monday 2.10

Cauchy sequence

(2) How to prove that a sequence $(s_n) \in \mathbb{R}$ converge w/o knowing its limit?

- monotone & bdd $\Rightarrow$ converges
- $\liminf s_n = \limsup s_n \in \mathbb{R} \Leftrightarrow$ converges
- converges $\Leftrightarrow$ Cauchy

idea: if $\lim s_n = s$, then for all suff large n, s_n is close to s, & hence for all suff large m, n, s_m & s_n are close to each other

def) a sequence (s_n) in a metric space (M, d) is a Cauchy sequence if for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ st $d(s_n, s_m) < \varepsilon$ for all $m, n > N$

$[m, n > N \Rightarrow d(s_n, s_m) < \varepsilon]$

note. suffices to consider $m > n$

thm) every convergent seq in a metric space is a Cauchy seq

pf) let (s_n) be a convergent seq in (M, d)

let $s = \lim s_n$. let $\varepsilon > 0$. Since $s_n \to s$, there exists $N \in \mathbb{N}$ st $\forall n > N$, $d(s_n, s) < \frac{\varepsilon}{2}$

Then for all $m, n > N$ we have $d(s_n, s_m) \le d(s_n, s) + d(s, s_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Thus (s_n) is a Cauchy sequence

thm) every Cauchy sequence $\in \mathbb{R}$ converges

Pf) let (s_n) be a Cauchy sequence $\in \mathbb{R}$

let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $|s_n - s_m| < \varepsilon \quad \forall m, n > N$

Then for all $n > N$, $|s_n - s_{N+1}| < \varepsilon$ and so $s_{N+1} - \varepsilon < s_n < s_{N+1} + \varepsilon$

Then for all $j > N$, $u_j = \inf\{s_n : n > N\} \ge s_{N+1} - \varepsilon$ and $v_j = \sup\{s_n : n > N\} \le s_{N+1} + \varepsilon$

$s_{N+1} - \varepsilon \le \liminf s_n \le \limsup s_n \le s_{N+1} + \varepsilon$

Hence, $\liminf s_n$ & $\limsup s_n$ are finite

also, for any $\varepsilon > 0$, $0 \le \limsup s_n - \liminf s_n \le 2\varepsilon$

Hence $\limsup s_n = \liminf s_n$

It follows that (s_n) converges $\square$

Corollary a sequence in $\mathbb{R}$ converges if and only if it is Cauchy

ex $s_n = \frac{\sin 1}{2} + \frac{\sin 2}{2^2} + \dots + \frac{\sin n}{2^n}$

is it Cauchy?

first, for any n, $|s_{n+1} - s_n| = \left| \frac{\sin(n+1)}{2^{n+1}} \right| \leq \frac{1}{2^{n+1}}$

For any $m>n$, $|s_n - s_m| =$

$$|s_n - s_{n+1} + s_{n+1} - s_{n+2} + \dots + s_{m-1} - s_m|$$

$$\leq |s_n - s_{n+1}| + |s_{n+1} - s_{n+2}| + \dots + |s_{m-1} - s_m| \leq \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \dots + \frac{1}{2^m}$$

Recall: $\frac{1}{2} + 1/2^2 + 1/2^3 + \dots = 1$

$1/2^{n+1} + \frac{1}{2^{n+2}} + \dots = 1/2^n$

$\therefore \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \dots + \frac{1}{2^m} \leq 1/2^n$

So for any $m>n$, $|s_n - s_m| \leq 1/2^n$

Let $\varepsilon > 0$. Let N be such that $\frac{1}{2^N} < \varepsilon$

Then for any $m>n>N$,

$|s_n - s_m| \leq \frac{1}{2^n} < 1/2^N < \varepsilon$

$\therefore$ our sequence is Cauchy & hence it converges

$|s_n - s| \leq 1/2^n$ for any n

(!) In a metric space, a Cauchy sequence does not necessarily converge

ex Consider $(0,1)$ w/ standard metric

& let $(s_n) = (1/n)$. (s_n) converges in $\mathbb{R} \Rightarrow$ is Cauchy in $\mathbb{R} \Rightarrow$ is Cauchy on $(0,1)$,

but does not have limit in $(0,1)$

(2) Consider $\mathbb{Q}$ with standard metric

Let (s_n) be a sequence of rational #s that converges to $\sqrt{2} \in \mathbb{R}$