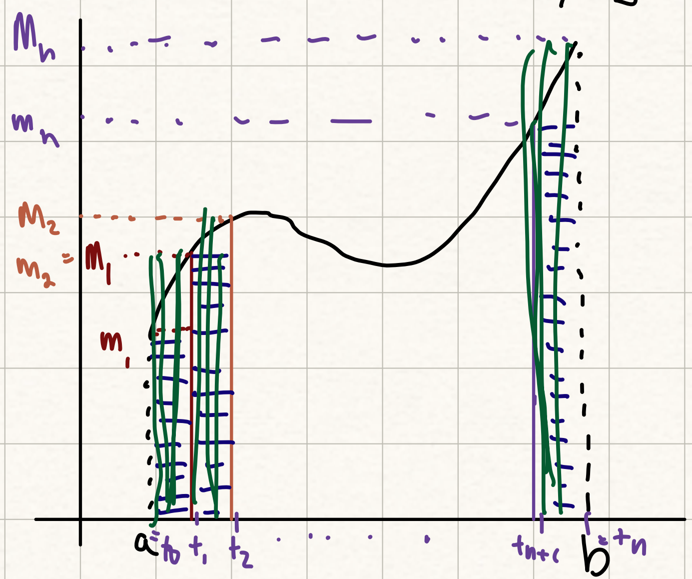
monday 4.21

Darboux Construction of the Integral

f - a bounded function on [a, b]



let $P = \{a = t_0 < t_1 < \dots < t_{n-1} < t_n = b\}$ be a partion of [a, b]

For each $k = 1 \dots n$, let

$$m_k = \inf\{f(x) : x \in [t_{k-1}, t_k]\}$$
$$M_k = \sup\{f(x) : x \in [t_{k-1}, t_k]\}$$

The lower Darboux sum of f with respect to P is

$$\sum_{k=1}^{n} m_k (t_k - t_{k-1})$$

The upper Darboux sum of f with respect to P is

$$\sum_{k=1}^{n} M_k (t_k - t_{k-1})$$

geometric meaning - sum of signed areas of rectangles

The upper Darboux integral of f over [a, b] is

$$U(f) = \inf\{U(f, P) : P \text{ is a partion of } [a, b]\}$$

The lower Darboux integral of f over [a, b] is

$$L(f) = \sup\{L(f, P) : P \text{ is a partion of } [a, b]\}$$

Note for any partition P of $[a,b]$

$$m(b-a) \leq L(f,P) \leq U(f,P) \leq M(b-a)$$

m: inf on $[a,b]$ $\qquad$ M: sup on $[a,b]$

$$L(f,P) = \sum_{k=1}^{n} m_k(t_k - t_{k-1}), \quad U(f,P) = \sum_{k=1}^{n} M_k(t_k - t_{k-1})$$

Claim For any partitions P and Q of $[a,b]$,

$$L(f,P) \leq U(f,Q) \quad (*)$$

and hence $L(f) \leq U(f)$

PROOF (OUTLINE)

- if Q is obtained by adding several points to P, then $L(f,Q) \geq L(f,P)$ and $U(f,Q) \leq U(f,P)$
- Now, for any P and Q
  $L(f,P) \leq L(f, P \cup Q) \leq U(f, P\cup Q) \leq U(f,Q)$ ($*$ holds)
- it follows (by ex. 4.8) in HW1) that
  $\sup_P L(f,P) \leq \inf_Q U(f,Q)$ ie $L(f) \leq U(f)$ $\square$

def If $L(f) = U(f)$, we say that f is (Darboux) integrable on $[a,b]$ and define $\int_a^b f(x)\,dx = \int_a^b f = L(f) = U(f)$

ex of a non-integrable function

f – salt and pepper function on $[a,b]$

let P be a partition of $[a,b]$. Then

$L(f,P) = 0$ and $U(f,P) = b-a$

$L(f) = 0 \qquad U(f) = b-a$

$L(f) \neq U(f)$

Criterion for integrability

a bounded function f on $[a,b]$ is integrable $\iff$ for each $\varepsilon>0$, there exists a partition P of $[a,b]$ st $U(f,P)-L(f,P)<\varepsilon$

PROOF ($\Leftarrow$) let $\varepsilon>0$. For P as $U(f,P)-L(f,P)<\varepsilon$

$U(f)\leq U(f,P)< L(f,P)+\varepsilon\leq L(f)+\varepsilon$

So $U(f)\leq L(f)+\varepsilon$ for any $\varepsilon>0$ and so $U(f)\leq L(f)$

Since $L(f)\leq U(f)$, $L(f)=U(f)$

($\Rightarrow$) Suppose f is integrable, ie $L(f)=U(f)$

Let $\varepsilon>0$. There exists P st $L(f,P)\geq L(f)-\varepsilon/2$

There exists Q st $U(f,Q)\leq U(f)+\varepsilon/2=L(f)+\varepsilon/2$

The partition $P\cup Q$ works (exercise)

Wednesday 4/23

integrable functions

thm/ every continuous function f on $[a,b]$ is integrable

PROOF discussion: let $\varepsilon>0$, want P st $U(f,P)-L(f,P)<\varepsilon$

$U(f,P)-L(f,P)=\sum_{k=1}^{n}(M_k-m_k)(t_k-t_{k-1})$ $\quad M_k-m_k<\frac{\varepsilon}{b-a}$

proof: Let $\varepsilon>0$. Since f is continuous on $[a,b]$, it is uniformly continuous

Hence, there is $\delta>0$ st $x,y\in[a,b]$ and $|x-y|<\delta$ imply $|f(x)-f(y)|<\frac{\varepsilon}{b-a}$ $(*)$

We take a partition $P=\{a=t_0<t_1<\dots<t_n=b\}$ st

$\text{mesh}(P)=\max\{t_k-t_{k-1}: k=1,\dots,n\}<\delta$

Since f is cont., it attains max & min on each $[t_{k-1},t_k]$ so

$m_k=\min_{[t_{k-1},t_k]} f$ and $M_k=\max_{[t_{k-1},t_k]} f$

By $(*)$, $M_k-m_k<\frac{\varepsilon}{b-a}$

Then, $U(f,P)-L(f,P)=\sum_{k=1}^{n}(M_k-m_k)(t_k-t_{k-1})\leq\sum_{k=1}^{n}\frac{\varepsilon}{b-a}|t_k-t_{k-1}|=$

$=\frac{\varepsilon}{b-a}\sum_{k=1}^{n}(t_k-t_{k-1})=\frac{\varepsilon}{b-a}(b-a)=\varepsilon$

Thus f satisfies the criterion for integrability & hence is integrable □

note: We showed that for *any* P w/ $\text{mesh}(P)<\delta$, $U(f,P)-L(f,P)<\varepsilon$

thm Every monotone function f on $[a,b]$ is integrable

PROOF for increasing - p. 280-281

note: a monotone f can have infinitely many discontinuities in $[a,b]$

Riemann Sums

Let f be bounded on $[a,b]$

Let $P=\{a=t_0<t_1<\ldots<t_n=b\}$ be a partition of $[a,b]$

a **Riemann sum** associated w/ P is a sum of the form

$$S=\sum_{k=1}^{n} f(x_k)(t_k-t_{k-1}) \text{ where } x_k\in[t_{k-1},t_k]$$

Since $m_k\le f(x_k)\le M_k$ $\quad L(f,P)\le S\le U(f,P)$

def We say that f is **Riemann integrable** on $[a,b]$ if there exists a number R st for each $\varepsilon>0$ there is $\delta>0$ st $|S-R|<\varepsilon$ for every Riemann sum associated w/ a partition P with $\text{mesh}(P)<\delta$

The number R is the **Riemann integral** of f over $[a,b]$

fact f is Riemann integrable $\iff$ it is Darboux integrable in which case the values of the integrals coincide (pf on 275-278)

Special Case: Suppose that f is continuous on $[a,b]$. Let $\varepsilon>0$. Then there is $\delta>0$ st for any partition P with $\text{mesh}(P)<\delta$, $U(f,P)-L(f,P)<\varepsilon$.

For any Riemann sum associated w/ such P,

$$L(f,P)\le S\le U(f,P) \text{ and also } L(f,P)\le \int_a^b f\le U(f,P)$$

Then, $|S-\int_a^b f|<\varepsilon$. So f is Riemann integrable and $\int_a^b f=R$

thm | Properties of the integral

Suppose that f and g are integrable on $[a,b]$. then

(i) cf and $f+g$ are also integrable, and

$$\int_a^b (cf) = c\int_a^b f \quad \text{and} \quad \int_a^b (f+g) = \int_a^b f + \int_a^b g$$

(ii) if $f(x) \le g(x)$ for all $x \in [a,b]$, then $\int_a^b f \le \int_a^b g$

(iii) $|f|$ is integrable, and $\left|\int_a^b f\right| \le \int_a^b |f| \le \int_a^b \sup_{[a,b]} |f| = (b-a) \sup_{[a,b]} f$

PROOF (outline)

(i) for the Riemann sums, we have $\mathcal{S}(cf) = c\,\mathcal{S}(f)$ and

$\mathcal{S}(f+g) = \mathcal{S}(f) + \mathcal{S}(g)$

(exercise)

(ii) if $f \le g$ on $[a,b]$, then for any P, $U(f,P) \le U(g,P)$

it follows that $\int_a^b f = U(f) \le U(g) = \int_a^b g$

(iii) for any P, $U(|f|, P) - L(|f|, P) \le U(f,P) - L(f,P)$

So if f satisfies the Criterion for integrabitily, so does $|f|$

$-|f| \le f \le |f| \Rightarrow -\int_a^b |f| \le \int_a^b f \le \int_a^b |f| \Rightarrow \left|\int_a^b f\right| \le \int_a^b |f|$

friday 4.25

The Fundemental Theorem of Calculus

thm | FTC part I

Suppose a function g is continuous on $[a,b]$, differentiable on (a,b), and g' is integrable on $[a,b]$.

Then $\int_a^b g' = g(b) - g(a)$ *

note: g' is integrable on $[a,b]$ means that its extension to $[a,b]$ is integrable. It does not matter how we define values at a and at b. Other, g' is defineed on $[a,b]$

applications: the integral of the rate of change = total change

* **PROOF** We will show that for any $\varepsilon>0$,

$\left|\int_a^b g' - (g(b)-g(a))\right| < \varepsilon$ (*)

it follows that $\int_a^b g' = g(b)-g(a)$

Let $\varepsilon>0$. Since g' is integrable, there is a partition $P=\{a=t_0<t_1<\cdots<t_n=b\}$ of $[a,b]$

st $U(g',P)-L(g',P)<\varepsilon$ ①

For each $k=1,\dots,n$ we apply the Mean Value Thm to g on $[t_{k-1},t_k]$

and obtain $x_k\in[t_{k-1},t_k]$ such that $g(t_k)-g(t_{k-1}) = g'(x_k)(t_k-t_{k-1})$

Then $g(b)-g(a) = \sum_{k=1}^n (g(t_k)-g(t_{k-1})) = \sum_{k=1}^n g'(x_k)(t_k-t_{k-1})$

This is a Riemann sum for g' associated with P

and so it is between $L(g',P)$ and $U(g',P)$

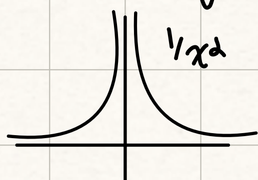
So, $L(g',P)\le g(b)-g(a)\le U(g',P)$ ②, Also, $L(g',P)\le \int_a^b g' \le U(g',P)$ ③

1,2,3 imply $\left|\int_a^b g' - (g(b)-g(a))\right|<\varepsilon$ * ◻

ex (1) $\int_1^2 x^2\,dx = \frac{x^3}{3}\Big|_1^2 = \frac{8}{3} - 1/3 = 7/3$

Here $g = x^3/3$, $g'(x)=x^2$ is cont. $\Rightarrow$ integrable on $[1,2]$

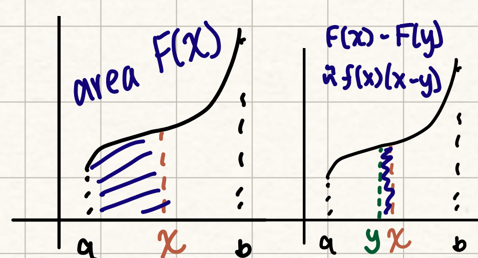
(2) $\int_{-1}^1 \frac{1}{x^2}\,dx \cancel{=} \cancel{-\frac{1}{x}\Big|_{-1}^1 =} -2$



FTC II Discussion

Let f be a non-neg cont. func on $[a,b]$

For $x\in[a,b]$, let $F(x)=\int_a^x f(t)\,dt$



? is F continuous? Yes (even if f is just integrable)

? is F differentiable on (a,b)?

$\frac{F(x)-F(y)}{x-y} \approx f(x)$ for y close to x } yes, and $F'=f$

thm FTC part 2

Let f be an integrable function on $[a,b]$. For $x \in [a,b]$, let $F(x) = \int_a^x f(t)\,dt$

Then, F is continuous on $[a,b]$ (i)

If f is continuous on $[a,b]$, then

F is differentiable on (a,b) and $F' = f$ (ii)

PROOF (i) Since f is integrable, it is bounded on $[a,b]$

So there is $M>0$ st $|f(x)| \le M$ for all $x \in [a,b]$

Then for any $x<y$ in $[a,b]$, we have

$$|F(y)-F(x)| = \left|\int_x^y f(t)\,dt\right| \le$$

$$\le \int_x^y |f(t)|\,dt \le \int_x^y M\,dt = M(y-x)$$

Let $\varepsilon>0$. Then for any $x<y$ in $[a,b]$ st $|x-y|<\frac{\varepsilon}{M}$, $|F(x)-F(y)|<\varepsilon$

So F is (uniformly) cont. on $[a,b]$

(ii) Suppose f is cont. on $[a,b]$

Convention: for $x>y$ $\int_x^y f = -\int_y^x f$

Let $x_0 \in (a,b)$. want: $F'(x_0) = \lim_{x\to x_0} \frac{F(x)-F(x_0)}{x-x_0} = f(x_0)$

$$\left|\frac{F(x)-F(x_0)}{x-x_0} - f(x_0)\right| = \left|\frac{1}{x-x_0}\int_x^{x_0} f(t)\,dt - \frac{1}{x-x_0}\int_x^{x_0} f(x_0)\,dt\right|$$

$$\left|\left(\frac{1}{x-x_0}\right)\int_x^{x_0}(f(t)-f(x_0))\,dt\right|$$

$$\le \frac{1}{\cancel{|x-x_0|}}\cdot\cancel{|x-x_0|}\cdot \sup\{|f(t)-f(x_0)| : t \text{ is between } x \text{ \& } x_0\}$$

as $x \to x_0$, the $\sup \to 0$ since f is cont at x_0

$\therefore F'(x_0)$ exists and equals $f(x_0)$ $\square$