monday 2.1

recall squeeze lemma

let $(a_n), (b_n), (c_n)$ be sequences in $\mathbb{R}$

if $a_n \le b_n \le c_n$ for all n and $\lim a_n = \lim c_n = s$, then $\lim b_n = s$

def) for any $n \in \mathbb{N}$, $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$

and $0! = 1$

def) for any $0 \le k \le n$, $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

thm) The Binomial theorem:

for any $a, b \in \mathbb{R}$ and $n \in \mathbb{N}$, $(a+b) = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k =$

$= \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b$

$a^n + n a^{n-1} b + \frac{n(n-1)}{2} a^{n-2} b + \ldots + b^n$

Corollary) for any $b > 0$ and $n \in \mathbb{N}$,

(i) $(1+b)^n \ge 1 + nb$

(ii) $(1+b)^n \ge \frac{n(n-1)}{2} b^2$

thm) (Basic examples to know)

(a) $\lim_{n\to\infty} \frac{1}{n^p} = 0$ for any $p > 0$

(b) $\lim_{n\to\infty} a^n = 0$ for any a with $|a| < 1$

(c) $\lim_{n\to\infty} n^{1/n} = 1$

(d) $\lim_{n\to\infty} a^{1/n} = 1$ for any $a > 0$

PROOFS $\frac{1}{n^p} - 0$ $(1/\varepsilon)^{1/p}$

(a) let $\varepsilon > 0$, let $N = (1/\varepsilon)^{1/p}$ so $1/n^p \to 0$

(b) let $\varepsilon > 0$, $|a^n - 0| < \varepsilon$

then $|a| < \frac{1}{1+b}$ for some $b > 0$

and so $|a^n| = \frac{1}{(1+b)^n} \ge \frac{1}{1+nb}$ by (i) for any $n \in \mathbb{N}$

for any $n \in \mathbb{N}$ $\quad 0 \le |a^n| \le 1/(1+nb) \le 1/nb$

$1/nb$ goes to 0 b/c in (a) if we let $p=1$ then $1/n \to 0$ & $\left(\frac{1}{n}\right)K \to K0 \therefore 0$

By the squeeze lemma $|a^n| \to 0$ $\therefore$ $a^n \to 0$

(c) let $\varepsilon > 0$, for any $n \in \mathbb{N}$, $n^{1/n} \ge 1$ and we have

$$|n^{1/n} - 1| < \varepsilon \iff n^{1/n} - 1 < \varepsilon \iff (n^{1/n})^n < (\varepsilon+1)^n \iff n < (1+\varepsilon)^n \le \frac{n(n-1)}{2}\varepsilon^2 \text{ (ii)}$$

$$\iff \frac{\cancel{n}(n-1)}{2}\varepsilon^2 > \cancel{n} \iff n > 2/\varepsilon^2 + 1$$

let $N = \frac{2}{\varepsilon^2} + 1$ Then it follows that for all $n > N$, $|n^{1/n} - 1| < \varepsilon$

Thus $n^{1/n} \to 1$

d) let $a > 1$, then for all $n > a$ we have

$1 \le a^{1/n} \le n^{1/n}$ $\therefore$ it follows from (c) & the squeeze lemma that $a^{1/n} \to 1$

let $0 < a < 1$, then $1/a > 1$ and so $\left(\frac{1}{a}\right)^{1/n} \to 1$

then $a^{1/n} = \frac{1}{(1/a)^{1/n}} \to 1/1 \to 1$

ex) $\lim\limits_{n\to\infty}(3n^2)^{1/n} = \lim\limits_{n\to\infty} 3^{1/n}\cdot n^{1/n}\cdot n^{1/n} = \lim\limits_{n\to\infty} 3^{1/n} \cdot \lim\limits_{n\to\infty} n^{1/n} \cdot \lim\limits_{n\to\infty} n^{1/n} = 1\cdot 1\cdot 1 = 1$

Remark Changing finitely many terms in a sequence does not affect its convergence/divergece, limit, (un) boundedness

Sequences diverging to $+\infty$ or $-\infty$

def We say that a sequence $(s_n) \in \mathbb{R}$ diverges to $+\infty$ and write $\lim s_n = +\infty$ if for any $K > 0$, there is $N \in \mathbb{R}(\mathbb{N})$ s.t. $s_n > K$ $\forall n > N$

$s_n \in (K, \infty)$

def We say that a sequence $(s_n) \in \mathbb{R}$ diverges to $-\infty$ and write $\lim s_n = -\infty$ if for any $K > 0$, there is $N \in \mathbb{R}(\mathbb{N})$ s.t. $s_n < K$ $\forall n > N$

$s_n \in (-\infty, K)$

ex) diverge to $+\infty$ $s_n = n, n^3 - n$

diverge to $-\infty$ $s_n = -n, -2^n$

unbounded but does not diverge to $-\infty$ or $+\infty$

$s_n = (-2)^n$ or $(-1)^n n$

Wednesday 2.5

Monotone sequences

def) a sequence in $\mathbb{R}$ is **increasing** if $s_{n+1} \geq s_n$ for all n

a sequence in $\mathbb{R}$ is **decreasing** if $s_{n+1} \leq s_n$ for all n

if (s_n) is increasing or decreasing, we call it **monotone** or monotonic

ex) increasing & bounded: $s_n = 1 - 1/2^n, -1/n$

increasing & unbounded: $s_n = n, n^2 - 1, \sqrt{n}$

decreasing & bounded: $s_n = 1/n, 1 + 1/2^n, ..$

decreasing & unbounded: $s_n = -n, -n^2, -2^n$

T/F? every increasing sequence is bdd below TRUE

$\Rightarrow$ yes! by s_1 BUT not necessarily above

& similarly, every decreasing sequence is bdd above by s_1

★ every bdd above inc seq converges & every bdd below dec seq converge

Thm) Every bounded monotone sequence in $\mathbb{R}$ converges

(1) if (s_n) is increasing and bdd above, then it converges

(2) if (s_n) is decreasing and bdd below, then it converges

PROOF of (1)

let (s_n) be increasing & bdd above

Since (s_n) is bdd above, its set of values, $S = \{s_n : n \in \mathbb{N}\}$ is bdd above. Then, by the completeness axiom, S has a supremum, $s \in \mathbb{R}$. We will show that $s = \lim s_n$

let $\varepsilon > 0$. Since $s - \varepsilon < s$, $s - \varepsilon$ is not an upper bound for S.

Hence, there is $N \in \mathbb{N}$ st $s_N > s - \varepsilon$

Since (s_n) is increasing for all $n > N$ we have $s_n \geq s_N > s - \varepsilon$

also, since $s = \sup S$, $s_n \leq s \ \forall n \in \mathbb{N}$

$s_n \in (s - \varepsilon, s]$ and so $|s_n - s| < \varepsilon$

$\therefore$ $s = \lim s_n$ $\square$

T/F? Every inc bdd above seq of $\mathbb{Q}$ has a limit in $\mathbb{Q}$ FALSE

COUNTER-EXAMPLE consider the decimal expansion of $\sqrt{2} = 1.a_1 a_2 a_3 \ldots$

& let $s_n = 1.a_1 a_2 a_3 \ldots a_n$

$s_n \in \mathbb{Q}$ for each n

(s_n) is increasing & bdd above by 2

$s_n \to \sqrt{2} \notin \mathbb{Q}$ in $\mathbb{R}$. It cannot converge to some $s \in \mathbb{Q}$ by uniqueness of limit

example of recursively defined sequence (s_n is defined in terms of s_{n-1} or several prev terms)

let $s_1 = 1/3$ and $s_n = \frac{1}{2}(s_{n-1}^2 + 1)$ $\forall n \geq 2$

does (s_n) converge?

$s_1 = 1/3$ $s_2 = 5/9 > 1/3$ $s_3 = \frac{1}{2}\left(\frac{25}{81} + 1\right) = 53/81 > 5/9$

Claim (1) (s_n) is increasing

(2) (s_n) is bounded ($s_n \leq 1 \ \forall n$)

(1) $s_{n+1} \geq s_n \iff 1/2(s_n^2 + 1) \geq s_n \iff s_n^2 - 2s_n + 1 \geq 0 \iff (s_n - 1)^2 \geq 0$ which is true for any $s_n \in \mathbb{R}$

note: $s_n \geq 0 \ \forall n$

(2) Proof by induction

BASE: $n = 1$, $s_1 = 1/3 < 1$

Ind: Suppose $s_n \leq 1$ for some $n \in \mathbb{N}$

Then $s_{n+1} = \frac{1}{2}(s_n^2 + 1) \leq 1/2(1^2 + 1) = 1$

Thus $s_n \leq 1$ $\forall n \in \mathbb{N}$

$\therefore$ Because (s_n) is incr & bdd above, it converges to some $s \in \mathbb{R}$

to find s, we pass the limit in & get

$s = \frac{1}{2}(s^2+1) \iff s^2 - 2s + 1 = 0 \iff s = 1$

Thus (s_n) conv to 1

now let $s_1 = 10$ instead of $1/3$

Then (s_n) is incr but no longer bdd $\therefore$ this sequence will not converge

(s_n) diverges to $+\infty$

Note: If (s_n) is increasing and not bdd above, then (s_n) diverges to $+\infty$

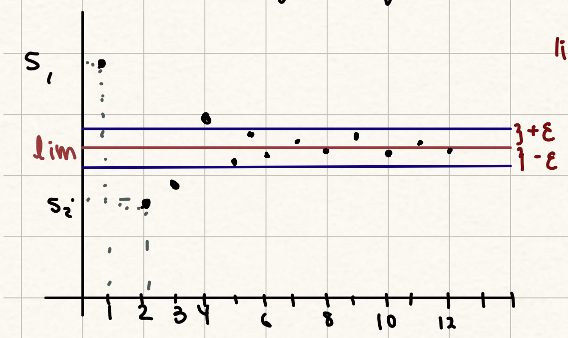
Pf: Since (s_n) is not bdd above, $\exists N \in \mathbb{N}$ st $s_N > K$, Then for all $n > N$ we have $s_n \geq s_N > K$ Thus $s_n \to +\infty$

Friday 2.7

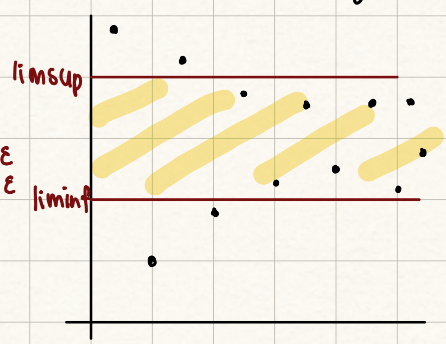
Liminf and Limsup

(limit inferior and limit superior)

Consider a convergent sequence



Bounded Divergent



after N,

$\liminf \leq s_n \leq \limsup$

Note: limiting behavior of (s_n) depends only on its "tail", $(s_n)_{n>N}$

let (s_n) be a bounded sequence in $\mathbb{R}$

for each $N \in \mathbb{N}$, let $u_N = \inf\{s_n : n > N\} = \inf\{s_{N+1}, s_{N+2}, \ldots\}$, and similarly take $v_N = \sup\{s_n : n > N\} = \sup\{s_{N+1}, s_{N+2}, \ldots\}$

Since (s_n) is bounded, ie $\{s_n : n \in \mathbb{N}\}$ is bdd,

the sequences (u_N) & (v_N) are bdd

(u_N) is increasing & (v_N) is decreasing

indeed, for each $N \in \mathbb{N}$, then $\{s_{N+2}, s_{N+3}, \dots\} \subseteq \{s_{N+1}, s_{N+2}, \dots\}$

and so $u_{N+1} = \inf\{s_{N+2}, \dots\} \leq \inf\{s_{N+1}, \dots\} = u_N$

& $v_{N+1} = \sup\{\text{—''—}\} \leq \sup\{\text{—''—}\} = v_N$

Then, (u_N) and (v_N) are monotone & bdd, hence they converge

def let (s_n) be a bdd sequence in $\mathbb{R}$

$$\liminf s_n = \lim_{N\to\infty} \inf\{s_n : n > N\} \quad \left(= \lim_{N\to\infty} u_N\right)$$

$$\limsup s_n = \lim_{N\to\infty} \sup\{s_n : n > N\} \quad \left(= \lim_{N\to\infty} v_N\right)$$

if (s_n) is not bdd above, $\limsup s_n = +\infty$

if (s_n) is not bdd below, $\liminf s_n = -\infty$

Note: $\liminf s_n \leq \limsup s_n$ since $u_N \leq v_N \ \forall n$

ex find liminf & limsup

(1) $s_n = \sin\frac{n\pi}{2}$, $n \in \mathbb{N}$

$(s_n) = (1, 0, -1, 0, 1, 0, -1 \dots)$

for each N, $u_N = -1$ so $\liminf s_n = -1$

—''— , $v_N = 1$, so $\limsup s_n = 1$

(2) $s_n = 1/n$, $n \in \mathbb{N}$ (s_n) converges

for each N, $u_N = \inf\{s_{N+1}, s_{N+2}, \dots\} = \inf\{\frac{1}{N+1}, \frac{1}{N+2}, \dots\} = 0$

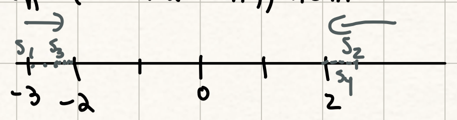
—''— , $v_N = \sup\{\frac{1}{N+1}, \frac{1}{N+2}, \dots\} = 1/N+1$

$\liminf s_n = 0$

$\limsup s_n = \lim_{N\to\infty} \frac{1}{N+1} = 0$

liminf = limsup

(3) $s_n = (-1)^n (2 + 1/n)$, $n \in \mathbb{N}$



$\liminf s_N = -2$

$\limsup s_N = 2$

(4) $s_n = (-1)^n n$

$\liminf s_n = -\infty$ $\limsup s_n = +\infty$

(5) $s_n = n$ $\limsup s_n = +\infty$ since not bdd above

$\liminf s_n = \lim_{N \to \infty} N+1 = +\infty$

thm let (s_n) be a sequence in $\mathbb{R}$

then (s_n) converges to $s \in \mathbb{R}$

iff $\liminf s_n = \limsup s_n = s$

Proof ($\Rightarrow$) Suppose that (s_n) conv to $s \in \mathbb{R}$

let $\varepsilon > 0$, then $\exists N \in \mathbb{N}$ st $s_n \in (s-\varepsilon, s+\varepsilon)$ $\forall n > N$

it follows that for all $m > N$, $s - \varepsilon \leq u_m \leq v_m \leq s + \varepsilon$

Then $s - \varepsilon \leq \liminf s_n \leq \limsup s_n \leq s + \varepsilon$

Since $\varepsilon > 0$ is arbitrary, we conclude that $\liminf s_n = \limsup s_n = s$

($\Leftarrow$) Suppose that $\liminf s_n = \limsup s_n = s$

let $\varepsilon > 0$. Since $\lim_{m \to \infty} u_m = s$ and $\lim_{m \to \infty} v_m = s$, then

$\exists N_1, N_2 \in \mathbb{N}$ st $\forall m > N_1$, $|u_m - s| < \varepsilon$ & hence $u_m > s - \varepsilon$

$\forall m > N_2$, $|v_m - s| < \varepsilon$ & hence $v_m < s + \varepsilon$

then, $\forall n > N_1 + 1$, $s_n > s - \varepsilon$, and $\forall n > N_2 + 1$, $s_n < s + \varepsilon$

let $N = \max\{N_1 + 1, N_2 + 1\}$

Then for all $n > N$, $s - \varepsilon < s_n < s + \varepsilon$

Thus $s_n \to s$ $\square$