monday 1.27

Sequences in metric spaces (M,d)

$(s_n)_{n=m}^{\infty}$, $s_n \in M$

ex) (1) $M=\mathbb{R}^2$ w/ euclidean dist $(s_n)_{n=1}^{\infty}$ where $s_n = \left(\frac{1}{n}, (-1)^n\right)$

(2) $M=\{a,b,c\}$ w/ discrete metric $(s_n)_{n\in\mathbb{N}} = (a,b,a,b,\dots)$

def) a sequence (s_n) in a metric space (M,d) converges to $s\in M$ if for each real number $\varepsilon>0$ there exists $N\in\mathbb{R}$ st.

(*) $d(s_n,s)<\varepsilon$ for all $n>N$ [$n>N$ implies $d(s_n,s)<\varepsilon$]

(*) means: $s_n \in B_\varepsilon(s)$ $\forall n>N$

$s_n \to s$ iff the sequence of real numbers $a_n := d(s_n,s)$ converges to 0

ex) (1) diverges

(2) diverges

(3) $\mathbb{R}^2$ w/ euclidean metric, $s_n = \left(\frac{(-1)^n}{n}, \frac{1}{n}\right)$ $n\geq 1$ converges to $(0,0)$

DISCUSSION $d(s_n,(0,0)) = \sqrt{\frac{1}{n^2}+\frac{1}{n^4}} \leq \sqrt{\frac{1}{n^2}+1/n^2} = \sqrt{\frac{2}{n^2}} = \frac{\sqrt{2}}{n} < \varepsilon \Leftrightarrow \frac{\sqrt{2}}{\varepsilon} < n$

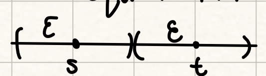
for all $n > \frac{\sqrt{2}}{\varepsilon}$ so take $N = \frac{\sqrt{2}}{\varepsilon}$

PROOF let $\varepsilon>0$, Let $N = \frac{\sqrt{2}}{\varepsilon}$, then for all $n>N$ we have $d(s_n,(0,0)) = \sqrt{\frac{1}{n^2}+1/n^4}$

$= \sqrt{\frac{1}{n^2}+1/n^4} \leq \sqrt{2\frac{1}{n^2}} = \frac{\sqrt{2}}{n} < \frac{\sqrt{2}}{N} = \frac{\sqrt{2}}{\sqrt{2}/\varepsilon} = \varepsilon$

so $d(s_n,(0,0))<\varepsilon$ for all $n>N$ thus $s_n\to(0,0)$ □

(?) Can a sequence in $\mathbb{R}$ have 2 limits? NO



$\varepsilon = \frac{|t-s|}{2}$

thm) The limit of a convergent sequence of $\mathbb{R}$ #s is unique, that is, if $s=\lim s_n$ and $t=\lim s_n$, then $s=t$

prf) (by contradiction)

let $s=\lim s_n$ & let $t=\lim s_n$. Suppose $s\neq t$. Then $|s-t|>0$. Let $\varepsilon = \frac{|s-t|}{2}$

Then $\varepsilon>0$ & by the def of a limit we have

there is N_1 st $|s_n-s|<\varepsilon$ $\forall n>N_1$ and

there is N_2 st $|s_n-s|<\varepsilon$ $\forall n>N_2$

let $n>\max\{N_1,N_2\}$, then $|s_n-s|<\varepsilon$ and $|s_n-t|<\varepsilon$

Using the triangle inequality we obtain

$|s-t| = |s-s_n + s_n - t| \le |s-s_n| + |s_n - t| < \varepsilon + \varepsilon = 2\varepsilon = |s-t|$

So $|s-t| < |s-t|$ which is impossible

Thus $s=t$ □

hw → in a metric space, proof is similar in HW, dist instead of abs

Bounded sequences

def a sequence (s_n) in $\mathbb{R}$ is bounded if its set of values is bounded

i.e. there exist $k, K \in \mathbb{R}$ st $k \le s_n \le K$ $\forall n$

equiv, there is $K' \in \mathbb{R}$ st $|s_n| \le K'$ $\forall n$

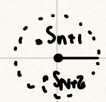
def a sequence (s_n) in (M,d) is bdd if ___ " ___

there is $x \in M$ and $r>0$ st $s_n \in B_r(x)$ $\forall n$

equiv, there is $x \in M$ & $r'>0$ st $d(x, s_n) \le r'$ $\forall n$

thm Every convergent sequence is a metric space (M,d) is bounded

proof let (s_n) be a convergent seq $\in (M,d)$ & let $s = \lim s_n$



let $\varepsilon = 1$, then there is $N \in \mathbb{N}$ st $d(s_n, s) < 1$ $\forall n > N$

let $K = \max\{1, d(s_1, s), d(s_2, s), \dots, d(s_N, s)\}$

Then, $d(s_n, s) \le K$ for all $n \in \mathbb{N}$

Thus (s_n) is bounded □

Wednesday 1.29 - see 9

Limit Theorems for sequences of real numbers

corollary if a sequence is not bounded, then it diverges : the contrapositive $\quad A \to B \equiv \neg B \to \neg A$

↳ **ex** $s_n = n, n, (-1)^n n, 2^n$ are unbounded → divergent

thm constant multiple (assuming (s_n) is seq of $\mathbb{R}$ #s)

if (s_n) converges to $s \in \mathbb{R}$, and $k \in \mathbb{R}$, then (ks_n) converges to ks i.e. $\lim(ks_n) = k\lim(s_n)$

idea: make $|ks_n - ks| < \varepsilon$ make $|s_n - s| = \varepsilon/|k|$ works if $k \ne 0$

PROOF: if $k=0$, then $ks_n = 0$ for all n & so $ks_n \to 0 = 0 \cdot s$

if $k \neq 0$, let $\varepsilon > 0$, let $\varepsilon' = \frac{\varepsilon}{|k|}$. (Then $\frac{\varepsilon}{|k|} > 0$)

Since $s_n \to s$, there is $N \in \mathbb{N}$ st $|s_n - s| < \varepsilon' = \frac{\varepsilon}{|k|}$ $\forall n > N$

Then for all $n > N$, $|ks_n - ks| = |k| \cdot |s_n - s| < |k| \cdot \frac{\varepsilon}{|k|} = \varepsilon$

Thus $ks_n \to ks$ $\square$

thm (sum) If (s_n) converges to s & (t_n) converges to t, then $(s_n + t_n)$ converges to $s+t$ ie $\lim(s_n + t_n) = \lim s_n + \lim t_n$

idea: want $|s_n + t_n - (s+t)| < \varepsilon \Leftrightarrow |(s_n - s) + (t_n - t)| \leq |s_n - s| + |t_n - t|$

make each $< \frac{\varepsilon}{2}$

PROOF let $\varepsilon > 0$ let $\varepsilon' = \varepsilon/2$

Since $s_n \to s$, there is N_1 st $|s_n - s| < \varepsilon' = \varepsilon/2$ $\forall n > N_1$

Since $t_n \to t$, there is N_2 st $|t_n - t| < \varepsilon' = \varepsilon/2$ $\forall n > N_2$

let $N = \max\{N_1, N_2\}$, Then for all $n > N$, $|s_n - s| < \frac{\varepsilon}{2}$ and $|t_n - t| < \frac{\varepsilon}{2}$ and

hence $|s_n + t_n - (s+t)| = |(s_n - s) + (t_n - t)| \leq |s_n - s| + |t_n - t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Thus for all $n > N$, $|(s_n + t_n) - (s+t)| < \varepsilon$

Therefore, $s_n + t_n \to s+t$ $\square$

thm (product) If (s_n) converges to s & (t_n) converges to t, then $(s_n \cdot t_n)$ converges to st ie $\lim(s_n \cdot t_n) = (\lim s_n) \cdot (\lim t_n)$

idea: $|s_n t_n - st| = |s_n t_n - s_n t + s_n t - st| \leq |s_n t_n - s_n t| + |s_n t - st| =$

b/c s_n bounded $K > 0$ $= \underbrace{|s_n|}_{\leq K} \underbrace{|t_n - t|}_{< \frac{\varepsilon}{2K}} + \underbrace{|s_n - s|}_{< \frac{\varepsilon}{2(|t|+1)}} \cdot |t| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

PROOF let $\varepsilon > 0$, Since s_n is convergent, it is bounded. Hence $\exists K > 0$ st $|s_n| \le K$ $\forall n$

let $\varepsilon_1 = \frac{\varepsilon}{2K}$ since $t_n \to t$ there is N_1 st $|t_n - t| < \frac{\varepsilon}{2K}$ $\forall n > N_1$

let $\varepsilon_2 = \frac{\varepsilon}{2(|t|+1)}$, since $s_n \to s$, there $\exists N_2$ st. $|s - s_n| < \frac{\varepsilon}{2(|t|+1)}$ $\forall n > N_2$

let $N = \max\{N_1, N_2\}$ Then $\forall n > N$ we have

$|s_n t_n - st| = |s_n t_n - s_n t + s_n t - st| \le |s_n|\,|t_n - t| + |s_n - s| \cdot |t| <$

$K \cdot \frac{\varepsilon}{2K} + \frac{\varepsilon}{2(|t|+1)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\therefore$ $s_n t_n \to st$ $\square$

quotient $\to$ idea $1/s_n \to 1/s$ & then quotient follows

thm if (s_n) converges to s, $s \neq 0$ and $s_n \neq 0$ for all n, then $\left(\frac{1}{s_n}\right)$ conv to $1/s$

idea: $\left|\frac{1}{s_n} - 1/s\right| = \left|\frac{s - s_n}{s_n \cdot s}\right| = \frac{|s - s_n|}{|s_n| \cdot |s|}$ want s_n to stay away from 0

lemma: For a seq (s_n) as in the thm, there is $m > 0$ st $|s_n| \ge m$ for all n

$\Rightarrow$ proof: let $\varepsilon = \frac{|s|}{2}$. Since $s_n \to s$, there is N st for all $n > N$, $|s_n - s| < \frac{|s|}{2}$

and hence $|s_n| \ge \frac{|s|}{2}$ (exercise)

let $m = \min\left\{\frac{|s|}{2}, |s_1|, \ldots, |s_N|\right\}$ then $|s_n| \ge m$ for all n & $m \ge 0$

PROOF By the lemma, there is $m > 0$ st $|s_n| \ge m$ $\forall n$

let $\varepsilon > 0$, let $\varepsilon' = \varepsilon \cdot m \cdot |s|$

Since $s_n \to s$, there is N st $\forall n > N$, $|s_n - s| < \varepsilon' = \varepsilon \cdot m|s|$, & hence

$\left|\frac{1}{s_n} - 1/s\right| = \frac{|s_n - s|}{|s_n| \cdot |s|} < \frac{\varepsilon m |s|}{m|s|} = \varepsilon$

$\therefore$ $\frac{1}{s_n} \to 1/s$ $\square$

thm (quotient) if (s_n) converges to s, $s \neq 0$ & $s_n \neq 0$ $\forall n$, & (t_n) conv to t

then (t_n / s_n) conv to t/s