2.7 Constructing new Vector Spaces from Old

example 2.33

- $\mathbb{F}^n$ is a $\mathbb{F}$-vector space
- the set of polynomials with coeff in $\mathbb{F}$, denoted $\mathbb{F}[x]$ is an $\mathbb{F}$ vector space
- the set of sequences $\mathbb{F}^\infty$ with entries in $\mathbb{F}$ is an $\mathbb{F}$-vector space
- the set of functions $\mathbb{F} \to \mathbb{F}$ is a vector space over $\mathbb{F}$
- for fields where we have a notion of continuity or smoothness, continuous functions, differentiable functions, or continuously differentiable func. form a subspace of the func. space

2.7.1 Products of Vector Spaces

def 2.34 | given 2 $\mathbb{F}$-vector spaces U and V, the product space $U \times V$ is a vector space as follows:

- the vectors consist of pairs (u, v) where $u \in U$ and $v \in V$

$(u_1, v_1) +_{U\times V} (u_2, v_2) = (u_1 +_U u_2, v_1 +_V v_2)$

$\lambda_{U\times V}(u, v) = (\lambda_U u, \lambda_V v)$

(the subscripts on operations specify which VS we're in)

lemma 2.36 | (basis of product space) given finite-dimensional $\mathbb{F}$-vector spaces U, V with bases $u_1 \dots u_m$ and $v_1 \dots v_m$ respectively, a basis for $U \times V$ is given by the set of pairs:

$$\{(u_i, 0_V), (0_U, v_j) \mid 1 \le i \le m, 1 \le j \le n\}$$

corollary 2.37 | if U and V are finite dimensional $\mathbb{F}$-vector spaces of dimension m and n respectively, then the dimension of the product $U \times V$ is $m+n$

2.8 Quotients of Vector Spaces

def 2.38 | (quotient of a vector space). Given an $\mathbb{F}$-vector space V with a subspace $U \le V$, the quotient space V/U has elements (as a set) $\{v + U, v \in V\}$ where we say $v_1 + U = v_2 + U$ if $v_1 - v_2 \in U$

the vector addition is defined:

$(v_1 + U) + (v_2 + U) = (v_1 + v_2) + U$

$\lambda(v + U) = \lambda v + U$

lemma 2.39 | Steinitz Exchange lemma

if $U \leq V$ is a subspace of a finite dimensional space V, then $\dim(V/U) = \dim(V) - \dim(U)$

thm 2.40 | given the surjective linear map $\alpha: V \to W$, the kernel $\ker(\alpha)$ is a subspace of V, and the image W is isomorphic to the quotient space:

$$V/\ker(\alpha) \cong W$$

indeed, any subspace and the corresponding quotient can be identified this way