Output Stabilization of Military UAV in the Unobservable Case

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Abstract— **The problem of dynamic output stabilization is a very general and important problem in control theory. This problem is completely solved in the case where the system under consideration is strongly observable, i.e. observable whatever the control function applied to the system. On the contrary, there is almost no theoretical result in the case where certain "unobservable inputs" do exist. Moreover, this is the "generic" situation.**

It turns out that the basic kinematic model of a HALE drone (when the only observable quantity is the distance to the target) falls in this bad class of systems. In this paper, we assume that we are given a smooth stabilizing feedback control law (this concept is properly defined in the paper), and we exhaust a class of observer systems that reconstruct asymptotically the full information, in such a way that the fully coupled system "observer-feedback" is (almost) globally asymptotically stable.

We provide some simulation results in the case of a slowly moving target.

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1. INTRODUCTION, KINEMATIC MODEL OF A HALE DRONE

A HALE drone (High Altitude, Long Endurance) is a drone meeting the following kinematic requirements:

- It moves at a fixed altitude, with constant speed (as a first approximation but this is of no importance in the present work)
- It can turn right and left with a certain minimum curvature radius.

Typical use of such drones is within the domain of military oversight or tracking of fixed or moving targets. Typical constants for such a HALE drone are velocity of 50 miles per hour, altitude 3 miles.

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Therefore, the kinematic equations are simply the following:

$$\dot{x} = \cos(\theta), \ \ \dot{y} = \sin(\theta), \ \ \dot{\theta} = u, \qquad -u_{max} \leq u \leq u_{max}, \tag{1}$$

which express that the drone moves on the (x, y) plane, with constant velocity 1, with minimum possible curvature radius r of trajectories being $r = \frac{1}{u\max}$.

This model is inspired from the Dubins model [1]. It has been discussed in a lot of papers, and we just give the elements of bibliography that are absolutely necessary in the present work ([1], [2]).

In a series of previous works [3] [4] [5] [6], we used this model, in the long-time perspective of autonomy of the drones in the context of military missions. The situation to day is quite elementary: drones are just driven from the headquarter by pilots, according to a prescribed flight plan. The main issue, for long flights in military missions is of course fuel consumption. It is why minimum time strategy is in general considered.

One could think that such a minimum time strategy is more or less obvious. This is false, as it has been long known. These issues are discussed in our paper [3]. See also [1], [4], [2]. In particular, the "optimal synthesis" is not even continuous.

In the paper, we address the problem of dynamic output stabilization (to the target) of the drone, considering minimum available continuous observations: the only information accessible by the pilot is the instantaneous distance of the drone to the target. We do not know if this situation makes sense from a practical point of view. However, it leads to a very interesting problem of dynamic-output stabilization, in presence of unobservability.

Although the problem of dynamic output stabilization has been studied a lot, one of the main results being for instance Coron's result [7], the HALE drone provides a nontrivial simple example of (semi-global) output stabilization using a state feedback law coupled with a nonlinear observer. See (among a number of authors) [8] for a general treatment of this problem in the case where no observability problem appears.

Here, we restrict to the case where the stabilizing feedback law is smooth. Such feedbacks do exist, and some explicit ones have been exhausted in our previous papers ([3] for instance). The problem of coupling an observer with the minimum-time strategy is still open. We will shortly discuss this at the end of the paper.

The paper is organized as follows:

- In the next section 2, we present the problem, and several of its reductions, for the purpose of output stabilization using observers, together with our output-stabilization result.
- In section 3, we present a sketch of the proof of this result, which is not at all elementary. We point out some difficulties and open questions. Details of the proof will appear in a forthcoming paper.
- In Section 4 we discuss the problem of the minimum-time strategy, pointing out the main difficulties and suggesting reasonable intermediate strategies.
- In section 5, we show some simulation results, including the case of a moving target. We present our conclusions and perspectives.

2. THE PROBLEM AND ITS REDUCTIONS, STATEMENT OF THE RESULT.

The problem, and the first reduction

We want to stabilize System 1 at the origin, but what does this mean, since the drone has constant velocity (or at least minimum velocity) and therefore is always moving? In fact, in the case of a fixed target, it is required that at the end of the mission the pilot turns around the target achieving a circle of minimum radius $r = \frac{1}{u_{\max}}$, waiting for the order to drop his charge.

To express this requirement, we use a "moving-frame" technique. We define the target (travelled counter-clockwise) $\mathcal{T}$ by:

$$\mathcal{T} = \{(x, y, \theta)/x = r\sin\theta, y = -r\cos\theta\} \tag{2}$$

and we set

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{3}$$

In these new coordinates, System 1 can be rewritten as:

$$\left\{ \frac{d}{dt} \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} u\,\tilde{y} + 1 \\ -u\,\tilde{x} \end{pmatrix} . \right. \tag{4}$$

This system 4 possesses two equilibria for $u = \pm u_{\max}$, namely $e_1 = (0, r)$, $e_2 = (0, -r)$.

They correspond to the target $\mathcal{T}$ being browsed counter-clockwise and clockwise respectively. If u is changed for $-u$, the two equilibria are exchanged so that the set of equilibria is unchanged. It means that we can indifferently consider one among the two equilibria positions.

Changing the $(\tilde{x}, \tilde{y})$ coordinates for $(\bar{x}, \bar{y}) = (\tilde{x}, \tilde{y} + r)$, the stabilization problem to $\mathcal{T}$ is reformulated in these variables as the stabilization problem to the submanifold $\{(\bar{x}, \bar{y}, \theta) | \bar{x} = \bar{y} = 0\}$. Equivalently, we will refer to the convergence of $(\bar{x}, \bar{y})$ to the point $(0, 0)$ of the reduced state space. The reader can easily check that the $(\bar{x}, \bar{y})$ obey the following equations:

$$\left\{ \begin{array}{l} \frac{d\bar{x}}{dt} = u\,\bar{y} + 1 - r\,u \\ \frac{d\bar{y}}{dt} = -u\,\bar{x} \end{array} \right. \tag{5}$$

Now, we shall consider for the three systems 1, 4, 5, the following **"minimum information output"**, i.e. the square distance to the target:

$$\rho^2 = x^2 + y^2 = \tilde{x}^2 + \tilde{y}^2 = \bar{x}^2 + (\bar{y} - r)^2 \tag{6}$$

With this minimum information, it is clear that System 1 is not even weakly observable in the sense of [9]: changing θ for $\theta + \theta_0$ leaves both the system and the observation invariant.

On their side, both systems 4, 5 are weakly observable (they are just the quotient system 1 by the "weak indistinguishability" relation from [9]), but they are not strongly observable in the sense of [8]: for the constant control $u \equiv 0$, $\tilde{x}$ (resp. $\bar{x}$) only can be reconstructed from the observation ρ^2: $\tilde{y}$ can be reconstructed up to sign only.

Therefore, **even after this first reduction**, in which the stabilization concept makes sense, and the system becomes weakly observable, observability problems still remain, and **the general theory of dynamic output stabilization does not apply**.

Remark 1: Besides [9], one can also consult [10], [11], [12] for the general theory of quotienting through unobservability.

Second reduction: embedding into a state-affine system

At this point, another important fact appears: this system 4 (resp. 5) can be embedded into a state-affine one: following [13], its "observation space" is finite-dimensional. Actually, setting $z = (z_1, z_2, z_3)$, $z_1 = x^2 + y^2 = \tilde{x}^2 + \tilde{y}^2$, $z_2 = \tilde{x}$, $z_3 = \tilde{y}$, and denoting the output by s, we get the following (state affine in the sense of [13]) system:

$$\begin{array}{l} \dot{z} = Az + uBz + b, \\ s = Cz,\ u \in [-u_{\max}, u_{\max}] \end{array} \tag{7}$$

with $A = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$.

Statement of our main result

The first very natural idea that comes to mind when facing such an output stabilization problem is the use of a classical Kalman filter or a Kalman-like observer, as in [8] for instance, but in this situation, it is seems that it is not a good idea: the system is not strongly observable, and hence it is not guaranteed that the Riccati matrix will remain invertible.

Moreover in this case, we will exhibit a simpler Luenberger-type observer that does the job, and we will prove the following theorem:

Theorem 1: For any smooth stabilizing feedback at the target (the origin of $\mathbb{R}^2$), for systems 4, 5, there is a Luenberger-type observer for system 7, such that the coupled closed loop system is asymptotically stable at the target, with an arbitrarily large basin of attraction..

Remark 2: 1. We were not able to prove that the coupling with a Kalman filter works. We were also unable up to now, to prove any general result of dynamic output stabilization for bilinear systems **that are not strongly observable** using Kalman-like observers. We claim that such a result is a very important challenge.

2. In the paper [3], we have exhausted a smooth stabilizing feedback control law for system 5 that may be used in the applications of our theorem, see Section 5.

3. It would be interesting to analyze the behavior of the coupling of the observer with the time optimal synthesis, that has been computed in [3] (see also [4]). Note that the minimum time optimal synthesis is not smooth (it is not even continuous).

4. See [5] for a similar example of a dynamic-output stabilization problem, where moreover the target point is not observable.

5. It is been long known (see for instance [14]) that global feedback stabilization plus strong observability does not in general imply the possibility of global dynamic output stabilization. Here, the situation is even worse: the system 4 is not even strongly observable. However, we obtain semi-global output stabilization.

3. Sketch of the proof of the main result

The Luenberger-type observer

Our observer is the following standard Luenberger-type observer for System 7:

$$\frac{d\hat{x}}{dt} = A\hat{x} + u\, B\hat{x} + b - K(C\hat{x} - s) \tag{8}$$

with $K^{'} = (\alpha, 2, 0)$ for some arbitrary $\alpha > 0$. Here $\hat{x}$ stands for the estimate of z in 7.

Therefore, the coupled system feedback-observer is (for system 4):

$$\begin{aligned} \frac{d\hat{x}}{dt} &= A\hat{x} + u(\hat{x})\, B\hat{x} + b - K(C\hat{x} - z_2^2 - z_3^2), \quad (9)\\ \frac{dz_2}{dt} &= z_3 u(\hat{x}) + 1,\\ \frac{dz_3}{dt} &= -z_2 u(\hat{x}). \end{aligned}$$

This system 9 meets also:

$$\begin{aligned} \frac{d\hat{x}}{dt} &= A\hat{x} + u(\hat{x})\, B\hat{x} + b - KC(\hat{x} - z), \quad (10)\\ \frac{dz}{dt} &= Az + u(\hat{x})\, Bz + b, \end{aligned}$$

In which the state z is not arbitrary: it lives inside the (invariant) manifold $\mathcal{Z} = \{z |\, z_1 = z_2^2 + z_3^2\}$.

Nevertheless, the solutions of 9 satisfy, in the "error-estimate" coordinates:

$$\begin{aligned} \frac{d\varepsilon}{dt} &= (A + u(\hat{x})\, B - KC)\varepsilon, \quad (11)\\ \frac{d\hat{x}_2}{dt} &= \hat{x}_3 u(\hat{x}) + 1 - 2C\varepsilon,\\ \frac{d\hat{x}_3}{dt} &= -\hat{x}_2 u(\hat{x}), \end{aligned}$$

where ε is the estimation error, $\varepsilon = \hat{x} - z$, and the function $u(.)$ is the stabilizing feedback law. In particular one could use the feedback law proposed in [3].

The question, in the remaining of this paper, is the stability of this system at $P^* \in \mathbb{R}^5$, where P^* is the point of coordinates $\{\varepsilon = 0, \hat{x}_2 = 0, \hat{x}_3 = -r\}$ corresponding to the control $u = u_{\max} = \frac{1}{r}$.

The proof of the (semi) global asymptotic stability of this coupled system 11 is in three steps: proof of local asymptotic stability, proof that bounded trajectories go to the target, proof that all trajectories starting in a given compact set are bounded.

Local asymptotic stability

This is more or less standard.

-At the target point $\{\varepsilon = 0, \hat{x}_2 = 0, \hat{x}_3 = -r\}$ the linearized system is lower triangular, the eigenvalues relative to the ε-part being all with negative real part (as can be easily checked, with $u(\hat{x}) = -u_{\max}$).

-the diagonal block of the linearization of the $\hat{x}$ part has some eigenvalues with strictly negative real part, but also may be zero real part, depending on the fact that the closed loop feedback system

$$\begin{aligned} \frac{d\hat{x}_2}{dt} &= \hat{x}_3 u(\hat{x}) + 1,\\ \frac{d\hat{x}_3}{dt} &= -\hat{x}_2 u(\hat{x}), \end{aligned}$$

is asymptotically stable without or with a center manifold $\mathcal{C}$.

-If $\mathcal{C}$ is nonempty, $\mathcal{C}$ is also a (stable) center manifold for the coupled system 11, since it is an invariant manifold inside the invariant manifold $\{\varepsilon = 0\}$ and all other eigenvalues have strictly negative real part.

Bounded trajectories converge to the target

First, along a bounded trajectory $(\varepsilon(t), \hat{x}(t))$ of System 11, $C\varepsilon(t)$ tends to zero:

$\frac{1}{2}\frac{d}{dt}(||\varepsilon||^2) = \varepsilon' A\varepsilon - \varepsilon' KC\varepsilon = -\alpha(\varepsilon_1)^2$. Therefore, $C\varepsilon(t) = \varepsilon_1(t)$ is a square summable function over $\mathbb{R}+$. Moreover, $C\varepsilon(t)$ has bounded derivative by 11 and by the fact that $|u(\hat{x})| \leq u_{\max}$. An $\mathbb{L}^2$ function with bounded derivative tends to zero. Looking at the $\hat{x}$-equation in 11, we see that in the ω-limit set Ω of the trajectory $(\varepsilon(t), \hat{x}_2(t), \hat{x}_3(t))$ we have

$$\frac{d\hat{x}_2}{dt} = \hat{x}_3 u(\hat{x}) + 1,$$
$$\frac{d\hat{x}_3}{dt} = -\hat{x}_2 u(\hat{x}),$$

This is the equation of the feedback system, which is globally asymptotically stable by assumption. Hence, in Ω (by the general fact of invariance of the ω-limit set), $\hat{x} \equiv (0, 0, r)$ and $u(\hat{x}) = -u_{\max}$. Now, plugging $u(\hat{x}) = -u_{\max}$ in the equation of ε, we see that $\varepsilon = 0$ in Ω. Therefore, the trajectory $(\varepsilon(t), \hat{x}(t))$ enters in finite time in the basin of attraction of P^*.

All semi-trajectories are bounded

This is the hardest point. We just sketch the proof.

As shown in section 3, we have:

$$\frac{1}{2}\frac{d}{dt}(||\varepsilon||^2) = \varepsilon' A\varepsilon - \varepsilon' KC\varepsilon = -\alpha(\varepsilon_1)^2$$

Therefore, ε is bounded, and $C\varepsilon(t) = \varepsilon_1(t)$ tends to zero. But, $\dot{\varepsilon}_1 = -\alpha\varepsilon_1 - 2\varepsilon_2$. Hence

$$\varepsilon_1(t) = e^{-\alpha t}\varepsilon_1(0) - 2\int_0^t e^{-\alpha(t-s)}\varepsilon_2(s)ds \qquad (12)$$

and

$$|\varepsilon_1(t)| \leq e^{-\alpha t}|\varepsilon_1(0)| + 2\int_0^t e^{-\alpha(t-s)}||\varepsilon(0)||ds$$
$$\leq e^{-\alpha t}|\varepsilon_1(0)| + \frac{2||\varepsilon(0)||}{\alpha}(1 - e^{-\alpha t}).$$

Therefore,

Claim 1: for $||\varepsilon(0)|| \leq A$ ($A > 0$, an arbitrary constant) $|\varepsilon_1(t)|$ can be made arbitrarily small in arbitrary short time, by increasing α.

Now, let us consider the equation of $\hat{x}$ in 11:

$$\frac{d\hat{x}_2}{dt} = \hat{x}_3 u(\hat{x}) + 1 - 2\varepsilon_1, \qquad (13)$$
$$\frac{d\hat{x}_3}{dt} = -\hat{x}_2 u(\hat{x}), \qquad (14)$$

Due to the claim, and due to inverse Lyapunov's theorems (see [15], [16] for instance), it is possible to prove that we can chose α large enough for the state $(\tilde{x}_1, \tilde{x}2)$ remains bounded, when starting from a prescribed arbitrarily large compact set. The boundedness of the state of the fully coupled system follows.

Remark 3: The main difficulty in the case of the coupling with a Kalman-filter-type observer is also here. And it is a challenging question. Actually, considering bounded trajectories only for the coupled system, the two other steps in the proof can be adapted. The same holds in the general situation of any state-affine system, provided that the target point is observable. The case of a non-observable target point looks extremely delicate, and is also the purpose of our current research.

4. The coupling with the minimum time strategy

Computing the minimum time strategy for the reduced system 4 is not that easy ([3] in the case of constant velocity and [4] in the varying velocity case). However, it is less difficult than computing the point to point minimum-time strategy for System 1, a beautiful classical result, see [1] and [2] for instance.

In fact, the problem of constructing the time optimal synthesis for 2-dimensional systems is completely solved (at least via some iterative formal algorithm) in [17]. We used this method to get our result. It is an extremely interesting exercise, since more or less all possible generic singularities occur in the computations. The phase portrait of the optimal synthesis is drawn on figure 1. See [3] for details.

Due to the nonsmoothness of the synthesis, our main result Theorem 1 does not apply, therefore the following natural questions come to mind:

- What happens when coupling this time-optimal synthesis with our observer? In particular in presence of noisy measurement?
- Obviously, starting far away from the target, most of the time the optimal trajectory is a straight line. It would be interesting to construct a smooth feedback with this feature (at least approximately).

These two questions are the purpose of our current research.

5. Results, conclusion and perspectives

We show two simulations in which we used the smooth stabilizing feedback control law from [3], relative to System 4. The first simulation is with large α ($\alpha = 30$), the second with $\alpha = 0.1$.

On the left side of the figures, the evolution of the drone 1 on the plane is represented, while on the right side, one can see the trajectory of the reduced system 4, together with the evolution of the corresponding state estimate with our observer.

The figure 4 shows a motion of the drone with a slowly moving target.

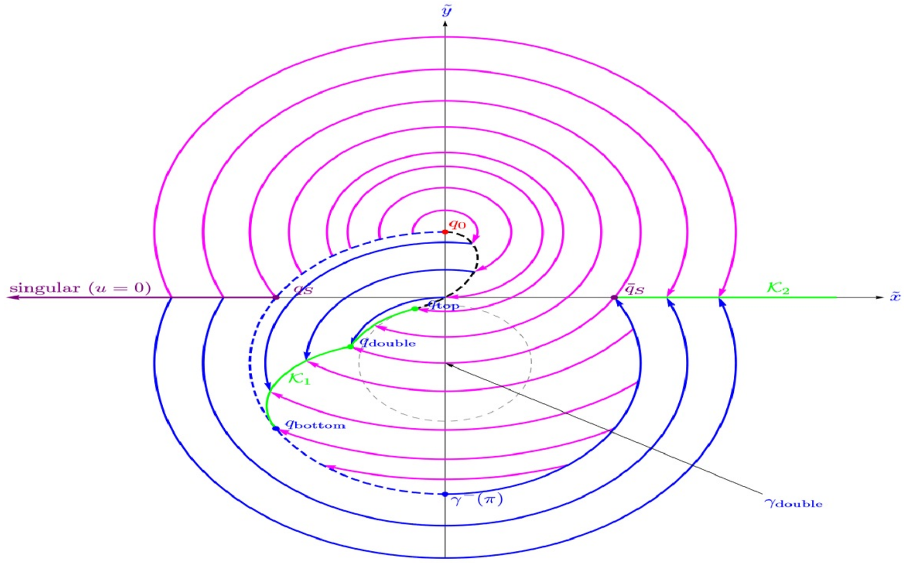

Figure 1. **The time optimal synthesis.**

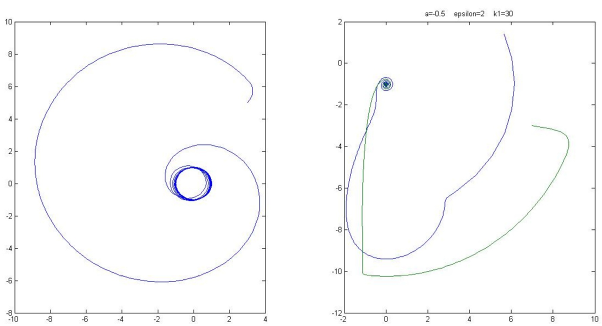

Figure 2. **Trajectory of the drone with alpha=30.**

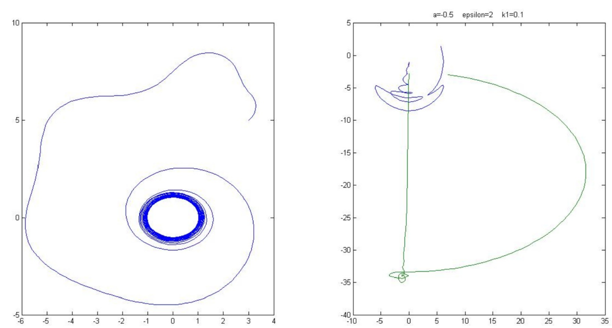

Figure 3. **Trajectory of the drone with alpha=0.1**

To finish, we would like to repeat that we consider as a challenge the question of practical output stabilization of control systems, in the unobservable case. Here we have treated a case where the target point is an observable point, but we refer to [5] for a case where the target control makes the system unobservable. It is particularly challenging to consider, for unobservable bilinear or bilinearizable systems, the coupling of a stabilizing feedback law with a Kalman-type observer. Up to now, we were not able to derive any (global or semi-global) result. Although, a local result is easy to prove provided that the target point is observable.

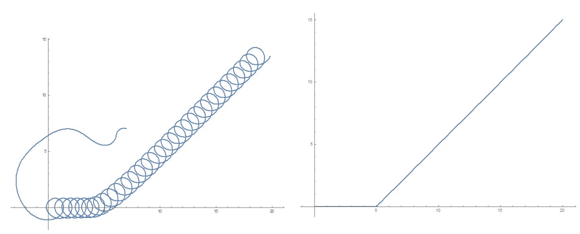

Figure 4. **Trajectory with a slowly moving target**

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