Wednesday 4/2

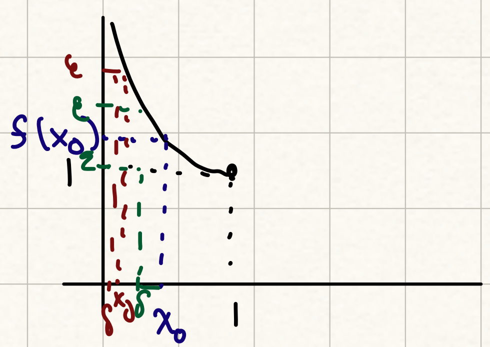
Uniform continuity (sec 19)

recall f is cont on a set $S \subseteq \text{dom}(f)$ if for any $x_0 \in S$, any $\varepsilon > 0$, there is $\delta > 0$ st $x \in S$ and $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

? Given $\varepsilon > 0$, is there $\delta > 0$ that works for all $x_0 \in S$?

$\Rightarrow$ in general, no

ex $f(x) = 1/x$ on $[0, 1]$



def let f be a function from a set $S \subseteq \mathbb{R}$ to $\mathbb{R}$. We say that f is uniformly continuous on S if for any $\varepsilon > 0$, there exists $\delta > 0$ st $x, y \in S$ and $|x - y| < \delta$ imply $|f(x) - f(y)| < \varepsilon$

note: $f(x) = 1/x$ is not unif. cont. on $(0, 1]$

? is it unif cont on $[a, 1]$ where $a > 0$?

take δ that works for a, and it should work for all $x_0 \in [0,1]$

ex Uniformly cont on $\mathbb{R}$: c, x, kx

Not unif cont on $\mathbb{R}$: x^2

note: if f is continuous on S
$\Uparrow$ $\not\Downarrow$
if f is unif cont on S

thm let f be a continuous function on $[a, b]$. Then f is uniformly continuous on $[a, b]$

note: if f is unif. cont. on a set S, then f is unif. cont. on any subset of S.

in particular, if f is unif. cont on $[a,b]$, then it is unif cont on (a,b)

ex $\frac{1}{x}$ is unif cont on $[a,1]$ by the thm, and hence on $(a,1]$ for any $a>0$

PROOF: By contradiction

let f be a continuous function on $[a,b]$. Suppose that f is <u>not</u> uniformly cont on $[a,b]$. This means there exists a $\varepsilon>0$ st for each $\delta>0$ there exists $x,y\in S$ st $|x-y|<\delta$ and $|f(x)-f(y)|\geq\varepsilon$. Take $\delta=1/n$, we see that for each $n\in\mathbb{N}$, there exists $x_n, y_n\in S$ such that $|x_n-y_n|<1/n$ and $|f(x_n)-f(y_n)|\geq\varepsilon$.

Since (x_n) is bounded, by Bolzano Weierstrass thm, it has subsequence (x_{n_k}) converging to some $x_0\in\mathbb{R}$. Since $[a,b]$ is closed and $x_{n_k}\in[a,b]$, $x_0\in[a,b]$.

Then, $y_{n_k}\to x_0$, since $|x_{n_k}-y_{n_k}|<1/n_k\to 0$. (explain w/ Δ inequality)

Since f is continuous at x_0, $\lim f(x_{n_k})=f(x_0)=\lim f(y_{n_k})$.

Hence $\lim(f(x_{n_k})-f(y_{n_k}))=0$, which contradicts $|f(x_{n_k})-f(y_{n_k})|\geq\varepsilon$ for all k. Thus f is unif. cont. on $[a,b]$

Compactness in terms of sequences

We showed that

if f is continuous on $[a,b]$ then

(1) f is bounded on $[a,b]$

(2) f attains its min and max values on $[a,b]$

(3) f is uniformly continuous on $[a,b]$

in the three proofs, we used

- every sequence in $[a,b]$ has a subsequence converging to a point in $[a,b]$

def a set S in $\mathbb{R}$, or more generally in a metric space (M,d) is called **compact** if every sequence in S has a subsequence converging to a point in S

note: (1,2,3) hold for any continuous function from a compact set S to $\mathbb{R}$

thm a set S in $\mathbb{R}$ is compact $\Longleftrightarrow$ it is closed and bounded (hw)

note: in metric spaces, in general $\Rightarrow$ but not $\Leftarrow$ (hw)

Friday 4/4

Limits of Functions

def let $a \in \mathbb{R}$ and suppose that a function f is defined on $J \setminus \{a\}$ for some open interval J containing a.

We say that a real number L is the limit of $f(x)$ as x tends to a and write $\lim_{x \to a} f(x) = L$ if ...

(i) for every sequence (x_n) in $J \setminus \{a\}$ converging to a we have $\lim_{n \to \infty} f(x_n) = L$

... equivalently ...

(ii) for each $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in J \setminus \{a\}$ and $|x - a| < \delta$ imply $|f(x) - L| < \varepsilon$,

... equivalently ...

(iii) for each $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - L| < \varepsilon$

remarks:

⇒ equivalence of first two definitions can be proved similarly to equivalence of 2 definitions of continuity; equiv of last 2 is easy to see

⇒ the limit, if it exists, is unique

⇒ f does not need to be defined at a

⇒ defining or changing a does not affect the limit

⇒ Suppose that f is defined on an open interval J containing a. Then f is continuous at a if and only if $\lim_{x \to a} f(x) = f(a)$

ex Let $f(x) = \frac{x^2 - 1}{x - 1}$. Let us consider $\lim_{x \to 1} f(x)$

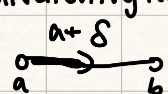
The function f is defined on $\mathbb{R} \setminus \{1\}$. On this set, $f(x) = \frac{(x-1)(x+1)}{(x-1)} = x + 1$

Then, using continuity of $x + 1$, we obtain $\lim_{x \to 1} f(x) = \lim_{x \to 1} (x + 1) = 1 + 1 = 2$

def Suppose that a function is defined on an interval (a, b). We write $\lim_{x \to a^+} f(x) = L$ if

(i) for every seq. (x_n) in (a, b) converging to a we have $\lim_{n \to \infty} f(x_n) = L$ ... equivalently ...

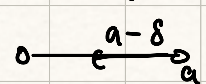
(ii) for each $\varepsilon > 0$ there exists $\delta > 0$ st $a < x < a + \delta$ implies $|f(x) - L| < \varepsilon$



def Suppose that a function is defined on an interval (c, a). We write $\lim_{x \to a^-} f(x) = L$ if

(i) for every seq. (x_n) in (c, a) converging to a we have $\lim_{n \to \infty} f(x_n) = L$ ... equivalently ...

(ii) for each $\varepsilon > 0$ there exists $\delta > 0$ st $a - \delta < x < a$ implies $|f(x) - L| < \varepsilon$



ex For the step function $f(x) = \begin{cases} 0 & x<0 \\ 1 & x \geq 0 \end{cases}$ $\lim_{x\to 0^-} f(x)=0$ and $\lim_{x\to 0^+} f(x)=1$

indeed, for every seq (x_n) in $(-\infty,0)$ converging to x has $\lim f(x_n)=0$

for every seq (y_n) in $[0,\infty)$ converging to x has $\lim f(y_n)=1$

ex for the salt and pepper function $f(x) = \begin{cases} 1 & x\in\mathbb{Q} \\ 0 & x\notin\mathbb{Q} \end{cases}$, $\lim_{x\to 0^+} f(x)$ does not exist

Let $x_n = 1/n$, then $x_n \in \mathbb{Q}$ and $x_n > 0$, $x_n \to 0$ and $\lim_{n\to\infty} f(x_n) = 1$

Let $y_n = \frac{\sqrt{2}}{n}$, then $y_n \notin \mathbb{Q}$ and $y_n > 0$, $y_n \to 0$ and $\lim_{n\to\infty} f(y_n) = 0$

Since there are 2 sequences in $(0,\infty)$ converging to 0 w/ different limits, $\lim_{x\to 0^+} f(x)$ DNE

thm Suppose that $a, L \in \mathbb{R}$ and a function f is defined on $J\setminus\{a\}$ for some open interval J containing a. Then $\lim_{x\to a} f(x) = L$ iff $\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$

PROOF $\Rightarrow$ is clear

$(\Leftarrow)$ Suppose that $\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$

Let $\varepsilon>0$. Then there exists $\delta_1>0$ and $\delta_2>0$ such that

$a-\delta_1 < x < a$ implies $|f(x)-L|<\varepsilon$ and

$a<x<a+\delta_2$ impiles $|f(x)-L|<\varepsilon$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then, $0<|x-a|<\delta$ implies $|f(x)-L|<\varepsilon$

corollary if at least one of the one-sided limits does not exist, or they are not equal, then $\lim_{x\to a} f(x)$ does not exist

note: in ex 2, $\lim_{x\to 0} f(x)$ DNE since $\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$

in ex 3, $\lim_{x\to 0} f(x)$ DNE since $\lim_{x\to 0^+} f(x)$ DNE

thm Suppose that $\lim_{x\to a} f(x) = L_1 \in \mathbb{R}$ and $\lim_{x\to a} g(x) = L_2 \in \mathbb{R}$. Then

(i) $\lim_{x\to a} (f+g)(x) = L_1 + L_2$

(ii) $\lim_{x\to a} (fg)(x) = L_1 L_2$

(iii) $\lim_{x\to a} \left(\frac{f}{g}\right)(x) = L_1/L_2$ provided $L_2 \neq 0$

PROOF OUTLINE - explain why defined on $J\setminus\{a\}$, then follows from corresponding thm for seq

Remark: This result also holds for one sided limits