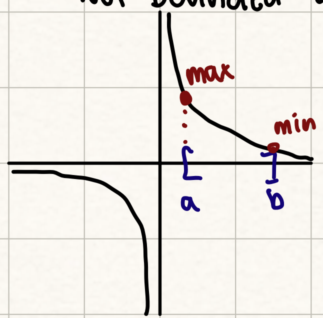
Monday 3/24

Properties of Continuous functions (Sec. 18)

def a func f is **bounded** if there is $K \in \mathbb{R}$ s.t. $|f(x)| \leq K$ for all $x \in dom(f)$

ex Bounded on natural domain: $\sin x$, $\cos x$, C

Not Bounded on natural domain: $x, x^2, \sqrt{x}, \ln x, e^x, 1/x, \ldots$



NOTE: for any $[a,b]$ in $\mathbb{R}\setminus\{0\}$, $1/x$ is bdd on $[a,b]$ and attains max & min $[a,b]$

thm let f be a continuous func on a closed interval $[a,b]$. Then

(i) f is bounded on $[a,b]$ &

(ii) f attains its maximum and minimum values on $[a,b]$ ie there exist $x_0, y_0 \in [a,b]$ st for all $x \in [a,b]$,

$f(x_0) \leq f(x) \leq f(y_0)$

PROOF (i) by contradiction

Suppose that f is not bdd on $[a,b]$

Then for each $n \in \mathbb{N}$ there is $x_n \in [a,b]$ st $|f(x_n)| > n$

By Bolzano-Weierstrass thm, since (x_n) is a bdd seq $\in \mathbb{R}$,

it has a subsequence (x_{n_k}) converging to some $x_0 \in \mathbb{R}$

Since $[a,b]$ is closed, $x_0 \in [a,b]$

Since f is cont on $[a,b]$ and hence at x_0, $f(x_{n_k}) \to f(x_0)$

This contradicts $|f(x_{n_k})| \to \infty$

So f is bdd $\square$

(ii) (for y_0 for x_0 is similar)

let $M = \sup\{f(x) : x \in [a,b]\}$

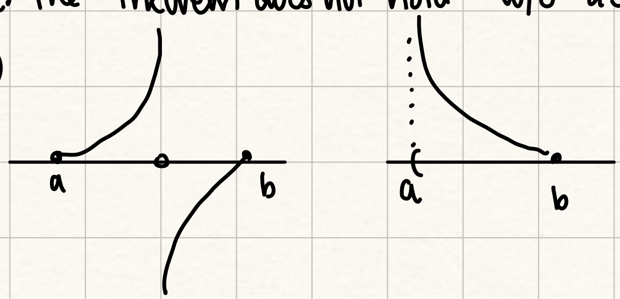
since f is bdd on $[a,b]$, $M \in \mathbb{R}$

By def of the supremum, for each $n \in \mathbb{N}$, $\exists\, y_n \in [a,b]$ st $\boxed{M - 1/n < f(y_n) \leq M}$ *

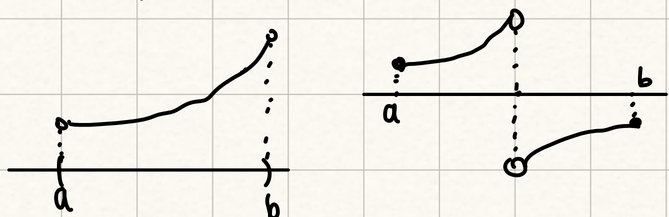
By Bolzano-Weierstrass, (y_n) has a convergent subsequence (y_{n_k}) and since $[a,b]$ is closed, $y_0 = \lim y_{n_k} \in [a,b]$. Since f is cont at y_0, $f(y_{n_k}) \to f(y_0)$. Also, from $(*)$ $f(y_{n_k}) \to M$. Thus $f(y_0) = M$ and so f assumes its max value

note: the theorem does not hold w/o assumptions

(i)



ii)



thm The Intermediate Value Theorem

Suppose that f is cont on an interval I

if $a<b$ are in I & y is between $f(a)$ and $f(b)$

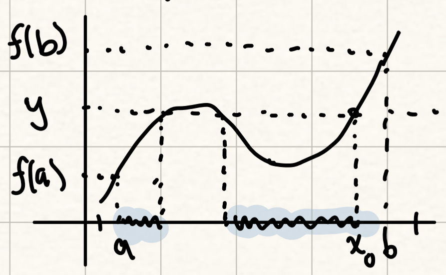
(ie $f(a)<y<f(b)$ or $f(b)<y<f(a)$)

Then there is $x \in (a,b)$ st $f(x)=y$

corollary If f is continuous on a closed interval $[a,b]$; then $f([a,b])$ is either a single pt or the closed interval $[m,M]$ where $m = \min\limits_{[a,b]} f$ and $M = \max\limits_{[a,b]} f$

PROOF Outline of IVT (8.2)

let $a<b$, $f(a)<f(b)$ and $f(a)<y<f(b)$



let $S = \{t \in [a,b] : f(t) < y\}$

$S \neq \emptyset$ since $a \in S$ and S is bdd above by b

By the completeness axiom, S has a $\sup \in \mathbb{R}$

denote $\sup S$ by x_0

Show: $f(x_0) \leq y$ and $f(x_0) \geq y$ (pick sequences)