Week 7

lemma 3.14 Let V be a vector space with basis $v_1, \dots, v_n$ with respect to which a linear map $\alpha: V \to V$ has an upper-triangular matrix M

Then, the eigenvalues are the diagonal entries

def 3.15 a diagonal matrix is a square matrix where every entry off the main diagonal is 0

def 3.16 Let $\alpha: V \to V$ be a linear map & let λ be an eigenvalue of α. The eigenspace of α corresponding to λ is defined as

$$E_\lambda(\alpha) = \ker(\alpha - \lambda \mathrm{Id})$$

lemma 3.17 if $\alpha: V \to V$ is a linear map w/ distinct eigenvalues $\lambda_1, \dots \lambda_m$ then the sum of eigenspaces

$$E_{\lambda_1}(\alpha) + E_{\lambda_2}(\alpha) + \dots + E_{\lambda_m}(\alpha)$$

is a direct sum

cor. 3.18 if $\lambda_1, \dots, \lambda_m$ are the eigenvalues of a linear map $\alpha: V \to V$ then

$$\dim(E_{\lambda_1}(\alpha)) + \dots + \dim(E_{\lambda_m}(\alpha)) \leq \dim(V)$$

def 3.19 a linear map $\alpha: V \to V$ is diagonalizable if, with respect to some basis of V, the matrix of α is diagonal

thm 3.20 if V is a finite-dimensional vector space and $\alpha: V \to V$ is a linear map, with eigenvalues $\lambda_1, \dots, \lambda_m$, then the following are equivalent:

(a) α is diagonalizable

(b) V has a basis of eigenvectors of α

(c) there exist one-dimensional subspaces $U_1, \dots, U_n$ of V, each of which is invariant under α such that $V = U_1 \oplus \dots \oplus U_n$

(d) V is a direct sum $V = E_{\lambda_1}(\alpha) \oplus \dots \oplus E_{\lambda_m}(\alpha)$

(e) $\dim(V) = \dim(E_{\lambda_1}(\alpha)) + \dots + \dim(E_{\lambda_m}(\alpha))$