monday 3.17

Continuous Functions (sec 17)

First, consider func from (a subset of) $\mathbb{R}$ to $\mathbb{R}$

def the domain of f, denoted $\mathrm{dom}(f)$, is the set where f is defined

if the domain is not specified, it is understood to be the natural domain

ie the largest set in $\mathbb{R}$ where f is a well defined, real valued func

ex (1) $f(x) = x^2 + 3x$ $\quad \mathrm{dom}(f) = \mathbb{R}$

(2) $f(x) = \frac{1}{x^3+1}$ $\quad \mathrm{dom}(f) = \mathbb{R} \setminus \{-1\} = (-\infty, -1) \cup (-1, \infty)$

(3) $f(x) = \sqrt{x^2 - 9}$ $\quad \mathrm{dom}(f) = \{x \in \mathbb{R} : |x| \geq 3\} = (-\infty, -3] \cup [3, \infty)$

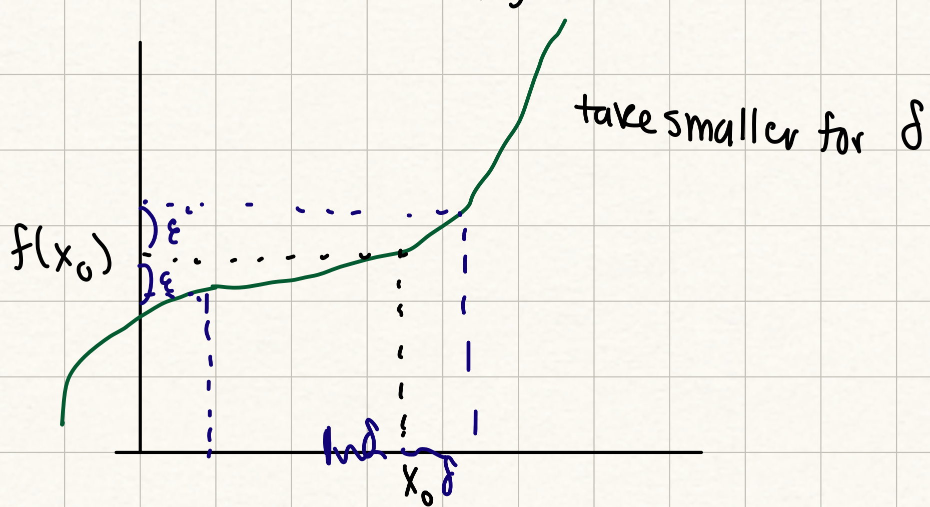
intuitively, f is continuous at $x_0 \in \mathrm{dom}(f)$ if for all $x \in \mathrm{dom}(f)$ close to x_0, $f(x)$ is close to $f(x_0)$

more precisely, we can make $f(x)$ as close to $f(x_0)$ as we .. ? with by taking x sufficiently close to x_0

def a function f is continuous at $x_0 \in \mathrm{dom}(f)$ if for every sequence (x_n) in $\mathrm{dom}(f)$ converging to x_0, (1)

$\lim f(x_n) = f(x_0)$ [ie $f(x_n)$ converges to $f(x_0)$]

def a func f is continuous @ $x_0 \in \mathrm{dom}(f)$ if for every $\varepsilon > 0$, there exists $\delta > 0$ st $x \in \mathrm{dom}(f)$ & $|x - x_0| < \delta$ imply that $|f(x) - f(x_0)| < \varepsilon$ (2)



ex Show that $f(x)=2x$ is continuous at $x_0=1$

... show $\mathrm{dom}(f)=\mathbb{R}$

PROOF 1 let (x_n) be a sequence converging to 1. By the constant multiple Thm for sequences..

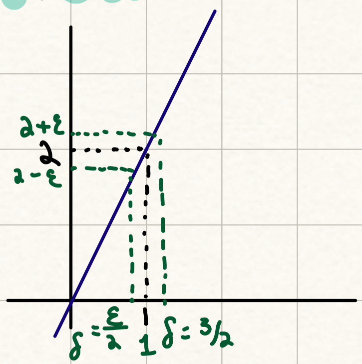
$\lim f(x_n) = \lim(2x_n) = 2\lim x_n = 2\cdot 1 = f(1)$

thus f is continuous @ 1

PROOF 2 let $\varepsilon>0$. let $\delta=\varepsilon/2$. Then for all x with $|x-1|<\delta$ we have $|f(x)-f(1)|=$

$|2x-2\cdot 1| = 2|x-1| < 2\delta = 2\cdot\frac{\varepsilon}{2} = \varepsilon$

Thus f is continuous at 1 ☐



thm a function f is continuous at $x_0\in\mathrm{dom}(f)$ if for every sequence (x_n) in $\mathrm{dom}(f)$ converging to x_0, $\lim f(x_n)=f(x_0)$ [ie $f(x_n)$ converges to $f(x_0)$]

AND

a func f is continuous @ $x_0\in\mathrm{dom}(f)$ if for every $\varepsilon>0$, there exists $\delta>0$ st $x\in\mathrm{dom}(f)$ & $|x-x_0|<\delta$ imply that $|f(x)-f(x_0)|<\varepsilon$

definition: 1 and 2 are equivalent, ie for $x_0\in\mathrm{dom}(f)$, (1) holds $\Longleftrightarrow$ (2) holds

PROOF (2) $\Rightarrow$ (1) Suppose 2 holds

let (x_n) be a sequence in $\mathrm{dom}(f)$ converging to x_0. let $\varepsilon>0$. By 2, there is $\delta>0$ st $x\in\mathrm{dom}(f)$ and $|x-x_0|<\delta$ imply $|f(x)-f(x_0)|<\varepsilon$

Since $x_n\to x_0$, there is N st for all $n>N$, $|x_n-x_0|<\delta$ and hence $|f(x_n)-f(x_0)|<\varepsilon$. Thus $|f(x_n)-f(x_0)|<\varepsilon$ for all $n>N$. So, $f(x_n)\to f(x_0)$

(1) $\Rightarrow$ (2) By contraposition. We will show that if 2 fails then 1 fails

Suppose (2) fails. ie. there exists $\varepsilon>0$ such that for every $\delta>0$, there is $x\in\mathrm{dom}(f)$ w/ $|x-x_0|<\delta$ st $|f(x)-f(x_0)|\geq\varepsilon$

take $\delta=1/n$, we see that for each $n\in\mathbb{N}$ there is $x_n\in\mathrm{dom}(f)$ with

$|x_n - x_0| < 1/n$ st $|f(x) - f(x_0)| \geq \varepsilon$

this sequence (x_n) converges to x_0, but $f(x_n) \not\to f(x_0)$

So (1) fails. Thus $(1) \Rightarrow (2)$ □

def if f is not continuous @ $x_0 \in dom(f)$ we say that f is discontinuous at x_0

this means that there is a sequence (x_n) in $dom(f)$ st $x_n \to x_0$ but $f(x_n) \not\to f(x_0)$

Wednesday 3/19

examples of continuous & discontinuous func

We say that f is continuous if f is cont. at each $x_0 \in dom(f)$

ex (1) $f(x) = x$ and $g(x) = c$ are cont on $\mathbb{R}$

(take $\delta = \varepsilon$ for $f(x) = x$)

(2) $f(x) = 1/x$ is cont on $\mathbb{R} \setminus \{0\}$

DISCUSSION: let $x_0 \neq 0$, let $\varepsilon > 0$; we need to find δ st $x \neq 0$ and $|x - x_0| < \delta$ implies $\left|\frac{1}{x} - \frac{1}{x_0}\right| < \varepsilon$

$\left|\frac{1}{x} - \frac{1}{x_0}\right| = \frac{|x - x_0|}{|x| \cdot |x_0|}$

δ may depend on ε and x_0 but not on x

We estimate $|x| \cdot |x_0|$ in terms of x_0. For any $|x - x_0| < \frac{1}{2}|x_0|$

We have $|x| > 1/2 |x_0|$ and so $|x| \cdot |x_0| > 1/2 |x_0|^2$

and $\left|\frac{1}{x} - \frac{1}{x_0}\right| < \frac{|x - x_0|}{\frac{1}{2}|x_0|^2}$

to make $< \varepsilon$ want $|x - x_0| < \varepsilon \cdot \frac{1}{2}|x_0|^2$

$\therefore$ We should take $\delta = \min\left\{\frac{1}{2}|x_0|, \varepsilon \cdot \frac{1}{2}|x_0|^2\right\}$

PROOF

let $x_0 \neq 0$ let $\varepsilon > 0$ let $\delta = \min\left\{\frac{1}{2}|x_0|, \varepsilon \cdot \frac{1}{2}|x_0|^2\right\}$

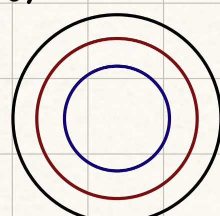
let x be st $|x - x_0| < \delta$; then $|x - x_0| < 1/2 |x_0|$ and hence $|x| > 1/2 |x_0|$

then $x \neq 0$ and $\left|\frac{1}{x} - \frac{1}{x_0}\right| = \frac{|x - x_0|}{|x| \cdot |x_0|} < \frac{|x - x_0|}{\frac{1}{2}|x_0|^2}$

Since $|x - x_0| < \varepsilon \cdot \frac{1}{2}|x_0|^2$, $\frac{|x - x_0|}{1/2 |x_0|^2} < \frac{\varepsilon \cdot 1/2 |x_0|^2}{1/2 |x_0|^2} < \varepsilon$

thus if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \varepsilon$;

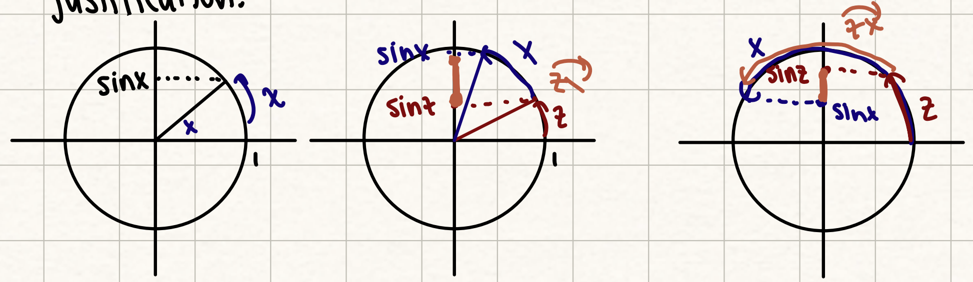
so f is cont at every $x_0 \neq 0$ □



(3) $f(x) = \sin x$ is cont on $\mathbb{R}$

claim: for any $x, z \in \mathbb{R}$ $|\sin x - \sin z| \leq |x - z|$

justification:



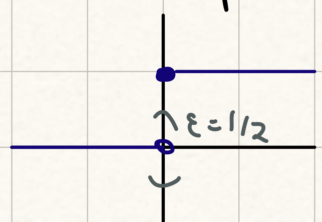
the projection of the arc $\overset{\frown}{zx}$ to the y-axis contains the interval w/ endpoints $\sin x$, $\sin z$

PROOF let $x_0 \in \mathbb{R}$ let $\varepsilon > 0$ let $\delta = \varepsilon$, then $|x - x_0| < \delta$ implies

$$|\sin x - \sin x_0| \leq |x - x_0| < \delta = \varepsilon$$

thus $\sin x$ is cont. @ every $x_0 \in \mathbb{R}$

(4) the step func $f(x) = \begin{cases} 0 & x < 0 \\ 1 & x \geq 0 \end{cases}$



f is cont. at every $x_0 \neq 0$

take $\delta = |x_0|$

f is discont. @ $x_0 = 0$

PROOF 1 let $x_n = -1/n$, Then $x_n \to 0$, but $\lim f(x_n) = 0 \neq 1 = f(0)$

PROOF 2 let $\varepsilon = 1/2$. let $\delta > 0$. let $x = -\delta/2$

Then $|x - 0| = \frac{\delta}{2} < \delta$

but $|f(x) - f(0)| = |0 - 1| = 1 > \varepsilon$

thus there is no δ for $\varepsilon = 1/2$

(5) the "salt and pepper" function

$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$ f is discontinuous at each $x_0 \in \mathbb{R}$

dense lines @ 1 & 0

PROOF let $x_0 \in \mathbb{R} \setminus \mathbb{Q}$. Then $f(x_0) = 0$. let $\varepsilon = 1/2$. let $\delta > 0$ since $\mathbb{Q}$ is dense in $\mathbb{R}$, there is a rational x in $(x_0 - \delta, x + \delta)$. For this x, $|x - x_0| < \delta$ but $|f(x) - f(x_0)| = |1 - 0| = 1 > \varepsilon$ so there is no δ for $\varepsilon = 1/2$ & hence f is disc. at x_0

let $x_0 \in \mathbb{Q}$. Then $f(x_0)=1$, let $\varepsilon = 1/2$, ~ using density of irrationals in $\mathbb{R}$ □

friday 3/21

Continuity of kf, $|f|$, $f+g$, fg, f/g $\overset{g \circ f}{\wedge}$ (see 17)

Thm if f is continuous at $x_0 \in \text{dom}(f)$ then kf, where $k \in \mathbb{R}$, and $|f|$ are also continuous at x_0

PROOF $\underline{kf}$ let (x_n) be a sequence in $\text{dom}(f)$ converging to x_0.

since f is continuous at x_0, $f(x_n) \to f(x_0)$

and it follows (by thm 9.2) that $(kf)(x_n) = k f(x_n) \to k f(x_0) = (kf)(x_0)$ □

$\underline{|f|}$ we will use $||a|-|b|| \leq |a-b|$

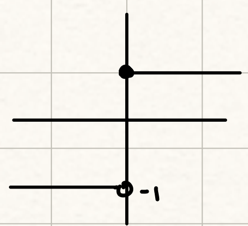
let $\varepsilon > 0$. Since f is continuous at x_0, there is $\delta > 0$ st

$x \in \text{dom}(f)$ and $|x-x_0| < \delta$ imply $|f(x)-f(x_0)| < \varepsilon$

then $||f(x_n)|-|f(x_0)|| \leq |f(x)-f(x_0)| < \varepsilon$

thus $|f|$ is continuous at x_0 □

ex let $f(x) = \begin{cases} -1 & x<0 \\ 1 & x \geq 0 \end{cases}$



f is discont. at $x_0=0$

but $|f| \equiv 1$ is cont. at $x_0=0$

ex f is discontinuous at every x_0 and $|f|$ is continuous at every x_0

$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \notin \mathbb{Q} \end{cases}$ $\qquad |f| \equiv 1$

thm If functions f and g are continuous at x_0, then

(i) $f+g$ is continuous at x_0

(ii) fg is continuous at x_0

(iii) f/g is continuous at x_0 if $g(x_0) \neq 0$

Note: The statement assumes that $x_0 \in \text{dom}(f)$ and $x_0 \in \text{dom}(g)$

$\text{dom}(f+g) = \text{dom}(fg) = \text{dom}(f) \cap \text{dom}(g)$

$\text{dom}(f/g) = \text{dom}(f) \cap \{x \in \text{dom}(g) : g(x) \neq 0\}$

PROOF (using results for limits of sequences)

(i, ii) – ex

(iii) let (x_n) be a sequence in $\text{dom}(f/g)$

since f and g are cont. at x_0,

$f(x_n) \to f(x_0)$ and $g(x_n) \to g(x_0)$

Since $g(x_0) \neq 0$ it follows by thm 9.6

that $\lim \left(\frac{f}{g}\right)(x_n) = \lim \frac{f(x_n)}{g(x_n)} = f(x_0)/g(x_0) = (f/g)(x_0)$ $\square$

Recall: $g(x) = c$ and $f(x) = x$ are continuous on $\mathbb{R}$

Corollary:
- for any $n \in \mathbb{N}$, x^n is continuous on $\mathbb{R}$
- any polynomial function $a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ is cont. on $\mathbb{R}$
- any rational func

  $f(x) = \frac{p(x)}{q(x)}$ where p and q are polynomials and $q \not\equiv 0$,

  is continuous on its domain, $\{x \in \mathbb{R} : q(x) \neq 0\}$

Recall: The composition of f and g is $(g \circ f)(x) = g(f(x))$

$$x \xrightarrow{f} f(x) \xrightarrow{g} g(f(x)) = g \circ f(x)$$

Usually $f \circ g \neq g \circ f$

$\text{dom}(g \circ f) = \{x \in \text{dom}(f) : f(x) \in \text{dom}(g)\}$

ex
$f(x) = x^3 - 8$ $\quad (g \circ f)(x) = \sqrt{x^3 - 8}$ $\quad \text{dom}(g \circ f) = [2, \infty)$

$g(x) = \sqrt{x}$ $\quad (f \circ g)(x) = (\sqrt{x})^3 - 8$ $\quad \text{dom}(f \circ g) = [0, \infty)$

thm if f is continuous at x_0 and g is continuous at $f(x_0)$ then $g \circ f$ is continuous at x_0

PROOF let (x_n) be a sequence in dom$(g \circ f)$ converging to x_0

since f is cont. at x_0, $f(x_n) \to f(x_0)$

since g is cont at $f(x_0)$, $g(f(x_n)) \to g(f(x_0))$

Thus, $g \circ f$ is cont. at x_0 □

corollary if f and g are continuous then $g \circ f$ is cont (on its domain)

examples

- $\cos x = \sin(x + \pi/2)$ Since $\sin x$ and $x + \pi/2$ are continuous on $\mathbb{R}$, cos is cont on $\mathbb{R}$
- $\tan x = \frac{\sin x}{\cos x}$ is cont at every $x_0 \neq \pi/2 + \pi k$
- $f(x) = x^3+5/x^2-4$ since this is rational func, cont where $x_0 \neq \pm 2$ $\mathbb{R} \setminus \{-2, 2\}$