GLV with explicit autotoxin dynamics

February 12, 2025

1 Notes on the models

The model version 1 with explicit dynamics for species-specific autotoxicity reads

$$\frac{dn_i}{dt} = n_i\Big(1 - \rho a_i - \sum_j C_{ij} n_j\Big), \tag{1}$$

$$\frac{da_i}{dt} = \beta n_i - \delta a_i, \tag{2}$$

with $C_{ii} = 1\ \forall i$, and $\langle C_{ij} \rangle = \mu/S$ and $\langle C_{ij} C_{kl} \rangle = \delta_{ik}\delta_{jl}\sigma^2/S$. Note that we could also consider the setting in which $C_{ii} = 0$ and the only contribution to the diagonal is given by the effect of the autotoxicity with no major consequences for the following after adjusting parameters accordingly also in the GLV case.

The GLV model to use for comparison should be

$$\frac{dn_i}{dt} = n_i\Big(1 - \frac{\rho\beta}{\delta} n_i - \sum_j C_{ij} n_j\Big), \tag{3}$$

with the same statistics as before for the C_{ij} and where, again, we can choose $C_{ii} = 0$ if we don't want the extra contribution of 1 on the diagonal. This choice of the model guarantees that in the case of fast dynamics of the autotoxicity with respect to the dynamics of the populations the two models coincide, as they should.

The equilibrium values n_i^* are formally the same in the two models. Indeed solving Eq. (2) at stationarity gives

$$a_i^* = \frac{\beta}{\delta} n_i^*, \tag{4}$$

and therefore we have for both models

$$n_i^* = \frac{1 - \sum_{j \neq i} C_{ij} n_j^*}{1 + \rho\beta/\delta}. \tag{5}$$

We can find the solution without integrating the dynamics, for example to check for feasibility, in both cases by solving the system

$$\sum_j \tilde{C}_{ij} n_j^* = 1, \tag{6}$$

with $\tilde{C}_{ij} = C_{ij}$ for $j \neq i$ and $\tilde{C}_{ii} = C_{ii} + \rho\beta/\delta$.

However, the Jacobian matrices, J, are different, leading to different stability properties of the two models. Here by J we mean the Jacobian evaluated at equilibrium. For the GLV model we have

$$J_{ij} = -C_{ij} n_i^*, \tag{7}$$

for $j \neq i$, and

$$J_{ii} = -(1 + \frac{\rho\beta}{\delta}) n_i^*. \tag{8}$$

For the model with explicit autotoxicity we have the following structure for the Jacobian

$$J = \begin{bmatrix} J^{nn} & J^{na} \\ J^{an} & J^{aa} \end{bmatrix}, \tag{9}$$

with hopefully intuitive notation. In the four blocks we have on the top left J^{nn}

$$J_{ij}^{nn} = -C_{ij} n_i^*, \tag{10}$$

$$J_{ii}^{nn} = -n_i^*, \tag{11}$$

in the top right J^{na}

$$J_{ij}^{na} = 0, \tag{12}$$

$$J_{ii}^{na} = -\rho n_i^*, \tag{13}$$

in the bottom left J^{an}

$$J_{ij}^{an} = 0, \tag{14}$$

$$J_{ii}^{an} = \beta, \tag{15}$$

and in the bottom right J^{aa}

$$J_{ij}^{aa} = 0, \tag{16}$$

$$J_{ii}^{aa} = -\delta. \tag{17}$$

Figure 1 represent a sketch of the Jacobian matrix for the model with autotoxicity with explicit dynamics.

The key insight is that in the Jacobian for the equivalent GLV model, the parameters enters only in the combination

$$\frac{\rho\beta}{\delta} \tag{18}$$

while in the Jacobian for the model with autotoxicity with explicit dynamics they appear separately in different parts of the block structure of the Jacobian. The stability properties can be different. In Fig. 2 we show the spectrum of the eigenvalues of the Jacobians of the two model compared. If $\rho = \delta/\beta$, in such a way that the GLV model has always the same behaviour, we see that changing β does not change the stability properties of the system while changing δ has a string effect. For low values (to be quantified) of δ, the model has worst stability properties with respect to GLV, with the largest eigenvalue closer to 0. For relatively higher values the model become, better, regarding stability properties Finally, for large values of δ, i.e. fast autotoxin dynamics, the system tend to coincide.

Here we focused on feasible and stable parts of the parameter space in order to avoid solving explicitly the dynamics to find the equilibrium values, but the consideration should be extendable closer to the instability threshold.

It is most probably possible to do analytical progress on the estimation of the position of the largest eigenvalue for the model with explicit autotoxin dynamics, thanks to the particular structure of the Jacobian and using the analytical insight for the GLV part.

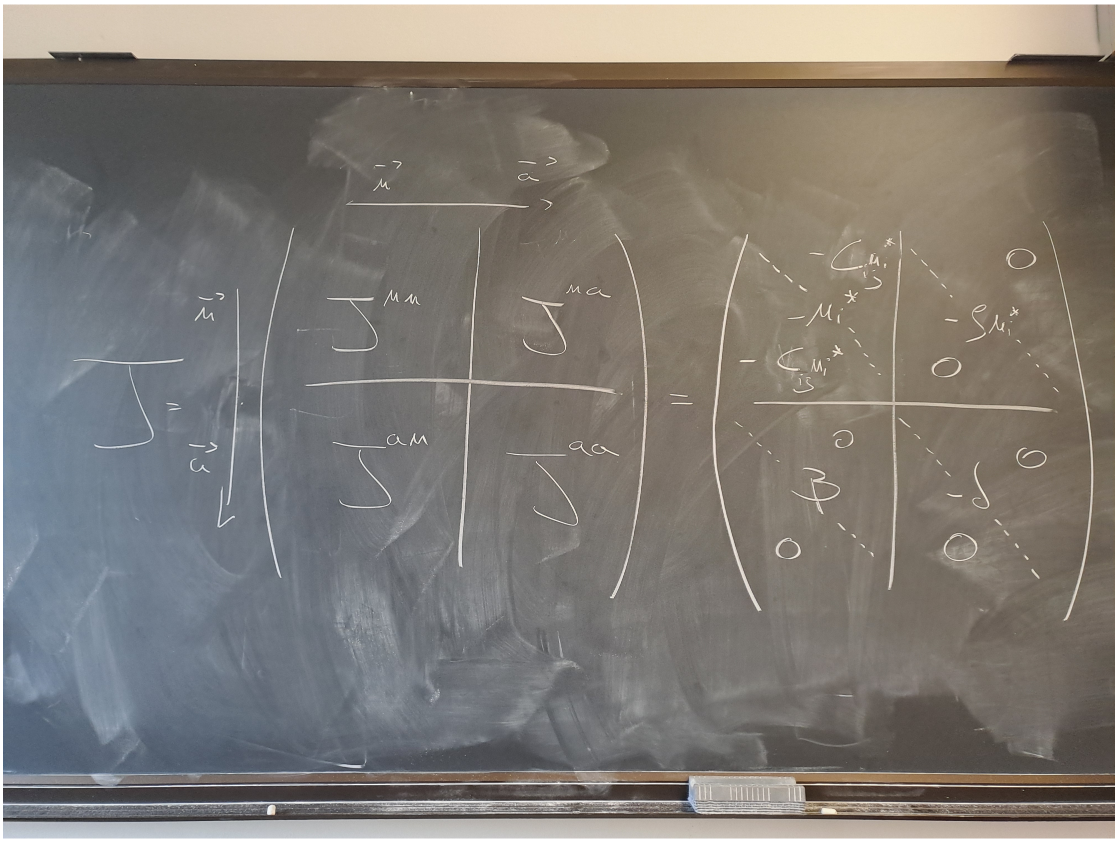

Figure 1: Sketch of the Jacobian matrix for the model with explicit autotoxicity

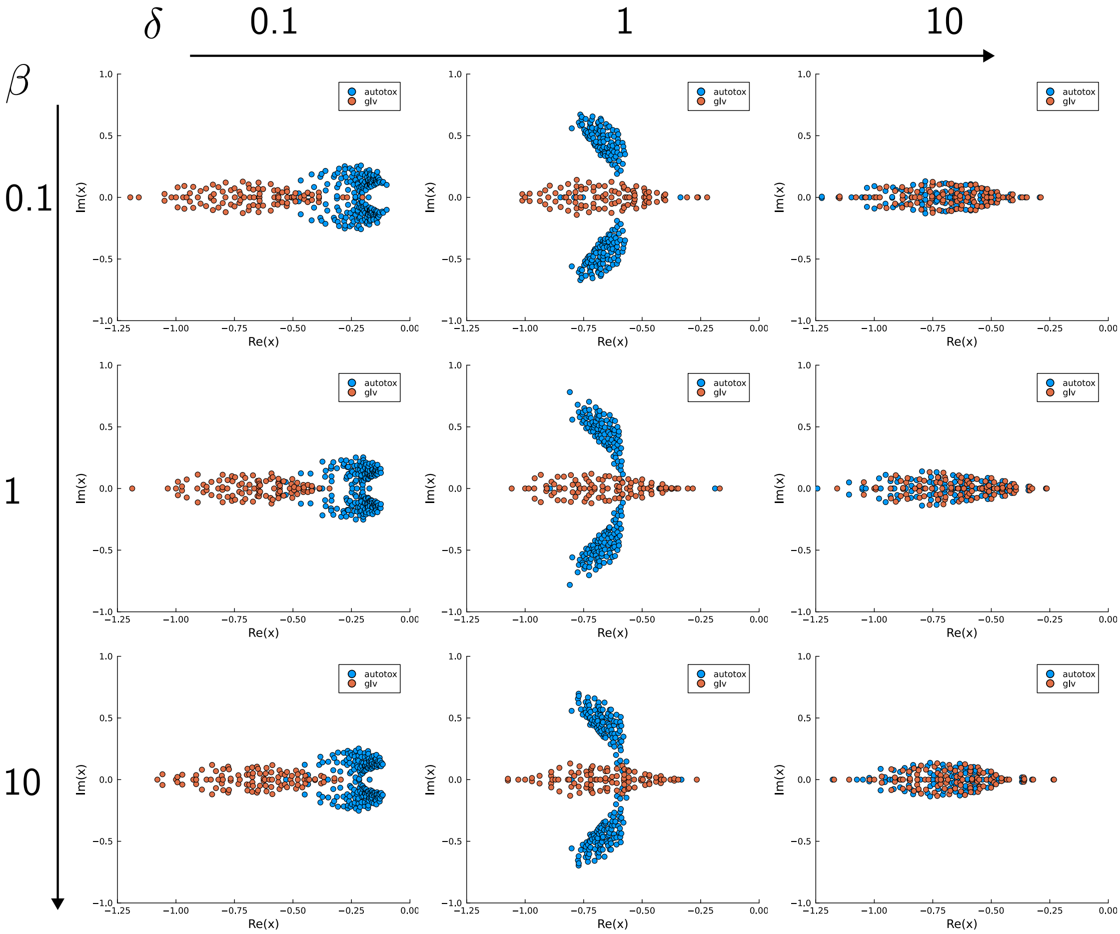

Figure 2: Spectrum of the eigenvalues for the Jacobian matrix of the GLV model with explicit dynamics for the autotoxin (blue) and the one for the equivalent GLV (orange). Parameters are, number of species $S = 100$, $\mu = 1$, $\sigma = 0.5$, $\rho = \delta/\beta$. Large negative outliers are not represented in the plots.