monday 1/13

natural #s: $\mathbb{N} = \{1, 2, 3, \ldots\}$

integers: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

rational #s: $\mathbb{Q}$ = set of all #s that can be written in the form $\frac{m}{n}$ where $m, n \in \mathbb{Z}$ & $n \neq 0$

$\Rightarrow$ 2 fractions $\frac{m}{n}$ and $\frac{m'}{n'}$ represent the same rational # iff $mn' = m'n$

thm) $\sqrt{2}$ is irrational

$\Rightarrow$ proof by contradiction

suppose $\sqrt{2}$ is rational

- then $\sqrt{2} = \frac{m'}{n'}$ for some $m', n' \in \mathbb{Z}$ where $n' \neq 0$
- divide m' & n' by their gcd, we obtain $\sqrt{2} = \frac{m}{n}$ where $m, n \in \mathbb{Z}$, $n \neq 0$, & $\gcd(m,n) = 1$
- square both sides to get $2 = \frac{m^2}{n^2}$ $\therefore$ $2n^2 = m^2$
- since 2 is a prime & 2 divides m^2, it must divide m $\therefore$ $m = 2k$ for some $k \in \mathbb{Z}$
- Thus $2n^2 = 4k^2 \rightarrow n^2 = 2k^2$ & it follows 2 divides n. thus 2 divides m & n
- This contradicts $\gcd(m,n) = 1$ $\therefore$ $\sqrt{2}$ is irrational

properties of rational #s

for any $a, b \in \mathbb{Q}$, $a+b \in \mathbb{Q}$ and $ab \in \mathbb{Q}$

(A1) $a + (b+c) = (a+b) + c$ $\quad \forall a, b, c \in \mathbb{Q}$ (associativity)

(A2) $a + b = b + a$ $\quad \forall a, b \in \mathbb{Q}$ (commutivity)

(A3) there $\exists$ an element $0 \in \mathbb{Q}$ st $0 + a = a$ $\quad \forall a \in \mathbb{Q}$ (identity)

(A4) For each $a \in \mathbb{Q}$ there is an element $-a \in \mathbb{Q}$ st $a + (-a) = 0$ (inverse)

(M1) $a(bc) = (ab)c$ $\quad \forall a, b, c \in \mathbb{Q}$ (associativity)

(M2) $ab = ba$ $\quad \forall a, b \in \mathbb{Q}$ (commutativity)

(M3) $\exists$ an element $1 \in \mathbb{Q}$ st $1 \cdot a = a$ $\quad \forall a \in \mathbb{Q}$ (identity)

(M4) for each $a \neq 0 \in \mathbb{Q}$, $\exists$ an element $a^{-1} \in \mathbb{Q}$ st $a \cdot a^{-1} = 1$ (inverse)

(DL) $a(b+c) = ab + ac$ $\quad \forall a, b, \in \mathbb{Q}$ (distributive)

def) a field is a set containing at least 2 elements w/ 2 operations, addition & multiplication, satisfying the 9 properties above

ordered properties

for any $a, b \in \mathbb{Q}$

(O1) either $a \leq b$ or $b \leq a$

(O2) if $a \leq b$ and $b \leq a$ then $a = b$

for any $a, b, c \in \mathbb{Q}$

(O3) if $a \leq b$ and $b \leq c$, then $a \leq c$

(O4) if $a \leq b$, then $a + c \leq b + c$

(O5) if $a \leq b$ and $0 \leq c$ then $ac \leq bc$

def] a field w/ an order relation satisfying O1 to O5 is called an ordered field

def] the absolute value of $a \in \mathbb{R}$ is $|a| \begin{cases} a & \text{if } a \geq 0 \\ -a & \text{if } a < 0 \end{cases}$

Basic properties of absolute value

i) $|a| \geq 0 \quad \forall a \in \mathbb{R}$

ii) $|ab| = |a| \cdot |b| \quad \forall a, b \in \mathbb{R}$

iii) $|a+b| \leq |a| + |b| \quad \forall a, b \in \mathbb{R}$

$\Rightarrow$ proof for iii

let $a, b \in \mathbb{R}$

$|a+b| =$ either $a+b$ or $-(a+b)$

Since $a = |a|$ or $-|a|$ $\quad -|a| \leq a \leq |a|$ $\quad$ & similarly $-|b| \leq b \leq |b|$

Adding the inequalities we get $\quad a+b \leq |a| + |b|$ $\quad$ & $-|a| - |b| \leq a+b$ $\quad$ which implies $-(a+b) \leq |a| + |b|$

it follows that $|a+b| \leq |a| + |b|$

cor] for any $n \in \mathbb{N}$ & any $a_1 \dots a_n \in \mathbb{R}$, $\quad |a_1 + \dots + a_n| \leq |a_1| + \dots + |a_n|$

def] for $a, b \in \mathbb{R}$, the distance between a and b is $dist(a,b) = |a-b|$

cor] for any $a, b, c \in \mathbb{R}$, $\quad dist(a,c) \leq dist(a,b) + dist(b,c$

$dist(a,c) = |a-c| = \left|(a-b) + (b-c)\right| \leq |a-b| + |b-c| = dist(a,b) + dist(b,c)$

thm] for any $a, b, c \in \mathbb{R}$, $\quad \left||a| - |b|\right| \leq |a-b|$ & in particular $|a| - |b| \leq |a-b|$

wednesday 1/15

Section 4 - Maximum & Minimum

def let S be a non-empty subset of $\mathbb{R}$ ($S \subseteq \mathbb{R}$ & $S \neq \emptyset$)

if S has the largest element M, we call M the maximum of S : $M = \max S$

... smallest ..., m, m $m = \min S$

ex (1) a finite non-empty set has a maximum & a min

(2) $\mathbb{R}$, $\mathbb{Q}$, & $\mathbb{Z}$ have no max & no min; $\mathbb{N}$ has min (1) & no max

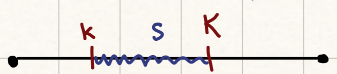
(3) $S = (a, b]$ no minimum, $\max S = b$

(4) $S = \{r \in \mathbb{Q} : 4 \leq r^2 \leq 5 \text{ and } r > 0\}$ $\min S = 2$ but no maximum

(5) $S = \{\frac{1}{2^n} : n \in \mathbb{N}\}$ $\max S = 1/2$ & no minimum

Upper & Lower Bounds

let $S \subseteq \mathbb{R}$ and $S \neq \emptyset$



(a) if a $\mathbb{R}$ num K satisfies $s \leq K$ $\forall s \in S$, we say that K is an upper bound for S.

if such $K \in \mathbb{R}$ exists, we say S is bounded above

(b) if a $\mathbb{R}$ num k satisfies $s \geq k$ $\forall s \in S$, we say that k is a lower bound for S

if such $k \in \mathbb{R}$ exists, we say S is bounded below

(c) S is bounded iff it is bounded above & below

note if S has a max, then it is bounded above

" min " bdd below

but a bdd set does not necessarily have a max or a min

ex) (1) $S = (a, b)$ is bdd; any $k \leq a$ is a lower bound for S & $K \geq b$ is an upper bdd for S; S has no max/min

(2) $S = \{\frac{1}{2^n} : n \in \mathbb{Z}\}$ which has a lower bdd of 0 but no upper bound

Supremum & infimum

Let S be a $\neq \emptyset \in \mathbb{R}$

(a) if S is bounded above, the least upper bound for S is called the supremum of S & denoted $\sup S$

(b) if S is bounded below, the greatest upper bound for S is called the infimum of S & denoted $\inf S$

ex) (1) if S has a max, then the supremum is the maximum ($\max S = \sup S$)
" min, " infinitum is the min ($\min S = \inf S$)

(2) $S = [2,4)$ $\min S = \inf S = 2$, $\sup S = 4$ w/ no maximum

(3) $S = \{2^n ; n \in \mathbb{Z}$ and $n \leq 5\}$ $\max S = \sup S = 32$, there is no min $\inf S = 0$

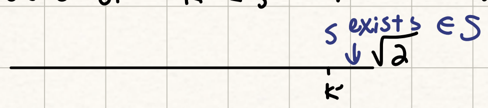
(4) $S = \{r \in \mathbb{Q} : r^2 \leq 2\} \subseteq \mathbb{R}$ no max or min, $\sup S = \sqrt{2}$ $\inf S = -\sqrt{2}$

(5) $S = \{a \in \mathbb{R} : r^2 \leq 2\}$ = max & min exist $\max S = \sup S = \sqrt{2}$ $\min S = \inf S = -\sqrt{2}$

note: we assume that all sets are $\subseteq \mathbb{R}$ unless specified otherwise

Let $S \subseteq \mathbb{R}$ & $S \neq \emptyset$

(a) Suppose S is bdd above
then $K = \sup S$ iff $s \leq K$ $\forall s \in S$ (is an upper bound) and if for each $K' < K$, there is $s \in S$ st $K' < s$ $\therefore$ K' is not an upper bound



(b) Similiar for $\inf S$ //

The Completeness Axiom

Every $S \neq \emptyset$ & $S \subseteq \mathbb{R}$ that is bounded above has a least upper bound
i.e. $\sup S$ exists & $\in \mathbb{R}$

Difference with $\mathbb{Q}$

It is not true that every nonempty $\in \mathbb{Q}$ that is bdd above has a least upper bound in $\mathbb{Q}$

ex) $\{r \in \mathbb{Q} : r^2 \le a\}$

is bdd above (ex by a) but there does not exist a least upper bound $\in \mathbb{Q}$

Corollary every nonempty subset $S \in \mathbb{R}$ that is bounded below has the greatest lower bound i.e. $\inf S \ \exists \in \mathbb{R}$

Outline: Consider the set $-S = \{-s ; s \in S\}$

- Since S is bounded below, by some m, $-S$ is bdd above by $-m$
- Then by completeness axiom $-S$ has a supremum, s_0
- Show that the $\inf S = -s_0$ (ex)

Friday 1.17

Symbols $+\infty$ (∞) & $-\infty$ - section 5

❗ $+\infty$ and $-\infty$ are not real numbers

Thms about real #s do not apply to them

We agree that for any a in the set $\mathbb{R} \cup \{\infty, -\infty\}$, $-\infty \le a \le \infty$

If S is not bdd above, we can write $\sup S = \infty$

" below, " $\inf S = -\infty$

Note: for any $s \in S$ $\inf S \le s \le \sup S$

and hence $\inf S \le \sup S$

back to section 4...

empty large tub w/ small spoon

archimedean Property

for any $a, b \in \mathbb{R}$ st $a > 0$ and $b > 0$, there $\exists$ $n \in \mathbb{N}$ s.t. $na > b$

corollary

(a) for any $a > 0$, there $\exists$ $n \in \mathbb{N}$ st $\frac{1}{n} < a$ (take $b = 1$)

(b) for any $b > 0$, there $\exists n \in \mathbb{N}$ st $n > b$ (take $a = 1$)

Proof (by contradiction)

Suppose that there $\exists a,b \in \mathbb{R}$ w/ $a>0$ & $b>0$ s.t. $\forall n \in \mathbb{N}$, $na \leq b$

This means b is an upper bound for the set $S = \{na : n \in \mathbb{N}\}$

By the completeness axiom, the set S has a supremum, $S_0 \in \mathbb{R}$

Since $a>0$, $S_0 < S_0 + a$ and hence $S_0 - a < S_0$. Therefore, $S_0 - a$ is not an upperbound for S. Then, there $\exists n_0 \in \mathbb{N}$ st $S_0 - a < n_0 a$. Then, $S_0 < (n_0+1)a$. Since $n_0 + 1 \in \mathbb{N}$, then $(n_0+1)a \in S$ and so S_0 is not an upper bound for S, which contradicts $S_0 = \sup S$

Therefore, archimedean property holds $\square$

Corollary $\Rightarrow$ Denseness of $\mathbb{Q} \in \mathbb{R}$

for any $a,b \in \mathbb{R}$ st $a<b$, there $\exists r \in \mathbb{Q}$ st $a<r<b$

i.e. every open interval $\in \mathbb{R}$ contains a $\mathbb{Q}$ number

Outline of proof let $a,b \in \mathbb{R}$, a,b

We will show that $\exists m,n \in \mathbb{Z}$ w/ $n>0$ st $a < \frac{m}{n} < b$ "a" "b"

ie $an < m < bn$. Since $b-a>0$, by arch prop, $\exists$ $n \in \mathbb{N}$ st $(b-a)n > 1$ and so $bn - an > 1$. (SKIP) Then, there $\exists m \in \mathbb{Z}$ between an & bn $\square$

Definition and Construction of $\mathbb{R}$

axiomatic definition of $\mathbb{R}$

$\mathbb{R}$ is an ordered field that satisfies the completeness axiom

? Why does such an object exist?

$\mathbb{R}$ can be constructed starting w/ $\mathbb{Q}$

Dede Kind cuts

Motivation: Suppose we already have $\mathbb{R}$

then, there is a 1 to 1 correspondence between real #s & certain subsets of $\mathbb{Q}$

$a \in \mathbb{R} \leftrightarrow a^* = \{r \in \mathbb{Q} : r < a\} \subseteq \mathbb{Q}$

note: $a \leq b$ iff $a^* \leq b^*$

let α be a $\subseteq \mathbb{Q}$ of the form

$\alpha = \{r \in \mathbb{Q} : r < a\}$ for some $a \in \mathbb{R}$ then

(i) $\alpha \neq \emptyset$ and $\neq \mathbb{Q}$

(ii) if $r \in \alpha$, $s \in \mathbb{Q}$ and $s \leq r$, then $s \in \alpha$ — if r in class, any smaller $\mathbb{Q} \in \alpha$

(iii) α does not contain a largest rational — follows from density of $\mathbb{R}$

But, we do not have $\mathbb{R}$ yet!

Subsets of $\mathbb{Q}$ that have properties (i, ii, & iii) are called Dedekind cuts

We define $\mathbb{R}$ as the set of all Dedekind cuts

Note: $s \in \mathbb{Q}$ corresponds to $s^* = \{r \in \mathbb{Q} : r < s\}$

Order: $\alpha \leq \beta$ if $\alpha \subseteq \beta$

Addition: $\alpha + \beta = \{r + s : r \in \alpha \text{ and } s \in \beta\}$

Multiplication: think thru cases of pos & neg ... HW

then show that it is an ordered field & satisfies completeness axiom