monday 4.28

integration and differentiation

FTC connects integration and differentiation

FTC I if g is continuous on $[a,b]$ and differentiable on (a,b) with g' integrable on $[a,b]$, then $\int_a^b g' = g(b) - g(a)$

FTC II if f is integrable on $[a,b]$, then
$F(x) = \int_a^x f(t)\,dt$ is cont. on $[a,b]$
if f is continuous on $[a,b]$, then
$F(x)$ is differentiable on (a,b) and $F' = f$

note: integration & differentiation are inverse operations in the following sense:

(1) Suppose f is continuous on $[a,b]$. Then,

$$f \xrightarrow{\text{integrate}} F(x) = \int_a^x f(t)\,dt \xrightarrow[\text{FTC II}]{\text{differentiate}} F'(x) = f(x)$$

(2) Suppose g is continuous on $[a,b]$ and differentiable on (a,b) with g' integrable on $[a,b]$. Then,

$$g(x) \xrightarrow{\text{differentiate}} g'(x) \xrightarrow{\text{integrate}} \int_a^x g'(t)\,dt = g(x) - \overbrace{g(a)}^{\text{constant}}$$

note: By FTC II, every continuous f on $[a,b]$ has an antiderivative, specifically, there is a cont. F on $[a,b]$ diff on (a,b) st $F' = f$

Every anti-derivative of f is of the form $F(x) + c$

$\int_a^x f(t)\,dt$ has value 0 at a.

By FTC I, if f is cont on $[a,b]$, then it has an antiderivate F and $\int_a^b f = F(b) - F(a)$

note: Some functions do <u>not</u> have an elementary antiderivative, ie their antiderivatives cannot be written in terms of elementary functions

ex: $f(x) = e^{-x^2}$, $\sin(x^2)$, $\frac{\sin x}{x}$, $1/\ln(x)$, $\sqrt{1-x^4}$, ...

? let f be cont. on $[a,b]$. Suppose we cannot find an antiderivative of f. How can we find $\int_a^b f$?

Taylor Series & integrate term by term

Riemann Sum

We can use Riemann sums to approximate $\int_a^b R$

We know: For each $\varepsilon>0$, there is $\delta>0$ st for any partition P of $[a,b]$ w/ $\text{mesh}(P)<\delta$,

* $\boxed{U(f,P)-L(f,P)<\varepsilon}$ & hence for any Riemann sum S associated w/ P $|S - \int_a^b f| < \varepsilon$

We can divide $[a,b]$ into subintervals of length $\frac{b-a}{n} < \delta$ and take the Riemann sum with $x_k = t_{k-1}$ or $x_k = t_k$ or $x_k = t_k + t_{k-1}/2$ (midpoint)

? let $\varepsilon>0$ be given, say $\varepsilon = \frac{1}{1000}$. What n should we take?

$$U(f,P)-L(f,P) = \sum_{k=1}^{n} (M_k - m_k)(t_k - t_{k-1}) = \frac{(b-a)}{n} \sum_{k=1}^{n} (M_k - m_k) \overset{?}{<} \varepsilon$$

$\Rightarrow$ monotone or differentiable, then how do you know
w/ bounded f'