wednesday 1.22

$\mathbb{R}$ are a metric space

Metric Spaces (notes)

Recall: the standard dist on $\mathbb{R}$ - $d(x,y) = |x-y|$

properties: i) $d(x,y) \geq 0 \ \forall x,y \in \mathbb{R}$ and $d(x,y)=0$ iff $x=y$

equiv. $d(x,x)=0 \ \forall x \in \mathbb{R}$ and $d(x,y)$ for any distinct $x,y \in \mathbb{R}$

ii) $d(x,y) = d(y,x) \ \forall x,y \in \mathbb{R}$

iii) $d(x,z) \leq d(x,y) + d(y,z) \ \forall x,y,z \in \mathbb{R}$

def let M be a nonempty set.

a metric or a dist func on M is a function $d: M \times M \to \mathbb{R}$ satisfying

i) $d(x,y) \geq 0 \ \forall x,y \in M$ and $d(x,y)=0$ iff $x=y$

ii) $d(x,y) = d(y,x) \ \forall x,y \in M$

iii) $d(x,z) \leq d(x,y) + d(y,z) \ \forall x,y,z \in M$ (Δ inequality)

The pair (M,d) is a metric space

ex (1) $\mathbb{R}$ with the standard metric $d(x,y) = |x-y|$ is a metric space

Remark: There are other metrics on $\mathbb{R}$, $d(x,y) = \dfrac{|x-y|}{1+|x-y|}$, $d(x,y) = \sqrt{|x-y|}$, ...

Distances on $\mathbb{R}^2 = \{(x_1, x_2) = x_1, x_2 \in \mathbb{R}\}$ (similarly on $\mathbb{R}^3 \dots \mathbb{R}^K$ $K \geq 2$)

2) $M = \Big(\mathbb{R}^2$, euclidean metric for $x=(x_1,x_2)$ & $y=(y_1,y_2)$ in $\mathbb{R}^2\Big)$

$\hookrightarrow d(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2}$

(ex Δ inequality)

(3) $M = \mathbb{R}^2$, $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$

"taxi cab metric"

proof Δ ineq; let $x,y,z \in \mathbb{R}^2$

$|x_1 - z_1| \leq |x_1 - y_1| + |y_1 - z_1|$ and $|x_2 - z_2| \leq |x_2 - y_2| + |y_2 - z_2|$

$d(x,z) \leq d(x,y) + d(y,z)$

(4) $M = \mathbb{R}^2$ $\quad d(x,y) = \max\{ |x_1 - y_1|, |x_2 - y_2| \}$ "maximum metric"

personal: for polar coords, is this also a metric space w/ polar distances?

(5) M - a nonempty set $\quad d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$ is a metric on M

$\triangle$ ineq) let $x,y,z \in M$

if $x = z$, then it easily holds

if $x \neq z$, then either $x \neq y$ or $y \neq z$, and so $\overbrace{d(x,y) = 1 \text{ or } d(y,z) = 1}^{\max 1}$

(if both then $x = z$)

then $d(x,y) = 1 \leq \max 1$

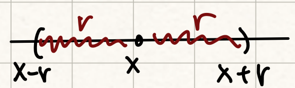
d is the discrete metric on M

(M,d) - a discrete metric space

def let (M,d) be a metric space, an open ball in (M,d) centered at $x \in M$ of radius $r > 0$ is the set $B_r(x) = \{ y \in M : d(x,y) < r \}$ where $r \in \mathbb{R}$

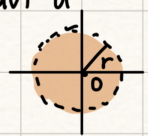
ex (1) $\mathbb{R}$ w/ usual dist

$B_r(x) = (x - r, x + r)$



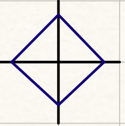
(2) $\mathbb{R}^2$, euclidean d

$B_r(0) \rightarrow$



(3) $\mathbb{R}^2$, taxi cab

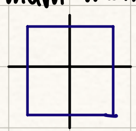
$B_r(0) =$



HW

(4) $\mathbb{R}^2$, maximum metric

$B_r(0) =$



(5) M, discrete metric

$B_r(x) = \begin{cases} M, & r > 1 \\ \{x\}, & 0 < r \leq 1 \end{cases}$

def a set S in a metric space (M,d) is bounded if S is contained in an open ball $B_r(x)$ for some $x \in M$ and $r>0$

$\Rightarrow$ it agrees w/ definition for $|\,|$

Friday 1.24

Sequences Sec 7 & 8

def a sequence of $\mathbb{R}$ #s is a function, s, from $\mathbb{N}$ to $\mathbb{R}$, or more generally, from a set $\{n \in \mathbb{Z} : n \geq m\}$ to $\mathbb{R}$

We write s_n for $s(n)$, and we write sequences as

$$(s_n)_{n=m}^{\infty} \quad \text{or} \quad (s_m, s_{m+1}, \dots)$$

for $(s_n)_{n=1}^{\infty}$, we can also write $(s_n)_{n \in \mathbb{N}}$

ex $(s_n)_{n=3}^{\infty}$ where $s_n = \frac{n+1}{n}$

the seq: $\left(\frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \dots\right)$ ★ order matters ★

its set of values: $\left\{\frac{4}{3}, \frac{5}{4}, \frac{6}{4}, \dots\right\}$ ★ order doesn't matter in set ★

ex $(s_n)_{n \in \mathbb{N}}$, $s_n = \sin\frac{n\pi}{2}$

The sequence: $(1, 0, -1, 0, 1, 0, -1, 0, \dots)$ is periodic

The set of values: $\{-1, 0, 1\}$

Limits of seq

intuitively, a sequence (s_n) converges to a $\mathbb{R}$ # s if s_n is close to s $\forall$ large n

more precisely, we can make (s_n) as close to s as we wish by taking n sufficiently large

def We say that a sequence (s_n) of real numbers converges to a real number $s \in \mathbb{R}$ if for each $\varepsilon > 0$ there $\exists N \in \mathbb{R}$ (*) such that (*) $|s_n - s| < \varepsilon \ \forall n > N$ (OR $n > N$ implies $|s_n - s| < \varepsilon$)

note: ε typically is very small, positive $\mathbb{R}$ number

if (s_n) converges, we write $\lim_{n\to\infty} s_n = s$ or $s_n \to s$ (as $n \to \infty$)

We say that the number s is the limit of (s_n)

Note (*) means that $\forall n \in \mathbb{N}$, $s_n \in (s-\varepsilon, s+\varepsilon)$

Note (*) we can write $n \in \mathbb{N}$ (by the archimedean prop)

ex $(s_n)_{n\in\mathbb{N}}$ where $s_n = \frac{n+2}{n} = 1 + \frac{2}{n} \implies \lim_{n\to\infty} s_n = 1$

let $\varepsilon = 1/10$ $\quad |s_n - s| = \frac{2}{n} < 1/10$ for all $n > 20$

so for $\varepsilon = 1/10$, we can take $N = 20$

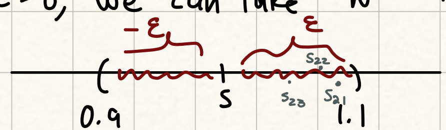
let $\varepsilon = 1/100$ $\quad |s_n - s| = \frac{2}{n} < \frac{1}{100}$ for all $n > 200$

so for $\varepsilon = 1/100$ we can take $N = 200$

let $\varepsilon > 0$ $\quad |s_n - s| = \frac{2}{n} < \varepsilon$ for all $n > 2/\varepsilon$

so for $\varepsilon > 0$, we can take $N = 2/\varepsilon$

$\varepsilon = \frac{1}{10}$



proof of $\lim_{n\to\infty} \frac{n+2}{n} = 1$

let $\varepsilon > 0$. let $N = \frac{2}{\varepsilon}$. Then for every $n > N$,

$$|s_n - s| = \left|\frac{n+2}{n} - 1\right| = \frac{2}{n} < \frac{2}{N} = \frac{2}{2/\varepsilon} = \varepsilon$$

so for all $n > N$, $|s_n - 1| < \varepsilon$. Thus $\lim_{n\to\infty} s_n = 1$ $\square$

general proof to prove that $\lim_{n\to\infty} s_n = s$ given $\varepsilon > 0$ we need to find N in terms of ε st

$|s_n - s| < \varepsilon \quad \forall n > N$

ex $s_n = \frac{1}{\sqrt{n}}$, $n \geq 1$ $\quad$ Prove that $\lim_{n\to\infty} s_n = 0$

DISCUSSION: let $\varepsilon > 0$, $|s_n - 0| = |s_n| = \frac{1}{\sqrt{n}} < \varepsilon$ for all $n > 1/\varepsilon^2$

PROOF. let $\varepsilon > 0$, let $N = 1/\varepsilon^2$, then for every $n > N$, $|s_n - 0| = \left|\frac{1}{\sqrt{n}}\right| < \left|\frac{1}{\sqrt{N}}\right| = \frac{1}{\sqrt{(1/\varepsilon)^2}} = \varepsilon$

so $|s_n - 0| < \varepsilon \quad \forall n > N$ thus $s_n \to 0$ $\square$

ex $s_n = \frac{n-2}{3n+5}$, $n \in \mathbb{N}$ $\lim_{n\to\infty} s_n = 1/3$

DISCUSSION: let $\varepsilon > 0$, $|s_n - 1/3| = \left|\frac{n-2}{3n+5} - 1/3\right| < \varepsilon \iff \left|\frac{3n-6-3n-5}{3(3n+5)}\right| = \frac{11}{3(3n+5)}$

$\iff 3n+5 > \frac{11}{3\varepsilon} \iff n > \frac{11}{9\varepsilon} - 5/3$

PROOF: let $\varepsilon > 0$, let $N = \frac{11}{9\varepsilon} - 5/3$, then $n > N$ means that $n > \frac{11}{9\varepsilon} - 5/3$

by the disc above, it follows that $\left|\frac{n-2}{3n+5} - 1/3\right| < \varepsilon$ therefore, $\lim_{n\to\infty} s_n = 1/3$ $\square$

def if (s_n) does not converge to any real number, we say that (s_n) **diverges**

ex n, n^2, $(-1)^n$, $\sin\left(\frac{n\pi}{4}\right)$, $(-1)^n(1+1/n)$

note: (s_n) does not converge to s means:

(neg) there $\exists \varepsilon > 0$ such that for any $N \in \mathbb{R}$, there exists $n > N$ with $|s_n - s| \geq \varepsilon$

$\hookrightarrow$ there are arbitrarily large n w/ $s_n \notin (s-\varepsilon, s+\varepsilon)$

$$\frac{7n-19}{3n+7} - \frac{7}{3} = \frac{21n-57-21n-49}{9n+21} = \frac{106}{9n+21} < \varepsilon$$

$106 < 9\varepsilon n + 21\varepsilon$

$\frac{106 - 21\varepsilon}{9\varepsilon}$