monday 4/7

more on limits and continuity

ex 1 $f(x) = \sin(1/x)$

$\text{dom}(f) = \mathbb{R} \setminus \{0\}$

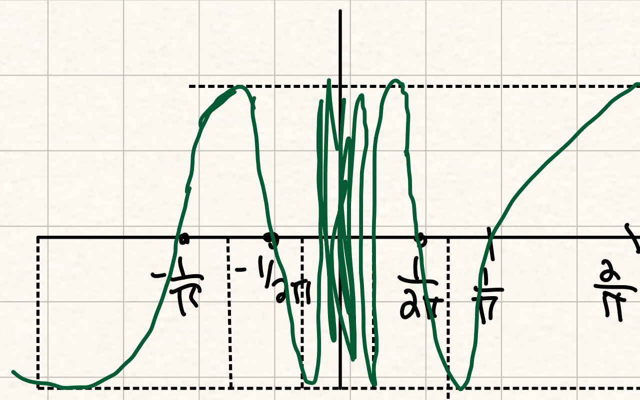
f is continuous on dom(f)

f is bounded: $|f(x)| \leq 1$ for all x

$f(x) = 0$ for $x = \frac{1}{n\pi}$ $\quad n \in \mathbb{Z} \setminus \{0\}$

$f(x) = 1$ for $x = \frac{1}{\frac{\pi}{2} + 2\pi n}$ $\quad n \in \mathbb{Z}$

$f(x) = -1$ for $x = \frac{1}{-\frac{\pi}{2} + 2\pi n}$ $\quad n \in \mathbb{Z}$



note: f is not uniformly cont on $(0,1]$

? Does $\lim_{x\to 0} f(x)$ exist? Does $\lim_{x\to 0^+} f(x)$ exist?

$\lim_{x\to 0^+}$ DNE. Consider $x_n = 1/\pi n$ and $y_n = \frac{1}{\pi/2 + 2\pi n}$

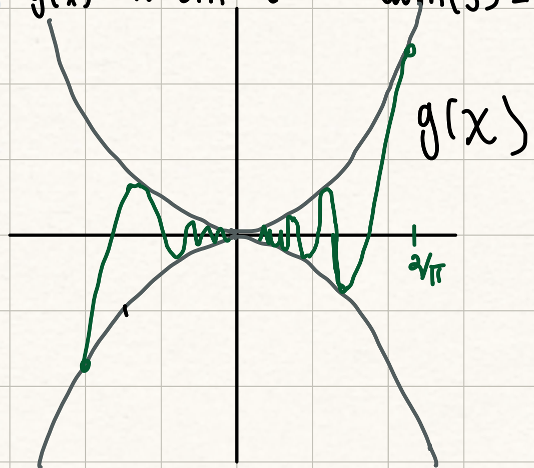
Then $x_n, y_n \in (0,\infty)$ for all n, $x_n \to 0$, $y_n \to 0$

$f(x_n) = 0$ and $f(y_n) = 1$ for all n

it follows that $\lim_{x\to 0} f(x)$ DNE

? Can we define $f(0)$ such that the resulting func is cont on $\mathbb{R}$? No!

ex $g(x) = x^2 \sin 1/x$ $\quad \text{dom}(f) = \mathbb{R}\setminus\{0\}$ $\quad |g(x)| \leq x^2$ for all $x \in \text{dom}(f)$



claim: $\lim_{x\to 0} g(x) = 0$

proof: let $\varepsilon > 0$, let $\delta = \sqrt{\varepsilon}$

Then for all x st $0 < |x - 0| < \delta$ we have

$|f(x) - 0| = |x^2 \sin \frac{1}{x}| \leq x^2 < \delta^2 = \varepsilon$

? Can we def value at 0 so that the resulting func is cont on $\mathbb{R}$. Yes,
set it to be 0

let $\tilde{g}(x) = \begin{cases} g(x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

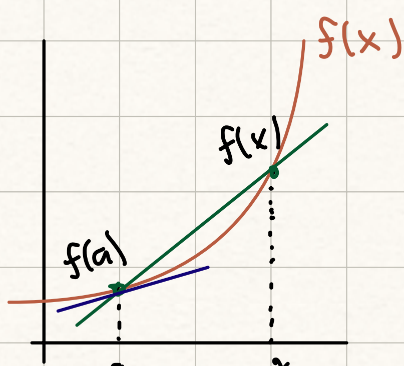
Then $\tilde{g}$ is cont on $\mathbb{R}$

The derivative

def Suppose that f is defined on an open interval containing a. We say that f is differentiable at a, or has a derivative at a, if $\displaystyle\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists and is finite. We call this finite limit the derivative of f at a & denote it $f'(a)$

note: denoting $x - a$ by h, we write:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$



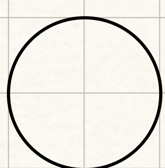
note: $\dfrac{f(x) - f(a)}{x - a}$ is the slope of the secant line

$$\lim_{x \to 0} \frac{\tilde{g}(x) - \tilde{g}(0)}{x - 0} = \frac{x^2 \sin 1/x}{x}$$

$$\lim_{x \to 0} x \sin \tfrac{1}{x} = 0$$

↑ exercise

$f'(a)$ is the slope of the tangent line to the graph of f at $(a, f(a))$



Wednesday 4/9

The derivative

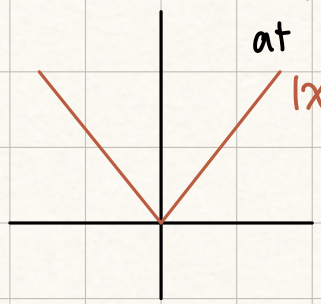
ex| for each $n \in \mathbb{N}$ consider $f(x) = x^n$. For any $a \in \mathbb{R}$, $f'(a) = na^{n-1}$

PROOF $n=1$ - check

$n \geq 2$ For all $x \in \mathbb{R}$, $x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \ldots + xa^{n-2} + a^{n-1})$

Then $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{x^n - a^n}{x-a} = \lim_{x\to a}(x^{n-1} + x^{n-2}a + \ldots + a^{n-1}) = na^{n-1}$

ex| $f(x) = |x|$ is not differentiable at $a=0$



PROOF $\lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^+} \frac{|x|}{x} = 1$

$\lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0^-} \frac{|x|}{-x} = -1$

limit DNE

Since $\lim_{x\to 0^+} \ldots \neq \lim_{x\to 0^-} \ldots$, $\lim_{x\to 0} \ldots$ DNE & $\therefore$ not differentiable

note: continuity does not imply differentiability

thm| if f is differentiable at a, then f is continuous at a

PROOF We will show that $\lim_{x\to a} f(x) = f(a)$, which means that f is continuous at a

$f(x) = (x-a) \cdot \frac{f(x)-f(a)}{x-a} + f(a)$

Since f is diff at a, $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a) \in \mathbb{R}$

As $x \to a$, $x-a \to 0$, and by the thm on limits of functions,

$\lim_{x\to a} f(x) = 0 \cdot f'(a) + f(a) = f(a)$ $\square$

Thm| Differentiation Rules

Suppose f and g are differentiable at a

Then so are cf, $f+g$, fg and f/g provided that $g(a) \neq 0$

The derivatives are:

(i) $(cf)'(a) = c f'(a)$

(ii) $(f+g)'(a) = f'(a) + g'(a)$

(iii) $(fg)'(a) = f'(a)g(a) + f(a)g'(a)$

(iv) $(f/g)'(a) = \dfrac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2}$ provided that $g(a) \neq 0$

PROOF Since f and g are differentiable at a, they are defined on an open interval $J \ni a$

(i, ii) exercise or pg 226

(iii) For all $x \in J \setminus \{a\}$

$$\frac{(fg)(x) - (fg)(a)}{x-a} = \frac{f(x)g(x) - f(a)g(a) + f(a)g(x) - f(a)g(x)}{x-a}$$

$$= \frac{f(x)-f(a)}{x-a} g(x) + f(a) \frac{g(x)-g(a)}{x-a} \quad (*)$$

Since g is diff at a, it is continuous at a, and so $\lim_{x \to a} g(x) = g(a)$

Taking $\lim_{x \to a}$ in $(*)$ we obtain (iii)

(iv) Since g is continuous at a and $g(a) \neq 0$, $g(x) \neq 0$ for all x in an open interval $I \subseteq J$ st $a \in I$

For all $x \in I$,

$$(f/g)(x) - (f/g)(a) = \frac{f(x)}{g(x)} - \frac{f(a)}{g(a)} = \frac{f(x)g(a) - f(a)g(x)}{g(x)g(a)}$$

$$= \frac{f(x)g(a) - f(a)g(x) + f(a)g(a) - f(a)g(a)}{g(x)g(a)} = \big((f(x) - f(a))g(a) - f(a)(g(x) - g(a))\big) \cdot \frac{1}{g(x)g(a)}$$

Dividing by $x-a$ & taking $\lim_{x \to a}$, we obtain

$$\frac{f'(a)g(a) - f(a)g'(a)}{(g(a))^2} \quad \square$$

thm Chain Rule

Suppose that f is differentiable at a and g is differentiable at $f(a)$. Then the composite function $g \circ f$ is differentiable at a and

$(g \circ f)'(a) = g'(f(a)) \cdot f'(a)$ $(*)$

DISCUSSION

$$\frac{g(f(x)) - g(f(a))}{x-a} = \frac{g(f(x)) - g(f(a))}{f(x) - f(a)} \cdot \frac{f(x) - f(a)}{x-a}$$

if there is an open interval $I \ni a$ st $f(x) \neq f(a)$ for all $x \in I$

CASE 1: Let (x_n) be a sequence in $I \setminus \{a\}$ conv. to a.

Since f is cont at a, $y_n = f(x_n) \to f(a)$

Since g is cont at $f(a)$, $g(y_n) \to g(f(a))$

Then $$\frac{g(f(x_n)) - g(f(a))}{x_n - a} = \frac{g(y_n) - g(f(a))}{y_n - f(a)} \cdot \frac{y_n - f(a)}{x_n - a} \to g'(f(a)) \cdot f'(a)$$

Since (x_n) was arbitrary, we obtain $(*)$

...

CASE 2: (OUTLINE) there x arbitrarily close to a st $f(x) = f(a)$

In this case, $f'(a) = 0$

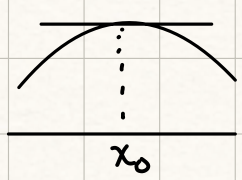
also, $g \circ f'(a) = 0$ see p229

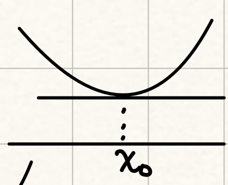
So $(g \circ f)'(a) = 0 = g'(f(a)) \cdot f'(a)$

Friday 4.11

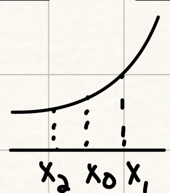
Zeros of the derivative

thm Suppose that f is defined on an open interval $I \ni x_0$. If f has its max or min values on I at x_0 and f is differentiable at x_0, then $f'(x_0) = 0$





$f'(x_0) > 0$



PROOF (by contraposition)

We show that if $f'(x_0) > 0$ or $f'(x_0) < 0$, then f does not have a max or a min at x_0

Suppose $f'(x) > 0$ (similar for <0). then, $\lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} > 0$.

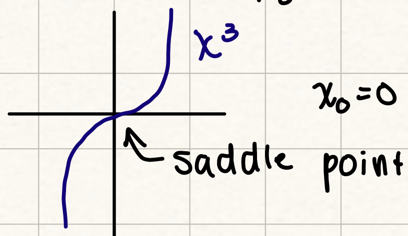
Then there exists $\delta > 0$ such that for all $x \in \underbrace{(x_0-\delta, x+\delta)}_{\subseteq I} \setminus \{x_0\}$,

$$\frac{f(x)-f(x_0)}{x-x_0} > 0.$$

Let $x_2 \in (x_0, x_0+\delta)$. Then $x_2 - x_0 > 0$ and so $f(x_2)-f(x_0) > 0$.
Let $x_1 \in (x_0-\delta, x_0)$. Then $x_1 - x_0 < 0$ and so $f(x_1)-f(x_0) < 0$
So $f(x_1) < f(x_0) < f(x_2)$. Thus $f(x_0)$ is neither max nor min. $\square$

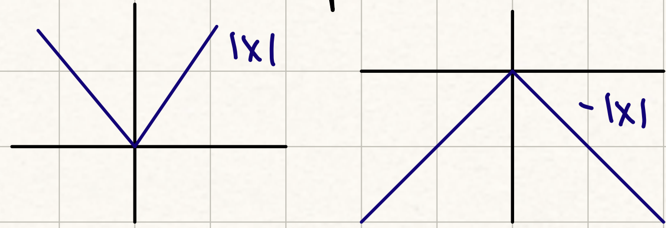
T/F) if f is diff on I and $x_0 \in I$ and $f'(x_0) = 0$, Then f has a local max or min at x_0

False...



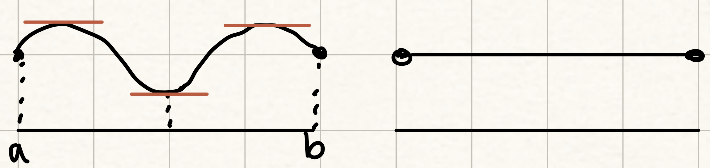
note: f can have a max or min at a point where it is not differentiable

ex $|x|$ or $-|x|$



thm| Rolles Theorem

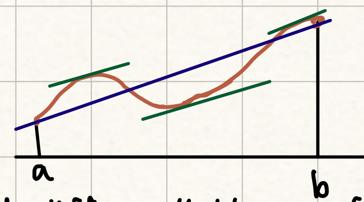
Suppose that f is continuous on $[a,b]$, differentiable on (a,b), and $f(a) = f(b)$. Then there exists $x_0 \in (a,b)$ such that $f'(x_0) = 0$



PROOF

Since f is continuous on $[a,b]$ it attains its max and min values on $[a,b]$.
If f attains its max or min at some $x_0 \in (a,b)$, then, by the previous therom, $f'(x_0) = 0$. Otherwise, f attains max and min values at the endpoints, and we have

$\max_{[a,b]} f = \min_{[a,b]} f = f(a) = f(b)$. It follows that f is constant on $[a,b]$ and so $f'(x_0)=0$ for every $x_0 \in (a,b)$ ◻



thm The Mean Value Thm (MVT)

Suppose that f is continuous on $[a,b]$ and differentiable on (a,b). Then there exists (at least one) $x_0 \in (a,b)$ such that

$$f'(x_0) = \frac{f(b)-f(a)}{b-a} \quad \text{ie}$$

$$f(b)-f(a) = f'(x_0)(b-a)$$

geometric meaning: the slope of the tangent line at $(x_0, f(x_0))$ equals the slope of the secant line through $(a, f(a))$ and $(b, f(b))$

x · time

an interpretation: the instantaneous rate of change at x_0 equals the average rate of change over $[a,b]$

PROOF Let L be the linear function whose graph is the straight line through $(a, f(a))$ and $(b, f(b))$ ($L(x) = ?$). Then, $L'(x) = \frac{f(b)-f(a)}{b-a}$

Let $g(x) = f(x) - L(x)$. Then, g is cont on $[a,b]$, differentiable on (a,b), and $g(a) = g(b)$. Then by Rolles thm, there is $x_0 \in (a,b)$ such that $g'(x_0) = f'(x_0) - L'(x_0) = 0$. Then $f'(x_0) = \frac{f(b)-f(a)}{b-a}$ ◻

Corollaries on MVT

Corollary 1 if f is differentiable on (a,b) and $f'(x)=0$ for all $x \in (a,b)$, then f is constant on (a,b)

PROOF Let $x_1 < x_2$ be in (a,b). Note. f is cont on (a,b)
Applying MVT to $[x_1, x_2]$ we obtain that there is $x_0 \in (x_1, x_2)$ st $f(x_2) - f(x_1) = f'(x_0)(x_2 - x_1) = 0(x_2 - x_1) = 0$

So $f(x_2) = f(x_1)$. Thus f is constant on (a,b)

Corollary 2 if f and g are differentiable on (a,b) and $f'(x) = g'(x)$ for all $x \in (a,b)$

then $f(x) = g(x) + C$

PROOF apply cor 1 to $h(x) = f(x) - g(x)$

def Let f be a function on an interval I.

f is increasing on I if

for any $x_1 < x_2$ in I, $f(x_1) \leq f(x_2)$

f is strictly increasing on I if

for any $x_1 < x_2$ in I, $f(x_1) < f(x_2)$

corollary 3 let f be differentiable on (a,b)

If $f'(x) \geq 0$, then f is increasing on (a,b)

If $f'(x) > 0$, then f is strictly increasing on (a,b)

PROOF in HW (converse in hw)

converse of 2nd is false b/c x^3 is strictly increase but $f'(0) = 0$