EXAM 1 Math312H

DEFINITIONS

- if $K \in \mathbb{R}$ satisfies $s \leq K$ for all $s \in S$, we say that K is an upper bound for S.
  ↳ if such $K \in \mathbb{R}$ exists, we say S is bounded above

- if $k \in \mathbb{R}$ satisfies $s \geq k$ for all $s \in S$, we say that k is a lower bound for S.
  ↳ if such $K \in \mathbb{R}$ exists, we say S is bounded below

- if S is bounded above, the least upperbound for S is called the supremum of S and is denoted $\sup S$

- if S is bounded below, the largest lower bound for S is called the infimum of S and is denoted $\inf S$

- The Completeness Axiom – every nonempty subset $S \subseteq \mathbb{R}$ that is bounded above has a least upper bound i.e. $\sup S \in \mathbb{R}$ exists

- Axiomatic definition of $\mathbb{R}$ – $\mathbb{R}$ is an ordered field that satisfies the completeness axiom

- Let M be a nonempty set
  a metric on M is a function $d: M \times M \to \mathbb{R}$ satisfying the following properties:
  i) $d(x,y) \geq 0$ for all $x,y \in M$ AND $d(x,y) = 0$ if and only if $x = y$
  ii) $d(x,y) = d(y,x)$ for all $x,y \in M$
  iii) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x,y,z \in M$ (triangle inequality)
  The pair (M,d) is a metric space

example metric spaces

Euclidean Metric = $(\mathbb{R}^2, \ d(x,y) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2})$

Taxi cab Metric = $(\mathbb{R}^2, \ d(x,y) = |x_1-y_1| + |x_2-y_2|)$

Maximum Metric = $(\mathbb{R}^2, \ d(x,y) = \max\{|x_1-y_1|, |x_2-y_2|\})$

Discrete Metric = M - a nonempty set $\left(M, \ d(x,y) = \begin{cases} 0 & \text{if } x=y \\ 1 & \text{if } x\neq y \end{cases}\right)$

We say that a sequence (s_n) of real numbers converges to $s \in \mathbb{R}$ if for each $\varepsilon > 0$, there exists $N \in \mathbb{R}$ such that $|s_n - s| < \varepsilon \ \forall n > N$

if (s_n) converges, we write $\lim_{n\to\infty} s_n = s$ or $s_n \to s$

We say that s is the limit of (s_n)

We say that a sequence (s_n) in a metric space (M,d) converges to $s \in M$ if for each $\varepsilon > 0$, there exists $N \in \mathbb{R}$ such that $d(s_n, s) < \varepsilon \ \forall n > N$

a sequence in $\mathbb{R}$ is bounded if its set of values is bounded

ie there exists $k, K \in \mathbb{R}$ such that $k \leq s_n \leq K$ for all n

equivalently, there is $K' \in \mathbb{R}$ such that $|s_n| \leq K'$ for all n

a sequence in (M,d) is bounded if its set of values is bounded

ie there exists $x \in M$ and $r > 0$ such that $s_n \in B_r(x)$ for all n

equivalently, there is $x \in M$ and $r' > 0$ such that $d(x, s_n) < r'$ for all n

We say a sequence $(s_n) \in \mathbb{R}$ diverges to $+\infty$ if for any $K > 0$, there $\exists N \in \mathbb{R}$ s.t. $s_n > K \ \forall n > N$

We say a sequence $(s_n) \in \mathbb{R}$ diverges to $-\infty$ if for any $K > 0$, there $\exists N \in \mathbb{R}$ s.t. $s_n < K \ \forall n > N$

a sequence (s_n) in $\mathbb{R}$ is increasing if $s_{n+1} \geq s_n$ for all $n \in \mathbb{N}$

a sequence (s_n) in $\mathbb{R}$ is decreasing if $s_{n+1} \leq s_n$ for all $n \in \mathbb{N}$

if (s_n) is increasing or decreasing, we call it monotone or monotonic

Let (s_n) be a bounded sequence $\in \mathbb{R}$

$\liminf s_n = \lim_{N \to \infty} \inf \{s_n : n > N\} = \lim_{N \to \infty} u_N$

$\limsup s_n = \lim_{N \to \infty} \sup \{s_n : n > N\} = \lim_{N \to \infty} v_N$

if (s_n) is not bounded above, $\limsup s_n = +\infty$

if (s_n) is not bounded below, $\liminf s_n = -\infty$

a sequence (s_n) in a metric space (M,d) is a Cauchy sequence if for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(s_n, s_m) < \varepsilon$ for all $m, n > N$

THEOREMS AND COROLLARIES

$\sqrt{2}$ is irrational

$|a+b| \leq |a| + |b|$ for all $a, b \in \mathbb{R}$

for any $n \in \mathbb{N}$ and any $a_1 \ldots a_n \in \mathbb{R}$, $|a_1 + \ldots + a_n| \leq |a_1| + \ldots + |a_n|$ (Δ inequality)

for any $a, b, c \in \mathbb{R}$, $\text{dist}(a,c) \leq \text{dist}(a,b) + \text{dist}(b,c)$

for any $a, b, c \in \mathbb{R}$, $||a| - |b|| \leq |a-b|$ or $|a| - |b| \leq |a-b|$

The Completeness Axiom – every nonempty subset $S \subseteq \mathbb{R}$ that is bounded above has a least upper bound i.e. $\sup S \in \mathbb{R}$ exists

$\Rightarrow$ every nonempty subset $S \in \mathbb{R}$ that is bounded below has a greatest lower bound ie $\inf S \in \mathbb{R}$ exists

Archimedian Property - for any $a, b \in \mathbb{R}$ such that $a>0$ and $b>0$, there $\exists n \in \mathbb{N}$ st $na > b$

for any $a>0$, there $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < a$

for any $b>0$, there $\exists n \in \mathbb{N}$ such that $n > b$

Denseness of $\mathbb{Q} \in \mathbb{R}$ - for any $a, b \in \mathbb{R}$ such that $a<b$, there exists $r \in \mathbb{Q}$ such that $a<r<b$ (ie every open interval in $\mathbb{R}$ contains a rational number)

Uniqueness of Limit - the limit of a convergent sequence of real numbers is unique, that is, if $s = \lim s_n$ and $t = \lim s_n$, then $s=t$

every convergent sequence in a metric space (M, d) is bounded

if a sequence is not bounded, then it diverges

⇒ limit theorems ⇒ assuming $(s_n) \in \mathbb{R}$

constant multiple - if (s_n) converges to $s \in \mathbb{R}$, and $k \in \mathbb{R}$, then (ks_n) converges to ks $(\lim(ks_n) = k \lim s_n)$

sum - if (s_n) converges to s and (t_n) converges to t, then $(s_n + t_n)$ converges to $s+t$

product - if (s_n) converges to s and (t_n) converges to t, then $(s_n t_n)$ converges to st

inverse - if (s_n) converges to s, $s \neq 0$ and $s_n \neq 0$ for all n, then $\left(\frac{1}{s_n}\right)$ converges to $1/s$

lemma - for a seq (s_n) as above, there is $m>0$ such that $|s_n| \geq m$ for all n

quotient - if (s_n) converges to s, $s \neq 0$ and $s_n \neq 0$ for all n, and (t_n) converges to t, then (t_n / s_n) converges to t/s

squeeze lemma - let $(a_n), (b_n), (c_n)$ be seq $\in \mathbb{R}$; if $a_n \leq b_n \leq c_n$ for all n & $\lim a_n = \lim c_n = s$, then $\lim b_n = s$

Binomial Theorem - for any $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$, $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$

for any $b > 0$ and $n \in \mathbb{N}$:

(i) $(1+b)^n \geq 1 + nb$

(ii) $(1+b)^n \geq \frac{n(n-1)}{2} b^2$

Basic Limit Examples

(a) $\lim_{n \to \infty} \frac{1}{n^p} = 0$ for any $p > 0$

(b) $\lim_{n \to \infty} a^n = 0$ for any a where $|a| < 1$

(c) $\lim_{n \to \infty} n^{1/n} = 1$

(d) $\lim_{n \to \infty} a^{1/n} = 1$ for any $a > 0$

Every bounded monotone sequence in $\mathbb{R}$ converges

(i) if (s_n) is increasing and bounded above, then it converges

(ii) if (s_n) is decreasing and bounded below, then it converges

let (s_n) be a sequence in $\mathbb{R}$, then (s_n) converges to $s \in \mathbb{R}$ if and only if $\liminf s_n = \limsup s_n = s$

every convergent sequence in a metric space is a Cauchy sequence

every Cauchy sequence in $\mathbb{R}$ converges

a sequence in $\mathbb{R}$ converges if and only if it is Cauchy

PROOFS TO KNOW

Uniqueness of Limit - the limit of a convergent sequence of real numbers is unique, that is, if $s = \lim s_n$ and $t = \lim s_n$, then $s = t$

PROOF BY CONTRADICTION

Let $s = \lim s_n$ and $t = \lim s_n$; Suppose $s \neq t \Rightarrow |s-t| > 0$; Let $\varepsilon = \frac{|s-t|}{2}$

Then $\varepsilon > 0$ and, by the definition of a limit:

$\Rightarrow$ there exists $N_1 \in \mathbb{N}$ such that $|s_n - s| < \varepsilon \quad \forall n > N_1$ and

$\Rightarrow$ there exists $N_2 \in \mathbb{N}$ such that $|s_n - t| < \varepsilon \quad \forall n > N_2$

let $n > \max\{N_1, N_2\}$ which implies $|s_n - s| < \varepsilon$ and $|s_n - t| < \varepsilon$

Using the triangle inequality we obtain:

$\Rightarrow |s-t| = |s - s_n + s_n - t| \leq |s - s_n| + |s_n - t| < \varepsilon + \varepsilon = 2\varepsilon = |s-t|$

This implies $|s-t| < |s-t|$ which is impossible; Thus, $s = t$ □

every convergent sequence in a metric space (M,d) is bounded

PROOF

Let (s_n) be a convergent sequence in (M,d) and let $s = \lim s_n$

Let $\varepsilon = 1$; then, there is $N \in \mathbb{N}$ such that $d(s_n, s) < 1$ for all $n > N$

Let $K = \max\{1, d(s_1, s), d(s_2, s), \dots, d(s_N, s)\}$

Then, $d(s_n, s) \leq K$ for all $n \in \mathbb{N}$

Thus, (s_n) is bounded □

sum - if (s_n) converges to s and (t_n) converges to t, then $(s_n + t_n)$ converges to $s+t$

PROOF Let $\varepsilon > 0$. Let $\varepsilon' = \varepsilon/2$

Since $s_n \to s$, there is N_1 such that $|s_n - s| < \varepsilon' = \varepsilon/2 \quad \forall n > N_1$

Since $t_n \to t$ there is N_2 such that $|t_n - t| < \varepsilon' = \varepsilon/2 \quad \forall n > N_2$

Let $N = \max\{N_1, N_2\}$; Then, $\forall n > N$, $|s_n - s| < \varepsilon/2$ and $|t_n - t| < \varepsilon/2$

hence, $|s_n + t_n - (s+t)| = |(s_n - s) + (t_n - t)| \leq |s_n - s| + |t_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Thus, for all $n > N$, $|(s_n + t_n) - (s+t)| < \varepsilon$; Therefore, $s_n + t_n \to s+t$

product - if (s_n) converges to s and (t_n) converges to t, then $(s_n t_n)$ converges to st

PROOF let $\varepsilon > 0$. Since (s_n) is convergent, it is bounded. Hence, $\exists K > 0$ st $|s_n| \le K \; \forall n$

Let $\varepsilon_1 = \frac{\varepsilon}{2K}$; Since $t_n \to t$ there is N_1 such that $|t_n - t| < \frac{\varepsilon}{2K} \quad \forall n > N_1$

Let $\varepsilon_2 = \frac{\varepsilon}{2(|t|+1)}$; since $s_n \to s$ there is N_2 such that $|s - s_n| < \frac{\varepsilon}{2(|t|+1)} \quad \forall n > N_2$

Let $N = \max\{N_1, N_2\}$; Then $\forall n > N$:

$|s_n t_n - st| = |s_n t_n - s_n t + s_n t - st| \le |s_n||t_n - t| + |s_n - s||t| < K \cdot \frac{\varepsilon}{2K} + \frac{\varepsilon}{2(|t|+1)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$s_n t_n \to st$

Every bounded monotone sequence in $\mathbb{R}$ converges

(i) if (s_n) is increasing and bounded above, then it converges

PROOF

let (s_n) be increasing & bounded

Since (s_n) is bounded above, its set of values, $S = \{s_n : n \in \mathbb{N}\}$ is bounded above.

Then by the completeness axiom, S has a supremum, $s \in \mathbb{R}$. We will show that $s = \lim s_n$

Let $\varepsilon > 0$. Since $s - \varepsilon < s$, $s - \varepsilon$ is not an upper bound for S

Hence, there is $N \in \mathbb{N}$ st $s_N > s - \varepsilon$

Since (s_n) is increasing for all $n > N$ we have $s_n \ge s_N > s - \varepsilon$

Also, since $s = \sup S$, $s_n \le s \quad \forall n \in \mathbb{N}$

$s_n \in (s - \varepsilon, s]$ and so $|s_n - s| < \varepsilon$

$\therefore \; s = \lim s_n \quad \square$

every convergent sequence in a metric space is a Cauchy sequence

PROOF Let (s_n) be a convergent sequence in (M, d)

let $s = \lim s_n$. let $\varepsilon > 0$. Since $s_n \to s$ there exists $N \in \mathbb{N}$ such that $\forall n > N$, $d(s_m, s) < \varepsilon/2$

Then for all $m, n > N$ we have $d(s_n, s_m) \le d(s_n, s) + d(s, s_m) < \varepsilon/2 + \varepsilon/2 = \varepsilon$

Thus (s_n) is a Cauchy sequence

TRUE OR FALSE

1. For any real numbers a and b $|a-b| \leq |a|+|b|$ TRUE

2. For any real numbers a and b $|a-b| \leq |a|-|b|$ FALSE

   counter-example: let $a=10$ $b=-10$: $|20| \leq 10-10$ ×

3. If $s<M$ for all $s \in S$ then $\sup S < M$ False

   $S = \{1-1/n, n \in \mathbb{N}\}$ where $\sup S = 1$ and $s<1$

4. If $m > \inf S$ then there exists $s \in S$ such that $s<m$ TRUE

5. Every non-empty bounded set in $\mathbb{Q}$ has a greatest lower bound $\in \mathbb{Q}$ FALSE

   ex; $S = \{1.a_1a_2 \ldots a_n \mid n \in \mathbb{N}$ such that $a_i \to \sqrt{2}\}$

6. If d is a metric on a non-empty set M, then for any $k \geq 0$, $K \geq 0$, Kd is also a metric on M

   FALSE let $(\mathbb{R}, \text{any } d)$, $x=5, y=10$ and $k=0$.. $0 \cdot d(5,10) = 0$ but $x \neq y$

7. In $\mathbb{R}^2$ with the maximum metric, for any $x=(x_1,x_2) \in \mathbb{R}^2$ and any $r>0$,

   $B_r(x) = (x_1-r, x_1+r) \times (x_2-r, x_2+r)$ TRUE

8. If $\lim s_n = 1$, then $.9 < s_n < 1.1$ for all n except finitely many TRUE

9. If for each $\varepsilon > 0$, there are infinitely many n st $|s - s_n| < \varepsilon$, then (s_n) converges to s FALSE

   ex $s_n = (-1)^n$ then let $s=1$, $|1-s_n| < \varepsilon \to$ there are infinitely many n but doesn't converge

10. If the set of values of a sequence is $\{\frac{1}{n}, n \in \mathbb{N}\}$, then the sequence converges FALSE

11. If $|s_n| \leq t_n$ for all n and $t_n \to 0$, then $s_n \to 0$ TRUE

12. If $s_n > a$ for all n, then $\lim s_n > a$ FALSE ex $s_n = 1/n$, $s=0$, let $a=0$ $s_n > 0$ but $0 \not> 0$

13. If $a \leq s_n \leq b$ for all n & (s_n) converges, then $a \leq \lim s_n \leq b$ TRUE

14. If (s_n/t_n) converges and (s_n) converges, then (t_n) also converges FALSE

    let $s_n = 0$ $\forall n$, and $t_n = (-1)^n$ Δ

15. Every bounded seq converges FALSE ex $(-1)^n$

16. For every unbounded seq (s_n), the seq $(|s_n|)$ diverges to $+\infty$ FALSE ~~TRUE~~ ?

17. Every bounded monotone seq converges TRUE

18. If $b = \limsup s_n$, then there is a number N st $s_n \leq b$ for all $n > N$ FALSE

    it is an open ball so it could be slightly above ex $s_n = (-1)^n (2 + 1/n)$,

19. If $\liminf s_n \in \mathbb{R}$ and $\liminf t_n \in \mathbb{R}$, then $\liminf(s_n t_n) = (\liminf s_n) \cdot (\liminf t_n)$ FALSE

ex $s_n = \begin{cases} 1/n & \text{if even} \\ -1 & \text{if odd} \end{cases}$ $t_n = \begin{cases} -1 & \text{if even} \\ 1/n & \text{if odd} \end{cases}$

$\therefore \liminf s_n = \liminf t_n = -1$, $\frac{-1}{n} \to 0$ $\therefore$ false

20. If $\liminf s_n \neq \limsup s_n$ then (s_n) diverges TRUE

21. Changing finiely many terms of a sequence does not affect its limsup TRUE

22-25, all sequences are in a metric space

22. If a sequence (s_n) converges to s & to t, then $s=t$ TRUE

23. If for each $K \in \mathbb{N}$, there $\exists N \in \mathbb{N}$ st $d(s_n, s) < 1/K$ for all $n > N$, then $s_n \to s$ TRUE

24. Every convergent sequence in a metric space is a Cauchy seq TRUE

25. In a discrete metric space, every Cauchy seq converges TRUE