exam 2

DEFINITIONS

Let f be a function from a set $S \subseteq \mathbb{R}$ to $\mathbb{R}$. We say that f is uniformly continuous on S if for any $\varepsilon > 0$, there exists $\delta > 0$ such that $x, y \in S$ and $|x-y| < \delta$ imply $|f(x) - f(y)| < \varepsilon$

Let $a \in \mathbb{R}$ and suppose that a function f is defined on $J \setminus \{a\}$ for some open interval J containing a
We say that a real number L is the limit of $f(x)$ as x tends to a and write $\lim_{x \to a} f(x) = L$ if...

(i) for every sequence (x_n) in $J \setminus \{a\}$ converging to a we have $\lim_{n \to \infty} f(x_n) = L$

... equivalently...

(ii) for each $\varepsilon > 0$, there exists $\delta > 0$ such that $x \in J \setminus \{a\}$ and $|x - a| < \delta$ imply $|f(x) - L| < \varepsilon$

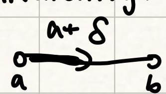
... equivalently...

(iii) for each $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - L| < \varepsilon$

Suppose that a function is defined on an interval (a, b). We write $\lim_{x \to a^+} f(x) = L$ if

(i) for every seq. (x_n) in (a, b) converging to a we have $\lim_{n \to \infty} f(x_n) = L$ ... equivalently...

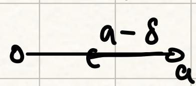
(ii) for each $\varepsilon > 0$ there exists $\delta > 0$ st $a < x < a + \delta$ implies $|f(x) - L| < \varepsilon$



Suppose that a function is defined on an interval (c, a). We write $\lim_{x \to a^-} f(x) = L$ if

(i) for every seq. (x_n) in (c, a) converging to a we have $\lim_{n \to \infty} f(x_n) = L$ ... equivalently...

(ii) for each $\varepsilon > 0$ there exists $\delta > 0$ st $a - \delta < x < a$ implies $|f(x) - L| < \varepsilon$



Suppose that f is defined on an open interval containing a. We say that f is differentiable at a, or has a derivative at a, if $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists and is finite. We call this finite limit the derivative of f at a and denote it $f'(a)$

Let f be a function on an interval I.

f is increasing on I if for any $x_1 < x_2$ in I, $f(x_1) \le f(x_2)$

f is decreasing on I if for any $x_1 < x_2$ in I, $f(x_1) \ge f(x_2)$

f is strictly increasing on I if for any $x_1 < x_2$ in I, $f(x_1) < f(x_2)$

f is strictly decreasing on I if for any $x_1 < x_2$ in I, $f(x_1) > f(x_2)$

Suppose f is differentiable at a. The linear approximation of f at $x = a$ is $f(x) = f(a) + f'(a)(x-a)$

Suppose f is n times differentiable on I.

$T_n(x) = \sum_{k=0}^{n} \frac{f^k(a)}{k!}(x-a)^k$ is the nth Taylor Polynomial of f at a

Let $P = \{a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b\}$ be a partition of $[a,b]$

For each $k = 1 \ldots n$, let

$m_k = \inf\{f(x) : x \in [t_{k-1}, t_k]\}$

$M_k = \sup\{f(x) ; x \in [t_{k-1}, t_k]\}$

The lower Darboux sum of f with respect to P is $\sum_{k=1}^{n} m_k(t_k - t_{k-1})$

The upper Darboux sum of f with respect to P is $\sum_{k=1}^{n} M_k(t_k - t_{k-1})$

The upper Darboux integral of f over $[a,b]$ is $U(f) = \inf\{U(f,P) \mid P \text{ is a partition of } [a,b]\}$

The lower Darboux integral of f over $[a,b]$ is $L(f) = \sup\{L(f,P) \mid P \text{ is a partition of } [a,b]\}$

If $L(f) = U(f)$, we say that f is (Darboux) integrable on $[a,b]$ and define $\int_a^b f(x)\,dx = \int_a^b f = L(f) = U(f)$

Let f be bounded on $[a,b]$. Let $P = \{a = t_0 < t_1 < \ldots < t_n = b\}$ be a partition of $[a,b]$

a Riemann sum associated with P is a sum of the form

$S = \sum_{k=1}^{n} f(x_k)(t_k - t_{k-1})$ where $x_k \in [t_{k-1}, t_k]$

THEOREMS AND COROLLARIES

if f is continuous on a closed interval $[a,b]$, then f is uniformly continuous on $[a,b]$

Suppose that $a, L \in \mathbb{R}$ and a function f is defined on $J \setminus \{a\}$ for some open interval J containing a. Then $\lim_{x \to a} f(x) = L$ iff $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$

if at least one of the one sided limits does not exist, or they are not equal, then $\lim_{x \to a} f(x)$ does not exist

Suppose that $\lim_{x \to a} f(x) = L_1 \in \mathbb{R}$ and $\lim_{x \to a} g(x) = L_2 \in \mathbb{R}$. Then...

(i) $\lim_{x \to a} (f+g)(x) = L_1 + L_2$

(ii) $\lim_{x \to a} (fg)(x) = L_1 L_2$

(iii) $\lim_{x \to a} (f/g)(x) = L_1 / L_2$ provided $L_2 \neq 0$

if f is differentiable at a point a, then f is continuous at a

Let f and g be functions that are differentiable at the point a. Each of the functions cf (c is constant), $f+g$, fg, and f/g is also differentiable at a, except f/g if $g(a)=0$ since f/g is not defined at a. The formulas are..

(i) $(cf)'(a) = c \cdot f'(a)$

(ii) $(f+g)'(a) = f'(a) + g'(a)$

(iii) $(fg)'(a) = f(a)g'(a) + f'(a)g(a)$

(iv) $(f/g)'(a) = \dfrac{g(a)f'(a) - f(a)g'(a)}{g^2(a)}$ if $g(a) \neq 0$

if f is differentiable at a and g is differentiable at $f(a)$, then the composite function $g \circ f$ is differentiable at a and we have:

$$(g \circ f)'(a) = g'(f(a))\, f'(a)$$

if f is defined on an open interval containing x_0, if f assumes its maximum or minimum at x_0, and if f is differentiable at x_0, then $f'(x_0) = 0$

Rolles Theorem

Let f be a continuous function on $[a,b]$ that is differentiable on $[a,b]$ and satisfies $f(a) = f(b)$. There exists (at least one) x in (a,b) such that $f'(x) = 0$

Mean Value Theorem

Let f be a continuous function on $[a,b]$ that is differentiable on (a,b). Then there exists (at least one) x in (a,b) such that $f'(x) = \frac{f(b) - f(a)}{b-a}$

Let f be a differentiable function on (a,b) such that $f'(x) = 0$ for all $x \in (a,b)$
Then f is a constant function on (a,b)

Let f and g be differentiable functions on (a,b) such that $f' = g'$ on (a,b)
Then there exists a constant c such that $f(x) = g(x) + c$ for all $x \in (a,b)$

Let f be a differentiable function on an interval (a,b). Then,

(i) f is strictly increasing if $f'(x) > 0$ for all $x \in (a,b)$;

(ii) f is strictly decreasing if $f'(x) < 0$ for all $x \in (a,b)$

(iii) f is increasing if $f'(x) \geq 0$ for all $x \in (a,b)$

(iv) f is decreasing if $f'(x) \leq 0$ for all $x \in (a,b)$

Let f be a one to one continuous function on an open interval I, and let $J = f(I)$
if f is differentiable at $x_0 \in I$ and if $f'(x_0) \neq 0$, then f^{-1} is differentiable at $y_0 = f(x_0)$ and $(f^{-1})'(y_0) = 1/f'(x_0)$

if f is a bounded function on $[a,b]$, and if P and Q are partitions of $[a,b]$, then $L(f,P) \leq U(f,Q)$

if f is a bounded function on $[a,b]$, then $L(f) \leq U(f)$

a bounded function f on $[a,b]$ is integrable if and only if for each $\varepsilon > 0$ there exists a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$

every monotonic function f on $[a,b]$ is integrable

every continuous function f on $[a,b]$ is integrable

Let f and g be integrable functions on $[a,b]$, and let c be a real #. Then,

(i) cf is integrable and $\int_a^b cf = c\int_a^b f$

(ii) $f+g$ is integrable and $\int_a^b (f+g) = \int_a^b f + \int_a^b g$

if f is integrable on $[a,b]$, then $|f|$ is integrable on $[a,b]$, and

$$\left|\int_a^b f\right| \leq \int_a^b |f|$$

if g is a continuous function on $[a,b]$ that is differentiable on (a,b) and if g' is integrable on $[a,b]$ then, $\int_a^b g' = g(b) - g(a)$

Let f be an integrable function on $[a,b]$. For x in $[a,b]$, let

$$F(x) = \int_a^x f(t)\,dt$$

Then F is continuous on $[a,b]$. If f is continuous at x_0 in (a,b), then F is differentiable at x_0 and $F'(x_0) = f(x_0)$

PROOFS

if f is differentiable at a point a, then f is continuous at a

We will show that $\lim_{x\to a} f(x) = f(a)$, which means that f is continuous at a

$$f(x) = (x-a)\cdot \frac{f(x)-f(a)}{x-a} + f(a)$$

Since f is differentiable at a, $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a) \in \mathbb{R}$

As $x\to a$, $x-a\to 0$, and by the thm on limits of functions,

$\lim_{x\to a} f(x) = 0\cdot f'(a) + f(a) = f(a)$ $\square$

Let f and g be functions that are differentiable at the point a. Each of the functions cf (c is constant), $f+g$, fg, and f/g is also differentiable at a, except f/g if $g(a)=0$ since f/g is not defined at a. The formulas are..

(i) $(cf)'(a) = c\cdot f'(a)$

(ii) $(f+g)'(a) = f'(a) + g'(a)$

(iii) $(fg)'(a) = f(a)g'(a) + f'(a)g(a)$ *

(iv) $(f/g)'(a) = \dfrac{g(a)f'(a) - f(a)g'(a)}{g^2(a)}$ if $g(a)\neq 0$

iii) Since f and g are differentiable at a, they are defined on an open interval $J \ni a$

For all $x \in J\setminus\{a\}$

$$\frac{(fg)(x)-(fg)(a)}{x-a} = \frac{f(x)g(x) - f(a)g(a) + f(a)g(x) - f(a)g(x)}{x-a}$$

$$= \frac{f(x)-f(a)}{x-a}\, g(x) + \frac{g(x)-g(a)}{x-a}\, f(a) \quad (*)$$

Since g is differentiable at a, it is continuous at a, and so $\lim_{x\to a} g(x) = g(a)$.

Taking $\lim_{x\to a}$ of $(*)$ we obtain (iii)

Let f be a differentiable function on an interval (a,b). Then,

(i) f is strictly increasing if $f'(x)>0$ for all $x\in(a,b)$;

(ii) f is strictly decreasing if $f'(x)<0$ for all $x\in(a,b)$

(iii) f is increasing if $f'(x)\geq 0$ for all $x\in(a,b)$

(iv) f is decreasing if $f'(x)\leq 0$ for all $x\in(a,b)$

i) Consider x_1, x_2 where $a<x_1<x_2<b$. By the Mean Value Theorem, for some $x\in(x_1,x_2)$ we have

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(x)>0$$

Since $x_2-x_1\geq 0$, we see $f(x_2)>f(x_1)$ $\therefore$ $f(x_2)-f(x_1)>0$.

The remaining cases are left □

every continuous function f on $[a,b]$ is integrable

DISCUSSION: Let $\varepsilon>0$; want P such that $U(f,P)-L(f,P)<\varepsilon$

$$U(f,P)-L(f,P)=\sum_{k=1}^{n}(M_k-m_k)(t_k-t_{k-1}) \quad \therefore \text{ want } \quad M_k-m_k<\frac{\varepsilon}{b-a}$$

PROOF: Let $\varepsilon>0$. Since f is continuous on $[a,b]$, it is uniformly continuous

Hence, there is $\delta>0$ such that $x,y\in[a,b]$ and $|x-y|<\delta$ imply $|f(x)-f(y)|<\frac{\varepsilon}{b-a}$ (*)

We take a partition $P=\{a=t_0<t_1<\ldots<t_n=b\}$ such that $\text{mesh}(P)=\max\{t_k-t_{k-1} \mid k=1\ldots n\}<\delta$

Since f is continuous, it attains max and min on each $[t_{k-1},t_k]$ so

$$m_k=\min_{[t_{k-1},t_k]} f \quad \text{and} \quad M_k=\max_{[t_{k-1},t_k]} f$$

By (*), $M_k-m_k<\frac{\varepsilon}{b-a}$

Then, $U(f,P)-L(f,P)=\sum_{k=1}^{n}(M_k-m_k)(t_k-t_{k-1})\leq\sum_{k=1}^{n}\frac{\varepsilon}{b-a}|t_k-t_{k-1}|=$

$\frac{\varepsilon}{b-a}\sum_{k=1}^{n}(t_k-t_{k-1})=\frac{\varepsilon}{b-a}(b-a)=\varepsilon$

Thus f satisfies the Criterion for integrability & hence is integrable □

if g is a continuous function on $[a,b]$ that is differentiable on (a,b) and if g' is integrable on $[a,b]$ then, $\int_a^b g' = g(b) - g(a)$

We will show that for any $\varepsilon > 0$,

$\left| \int_a^b g' - (g(b) - g(a)) \right| < \varepsilon$ (*)

It follows that $\int_a^b g' = g(b) - g(a)$

Let $\varepsilon > 0$. Since g' is integrable, there is a partition $P = \{a = t_0 < t_1 < \dots < t_n = b\}$ such that $U(g', P) - L(g', P) < \varepsilon$ ①

for each $k = 1, \dots, n$ we apply the mean value theorem to g on $[t_{k-1}, t_k]$ and obtain $x_k \in [t_{k-1}, t_k]$ such that $g(t_k) - g(t_{k-1}) = g'(x_k)(t_k - t_{k-1})$

Then, $g(b) - g(a) = \sum_{k=1}^{n} (g(t_k) - g(t_{k-1})) = \sum_{k=1}^{n} g'(t_k)(t_k - t_{k-1})$

This is a Riemann sum for g' associated with P

and so it is between $L(g', P)$ and $U(g', P)$

So, $L(g', P) \le g(b) - g(a) \le U(g', P)$ (2), Also $L(g', P) \le \int_a^b g' < U(g', P)$ (3)

(1) (2) and (3) imply $\left| \int_a^b g' - (g(b) - g(a)) \right| < \varepsilon$ (*) □

TRUE OR FALSE

(1) Any continuous function on a bounded interval is uniformly continuous **False**

(2) If $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ exist, then $\lim_{x \to a} f(x)$ exists **False**

$f(x) = \begin{cases} -1 & x \ge 0 \\ 1 & x > 0 \end{cases}$ $\therefore$ $\lim_{x \to 0^+} f(x) = 1$ $\lim_{x \to 0^-} f(x) = -1$ $\lim_{x \to 0} f(x)$ DNE □

(3) if f is defined on $(a-1, a+1)$ and continuous at a, then $\lim_{x \to a} f(x)$ exists **TRUE**

(4) If f is continuous and bounded on (a,b), then $\lim_{x \to a^+} f(x)$ exists **FALSE**

ex $f(x) = \sin(1/x)$ and $a = 0$ $b = 1$

(5) if f is continuous on (a,b), then f is differentiable on (a,b) **FALSE**

$f(x) = |x|$ and is cont on $(-1, 1)$ but not diff at $x = 0$

(6) If f and $f+g$ are diff at a, then g is diff at a **TRUE**

(7) If f and $f+g$ are differentiable at a and $g(a) \neq a$, then f/g is diff at a TRUE

(8) if f is diff at a and g is diff at $f(a)$, then $g \circ f$ is diff at a TRUE

(9) if f is diff on (a,b) and $f'(x_0)=0$ for some x_0 in (a,b), then f has a local max/min at x_0 FALSE

$f(x)=x^3$ has $f'(x_0)=0$ when $x_0=0$, but is not max/min

(10) if f is cont on $[a,b]$ and diff on (a,b), then there is $x_0 \in (a,b)$ st $f(b)-f(a)=f'(x_0)(b-a)$ TRUE (MVT)

(11) if f is diff & strictly dec. on (a,b), then $f'(x)<0$ for all $x \in (a,b)$ FALSE

ex $f(x)=-x^3$ is strictly dec but $f'(0)=0$

(12) if $f'(x)=g'(x)$ for all x in an open interval I, then $f(x)=g(x)$ for all $x \in I$ FALSE

let $g(x)=x$ and $f(x)=10+x$. $f'(x)=g'(x)$ for all x in $(-10,10)$ but $f(x) \neq g(x)$

(13) Any continuous function from $\mathbb{R}$ onto $\mathbb{R}$ has an inverse False

$f(x)=x^2$ but no inverse b/c not one to one

(14) if f is diff on an open interval I and $f'(x)>0$ for all $x \in I$, then the inverse func f^{-1} is diff on $J=f(I)$ TRUE ?

(15) if f is differentiable at a, then $f(x) \approx f(a)+f'(a)(x-a)$ for all x close to a TRUE linear approx

(16) if f is 5 times differentiable on $(-1,1)$ & $T_5(x)$ is the 5^{th} Taylor polynomial of f at 0, then $T_5''(0)=f''(0)$ TRUE

(17) for any bounded function f on $[a,b]$ & any partition P of $[a,b]$, $L(f,P)<U(f,P)$ FALSE

ex $f(x)=c$ where c is constant on $[a,b]$ then for any partition

$L(f,P)=U(f,P)=c(b-a)$

(18) for any bounded func f on $[a,b]$, any partition P of $[a,b]$, & any Riemann sum S associated w/ P, $L(f,P) \leq S \leq U(f,P)$ TRUE

(19) for any integrable function f on $[a,b]$ & any partition P of $[a,b]$

$L(f,P) \leq \int_a^b f \leq U(f,P)$ TRUE

(20) every bounded func f on $[a,b]$ is integrable FALSE salt & pepper func

(21) if f is cont on $[2,5]$, then $g(x)$ is integrable on $[2,5]$ TRUE

(22) if f is integrable on $[a,b]$ and $m \leq f(x) \leq M$ on $[a,b]$, then $m \leq \int_a^b f \leq M$

FALSE cx $f(x)=1$ on $[0,10]$, $f(x) \leq 1 \; \forall x$ but $\int_0^{10} dx = 10 > 1$

(23) if f and g are increasing on $[a,b]$, then $f-g$ is integrable on $[a,b]$ TRUE

(24) if f is differentiable on $(0,3)$, f' is cont on $(0,3)$, and $f(1)=f(2)$, then $\int_1^2 f' = 0$

TRUE, thm of calc

(25) if f is cont on $[a,b]$ then there exists a func F on $[a,b]$ st $F'(x)=f(x)$ for all $x \in (a,b)$ TRUE