wednesday 2.19

Complete Metric Spaces

def) a metric space (M,d) is called complete if for every Cauchy sequence in (M,d) converges to some element of M

ex) (1) $\mathbb{R}$ w/ the standard metric is a complete metric space

(2) $\mathbb{Q}$ w/ the standard metric is not a complete metric space

(3) $(0,1]$ w/ the standard metric is not a complete metric space

(4) $[a,\infty)$ is complete

PROOF let (s_n) be a Cauchy sequence in $[a,\infty)$.

Then (s_n) is also a Cauchy seq $\in \mathbb{R}$

& hence it converges to some $s \in \mathbb{R}$

Since $s_n \geq a$ for all n $s = \lim s_n \geq a$

Thus $s \in [a,\infty)$ and so (s_n) converges to a point in $[a,\infty)$

(5) a discrete metric space is a complete metric space

(6) $\mathbb{R}^2$ w/ Euclidean metric is a complete metric space

? could we say every valid metric on $\mathbb{R}$ converges since we know every Cauchy seq converges in $\mathbb{R}$? no

PROOF let $s_n = (a_n, b_n)$ be a Cauchy seq $\in \mathbb{R}^2$. Then (a_n) & (b_n) are Cauchy seq $\in \mathbb{R}$ ($\Leftarrow$ exercise)

Then (a_n) & (b_n) conv to some a & $b \in \mathbb{R}$

and hence (s_n) converges to (a,b)

Construction of $\mathbb{R}$ from $\mathbb{Q}$ using Cauchy sequences

Informally, we can view $\mathbb{R}$ as the set of all limits of Cauchy sequences of rational numbers

Or we can take $\mathbb{Q}$ & add the limits of Cauchy sequences in $\mathbb{Q}$

Note: many sequences converge to the same number

We say that 2 Cauchy sequences (s_n) & (t_n) of rational numbers are equivalent if $d(s_n, t_n) = |s_n - t_n| \to 0$ as $n \to \infty$

We can define $\mathbb{R}$ as the set of equivalence classes of Cauchy sequences of rational #s

Subsequences (Sec 11)

Def let (s_n) be a sequence in $\mathbb{R}$ or in (M,d), a subsequence of (s_n) is a sequence $(s_{n_k})_{k \in \mathbb{N}}$, where $n_k \in \mathbb{N}$ and $n_1 < n_2 < n_3 < \ldots$

ex $(s_9, s_{12}, s_{40}, s_{99}, \ldots)$

$n_1 = 9 \quad n_2 = 12 \quad n_3 = 14 \quad n_4 = 99$

ex $s_n = 1/n$, $n \in \mathbb{N}$

subsequences:

$(1, 1/3, 1/5, 1/7, 1/9, \ldots)$ $\quad s_{n_k} = \frac{1}{2k-1}$ $\quad n_k = 2k-1$

$(1/3, 1/9, 1/27, 1/81, \ldots)$ $\quad s_{n_k} = \frac{1}{3^k}$ $\quad n_k = 3^k$

NOT Subsequenes:

$(1, 1, \ldots)$ even $(1, 1, 1/2, 1/3 \ldots)$

$(\frac{1}{10}, 1/9, 1/20, 1/19, \ldots)$ ← changed order

ex $s_n = \sin \frac{n\pi}{2}$, $n \in \mathbb{N}$

$(s_n) = (1, 0, -1, 0, \ldots)$

subsequences:

$(1, 1, 1, \ldots)$

$(1, 0, 1, 0, 1, 0, \ldots)$

$(1, 0, 1, 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 0, \ldots)$

thm If (s_n) in (M, d) converges, then all its subsequences converge to the same limit

PROOF let $s = \lim(s_n)$ & let $(s_{n_k})_{k \in \mathbb{N}}$ be a subsequence of $(s_n)_{n \in \mathbb{N}}$

For any $k \in \mathbb{N}$, $n_k \geq k$ (exercise, proof by induction)

let $\varepsilon > 0$, Since $s_n \to s$, there is N st $d(s_n, s) < \varepsilon$ $\forall n > N$

Since $n_k \geq k$ for all $k \in \mathbb{N}$, it follows that

$d(s_{n_k}, s) < \varepsilon$ $\forall k > N$

Thus, $s_{n_k} \to s$ $\square$

Corolary If (s_n) has a diverging subsequence or 2 subsequences converging to different limits, then (s_n) diverges

friday 2.21

ex $s_n = \cos\left(\frac{n\pi}{100}\right)$

$s_{200k} = \cos\left(\frac{200k\pi}{100}\right) = \cos(2\pi k) = 1$ for all $k \in \mathbb{N}$

So the Subseq s_{200k} conv to 1

$s_{100+200k} = \cos\left(\frac{(100+200k)\pi}{100}\right) = \cos(\pi + 2\pi k) = -1$ for all $k \in \mathbb{N}$

so the subsequence $s_{100+200k}$ conv to -1

Since (s_n) has 2 subsequences converging to different limits, it diverges

T/F Every sequence in $\mathbb{R}$ has a convergent subsequence False

$s_n = n \to$ every subseq, diverges to $+\infty$

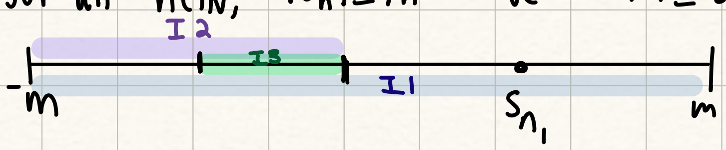
thm Bolzano-Weierstrass Theorem - every bounded sequence $\in \mathbb{R}$ has a convergent subsequence

PROOF let (s_n) be a bdd sequence in $\mathbb{R}$

We will construct a Cauchy sequence of (s_n) inductively

Since (s_n) is bdd, there exists $m > 0$ such that

for all $n \in \mathbb{N}$, $|s_n| \leq m$ ie $-m \leq s_n \leq m$



We let $I_1 = [-M, M]$ and $n_1 = 1$

We divide I_1 into halves, denoting a half w/ ∞ly many terms of (s_n) by I_2, & choose $n_2 > N_1$ st $s_{n_2} \in I_2$

Suppose $I_1 > I_2 > \dots > I_k$ and $n_1 < n_2 < \dots < n_k$ st $s_{n_k} \in I_k$ are chosen

We divide I_k into halves, denote a half w/ ∞ly many terms of (s_n) by I_{k+1} & choose $n_{k+1} > n_k$ st $s_{n_{k+1}} \in I_{k+1}$

The resulting subsequence (s_{n_k}) is Cauchy

Indeed, for each $N \in \mathbb{N}$, $s_{n_k} \in I_N$ for all $k \geq N$

and $|I_N| = \frac{2M}{2^{N-1}} \to 0$ as $N \to \infty$

(explain)

It follows that (s_{n_k}) converges □

Limits of Subsequences

def) a subsequential limit of (s_n) is the limit of a subsequence of (s_n)

it can be a real number, $-\infty$, or ∞

ex) $s_n = \cos\left(\frac{n\pi}{4}\right) + 1/n$

subseq limits: $1, \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}, -1$

$s_n = (-n)^n$

subseq limits: $\infty, -\infty$

thm) let (s_n) be a bounded sequence $\in \mathbb{R}$

then $\limsup s_n$ is the largest of its subsequential limits, & $\liminf$ is the smallest

PROOF for limsup. let (s_n) be a bdd sequence $\in \mathbb{R}$ & let $b = \limsup s_n$

(1) for each $\varepsilon > 0$, there is $N \in \mathbb{N}$ st $s_n < b + \varepsilon$ $\forall n > N$

(2) For each $\varepsilon > 0$, there are ∞ly many n st $s_n > b - \varepsilon$

(3) There is a subsequence (s_{n_k}) conv to b

We construct (s_{n_k}) st $|s_{n_k} - b| < 1/k$ for all k inductively

it follows that $s_{n_k} \to b$ (explain)

By (1) & (2), for each $\varepsilon > 0$ there are ∞ly many n st $s_n \in (b-\varepsilon, b+\varepsilon)$

We take n_1 st $|s_{n_1} - b| < 1$

Suppose $n_1 < n_2 < \dots < n_k$ are chosen st $|b - s_{n_i}| < 1/i$ for all $i = 1, \dots, k$

Since there are ∞ many n st $|s_n - b| < 1/k+1$, we can choose

$n_{k+1} > n_k$ st $|s_{n_{k+1}} - b| < 1/k+1$

(4) (s_n) has no subsequence (s_{n_k}) whose limit $\lim s_{n_k} > b$

exercise

? How many subsequential limits can a sequence $(s_n) \in \mathbb{R}$ have?

$(1, 2, \dots, N, 1, 2, \dots, N, \dots)$ has exactly N subseq limits

? Can (s_n) have infinitely many subseq limits?

ex) of a sequence st every $s \in \mathbb{R}$ is its subseq limit.

We construct (s_n) that it contains every rational number

$$\begin{array}{cccccccc}
0 & 1 & -1 & 2 & -2 & 3 & -3 & \dots \\
 & 1/2 & -1/2 & 2/2 & -2/2 & 3/2 & -3/2 & \dots \\
 & 1/3 & -1/3 & 2/3 & -2/3 & 3/3 & -3/3 & \dots \\
 & 1/4 & \dots & & & & & \\
 & \vdots & & & & & &
\end{array}$$

similar to (3) in earlier proof, use density of $\mathbb{Q}$

∞ and $-\infty$ are also its subseq limits

So the set of all subseq limits of (s_n) is $\mathbb{R} \cup \{\infty, -\infty\}$