Dirichlet's Theorem on Arithmetic Progressions

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Abstract

Dirichlet's theorem on arithmetic progressions states that given q,$l \in \mathbb{Z}$ such that $\gcd(q, l) = 1$, there are infinitely many primes of the form $q + kl$ where $k \in \mathbb{Z}$. Given $q, l \in \mathbb{N}$, we define $\delta_l(n) : \mathbb{Z} \to \{0, 1\}$ to take 1 if $n \equiv l \bmod q$ and 0 otherwise and note that

$$\sum_{p \equiv l \bmod q} \frac{1}{p^s} = \sum_p \frac{\delta_l(p)}{p^s} = \frac{1}{\phi(q)} \sum_{p \text{ does not divide } q} \frac{1}{p^s} + \frac{1}{\phi(q)} \sum_{\chi \neq \chi_0} \overline{\chi(l)} \sum_p \frac{\chi(p)}{p^s},$$

where χ is the Dirichlet character and χ_0 is a trivial Dirichlet character. The sum $\sum_{p \text{ does not divide } q} \frac{1}{p^s}$ diverges to ∞ as $s \to 1+$ and the one on the right is finite, proving that $\sum_{p \equiv l \bmod q} \frac{1}{p^s}$, proving Dirichlet's theorem.

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Contents

Chapter 1

Preliminaries

We shall introduce a few preliminaries here which will list some of the theorems and facts used throughout this thesis.

1. Let G be a finite Abelian group. Then a character on G is a function $e : G \to S^1$(the set of complex numbers of modulus 1) such that

$$\forall a, b \in G, e(a)e(b) = e(ab).$$

   We denote the set of characters of G, $\hat{G}$.

2. We define $\hat{f} : G \to \mathbb{C}$ by

$$\hat{f}(e) = \frac{1}{|G|} \sum_{a \in G} f(a)\overline{e(a)}.$$

   This $\hat{f}(e)$ is said to be the Fourier coefficient of f with respect to e.

3. Suppose G is a finite Abelian group and $f : G \to \mathbb{C}$ is a function. Then we can write f as

$$f = \sum_{e \in \hat{G}} \hat{f}(e)e.$$

The sum on the right hand side is called the Fourier series of f. (The characters e form an orthonormal basis for the vector space V of functions from G to $\mathbb{C}$, so we can write $f = \sum_{e \in \hat{G}} c_e e$. An inner product on V is defined by $(f, g) = \frac{1}{|G|} \sum_{a \in G} f(a)\overline{g(a)}$ and we have $(f, e) = c_e$ which gives us the result.)

4. $\{f_n\}$ is a sequence of functions, differentiable on (a, b) such that $\{f_n(x_0)\}$ is convergent for some $x_0 \in [a, b]$. If $\{f_n'\}$ converges uniformly on (a, b), then $\{f_n\}$ converges to f uniformly on (a, b) and $f'(x) = \lim_{n \to \infty} f_n'(x)$.

5. A multiplicative function $f : \mathbb{N} \to \mathbb{C}$ is one which satisfies the condition that $\forall a, b \in \mathbb{N}$ such that $\gcd(a, b) = 1$, $f(a)f(b) = f(ab)$. If f is multiplicative, then so is $F(n) = \sum_{d|n} f(d)$.

6. Let f be a multiplicative arithmetic function such that the series $\sum f(n)$ be absolutely convergent. Then the sum of the series can expressed as an absolutely convergent infinite product,

$$\sum_{n=1}^{\infty} f(n) = \prod_p \{1 + f(p) + f(p^2) + \dots\},$$

where the product is taken over all primes. If f is completely multiplicative (i.e. for all $a, b \in \mathbb{N}$, $f(a)f(b) = f(ab)$) then we have

$$\sum_{n=1}^{\infty} f(n) = \prod_p \frac{1}{1 - f(p)}.$$

See [2] for a proof.

7. The characters of a finite abelian group G form a basis for the vector space of functions on G whose dimension is $|G|$. So, the number of characters is finite.

Chapter 2

A few identities

Definition: For $|z| < 1$, define $\log_1 \dfrac{1}{1-z} = \displaystyle\sum_{k=1}^{\infty} \frac{z^k}{k}$.

We note that the radius of convergence of the above power series is 1, so it converges for $|z| < 1$. It can thus be differentiated term by term within the region $|z| < 1$.

Theorem 2.1: $\exp\left(\log_1 \frac{1}{1-z}\right) = \frac{1}{1-z}$.

Proof: Write $z = re^{i\theta}$ where $\theta \in [0, 2\pi]$ and $0 < r < 1$. We define

$$f(r) = (1 - re^{i\theta})e^{\log_1 \frac{1}{1-re^{i\theta}}} = (1 - re^{i\theta})e^{\sum_{k=1}^{\infty} \frac{(re^{i\theta})^k}{k}}$$

We differentiate $f(r)$ to get $f'(r) = -e^{i\theta}e^{\sum_{k=1}^{\infty} \frac{(re^{i\theta})^k}{k}} + (1 - re^{i\theta})(e^{\sum_{k=1}^{\infty} \frac{(re^{i\theta})^k}{k}})' = -e^{i\theta}e^{\sum_{k=1}^{\infty} \frac{(re^{i\theta})^k}{k}} + (1 - re^{i\theta})e^{i\theta}\displaystyle\sum_{k=1}^{\infty}(re^{i\theta})^{k-1}e^{\sum_{k=1}^{\infty} \frac{(re^{i\theta})^k}{k}}$

Since $\displaystyle\sum_{k=1}^{\infty}(re^{i\theta})^{k-1} = \frac{1}{1-re^{i\theta}}$, we see that $f'(r) = 0$ and hence $f(r) = c$ for some constant c. We put $r = 0$ and see that $f(0) = 1$ and hence $c = 1$.

Theorem 2.2: If $|z| < 1$, $\log_1 \frac{1}{1-z} = z + E(z)$, where $|E(z)| \le |z|^2$ if $|z| < \frac{1}{2}$.

Proof: Let $E(z) = \log_1 \frac{1}{1-z} - z$. Thus $|E(z)| = \left|\displaystyle\sum_{k=2}^{\infty} \frac{z^k}{k}\right| \le \displaystyle\sum_{k=2}^{\infty} |\frac{z^k}{k}| \le \displaystyle\sum_{k=2}^{\infty} \left|\frac{z^k}{2}\right| = \dfrac{|z|^2}{2}\dfrac{1}{1-|z|}$. Since $|z| < \frac{1}{2}$, $\frac{1}{1-|z|} < 2$ and hence $|E(z)| \le |z|^2$.

Theorem 2.3: For $|z| < \frac{1}{2}$, $|\log_1 \frac{1}{1-z}| \le 2|z|$.

Proof Let $\log_1 \frac{1}{1-z} = y(z)$.

$y(z) = z + E(z) \implies y(z)/z = 1 + E(z)/z \implies |\frac{y(z)}{z}| = |1 + \frac{E(z)}{z}| \leq 1 + |\frac{E(z)}{z}| \leq 1 + |z| \leq 2$ which proves our claim.

Theorem 2.4: If $\sum |a_n|$ converges and $a_n \neq 1$ for all n, then $\displaystyle\prod_{n=1}^{\infty} \frac{1}{1-a_n}$ converges. Also, the product is non-zero.

Proof Since $\sum |a_n|$ converges, $|a_n|$ and hence $a_n \to 0$. Therefore, for sufficiently large N, $|a_n| < \frac{1}{2}$ for all $n \geq N$. so, without loss of generality, assume that $|a_n| < 1/2$

$$\prod_{1}^{k} \frac{1}{1-a_n} = \prod_{1}^{k} \exp\left(\log_1 \frac{1}{1-a_n}\right) = \exp\left(\sum_{1}^{k} \log_1 \frac{1}{1-a_n}\right)$$

Note that $|\log_1 \frac{1}{1-a_n}| \leq 2|a_n|$ for $n \geq N$ as $|a_n| < 1/2$ and by the comparison test, $\displaystyle\sum_{n=1}^{\infty} \log_1 \frac{1}{1-a_n}$ converges to say, A. Since the exponential function $\exp : \mathbb{C} \to \mathbb{C}$ is continuous, $\exp(\displaystyle\sum_{n=1}^{\infty} \log_1 \frac{1}{1-a_n}) \to \exp(A) \neq 0$

2.1 Estimates of some sums and products

Theorem 2.5: For $N \in \mathbb{N}$, $\displaystyle\sum_{1 \leq n \leq N} \frac{1}{n} = \log N + O(1) = \log N + \gamma + O(\frac{1}{N})$ and this γ is called the Euler's constant.

Proof Define $\gamma_n = \displaystyle\frac{1}{n} - \int_n^{n+1} \frac{\mathrm{d}x}{x}$. Since $1/x$ is decreasing on $[n, n+1]$, $1/n \geq \displaystyle\int_n^{n+1} \frac{\mathrm{d}x}{x} \geq 1/(n+1)$. Therefore, $0 \leq \gamma_n \leq \frac{1}{n} - \frac{1}{n+1} \leq \frac{1}{n^2}$ and by the comparison test, $\sum \gamma_n$ converges to some γ.

Write $\gamma = \displaystyle\sum_{n=1}^{\infty} \gamma_n = \sum_{n=1}^{N} \gamma_n + \sum_{n=N+1}^{\infty} \gamma_n = \sum_{n=1}^{N} \frac{1}{n} - \int_1^{N+1} \frac{\mathrm{d}x}{x} + \sum_{n=N+1}^{\infty} \gamma_n = \sum_{n=1}^{N} \frac{1}{n} - \int_1^{N} \frac{\mathrm{d}x}{x} + \sum_{n=N+1}^{\infty} \gamma_n - \int_N^{N+1} \frac{\mathrm{d}x}{x}$

From this, we get $\gamma + \log N - \displaystyle\sum_{n=1}^{N} \frac{1}{n} = \sum_{n=N+1}^{\infty} \gamma_n - \int_N^{N+1} \frac{\mathrm{d}x}{x}$ and we will show that each term on the right is $O(1/N)$.

$\int_N^{N+1} \frac{\mathrm{d}x}{x} \leq 1/N$ and so it is $O(1/N)$. Also, $\sum_{n=N+1}^{\infty} \gamma_n \leq \sum_{n=N+1}^{\infty} \frac{1}{n^2} \leq \int_N^{\infty} \frac{\mathrm{d}x}{x^2} = 1/N$ which makes this sum $O(1/N)$.

Theorem 2.6: If N is a natural number, then

$$\sum_{n=1}^{N} \frac{1}{n^{\frac{1}{2}}} = 2N^{\frac{1}{2}} + c + O(1/N^{\frac{1}{2}})$$

for some constant c.

The proof is a repetition of the previous proposition.

Before we prove the next theorem, let us consider a function $F : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$. The sum $S_N = \sum\sum F(m,n)$ where the sum is taken over pairs of (m,n) where $mn \leq N$.

We can find S_N in three ways:

1. Along hyperbolas:
$$S_N = \sum_{1\leq k\leq N} \left(\sum_{nm=k} F(m,n) \right)$$

2. Vertically:
$$S_N = \sum_{1\leq m\leq N} \left(\sum_{1\leq n\leq N/m} F(m,n) \right)$$

3. Horizontally:
$$S_N = \sum_{1\leq n\leq N} \left(\sum_{1\leq m\leq N/m} F(m,n) \right)$$

Theorem 2.7: If $k \in \mathbb{N}$, then $\frac{1}{N}\sum_{k=1}^{N} d(k) = \log N + O(1) = \log N + (2\gamma - 1) + O(1/N^{\frac{1}{2}})$ where $d(k)$ is the number of positive divisors of k.

Proof Define $S_N = \sum_{k=1}^{N} d(k) = \sum_{k=1}^{N} \sum_{nm=k} 1 = \sum_{1\leq m\leq N} \sum_{1\leq n\leq N/m} 1$(Summing up vertically) $= \sum_{1\leq m\leq N} \lfloor \frac{N}{m} \rfloor = \sum_{1\leq m\leq N} (\frac{N}{m} + O(1)) = N \sum_{1\leq m\leq N} + NO(1) \implies \frac{S_N}{N} = \log N + O(1)$

The harder part is to prove the second one.

We can divide the sum S_N into three sums: $S_I = \sum_{1\le m<N^{\frac12}} \sum_{N^{\frac12}<n\le N/m} 1$, $S_{II} = \sum_{1\le m\le N^{\frac12}} \sum_{1\le n\le N^{\frac12}} 1$ and $S_{III} = \sum_{N^{\frac12}<m\le N/n} \sum_{1\le n<N^{\frac12}} 1$ i.e. $S_I + S_{II} + S_{III} = S_N$. By symmetry, $S_I = S_{III}$ and we have $S_N = S_I + S_{II} + S_{III} = 2(S_I + S_{II}) - S_{II}$. Note that $S_I + S_{II} = \sum_{1\le m\le N^{\frac12}} \sum_{1\le n\le N/m} 1 = \sum_{1\le m\le N^{\frac12}} \lfloor N/m \rfloor = \sum_{1\le m\le N^{\frac12}} (N/m + O(1)) = N \sum_{1\le m\le N^{\frac12}} 1/m + O(N^{\frac12}) = N\log N^{\frac12} + N\gamma + O(N^{\frac12})$(using Theorem 2.5)

Also, $S_{II} = N$ and so have our result.

Theorem 2.8: For every $s > 1$, we have

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}},$$

where the product is taken over all primes.

Proof Consider (6) in chapter 1, preliminaries. Take $f(n) = \frac{1}{n^s}$ and we are done.

Theorem 2.9

$$\sum_p \frac{1}{p} \text{ diverges.}$$

Proof Note that

$$\prod_p \frac{1}{1-p^{-s}} = \zeta(s)$$

implies $\log\zeta(s) = \log\prod_p \frac{1}{1-p^{-s}}$. Then $\lim_{N\to\infty} \prod_{p\le N} \log\frac{1}{1-p^{-s}} = \log\prod_p \frac{1}{1-p^{-s}}$.

So, we have $\lim_{N\to\infty} \log\prod_{p\le N} \frac{1}{1-p^{-s}} = \lim_{N\to\infty} \sum_{p\le N} \log\left(\frac{1}{1-p^{-s}}\right) = \sum_p \log\left(\frac{1}{1-p^{-s}}\right) = \log\prod_p \frac{1}{1-p^{-s}}$

Simplifying, we get $\log\zeta(s) = -\sum_p \log(1-\frac{1}{p^s}) = -\sum_p(-\frac{1}{p^s} + E_p)$ where $|E_p| \le \frac{1}{p^{2s}}$. Summing it, we get $\log\zeta(s) = \sum_{p \text{ prime}} \frac{1}{p^{2s}} + O(1)$ as $s \to 1+$ by comparing $\sum 1/p^{2s}$ with the convergent sum $\sum 1/n^s$ as $s > 1$.

We now wish to show that $\zeta(s)$ diverges as $s \to 1+$.

We consider a sequence $s_n \to 1$, $s_n > 1$.

We see that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} \geq \sum_{n=1}^{M} \frac{1}{n^s}$ for every natural number M.

In particular, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{s_n}} \geq \sum_{n=1}^{M} \frac{1}{n^{s_n}}$ and so $\displaystyle\liminf_{k\to\infty} \sum_{n=1}^{\infty} \frac{1}{n^{s_k}} \geq \liminf_{k\to\infty} \sum_{n=1}^{M} \frac{1}{n^{s_k}} = \sum_{n=1}^{M} \frac{1}{n}$.

Since this works for any $M \in \mathbb{N}$, we infer that $\displaystyle\liminf_{k\to\infty} \sum_{n=1}^{\infty} \frac{1}{n^{s_k}} \geq \sum_{n=1}^{\infty} \frac{1}{n}$.

Also, recall that $\sum 1/n$ diverges.

As a result, $\zeta(s_n)$ diverges as $s_n \to 1+$, so $\log \zeta(s_n)$ diverges as $s_n \to 1+$ and hence $\displaystyle\liminf_{n\to\infty} \sum_{p} \frac{1}{p^{s_n}}$ diverges and since $\displaystyle\liminf_{n\to\infty} \sum_{p} \frac{1}{p^{s_n}} \leq \sum_{p} \frac{1}{p}$, we get that $\displaystyle\sum_{p} \frac{1}{p}$ diverges.

Chapter 3

$L-$ functions and their properties

3.1 Some properties of $L-$ functions

Before we define $L-$ functions, we define Dirichlet characters. Recall that given a group G, a character on G is a function $e : G \to S^1$ such that for all $a, b \in G$, $e(a)e(b) = e(ab)$.

Let $q \in \mathbb{N}$ and $G = Z^*(q)$ (the group of units mod$q$). A Dirichlet character modulo q is a function $\chi : \mathbb{Z} \to \mathbb{C}$ such that $\chi(n) = \begin{cases} e(n) & \text{if } n \in G \\ 0 & \text{otherwise} \end{cases}$

An $L-$function for a fixed Dirichlet character, χ modulo q and $s > 0$ is a function $L(s, \chi) = \displaystyle\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$. Throughout the rest of the text, we will only use Dirichlet characters modulo q.

Theorem 3.1: For $s > 1$, $L(s, \chi) = \displaystyle\prod_p \frac{1}{1 - \chi(p)p^{-s}}$ where the product is taken over primes.

Proof Consider (6) in chapter 1, preliminaries. Take $f(n) = \frac{\chi(n)}{n^s}$ and we are done.

Theorem 3.2: Suppose, χ_0 is the trivial Dirichlet character given by

$$\chi_0 = \begin{cases} 1 & \gcd(n,q)=1 \\ 0 & \text{otherwise} \end{cases}$$

and $q = p_1^{a_1} \dots p_N^{a_N}$ is the prime factorisation of q. Then $L(s,\chi_0) = (1-p_1^{-s})\dots(1-P_N^{-s})\zeta(s)$. Therefore $L(s,\chi_0) \to \infty$ as $s \to 1+$.

Proof By the previous theorem, $L(s,\chi) = \prod_p \frac{1}{1-\chi(p)p^{-s}}$. Let $q = p_1^{a_1}\dots p_N^{a_N}$, $q > 1$. Notice that $\chi(p_k) = 0$, $1 \leq k \leq N$. So,

$$L(s,\chi_0) = \prod_{p \neq p_1,\dots,p_N} \frac{1}{1-\chi_0(p)p^{-s}} = \prod_{p \neq p_1,\dots,p_N} \frac{1}{1-p^{-s}}$$

which shows that $L(s,\chi_0)((1-p_1^{-s})\dots,(1-p_N^{-s}))^{-1} = \zeta(s) \to \infty$ as $s \to 1+$ which proves our claim.

Theorem 3.3: Let χ be a non-trivial Dirichlet character modulo q. Then

$$|\sum_{n=1}^{k} \chi(n)| \leq q$$

for any $k \in \mathbb{N}$.

Proof Let $S = \sum_{n=1}^{q} \chi(n)$. Since χ non-trivial, we can choose a such that $\chi(a) \neq 1, 0$. The product $S\chi(a) = \sum_{n=1}^{q} \chi(n)\chi(a) = \sum_{n=1}^{q} \chi(na) = \sum_{n=1}^{q} \chi(n) = S$ and so $S = 0$. Given $k \in \mathbb{N}$, we can write $k = aq + b, 0 \leq b < q$. So,

$$\left|\sum_{n=1}^{k} \chi(n)\right| = \left|\sum_{n=1}^{qa} \chi(n) + \sum_{aq<n\leq aq+b} \chi(n)\right| = \left|\sum_{aq<n\leq aq+b} \chi(n)\right| \leq \sum_{aq<n\leq aq+b} |\chi(n)| \leq b < q.$$

Theorem 3.4: Let χ be a non-trivial Dirichlet character. Then the series $\sum_{n=1}^{\infty} \frac{X(n)}{n^s}$ converges for $s > 0$ and satisfies

(i) $L(s,\chi)$ is continuously differentiable on $0 < s < \infty$ and

(ii) There exists a constant c such that

$$L(s,\chi) = 1 + O(e^{-cs})$$

and

$$L'(s,\chi) = O(e^{-cs}).$$

Proof We begin by showing that $\sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$ is convergent for $s > 0$.

Define $s_n = \sum_{k=1}^{n} \chi(k)$ and $s_0 = 0$. By Theorem 3.3, $|s_n| \leq q$.

$\sum_{n=1}^{N} \frac{\chi(n)}{n^s} = \sum_{n=1}^{N} \frac{s_n - s_{n-1}}{n^s} = \sum_{n=1}^{N} (\frac{s_n}{n^s} - \frac{s_n}{(n+1)^s}) + \frac{s_N}{N^s}$ and so $|\sum_{n=1}^{N} \frac{\chi(n)}{n^s}| \leq q \sum_{n=1}^{N-1} |\frac{1}{n^s} - \frac{1}{(n+1)^s}| + |\frac{s_N}{N^s}|$. Define $a_n = \frac{s_n}{n^s} - \frac{s_n}{(n+1)^s}$

Now, consider $f(x) = x^{-s}$ on $\mathbb{R}$. $f'(x) = -sx^{-s-1}$, so applying the mean value theorem, $|f(n+1) - f(n)| = sc^{-s-1}$ where $c \in (n, n+1)$ which proves that $|a_n| \leq qsn^{-s-1}$ which implies that $\sum a_n < \infty$ by the comparison test.

As a result, $\sum_{n=1}^{N} \frac{\chi(n)}{n^s}$ is convergent ($\left|\frac{s_N}{N^s}\right| \leq \left|\frac{q}{N^s}\right|$ which goes to 0 as $N \to \infty$)

Define $f_n : (0,\infty) \to \mathbb{C}$ by $f_n(s) = \sum_{k=1}^{n} \frac{\chi(k)}{k^s}$ and $f(s) := L(s,\chi)$. We note that $f_n(s) \to f(s)$ pointwise and that $f_n(s)$ converges for $s > 0$ by what we proved earlier.

We wish to show that $\{f_n'\}$ converges uniformly.

$f_N'(s) = -\sum_{n=1}^{N} (\log n) \frac{\chi(n)}{n^s} = -\sum_{n=1}^{N-1} s_n \left(\frac{\log n}{n^s} - \frac{\log(n+1)}{(n+1)^s}\right) - \log N \frac{s_N}{N^s}$. The term on the right goes to 0 as $N \to \infty$.

We now consider

$$s_k \left[-k^{-s} \log k + (k+1)^{-s} \log(k+1)\right]$$

Let $g(x) = x^{-s} \log x$. Then, $g'(x) = -sx^{-s-1} \log x + x^{-s-1} = x^{-s-1}(-s \log x + 1)$. Using the mean value theorem, $\exists x \in (k, k+1)$ such that $|g(k+1) - g(k)| = |g'(x)|$.

We note that for $s > \delta > 0$, $|g'(x)| = |x^{-s-1}(-s\log x + 1)| \leq k^{-\delta-1}|(-s\log x + 1)|$

The result of the product is $O(-k^{\delta/2-1})$ which proves our claim.

Now, $L(s,\chi) = 1 + \sum_{k=2}^{\infty} \frac{\chi(k)}{k^s}$ which means that $|L(s,\chi) - 1| = |\sum_{k=2}^{\infty} \frac{\chi(k)}{k^s}| \leq \sum_{k=2}^{\infty} \frac{1}{k^s}$.

We claim that $\sum_{k=2}^{\infty} \frac{1}{k^s} \to 0$ as $s \to \infty$.

$\sum_{k=2}^{\infty} \frac{1}{k^s} = (\frac{1}{2^s} + \frac{1}{3^s}) + (\frac{1}{4^s} + \frac{1}{5^s}) + \cdots \leq \frac{2}{2^s} + \frac{2}{4^s} + \frac{2}{6^s} + \cdots = \frac{2}{2^s}(\sum_{k=1}^{\infty} \frac{1}{k^s})$ and $\sum_{k=1}^{\infty} \frac{1}{k^s}$ converges to say C since s is large. Therefore, $\sum_{k=2}^{\infty} \frac{1}{k^s} \leq 2C/2^s = 2Ce^{-cs}$ which proves our claim that $L(s,\chi) = 1 + O(e^{-cs})$ for some constant c.

$L'(s,\chi) = -\sum_{k=2}^{n} \frac{\log k\chi(k)}{k^s}$ and so $|L(s,\chi)| \leq \sum_{k=2}^{n} \frac{\log k)}{k^s} \leq \sum_{k=2}^{n} \frac{1}{k^s}$ and so $L'(s,\chi) = O(e^{-cs})$.

Theorem 3.5: Let $\delta_l : \mathbb{Z} \to \mathbb{Z}$ by

$$\delta_l(n) = \begin{cases} 1 & \text{if n} \equiv l \bmod q \\ 0 & \text{otherwise} \end{cases}$$

We claim that

$$\delta_l(n) = \frac{1}{\phi(q)} \sum_{\chi} \overline{\chi(l)}\chi(n).$$

Proof We restrict this function to the group $\mathbb{Z}^*(q)$ and expand this function in a Fourier series as

$$\delta_l(n) = \sum_{e \in \hat{G}} \hat{\delta}_l(e)e(n)$$

where

$$\hat{\delta_l(n)} = \frac{1}{|G|} \sum_{m \in G} \delta_l(m)\overline{e(m)} = \frac{1}{|G|}\overline{e(l)}$$

which allows us to write

$$\delta_l(n) = \frac{1}{|G|} \sum_{e \in \hat{G}} \overline{e(l)}e(n)$$

Now, any Dirichlet character on G will be of the form $\chi(m) = e(m)$ if $\gcd(m,q) = 1$ and 0 otherwise. Since $l \in G$, l and q are coprime and hence we get $\chi(l) = e(l)$ and for the same reason $\chi(m) = e(m)$ when m and q are coprime. If not, $\chi(m) = 0$. So we can rewrite our sum as

$$\delta_l(n) = \frac{1}{\phi(q)} \sum_{\chi} \overline{\chi(l)}\chi(n)$$

We define for $s > 1$,

$$\log_2 L(s,\chi) = -\int_s^\infty \Re\frac{L'(t,\chi)}{L(t,\chi)}\mathrm{d}t - i\int_s^\infty \Im\frac{L'(t,\chi)}{L(t,\chi)}\mathrm{d}t,$$

where $\Re(z)$ and $\Im(z)$ denote the real and complex parts of z.

For now, we assume that the integrals are convergent and derivative of $\log_2 L(s,\chi)$ is $L(s,\chi)$. Those calculations are done in chapter 4.

Theorem 3.6: If $s > 1$, then

$$\exp(\log_2 L(s,\chi)) = L(s,\chi).$$

Further

$$\log_2 L(s,\chi) = \sum_p \log_1\left(\frac{1}{1-\chi(p)/p^s}\right).$$

Proof Differentiating $\exp(-\log_2 L(s,\chi))L(s,\chi)$ we get

$$-\frac{L'(s,\chi)}{L(s,\chi)}\exp(-\log_2 L(s,\chi))L(s,\chi) + \exp(-\log_2 L(s,\chi))L'(s,\chi)$$

which is 0 (by Theorem 3.1, $L(s,\chi) \neq 0$). Therefore, $\exp(-\log_2 L(s,\chi))L(s,\chi) = c$ for some constant c.

We claim that $\exp(-\log_2 L(s,\chi)) \to 1$ as $s \to \infty$. ($|L(s,\chi)| \to 1$, so that would imply that $\exp(-\log_2 L(s,\chi))L(s,\chi) \to 1$ as $s \to \infty$.

For large s, since $t > s$, $|\frac{L'(t,\chi)}{L(t,\chi)}| = O(e^{-ct})$. So we get that $|\displaystyle\int_s^\infty \Re\Big(\frac{L'(t,\chi)}{L(t,\chi}\Big)\mathrm{d}t| \leq \int_s^\infty \Big|\frac{L'(t,\chi)}{L(t,\chi)}\Big|\mathrm{d}t \leq \int_s^\infty e^{-et}\mathrm{d}t = \frac{e^{-cs}}{c} \to 0$ as $s \to \infty$, where the real part of complex number z is $\Re(z)$. Therefore, $\log_2 L(s,\chi) \to 0$ as $s \to \infty$ which proves our claim above.

We claim that $\displaystyle\sum_p \log_1\left(\frac{1}{1-\chi(p)/p^s}\right)$ converges. By Theorem 2.3, we get $\displaystyle\left|\log_1(\frac{1}{1-\chi(p)/p^s})\right| \leq \left|\frac{\chi(p)}{p^s}\right| \leq \frac{1}{p^s}$ for large p since for large p, $\left|\frac{\chi(p)}{p^s}\right| \leq 1/p^s$ and $\displaystyle\sum_p \frac{1}{p^s} \leq \sum_n \frac{1}{n^s} < \infty$ and so by the comparison test, $\displaystyle\sum_p \log_1\left(\frac{1}{1-\chi(p)/p^s}\right)$ converges.

So, we take $\displaystyle\exp\left(\sum_p \log_1(\frac{1}{1-\chi(p)/p^s})\right) = \prod_p \exp\left(\log_1(\frac{1}{1-\chi(p)/p^s})\right) = \prod_p \frac{1}{1-\chi(p)/p^s}$ (Using Theorem 3.1, this is) $= L(s,\chi) = \exp(\log_2 L(s,\chi))$, so there exists $M(s) \in \mathbb{Z}$ such that $\displaystyle\log_2 L(s,\chi) - \sum_p \log_1(\frac{1}{1-\chi(p)/p^s}) = 2\pi i M(s)$

We will show that that both $\log_2 L(s,\chi)$ and $\displaystyle\sum_p \log_1\left(\frac{1}{1-\chi(p)/p^s}\right)$ are both continuous functions of s.

Let $s_n \to s, s_n > 0$. Consider the difference $\displaystyle|\int_s^\infty \Re\Big(\frac{L'(t,\chi)}{L(t,\chi)}\mathrm{d}t - \int_{s_n}^\infty \Re\Big(\frac{L'(t,\chi)}{L(t,\chi)}\Big)\mathrm{d}t| = \Big|\int_s^{s_n} \Re\Big(\frac{L'(t,\chi)}{L(t,\chi)}\Big)\mathrm{d}t\Big| \leq \int_s^{s_n} \Big|\frac{L'(t,\chi)}{L(t,\chi)}\Big|\mathrm{d}t$

By the property of Riemann integrability, we get that the integral on the right goes to 0 as $s_n \to s$ (using the property that for integrable f, $g(s) = \displaystyle\int_a^s f(x)\mathrm{d}x$ is continuous). Similarly, the imaginary part of $\log_2 L(s,\chi)$ is also continuous in s and so $\log_2 L(s,\chi)$ is continuous.

Now, recall that the number of primes is infinite and an infinite subset of $\mathbb{N}$, hence countable. So we can label the primes as p_1, p_2 and so on. In fact, we can use the calculations above to note that the sequence of functions, $\{f_n\}$ defined by $f_n(s) = \displaystyle\sum_{k=1}^n \log_1 \frac{1}{1-\chi(p_k)/p_k^s}$ on $(1,\infty)$ is uniformly convergent to $\displaystyle\sum_p \log_1(\frac{1}{1-\chi(p)/p^s})$ and since each of these f_n's is continuous, we are done.

As we proved earlier, we have $\log_2 L(s, \chi) - \sum_p \log_1 \left(\frac{1}{1 - \chi(p)/p^s} \right) = 2\pi i M(s)$. As $s \to \infty$, both the expressions on the LHS $\to 0$ and hence $M(s) = 0$.

Lemma 3.1 The number of Dirichlet characters modulo q is finite.

Proof Since the number of characters is finite on $G = \mathbb{Z}^*(q)$, by chapter 1, preliminary 7, the number of Dirichlet characters is also finite.

3.2 $L(1,\chi) \neq 0$ for non-trivial, complex Dirichlet character χ

Here, we use proof by contradiction to prove our result.

Lemma 3.2 If $s > 1$, then we have

$$\prod_{\chi} L(s,\chi) \geq 1$$

where the product is over all Dirichlet characters. In particular, the product is real.

Proof: We know by Theorem 3.6 that

$$L(s,\chi) = \exp\left(\sum_{p} \log_1\left(\frac{1}{1-\chi(p)/p^s}\right)\right)$$

and hence we can write $\prod_{\chi} L(s,\chi) = \exp\left(\sum_{\chi}\sum_{p} \log_1\left(\frac{1}{1-\chi(p)/p^s}\right)\right) = \exp\left(\sum_{\chi}\sum_{p}\sum_{k=1}^{\infty} \frac{1}{k}\frac{\chi(p^k)}{p^{ks}}\right)$ which can be written as

$$\exp\Big(\sum_{p}\sum_{k=1}^{\infty}\sum_{\chi} \frac{\chi(p^k)}{kp^{ks}}\Big).$$

We know that if $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} |a_{jk}| < \infty$, then $\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} a_{jk} = \sum_{k=1}^{\infty}\sum_{j=1}^{\infty} a_{jk}$. To check that exchange the order of summation as above is valid, since the number of Dirichlet characters modulo q is finite, it is sufficient to check for a fixed χ, $\sum_{p}\sum_{k=1}^{\infty} |\frac{\chi(p^k)}{kp^{ks}}| < \infty$

Note that $|\frac{\chi(p^k)}{kp^{ks}}| \leq \left|\frac{1}{kp^{ks}}\right| \leq \left|\frac{1}{p^{ks}}\right|$ and $\sum_{k=1}^{\infty} \left|\frac{1}{p^{ks}}\right| = \frac{1}{p^s-1}$. Next, we see that by comparing with $\sum \frac{1}{p^s}$, $\sum_{p} \frac{1}{p^s-1}$ converges which proves our claim.

Taking $l = 1$ in Theorem 3.5, we get $\sum_{\chi} \chi(p^k) = \phi(q)\delta_1(p^k)$ and so we get

$$\prod_{\chi} L(s, \chi) = \exp\left(\phi(q) \sum_{p} \sum_{k=1}^{\infty} \frac{\delta_1(p^k)}{kp^{ks}}\right).$$

In fact the calculation above allows us to say that the double-sum inside is finite. Also δ_1 gives a positive value and hence the product of these L functions at s is non-negative and real.

Lemma 3.3 The following properties hold:

1. If $L(1, \chi) = 0$, then $L(1, \overline{\chi}) = 0$.

2. If χ is a non-trivial Dirichlet character and $L(1, \chi) = 0$, then $|L(s, \chi)| \leq C|s-1|$ when $1 \leq s \leq 2$.

3. For the trivial Dirichlet character χ_0, we have

$$|L(s, \chi_0)| \leq \frac{C}{s-1}$$

   when $1 < s \leq 2$.

Proof The first one follows from

$$\overline{\sum_{n=1}^{\infty} a_n} = \sum_{n=1}^{\infty} \overline{a_n}.$$

For the second one, recall that $L(s, \chi)$ is C^1. If $s = 1$, we are done. If not, by the mean value theorem, there exists $c \in (1, s)$, $|L(s, \chi) - L(1, \chi)| = |L'(c, \chi)||s - 1|$. Now, $1 \leq s \leq 2$ and since $L'(s, \chi)$ is continuous, on the compact interval $[1, 2]$, there exists $M > 0$ such that $|L'(c, \chi)| \leq M$ which proves our claim.

For (3), recall from theorem 3.2 that

$$L(s, \chi_0) = (1 - p_1^{-s}) \ldots (1 - p_N^{-s})\zeta(s)$$

where $p_1^{a_1} \dots p_N^{a_N}$ is the prime factorisation of q. We estimate $\zeta(s)$.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \leq 1 + \int_1^{\infty} \frac{\mathrm{d}x}{x^s} = 1 + \frac{1}{s-1} \leq k\frac{1}{s-1}$$

for large k and this shows (3) is true.

Now, we prove that $L(s, \chi_1) \neq 0$. Suppose not. Then $L(1, \chi_1) = 0$.

We can write $\prod_{\chi} L(s, \chi) = L(s, \chi_1)L(s, \overline{\chi_1})L(s, \chi_0)K$ since the rest of the product converges. As a result, $|L(s, \chi_1)||L(s, \overline{\chi_1})|L(s, \chi_0)| \leq C|s-1|$ which goes to 0 as $s \to 1+$ and so the whole product goes to 0 which contradicts Lemma 3.1. So, $L(1, \chi) \neq 0$.

3.3 $L(1,\chi) \neq 0$ for non-trivial, real Dirichlet character χ

To prove theorem, we will require a couple of lemmas.

Lemma 3.4 $\sum_{n|k} \chi(n) \geq 0$ for all k. In fact, it is ≥ 1 if $k = l^2$ for some $l \in \mathbb{Z}$.

Proof We recall that if $f(n)$ is multiplicative, so is $\sum_{d|n} f(d)$. Note that $\chi(n)$ is multiplicative, so we ought to compute $\sum_{n|p^a} \chi(n)$ where p is a prime. We see that $\sum_{n|p^a} \chi(n) = 1 + \chi(p) + \cdots + \chi(p^a)$ and the sum is equal to

$$\begin{cases} a+1 & \chi(p) = 1 \\ 1 & \chi(p) = -1 \text{ and } a \text{ is even} \\ 0 & \chi(p) = -1 \text{ and } a \text{ is odd} \\ 1 & \text{if } \chi(p) = 0, \text{ that is } p|q \end{cases}$$

For k square, a is even and our claim is proved.

Lemma 3.5 For all naturals $0 < a < b$ we have

1. $$\sum_{n=a}^{b} \frac{\chi(n)}{n^{\frac{1}{2}}} = O(a^{\frac{-1}{2}}).$$

2. $$\sum_{n=a}^{b} \frac{\chi(n)}{n} = O(a^{-1}).$$

Proof Once again, we define $s_n = \sum_{n=1}^{k} \chi(k)$ and $|s_n| \leq q$ for all n. Then $\sum_{n=a}^{b} \frac{\chi(n)}{n^{\frac{1}{2}}} = \sum_{n=a}^{b-1} s_n \left[n^{\frac{-1}{2}} - (n+1)^{\frac{-1}{2}} \right] + O(a^{\frac{-1}{2}})$.

We will prove that $\sum_{n=a}^{b-1} s_n \left[n^{\frac{-1}{2}} - (n+1)^{\frac{-1}{2}}\right]$ is $O(a^{\frac{-1}{2}})$. For that, we notice $\left|\sum_{n=a}^{b-1} s_n[n^{\frac{-1}{2}} - (n+1)^{\frac{-1}{2}}]\right| \leq q \sum_{n=a}^{b-1} \left[n^{\frac{-1}{2}} - (n+1)^{\frac{-1}{2}}\right]$.

Now, consider the function $f(x) = x^{\frac{-1}{2}}$ and note that $f'(x) = \frac{-1}{2}x^{\frac{-3}{2}}$ and by the mean value theorem, $|f(n+1) - f(n)| = \frac{1}{2}x^{\frac{-3}{2}}$ for some $x \in (n, n+1)$ which means that each term in the above sum $\leq \frac{1}{2}n^{\frac{-3}{2}}$ and so the above sum is $O(\sum_{n=a}^{\infty} n^{\frac{-3}{2}})$ and we will show that this sum itself is $O(a^{\frac{-1}{2}})$.

Consider the function $f(x) = x^{\frac{-3}{2}}$. We note that $\sum_{n=a}^{\infty} n^{\frac{-3}{2}} \leq \int_a^\infty f(x)\mathrm{d}x$. The integral is $\frac{1}{2}a^{\frac{-1}{2}}$ which proves our claim. The second part has a very similar proof.

Let χ be a non-trivial Dirichlet character. Given such a character, we let

$$F(m,n) = \frac{\chi(n)}{(nm)^{\frac{1}{2}}}$$

and

$$S_N = \sum\sum F(m,n)$$

where the sum is taken over all integers $m, n \geq 1$ such that $mn \leq N$.

The two lemmas above lead to the following proposition:

Theorem 3.7: The following statements are true:

1. $S_N \geq c \log N$ for constant c.

2. $S_N = 2N^{\frac{1}{2}}L(1,\chi) + O(1)$.

Proof As before, we divide our sum S_N into three parts, S_I, S_{II} and S_{III}. Summing vertically, $S_I = \sum_{m<N^{\frac{1}{2}}} \frac{1}{m^{\frac{1}{2}}} \left(\sum_{N^{\frac{1}{2}}<n\leq N/m} \chi(n)/n^{\frac{1}{2}} \right)$.

$\sum_{N^{1/2}<n\leq N/m} \frac{\chi(n)}{n^{\frac{1}{2}}} = O((\lfloor N^{\frac{1}{2}} \rfloor + 1)^{\frac{-1}{2}})$ by Lemma 3.5 and $\lfloor N^{\frac{1}{2}} \rfloor + 1 \geq N^{\frac{1}{2}}$ and raising both the sides to power $-1/2$, the inequality flips and so the sum above is actually $O(N^{\frac{-1}{4}})$.

$\left(\sum_{N^{\frac{1}{2}} < n \leq N/m} \chi(n)/n^{\frac{1}{2}} \right)$ is $O(N^{\frac{-1}{4}})$. Now, $\sum_{1 < m \leq N^{1/2}} \frac{1}{m^{1/2}} = 2(\lfloor N^{1/4} \rfloor + 1) + c + O(1/N^{1/2}) \leq jN^{1/4} + c + O(1/N^{1/2})$ for large j. Because of that, we see that $S_I = O(1)$.

Then we sum horizontally to get $S_I + S_{II} = \sum_{1 \leq n \leq N^{\frac{1}{2}}} \frac{\chi(n)}{n^{\frac{1}{2}}} \left(\sum_{m \leq N/n} \frac{1}{m^{\frac{1}{2}}} \right)$

(which using theorem 2.6 can be written as)

$\sum_{1 \leq n \leq N^{\frac{1}{2}}} \frac{\chi(n)}{n^{\frac{1}{2}}} \{2(N/n)^{\frac{1}{2}} + c + O((n/N)^{\frac{1}{2}}\}$.

So, we can write $S_I + S_{II} = 2N^{\frac{1}{2}} \sum_{1 \leq n \leq N^{\frac{1}{2}}} \frac{\chi(n)}{n} + c \sum_{1 \leq n \leq N^{\frac{1}{2}}} \frac{\chi(n)}{n^{\frac{1}{2}}} + O(1)$.

The sum $\sum_{1 \leq n \leq N^{\frac{1}{2}}} \frac{\chi(n)}{n} = \sum_{1}^{\infty} \frac{\chi(n)}{n} - \sum_{N^{\frac{1}{2}} < n < \infty} \frac{\chi(n)}{n}$. The second sum on the RHS is $O(1)$ by lemma 3.4 while the first on the right hand side is $2N^{\frac{1}{2}} L(1, \chi)$.

By lemma 3.4 (ii), the sum $\sum_{1 \leq n \leq N^{\frac{1}{2}}} \frac{\chi(n)}{n^{\frac{1}{2}}}$ is $O(1)$ which implies that $S_{II} + S_{III} = 2N^{\frac{1}{2}} L(1, \chi) + O(1)$.

Summing S_I and $S_{II} + S_{III}$, we get the desired result.

Now we show the first part of the theorem i.e $S_N \geq c \log N$ for some constant $c > 0$.

We sum up along hyperbolas: $\sum_{k=1}^{N} \sum_{nm=k} \frac{\chi(n)}{(nm)^{\frac{1}{2}}} = \sum_{k=1}^{N} \frac{1}{k^{\frac{1}{2}}} \sum_{n|k} \chi(n)$. Using lemma 3.3, we have $S_N \geq \sum_{k=l^2, l \leq N^{\frac{1}{2}}} \frac{1}{k^{\frac{1}{2}}} = \frac{1}{2} \log N + O(1)$(Using Theorem 2.5) By the archimedian property, there exists large $j \in \mathbb{N}$ such that $1/j \log N \leq 1/2 \log N + O(1)$ which proves our claim.

To prove that $L(1, \chi) \neq 0$, we note that if $L(1, \chi) = 0$ then $S_N = O(1)$ i.e. some constant. However, $S_N \geq c \log N$ for some constant $c > 0$ implies that S_N is not bounded above, a contradiction.

Chapter 4

Endgame

Now, we wish to come to the proof of the main theorem.

Let χ be a non-trivial Dirichlet character and $s > 1$. We define

$$\log_2 L(s,\chi) = -\int_s^\infty \Re\frac{L'(t,\chi)}{L(t,\chi)}\mathrm{d}t - i\int_s^\infty \Im\frac{L'(t,\chi)}{L(t,\chi)}\mathrm{d}t.$$

We know that $L(s,\chi) = \prod_p \frac{1}{1-\chi(p)p^{-s}}$ and by theorem 2.4, $L(s,\chi) \neq 0$.

To prove the convergence of the integral, we note that $\frac{L'(t,\chi)}{L(t,\chi)} = O(e^{-ct})$. Recall that $L(t,\chi) = 1 + O(e^{-ct})$ so for large t, we have $|L(t,\chi)| \geq \frac{1}{2}$ and hence $\left|\frac{L'(t,\chi)}{L(t,\chi)}\right| \leq 2|L'(t,\chi)|$ which makes it $O(e^{-ct})$. Now, by continuity of both the numerator and denominator, we get that $\frac{L'(t,\chi)}{L(t,\chi)}$ is continuous on $[s,t]$ and hence its real and imaginary parts $\Re\frac{L'(t,\chi)}{L(t,\chi)}$ and $\Im\frac{L'(t,\chi)}{L(t,\chi)}$ are continuous and hence Riemann integrable. For large enough k, we can split the integral as $\int_s^N \Re\frac{L'(t,\chi)}{L(t,\chi)}\mathrm{d}t = \int_s^k \Re\frac{L'(t,\chi)}{L(t,\chi)}\mathrm{d}t + \int_k^N \Re\frac{L'(t,\chi)}{L(t,\chi)}\mathrm{d}t$.

The first integral is a real number. We need to check the convergence of the second one as $N \to \infty$. To do that, we see that the real and complex parts of the following integrals converge

$$\left|\int_k^N \Re\left(\frac{L'(t,\chi)}{L(t,\chi)}\right)\mathrm{d}t\right| \leq \int_k^N \left|\frac{L'(t,\chi)}{L(t,\chi)}\right|\mathrm{d}t \leq \int_k^N e^{-ct}\mathrm{d}t < \infty$$

and

$$\left|\int_k^N \Im\frac{L'(t,\chi)}{L(t,\chi)}\mathrm{d}t\right| \leq \int_k^N \left|\frac{L'(t,\chi)}{L(t,\chi)}\right|\mathrm{d}t \leq \int_k^N e^{-ct}\mathrm{d}t < \infty$$

so $\log_2 L(s,\chi) < \infty$.

We can now prove that the derivative of $\log_2 L(s,\chi)$ is $L(s,\chi)$. Consider the real part of $\log_2 L(s,\chi)$. As before, for large enough k, we can write $\displaystyle\int_s^N \Re\frac{L'(t,\chi)}{L(t,\chi)}\mathrm{d}t = \int_s^k \Re\frac{L'(t,\chi)}{L(t,\chi)}\mathrm{d}t + \int_k^N \Re\frac{L'(t,\chi)}{L(t,\chi)}\mathrm{d}t$. The second integral is a constant as $N \to \infty$, while the derivative of the Riemann integral, $\displaystyle\int_s^k \Re\frac{L'(t,\chi)}{L(t,\chi)}\mathrm{d}t$ is $\Re\left(\frac{L'(t,\chi)}{L(t,\chi)}\right)$ by the fundamental theorem of calculus. The imaginary part can be dealt with likewise. This proves our claim that the derivative of $\log_2 L(s,\chi)$ is $L(s,\chi)$.

Theorem 4.1: $\displaystyle\sum_p \frac{\chi(p)}{p^s}$ remains bounded as $s \to 1+$ for non-trivial Dirichlet character χ.

Proof We can write

$$\log_1\left(\frac{1}{1-\chi(p)/p^s}\right) = \sum_p \frac{\chi(p)}{p^s} + O\left(\sum_p \frac{1}{p^{2s}}\right) = \sum_p \frac{\chi(p)}{p^s} + O(1)$$

If $L(1,\chi) \neq 0$, then $\log_2 L(s,\chi)$ remains bounded as $s \to 1+$ since $\log_2 L(s,\chi)$ is continuous on $(0,\infty)$, so it is bounded on $[1,2]$ which means $\displaystyle\sum_p \frac{\chi(p)}{p^s}$ remains bounded as $s \to 1+$.

Theorem 4.2 (Dirichlet's Theorem): Given $k, l \in \mathbb{N}$ such that $\gcd(k,l) = 1$, there are infinitely many primes in the sequence $\{l + kq\}_{k \in \mathbb{N}}$.

Proof We can write

$$\sum_{p\equiv l \bmod q} \frac{1}{p^s} = \sum_p \frac{\delta_l(p)}{p^s} = \frac{1}{\phi(q)} \sum_\chi \overline{\chi(l)} \sum_p \frac{\chi(p)}{p^s}$$

which can be split into

$$\frac{1}{\phi(q)} \sum_p \frac{\chi_0(p)}{p^s} + \frac{1}{\phi(q)} \sum_{\chi\neq\chi_0} \overline{\chi(l)} \sum_p \frac{\chi(p)}{p^s}$$

Take $s_n \to 1, s_n > 1$.

The sum on right is finite as $s \to 1+$ by Theorem 4.1 but the sum on the left diverges: $\displaystyle\sum_p \frac{1}{p^{s_n}} \geq \sum_{p\leq N} \frac{1}{p^{s_n}}$, so we have $\displaystyle\liminf_{n\to\infty} \sum_p \frac{1}{p^{s_n}} \geq \liminf_{n\to\infty} \sum_{p\leq N} \frac{1}{p^{s_n}} = \sum_{p\leq N} \frac{1}{p}$ and since this holds for every natural number N, we have $\liminf_{n\to\infty} \sum_p \frac{1}{p^{s_n}} \geq \sum_p \frac{1}{p}$ which diverges, proving that $\displaystyle\sum_p \frac{1}{p^s}$ diverges as $s \to 1+$. Now, $\displaystyle\sum_p \frac{\chi_0(p)}{p^s} = \sum_{p \text{ does not divide } q} \frac{\chi_0(p)}{p^s}$(otherwise $\chi_0(p) = 0$) $= \displaystyle\sum_{p \text{ does not divide } q} \frac{1}{p^s}$. Let the sum of reciprocals of primes dividing q be A (it is a real number A since such primes are finite). Then adding, we get

$$\sum_{p \text{ does not divide } q} \frac{1}{p^s} + A = \sum_p \frac{1}{p^s}$$

which thus converges as $s \to 1+$, a contradiction to what we showed above.

If the number of primes in the sequence $l + kq$ were finite, the sum $\displaystyle\lim_{s\to 1+} \sum_{p\equiv l \bmod q} \frac{1}{p^s}$ would be finite as well. Since the sum diverges, there are infinitely many primes in that sequence.

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