converges if $|r|<1$ and its sum is $\frac{a}{1-r}$

(ii) diverges if $|r|\geq 1$

Alternating Series Theorem - if sequence (a_n) is decreasing, $a_n\geq 0$ for all n, and $\lim a_n=0$, then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}a_n=a_1-a_2+a_3-a_4+\ldots$ converges

Moreover, for each $n\in\mathbb{N}$:

$$\left|\sum_{k=n+1}^{\infty}(-1)^{k+1}a_k\right|\leq a_n$$

ie $|s-s_n|\leq a_n$

For a power series $\sum_{n=0}^{\infty}a_nx^n$, let $\beta=\limsup|a_n|^{1/n}$

(i) if $\beta=0$, then the series converges absolutely for all $x\in\mathbb{R}$

(ii) if $\beta=+\infty$, then the series converges for $x=0$ only

(iii) else, then the series converges absolutely for $|x|<1/\beta$ and div. for $|x|>1/\beta$

$R=1/\beta$ is called the radius of convergence of the series let f be a real valued function whose domain is a subset of $\mathbb{R}$. Then f is continuous at x_0 in dom(f) if and only if for each $\varepsilon > 0$, there exists $\delta > 0$ st $x \in \text{dom}(f)$ and $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$

let f be a real valued function whose domain is a subset of $\mathbb{R}$. If f is continuous at x_0 in dom(f), then $|f|$ and kf, $k \in \mathbb{R}$, are continuous at x_0

let f and g be real-valued functions that are continuous at x_0 in $\mathbb{R}$, then

(i) $f+g$ is continuous at x_0

(ii) fg is continuous at x_0

(iii) f/g is continuous at x_0 if $g(x_0) \neq 0$

if f is continuous at x_0 and g is continuous at $f(x_0)$, then the composite function $g \circ f$ is continuous at x_0

let f be a continuous real valued function on a closed interval $[a,b]$. Then f is a bounded function. Moreover, f assumes its maximum and minimum values on $[a,b]$: that is, there exists $x_0, y_0 \in [a,b]$ such that $f(x_0) \leq f(x) \leq f(y_0)$ for all $x \in [a,b]$

intermediate value theorem - suppose that f is a continuous function on an interval I if $a < b$ are in I and y is between $f(a)$ and $f(b)$, then there exists $x \in (a,b)$ such that $f(x) = y$

PROOFS

1 **Comparison Test** - let $\sum a_n$ be a series where $a_n \geq 0$ for all n
(i) if $\sum a_n$ converges and $|b_n| \leq a_n$ for all n, then $\sum b_n$ converges
(ii) if $\sum a_n = +\infty$ and $b_n \geq a_n$ for all n, then $\sum b_n = +\infty$

PROOF

(i) Since $|b_n| \leq a_n$ for all n, for any $n \geq m$

$$(*) \left| \sum_{k=m}^{n} b_k \right| \leq \sum_{k=m}^{n} |b_k| \leq \sum_{k=m}^{n} a_k = \left| \sum_{k=m}^{n} a_k \right|$$

Since $\sum a_k$ converges, it satisfies the Cauchy Criterion, and it follows by (*) that $\sum b_k$ and $\sum |b_k|$ also satisfy the Cauchy Criterion, and hence they converge □

(ii) let s_n and t_n be partial sums for $\sum a_n$ and $\sum b_n$ respectively
Since $\sum a_n = \infty$, $s_n \to \infty$ and since $t_n \geq s_n$ for each n, it follows that $t_n \to +\infty$
$\therefore \sum b_n = +\infty$

Root Test - let $\sum a_n$ be a series and let $\alpha = \limsup |a_n|^{1/n}$. The series $\sum a_n$
(i) converges absolutely if $\alpha < 1$
(ii) diverges if $\alpha > 1$
(iii) if $\alpha = 1$ we get no information

PROOF

(i) let $\alpha < 1$. We take β such that $\alpha < \beta < 1$
Since $\alpha = \limsup |a_n|^{1/n}$ and $\beta > \alpha$, there is N such that for all $n > N$, $|a_n^{1/n}| < \beta$, and so $|a_n| < \beta^n$
Since $|\beta| < 1$, the geometric series $\sum B^n$ converges, and hence, by the comparison test, $\sum a_n$ converges absolutely

(ii) Suppose $\alpha = \limsup |a_n|^{1/n} > 1$, then there are infinitely many n such that $|a_n|^{1/n} > 1$ and hence $|a_n| > 1$, it follows that $a_n \not\to 0$ and hence $\sum a_n$ diverges

(iii) for each of the series $\sum 1/n$ and $\sum 1/n^2$, $\alpha = 1$ but $\sum 1/n$ diverges and $\sum 1/n^2$ converges, so $\alpha = 1$ does not guarantee convergence or divergence

Let f be a real valued function whose domain is a subset of $\mathbb{R}$. Then f is continuous at x_0 in dom(f) if and only if for each $\varepsilon>0$, there exists $\delta>0$ st $x \in$ dom(f) and $|x-x_0|<\delta$ implies $|f(x)-f(x_0)|<\varepsilon$

PROOF (a) $\Rightarrow$ (i) Suppose a holds

Let (x_n) be a sequence in dom(f) converging to x_0. Let $\varepsilon>0$. By a, there is $\delta>0$ such that $x\in$ dom(f) and $|x-x_0|<\delta$ imply $|f(x)-f(x_0)|<\varepsilon$. Since $x_n \to x_0$ there is N such that for all $n>N$, $|x_n - x_0|<\delta$ and hence $|f(x_n)-f(x_0)|<\varepsilon$

Thus, $|f(x_n)-f(x_0)|<\varepsilon$ for all $n>N$. So, $f(x_n)\to f(x_0)$

(1)$\Rightarrow$(2) By contraposition. Suppose (a) fails i.e. there exists $\varepsilon>0$ such that for every $\delta>0$, there is $x\in$ dom(f) with $|x-x_0|<\delta$ such that $|f(x)-f(x_0)|\geq\varepsilon$.

Take $\delta = 1/n$. We see that for each $n\in\mathbb{N}$ there is $x_n\in$ dom(f) with $|x_n - x_0|<1/n$ such that $|f(x_n)-f(x_0)|\geq\varepsilon$; this sequence converges to x_0, but $f(x_n)\not\to f(x_0)$; so 1 fails, thus (1) $\Rightarrow$ (2)

Let f be a continuous real valued function on a closed interval $[a,b]$. (i) Then f is a bounded function. Moreover, (ii) f assumes its maximum and minimum values on $[a,b]$; that is, there exists $x_0, y_0 \in [a,b]$ such that $f(x_0)\leq f(x)\leq f(y_0)$ for all $x\in[a,b]$

PROOF

(i) by contradiction

Suppose that f is not bounded on $[a,b]$

Then, for each $n\in\mathbb{N}$, there is $x_n\in[a,b]$ such that $|f(x_n)|>n$

By Bolzono Weierstrass, since (x_n) is a bdd sequence in $\mathbb{R}$, it has a sub seq. (x_{n_k}) converging to some $x_0\in\mathbb{R}$. Since $[a,b]$ is closed, $x_0\in[a,b]$

Since f is continuous on $[a,b]$ and hence at x_0, $f(x_{n_k})\to f(x_0)$

This contradicts $|f(x_{n_k})|\to\infty$; so f is bounded $\square$

(ii) Let $M = \sup\{f(x) : x \in [a,b]\}$

Since f is bdd on $[a,b]$, $M \in \mathbb{R}$

By the def of supremum, for each $n \in \mathbb{N}$, there exists $y_n \in [a,b]$ such that

$$M - \frac{1}{n} < f(y_n) \le M \quad *$$

By Bolzano Weierstrass, (y_n) has a convergent subsequence (y_{n_k}) and since $[a,b]$ is closed, $y_0 = \lim y_{n_k} \in [a,b]$. Since f is continuous at y_0, $f(y_{n_k}) \to f(y_0)$

Also, from $(*)$ $f(y_{n_k}) \to M$. Thus, $f(y_0) = M$ and so f assumes its max value

TRUE OR FALSE

(1) a sequence (s_n) can have infinitely many subsequential limits? TRUE (from hw 5.4)

(2) There is no sequence (s_n) such that $|s_n| < 2$ for all n and every subsequence of (s_n) diverges TRUE (Bolzano Weierstrass)

(3) if $\liminf s_n = a$, then for every convergent subsequence (s_{n_k}) of (s_n), we have $\lim s_{n_k} \ge a$ TRUE (thm from week 6)

(4) If a sequence (a_n) converges, then the series $\sum a_n$ converges FALSE

let $a_n = 1/n$; $a_n \to 0$ but $\sum a_n$ diverges

(5) If for a series of nonnegative numbers the partial sums satisfy $s_n \le 100$ for all n, then the series converges TRUE

(6) if $\sum a_n$ diverges, $a_n \ge 0$ for all n, and $b_n \ge 0$ for all n, then $\sum (a_n + b_n)$ diverges true

(7) if a series $\sum a_n$ converges, then for every $\varepsilon > 0$, there exists n st $\left|\sum_{k=n}^{2n} a_k\right| < \varepsilon$ true (Cauchy Criterion)

(8) if a series is convergent, then it is absolutely convergent FALSE

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges but does not converge absolutely

(9) if $a_n \le b_n$ for all n and $\sum b_n$ converges, then $\sum a_n$ converges False

$a_n = -1$ for all n and $b_n = 1/n^2$, then $a_n \le b_n$ $\forall n$ but $\sum a_n$ diverges

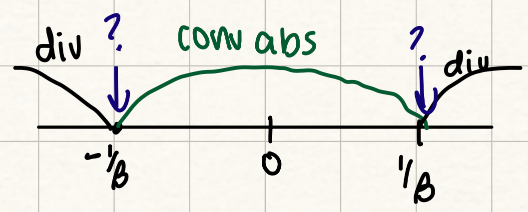
(10) if $a_n \ne 0$ and $\left|\frac{a_{n+1}}{a_n}\right| < 1$ for all n, then the series $\sum a_n$ converges FALSE

$a_n = 1/n$

(11) if $\lim a_n = 0$ then $\sum_{n=1}^{\infty} (-1)^n a_n$ converges False

let $a_n = (\frac{1}{n})(-1)^{n+1}$

(12) A power series $\sum_{n=0}^{\infty} a_n x^n$ cannot conv at $x=2$ and diverge at $x=1$ TRUE



(13) if $\limsup |a_n|^{1/n} = 3$, then the power series $\sum_{n=0}^{\infty} a_n x^n$ conv for every $x \in (-3, 3)$ False $R = 1/3$ so 3 diverges

(14) every nonzero integer has exactly 2 decimal expansions TRUE

(15) every non-zero rational func has exactly 2 dec exp False

only if it can be in the form $1/2^m 5^n$ where $n, m \in \mathbb{N} \cup \{0\}$ ex $1/3$

(16) if a subset of a set is uncountable, then X is uncountable TRUE

(17) in any metric space, a set that contains a single point is closed TRUE

(18) in $\mathbb{R}$ w/ the standard metric, the complement of every countable set is open FALSE

ex. $\mathbb{Q}$ is a countable set but $\mathbb{R}/\mathbb{Q}$ is not open b/c of density of rationals

(19) if a function f is continuous on $\mathbb{R}$ & $f(x_0) = c > 0$, then there is an open intervel containing x_0 such that $f(x_0) > c/2$ for every x in this intervel TRUE

(20) if a func f is discontinuous at $x_0 \in \text{dom}(f)$, then for every sequence $(x_n) \in \text{dom}(f)$ converging to x_0, the sequence $(f(x_n))$ diverges False

$$f(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

(21) if f is continuous on $\mathbb{R}$, then $g(x) = \sin x / (f(x))^2 + 1$ is continuous on $\mathbb{R}$ True

(22) the domain of the composition $g \circ f$ is $\text{dom}(f) \cap \text{dom}(g)$ FALSE

$\text{dom}(g \circ f) = \{x \in \text{dom}(f) : f(x) \in \text{dom}(g)\}$; $f(x) = 0$ $g(x) = 1/x$

(23) if f is a rational function, then f is bounded on any closed interval $[a,b] \subset \text{dom}(f)$ TRUE

(24) if f is defined on $[a,b]$, $f(a) = -1$ and $f(b) = 1$ then $f(c) = 0$ for some $c \in (a,b)$ FALSE

$a = -1$ $b = 1$ $f(x) = \begin{cases} -1 & x \leq 0 \\ 1 & x > 0 \end{cases}$ since f is not continuous

(25) for any continuous function f on $\mathbb{R}$ and any interval (a,b), the set $f((a,b))$ is either a single pt or an open interval FALSE ?

$f(x) = x^2$ then $f((-1,1)) =$