2.2 Systems of Equations

Corollary 2.5 (of rank-nullity) Given a system of n linear equations in m variables where the constant term is 0, there is always a nontrivial solution if $m>n$. Moreover, the dimension of the solution space of at least $m-n$

def 2.16 Let $\alpha: V \to W$ be a linear map. Let $v_1, \dots, v_n$ be a basis for V and let $w_1, \dots, w_m$ be a basis for W. Then, for each i, $\alpha(v_i)$ can be written uniquely as a linear combination of the w_j, $\alpha(v_i) = \sum_j \lambda_{ij} w_j$. We define $M(\alpha)$ to be the mxn matrix with entry in row i and column j given by λ_{ij}

remark 2.17 note that this depends on our choice of basis for V & for W

def 2.19 Given matrices A & B of appropriate dimensions, the product AB has entry in row i & column j given by

$$\sum_r A_{j,r} B_{r,i}$$

2.4 Invertible Linear Map

def 2.20 For a linear map $\alpha: V \to W$, a linear map $\beta: W \to V$ is an inverse to α if, for every $v \in V$, $\beta(\alpha(v)) = v$ and for every $w \in W$, $\alpha(\beta(w)) = w$

$\alpha \circ \beta = Id_W$ & $\beta \circ \alpha = Id_V$

If a map α has an inverse, we say it is invertible

def 2.21 a linear map $\alpha: V \to W$ is invertible if it has an inverse

lemma 2.22 if β_1 and β_2 are inverses for α, then $\beta_1 = \beta_2$

lemma 2.23 a linear map $\alpha: V \to W$ is invertible iff it is injective & surjective

2.5 Isomorphisms of Vector Spaces

def 2.24 two vector spaces V and W are isomorphic if there is an invertible linear map $\alpha: V \to W$

2.6 Duality and the space of Linear Maps

def 2.26 Let V, W be $\mathbb{F}$-vector spaces and let $\lambda \in \mathbb{F}$. Let $\alpha: V \to W$ be a linear map. Then, $\lambda \cdot (\alpha)$ is the linear map such that, for each $v \in V$, $(\lambda \cdot \alpha)(v) = \lambda \cdot \alpha(v)$

def 2.27 | Let V, W be $\mathbb{F}$-vector spaces. Then $L(V,W)$ is the set of all linear maps from V to W. The zero is the map that sends everything in V to 0_W. Addition & scalar multiplication are those previously defined

def 2.28 | for a given vector space V, $L(V,V)$ is the space of endomorphisms from V to itself

def 2.29 | for a given $\mathbb{F}$ vector space V, $L(V,\mathbb{F})$ is called the dual space, and often denoted V^*. Linear maps from V to $\mathbb{F}$ are sometimes called linear functionals

ex 2.30 | the vector space $L(\mathbb{R}^3, \mathbb{R})$ has a basis given by linear maps $\alpha_1, \alpha_2, \alpha_3$ where

$$\alpha_1\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \qquad \alpha_2\begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \qquad \alpha_3\begin{pmatrix} x \\ y \\ z \end{pmatrix} = z$$

the basis we pick for V^* is similar to our basis for V when V is finite dimensional

thm 2.31 | for a finite dimensional vector space V over a field $\mathbb{F}$, the dual space V^* is isomorphic to V

def 2.32 | given a basis $\{v_1, \dots, v_n\}$ for a vector space V, the dual basis $\{v_1^*, \dots, v_n^*\}$ is the set of linear functionals where $v_i^*(v_j) = 1$ if $i=j$ and 0 otherwise