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Thèse de doctorat

Singularités en géométrie sous-riemannienne

Singularities in sub-Riemannian geometry

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Résumé

On étudie le lien entre la géométrie des variétés sous-riemanniennes et des singularités qui y apparaissent. Les approches employées proviennent de la théorie géométrique du contrôle et des méthodes perturbatives.

Les théorèmes de prolongement de Whitney assurent l'existence de prolongements réguliers globaux à des jets d'applications définis sur des fermés sous la condition minimale de cohérence des développements de Taylor. Nous étudions ici le cas des courbes horizontales de classe C^1 dans les variétés sous-riemanniennes. La cohérence des jets tient en l'uniformité d'une dérivabilité de Pansu généralisée aux courbes horizontales dans les variétés par des méthodes d'estimations uniformes de la distance. Si la variété est supposée équirégulière, on obtient que la propriété de prolongement est vérifiée sous une hypothèse de non-singularité de l'application point-final du groupe de Carnot obtenu par approximation nilpotente nommée pliabilité forte. On peut étendre ce résultat à des variétés quelconques via désingularisation, ce qui prouve le théorème de Whitney C^1 horizontal pour toutes les variétés sous-riemanniennes de pas 2. Cette analyse trouve une application avec les théorèmes d'approximation de Lusin et la 1-rectifiabilité.

On étudie la géométrie de la caustique de l'appplication exponentielle des variétés sous-riemanniennes de contact de dimension au moins 5. On utilise des méthodes perturbatives similares aux cas 3D et quasi contact pour obtenir des développements asymptotiques du temps conjugué, lié à la longueur des courbes localement minimisantes perdant leur optimalité dans un petit voisinage de leur point de départ. Cette méthode permet d'entrevoir que le temps conjugué de l'approximation nilpotente est une approximation du temps conjugué. Par rapport au classique cas 3D, on observe grâce à cette analyse des comportments originaux que nous décrivons. Ces différences peuvent notamment être imputées à la présence de paramètres dans le tangent métrique des variétés de contact à partir de la dimension 5.

La structure hamiltonienne de l'équation des géodésiques garantit la lagrangiennité des singularités de l'exponentielle. En s'appuyant sur notre étude de son lieu singulier aux points génériques, cela permet de prouver la stabilité et de classifier les points de la caustique sous-riemannienne parmi les singularités lagrangiennes stables dans le cas spécifique de la dimension 5

Nous proposons une construction permettant d'identifier un champ de direction en tout point avec la droite bissectant un couple de champs de vecteurs sur une surface munie d'une structure conforme. La topologie des couples de champs de vecteurs apparaît comme naturelle pour étudier la stabilité structurelle des singularités de champs de directions. Les singularités du champ de direction ainsi obtenues coïncident avec les zéros des champs de vecteurs et ont génériquement un indice $\pm 1/2$. Les singularités génériques stables de ces champs sont de plus topologiquement équivalentes aux singularités darbouxiennes classiques des ombilics des directions de courbure principale.

Abstract

We study the link between the geometry of sub-Riemannian manifolds and singularities that appear in this context. The favored approaches belong to geometric control theory and perturbative methods.

Whitney extension theorems ensure the existence of global regular extensions of jets of maps defined on closed subsets, under a compatibility assumption with Taylor expansions. We focus here on the case of C^1 horizontal curves in sub-Riemannian manifolds. The compatibility hypothesis takes the form of a uniform Pansu differentiability, generalized to horizontal curves in sub-Riemannian manifolds via distance estimates. If the manifold is assumed to be equiregular, we observe that the extension property holds if the endpoint map in the Carnot group obtained by nilpotent approximation is not too singular. Via desingularization, we can go beyond the equiregular assumption in some cases. As a result, we prove that the C^1 horizontal Whitney extension theorem holds for all step-2 sub-Riemannian manifolds. This analysis finds an application with Lusin approximation theorems and 1-countable rectifiability.

We study the geometry of the caustic of the exponential map of contact sub-Riemannian manifolds of dimension greater or equal to 5. We use perturbative methods similar to the 3D contact and 4D quasi-contact cases to obtain asymptotic expansions of the conjugate time, which relates to the length of locally-length-minimizing curves losing optimality in a small neighborhood of their starting point. This allows to prove that the conjugate time of the nilpotent approximation is an approximation of the conjugate time. In regard of the classic 3D case, we obtain new original behaviors that we describe. These differences can be attributed to the presence of parameters in the metric tangent of contact manifolds in dimension strictly more than 3.

The Hamiltonian structure of the geodesic equation implies that singularities of the exponential are also Lagrangian. Relying on our study of the conjugate locus of the exponential at generic points, we can prove the stability and classify the singularities of the sub-Riemannian caustic among stable Lagrangian singularities in the specific case of dimension 5.

We propose a construction allowing to identify a line field at any point with the line bisecting a pair of vector fields on a surface endowed with a conformal structure. The topology of pairs of vector fields then appears natural to study the structural stability of singularities of line fields. Singularities of line fields obtained via this construction generically have index $\pm 1/2$ and coincide with the zeros of the pairs vector fields they bisect. Furthermore, stable generic singularities of these line fields are topologically equivalent to the classical Darbouxian umbilic singularities of lines of principal curvature.

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Chapitre 1

Introduction

On se propose ici de présenter les résultats développés dans la suite de cette thèse où nous explorons le lien entre la géométrie sous-riemannienne et la théorie des singularités. Après de brefs préliminaires généraux en section 1.1, nous nous consacrons aux contributions qui sont apportées dans chaque chapitre.

Nous présentons en section 1.2 le sujet des prolongements de Whitney sous-riemanniens qui font l'objet du chapitre 3. La section 1.3 constitue une introduction aux deux chapitres 4 et 5. Le premier offre une approche générale sur la géométrie du lieu conjugué sous-riemannien de contact, et le second est une application de ces résultats à l'étude de la stabilité de la caustique sous-riemannienne dans le cas particulier de la dimension 5. Ce chapitre se clôt par la section 1.4 sur les motivations et les techniques développées dans le cadre de la modélisation des singularités de champs de directions, que l'on peut retrouver en détail dans le chapitre 6 final.

Contents

1.1 Préliminaires de géométrie sous-riemannienne

Nous débutons ici par des rappels généraux concernant la géométrie sous-riemannienne. Les notions essentielles à la lecture des différents chapitres y sont résumées en soulignant leur rôle dans la suite de cette thèse. Pour plus de détails, nous renvoyons le lecteur à quelques ouvrages généraux de géométrie sous-riemannienne [ABB16, BBS16, Mon06, Rif14], ainsi qu'aux références citées dans les sections qui suivent.

1.1.1 Variétés sous-riemanniennes

Premières notions. On donne ici la définition de variété sous-riemannienne présente dans [ABB16].

Définition 1.1.1. Soit M une variété différentielle lisse et connexe. Une *structure sous-riemannienne sur* M est la donnée d'un couple (U, f) tel que

(i) U est un fibré euclidien de base M, *i.e.*, pour tout $q \in M$, la fibre U_q est un espace vectoriel muni d'un produit scalaire $(\cdot \mid \cdot)_q$ lisse par rapport à q. En particulier, la dimension de U_q ne dépend pas de q.

(ii) L'application $f : U \to TM$ est un morphisme de fibrés vectoriels, *i.e.*, f est linéaire sur les fibres et le diagramme

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[dr, "\pi_U"'] & TM \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

est commutatif (avec $\pi_U : U \to M$ et $\pi : TM \to M$ les projections canoniques).

(iii) L'ensemble des *champs de vecteurs horizontaux* $\Delta = \{f \circ \sigma \mid \sigma : M \to U \text{ section lisse}\}$ est une famille de champs de vecteurs génératrice par crochets de Lie, *i.e.*, l'algèbre de Lie engendrée par Δ coïncide avec T_qM en tout point $q \in M$.

On appelle alors *variété sous-riemannienne* un triplet (M, U, f) tel que M soit une variété lisse munie de la structure sous-riemannienne (U, f). La *distribution* de cette variété est la famille de sous-espaces

$$(\Delta_q)_{q \in M} \text{ où } \Delta_q = f(U_q) \subset T_qM.$$

La dimension de Δ_q est appellée *rang de la distribution en q.*

Cette définition générale nous est notamment utile lors de l'introduction d'applications entre variétés sous-riemanniennes (pour définir des relations du type projection et relèvements par exemple, cf chapitre 3). Toutefois, quand ce n'est pas ambigu, on préfère désigner la variété (M, U, f) par (M, Δ, g), où g est la forme bilinéaire définie ci-dessous.

Définition 1.1.2. Pour tout $q \in M$, soit g_q la forme bilinéaire sur Δ_q définie par identité de polarisation à partir de sa valeur sur la diagonale : pour $v \in \Delta_q$

$$g_q(v, v) = \inf\{(u \mid u)_q \mid f(q, u) = (q, v), (q, u) \in U\}.$$

Une courbe lipschitzienne $\gamma : I \to M$ est dite *horizontale* s'il existe une application $u : I \to U$ mesurable et essentiellement bornée telle que $\pi_U(u) = \gamma$ et $\dot{\gamma}(t) = f(\gamma(t), u(t))$ pour presque tout $t \in I$. On appelle la fonction u le *contrôle* associé à γ. On définit la longueur $l(\gamma)$ d'une telle courbe horizontale γ par

$$l(\gamma) = \int_I g(\dot{\gamma}(t), \dot{\gamma}(t))^{1/2} \, \mathrm{d}t.$$

Cette longueur nous permet de définir la *distance de Carnot-Caratheodory entre deux points* $q_0, q_1 \in M$:

$$\mathrm{d}_{\mathrm{SR}}(q_0, q_1) = \inf \{l(\gamma) \mid \gamma : [0,1] \to M \text{ horizontale}, \gamma(0) = q_0, \gamma(1) = q_1\}.$$

Le théorème de Chow-Raschevskii affirme que deux points de M sont toujours reliés par une courbe horizontale de longueur finie et que $(M, \mathrm{d}_{\mathrm{SR}})$ est un espace métrique partageant la topologie de M (en tant que variété différentielle). On peut noter également que deux points suffisamment proches d'une variété sous-riemannienne quelconque sont toujours reliés par une courbe minimisante. Ce résultat s'étend à tout couple de points si la variété est supposée complète (cf [ABB16, Théorème 3.40]).

Courbes minimisantes.

Définition 1.1.3. Soit (M, U, f) une variété sous-riemannienne et Ω un ouvert de M. Une *base orthonormée de la distribution sur* Ω est une famille de champs de vecteurs $(X_1, \ldots, X_m)$ tels qu'il existe une base orthonormée lisse $(e_1, \ldots, e_m)$ du fibré euclidien sur $\pi^{-1}(\Omega)$ satisfaisant pour tout $q \in \Omega$ et $1 \leq j \leq m$

$$X_j(q) = f(q, e_j).$$

Grâce au choix d'une base orthonormée $(X_1, \ldots, X_m)$, il est possible de décrire une courbe horizontale $\gamma : [0,1] \to \Omega$ comme une solution de l'équation différentielle ordinaire

$$\dot{\gamma}(t) = \sum_{i=1}^{m} u_i(t) X_i(\gamma(t)) \qquad \text{p.p. } t \in (0,1), \tag{1.1}$$

où le contrôle u satisfait la décomposition $\sum_{j=1}^m u_j e_j$ dans la base $(e_j)_{1\leq j\leq m}$. On identifie alors l'ensemble des contrôles de courbes horizontales avec un ensemble $\mathcal{U} \subset L^\infty((0,1),\mathbb{R}^m)$ correspondant à leur écriture dans cette base. On peut alors définir l'*application point-final en* q, $E_q : \mathcal{U} \to M$, qui associe à $u \in \mathcal{U}$ le point $\gamma_u(1)$, où γ_u est la solution de (1.1) avec condition initiale $\gamma(0) = q$.

Un des intérêts de l'application point-final est qu'elle permet la description des courbes minimisant la distance comme les solutions d'un problème d'optimisation sous contraintes. Commençons par remarquer que si la courbe $\gamma : [0,1] \to \Omega$ a une vitesse constante et minimise la distance entre $\gamma(0) = q_0$ et $\gamma(1) = q_1$, elle minimise également l'énergie sous-riemannienne

$$\int_0^1 g(\dot\gamma(t), \dot\gamma(t))\,\mathrm{d}t.$$

Lorsque l'on considère une courbe $\gamma_u : [0,1] \to M$ solution de (1.1) avec condition initiale $\gamma_u(0) = q_0$, son énergie s'exprime alors en fonction de la norme L^2 de son contrôle (la base $(X_1, \ldots, X_m)$ étant orthonormée)

$$\int_0^1 g(\dot\gamma_u(t), \dot\gamma_u(t))\,\mathrm{d}t = \|u\|_{L^2}^2 =: J(u).$$

La courbe minimisant la distance entre q_0 et q_1 devient alors la courbe dont le contrôle est solution du problème d'optimisation

$$\min\{J(u) \mid u \in \mathcal{U}, E_{q_0}(u) = q_1\}.$$

Par application du théorème des multiplicateurs de Lagrange, il existe $(\lambda, \mu) \in T_{q_1}^* M \times \mathbb{R}$ non nul tel qu'à l'optimal $\bar u$,

$$\lambda \cdot D_{\bar u} E_{q_0} = \mu D_{\bar u} J.$$

Deux cas de figure peuvent alors se produire. Les solutions $\bar u$ pour lesquelles $\mu = 0$ correspondent à des points singuliers de E_{q_0} et cela nous en apprend peu sur les courbes minimisantes. En revanche, les solutions $\bar u$ pour lesquelles μ est non nul mènent à des systèmes d'équations différentielles ordinaires dont les courbes minimisantes sont solution. Il s'agit d'une forme sous-riemannienne de l'équation des géodésiques usuelle.

Trajectoires hamiltoniennes. Il nous est très utile d'introduire le point de vue hamiltonien pour en dire plus sur le problème des courbes minimisantes. Soit

$$\begin{array}{rccc} H: & T^*M & \longrightarrow & M \\ & (p,q) & \longmapsto & \dfrac{1}{2} \displaystyle\max_{v \in \Delta_q \setminus \{0\}} \dfrac{\langle p, v\rangle^2}{g_q(v,v)} \end{array}$$

le hamiltonien sous-riemannien, dont on note $\vec{H}$ le champ hamiltonien associé. Ce hamiltonien contient une grande part de l'information de la structure sous-riemannienne et il est particulièrement lié à la géométrie des courbes minimisantes.

En effet, l'analyse effectuée sur les courbes minimisantes donne l'analogue sous-riemannien du Principe du maximum de Pontryagin, qui peut être exprimé ici sous forme hamiltonienne.

Théorème 1.1.4 (Principe du maximum de Pontryagin). *Soit $\gamma : [0,1] \to M$ une courbe horizontale à vitesse constante minimisant la distance entre $\gamma(0)$ et $\gamma(1)$.*

Il existe une courbe lipschitzienne $\lambda : [0,1] \to T^*M$, telle que $\lambda(t) \in T^*_{\gamma(t)}M \setminus \{0\}$ pour presque tout $t \in [0,1]$, et l'un des deux cas suivants est vérifié :

(N) *Soit $H(\lambda(t)) \neq 0$ et $\dot{\lambda}(t) = \vec{H}(\lambda(t))$ pour tout $t \in [0,1]$.*

(A) *Soit $H(\lambda(t)) = 0$ pour tout $t \in [0,1]$.*

Les courbes minimisantes correspondant au cas **(N)** sont appelées extrémales normales, alors que celles correspondant au cas **(A)** sont appelées extrémales anormales.

Les extrémales normales jouent pour nous un rôle principalement dans le chapitre 4 (voir également section 1.3), où nous utilisons le principe du maximum de Pontryagin pour déterminer l'équation des géodésiques. En effet, lorsque l'on munit un ouvert Ω de M d'une base orthonormée de la distribution $(X_1, \ldots, X_m)$, le hamiltonien sous-riemannien prend la forme

$$H(p,q) = \frac{1}{2} \sum_{i=1}^{m} \langle p, X_i(q) \rangle^2.$$

Si de plus on munit T^*M de coordonnées canoniques $(p,q) : T^*M \to \mathbb{R}^{2d}$, la courbe $t \mapsto \lambda(t) = (p(t), q(t))$ est solution du système hamiltonien

$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p} = \displaystyle\sum_{i=1}^{m} {}^t p \, X_i(q) \; X_i(q), \\ \dot{p} = -\dfrac{\partial H}{\partial q} = -\displaystyle\sum_{i=1}^{m} {}^t p \, X_i(q) \; {}^t p \, D_q X_i(q). \end{cases}$$

Les courbes anormales en revanche jouent un rôle dans le Chapitre 3, où elles interviennent en tant que valeurs singulières de l'application point-final. En effet, les courbes singulières sont potentiellement des obstructions aux prolongements de Whitney (voir Section 1.2). Le principe du maximum ne permet pas d'en apprendre davantage sur les courbes anormales. Toutefois, la théorie des conditions d'optimalité d'ordre 2 (cf [AS96, AS04]), ainsi nommées par opposition au principe du maximum, une condition d'optimalité d'ordre 1, nous permet d'étudier le cas des courbes singulières plus précisément.

1.1.2 Approximation nilpotente

Groupes de Carnot. L'approximation nilpotente est un des outils centraux des méthodes employées dans cette thèse. Pour la présenter, il nous faut considérer une classe de variétés sous-riemanniennes y jouant un rôle central.

Définition 1.1.5. Soit $\mathfrak{g}$ une algèbre de Lie. Pour tout $i \in \mathbb{N}^*$, on note $\mathfrak{g}^1 = \mathfrak{g}$ et $\mathfrak{g}^{i+1} = [\mathfrak{g}, \mathfrak{g}^i]$. L'algèbre de Lie $\mathfrak{g}$ est dite *nilpotente* s'il existe $k \in \mathbb{N}^*$ tel que $\mathfrak{g}^k = \{0\}$. Elle est en outre dite *stratifiée* si elle admet une famille $(\mathfrak{h}^k)_{k\in\mathbb{N}^*}$ de sous espaces de $\mathfrak{g}$ tels que

$$\mathfrak{h}^{i+j} = \left[\mathfrak{h}^i, \mathfrak{h}^j\right] \qquad \text{et} \qquad \mathfrak{g}^i = \mathfrak{g}^{i+1} \oplus \mathfrak{h}^i, \qquad \forall i, j \in \mathbb{N}^*.$$

Un *groupe de Carnot* est un groupe de Lie $\mathbb{G}$ simplement connexe dont l'algèbre de Lie $\mathfrak{g}$ des vecteurs invariants à gauche est nilpotente et stratifiée.

Par translation à gauche, on associe à l'espace $\mathfrak{h} = \mathfrak{h}^1$ un fibré vectoriel sur $\mathbb{G}$, générateur par crochets de Lie. Le choix de d'une métrique euclidienne sur $\mathfrak{h}$ qui soit invariante à gauche implique alors le choix d'une structure sous-riemannienne sur $\mathbb{G}$, que l'on qualifie de *structure sous-riemannienne invariante à gauche.*

Les groupes de Carnot munis de structures invariantes à gauche ont un grand intérêt comme modèles d'espaces sous-riemanniens, car à la structure métrique s'ajoutent leurs symétries. En plus de l'invariance à gauche, les groupes de Carnot sont munis d'une dilatation intrinsèque pour laquelle la distance est homogène. En effet pour tout $\lambda \in \mathbb{R}$ on peut définir δ_λ la *dilatation isotrope de coefficient* λ telle que, en notant $\exp : \mathfrak{g} \to \mathbb{G}$ l'exponentielle du groupe de Lie,

$$\begin{array}{rccc} \delta_\lambda : & \mathbb{G} & \longrightarrow & \mathbb{G} \\ & g & \longmapsto & \exp\left(\lambda \exp^{-1}(g)\right). \end{array}$$

Pour un champ de vecteurs horizontal X invariant à gauche, on a $\delta_{\lambda *}X = \lambda X$ et par conséquent il en va de même pour la longueur des courbes horizontales. Ainsi $\mathrm{d_{SR}}\left(\delta_\lambda(g), \delta_\lambda(g')\right) = |\lambda| \mathrm{d_{SR}}\left(g, g'\right)$ pour tout $g, g' \in \mathbb{G}$ et tout $\lambda \in \mathbb{R}$.

Pour nous, les groupes de Carnot occupent une position centrale car d'un point de vue métrique, leur rôle est similaire à celui des espaces euclidiens en géométrie riemannienne, essentiel aux méthodes perturbatives. On va s'intéresser à expliquer ce lien via l'introduction des outils de l'approximation nilpotente.

Coordonnées privilégiées. Soit (M, Δ, g) une variété sous-riemannienne. Pour tout $i \in \mathbb{N}^*$, on note $\Delta^1 = \Delta$ et $\Delta^{i+1} = \Delta^i + [\Delta, \Delta^i]$. L'hypothèse de génération par crochets de Lie implique qu'en tout $q \in M$ il existe un entier $r > 0$ appelé *pas de la distribution en* q tel que $\Delta_q^{r-1} \subsetneq \Delta_q^r = T_qM$. Le vecteur d'entiers croissants $\left(\dim \Delta_q^1, \ldots, \dim \Delta_q^r\right) = (n_1, \ldots, n_r)$ est appelé *vecteur de croissance en* q. Si le vecteur de croissance est constant sur un voisinage de q, le point q est dit *régulier*, et si tous les points de M sont réguliers, alors (M, Δ, g) est dite *équirégulière.*

Le *vecteur des poids de la structure en* q correspond au vecteur croissant $w \in \mathbb{N}^d$ tel que $w_j = s$ si $n_s < j \leq n_{s+1}$ (avec $n_0 = 0$). Pour Ω un ouvert de M, des coordonnées au point $q \in M$, $(x_1, \ldots, x_d) : \Omega \to \mathbb{R}^d$ sont dites *privilégiées* si

$$\sup\left\{s \in \mathbb{R} \mid x_j(q') = O\left(\mathrm{d_{SR}}\left(q, q'\right)^s\right)\right\} = w_j, \qquad 1 \leq j \leq d.$$

De telles coordonnées ont la propriété d'être *linéairement adaptées en* q, c'est-à-dire que pour tout $1 \leq j \leq d$, $\mathrm{d}x_j\left(\Delta_q^{w_j}\right) \neq 0$, alors que $\mathrm{d}x_j\left(\Delta_q^{w_j-1}\right) = 0$.

Dans ces coordonnées, une variété sous-riemannienne "ressemble" à un groupe de Carnot. Elles permettent de décrire le comportement de la métrique sous-riemannienne à petite échelle. En effet on peut déjà donner une expression du théorème "ball-box", le plus célèbre représentant des théorèmes d'estimation de la distance sous-riemannienne. (Sa preuve requiert toutefois l'introduction d'outils plus avancés dérivant des coordonnées adaptées.)

Théorème 1.1.6 (théorème ball-box)**.** *Soit $q \in M$ et soit $\Phi_q : \Omega \to \mathbb{R}^d$ un système de coordonnées privilégiées en q. Il existe $C_q, \varepsilon_q > 0$ tels que toute boule sous-Riemannienne $B_{\mathrm{SR}}(q,\varepsilon)$ de rayon $\varepsilon < \varepsilon_q$ et centrée en q satisfait l'imbrication*

$$\mathrm{Box}(\varepsilon/C_q) \subset \Phi_q\left(B_{\mathrm{SR}}(q,\varepsilon)\right) \subset \mathrm{Box}(C_q\varepsilon),$$

où $\mathrm{Box}(\varepsilon) = [-\varepsilon^{w_1}, \varepsilon^{w_1}] \times \cdots \times [-\varepsilon^{w_d}, \varepsilon^{w_d}]$.

Ces coordonnées permettent également d'introduire les dilatations sur les variétés sous-riemanniennes. Soit $q \in M$ et soit $\Phi_q : \Omega \to \mathbb{R}^d$ un système de coordonnées privilégiées en q. Pour tout $\lambda > 0$, on peut définir $\delta_\lambda^q : \Omega \to \mathbb{R}^d$ la *dilatation d'ordre λ centrée en q*, telle que

$$\delta_\lambda^q \circ \Phi_q^{-1}(x_1, \ldots, x_d) = \left(\lambda^{w_1}x_1, \ldots, \lambda^{w_i}x_i, \ldots, \lambda^{w_d}x_d\right).$$

Approximation nilpotente. Mesurer les effets des dilatations sur la distance est essentiel pour en construire des estimations. Elles permettent immédiatement de définir l'approximation nilpotente d'un champ de vecteurs horizontal.

Proposition 1.1.7. *Soit (M, Δ, g) une variété sous-riemannienne de dimension d. Soit $q \in M$, soit Ω un voisinage ouvert de q et soit $\Phi_q : \Omega \to \mathbb{R}^d$ un système de coordonnées privilégiées en q. Pour tout champ de vecteurs horizontal $X \in \Delta$, il existe un champ de vecteurs $\widehat{X}$ sur $\mathbb{R}^d$ appelé* approximation nilpotente de X en q*, tel que la famille de champs de vecteurs sur $\mathbb{R}^d$*

$$\left(\frac{1}{\lambda}\delta^q_{\lambda *}X\right)_{\lambda>0}$$

converge uniformément sur tout compact vers $\widehat{X}$ lorsque $\lambda \to \infty$.

On remarque qu'avec cette méthode on capture la partie de X qui est homogène d'ordre 1 par rapport à la dilatation, comme les champs de vecteurs horizontaux invariants à gauche d'un groupe de Carnot.

On peut utiliser l'approximation nilpotente des champs de vecteurs horizontaux pour définir l'approximation nilpotente d'une variété sous-riemannienne. En effet soit $(X_1, \ldots, X_m)$

une base orthonormée de (M, Δ, g), et soit $\left(\widehat{X}_1, \ldots, \widehat{X}_m\right)$ une approximation nilpotente de la base orthonormée de (M, Δ, g) en $q \in M$. Soit $\widehat{\Delta}$ le fibré vectoriel de $\mathbb{R}^d$ tel que pour tout $x \in \mathbb{R}^d$, Δ_x est engendré par la famille de vecteurs $\left(\widehat{X}_1(x), \ldots, \widehat{X}_m(x)\right)$, et soit $\widehat{g}$ la métrique sur $\widehat{\Delta}$ pour laquelle $\left(\widehat{X}_1, \ldots, \widehat{X}_m\right)$ est une base orthonormée. Une *approximation nilpotente de* (M, Δ, g) *en* q est alors la variété sous-riemannienne représentée par le triplet $\left(\mathbb{R}^d, \widehat{\Delta}, \widehat{g}\right)$ dont on note la distance $\widehat{\mathrm{d}}_{\mathrm{SR}}$.

Cette construction semble dépendre du choix initial des coordonnées privilégiées, cependant l'intérêt de cette construction peut être résumé ainsi.

Théorème 1.1.8. *Soit* (M, Δ, g) *une variété sous-riemannienne, et* $q \in M$*. Il existe un groupe de Carnot* $\mathbb{G}_q$ *agissant sur* $\mathbb{R}^d$*, dont on note* $\mathbb{H}_q$ *le groupe d'isotropie en* 0 *(i.e.* $g \in \mathbb{H}_q$ *si* $g \cdot 0 = 0$*), et une distance sous-riemannienne* $\bar{d}$ *sur* $\mathbb{G}_q/\mathbb{H}_q$ *tels que toute approximation nilpotente* $\left(\mathbb{R}^d, \widehat{\Delta}, \widehat{g}\right)$ *de* (M, Δ, g) *en* q *est isométrique à* $\left(\mathbb{G}_q/\mathbb{H}_q, \bar{d}\right)$*.*

De plus, si q *est un point régulier, il existe un tel* $\mathbb{G}_q$ *pour lequel* $\mathbb{H}_q$ *est réduit à l'élément neutre.*

Remarque 1.1.9. Il est possible d'obtenir une construction explicite de $\mathbb{G}_q$ (en partie grâce l'approche algorithmique sous-jacente à [Bel96, Jea01, Jea14]). En effet pour $\left(\mathbb{R}^d, \widehat{\Delta}, \widehat{g}\right)$ une approximation nilpotente de (M, Δ, g) en q, le groupe $\mathbb{G}_q$ est engendré par la famille de difféomorphismes $\left(\mathrm{e}^{t\widehat{X}_i}\right)_{1\leq i\leq m}$. De plus, si le point q est régulier, ce groupe correspond au choix pour lequel $\mathbb{H}_q$ est trivial.

L'adjectif "nilpotent" prend plus de sens car la structure de $\left(\mathbb{R}^d, \widehat{\Delta}, \widehat{g}\right)$ est celle d'un quotient de groupes nilpotents (et même de groupes de Carnot), et dans le cas de points réguliers, on a exactement une structure de groupe de Carnot. L'aspect approximation peut être illustré par l'une des multiples estimations de la distance sous-riemannienne.

Théorème 1.1.10 ([Jea14, Theorem 2.2])**.** *Sur un voisinage de* p *dans* M*,*

$$\mathrm{d}_{\mathrm{SR}}(p, q) = \widehat{\mathrm{d}}_{\mathrm{SR}}(p, q)\left(1 + O\left(\widehat{\mathrm{d}}_{\mathrm{SR}}(p, q)\right)\right).$$

Il est bien sûr possible d'être plus précis et des applications de ces méthodes se trouvent en particulier dans les sections 3.2 et 3.5.

Pour finir, il est possible d'observer que l'espace métrique $\left(\mathbb{G}_q/\mathbb{H}_q, \bar{d}\right)$ remplit le rôle d'espace tangent, au sens métrique de Gromov [Bel96]. C'est-à-dire que la famille d'espaces métriques $M_\lambda = (M, \lambda \mathrm{d}_{\mathrm{SR}})$, pointée en q, converge pour la distance de Gromov–Hausdorff vers $\left(\mathbb{G}_q/\mathbb{H}_q, \bar{d}\right)$, pointé en 0, lorsque λ tend vers l'infini. Ceci vaut à $\left(\mathbb{G}_q/\mathbb{H}_q, \bar{d}\right)$ l'appellation de tangent métrique ([Jea14]) ou tangent non-holonome ([ABB16]).

De ce constat nous pouvons tirer un éclairage sur les méthodes employées dans cette thèse. Dans la géométrie riemannienne, le tangent métrique d'une variété de dimension

d est toujours l'espace euclidien de dimension d. En géométrie sous-riemannienne en revanche, il y a une très grande variété d'espace tangents possibles. Si une variété sous-riemannienne n'est pas equirégulière, cela implique qu'il existe deux points de la variété pour lesquels leurs tangents ne sont pas difféomorphes. Mais même sous la contrainte d'equirégularité, les groupes de Carnot pouvant dépendre de paramètres, les tangents métriques en deux points distincts d'une variété equirégulière peuvent ne pas être isométriques.

Dans le contexte des singularités, qu'il s'agisse des singularités spatiales de l'application exponentielle sous-riemannienne ou des singularités des prolongements et de l'application point-final, nous faisons abondamment usage de la structure infinitésimale des variétés sous-riemanniennes. Nous essayons par deux approches d'exploiter notre connaissance du tangent pour l'appliquer à la variété. Dans le cas des singularités de l'exponentielle nous employons des formes normales et des méthodes perturbatives faisant ressortir les similarités de structure entre la variété et son tangent à petite échelle. En revanche, dans le cas des prolongements nous utilisons des estimées de la distance et nous cherchons des propriétés supportant le passage du tangent à la variété.

1.2 Prolongements de Whitney

Les variétés sous-riemanniennes constituent une classe d'espaces métriques à la fois variés et structurés, ce qui en fait de bons modèles d'espaces dans le domaine de la théorie géométrique de la mesure. Cela nous permet notamment d'appliquer à des problèmes métriques des outils provenant du contrôle géométrique et de l'approximation nilpotente. Nous employons cette approche sur le problème des prolongements de Whitney, et nous allons voir comment ce problème trouve une interprétation classique en tant que problème de contrôlabilité au voisinage de courbes singulières.

1.2.1 Prolongements de Whitney de courbes

Les théorèmes de prolongement de Whitney sont une famille de résultats d'analyse dont la version euclidienne est due à Whitney [Whi34]. Pour une application continue sur un fermé $K \subset \mathbb{R}^d$ à valeur dans les jets d'ordre k, $f : K \to \mathrm{Jet}^k(\mathbb{R}^n)$, le théorème de Whitney affirme l'existence d'une fonction C^k coïncidant avec les jets donnés par f sous l'hypothèse que la valeur des jets soit compatible avec les théorèmes de Taylor. C'est cette hypothèse que l'on nomme *condition de Whitney*.

Théorème 1.2.1 (théorème de prolongement de Whitney euclidien C^1)**.** *Soit K une fermé de $\mathbb{R}^n$ et $(f, L) : K \to \mathbb{R}^m \times L(\mathbb{R}^n, \mathbb{R}^m)$ continu. S'il existe $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ tel que*

$\omega(t) = o(t)$ en 0^+ et pour tout $x, y \in K$,

$$|f(y) - f(x) - L(x) \cdot (y - x)|_{\mathbb{R}^m} \leq \omega(|x - y|_{\mathbb{R}^n}) \tag{1.2}$$

alors il existe $F : \mathbb{R}^n \to \mathbb{R}^m$, $F \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, tel que

$$F_{|K} = f \quad \text{et} \quad DF_{|K} = L.$$

Les premières généralisations de ce théorèmes au contexte sous-riemannien concernent les applications à valeurs réelles définies sur les groupes de Carnot. Dans un premier lieu on doit le cas des applications C^1 à valeurs réelles sur le groupe de Heisenberg à Franchi–Serapioni–Serra Cassano [FSSC01], qui fut ensuite généralisé au cas des applications C^k à valeurs réelles sur les groupes de Carnot par Pupyshev–Vodopyanov dans [VP06].

Le cas des applications horizontales C^1 de $\mathbb{R}$ dans les groupes de Carnot est plus tardif et dû à Zimmermann pour le cas Heisenberg [Zim18], et Juillet–Sigalotti pour le cas général [JS17]. On souhaite poursuivre cette démarche en tentant d'étendre ces résultats au cas des courbes horizontales $\gamma : \mathbb{R} \to M$ dans des variétés sous-riemanniennes (M, Δ, g).

Dans les cas qui précèdent, il est nécessaire de d'employer la dérivée de Pansu [Pan89] d'une application horizontale pour généraliser la condition (1.2). En lieu et place d'application linéaire, la dérivée d'une application entre groupes de Carnot est un *homomorphisme de groupe homogène* $L : \mathbb{G}_1 \to \mathbb{G}_2$ entre les deux groupes de Carnot $\mathbb{G}_1$ et $\mathbb{G}_2$, c'est-à-dire un morphisme de groupes tel que $L \circ \delta_\lambda^{\mathbb{G}_1} = \delta_\lambda^{\mathbb{G}_2} \circ L$, avec $\delta_\lambda^{\mathbb{G}_i}$ la dilatation de coefficient $\lambda \in \mathbb{R}$ associée au groupe $\mathbb{G}_i$, $i \in \{1, 2\}$. Ainsi en notant $\mathrm{d}_{\mathbb{G}_i}$ la distance sous riemannienne associée au groupe $\mathbb{G}_i$, $i \in \{1, 2\}$, la condition de Whitney prend la forme suivante (cf [VP06, JS17]) :

Soit K un fermé de $\mathbb{G}_1$ et $f : K \to \mathbb{G}_2$. On suppose qu'il existe une application qui associe à tout $g \in K$ un homomorphisme de groupe homogène $L(g) : \mathbb{G}_1 \to \mathbb{G}_2$. La *condition de Whitney C^1_H* est vérifiée par (f, L) si (f, L) est continu et s'il existe $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ tel que $\omega(t) = o(t)$ en 0^+ et pour tout $g, h \in K$

$$\mathrm{d}_{\mathbb{G}_2}\left(f(h), f(g) \cdot L(g)(g^{-1} \cdot h)\right) \leq \omega(\mathrm{d}_{\mathbb{G}_1}(g, h)). \tag{1.3}$$

La dérivée de Pansu se limite malheureusement aux groupes de Carnot et des généralisation de cette idée sont à apporter (par exemple, cf [Vod07] pour une proposition de généralisation utilisant l'approximation nilpotente). Une observation capitale est que dans le cas où le groupe de Carnot $\mathbb{G}_1$ est juste $\mathbb{R}$ euclidien, un homomorphisme homogène de groupes entre $\mathbb{R}$ et $\mathbb{G}_2$ prend la forme $L(t) = \exp_{\mathbb{G}_2}(tX)$ où X est un vecteur invariant à gauche de $\mathbb{G}_2$.

Pour une courbe horizontale $\gamma : \mathbb{R} \to \mathbb{G}_2$ de classe C^1, la dérivée $\dot\gamma(t)$ est donnée pour tout $t \in \mathbb{R}$ par un champ de vecteurs invariant à gauche $X(t)$. Ainsi la condition (1.3) prend la forme

$$\mathrm{d}_{\mathbb{G}_2}\left(f(t), f(s) \cdot \exp_{\mathbb{G}_2}\left[(t - s)\, X(s)\right]\right) \leq \omega(|t - s|). \tag{1.4}$$

Soit (M, Δ, g) une variété sous-riemannienne de dimension d. Dans le cas des courbes horizontales $\gamma : \mathbb{R} \to M$, on souhaiterait alors remplacer la dérivée de Pansu par le flot d'un champ de vecteurs horizontal. Cependant, contrairement au cas des champs de vecteurs invariants à gauche dans un groupe de Carnot, il n'y a pas d'extension intrinsèque au vecteur $\dot{\gamma}(t) \in T_{\gamma(t)}M$.

La solution que nous choisissons est d'employer une base orthonormée de la distribution pour définir cette extension. Soit $(X_1, \ldots, X_m)$ une base orthonormée de la distribution sur l'ouvert $\Omega \subset M$. Pour tout $q \in \Omega$ et tout $v \in \Delta_q$, il existe $(u_i)_{1 \leq i \leq m}$ tel que $\sum_{i=1}^m u_i X_i(q) = v$. Dans la suite, on note $X_u = \sum_{i=1}^m u_i X_i$. Quitte à restreindre l'ensemble de définition de la courbe, on suppose également que la courbe demeure dans Ω. Par ailleurs, dans le cas d'une variété sous-riemannienne générale (M, U, f) (cf définition 1.1.1), une courbe horizontale $\gamma : \mathbb{R} \to M$ est dite C^1_H s'il existe un contrôle $u : \mathbb{R} \to M$ continu tel que $\gamma = \pi_U(u)$ et $\dot{\gamma} = f(\gamma, u)$.

Définition 1.2.2. Soit K un fermé de $\mathbb{R}$, $(f, L) : K \to M \times TM$ continu tel que $L(t) \in \Delta_{f(t)}$ pour tout $t \in K$. Le couple (f, L) satisfait la *condition de Whitney C^1_H sur K* s'il existe $u : K \to \mathbb{R}^m$ continu et $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ tel que $X_{u(s)}(f(s)) = L(s)$ pour tout $s \in K$, $\omega(t) = o(t)$ en 0^+, et

$$\mathrm{d}_{\mathrm{SR}}\left(f(t), \mathrm{e}^{(t-s)X_{u(s)}} f(s)\right) \leq \omega(|t-s|) \qquad \text{pour tout } t, s \in K.$$

Cette définition semble impliquer que satisfaire la condition de Whitney C^1_H dépend du choix de la base $(X_1, \ldots, X_m)$ et soit une propriété de u et non de L. La majorité du travail effectué dans les sections 3.2.3 et 3.3.1 est consacrée à prouver qu'il s'agit bien d'une propriété vérifiée par le couple (f, L) indépendemment du choix de la base.

On peut également remarquer que pour toute courbe C^1_H $\gamma : \mathbb{R} \to M$, et pour tout fermé K de $\mathbb{R}$, le couple $(\gamma_{|K}, \dot{\gamma}_{|K})$ satisfait la condition de Whitney C^1_H sur K. Cela confirme qu'elle remplit le rôle de condition de compatibilité avec les développements de Taylor. Ceci justifie la définition suivante, où le théorème de Whitney est vu comme une propriété métrique d'une variété sous-riemannienne.

Définition 1.2.3. La *propriété de prolongement C^1_H* est vérifiée par (M, Δ, g) si pour tout fermé $K \subset \mathbb{R}$ et tout couple (f, L) satisfaisant la condition de Whitney C^1_H sur K, il existe une courbe $\gamma : \mathbb{R} \to M$ C^1_H telle que

$$\gamma_{|K} = f, \qquad \dot{\gamma}_{|K} = L.$$

Dans ce problème on bénéficie à la fois du caractère local des prolongements de Whitney et de la topologie particulière de $\mathbb{R}$, deux points qui ne sont pas immédiatement apparents.

En effet, si $t_0 < t_1$ sont deux points de $\mathbb{R}$ et $(f_0, L_0), (f_1, L_1) \in M \times \Delta$ sont deux jets de courbes horizontales, trouver une courbe $\gamma : \mathbb{R} \to M$ horizontale et C^1 telle que

$(\gamma(t_0), \dot\gamma(t_0)) = (f_0, L_0)$ et $(\gamma(t_1), \dot\gamma(t_1)) = (f_1, L_1)$ est un simple problème de contrôlabilité.

Comme le complémentaire d'un fermé de $\mathbb{R}$ est une union dénombrable d'intervalles ouverts, l'existence d'un prolongement continu et horizontal presque partout est obtenue en complétant la courbe sur chaque intervalle. La difficulté est de prouver qu'il existe un tel prolongement qui soit C^1, car la continuité de la dérivée peut être mise en défaut aux points d'accumulation du bord de K, là où ont lieu les collages avec les courbes complétant f.

C'est ainsi que l'approximation nilpotente entre en jeu. Aux points d'accumulation de $f(K)$, le couple (f, L) peut être transposé dans le tangent métrique, et c'est là que nous trouvons des conditions de contrôlabilité suffisamment fortes pour affirmer que la courbe peut admettre un prolongement global C^1.

Afin de pouvoir employer des estimations de la distance sous-riemannienne, nous sommes contraints de nous concentrer sur le cas équirégulier dans un premier temps.

1.2.2 Le cas équirégulier

Le passage à l'approximation nilpotente. On suppose ici que (M, Δ, g) est une variété sous-riemannienne équirégulière (cf section 1.1.2). On rappelle que dans ce cas, la croissance des crochets ne dépend pas de la position et que le tangent métrique à la variété a une structure de groupe de Carnot en tout point.

Puisque la condition de Whitney sur L doit être exprimée dans une base orthonormée $(X_1, \dots, X_m)$ de la distribution mais n'en dépend pas, on suppose dans la suite que cette base est fixée et on parle indifféremment de $L(t) \in \Delta_{f(t)}$ et $u(t) \in \mathbb{R}^m$ tel que $X_{u(t)} = L(t)$, pour tout $t \in K$.

Nous avons besoin ici d'employer des estimées uniformes de la distance sous-riemannienne, c'est pourquoi nous introduisons sur un ouvert $\Omega \subset M$ un *système de coordonnées privilégiées continu*

$$\Phi : (p, q) \longmapsto \Phi_p(q) \in \mathbb{R}^d,$$

une application continue telle que Φ_p est un système de coordonnées privilégiées en p, pour tout $p \in \Omega$.

Avec de telles coordonnées on peut introduire l'équipement habituel de l'approximation nilpotente, cette fois en conservant la continuité par rapport à la position. Pour $p \in \Omega$, on a toujours δ^p_λ la dilatation centrée en p et nous utilisons la convergence uniforme sur tout compact

$$\varepsilon \delta^p_{\frac{1}{\varepsilon}*} X_u \to \widehat{X}^{(p)}_u, \qquad \varepsilon \to 0, \tag{1.5}$$

et des estimées uniformes de la distance pour donner une caractérisation de la condition de Whitney C^1_H. Celle-ci fait le lien entre la condition de Whitney, les points accumulation dans K et l'approximation nilpotente.

Proposition 1.2.4. *Le couple $(f,u) : K \to M \times \mathbb{R}^m$ satisfait la condition de Whitney C^1_H sur K si et seulement si pour tout $l \in K$, pour toutes suites convergentes $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ dans K telles que $a_n < b_n$ et $a_n, b_n \to l \in K$, on a*

$$\lim_{n\to\infty} \delta^{f(a_n)}_{\frac{1}{b_n-a_n}}(f(b_n)) = \mathrm{e}^{\widehat{X}_{u(l)}}(0).$$

Malgré la variation du centre de la dilatation, grâce à (1.5) nous avons rapidement la convergence sur tout compact

$$(b_n - a_n)\delta^{f(a_n)}_{\frac{1}{b_n-a_n}*}X_{u(a_n)} \xrightarrow[n\to\infty]{} \widehat{X}^{(f(l))}_{u(l)}.$$

La condition de Whitney assure (grâce des estimations de la distance) alors que la distance entre

$$\delta^{f(a_n)}_{\frac{1}{b_n-a_n}}(f(b_n)) \quad \text{et} \quad \delta^{f(a_n)}_{\frac{1}{b_n-a_n}}\left(\mathrm{e}^{(b_n-a_n)X_{u(a_n)}}(f(a_n))\right)$$

tend vers 0 lorsque n tend vers l'infini (cf Figure 1.1). Comme

$$\delta^{f(a_n)}_{\frac{1}{b_n-a_n}}\left(\mathrm{e}^{(b_n-a_n)X_{u(a_n)}}(f(a_n))\right) = \mathrm{e}^{(b_n-a_n)\delta^{f(a_n)}_{\frac{1}{b_n-a_n}*}X_{u(a_n)}}$$

on obtient le résultat annoncé.

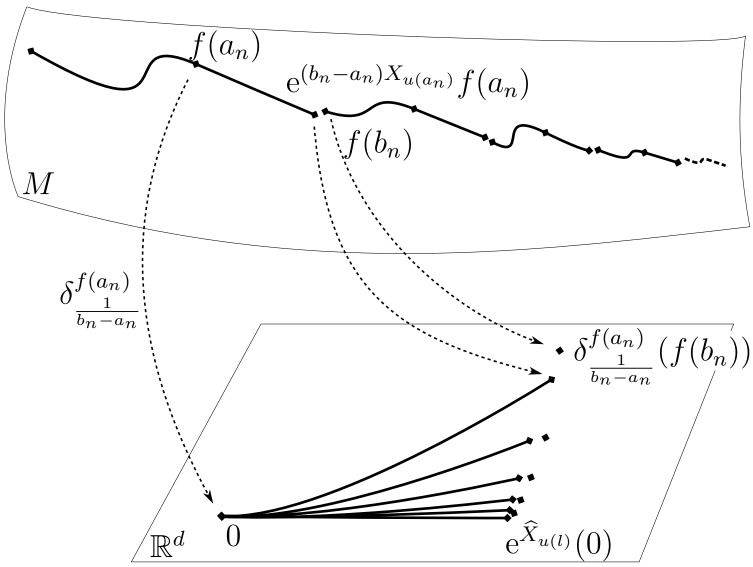


Figure 1.1 – Représentation de l'action des dilatations sur une courbe satisfaisant la condition de Whitney. Une fois dans $\mathbb{R}^d$, les dilatations des courbes de flot convergent vers la courbe droite $\mathrm{e}^{\widehat{X}_{u(l)}}(0)$. La condition de Whitney assure que la distance entre le point final des courbes de flot $\mathrm{e}^{(b_n-a_n)X_{u(a_n)}}f(a_n)$ et leur cible $f(b_n)$ converge suffisamment rapidement vers 0 pour conserver cette convergence après l'éclatement.

On peut tirer une interprétation (attendue) de ce résultat : la condition de Whitney impose aux points de la courbe de s'aligner avec la dérivée à l'échelle infinitésimale, un comportement qui s'observe dans le tangent métrique (encore une fois, cf Figure 1.1).

Les applications point-final. Ces observations justifient de s'intéresser à l'application point-final de l'approximation nilpotente aux points d'accumulation de la courbe. En particulier son ouverture : dans le cas des groupes de Carnot, c'est l'absence d'ouverture de l'application point-final qui permet de construire une courbe satisfaisant la condition de Whitney mais n'admettant pas d'extension C^1_H (cf [JS17]).

Considérons $(\widehat{X}_1^{(q)}, \dots, \widehat{X}_m^{(q)})$ l'approximation nilpotente en q de la base orthonormée et le groupe de Carnot $\mathbb{G}_q$ muni de la structure invariante à gauche qui lui est associée.

Pour tout $u \in L^1([0,1], \mathbb{R}^m)$ on note γ_u la courbe telle que

$$\gamma_u(0) = 0_{\mathbb{G}_q}, \qquad \dot{\gamma} = \widehat{X}_u(\gamma) \text{ p.p.}$$

L'application point-final $\widehat{E}_q : L^1([0,1], \mathbb{R}^m) \to \mathbb{G}_q$ est alors définie par la relation

$$\widehat{E}_q(u) = \gamma_u(1).$$

(Dans ce cas $\widehat{E}_q$ est l'application point-final de $\mathbb{G}_q$ au point $0_{\mathbb{G}_q}$ telle qu'elle a été définie en section 1.1.1.)

Ainsi, la proposition 1.2.4 implique que pour tout $l \in K$, pour toutes suites convergentes $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ dans K telles que $a_n < b_n$ et $a_n, b_n \to l \in K$, on a la convergence

$$\delta^{f(a_n)}_{\frac{1}{b_n - a_n}}(f(b_n)) \to \widehat{E}_{f(l)}(u(l)).$$

Pour tout $n \in \mathbb{N}$ on considère la structure sous riemannienne (Δ_n, g_n) sur $\mathbb{R}^d$ donnée par la base orthonormée

$$\left((b_n - a_n)\delta^{f(a_n)}_{\frac{1}{b_n - a_n}*} X_i\right)_{1 \le i \le m}.$$

En notant E^n son application point-final en 0, on a $E^n \to \widehat{E}_{f(l)}$ uniformément sur tout compact. Par cette transformation, il apparaît que prolonger (f, L) nécessite que l'on puisse trouver pour tout n un contrôle v_n continu sur $[0,1]$ tel que

$$E^n(v_n) = \delta^{f(a_n)}_{\frac{1}{b_n - a_n}}(f(b_n)), \qquad v_n(0) = u(a_n) \quad \text{ et } \quad v_n(1) = u(b_n), \quad \forall n \in \mathbb{N}.$$

Comme $u(a_n), u(b_n) \to u(l)$, $E^n(v_n) \to \widehat{E}_{f(l)}(u(l))$, l'existence d'un prolongement C^1 implique l'existence d'une telle suite de contrôles $(v_n)_{n\in\mathbb{N}}$ convergeant vers $u(l)$.

On en vient à l'intuition suivante : une obstruction à l'existence d'un prolongement serait alors un défaut d'ouverture en 0 de l'application

$$\begin{array}{ccc} \{v \in C^0([0,1], \mathbb{R}^m) \mid v(0) = 0\} & \longrightarrow & \mathbb{G}_q \times \mathbb{R}^m \\ v & \longmapsto & \left(\widehat{E}_q(u(l) + v), v(1)\right). \end{array}$$

1.2.3 La pliabilité forte aux points réguliers

Pliabilité, forte ou non. Dans le cas d'un groupes de Carnot $\mathbb{G}$, il faut observer que la construction de la section précédente est tautologique par la propriété d'homogénéité des champs de vecteurs via dilatation. En effet, pour tout $1 \leq i \leq m$, les éléments de la base orthonormée satisfont

$$X_i = (b_n - a_n)\delta^{f(a_n)}_{\frac{1}{b_n - a_n}*} X_i = \widehat{X}_i, \qquad \forall n \in \mathbb{N},$$

et les applications point-final satisfont $E_{0_\mathbb{G}} = E^n = \widehat{E}$. Ces observations mènent à l'introduction de la notion de pliabilité d'un champ de vecteurs invariant à gauche.

Définition 1.2.5 (Pliabilité [JS17])**.** Soit $\mathbb{G}$ un groupe de Carnot et $(X_1, \ldots, X_m)$ une base orthonormée d'une structure sous-riemannienne invariante à gauche sur $\mathbb{G}$. Le champ de vecteurs invariant à gauche X_{u_0}, $u_0 \in \mathbb{R}^m$, est *pliable* si l'application

$$\begin{array}{ccc} \mathcal{C}_0 & \longrightarrow & \mathbb{G} \times \mathbb{R}^m \\ v & \longmapsto & (E_0(u_0 + v), v(1)) \end{array}$$

est ouverte en 0, avec $\mathcal{C}_0 = \{v \in C^0([0,1], \mathbb{R}^m) \mid v(0) = 0\}$.

On peut observer que l'absence d'ouverture de $E_{0_\mathbb{G}}$ pour certains u_0, c'est-à-dire l'existence de champs de vecteurs non pliables, permet la construction de courbes n'admettant pas de prolongement tout en satisfaisant les hypothèses de la condition de Whitney C^1_H. Les conséquence de cette analyse suivent.

Théorème 1.2.6 (Juillet–Sigalotti [JS17])**.** *La propriété de prolongement C^1_H est satisfaite sur le groupe de Carnot $\mathbb{G}$ si et seulement si tous les champs de vecteurs horizontaux invariants à gauche sont pliables.*

Remarque 1.2.7. Le nom “pliabilité” fait référence à la non-rigidité des courbes. Une courbe $\gamma : [a, b] \to \mathbb{G}$ C^1 horizontale est rigide au sens de Bryant–Hsu [BH93] s'il existe un voisinage $\mathcal{V}$ de γ dans $C^1_H([a, b], \mathbb{G})$ tel que si $\tilde{\gamma} \in \mathcal{V}$, $\tilde{\gamma}(a) = \gamma(a)$ et $\tilde{\gamma}(b) = \gamma(b)$, alors $\tilde{\gamma}$ est une reparamétrisation de γ.

Si X est un champ de vecteur invariant à gauche et que la courbe $t \mapsto \mathrm{e}^{tX}$ est rigide alors X n'est pas pliable. Puisqu'il existe des groupes de Carnot admettant des courbes de cette forme et rigide [GK95], il existe des groupes de Carnot ne satisfaisant la propriété de prolongement C^1_H.

En s'inspirant du cas des groupes de Carnot, nous pouvons proposer une condition suffisante sur $\widehat{E}_q$ pour avoir l'existence des prolongements lorsque (M, Δ, g) est equirégulière. À présent les applications E^n et $\widehat{E}_q$ sont différentes mais nous avons la convergence uniforme sur tout compact de $(E^n)_{n\in\mathbb{N}}$ vers $\widehat{E}_q$. Ce qui est attendu de la condition suffisante est qu'elle soit suffisamment robuste pour assurer l'ouverture des E^n à partir d'un certain rang.

Clairement si $v \mapsto \Big(\widehat{E}_q(u_0+v), v(1)\Big)$ est une submersion à 0, on obtiendra l'ouverture des E^n, mais il existe de très nombreux cas où cela n'est pas vérifié. En s'inspirant des méthodes sur l'ouverture locale de l'application point-final ($F : \mathcal{U} \to M$ est localement ouverte en u si $F(O_u)$ est un voisinage de $F(u)$ pour tout voisinage ouvert $O_u \subset \mathcal{U}$ de u, cf [AS04, Chapitre 20]) il est possible de donner une condition suffisante valable dans une grande variété de cas où $\widehat{E}_q$ est critique en u_0.

Définition 1.2.8 (Pliabilité forte)**.** Le couple $(q, u_0) \in M \times \mathbb{R}^m$ est *fortement pliable* si pour tout $\eta > 0$ il existe $v \in \mathcal{C}_0$, $\|v\|_\infty < \eta$, tel que l'application

$$\begin{array}{rccc} \mathcal{F}_q^{u_0} : & \mathcal{C}_0 & \longrightarrow & \mathbb{G}_q \times \mathbb{R}^m \\ & v & \longmapsto & \Big(\widehat{E}_q(u_0+v), v(1)\Big) \end{array}$$

est une submersion en v et $\mathcal{F}_q^{u_0}(v) = \mathcal{F}_q^{u_0}(0)$.

Pour étudier la forte pliabilité d'un couple (q, u_0), on peut maintenant se tourner vers des méthodes plus classiques de la théorie du contrôle géométrique. On note

$$\begin{array}{rccc} G_q^{u_0} : & L^\infty([0,1], \mathbb{R}^m) & \longrightarrow & \mathbb{G}_q \\ & v & \longmapsto & \widehat{E}_q(u_0+v). \end{array}$$

Il est déjà possible de se restreindre à l'étude de la fonction $G_q^{u_0}$ pour étudier la pliabilité forte.

Lemme 1.2.9. *Le couple (q, u_0) est fortement pliable si et seulement si pour tout $\eta > 0$ il existe $v \in L^\infty([0,1], \mathbb{R}^m)$ tel que $\|v\|_{L^\infty} < \eta$, $G_q^{u_0}(v) = G_q^{u_0}(0)$ et $G_q^{u_0}$ est une submersion en v.*

Comme évoqué précédemment, le cas le plus immédiat est celui pour lequel $G_q^{u_0}$ est une submersion en 0.

Proposition 1.2.10. *Si 0 est une valeur régulière de $G_q^{u_0}$ alors (q, u_0) est fortement pliable.*

Toutefois si 0 n'est pas une valeur régulière de $G_q^{u_0}$, il existe des méthodes classiques pour étudier les valeurs singulières d'une application point-final. Il s'agit maintenant d'étudier des conditions d'ordre 2 sur la hessienne de $G_q^{u_0}$ en 0 (cf [AS04, Chapitre 20]).

Le point de vue de la contrôlabilité permet d'interpréter ces techniques. Si $D_0 G_q^{u_0}$ n'est pas surjectif, il est impossible d'affirmer qu'on puisse couvrir un voisinage de $G_q^{u_0}(0)$ avec des contrôles de petite norme L^∞. On va alors chercher à savoir si la hessienne de

$G_q^{u_0}$ en 0 a une forme qui elle permet de couvrir un voisinage (en quelque sorte par une approximation d'ordre 2 de l'application point-final). Plus précisément, s'il est possible d'assurer que pour tout élément non nul λ de $\mathrm{im} D_0 G_q^{u_0 \perp}$, l'indice (négatif) de la forme quadratique $\lambda \mathrm{Hess}_0 G_q^{u_0}$ sur $\ker D_0 G_q^{u_0}$, est plus grand que le corang de $G_q^{u_0}$ en 0, alors on a la preuve de l'existence de valeurs régulières pour G arbitrairement proches de 0.

Nous pouvons donner des critères assez généraux pour vérifier que cette condition est remplie. Il s'agit de tester si la courbe singulière correspondant à $G_q^{u_0}(0)$, c'est-à-dire $t \mapsto \mathrm{e}^{\widehat{X}_{u_0}}(0)$, n'est pas la projection d'une courbe de Goh ou une courbe de Legendre (faible) (cf [AS96]). Si ce n'est effectivement pas le cas, alors l'indice de $\lambda \mathrm{Hess}_0 G_q^{u_0}$ est $+\infty$ pour tout $\lambda \in \mathrm{im} D_0 G_q^{u_0 \perp}$ non nul. Ces observations mènent au diagramme d'inclusion illustré dans la figure 1.2.

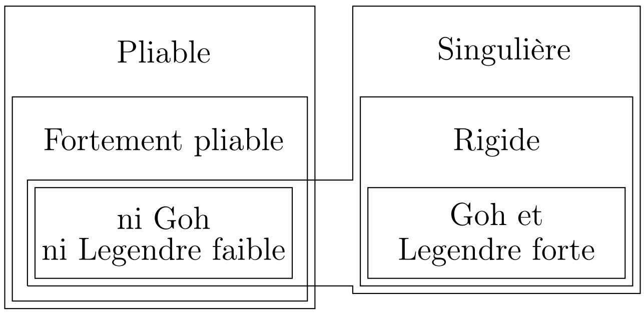


FIGURE 1.2 – Diagramme d'inclusion de différentes familles de courbes horizontales en lien avec la singularité de l'application point-final et la pliabilité.

De ce constat on peut tirer plusieurs critères à tester sur la variété (M, Δ, g) pour prouver la pliabilité forte de tout couple (q, u). On obtient notamment le résultat suivant pour les distributions de pas 2.

Proposition 1.2.11. *Si (M, Δ, g) est une variété sous-riemmannienne équirégulière de pas 2, alors tous les couples (q, u_0) sont fortement pliables.*

Un théorème de prolongement de Whitney. Comme annoncé, la pliabilité forte permet de construire des prolongements C^1 et horizontaux.

Théorème 1.2.12. *Soit (M, Δ, g) une variété sous-riemannienne équirégulière. Si tout couple $(q, u) \in M \times \mathbb{R}^m$ est fortement pliable alors la propriété de prolongement C_H^1 est vérifiée par (M, Δ, g).*

Soit (f, L) satisfaisant la condition de Whitney sur K. Pour des raisons de contrôlabilité, il nous est possible d'affirmer pour tout $(a, b) \subset \mathbb{R} \setminus K$, $a, b \in K$, l'existence d'un contrôle continu $v : [a, b] \to \mathbb{R}^m$ tel que

$$X_{v(a)} = L(a), \qquad X_{v(b)} = L(b) \quad \text{et} \quad E_{f(a)}\left(v\left[a + (b-a) \cdot\right]\right) = f(b). \tag{1.6}$$

On emploie la pliabilité forte pour affirmer que des prolongements v existent sur chaque composante connexe de K^c de telle sorte que le prolongement global de L soit continu.

Pour $(a,b) \subset K^c$, $a,b \in K$, $\eta > 0$, on introduit l'ensemble $\mathcal{P}_\eta([a,b]) \subset C^0([a,b], B_{\mathbb{R}^m}(0,\eta))$. C'est l'ensemble des contrôles de norme L^∞ bornée par η, tel que $u(a)+v$ est un prolongement admissible sur $[a,b]$, c'est-à-dire satisfaisant (1.6), pour tout $v \in \mathcal{P}_\eta([a,b])$.

La controlabilité assure que

$$\inf\{\eta > 0 \mid \mathcal{P}_\eta([a,b]) \neq \emptyset\} < +\infty.$$

pour tout $(a,b) \subset \mathbb{R} \setminus K$, $a,b \in K$. La pliabilité forte, elle, assure le fait suivant.

Lemme 1.2.13. *Sous les hypothèses du théorème 1.2.12, soit $(a_n)_n$ et $(b_n)_n$ deux suites dans ∂K telles que $(a_n,b_n) \subset \mathbb{R} \setminus K$ pour tout $n \in \mathbb{N}$. Si $\lim a_n = \lim b_n = l \in \mathbb{R}$ alors*

$$\inf\{\eta > 0 \mid \mathcal{P}_\eta([a_n,b_n]) \neq \emptyset\} \to 0.$$

La méthode employée est commune aux parties précédentes. On applique une dilatation centrée en $f(a_n)$ de facteur $\frac{1}{b_n-a_n}$. Ceci permet de retourner à la limite vers un problème de contrôlabilité dans le nilpotent. La suite $(\mathcal{F}^n)_{n\in\mathbb{N}}$ des applications définies sur $\mathcal{C}_0 = \{v \in C^0([0,1],\mathbb{R}^m) \mid v(0)=0\}$ par

$$\mathcal{F}^n(v) = \left(\delta^{f(a_n)}_{\frac{1}{b_n-a_n}}\left(E_{f(a_n)}(u(a_n)+v\right), v(1)\right)$$

converge uniformément vers $\mathcal{F}^{u_0}_q$, de façon similaire à ce que l'on a vu précédemment.

Par définition des $\mathcal{F}^n$, si $v_n \in C^0([a,b], B_{\mathbb{R}^m}(0,\eta))$, $v_n(0)=0$, et

$$\mathcal{F}_n(v_n) = \left(\delta^{f(a_n)}_{\frac{1}{b_n-a_n}}(f(b_n)), u(b_n)-u(a_n)\right)$$

alors $v_n \in \mathcal{P}_\eta([a_n,b_n])$. Par hypothèse, l'application $\mathcal{F}^{u_0}_q$ possède des valeurs régulières arbitrairement proches de 0, et $\mathcal{F}^n(v_n) \to \mathcal{F}^{u_0}_q(0)$. Par un argument de degré topologique s'appuyant sur la convergence uniforme de $(\mathcal{F}_n)_{n\in\mathbb{N}}$ (cf appendice 3.A), on obtient alors que pour tout $\eta > 0$, $\mathcal{P}_\eta([a_n,b_n])$ est non vide à partir d'un certain rang. Ceci prouve le lemme 1.2.13 et le théorème 1.2.12.

1.2.4 Implications du cas équirégulier

Aux points non réguliers. L'analyse qui précède pourrait éventuellement s'étendre au cas des variétés sous-riemanniennes non équirégulières, toutefois les méthodes d'estimation de la distance sous-riemannienne que nous avons employées sont réservées au cas équirégulier. On peut malgré tout étendre notre résultat en employant des techniques de désingularisation.

Le principe de la désingularisation tient en deux observations. La première est que la structure de la projection $\mathbb{G}_q \to \mathbb{G}_q/\mathbb{H}_q$ sur le tangent métrique à (M, Δ, g) en q peut être répliquée sur M. En effet pour $d' = \dim \mathbb{G}_q$, il existe une projection $\pi : \widetilde{M} = M \times \mathbb{R}^{d'-d} \to M$ et une structure sous-riemannienne $(\widetilde{M}, \widetilde{\Delta}, \widetilde{g})$, de même rang m et même pas r, telles que

— Pour toute base orthonormée $(\widetilde{X}_1, \ldots, \widetilde{X}_m)$ de $(\widetilde{M}, \widetilde{\Delta}, \widetilde{g})$, la famille

$$(\pi_* \widetilde{X}_1, \ldots, \pi_* \widetilde{X}_m)$$

est une base orthonormée de (M, Δ, g).

— Le tangent métrique à $(\widetilde{M}, \widetilde{\Delta}, \widetilde{g})$ en $(q, 0)$ est isométrique à $\mathbb{G}_q$.

La seconde observation est qu'il est possible de choisir $\mathbb{G}_q = N_{m,r}$, le groupe libre de rang m et de pas r (cf [Jea01, Jea14]). L'intérêt de ce choix est qu'il assure que $(\widetilde{M}, \widetilde{\Delta}, \widetilde{g})$ est régulière en q.

C'est cette construction (locale) que l'on nomme relèvement équirégulier de (M, Δ, g).

Proposition 1.2.14. *Soit $(\widetilde{M}, \widetilde{\Delta}, \widetilde{g})$ un relèvement équirégulier de (M, Δ, g) sur l'ouvert Ω. Si $(f, X_u) : K \to \Omega \times TM$ satisfait la condition de Whitney C^1_H sur K, il existe un relèvement continu $(\widetilde{f}, \widetilde{X}_u) : K \to \widetilde{M} \times T\widetilde{M}$ de (f, X_u) satisfaisant (dans la variété $(\widetilde{M}, \widetilde{\Delta}, \widetilde{g})$) la condition de Whitney C^1_H sur K.*

Il faut remarquer que lorsque l'on possède un relèvement de la variété, un relèvement d'une courbe absolument continue peut être obtenu en intégrant sa dérivée dans la structure relevée, après avoir choisit un point de départ. L'idée de la preuve est simplement de choisir un prolongement continu et horizontal de f, arbitrairement sélectionné pour que son relèvement construit de la sorte respecte encore la condition de Whitney C^1_H.

Par conséquent, comme toute variété sous-riemannienne de pas 2 admet un relèvement équirégulier de pas 2, on obtient la conclusion suivante.

Corollaire 1.2.15. *Toutes les variétés sous-riemanniennes de pas 2 vérifient la propriété de prolongement C^1_H.*

Application aux approximations de Lusin et à la rectifiabilité. Le théorème d'approximation de Lusin est un résultat souvent associé au théorème de Whitney. En effet il est fréquemment observé que l'utilisation du théorème de Rademacher en conjonction avec celui de Whitney permet classiquement de prouver le théorème de Lusin [LDS16, Spe16, JS17, Zim18].

Nous adaptons donc un théorème de Rademacher pour les courbes dans les variétés sous-riemanniennes dû à Vodpyanov [Vod06] et nous pouvons prouver le théorème d'approximation de Lusin pour les courbes horizontales.

Proposition 1.2.16 (Approximation de Lusin d'une courbe horizontale)**.** *Soit* (M, Δ, g) *une variété sous-riemannienne vérifiant la propriété de prolongement* C^1_H *et soit* $\gamma : [a, b] \to M$ *une courbe absolument continue horizontale. Pour tout* $\varepsilon > 0$ *il existe un compact* $K \subset [a, b]$ *de mesure* $\mathcal{L}([a, b] \setminus K) < \varepsilon$ *et une courbe* $\gamma_1 : [a, b] \to M$ *de classe* C^1_H *telle que* γ *et* γ_1 *coïncident sur* K*.*

Ce résultat trouve un corollaire immédiat dans la théorie de la 1-rectifiabilité dans les variétés sous-riemanniennes ([LDS16]). Pour rappel, un ensemble $E \subset M$ est dit 1-rectifiable s'il existe une famille dénombrable de courbes lipschitziennes $f_k : \mathbb{R} \to M$, $k \in M$, telles que la mesure de Hausdorff résiduelle $\mathcal{H}^1(E \setminus \cup_k f_k(\mathbb{R}))$ est nulle.

Corollaire 1.2.17 (1-rectifiabilité)**.** *Soit* (M, Δ, g) *une variété sous-riemannienne satisfaisant la propriété de prolongement* C^1_H*. Si* E *est un sous ensemble* 1*-rectifiable de* M*, alors il existe une famille dénombrable de courbes* $f_k : \mathbb{R} \to M$ *de classe* C^1_H *telles que* $\mathcal{H}^1 (E \setminus \cup_k f_k(\mathbb{R})) = 0$*.*

1.3 La caustique des variétés de contact sous-riemanniennes

Définition 1.3.1. Une variété sous-riemannienne de contact est une variété sous-riemannienne lisse (M, Δ, g) de dimension $2n + 1$, $n \geq 1$, de rang $2n$ et de pas 2. La distribution étant non intégrable, elle coïncide (localement) avec le noyau d'une 1-forme de contact $\omega \in \Lambda^1 M$ telle que $\omega \wedge (\mathrm{d}\omega)^n \neq 0$.

Les variétés sous-riemanniennes de contact forment une classe de variétés possédant plus de structure mais offrant malgré tout une grande variété de phénomènes à étudier. Notamment, l'étude des singularités de l'exponentielle est un problème sous-riemannien classique en raison de ses connexions avec la géométrie des variétés sous-riemanniennes (cf, par exemple, [DV09, Hug95]). Dans le cas des variétés de contact de dimension 3, ce problème trouve une résolution satisfaisante via une approche perturbative particulièrement fructueuse [Agr96, ACGZ00, EAGK96].

Des outils ont été mis en place pour aborder le cas des variétés de contact de dimension supérieure, mais l'analyse n'a pas encore été menée en dehors de certains cas particuliers. Une des richesses de ce cas vient du fait qu'à partir de la dimension 5, on ne peut plus assurer que les tangents métriques en deux points distincts d'une même variété sous-riemannienne de contact soient isométriques. On s'intéresse donc à produire une première étude des propriétés géométriques à petite échelle des ces variétés, via la construction d'une approximation de l'exponentielle sous-riemannienne et de son lieu conjugué, qui y jouent un rôle prépondérant.

1.3.1 Les groupes de Heisenberg

Une variété sous-riemannienne de contact étant par définition équirégulière, l'étude des groupes de Carnot de contact s'impose.

Pour tout entier $n \geq 1$, on note $\mathbb{H}_{2n+1}$ le groupe de Heisenberg (réel) de dimension $2n+1$. Pour rappel (voir, par exemple, [BBS16, Gav77, LR17]), il s'agit du groupe de Lie admettant dans $\mathrm{GL}_{n+2}(\mathbb{R})$ la représentation matricielle suivante, $\mathbb{H}_{2n+1} = \{m(x,y,z) \mid (x,y,z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}\}$, avec

$$m(x,y,z) = \begin{pmatrix} 1 & x_1 & x_2 & \cdots & x_n & z - \frac{1}{2}\sum_{i=1}^n x_i y_i \\ & 1 & & (0) & & y_1 \\ & & \ddots & & & y_2 \\ & (0) & & 1 & & \vdots \\ & & & & & y_n \\ & & & & & 1 \end{pmatrix}.$$

En notant $E_{i,j}$ la matrice de taille $(n+2)\times(n+2)$ dont le seul coefficient non nul est un 1 en position (i,j), on définit $X_i = E_{1,i+1}$, $Y_i = E_{n+2,i+1}$, pour tout $1 \leq i \leq n$, et $Z = E_{1,n+2}$. L'algèbre de Lie $\mathfrak{h}_{2n+1}$ de $\mathbb{H}_{2n+1}$ admet dans cette représentation la base $\{X_i, Y_i, Z \mid 1 \leq i \leq n\}$.

En remarquant que pour tout $1 \leq i,j \leq n$,

$$[X_i, X_j] = [Y_i, Y_j] = [X_i, Z] = [Y_i, Z] = 0 \qquad \text{et} \qquad [X_i, Y_j] = \delta_{ij},$$

on retrouve alors le fait que le groupe de Lie $\mathbb{H}_{2n+1}$ est un groupe de Carnot de pas 2 et de strate d'ordre 1 engendrée par $\{X_i, Y_i \mid 1 \leq i \leq n\}$. La distribution Δ est obtenue par translation à gauche des vecteurs de la strate 1 de $\mathfrak{h}_{2n+1}$.

En considérant $Z^* : \mathfrak{h}_{2n+1} \to \mathbb{R}$ la forme linéaire duale de Z sur $\mathfrak{h}_{2n+1}$, elle admet par translation à gauche une extension $\omega \in \Lambda^1\mathbb{H}_{2n+1}$ qui est une forme de contact annulant la distribution Δ sur $\mathbb{H}_{2n+1}$.

Soit $Q : \mathrm{Vect}\{X_i, Y_i \mid 1 \leq i \leq n\} \to \mathbb{R}$ une forme quadratique définie positive. Par translation à gauche de Q on définit sur $\mathbb{H}_{2n+1}$ une structure sous-riemannienne invariante à gauche $(\mathbb{R}^{2n+1}, \Delta, g)$ telle que $g_0 = Q$.

Proposition 1.3.2. *Pour tout $n \geq 1$, $\mathbb{H}_{2n+1}$ est le seul groupe de Carnot de dimension $2n+1$ qui soit une variété sous-riemannienne de contact une fois muni d'une structure sous-riemannienne invariante à gauche.*

Du point de vue différentiel, pour tout $n \geq 1$ il n'y a qu'un seul groupe de Carnot de dimension $2n+1$ qui soit également une variété de contact. En revanche, du fait de l'introduction d'éléments métriques, il n'y a pas en général unicité à isométrie près du groupe de Carnot contact de dimension $2n+1$ muni d'une structure sous-riemannienne invariante à gauche.

On introduit alors des invariants métriques qui permettent de classifier ces espaces. Soit $(\mathbb{R}^{2n+1}, \Delta, g)$ une variété sous-riemannienne invariante à gauche pour la structure de groupe de $\mathbb{H}_{2n+1}$. Soit ω une 1-forme de contact telle que $\ker \omega = \Delta$. En tout point $q \in \mathbb{R}^{2n+1}$, il existe une application linéaire $A(q) : \Delta_q \to \Delta_q$ antisymétrique pour g_q telle que pour tout $q \in \mathbb{R}^{2n+1}$ et pour tout $X, Y \in \Delta$,

$$\mathrm{d}\omega(X,Y)(q) = g_q(A(q)X(q), Y(q)).$$

On peut remarquer que $A(q)$ ne dépend pas du choix de ω mais bien uniquement de g_q et Δ_q. En effet, pour toute fonction lisse α à valeurs réelles strictement positives, $\alpha\omega$ est aussi une forme de contact telle que $\ker \alpha\omega = \Delta$, mais ceci implique que pour tout $X, Y \in \Delta$,

$$\mathrm{d}\alpha\omega(X,Y)(q) = \mathrm{d}\omega(X,Y)(q) + \mathrm{d}\alpha \wedge \omega(X,Y)(q) = \mathrm{d}\omega(X,Y)(q).$$

Comme $\omega \wedge (\mathrm{d}\omega)^n \neq 0$, il existe n réels strictement positifs $(b_1, \ldots, b_n)$ tels que le spectre de $A(q)$ soit $\{\pm ib_1, \cdots \pm ib_n\}$ (par invariance à gauche, les valeurs propres de la matrice $A(q)$ sont indépendantes de la position).

Pour tout vecteur $(b_1, \ldots, b_n)$ dans $(\mathbb{R}_+^*)^n$ il existe un représentant classique de cette structure pour laquelle A admet les valeurs propres $\{\pm ib_1, \cdots \pm ib_n\}$. Il s'agit de la famille de vecteurs $(X_i)_{1 \le i \le 2n}$ définie sur $\mathbb{R}^{2n+1}$, muni de coordonnées $(x, z) \in \mathbb{R}^{2n} \times \mathbb{R}$, tels que pour tout $1 \le i \le 2n$

$$X_{2i-1} = \partial_{x_{2i-1}} + \frac{b_i}{2} x_{2i} \partial_z, \qquad X_{2i} = \partial_{x_{2i}} - \frac{b_i}{2} x_{2i-1} \partial_z. \tag{1.7}$$

On note $\mathbb{H}_{2n+1}(b_1, \ldots, b_n)$ l'espace métrique $(\mathbb{R}^{2n+1}, \Delta, g)$ pour lequel $(X_i)_{1 \le i \le 2n}$ est une base orthonormée de la distribution. C'est une structure invariante à gauche pour $\mathbb{H}_{2n+1}$ qui nous permet de mettre en lumière que le vecteur $(b_1, \ldots, b_n)$ définit un invariant métrique des structures de contact invariantes à gauche.

Théorème 1.3.3. *Toute structure sous-riemannienne de contact invariante à gauche sur $\mathbb{H}_{2n+1}$ est isométrique à un unique $\mathbb{H}_{2n+1}(b_1, \ldots, b_n)$ tel que $1 = b_1 \ge b_2 \ge \cdots \ge b_n > 0$.*

Corollaire 1.3.4. *Il existe une unique structure sous-riemannienne de contact invariante à gauche sur $\mathbb{H}_3$ (à isométrie près).*

Les groupes de Heisenberg constituant les tangents métriques aux variétés sous-riemanniennes de contact, nous allons voir le rôle de ces observations dans l'étude du lieu conjugué.

1.3.2 Géodésiques sous-riemanniennes

Considérations symplectiques. Soit (M, Δ, g) une variété sous-riemannienne dimension d. On s'intéresse ici à employer des outils provenant de la structure symplectique de T^*M pour tirer du principe du maximum des informations sur les courbes minimisantes dans le cas contact. On note σ la structure symplectique canonique sur T^*M et on rappelle que l'on note

$$\begin{array}{rrcc} H: & T^*M & \longrightarrow & M \\ & (p,q) & \longmapsto & \dfrac{1}{2} \max\limits_{v \in \Delta_q \setminus \{0\}} \dfrac{\langle p, v\rangle^2}{g_q(v,v)} \end{array}$$

le hamiltonien sous-riemannien.

Pour une variété N et pour $\Omega \in \Lambda^2 N$ une 2-forme, une courbe lipschitzienne $\gamma : [0,T] \to N$ est caractéristique pour Ω si pour presque tout $t \in [0,T]$, $\dot\gamma(t) \in \ker \Omega_{\gamma(t)}$. Cette définition permet de rassembler les courbes minimisantes sous le chapeau des courbes caractéristiques de σ après restriction de son ensemble de définition.

Proposition 1.3.5 ([ABB16, propositions 4.31-4.34])**.** *Soit H le hamiltonien sous-riemannien et soit $c \geq 0$ une valeur régulière de H. La courbe γ est caractéristique pour $\sigma_{|H^{-1}(c)}$ si et seulement si γ est la reparamétrisation d'une courbe extrémale à valeurs dans $H^{-1}(c)$.*

En particulier c'est une extrémale normale si $c > 0$ et anormale si $c = 0$.

Dans le cas d'une structure sous-riemannienne équirégulière, 0 est toujours une valeur régulière pour H car $H^{-1}(0)$ est l'annulateur de la distribution, la sous-variété de dimension $d+1$ de T^*M orthogonale à Δ au sens de la dualité entre TM et T^*M.

Si cette structure est de contact, il existe un entier $n \geq 1$ tel que $d = 2n+1$ et $\omega \in \Lambda^1 M$ tel que $\omega(\Delta) = 0$ et $\omega \wedge (\mathrm{d}\omega)^n \neq 0$ (au moins localement). Les éléments de $H^{-1}(0)$ sont alors proportionnels à ω dans les fibre de T^*M. Ceci implique que la restriction à $H^{-1}(0)$ de la 1-forme de Liouville s est proportionnelle à ω : il existe une application à valeur réelle $\alpha : M \to \mathbb{R}$ telle que $s_{|H^{-1}(0)} = \alpha\omega$. Comme $\mathrm{d}s = \sigma$, on obtient

$$\sigma_{|H^{-1}(0)} = \mathrm{d}\alpha \wedge \omega + \alpha\, \mathrm{d}\omega.$$

Ces observations sont cruciales car elles permettent de vérifier, par hypothèse de non-intégrabilité de la distribution de contact,

$$\left(\sigma_{|H^{-1}(0)}\right)^{n+1} = (n+1)\, \mathrm{d}\alpha \wedge \omega \wedge (\mathrm{d}\omega)^n \neq 0.$$

Comme la dimension de $H^{-1}(0)$ est $2n+2$, cette équation affirme que le noyau de $\sigma_{|H^{-1}(0)}$ est réduit à $\{0\}$ en tout point.

Par application de la proposition 1.3.5, on en déduit le résultat suivant sur les extrémales anormales dans les variétés sous-riemanniennes de contact.

Proposition 1.3.6. *Soit (M, Δ, g) une variété sous-riemannienne de contact. Les extrémales anormales sont les projections de courbes stationnaires dans T^*M.*

Il nous reste donc à traiter le cas des extrémales normales, qui elles sont solutions d'un système hamiltonien autonome.

Champs hamiltoniens. Soit M une variété différentielle, on rappelle que pour toute fonction $a \in C^\infty(T^*M, \mathbb{R})$, on note $\vec{a}$ le champ de vecteurs hamiltonien associé à a sur T^*M. On peut noter qu'il est défini par la relation

$$\sigma(\cdot, \vec{a}) = \mathrm{d}a.$$

En coordonnées canoniques (p, x) sur T^*M, $\sigma = \sum_{i=1}^{d} \mathrm{d}p_i \wedge \mathrm{d}x_i$ et $\vec{a}$ prend la forme classique

$$\vec{a} = \sum_{i=1}^{d} \frac{\partial a}{\partial p_i}\frac{\partial}{\partial x_i} - \frac{\partial a}{\partial x_i}\frac{\partial}{\partial p_i}.$$

Dans le cas des variétés sous-riemanniennes, les extrémales normales sont les projections des courbes de flot de $\vec{H}$, mais il convient d'introduire une forme différente de l'équation des courbes intégrales de $\vec{H}$. En notant $\{\cdot, \cdot\} : C^\infty(T^*M, \mathbb{R}) \times C^\infty(T^*M, \mathbb{R}) \to C^\infty(T^*M, \mathbb{R})$ le crochet de Poisson sur T^*M, on rappelle que pour tout $a, b \in C^\infty(T^*M, \mathbb{R})$,

$$\{a, b\} = \vec{a} \cdot b.$$

Ainsi pour $t \mapsto \lambda(t) = \mathrm{e}^{t\vec{a}}\lambda(0)$, une courbe intégrale du champ hamiltonien $\vec{a}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} b(\lambda(t)) = \{a, b\}(\lambda(t)).$$

On peut ainsi donner une expression du flot de $\vec{H}$ sur des coordonnées plus adaptées à l'analyse de ses trajectoires dans le cas d'une variété (M, Δ, g) de contact. Soit $(X_1, \ldots, X_{2n})$ une base orthonormée de (Δ, g), et soit X_0 un champ de vecteurs transverses à la distribution tel que $(X_1(q), \ldots, X_{2n}(q), X_0(q))$ soit une base de T_qM en tout $q \in M$ (ou sur un ouvert de M s'il n'existe pas de base globale).

La famille $(X_i)_{0 \le i \le 2n}$ nous permet de définir des coordonnées sur les fibres de T^*M. Pour tout $(p, q) \in T^*M$, on note

$$h_i(p, q) = \langle p, X_i(q) \rangle, \qquad 0 \le i \le 2n.$$

Par définition de H, on a maintenant

$$H(p, q) = \frac{1}{2}\sum_{i=1}^{2n} h_i^2(p, q) \qquad \forall (p, q) \in T^*M,$$

et l'équation hamiltonienne des extrémales normales devient en coordonnées (x, h)

$$\begin{cases} \dot{x} = \sum_{i=1}^{2n} h_i X_i(x), \\ \dot{h}_i = \{H, h_i\} \qquad \forall\, 0 \leq i \leq 2n. \end{cases}$$

En appliquant des propriétés classiques du crochet de Poisson, on peut raffiner cette expression. Le crochet de Poisson étant une dérivation vis à vis de chaque argument,

$$\dot{h}_i = \left\{ \frac{1}{2} \sum_{j=1}^{2n} h_j^2, h_i \right\} = \sum_{j=1}^{2n} h_j \{h_j, h_i\}.$$

Par ailleurs, pour deux champs de vecteurs X_i et X_j, $0 \leq i, j \leq 2n$, on a la propriété fondamentale des crochets de Poisson,

$$\{h_i, h_j\}(p, q) = \langle p, [X_i, X_j]\,(q) \rangle\,, \qquad (p, q) \in T^*M.$$

Ainsi, en notant $\left(c_{ij}^k\right)_{0 \leq i,j,k \leq 2n} : M \to \mathbb{R}$ la famille d'applications lisses telles que

$$[X_i, X_j] = \sum_{k=0}^{2n} c_{ij}^k X_k,$$

nous obtenons l'équation des géodésiques que nous étudions dans cette thèse,

$$\begin{cases} \dot{x} = \sum_{i=1}^{2n} h_i X_i(x), \\ \dot{h}_i = h_0 \sum_{j=1}^{2n} c_{ji}^0 h_j + \sum_{j=1}^{2n} \sum_{k=1}^{2n} c_{ji}^k h_j h_k, \qquad \forall\, 0 \leq i \leq 2n. \end{cases} \tag{1.8}$$

Le cas Heisenberg. En ce qui concerne les groupes de Heisenberg, il est possible de donner une expression exacte des solutions de (1.8) grâce à la forme normale (1.7). On choisit alors comme complément à la base $(X_i)_{1 \leq i \leq 2n}$ le champ $X_0 = \partial_z$.

On calcule immédiatement les crochets $[X_i, X_j] = 0$ s'il n'existe pas de $k \in \mathbb{N}$, $1 \leq k \leq n$, tel que $\{i, j\} = \{2k-1, 2k\}$, et $[X_{2i-1}, X_{2i}] = -[X_{2i-1}, X_{2i}] = -b_i$ dans le cas contraire. En notant la matrice $\bar{J} = \left(c_{ji}^0\right)_{1 \leq i,j \leq 2n}$, on obtient alors la matrice diagonale par blocs

$$\bar{J} = \begin{pmatrix} 0 & b_1 & & & & & \\ -b_1 & 0 & & & & (0) & \\ & & 0 & b_2 & & & \\ & & -b_2 & 0 & & & \\ & & & & \ddots & & \\ & (0) & & & & 0 & b_n \\ & & & & & -b_n & 0 \end{pmatrix},$$

et l'équation (1.8) devient (en notant h le vecteur $(h_1,\dots,h_{2n})$)

$$\begin{cases} \dot{x} = h, \\ \dot{z} = \sum_{i=1}^{n} \frac{b_i}{2} \left(h_{2i-1}x_{2i} - h_{2i}x_{2i-1} \right), \\ \dot{h} = h_0 \bar{J} h, \\ \dot{h}_0 = 0. \end{cases}$$

Ainsi, l'extrémale normale de covecteur initial $\left(\bar{h}_1,\dots,\bar{h}_{2n},\bar{h}_0\right)$ partant de 0 (on peut obtenir les autres solutions par translation à gauche dans le groupe de Heisenberg) admet l'équation paramétrée, pour tout $i \in \mathbb{N}$, $1 \le i \le n$, (avec $\bar{h}_0 \neq 0$, sinon elle est stationnaire)

$$\begin{aligned} x_{2i-1}(t) &= \frac{1}{b_1\bar{h}_0}\left[\bar{h}_{2i-1}\sin(b_i\bar{h}_0 t) + \bar{h}_{2i}(1-\cos(b_i\bar{h}_0 t))\right], \\ x_{2i}(t) &= \frac{1}{b_1\bar{h}_0}\left[\bar{h}_{2i-1}(\cos(b_i\bar{h}_0 t)-1) + \bar{h}_{2i}\sin(b_i\bar{h}_0 t)\right], \\ z(t) &= \sum_{i=1}^{n}\frac{\left(\bar{h}_{2i-1}^2+\bar{h}_{2i}^2\right)}{2b_i\bar{h}_0^2}(b_i\bar{h}_0 t - \sin(b_i\bar{h}_0 t)), \\ h_{2i-1}(t) &= \bar{h}_{2i-1}\cos(b_i\bar{h}_0 t) + \bar{h}_{2i}\sin(b_i\bar{h}_0 t), \\ h_{2i}(t) &= \bar{h}_{2i}\cos(b_i\bar{h}_0 t) - \bar{h}_{2i-1}\sin(b_i\bar{h}_0 t), \\ h_0(t) &= \bar{h}_0. \end{aligned} \tag{1.9}$$

On peut remarquer qu'il existe une infinité de paramétrisations produisant la courbe minimisant la longueur à $(\bar{x}_1,\dots,\bar{x}_{2n},\bar{z})$. On peut donc se limiter au choix de covecteurs tels que $\sum_{i=1}^{2n} h_i^2 = 1$ ou de géodésiques au temps 1, ce qui la rend unique dans la majorité des cas.

1.3.3 La caustique sous-riemannienne

Lieu conjugué. Soit (M,Δ,g) une variété sous-riemannienne de contact de dimension $2n+1$, $n \ge 1$, soit $q_0 \in M$. Dans cette section, $\pi : T^*M \to M$ désigne la projection canonique du fibré cotangent. Pour tout $q_0 \in M$, on note

$$\begin{array}{cccc} \mathcal{E}_{q_0}^1 : & T_{q_0}^*M & \longrightarrow & M \\ & p & \longmapsto & \pi\left(\mathrm{e}^{\vec{H}}(p,q_0)\right) \end{array}$$

l'exponentielle sous-riemannienne (au temps 1), qui associe au covecteur $p \in T_{q_0}^*M$ le point final au temps 1 de l'extrémale normale ayant pour point de départ q_0 et covecteur initial p. Les *points conjugués* sont les valeurs singulières de $\mathcal{E}_{q_0}^1$. On peut en dire plus en observant cette propriété fondamentale du flot hamiltonien.

Proposition 1.3.7. *Soit* $(p,q) \in T^*M$. *Pour tout* $\mu \in \mathbb{R}$,

$$\mathrm{e}^{\mu\vec{H}}(p,q) = \mathrm{e}^{\vec{H}}(\mu\, p, q).$$

De ce fait, on peut définir l'exponentielle au temps t telle que $\mathcal{E}^t_{q_0}(p) = \mathcal{E}^1_{q_0}(tp)$. Le *temps conjugué à* $p \in T^*_{q_0}M$ est alors $t_c(p) = \inf\{t > 0 \mid \mathcal{E}^t_{q_0}$ singulière en $p\}$ et le (premier)*lieu conjugué en* q_0 est l'ensemble des points

$$\left\{\mathcal{E}^{t_c(p)}_{q_0}(p) \mid p \in \mathcal{C}_{q_0}(1/2)\right\}.$$

On fait également référence à cet ensemble sous le nom de *caustique sous-riemannienne.* On peut noter que pour tout $p \in T^*_{q_0}M$, le temps $t_c(p)$ correspond à la date à laquelle la géodésique de covecteur initial p perd son optimalité locale.

Pour finir, rappelons la définition de la *longueur conjuguée* $l_c(p)$, $p \in T^*_{q_0}M$. Pour tout $p \in T^*_{q_0}M$, il s'agit de la longueur de la courbe $t \mapsto \mathcal{E}^t_{q_0}(p)$ définie sur $[0, t_c(p)]$. La longueur conjuguée a l'avantage d'avoir une définition intrinsèque vis à vis des points de la caustique. En effet pour tout $p \in T^*_{q_0}M$, on a par la propriété de dilatation du flot hamiltonien que $t_c(p) = \mu t_c(\mu p)$ pour tout $\mu > 0$ alors que $\mathcal{E}^{t_c(\mu p)}_{q_0}(\mu p) = \mathcal{E}^{t_c(p)}_{q_0}(p)$. En revanche $l_c(\mu p) = l_c(p)$ pour tout $\mu > 0$ et ne dépend donc que du point $\mathcal{E}^{t_c(p)}_{q_0}(p)$. On peut remarquer que si $p \in T^*_{q_0}M$ vérifie $H(p, q_0) = 1/2$, l'extrémale normale est paramétrisée à vitesse 1 et $l_c(p) = t_c(p)$.

Dans la suite de cette section, on étudie le lieu conjugué à un point $q_0 \in M$ fixé. Pour tout $c \geq 0$, on note $\mathcal{C}_{q_0}(c) \subset T^M_{q_0}$ l'ensemble des $p \in T^*_{q_0}M$ tels que $H(p, q_0) = c$. On peut remarquer que dans le cas contact, $\mathcal{C}_{q_0}(0)$ est une droite et si $c > 0$, l'ensemble $\mathcal{C}_{q_0}(c)$ possède la topologie du cylindre $\mathbb{S}^{2n-1} \times \mathbb{R}$. Par abus de notation on emploie également, pour $V \subset \mathbb{R}^+$, $\mathcal{C}_{q_0}(V) = \cup_{c\in V}\mathcal{C}_{q_0}(c)$.

Cas Heisenberg. On a rappelé dans la section précédente que les extrémales normales admettent comme paramétrisation (1.9). Il est alors possible de calculer le lieu conjugué dans ce cas. En effet, l'extrémale normale de covecteur initial $(\bar{h}, \bar{h}_0) \in \mathbb{R}^{2n} \times \mathbb{R}$ partant de 0 peut se noter

$$h(t) = \mathrm{e}^{\bar{h}_0 t \bar{J}}\bar{h} \qquad \text{et} \qquad x(t) = \frac{1}{\bar{h}_0}\bar{J}^{-1}\left(\mathrm{e}^{\bar{h}_0 t\bar{J}} - I_{2n}\right)\bar{h}.$$

Donc $D_h x(t) = \mathrm{e}^{\bar{h}_0 t\bar{J}} - I_{2n}$ qui est une matrice inversible tant que $\bar{h}_0 t \neq \frac{2\pi}{b_i}$, pour tout $1 \leq i \leq n$. On définit alors $Z = \{2k\pi/b_i \mid k \in \mathbb{N}, 1 \leq i \leq n\}$. On peut alors observer ce fait sur les points singuliers de $\mathcal{E}^t_0 : (\bar{h}, \bar{h}_0) \mapsto (x(t), z(t))$ (cf section 4.3.2).

Proposition 1.3.8. *On suppose que* $b_1 \geq b_2 \geq \cdots \geq b_n$. *Pour tout* $p = (\bar{h}, \bar{h}_0) \in H^{-1}(1/2)$, *pour tout* $t \in \left(0, \frac{2\pi}{b_1\bar{h}_0}\right)$, *on a* $\det(\mathrm{Jac}_p\mathcal{E}^t_0) \neq 0$. *On en déduit que pour tout* $p = (\bar{h}, \bar{h}_0) \in \mathcal{C}_0(1/2)$,

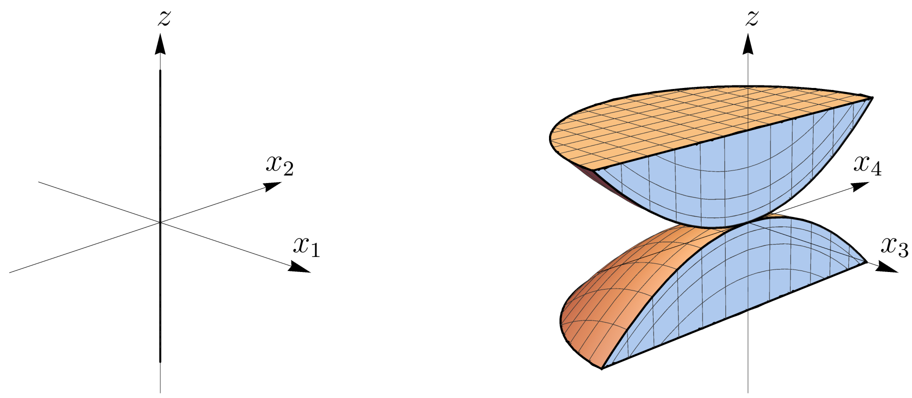
$$t_c(p) = \frac{2\pi}{b_1\bar{h}_0},$$

et le lieu conjugué à 0 *de* $\mathbb{H}_{2n+1}(b_1, \ldots, b_n)$ *est donnée par l'ensemble*

$$\left\{ \left(x\left(\frac{2\pi}{b_1 \bar{h}_0}\right), z\left(\frac{2\pi}{b_1 \bar{h}_0}\right) \right) \mid (\bar{h}, \bar{h}_0) \in \mathcal{C}_0(1/2) \right\}.$$

On remarque que dans le cas 3D, ce résultat permet de vérifier que le lieu conjugué est la droite $\{x_1 = x_2 = 0\}$. En dimension $2n+1 \geq 5$ en revanche, la géométrie de la caustique sous-riemannienne dépend de la valeur des b_i. Si tous les b_i sont égaux, le lieu conjugué est encore la droite $\{x_1 = \cdots = x_{2n} = 0\}$. Par contre s'il existe deux i, j distincts, $1 \leq i, j \leq n$, tels que $b_i \neq b_j$, alors le lieu conjugué n'est réduit à 0 que dans les coordonnées (x_{2i-1}, x_{2i}) telles que $b_i = \max_j b_j$ (voir figure 1.3).

On retrouve également le célèbre fait que le lieu conjugué contient le point 0 dans sa fermeture.



(a) Section dans le sous-espace (x_1, x_2, z)

(b) Section dans le sous espace $(x_3, x_4, z) \cap \{x_3 < 0\}$

FIGURE 1.3 – Représentation du lieu conjugué d'un groupe de Heisenberg 5D (avec des valeurs propres $b_1 > b_2$).

Cas tridimensionel. Dans le cas d'une variété sous-riemannienne de contact (M, Δ, g) de dimension 3, on ne peut pas attendre des résultats globaux similaires à ceux vus précédemment dans le cas des groupes de Heisenberg. On emploie donc une approche différente. Le lieu conjugué à q_0 contenant q_0 dans sa fermeture, on se concentre sur la géométrie du lieu conjugué à proximité du point de départ. On s'appuie sur l'observation suivante (cf proposition 5.2.2 pour une preuve de ce fait dans le cas général).

Proposition 1.3.9. *Soit* (M, Δ, g) *une variété sous-riemannienne de contact de dimension* 3*. Pour tout* $A > 0$*, il existe* $\varepsilon > 0$ *tel que pour tout covecteur initial* $(h, h_0) \in \mathcal{C}_{q_0}(1/2)$*,* $t_c(h, h_0) < \varepsilon$ *implique* $|h_0| > A$*.*

Comme $t_c(p)$ est un majorant de la distance à $\mathcal{E}_{q_0}^{t_c(p)}(p)$ lorsque $p \in \mathcal{C}_{q_0}(1/2)$, cela implique que les points conjugués à q_0 proches de q_0 correspondent à des covecteurs initiaux $p = (h, h_0)$ tels que $|h_0|$ tend vers l'infini. On se concentre alors sur le problème suivant : calculer des asymptotiques sur $t_c(p)$ et $\mathcal{E}_{q_0}^{t_c(p)}(p)$ par rapport à $|h_0|$ pour obtenir une description du lieu conjugué dans un petit voisinage de q_0.

Dans cette optique, on a le résultat suivant.

Théorème 1.3.10 ([EAGK96, Agr96]). *Soit (M, Δ, g) une variété sous-riemannienne de contact 3D, et soit $q_0 \in M$. Sur un voisinage de q_0, il existe une base orthonormée (X_1, X_2) de (Δ, g) et (x_1, x_2, z), des coordonnées centrées en q_0 telles que $X_i(q_0) = \partial_{x_i}$, permettant d'écrire les développements asymptotiques suivants. En notant $p_0 = (\cos\theta, \sin\theta, h_0) \in \mathcal{C}_{q_0}(1/2)$, on a pour $|h_0| \to \infty$*

$$t_c(p_0) = \frac{2\pi}{|h_0|} - \frac{\pi\kappa}{|h_0|^3} + O\left(\frac{1}{|h_0|^4}\right)$$

et

$$\mathcal{E}_{q_0}^{t_c(p_0)}(p_0) = \operatorname{sign}(h_0)\frac{\pi}{h_0^2}(0,0,1) + \frac{2\pi\chi}{h_0^3}\left(-\sin^3\theta, \cos^3\theta, 0\right) + O\left(\frac{1}{|h_0|^4}\right),$$

où κ et χ sont des invariants de structure dépendant de q_0.

De plus, aux points tels que $\chi \neq 0$, le lieu de coupure admet le développement asymptotique

$$t_{\text{coup}}(\theta, h_0) = \frac{2\pi}{|h_0|} - \frac{\pi\kappa + 2\chi\sin^2\theta}{|h_0|^3} + O\left(\frac{1}{|h_0|^4}\right)$$

et

$$\mathcal{E}_{q_0}^{t_{\text{coup}}(p_0)}(\theta, h_0) = \operatorname{sign}(h_0)\frac{\pi}{h_0^2}(0,0,1) + \frac{2\pi\chi}{h_0^3}(\cos\theta, 0, 0) + O\left(\frac{1}{|h_0|^4}\right).$$

(Voir figure 1.4.)

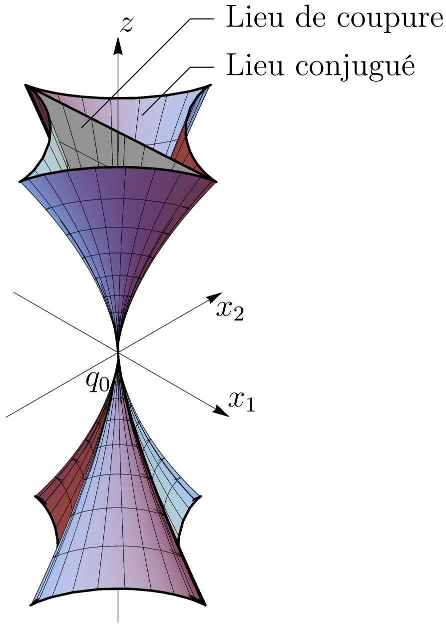
Les preuves de ce théorème dans [EAGK96] et [Agr96] sont différentes mais reposent sur le même principe. Lorsque la distance à q_0 est très faible, la métrique peut s'écrire comme un perturbation de la métrique obtenue par approximation nilpotente en q_0. L'exponentielle sous-riemannienne peut donc être calculée en donnant une approximation du flot hamiltonien prenant comme fondement le flot hamiltonien dans le groupe de Heisenberg.

La méthode employée dans [EAGK96] s'appuie en particulier sur la construction de coordonnées et d'une base orthonormale de la distribution accentuant ces similarités.

Théorème 1.3.11 ([EAGK96]). *Soit (M, Δ, g) une variété sous-riemannienne de contact de dimension 3. Pour tout $q_0 \in M$ il existe (sur un voisnage de q_0) des coordonnées (x, y, z) centrées en q_0 et (F, G) une base orthonormée de Δ telles que $F(0) = \partial_x$, $G(0) = \partial_y$, $[F, G](0) = \partial_z$ et*

$$\begin{cases} F = (1 + y^2\beta)\partial_x - xy\beta\partial_y + \dfrac{y}{2}(1+\gamma)\partial_z, \\ G = -xy\beta\partial_x + (1 + x^2\beta)\partial_y - \dfrac{x}{2}(1+\gamma)\partial_z, \end{cases}$$

FIGURE 1.4 – Le lieu conjugué et le lieu de coupure contact 3D aux points q_0 tels que $\chi(q_0) \neq 0$.
On peut notamment remarquer sur cette figure que les sections à z constant de la caustique ont la forme astroïdale caractéristique des singularités stables de fronts d'onde.
On observe également la croissance du rayon de l'astroïde en $z^{3/2}$. C'est une particularité de la caustique sous-riemannienne de contact 3D qui fait de la direction ∂_z ainsi choisie une direction particulière, transverse à la distribution (engendrée par $\partial_{x_1}, \partial_{x_2}$ en q_0) et tangente au lieu conjugué en q_0.



où β et γ sont des applications lisses de $\mathbb{R}^3$ dans $\mathbb{R}$ telles que

$$\beta(0,0,z) = \gamma(0,0,z) = \partial_x\gamma(0,0,z) = \partial_y\gamma(0,0,z) = 0.$$

Dans ces coordonnées, (F, G) est directement écrite sous la forme d'une perturbation de la forme normale d'une structure sous-riemannienne invariante à gauche sur le groupe de Heisenberg.

Cette étude a une conséquence indirecte mais essentielle dans l'étude de la géométrie à petite échelle des variétés sous-riemanniennes de contact. Le front obtenu par approximation nilpotente de l'exponentielle est une approximation extrêmement précise.

Proposition 1.3.12. *Soit (M, Δ, g) une variété sous-riemannienne de contact de dimension 3. Soit $q_0 \in M$. Il existe des coordonnées privilégiées centrées en q_0 telles que, en notant $\widehat{\mathcal{E}}$ l'exponentielle sous-riemannienne en 0 de la structure de contact obtenue par approximation nilpotente, pour tout $(h, h_0) \in \mathcal{C}_{q_0}(1/2)$, lorsque $|h_0| \to \infty$*

$$\mathcal{E}_{q_0}^{t/h_0}((h,h_0)) = \widehat{\mathcal{E}}^{t/h_0}((h,h_0)) + O\left(\frac{1}{h_0^3}\right).$$

En remarquant que $\left|\mathcal{E}_{q_0}^{t/h_0}((h,h_0))\right| \propto |h_0|^{-1}$, il existe donc des coordonnées pour lesquelles il possible de gagner un ordre d'approximation avec l'approximation nilpotente. Ceci fait notamment de la géométrie de l'approximation nilpotente dans le cas des variétés sous-riemanniennes de contact 3D une meilleure approximation qu'on ne pouvait le prévoir a priori.

1.3.4 En dimension supérieure à 3

Nouveautés en dimension supérieure. L'étude de l'exponentielle sous-riemannienne et de son lieu conjugué constituent la première étape vers la compréhension de la géométrie à petite échelle des variétés sous-riemanniennes de contact de dimension supérieure à 3. Nous avons l'ambition d'étendre les approches perturbatives employées en dimension 3 pour poser les jalons de cette analyse.

Soit (M, Δ, g) une variété sous-riemannienne de contact de dimension $2n+1$, $n \geq 2$. Comme exposé dans les sections précédentes, la géométrie de (M, Δ, g) est grandement liée à son approximation nilpotente. L'approximation nilpotente de M en q_0 est donnée par une structure invariante à gauche sur le groupe de Heisenberg $\mathbb{H}_{2n+1}$. Lorsque $n \geq 2$, le théorème 1.3.3 affirme qu'il existe une infinité de telles structures qui sont non isométriques. Ainsi, en tout $q_0 \in M$ il existe une famille $(b_1(q_0), \dots, b_n(q_0))$ de réels positifs (unique à multiplication par un réel positif et à réarrangement près) tels que l'approximation nilpotente en q_0 de (M, Δ, g) est donnée par $\mathbb{H}_{2n+1}(b_1(q_0), \dots b_n(q_0))$. On supposera par la suite que $b_1(q_0) \dots b_n(q_0) = \frac{1}{n!}$. (On pourrait récupérer les $(b_i)_{1 \leq i \leq n}$ sans avoir à passer par l'approximation nilpotente, de la même manière qu'on les construit dans la section 1.3.1, voir en particulier la section 4.1.)

Comme on l'a vu dans la section 1.3.3 la valeur de la famille des $(b_i)_{1 \leq i \leq n}$ influence grandement la géométrie du lieu conjugué, c'est pourquoi on fait le choix de se restreindre au cas des $(b_i)_{1 \leq i \leq n}$ tels que $b_1 > b_2 \geq \dots \geq b_n$. Ce choix se justifie par des arguments de généricité, car le sous-ensemble de M sur lequel il existe deux invariants b_i, b_j, $i \neq j$, tels que $b_i = b_j$ est un sous-ensemble stratifié de codimension 3 de M dans le cas d'une structure générique (cf [Cha02]). Sous cette hypothèse de généricité, on ne considère dans la suite que des points dans le complémentaire de cet ensemble stratifié.

De la même manière que cela est fait en dimension 3, nous choisissons d'employer des méthodes perturbatives pour tirer partie de nos connaissances de l'approximation nilpotente via l'usage de formes normales. Nous employons un analogue du théorème 1.3.11 dans le cas de variétés sous-riemanniennes de contact de dimension quelconque, provenant de [AG01] (nous en rappelons les éléments principaux dans l'appendice 4.A). Pour tout $q_0 \in M$, ce théorème nous fournit sur un voisinage de q_0 des coordonnées $(x_1, \dots, x_{2n}, z)$: $M \to \mathbb{R}^{2n+1}$ et une base orthonormée $(X_1, \dots, X_{2n})$ de la distribution tirant profit des symétries de la structure de contact et de sa similarité à $\mathbb{H}_{2n+1}(b_1, \dots, b_n)$. Nous faisons référence à ce cadre par le nom forme normale de Agrachev–Gauthier.

Nous cherchons à obtenir une approximation du lieu conjugué sous-riemannien au voisinage du point de départ de la caustique. Grâce à la forme normale de Agrachev–Gauthier, nous pouvons étendre la proposition 1.3.9, qui montre que les points conjugués à $q_0 \in M$ proches de q_0 correspondent aux covecteurs initiaux $(h, h_0) \in \mathcal{C}_{q_0}(1/2)$ pour lesquels $|h_0|$ tend vers l'infini.

L'essentiel dans un premier temps est donc d'obtenir une approximation de $t_c((h, h_0))$ lorsque $h_0 \to \infty$, l'approximation du lieu conjugué s'obtient ensuite aisément par évaluation de l'exponentielle au temps conjugué. Grâce à la propriété de dilatation du champ hamiltonien (cf proposition 1.3.7), une manière pour nous de calculer le temps conjugué à q_0 pour le covecteur p_0 est alors de trouver le premier temps $t > 0$ d'annulation de

$$\Phi(p_0, t) = \det\left(\frac{\partial \mathcal{E}^t_{q_0}}{\partial h_1}, \ldots, \frac{\partial \mathcal{E}^t_{q_0}}{\partial h_{2n}}, \frac{\partial \mathcal{E}^t_{q_0}}{\partial h_0}\right).$$

De l'identité $\Phi(p_0, t_c(p_0)) = 0$ nous pouvons alors tirer les informations dont nous avons besoin pour calculer une approximation de $t_c((h, h_0))$ lorsque $h_0 \to \infty$.

Temps conjugué. Dans ce cas, introduisons $\eta = 1/h_0$. Un difficulté qui peut apparaître est le fait (attendu) que $t_c((h, \eta^{-1})) \to 0$ quand $\eta \to 0$. Dans le cas de l'approximation nilpotente, on rappelle que l'on avait exactement (par notre choix $b_1 > b_2 \geq \cdots \geq b_n$)

$$t_c(h, \eta) = \frac{2\pi}{b_1}\eta.$$

Ainsi on fait le choix de reparamétrer le temps avec $t = \eta\tau$, de sorte que τ_c le temps conjugué pour la variable τ soit fixe dans l'approximation nilpotente. On introduit alors $F : \mathbb{R} \times \mathbb{R}^{2n} \times \mathbb{R} \to M$ telle que

$$F(\tau, h, \eta) = \mathcal{E}^{\eta\tau}_{q_0}((h, \eta^{-1}))$$

et

$$\Phi((h, \eta^{-1}), \eta\tau) = \det\left(\frac{\partial F}{\partial h_1}, \ldots, \frac{\partial F}{\partial h_{2n}}, -\eta^2\frac{\partial F}{\partial \eta} + \eta\tau\frac{\partial F}{\partial \tau}\right).$$

On peut vérifier que F est lisse vis à vis de η en 0, ce qui permet d'exprimer F en fonction de η sous la forme d'une série entière définie par développements limités. On note formellement $F(\tau, h, \eta) = \sum_{k=0}^{\infty} \eta^k F^{(k)}(\tau, h)$ et on peut en calculer les premiers termes non nuls $F^{(1)}$ et $F^{(2)}$. En notant $\widehat{\mathcal{E}}$ l'exponentielle en 0 de l'approximation nilpotente, on observe alors

$$\begin{aligned} \widehat{\mathcal{E}}^{\eta\tau}((h, \eta^{-1}))_i &= \eta F^{(1)}_i(\tau, h), \qquad\qquad \forall\, 1 \leq i \leq 2n, \\ \widehat{\mathcal{E}}^{\eta\tau}((h, \eta^{-1}))_{2n+1} &= \eta^2 F^{(2)}_{2n+1}(\tau, h) \end{aligned}$$

Ceci implique que la première approximation du temps conjugué est le temps conjugué nilpotent. On observe toutefois une propriété des temps conjugués dans l'approximation nilpotente qui a son importance ici.

Proposition 1.3.13. *Soit $b_1 > b_2 \geq \cdots \geq b_n$. On considère $(M, \Delta, g) = \mathbb{H}_{2n+1}(b_1, \ldots, b_n)$. Pour tout covecteur dans $p_0 \in \mathcal{C}_0(1/2)$ on note $t'_c(p_0)$ le second temps conjugué à 0 pour le covecteur initial p_0, c'est-à-dire l'infimum des $t > t_c(p_0)$ tels que $\mathcal{E}^t_0(p_0)$ est conjugué* *à 0. Alors pour $p_0 = (r_1\cos\theta_1, r_1\sin\theta_1, h_3, \dots, h_{2n}, h_0) \in \mathcal{C}_0(1/2)$, il existe une fonction $f(h_3, \dots, h_{2n})$ telle que $t_c'(p_0)$ admet le développement asymptotique lorsque $r_1 \to 0^+$:*

$$t_c'(p_0) = \frac{2\pi}{b_1 h_0} + r_1^2 \frac{f(h_3, \dots, h_{2n})}{h_0} + o(r_1^2).$$

Ainsi, il apparaît que dans le cas nilpotent, pour des covecteurs de la forme

$$(0, 0, h_3, \dots, h_{2n}, h_0)$$

le temps conjugué nilpotent $\frac{2\pi}{b_1 h_0}$ est un zéro double de la fonction Φ. Or un zéro double est instable par perturbation, il peut demeurer double après perturbation mais également se séparer en deux zéros simples distincts ou même disparaître.

Les temps conjugués doubles ne sont pas une nouveauté dans le contexte de structures sous-riemanniennes équirégulières de pas 2, puisque ce phénomène se retrouve notamment dans les structures quasi-contact de dimension 4 [Cha02]. On emploie ici une approche différente de [Cha02] en éclatant la singularité en $h_1 = h_2 = 0$. En effet, nous recherchons des perturbations d'ordre $1/h_0^2$ du temps conjugué nilpotent $\frac{2\pi}{b_1 h_0}$. On sépare donc $\mathcal{C}_{q_0}(1/2)$ en deux domaines, selon que $h_1^2 + h_2^2 > \varepsilon$ ou $h_1^2 + h_2^2 < \varepsilon$, pour un $\varepsilon \in (0, 1)$ arbitraire. On va donc considérer le second cas de covecteurs de la forme, lorsque $h_0 \to +\infty$,

$$\left(\frac{h_1}{\sqrt{h_0}}, \frac{h_2}{\sqrt{h_0}}, h_3, \dots, h_{2n}, h_0\right).$$

Remarque 1.3.14. Le principe de cette méthode est de limiter au maximum l'introduction dans les calculs de termes provenant d'ordres supérieurs de l'exponentielle. On parvient ici à se limiter à des invariants d'ordre 2 des jets de la métrique. La méthode générale, telle qu'exécutée dans [EAGK96, Cha02], reviendrait à calculer des expressions plus précises de l'exponentielle, en introduisant des termes provenant des jets d'ordre 3. Étant donnée la croissance du nombre d'invariant à chaque degré d'approximation, cela aurait été un frein à l'exécution des calculs.

Cette approche mène au résultat de développement asymptotique suivant.

Théorème 1.3.15. *Soit (M, Δ, g) une variété sous-riemannienne de contact de dimension $2n+1$, $n \geq 2$, et soit $q_0 \in M$ tel que $b_1(q_0) > b_2(q_0) \geq \dots \geq b_n(q_0)$.*

Pour tout $\varepsilon \in (0, 1)$, soit $S(\varepsilon) = \{(h_1, \dots, h_{2n}, h_0) \in \mathcal{C}_{q_0}(1/2) \mid h_1^2 + h_2^2 < \varepsilon^2\}$. Il existe quatre applications $c_1, c_2, c_3, c_4 : \mathbb{R}^{2n} \to \mathbb{R}$ telles que pour tout $\varepsilon \in (0, 1)$

— *Si $p_0 = (h, h_0) \in \mathcal{C}_{q_0}(1/2) \setminus S(\varepsilon)$, lorsque $h_0 \to +\infty$*

$$t_c(h, h_0) = \frac{2\pi}{b_1 h_0} + \frac{1}{h_0^2} t_c^{(2)}(h) + O\left(\frac{1}{h_0^3}\right)$$

où $t_c^{(2)}$ est solution de

$$X(h_1^2 + h_2^2) + c_4(h) = 0.$$

— *Lorsque* $h_0 \to +\infty$, *le développement*

$$t_c\left(\frac{h_1}{\sqrt{h_0}}, \frac{h_2}{\sqrt{h_0}}, h_3, \ldots, h_{2n}, h_0\right) = \frac{2\pi}{b_1 h_0} + O\left(\frac{1}{h_0^2}\right)$$

ne vaut que si l'équation polynomiale quadratique

$$X^2 c_1(h) + X\left[(h_1^2 + h_2^2) + c_2(h)\right] + [c_3(h) + c_4(h)] = 0.$$

admet des solutions réelles. Dans ce cas on note $\widetilde{t}_c^{(2)}$ *la plus petite de ses solutions réelles, et pour* $p_0 = \left(\frac{h_1}{\sqrt{h_0}}, \frac{h_2}{\sqrt{h_0}}, h_3, \ldots, h_{2n}, h_0\right) \in S(\varepsilon)$, *lorsque* $h_0 \to +\infty$,

$$t_c\left(\frac{h_1}{\sqrt{h_0}}, \frac{h_2}{\sqrt{h_0}}, h_3, \ldots, h_{2n}, h_0\right) = \frac{2\pi}{b_1 h_0} + \frac{1}{h_0^2}\tilde{t}_c^{(2)}(h_1, \ldots, h_{2n}) + O\left(\frac{1}{h_0^3}\right).$$

Caustique sous-riemannienne. Ces calculs permettent d'obtenir une approximation du lieu conjugué représenté dans les figures qui suivent. La figure 1.5 permet d'avoir un aperçu de la nouvelle géométrie du lieu conjugué dans les dimensions (x_1, x_2, z) où elle est considérablement modifiée par rapport au cas nilpotent (cf figure 1.3), contrairement aux coordonnées $(x_3, \ldots, x_{2n})$. On donne également une représentation schématique des lieux conjugués nilpotents (figure 1.6) et du cas général (figure 1.7) pour en comparer la géométrie globale.

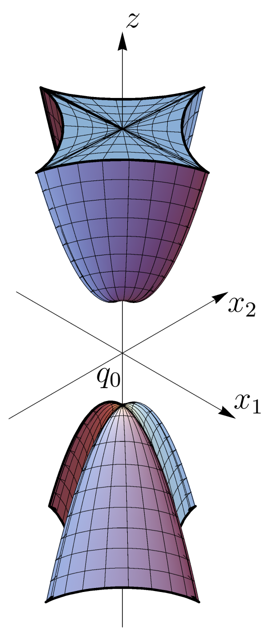
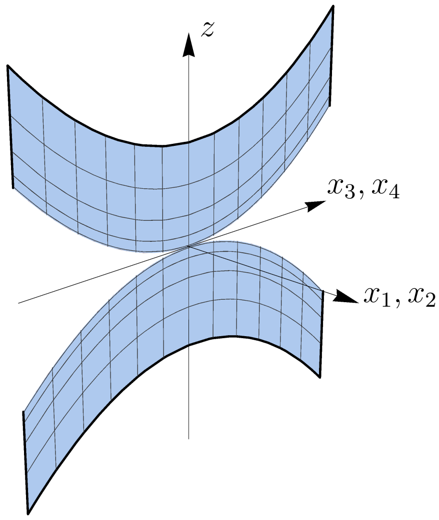


FIGURE 1.5 – Section du lieu conjugué en 5D par $(x_3, x_4) \neq (0,0)$ fixé et $z \in [-1, 1]$ (représentée sans éclatement de la singularité en $x_1 = x_2 = 0$.

On peut remarquer que certaines propriétés géométriques de la caustique sont perdues. Lorsque les covecteurs initiaux sont tels que $(h_3, h_4) \neq (0,0)$, le lieu conjugué apparaît séparé en deux composantes connexes.
On a parfois une section astroïdale, mais cela dépend de la valeur du couple (h_3, h_4) (cf figure 5.1 dans le chapitre 5 pour plus de précisions). Le rayon de la coupe transverse n'évolue plus en $z^{3/2}$. En effet pour $r_1 = (h_1^2 + h_2^2)^{1/2}$ et $r_2 = (h_3^2 + h_4^2)^{1/2}$, il existe des constantes $c_1, c_2 > 0$ telles que $F_5^{(2)}(\tau_c) = \eta^2(c_1 r_1^2 + c_2 r_2^2)$, alors que $F_i^{(2)}(\tau_c) \propto \eta^2 r_1^2 r_2$ pour $i \in \{1, 2\}$.

Remarque 1.3.16. Comme le montre le théorème 1.3.15 le temps conjugué n'est pas toujours une perturbation du temps conjugué nilpotent $2\pi/(b_1 h_0)$. On ne traitera pas ici des points conjugués associés à de tels covecteurs. Dans la suite, on se restreint donc aux points singuliers $\mathcal{E}_{q_0}^{t_c(p)}(p)$ pour lesquels $t_c(p) \sim 2\pi/(b_1 h_0)$.

FIGURE 1.6 – Représentation schématique du lieu conjugué d'un groupe de Heisenberg 5D (avec des valeurs propres distinctes)



Pour discuter la géométrie globale du lieu conjugué, on emploie par la suite une représentation schématique rassemblant x_1, x_2 d'une part et x_3, x_4 d'autre part. La représentation du lieu conjugué pour le groupe de Heisenberg est visible en figure 1.6.

Stabilité sous-riemannienne en dimension 5. Soient (M, Δ, g) et $(M, \widetilde{\Delta}, \widetilde{g})$ deux structures sous-riemanniennes de contact sur M, dont on note H et $\widetilde{H}$ les hamiltoniens. Soit $(p_0, q_0) \in T^*M$. Les exponentielles sous-riemanniennes $\mathcal{E}^1_{q_0}$ et $\widetilde{\mathcal{E}}^1_{q_0}$ des structures respectives (M, Δ, g) et $(M, \widetilde{\Delta}, \widetilde{g})$ sont *sous-riemanniennes équivalentes en* p_0 s'il existe deux voisinages $\Omega, \widetilde{\Omega}$ de (p_0, q_0) et deux applications $\phi : \pi(\Omega) \to \pi(\Omega')$, $\Phi : \Omega \to \Omega'$ telles que, pour σ la forme symplectique canonique, π la projection canonique du fibré T^*M

$$\Phi^*\sigma = \sigma, \qquad \pi \circ \Phi = \phi \circ \pi \quad \text{et} \quad \Phi \circ e^{\vec{H}} = e^{\vec{\widetilde{H}}}.$$

L'intérêt de ces considérations pour nous est le suivant : si deux exponentielles sont sous-riemanniennes équivalentes en un point, leurs caustiques sont difféomorphes (au voisinage de l'image de ce point). L'exponentielle sous-riemannienne $\mathcal{E}^1_{q_0}$ est *sous-riemannienne-stable en* p_0 s'il existe un voisinage V de p_0 et un voisinage W de $\mathcal{E}^1_{q_0|V}$ dans la classe des applications exponentielles sous-riemanniennes en q_0 pour la topologie de Whitney C^∞ tel que tous les éléments de W sont sous-riemannienne-équivalentes à $\mathcal{E}^1_{q_0}$ en p_0.

On peut observer que la stabilité sous-riemannienne est la restriction à la classe des applications exponentielles de la stabilité lagrangienne (cf [AGnZV85, Ben85, IFR$^+$16]). En effet l'exponentielle sous-riemannienne étant la projection du flot hamiltonien sous-riemannien, elle possède une structure naturelle d'application lagrangienne. Un fait connu de la stabilité des applications lagrangiennes est qu'en dimension inférieure ou égale à 5 il existe un nombre fini de classes d'équivalence pour la stabilité lagrangienne (voir par exemple le résumé dans [BBCN16, théoreme 2]).

Théorème 1.3.17. *Une application lagrangienne générique de $\mathbb{R}^5$ dans $\mathbb{R}^5$ n'a que des singularités stables, de type $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$, $\mathcal{A}_5$, $\mathcal{A}_6$, $\mathcal{D}_4^\pm$, $\mathcal{D}_5^\pm$, $\mathcal{D}_6^\pm$ et $\mathcal{E}_6^\pm$.*

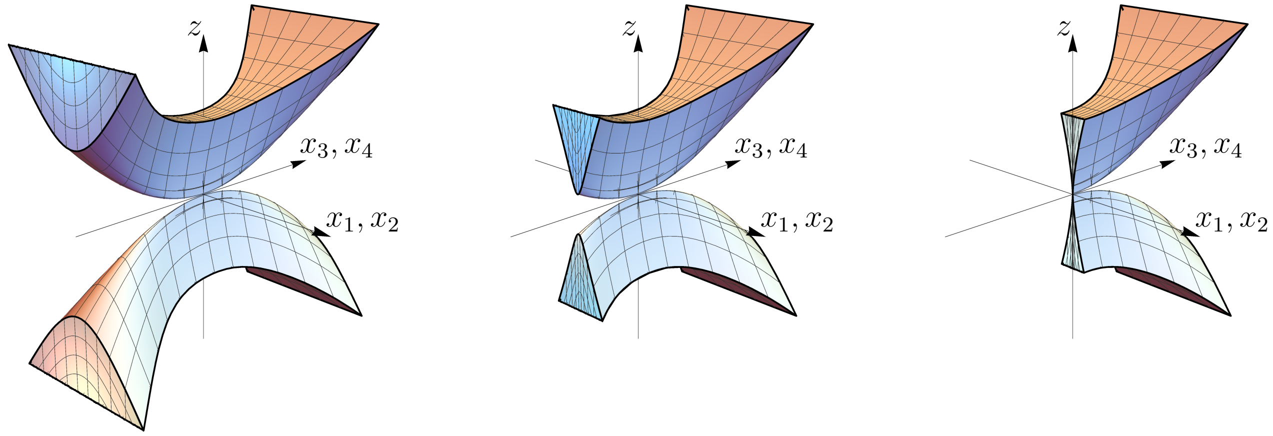


FIGURE 1.7 – Représentation schématique de la caustique d'une structure de contact 5D (avec des valeurs propres différentes). On peut observer que le lieu conjugué est pincé en q_0.

On donne un équivalent de ce résultat dans le cas des variétés sous-riemanniennes de contact, qui permet d'affirmer la stabilité de la caustique sous-riemannienne en dehors du point d'origine q_0.

Théorème 1.3.18. *Soit (M, Δ, g) une variété sous-riemannienne de contact de dimension 5 générique. Pour tout q_0 dans le complémentaire d'un sous-ensemble stratifié de codimension 1 il existe un voisinage V_{q_0} de q_0 tel que pour tout ouvert U contenant q_0, l'intersection de la caustique avec $V_{q_0} \setminus U$ est (sous-riemannienne-)stable et ne possède que des singularités lagrangiennes de type $\mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{D}_4^+$ et $\mathcal{A}_5$.*

Remarque 1.3.19. On étend ici les résultats précédents sur la stabilité du lieu conjugué pour les singularités de structures contact 3D et quasi-contact 4D. Sous un même hypothèse de restriction au complémentaire d'un sous-ensemble stratifié de codimension 1, on a la stabilité de la caustique sous-riemannienne sur un voisinage ouvert de q_0 (privé d'un petit voisinage de q_0), et les singularités sont de type $\mathcal{A}_2, \mathcal{A}_3$ en contact 3D [Agr96, EAGK96] et $\mathcal{A}_2, \mathcal{A}_3, \mathcal{D}_4^+$ en quasi-contact 4D [Cha02].

La preuve de ce fait repose sur une classification des singularités de l'exponentielle sous-riemannienne. On s'appuie sur le caractère lagrangien de l'exponentielle.

Proposition 1.3.20. *Soit (M, Δ, g) une variété sous-riemannienne de contact et soit $(p_0, q_0) \in T^*M$. Si $\mathcal{E}_{q_0}$ est Lagrange-stable en p_0 alors elle est sous-riemannienne stable en p_0.*

La classification s'effectue sur trois domaines, correspondant aux trois cas : (h_1, h_2) petits, (h_3, h_4) petits et leur complémentaire. L'introduction d'un troisième domaine ((h_3, h_4) petits) par rapport au problème de l'étude du temps conjugué tient à la géométrie du lieu conjugué. En effet le pincement du lieu conjugué en q_0 (cf figure 1.7) implique l'annihilation d'un ordre d'approximation, de manière similaire au cas 3D. On fait alors le choix de le traiter par éclatement, pour des raisons similaires au cas (h_1, h_2) petits.

Les domaines de classification et les singularités qui y sont observées sont représentés sur la figure 1.8.

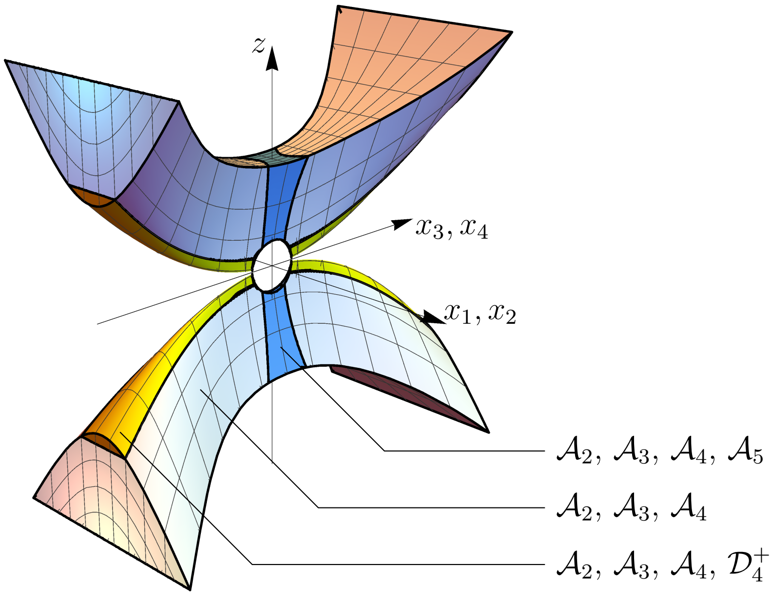


FIGURE 1.8 – Représentation schématique des trois domaines de la caustique étudiée d'où résulte la classification.

1.3.5 L'influence des invariants

Une des motivations de l'étude que nous venons d'exposer est de comprendre les similarités qui existent entre le cas 3D aujourd'hui bien compris et les variétés sous-riemanniennes de contact de dimension supérieures. Cette étude révèle que la géométrie des variétés de dimension supérieure est au final assez différente du fait de l'existence d'invariants du tangent métrique qui brisent la symétrie qui existait dans la structure 3D. Nous présentons ce fait par deux discussions.

Approximations de l'exponentielle. Comme nous le soulignions dans le commentaire de la proposition 1.3.12, dans le cas 3D, l'exponentielle sous-riemannienne est bien approchée par l'exponentielle en 0 de l'approximation nilpotente, moyennant le choix de coordonnées s'appuyant sur les symétrie du groupe de Heisenberg. L'asymptotique du lieu conjugué en dimension supérieure à 3 semble être en contradiction avec cette observation.

En 3D, on peut relier l'existence de coordonnées telles que

$$\mathcal{E}_{q_0}^{t/h_0}(h, h_0) = \widehat{\mathcal{E}}^{t/h_0}(h, h_0) + O(1/h_0^3) \tag{1.10}$$

au fait que le temps conjugué admet dans ces coordonnées un développement asymptotique de la forme
$$t_c(h, h_0) = \frac{2\pi}{h_0} + O(1/h_0^3).$$
En effet si on note $t_c^{(2)}$ tel que $t_c(h, h_0) = \frac{2\pi}{h_0} + \frac{t_c^{(2)}}{h_0^2} + O(1/h_0^3)$,

$$\mathcal{E}_{q_0}^{t_c}(h, h_0) = \widehat{\mathcal{E}}^{2\pi/h_0}(h, h_0) + \frac{t_c^{(2)}}{h_0^2}\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\mathcal{E}}^{2\pi/h_0} + O(1/h_0^3).$$

Le lieu conjugué étant tangent droite transverse à la distribution, on a alors que le terme $t_c^{(2)}\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\mathcal{E}}^{2\pi/h_0}$ doit être nul en choisissant des coordonnées (x, z) telles que l'axe des z porte la droite tangente en q_0 au lieu conjugué. En d'autres termes cela impose que $t_c^{(2)} = 0$.

De la géométrie du lieu conjugué on peut donc tirer des informations sur le comportement asymptotique de l'exponentielle. On s'inspire de cette observation pour prouver que ce comportement ne se reproduit pas en dimension plus grande.

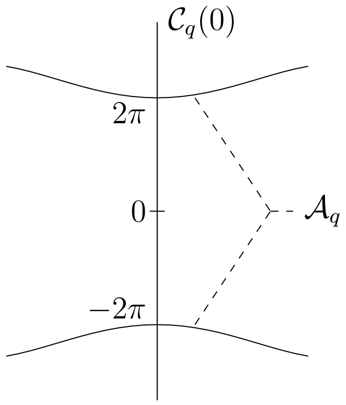
En dimension plus grande que 3 on ne peut pas attendre qu'il existe une droite tangente au lieu conjugué à cause de la géométrie du lieu conjugué nilpotent. On considère alors une propriété géométrique qui joue un rôle dual. Soit (M, Δ, g) une variété sous-riemannienne de contact et soit $q \in M$. On définit $\mathcal{A}_q \subset T_q^*M$ par

$$\mathcal{A}_q = \overline{\{t_c(p)\, p \mid p \in T_q^*M, H(p, q) > 0\}}.$$

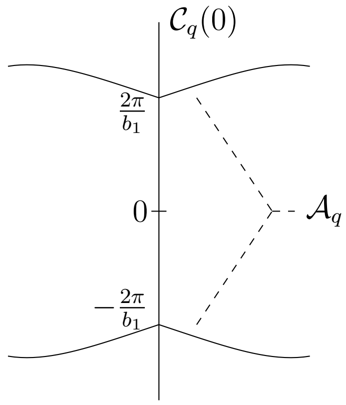
Il s'agit de l'ensemble des points singuliers pour l'exponentielle au temps 1 ne correspondant pas à une extrémale anormale. L'ensemble $\mathcal{A}_q$ est une hypersurface plongée, telle que pour $r > 0$ suffisamment petit, $A_q \cap \mathcal{C}_q([0, r])$ possède exactement deux composantes connexes (voir figure 1.9). En particulier l'intersection de $\mathcal{A}_q$ avec $\mathcal{C}_q(0)$ est réduite à deux points symétriques par rapport au 0 de T_q^*M que l'on note p_+ et p_- de coordonnées $(0, \pm\frac{2\pi}{\max b_i})$ (avec $\max b_i = 1$ dans le cas 3D). Dans le cas nilpotent par exemple, $\mathcal{A}_q$ est l'union des hyperplans $\{h_0 = \pm\frac{2\pi}{\max b_i}\}$.

L'analogue du raisonnement sur la géométrie en q_0 du lieu conjugué revient à considérer la géométrie de $\mathcal{A}_q$ en p_+ (ou p_- de façon équivalente). La propriété sur l'existence de développements asymptotiques semblables à (1.10) revient à prouver que l'hypersurface $\mathcal{A}_q$ est tangente à un hyperplan en p_+, ce qui est toujours vrai en dimension 3, et faux en général en dimension supérieure. (De plus, l'orthogonal dans T_qM du tangent à $\mathcal{A}_q$ en p^+ est la droite tangente au lieu conjugué.)

On peut chercher à justifier cette différence avec l'observation suivante. Dans le cas nilpotent 3D, la symétrie naturelle de la géométrie du front d'onde du groupe de Heisenberg est SO(2). Pour tout $\theta \in$ SO(2), l'action de θ commute avec l'exponentielle $\theta \cdot \widehat{\mathcal{E}}_0(p) = \widehat{\mathcal{E}}_0(\theta \cdot p)$, à condition qu'elle laisse stable ∂_z d'un côté et $\mathcal{C}_q(0)$ de l'autre. Ainsi au temps conjugué, l'action SO(2) est bien celle qui implique la conjugaison des points.



(a) Cas 3D



(b) Cas 5D

FIGURE 1.9 – Représentation schématique de l'ensemble $\mathcal{A}_q$, lisse à son intersection avec $\mathcal{C}_q(0)$ uniquement dans le cas 3D.

La généralisation de ce fait en dimension supérieure revient ici à considérer l'action du groupe $\mathrm{SO}(2)^n$, car à chaque invariant b_i correspond une rotation dans les coordonnées (x_{2i-1}, x_{2i}). Au temps conjugué, une trace du covecteur initial est conservée dans les coordonnées (x_{2i-1}, x_{2i}) telles que b_i n'est pas maximale. Ceci implique la non planéité de $\mathcal{A}_q$ en p^+ et p^-.

Si $\mathcal{A}_q$ admet un plan tangent à son intersection avec $\mathcal{C}_q(0)$, la symétrie naturelle devient en revanche l'action de $\mathrm{SO}(2n)$ dans T_q^*M qui laisse stable $\mathcal{C}_q(0)$. Il est clair que si $n = 1$, $\mathrm{SO}(2n) = \mathrm{SO}(2)^n$, mais en dimension supérieure, cela revient à imposer $b_1 = \cdots = b_n$.

Complémentaires intrinsèques. L'ensemble de ces considérations trouve un écho dans la construction de champs de vecteurs transverses à la distribution.

Soit (M, Δ, g) une variété sous-riemannienne de contact de dimension $2n+1$. Il existe (localement) une 1-forme de contact ω telle que $\omega(\Delta) = 0$ et $\omega \wedge (\mathrm{d}\omega)^n \neq 0$. Pour toute application lisse $f : M \to \mathbb{R} \setminus \{0\}$, la 1-forme $f\omega$ est également une 1-forme de contact. La forme de contact ω peut être alors caractérisée par la fonction lisse $\alpha : M \to \mathbb{R} \setminus \{0\}$ telle que $(\mathrm{d}\omega_{|\Delta})^n = \alpha \,\mathrm{vol}_g$, où vol_g est la forme volume induite par g sur Δ.

Le noyau de la 2-forme $\mathrm{d}\omega$ définit en tout point un sous-espace de dimension 1 de l'espace tangent et tout champ de vecteurs non nul X_0 tel que $\mathrm{d}\omega(X_0, \cdot) = 0$ est caractérisé par la fonction lisse $\beta : M \to \mathbb{R} \setminus \{0\}$ telle que $\omega(X_0) = \beta$ (le champ de Reeb correspond au cas $\beta = 1$).

On peut considérer le champ X_0 comme intrinsèque si α et β sont fonctions des invariants du tangent métrique. En dimension 3, il n'y a pas d'invariant dépendant de la position, ainsi α, β constants est le seul choix possible. Dans ce cas, la direction donnée par le noyau de $\mathrm{d}\omega$ est la droite tangente au lieu conjugué à son point de départ.

En dimension supérieure, les invariants du tangent métrique (c'est-à-dire les $(b_i)_{1\leq i\leq n}$) dépendent de la position. Alors le vecteur X_0 peut être arbitrairement fixé en un point par choix des fonctions α et β. Géométriquement, on observe bien qu'en dimension supérieure à 3, aucune direction intrinsèque n'apparaît comme canonique par des arguments de géométrie du lieu conjugué. Ni le lieu conjugué ni son antécédent ne permettent de discriminer une telle direction. Cette observation se retrouve dans d'autres aspects de la géométrie des variétés sous-riemanniennes de contact.

Dans [BNR17], les auteurs comparent deux constructions concurrentes de laplaciens sous-riemanniens. L'une via le choix d'une forme volume vol sur la variété et l'autre via le choix d'un complémentaire $\mathbf{c}$ de la distribution. On note alors L_{vol} et $L^{\mathbf{c}}$ les laplaciens obtenus par ces constructions respectives. On y trouve l'observation suivantes concernant les variétés de contact.

Soit ω une 1-forme de contact pour (M,Δ,g) et soit X_0 un champ de vecteurs transverse à la distribution. On définit $\omega' = \frac{\mathrm{d}\omega(X_0,\cdot)}{\omega(X_0)}$ et $g = \frac{\mathrm{d}\omega'\wedge\omega\wedge(\mathrm{d}\omega)^{n-1}}{\omega\wedge(\mathrm{d}\omega)^n}$. Si $\mathrm{d}\omega' = \mathrm{d}g\wedge\omega + g\,\mathrm{d}\omega$, alors ω' est une forme de contact dont X_0 est le Reeb, et il existe un unique volume vol (à multiplication par une constante près) tel que $L_{\mathrm{vol}} = L^{\mathbf{c}}$ avec $\mathbf{c}$ engendré par $\{X_0\}$. Le choix arbitraire d'un complémentaire de la distribution n'est donc pas non plus entravé par l'existence d'un volume tel que $L_{\mathrm{vol}} = L^{\mathbf{c}}$.

1.4 Singularités de champs de directions

La géométrie sous-riemannienne trouve une application dans la modélisation des sensibilités des neurones du cortex visuel V1, représentée comme une structure sous-riemannienne de contact 3D sur $\mathbb{R}^2\times\mathbb{P}^1$. L'organisation des neurones dans le cortex visuel ne satisfait toutefois pas cette géométrie, qui admet plutôt une modélisation mathématique comme un champ de directions où les singularités sont très nombreuses. L'apparition récurrente de certaines configurations aux singularités à travers différents domaines d'applications invite à étudier la stabilité de ces configurations. La définition naturelle des champs de direction ne permet pas une approche convenable de ce problème, c'est pourquoi nous proposons une construction alternative permettant de modéliser et d'étudier ces singularités.

1.4.1 Les champs de directions et leurs singularités

Géométrie sous-riemannienne, vision et champs de directions. On doit à Hubel et Wiesel [HW62] l'observation que le cortex visuel V1 de certains mammifères est muni de neurones sensibles à la fois aux positions et aux orientations. Ceci signifie l'existence d'une structure sous-jacente aux neurones, dont la sensibilité peut-être représentée par des points de l'espace $PT\mathbb{R}^2 = \mathbb{R}^2\times\mathbb{P}^1$, le fibré des directions du plan. Les connexions préférentielles entre neurones sensibles aux positions et orientations proches sont ensuite modélisées par une structure sous-riemannienne de contact sur $PT\mathbb{R}^2$ [CS06, Pet08]. Il s'agit de la classique structure engendrée par la base orthonormée (avec $\beta > 0$ un paramètre de la distance)

$$X_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 \\ 0 \\ \beta \end{pmatrix}.$$

Au niveau de l'agencement spatial des neurones dans le cortex visuel, on observe qu'ils ne sont pas répartis selon cette structure mais au contraire selon une structure en colonnes, où deux neurones de même position mais de profondeurs différentes partagent la même sensibilité. Ainsi la répartition des sensibilités aux orientations peut également être modélisée par un champ de directions, une section du fibré des directions de $\mathbb{R}^2$, qui associe à chaque point de $\mathbb{R}^2$ une direction dans $\mathbb{P}^1$ (voir figure 1.10).

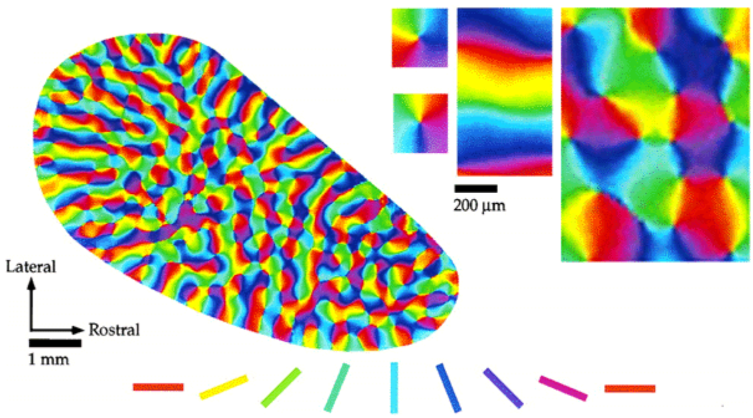


FIGURE 1.10 – Cartographie des orientations préférentielles du cortex visuel V1 obtenue par imagerie médicale [BZSF97].

Les travaux de cartographie des sensibilités aux orientations mettent en évidence une structure particulière. On observe une haute densité de singularités où toutes les orientations s'accumulent autour d'un point. En outre, deux types de singularités sont repérés, les points d'arrêt et les points triples, correspondant à deux chiralités possibles pour l'organisation des orientations autour d'une singularité.

Champs de directions. On doit à Hopf [Hop03] l'une des premières études des champs de directions et de leurs singularités en dehors de leurs domaines d'application historiques. Il définit un champ de directions comme l'application qui associe à chaque point d'une variété M une droite tangente à M en ce point. Une singularité d'un champ de directions est alors un point de M où l'application n'admet pas de prolongement continu. Cette définition initiale permet tout de même d'étendre la notion d'indice de singularités de champs de vecteurs au cas de champs de direction. Pour rappel, pour L un champ de directions continu sur un voisinage de $p \in M$, privé de p, l'*indice de L en p* est un réel quantifiant de la variation d'angle de L en tournant autour de p. Pour X un champ de vecteurs continu et non-singulier en p et C une courbe simple fermée entourant p dans un voisinage suffisamment petit, l'indice admet l'expression

$$\mathrm{ind}_p L = \frac{1}{2\pi}\delta_C \angle[X, L],$$

où l'on note $\angle[X, L]$ l'angle orienté entre X et L (dans $[0, \pi)$), et $\delta_C \angle[X, L]$ la variation totale de l'angle entre X et L le long de C. Dans le cas d'un champ de vecteurs continu, l'indice est toujours entier, et dans le cas d'un champ de directions continu sur un voisinage de p privé de p, c'est un demi-entier.

L'étude de Hopf permet l'extension de son théorème liant l'indice totale et la caractéristique d'Euler d'une surface.

Théorème 1.4.1 (Poincaré-Hopf)**.** *Soit (M, g) une variété riemannienne 2D, orientable et compacte, de caractéristique d'Euler $\chi(M)$, et soit L un champ de directions sur M dont les singularités sont isolées. Soit z_L l'ensemble des singularités de L, on a*

$$\sum_{p \in z_L} \mathrm{ind}_p L = \chi(M).$$

Par analogie avec les champs de vecteurs, on emploie une définition alternative plus précise des champs de directions. Soit M une variété de dimension 2. Le *fibré des directions* PTM est le fibré des droites tangentes à M en tout point, qui peut être obtenu par projectivisation PT_pM de l'espace vectoriel T_pM en tout $p \in M$. Par analogie avec les champs de vecteurs, on nomme alors *champ de direction* sur M une section du fibré PTM. Un champ de directions lisse est alors une section lisse du fibré PTM.

Une conséquence du théorème 1.4.1 sur les champs de vecteurs est qu'un champ de vecteurs continu doit s'annuler au moins une fois sur une variété compacte de caractéristique d'Euler non nulle. L'analogue de cette observation pour les champs de directions est qu'il n'existe pas de section continue de PTM si $\chi(M) \neq 0$. Autrement dit, la définition d'un champ de directions régulier en tant que section du fibré des directions d'une variété réclame l'exclusion *a priori* de ses singularités de son domaine de définition. C'est cette observation fondamentale qui motive l'introduction des éléments présentés dans la section suivante.

Outil de modélisation. Bien que les champs de directions et leurs singularités soient plus délicats à définir que les champs de vecteurs, leur utilité en tant qu'outil de modélisation se justifie par leur apparition dans de nombreux domaines qui vont au delà des structures singulières dans le cortex visuel.

De façon similaire aux courbes intégrales de champs de vecteur, on définit les *variétés intégrales* d'un champ de directions L sur M comme l'ensemble des courbes feuilletant la variété M et de tangent donné par L en tout point. Les variétés intégrales et leurs singularités sont observables dans plusieurs domaines d'application classiques tels que les empreintes digitales [Pen79, KW87], les cristaux liquides [Cha92, Pro95] et plus récemment l'optimisation de structure [PT08] (cf également la thèse de P. Geoffroy). (Voir figure 1.11.)

(a) Cristaux liquides en phase nématique.

(b) Empreintes digitales.

(c) Optimisation de structure avec matériau lattice.

FIGURE 1.11 – Singularités telles qu'observées dans les cristaux liquides[1], les empreintes digitales et optimisation de structure[2].

Un élément remarquable traversant ces applications est la prédominance des singularités d'indice $\pm 1/2$ (cf figure 1.12). Du point de vue de la modélisation géométrique des singularités de champs de directions, cela renvoie aux concepts de stabilité et généricité de l'école de Thom [Tho72], seules des singularités stables face à des petites perturbations peuvent être observées dans la nature.

Directions principales de courbure. Un domaine où les champs de directions et leurs singularités ont longtemps fait l'objet d'un grand intérêt est celui de la géométrie gaussienne des surfaces [Cay63, Por01]. En tout point d'une surface S immergée dans $\mathbb{R}^3$, les directions de courbure principales définissent chacune un champ de directions (orthogonaux entre eux). Les ombilics de la surface S sont les points pour lesquels les deux courbures principales sont égales, et correspondent à des singularités des champs de courbure principale (voir figure 1.13).

1. images de singularités de cristaux liquides en phases nématiques gracieusement fournies par S.J. DeCamp.

2. images de singularités dans la structure d'un poteau gracieusement fournies par P. Geoffroy.

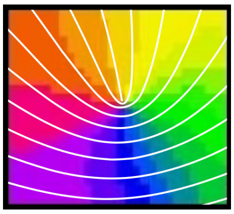
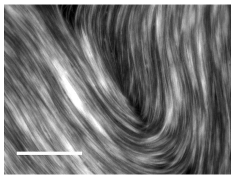
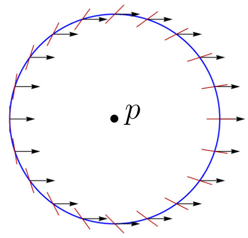
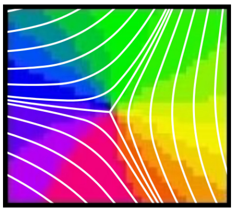
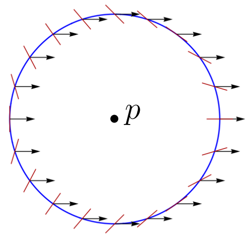
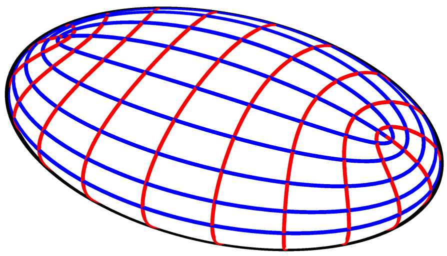
Cortex visuel	Empreintes digitales	Cristaux liquides	Indice $\pm 1/2$
			
			

FIGURE 1.12 – Les singularités de champs directions, telles qu'elles apparaissent dans la nature comportent en majorité des singularités d'indice $\pm 1/2$.

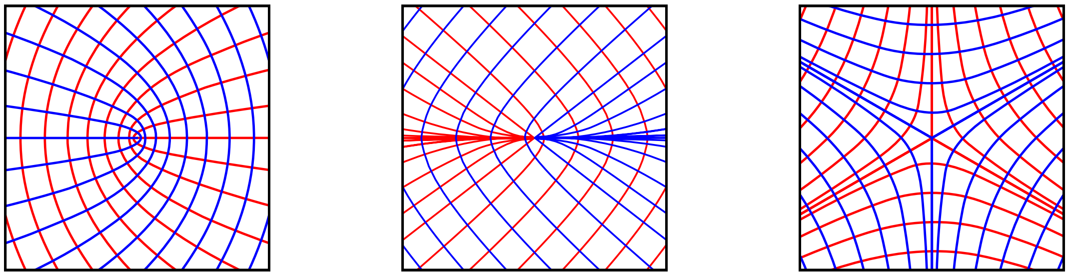
FIGURE 1.13 – Variétés intégrales des directions de courbure principales sur un ellipsoïde triaxial. Cette surface possède quatre ombilics correspondant toutes à des singularités d'indice 1/2.



Les champs de directions de courbure principale sont munis d'une topologie naturelle donnée par l'immersion $\alpha : M \to S \subset \mathbb{R}^3$. Trois singularités, *Lemon*, *Monstar*, *Star* (cf [BH77] et la figure 2.14), sont identifiées par Darboux dans [Dar96]. La stabilité structurelle des singularités (au sens des homéomorphismes entre lignes de champs) est prouvée par Gutierrez et Sotomayor [SG82] (voir également [SG08]).

1.4.2 Singularités de champs bissecteurs

Bissection de champs de vecteurs. Comme on l'a évoqué dans la section précédente, on aimerait se munir d'outils permettant une étude de la stabilité des singularités de champs de direction. La définition d'une section du fibré des directions impose, contrairement aux champs de vecteurs, de retirer le lieu singulier d'un champ de directions de son ensemble de définition. Dans ce cas la topologie naturelle des champs de direction ne permet pas la perturbation des positions des singularités d'un champ. On introduit une construction inspirée des champs de vecteurs qui permet définir et étudier plus simplement



(a) Lemon (b) Monstar (c) Star

FIGURE 1.14 – Représentation des intégrales des directions de courbure maximales au voisinage des trois singularités stables.

des champs de directions, leurs singularités et leur topologie.

On considère dans la suite une variété différentielle lisse M de dimension 2, munie d'une structure riemannienne g. On peut remarquer que seule la structure conforme donnée par g est nécessaire ici, car elle nous permet de mesurer des angles. Un orientation locale est nécessaire pour considérer des angles modulo 2π mais il n'est pas nécessaire de supposer M orientable.

Définition 1.4.2. Soit (M, g) une variété Riemannienne 2D. Un *proto champ de directions* est un couple (X, Y) de champs de vecteurs sur M.

Si z_X et z_Y sont les zeros de X et Y. Le *champ de directions associé à* (X, Y), noté $B(X, Y)$, est la section de $PT(M \backslash (z_X \cup z_Y))$ définie en $p \in M \backslash (z_X \cup z_Y)$ comme la droite $B(X(p), Y(p))$ de T_pM qui bissecte $(X(p), Y(p))$ pour la métrique $g(p)$.

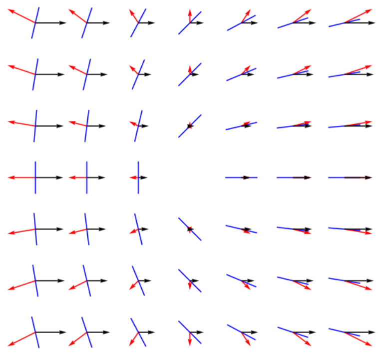
Les singularités de X et Y coïncident ainsi avec les singularités de $B(X, Y)$. On a également pourvu l'ensemble des champs de directions obtenus par bissection d'une topologie naturelle, la topologie de Whitney C^k sur les couples de champs de vecteurs. On peut justifier du bien fondé de cette définition avec les observations suivantes.

Proposition 1.4.3. *Soit (M, g) une variété Riemannienne 2D, K un ensemble fermé de M et L une section lisse de $PT(M \backslash K)$. Il existe des champs de vecteurs lisses X et Y tels que $L = B(X, Y)$.*

Proposition 1.4.4. *Soit (X, Y) un proto champ de directions sur (M, g). Étant donné un point isolé p de $z_X \cup z_Y$, on a $\mathrm{ind}_p(B(X, Y)) = \frac{1}{2}(\mathrm{ind}_p(X) + \mathrm{ind}_p(Y))$. (Voir figure 1.15)*

En conséquence, les champs de directions obtenus par bissection ont les comportements attendus de champs de direction classique, mais sont bien munis d'une définition et topologie permettant l'étude de la stabilité des singularités.

FIGURE 1.15 – La bissection d'un champ de vecteurs ayant une singularité d'indice 1 par un champ de vecteurs n'en ayant pas forme un champ de directions ayant une singularité d'indice 1/2.



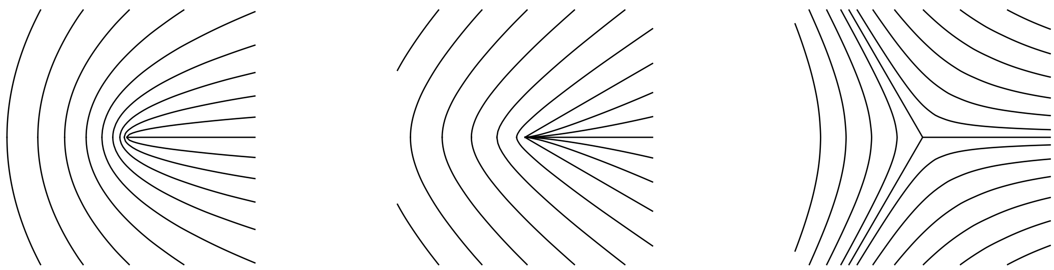
Stabilité et généricité. De la topologie de Whitney C^1 sur les couples de champs de vecteurs, on obtient sans effort un premier résultat encourageant dans la classification des singularités de champs de direction.

Proposition 1.4.5. *Le champ bissecteur d'un couple (X, Y) générique ne possède que des singularités isolées d'indice $\pm 1/2$.*

Par analogie avec les champs de vecteurs, on choisit de s'intéresser à la stabilité structurelle des champs bissecteurs à leurs singularités. Un proto-champ de directions est *localement structurellement stable en $p \in M$* si pour tout voisinage V_p de p il existe un voisinage $V_{(X,Y)}$ de (X, Y) pour la topologie de Whitney C^1 tel que pour tout $(X', Y') \in V_{(X,Y)}$, $B(X, Y)$ en p est topologiquement équivalent à $B(X', Y')$ en $p' \in V_p$ (au sens des homéomorphismes envoyant les variétés intégrales sur les variétés intégrales). Nous sommes contraints de définir ces notions topologiques sur les couples (X, Y) à cause de la non injectivité de l'opération de bissection.

Le travail du chapitre est essentiellement consacré à prouver le théorème suivant.

Théorème 1.4.6. *Génériquement vis à vis de la topologie de Whitney sur les couples de vecteurs, le proto-champ de directions (X, Y) est localement structurellement stable et ne possède que des singularités de type Lemon, Monstar et Star.*



(a) Lemon (b) Monstar (c) Star

Les méthodes employées pour prouver ce théorème résultent en grande partie des méthodes classiques de la stabilité structurelle des champs de vecteur en dimension 2 (voir, par exemple, [DLA06]). L'approche est similaire, nous prouvons dans un premier temps que le linéarisé d'un proto-champ de directions possède génériquement une singularité d'un des trois types cités dans le théorème 1.4.6. Ensuite nous prouvons que dans le cas d'un couple générique, tous les éléments d'un petit voisinage doivent avoir le même linéarisé (ou suffisamment proche). Cette propriété permet de construire à la main un homéomorphisme qui envoie les variété intégrales de deux singularités du même type les unes sur les autres.

Concernant l'étape linéaire, en supposant en coordonnées que $X(0) = 0$ et $Y(0) \neq 0$ (c'est le cas générique), on s'intéresse au bissecteur du proto-champ $L(p) = (D_0X \cdot p, Y(0))$ et à la fonction

$$\varphi(\theta) = \angle\left[e_1, B(L)(\cos\theta, \sin\theta)\right]$$

(avec $e_1 = (1, 0)$). La fonction φ mesure l'angle entre le champs bissecteur de L au point $(\cos\theta, \sin\theta)$ et l'horizontale (arbitraire) $(1, 0)$. En considérant les points fixes de φ et leur attractivité, on peut alors discriminer entre les cas du théorème 1.4.6.

Concernant l'étape de construction des homomorphismes, on remarque que la bissection aux singularités peut être comparée à l'action de l'application racine carrée $z \mapsto z^{1/2}$ dans $\mathbb{C}$. Une idée est alors d'effectuer l'opération inverse en construisant une opération carré sur les champs directions, permettant de récupérer des champs de vecteurs dont on prouve la stabilité. On couple cette observation avec un éclatement de la singularité, qui permet de calculer les homéomorphismes nécessaires aux équivalences topologiques de manière classique.

Remarque 1.4.7 (Sur la stabilité des singularités Monstar). On peut s'étonner de trouver trois types de singularités stables alors que dans la plupart des applications deux seulement sont relevées. On peut déjà relever que la stabilité structurelle n'est pas la plus proche des applications étant donné que la limite de résolution des observation ne permet pas toujours de différencier une singularité Monstar d'une singularité Lemon.

On peut toutefois contourner cette difficulté en considérant l'analogue de la fonction φ définie plus haut. La fonction φ permet effectivement de discerner la distribution des orientations au voisinage de la singularité, mais également de remarquer qu'il n'y a pas de différence essentielle entre les deux singularités stables d'indice positif. Leur différence tient en grande partie à l'homogénéité de la répartition des angles autour de la singularité (cf figure 6.5 dans le chapitre 6).

Chapitre 2

Introduction

We present here the results developed in this thesis where we explore the link between sub-Riemannian geometry and singularity theory. After a brief reminder of some of the key elements of the theory in Section 2.1, we devote this chapter to the contributions presented in the following chapters.

In Section 2.2 we present the subject of sub-Riemannian Whitney extension theorems, the main matter of Chapter 3. Section 2.3 is an introduction to both Chapters 4 and 5. The former presents a general approach to the geometry of the sub-Riemannian contact conjugate locus, and the latter is an application of these results to the study of the stability of the sub-Riemannian caustic in dimension 5. We end this introductory chapter with Section 2.4 on the motivations and techniques developed in the modeling of singularities of line fields, that are explained in details in the final Chapter 6.

Contents

2.1 Sub-Riemannian geometry preliminaries

We start off by presenting some key aspects of sub-Riemannian geometry. Notions that are essential to the comprehension of the chapters are recalled here, and we try to highlight their role in the following of this thesis. For more details, the reader is referred to books on sub-Riemannian geometry offering a global overview of the field [ABB16, BBS16, Mon06, Rif14], as well as the references in the following sections.

2.1.1 Sub-Riemannian manifolds

First notions. We give the definiton of a sub-Riemannian manifold as it can be found in [ABB16].

Definition 2.1.1. Let M be a smooth connected manifold. A *sub-Riemannian structure* on M is a pair (U, f) where

i. U is a Euclidean bundle with base M and Euclidean fiber U_q, *i.e.*, for every $q \in M$, U_q is a vector space endowed with a scalar product $(\cdot \mid \cdot)_q$ smooth with respect to q. In particular, the dimension of U_q is constant with respect to $q \in M$.

ii. $f : U \to TM$ is a smooth map that is a morphism of vector bundles, *i.e.*, f is linear on fibers and the diagram

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[dr, "\pi_U"'] & TM \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

is commutative (with $\pi_U : U \to M$ and $\pi : TM \to M$ the canonical projections).

iii. The set of *horizontal vector fields* $\Delta = \{f(\sigma) \mid \sigma : M \to U \text{smooth section}\}$ is a Lie bracket-generating family of vector fields.

A *sub-Riemannian manifold* is then a triple (M, U, f) where M is a smooth manifold endowed with a sub-Riemannian structure (U, f). The *distribution* of this manifold is the family of subspaces

$$(\Delta_q)_{q \in M} \text{ where } \Delta_q = f(U_q) \subset T_q M,$$

and $\dim \Delta_q$ is called the *rank of the sub-Riemannian structure at q*.

This general purpose definition is especially useful when considering maps between sub-Riemannian manifolds (for instance, one can define projections and lifts, see Chapter 3). However, when possible, we chose to rather use the notation (M, Δ, g) when referencing (M, U, f), where g is the bilinear form defined below.

Definition 2.1.2. For all $q \in M$, let g_q be the quadratic form on Δ_q obtained by polarization identity form its value on the diagonal: for all $v \in \Delta_q$,

$$g_q(v, v) = \inf\{(u \mid u)_q \mid f(q, u) = (q, v), (q, u) \in U\}.$$

A Lispchitz-continuous curve $\gamma : I \to M$ is said to be *horizontal* if there exists a measurable and essentially bounded map $u : I \to U$ such that $\pi_U(u) = \gamma$ and $\dot\gamma(t) = f(\gamma(t), u(t))$ for almost every $t \in I$. The map u is called the *control* associated with γ. Furthermore, the curve γ is said to be C^1_H if the associated control u is continuous.

The length $l(\gamma)$ of a horizontal curve γ is defined by

$$l(\gamma) = \int_I g(\dot\gamma(t), \dot\gamma(t))^{1/2}\,\mathrm{d}t.$$

With this length we define the *Carnot-Caratheodory distance between two points* $q_0, q_1 \in M$:

$$\mathrm{d}_{\mathrm{SR}}(q_0, q_1) = \inf\left\{l(\gamma) \mid \gamma : [0, 1] \to M \text{ horizontale}, \gamma(0) = q_0, \gamma(1) = q_1\right\}.$$

The Chow-Raschevskii Theorem states that any two points of M are connected by a finite length horizontal curve and that $(M, \mathrm{d}_{\mathrm{SR}})$ is a metric space sharing the topology of M as a differential manifold. It is worth pointing out that two points of a sub-Riemannian manifold are always connected by a length-minimizing curve, provided they are close enough. This property holds for any two points if the manifold is assumed to be complete (see [ABB16, Theorem 3.40]).

Length-minimizing curves.

Definition 2.1.3. Let (M, U, f) be a sub-Riemannian manifold and Ω be an open subset of M. An *orthonormal frame of the distribution on* Ω is a family of vector fields $(X_1, \dots, X_m)$ such that there exists a smooth orthonormal frame $(e_1, \dots, e_m)$ of the Euclidean fiber bundle on $\pi^{-1}(\Omega)$ satisfying for all $q \in \Omega$ and all $1 \le j \le m$

$$X_j = f(q, e_j).$$

Thanks to the choice of an orthonormal frame $(X_1, \dots, X_m)$, it is possible to consider a horizontal curve $\gamma : [0, 1] \to M$ as a solution of the ordinary differential equation

$$\dot\gamma(t) = \sum_{i=1}^{m} u_i(t) X_i(\gamma(t)) \qquad \text{p.p. } t \in (0, 1), \tag{2.1}$$

where the control u satisfies the decomposition $\sum_{i=1}^{m} u_j e_j$ in the basis $(e_j)_{1\leq j\leq m}$. We identify the set of controls of horizontal curves with a set $\mathcal{U} \subset L^\infty((0,1),\mathbb{R}^m)$ according to this basis decomposition. We can then define the *endpoint map at q*, $E_q : U \to M$, that maps $u \in \mathcal{U} \subset L^2([0,1],\mathbb{R}^m)$ to $\gamma_u(1)$, where γ_u is the solution to (2.1) with initial condition $\gamma(0) = q$.

One interest of the endpoint map is that it allows to describe length-minimizing curves as solutions of an optimization problem under constraints. First, notice that if the curve $\gamma : [0,1] \to M$ has constant speed and minimizes the distance between $\gamma(0) = q_0$ and $\gamma(1) = q_1$, it also minimizes the sub-Riemannian energy functional

$$\int_0^1 g(\dot\gamma(t),\dot\gamma(t))\,\mathrm{d}t.$$

Since a curve $\gamma_u : [0,1] \to M$ is solution to (2.1) with initial condition $\gamma_u(0) = q_0$, its energy can be expressed in terms of the L^2-norm of its control (the frame $(X_1,\dots,X_m)$ is orthonormal)

$$\int_0^1 g(\dot\gamma_u(t),\dot\gamma_u(t))\,\mathrm{d}t = \|u\|_{L^2}^2 =: J(u).$$

The curve minimizing the distance between q_0 and q_1 is then the curve which control is solution to the optimization problem

$$\min\{J(u) \mid u \in \mathcal{U}, E_{q_0}(u) = q_1\}.$$

As a consequence of Lagrange Multipliers Theorem, there exist non-zero $(\lambda,\mu) \in T^*_{q_1}M \times \mathbb{R}$ such that for $\bar u$ optimal,

$$\lambda \cdot D_{\bar u}E_{q_0} = \mu D_{\bar u}J.$$

Then two cases emerge. Either $\mu = 0$ and the corresponding $\bar u$ is a singular point for E_{q_0}, which happen to not say much about γ_u, or $\mu \neq 0$ and the equation leads to an ordinary differential equation, fitting the role of sub-Riemannian geodesic equation satisfied by the minimizing curve .

Hamiltonian trajectories. It proves useful to introduce the Hamiltonian point of view to say more on the length minimization problem. Let

$$\begin{array}{rccc} H: & T^*M & \longrightarrow & M \\ & (p,q) & \longmapsto & \dfrac{1}{2}\displaystyle\max_{v\in\Delta_q\setminus\{0\}} \frac{\langle p,v\rangle^2}{g_q(v,v)} \end{array}$$

be the *sub-Riemannian Hamiltonian*, and let $\vec{H}$ be its associated Hamiltonian vector field. This Hamiltonian holds a lot of the information of the sub-Riemannian structure and is deeply linked with the geometry of length-minimizing curves.

Indeed, the analysis we provided on length minimizing curves leads to the sub-Riemannian version of the Pontryagin Maximum Principle, that we can express in its Hamiltonian form.

Theorem 2.1.4 (Pontryagin Maximum Principle)**.** *Let $\gamma : [0,1] \to M$ be a horizontal constant speed curve minimizing the distance between $\gamma(0)$ and $\gamma(1)$.*

There exists a Lispchitz-continuous curve $\lambda : [0,1] \to T^*M$ such that $\lambda(t) \in T^*_{\gamma(t)}M \setminus \{0\}$ for almost every $t \in [0,1]$, and one of the following cases holds:

(N) *Either $H(\lambda(t)) \neq 0$ and $\dot{\lambda}(t) = \vec{H}(\lambda(t))$ for all $t \in [0,1]$.*

(A) *Or $H(\lambda(t)) = 0$ for all $t \in [0,1]$.*

Minimizing curves corresponding to case **(N)** are called *normal extremals*, while curves corresponding to case **(A)** are called *abnormal extremals.*

Normal extremals predominantly play a role in Chapter 4 (see also Section 2.3), where we use the Pontryagin maximum principle to determine the geodesic equation. Indeed, once we endow an open subset Ω of M with an orthonormal frame of the distribution $(X_1, \dots, X_m)$, the sub-Riemannian Hamiltonian becomes

$$H(p,q) = \frac{1}{2} \sum_{i=1}^{m} \langle p, X_i(q) \rangle^2.$$

Furthermore, if we endow T^*M with canonical coordinates $(p,q) : T^*M \to \mathbb{R}^{2d}$, the curve $t \mapsto \lambda(t) = (p(t), q(t))$ is a solution of the Hamiltonian

$$\begin{cases} \dot{q} = \dfrac{\partial H}{\partial p} = \displaystyle\sum_{i=1}^{m} {}^t p\, X_i(q)\ X_i(q), \\ \dot{p} = -\dfrac{\partial H}{\partial q} = -\displaystyle\sum_{i=1}^{m} {}^t p\, X_i(q)\ {}^t p\, D_q X_i(q). \end{cases}$$

On the other hand, abnormal curves play a role in Chapter 3, where they play a role as singular points of the endpoint map. Indeed, singular curves appear to be potential obstruction to Whitney extensions (see Section 2.2). We cannot learn much more on abnormal extremals from the Pontryagin Maximum Principle. However, second order optimality conditions (see [AS96, AS04]), named in contrast with the Maximum Principle, an order 1 optimality condition, allow us to precisely study such singular curves.

2.1.2 Nilpotent approximation

Carnot groups. Nilpotent approximation is one of the key aspects of the methods employed in this thesis. To present it, we must first consider a class of sub-Riemannian manifolds playing a major role in its construction.

Definition 2.1.5. Let $\mathfrak{g}$ be a Lie algebra. For all $i \in \mathbb{N}^*$, we set $\mathfrak{g}^1 = \mathfrak{g}$ and $\mathfrak{g}^{i+1} = [\mathfrak{g}, \mathfrak{g}^i]$. The Lie algebra $\mathfrak{g}$ is said to be *nilpotent* if there exists $k \in \mathbb{N}^*$ such that $\mathfrak{g}^k = \{0\}$. In addition, it is said to be *stratified* if there exists a family $(\mathfrak{h}^k)_{k\in\mathbb{N}^*}$ of subspaces of $\mathfrak{g}$ such that

$$\mathfrak{h}^{i+j} = \left[\mathfrak{h}^i, \mathfrak{h}^j\right] \qquad \text{and} \qquad \mathfrak{g}^i = \mathfrak{g}^{i+1} \oplus \mathfrak{h}^i, \qquad \forall i, j \in \mathbb{N}^*.$$

A *Carnot group* is a simply connected Lie group $\mathbb{G}$ such that its Lie algebra $\mathfrak{g}$ of left-invariant vectors is nilpotent and stratified.

By left translation, we associate with the subspace $\mathfrak{h} = \mathfrak{h}^1$ of $\mathfrak{g}$ a vector bundle on $\mathbb{G}$ that is Lie bracket generating. The choice of a left-invariant Euclidean metric on $\mathfrak{h}$ implies the choice of a sub-Riemannian structure on $\mathbb{G}$ that we call *left-invariant sub-Riemannian structure.*

Carnot groups endowed with left-invariant sub-Riemannian structures are very interesting as model spaces for sub-Riemannian manifolds, since the group symmetries add to the metric structure. In addition to the left-invariant symmetry, Carnot groups are endowed with an intrinsic dilation for which the distance is homogeneous. For all $\lambda \in \mathbb{R}$, we set δ_λ to be the *isotropic dilation of rate* λ such that, with $\exp : \mathfrak{g} \to \mathbb{G}$ the Lie group exponential,

$$\begin{array}{rccc} \delta_\lambda : & \mathbb{G} & \longrightarrow & \mathbb{G} \\ & g & \longmapsto & \exp\left(\lambda \exp^{-1}(g)\right). \end{array}$$

For any left invariant horizontal vector field X, one has $\delta_{\lambda *}X = \lambda X$ and thus the same goes for the length of horizontal curves. Hence $\mathrm{d_{SR}}\left(\delta_\lambda(g), \delta_\lambda(g')\right) = |\lambda| \mathrm{d_{SR}}\left(g, g'\right)$ for all $g, g' \in \mathbb{G}$ and all $\lambda \in \mathbb{R}$.

For us, Carnot groups play a key role because from the metric point of view, they occupy a position similar to the one of Euclidean spaces in Riemannian geometry, essential to perturbative methods. We will explain this link by introducing the tools from nilpotent approximation.

Privileged coordinates. Let (M, Δ, g) be a sub-Riemannian manifold. For all $i \in \mathbb{N}^*$, we denote by $\Delta^1 = \Delta$ and $\Delta^{i+1} = \Delta^i + [\Delta, \Delta^i]$. The Lie bracket generation hypothesis implies that for all $q \in M$ there exists an integer $r > 0$ called *step of the distribution at* q such that $\Delta_q^{r-1} \subsetneq \Delta_q^r = T_qM$. The vector of increasing integers $\left(\dim \Delta_q^1, \ldots, \dim \Delta_q^r\right) = (n_1, \ldots, n_r)$ is called *growth vector at* q. If the growth vector is constant on a neighborhood of q, the point q is said to be *regular*, and if all points of M are regular, then (M, Δ, g) is said to be *equiregular.*

The *weight vector at* q is the non-decreasing vector $w \in \mathbb{N}^d$ such that $w_j = s$ if $n_s < j \leq n_{s+1}$ (with $n_0 = 0$). With Ω an open subset of M, coordinates at $q \in M$, $(x_1, \ldots, x_d) : \Omega \to \mathbb{R}^d$ are said to be *privileged* if

$$\sup\left\{s \in \mathbb{R} \mid x_j(q') = O\left(\mathrm{d_{SR}}\left(q, q'\right)^s\right)\right\} = w_j, \qquad 1 \leq j \leq d.$$

Such coordinates have the property of being *linearly adapted at* q, that is, for all $1 \leq j \leq d$, $\mathrm{d}x_j\left(\Delta_q^{w_j}\right) \neq 0$, while $\mathrm{d}x_j\left(\Delta_q^{w_j-1}\right) = 0$.

In these coordinates, a sub-Riemannian manifold "looks like" a Carnot group. They allow to describe the behavior of the metric at small scale. For instance, we can already enunciate a version of the "Ball-Box" Theorem, probably the most famous instance of a sub-Riemannian distance estimates theorem. (Its proof requires more advanced tools from privileged coordinates however.)

Theorem 2.1.6 (Ball-Box Theorem)**.** *Let* $q \in M$ *and let* $\Phi_q : \Omega \to \mathbb{R}^d$ *be a system of privileged coordinates at* q*. There exist* $C_q, \varepsilon_q > 0$ *such that any sub-Riemannian ball* $B_{\mathrm{SR}}(q,\varepsilon)$ *of radius* $\varepsilon < \varepsilon_q$ *and centered at* q *satisfies the nesting*

$$\mathrm{Box}(\varepsilon/C_q) \subset \Phi_q\left(B_{\mathrm{SR}}(q,\varepsilon)\right) \subset \mathrm{Box}(C_q\varepsilon),$$

where $\mathrm{Box}(\varepsilon) = [-\varepsilon^{w_1}, \varepsilon^{w_1}] \times \cdots \times [-\varepsilon^{w_d}, \varepsilon^{w_d}]$.

These coordinates also allow to introduce dilations on sub-Riemannian manifolds. Let $q \in M$ and let $\Phi_q : \Omega \to \mathbb{R}^d$ be a system of privileged coordinates at q. For all $\lambda > 0$, we set $\delta_\lambda^q : \Omega \to \mathbb{R}^d$ to be the *dilation of rate* λ *centered at* q, such that

$$\delta_\lambda^q \circ \Phi_q^{-1}(x_1, \ldots, x_d) = \left(\lambda^{w_1}x_1, \ldots, \lambda^{w_i}x_i, \ldots, \lambda^{w_d}x_d\right).$$

Nilpotent approximation. Measuring the effects of dilations on the distance is essential to build estimates. We can start by defining the nilpotent approximation of a horizontal vector field.

Proposition 2.1.7. *Let* (M, Δ, g) *be a sub-Riemannian manifold of dimension* d*. Let* $q \in M$*, let* Ω *be an open neighborhood of* q *and let* $\Phi_q : \Omega \to \mathbb{R}^d$ *be a system of privileged coordinates at* q*. For any horizontal vector field* $X \in \Delta$ *there exists a vector field* $\widehat{X}$ *on* $\mathbb{R}^d$ *called* nilpotent approximation of X at q*, such that the family of vector fields on* $\mathbb{R}^d$

$$\left(\frac{1}{\lambda}\delta_{\lambda *}^q X\right)_{\lambda>0}$$

locally uniformly converges towards $\widehat{X}$ *as* $\lambda \to \infty$*.*

Notice that with this method, we capture the part of X that is homogeneous of order 1 with respect to the dilation, similarly to what happens with horizontal left-inavariant vector fields on a Carnot group.

We can then use the nilpotent approximation of horizontal vector fields to define the nilpotent approximation of sub-Riemannian manifolds. Indeed let $(X_1, \ldots, X_m)$ be an orthonormal frame of (M, Δ, g), and let $\left(\widehat{X}_1, \ldots, \widehat{X}_m\right)$ be a sub-Riemannian approximation of this frame at $q \in M$. Let $\widehat{\Delta}$ be the vector bundle of $\mathbb{R}^d$ such that for all $x \in \mathbb{R}^d$,

Δ_x is generated by the vectors $\left(\widehat{X}_1(x), \ldots, \widehat{X}_m(x)\right)$, and let $\widehat{g}$ be the Euclidean metric on $\widehat{\Delta}$ for which $\left(\widehat{X}_1, \ldots, \widehat{X}_m\right)$ is an orthonormal frame. A *nilpotent approximation of* (M, Δ, g) *at* q is the sub-Riemannian manifold represented by the triple $\left(\mathbb{R}^d, \widehat{\Delta}, \widehat{g}\right)$, with sub-Riemannian distance denoted by $\widehat{\mathrm{d}}_{\mathrm{SR}}$.

This construction seems to rely on the initial choice of coordinates, however the point of this method can be summarized as follows.

Theorem 2.1.8. *Let* (M, Δ, g) *be a sub-Riemannian manifold, and let* $q \in M$. *There exists a Carnot group* $\mathbb{G}_q$ *acting on* $\mathbb{R}^d$, *with* $\mathbb{H}_q$ *its isotropy group at* 0 *(i.e.* $g \in \mathbb{H}_q$ *if* $g \cdot 0 = 0$*), and a sub-Riemannian distance* $\bar{d}$ *on* $\mathbb{G}_q/\mathbb{H}_q$ *such that any nilpotent approximation* $\left(\mathbb{R}^d, \widehat{\Delta}, \widehat{g}\right)$ *of* (M, Δ, g) *at* q *is isometric to* $\left(\mathbb{G}_q/\mathbb{H}_q, \bar{d}\right)$.

In addition, if q *is regular, there exists such a* $\mathbb{G}_q$ *for which* $\mathbb{H}_q$ *is reduced to the identity element.*

Remark 2.1.9. It is possible to give an explicit construction of $\mathbb{G}_q$ (in parts thanks to the algorthmic approach unerlying in [Bel96, Jea01, Jea14]). Indeed for $\left(\mathbb{R}^d, \widehat{\Delta}, \widehat{g}\right)$ a nilpotent approximation of (M, Δ, g) at q, the group $\mathbb{G}_q$ is generated by the family of diffeomorphisms $\left(\mathrm{e}^{t\widehat{X}_i}\right)_{1 \le i \le m}$. Furthermore, if q is regular, this group corresponds to the choice for which $\mathbb{H}_q$ is trivial.

The "nilpotent" adjective makes more sense now, one can see that the structure of $\left(\mathbb{R}^d, \widehat{\Delta}, \widehat{g}\right)$ is a structure of quotient of nilpotent groups (and even Carnot groups), and at regular points, it is exactly a Carnot group structure. To comment on the approximation aspect, let us illustrate this property with another sub-Riemannian distance estimate result.

Theorem 2.1.10 ([Jea14, Theorem 2.2])**.** *On a neighborhood of* p *in* M,

$$\mathrm{d}_{\mathrm{SR}}(p, q) = \widehat{\mathrm{d}}_{\mathrm{SR}}(p, q)\left(1 + O\left(\widehat{\mathrm{d}}_{\mathrm{SR}}(p, q)\right)\right).$$

Of course it is possible to be more precise, and applications of these methods can be found in particular in Sections 3.2 and 3.5.

In closing, it is possible to observe that the metric space $\left(\mathbb{G}_q/\mathbb{H}_q, \bar{d}\right)$ plays the part of a tangent space, in the metric sense of Gromov [Bel96]. That is to say, the family of metric spaces $M_\lambda = (M, \lambda \mathrm{d}_{\mathrm{SR}})$, pointed at q, converges for the Gromov–Hausdorff distance towards $\left(\mathbb{G}_q/\mathbb{H}_q, \bar{d}\right)$, pointed at 0, as λ tends to infinity. This earns $\left(\mathbb{G}_q/\mathbb{H}_q, \bar{d}\right)$ the name of metric tangent ([Jea14]) or non-holonomic tangent ([ABB16]).

This analysis allows us to shed a light on the methods employed in this thesis. In Riemannian geometry, the metric tangent of a dimension d manifold is always the Euclidean space of dimension d. In sub-Riemannian geometry however, there is a wide variety of possible tangents. If a sub-Riemannian manifold is not equiregular, this implies that there exist two points of the manifold at which the tangent are not diffeomorphic. Even under equiregularity constraints, since Carnot groups can be parameter dependent, the metric tangent at two point of an equiregular manifold can be not isometric.

In the context of singularities, be it spatial singularities of the sub-riemannian exponential map or singularities of the endpoint map in the context of extensions, we abundantly use the infinitesimal structure of sub-Riemannian manifolds. We try two different approaches to apply our knowledge of the metric tangent to our advantage. For singularities of the exponential, we use normal forms and perturbative methods to highlight the similarities between a small scale sub-Riemannian structure and it tangent. On the other hand, in the case of extensions, we rather use distance estimates and look for metric properties on the tangent that are robust enough to be transmitted to the manifold.

2.2 Whitney extensions

Sub-Riemannian manifold form a class of metric spaces that are both diverse and structured, which makes them good model spaces for geometric measure theory. This allows to use tools and techniques from geometric control theory and nilpotent approximation in the context of metric problems. We use this approach on the subject of Whitney extensions, and we will see how this problem finds an interpretation in terms of a classical controlability issue near singular curves.

2.2.1 Whitney extension theorem for curves

We owe to Whitney the initial Euclidean version of a family of analysis results called Whitney Extension Theorems [Whi34]. For a continuous map defined on a closed set K of $\mathbb{R}^d$ and valued in the space of jets of order k, $f : K \to \mathrm{Jet}^k(\mathbb{R}^n)$, the Whitney Extension Theorem states the existence of a C^k map coinciding with the jets given by f under the assumption that the values of the jets are compatible with Taylor expansion theorems. We call this hypothesis the *Whitney condition.*

Theorem 2.2.1 (Eclidean C^1 Whitney extension theorem). *Let K be a closed subset of $\mathbb{R}^n$ and $(f, L) : K \to \mathbb{R}^m \times L(\mathbb{R}^n, \mathbb{R}^m)$ continuous. If there exists $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\omega(t) = o(t)$ at 0^+ and for all $x, y \in K$,*

$$|f(y) - f(x) - L(x) \cdot (y - x)|_{\mathbb{R}^m} \leq \omega(|x - y|_{\mathbb{R}^n}) \tag{2.2}$$

then there exists $F : \mathbb{R}^n \to \mathbb{R}^m$, $F \in C^1(\mathbb{R}^n, \mathbb{R}^m)$, such that

$$F_{|K} = f \quad \text{and} \quad DF_{|K} = L.$$

First generalizations of this theorem in the context of sub-Riemannian geometry regard real valued maps on Carnot groups. Initially the case of C^1 real valued maps on the Heisenberg group was tackled by Franchi–Serapioni–Serra Cassano [FSSC01], and later generalized to the case of C^k real valued maps on Carnot groups by Pupyshev–Vodopyanov in [VP06].

The case of horizontal C^1 maps from $\mathbb{R}$ to Carnot groups came later, Zimmermann solving the case of the Heisenberg group [Zim18], and Juillet–Sigalotti the case of arbitrary Carnot groups [JS17]. We wish to carry on this work by extending these results to the case of horizontal curves $\gamma : \mathbb{R} \to M$ in sub-Riemannian manifolds (M, Δ, g).

In these cases, it is necessary to use the Pansu derivative [Pan89] of a horizontal map to generalize the condition (2.2). Instead of a linear map, the derivative of a map between Carnot groups is a *homogeneous group homomorphism* $L : \mathbb{G}_1 \to \mathbb{G}_2$ between two Carnot groups $\mathbb{G}_1$ and $\mathbb{G}_2$, that is, a group homomorphism such that $L \circ \delta_\lambda^{\mathbb{G}_1} = \delta_\lambda^{\mathbb{G}_2} \circ L$, where $\delta_\lambda^{\mathbb{G}_i}$ denotes the dilation of coefficient $\lambda \in \mathbb{R}$ associated with the group $\mathbb{G}_i$, $i \in \{1, 2\}$. Thus, denoting $\mathrm{d}_{\mathbb{G}_i}$ the sub-Riemannian distance associated with each group $\mathbb{G}_i$, $i \in \{1, 2\}$, the Whitney condition takes the following form (see [VP06, JS17]):

Let K be a closed subset of $\mathbb{G}_1$ and $f : K \to \mathbb{G}_2$. We assume there exists L that maps every $g \in K$ to a homogeneous group homomorphism $L(g) : \mathbb{G}_1 \to \mathbb{G}_2$. The C^1_H *Whitney condition* is satisfied by (f, L) if (f, L) is continuous and there exists $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\omega(t) = o(t)$ at 0^+ and for all $g, h \in K$

$$\mathrm{d}_{\mathbb{G}_2}\left(f(h), f(g) \cdot L(g)(g^{-1} \cdot h)\right) \leq \omega(\mathrm{d}_{\mathbb{G}_1}(g, h)). \tag{2.3}$$

The Pansu derivative is limited to Carnot groups and generalizing this concept is necessary (for instance, see [Vod07] for a possible extension using nilpotent approximation) A crucial observation is that in the case where $\mathbb{G}_1$ is the Euclidean space $\mathbb{R}$, a homogeneous group homomorphism between $\mathbb{R}$ and $\mathbb{G}_2$ can be written as $L(t) = \exp_{\mathbb{G}_2}(tX)$ where X is a left invariant vector field on $\mathbb{G}_2$. Therefore, condition (1.3) takes the form

$$\mathrm{d}_{\mathbb{G}_2}\left(f(t), f(s) \cdot \exp_{\mathbb{G}_2}\left[(t - s)\, X(s)\right]\right) \leq \omega(|t - s|). \tag{2.4}$$

Let (M, Δ, g) be a sub-Riemannian manifold of dimension d. When considering horizontal curves $\gamma : \mathbb{R} \to M$, it would seem appropriate to replace the Pansu derivative with the flow of a horizontal vector field. However, unlike left invariant vector fields in a Carnot group, there is no intrinsic vector field extension of the vector $\dot\gamma(t) \in T_{\gamma(t)}M$.

We choose to introduce an orthonormal frame to define this extension. Let $(X_1, \dots, X_m)$ be an orthonormal frame of the distribution on the open set $\Omega \subset M$. For all $q \in \Omega$, and all $v \in \Delta_q$, there exist $(u_i)_{1 \leq i \leq m}$ such that $\sum_{i=1}^m u_i X_i(q) = v$. In the following, we denote by $X_u = \sum_{i=1}^m u_i X_i$. Up to restriction of the domain of the curve, we also assume the curve stays in Ω. Furthermore, in the case of a general sub-Riemannian manifold (M, U, f)

(see Definition 2.1.1), a horizontal curve $\gamma : \mathbb{R} \to M$ is said to be C^1_H if there exists a continuous control $u : \mathbb{R} \to M$ such that $\gamma = \pi_U(u)$ and $\dot{\gamma} = f(\gamma, u)$

Definition 2.2.2. Let K be a closed subset of $\mathbb{R}$, $(f, L) : K \to M \times TM$ be continuous such that $L(t) \in \Delta_{f(t)}$ for all $t \in K$. The pair (f, L) satisfies the C^1_H *Whitney condition on* K if there exist $u : K \to \mathbb{R}^m$ continuous and $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that $X_{u(s)}(f(s)) = L(s)$ for all $s \in K$, $\omega(t) = o(t)$ at 0^+, and

$$\mathrm{d}_{\mathrm{SR}}\left(f(t), \mathrm{e}^{(t-s)X_{u(s)}} f(s)\right) \leq \omega(|t-s|) \quad \text{for all } t, s \in K.$$

This definition seems to imply that satisfying the C^1_H Whitney condition might depend on the choice of frame $(X_1, \ldots, X_m)$ and that it would rather be a property of u and not L. The majority of the work in Sections 3.2.3 and 3.3.1 is dedicated to proving that it is indeed a property of the pair (f, L), independently of the choice of frame.

One can notice that for any C^1_H curve $\gamma : \mathbb{R} \to M$, and for all closed subset K of $\mathbb{R}$, the pair $(\gamma_{|K}, \dot{\gamma}_{|K})$ satisfies the C^1_H Whitney condition on K. This confirms that this condition plays the role of a compatibility condition with Taylor expansions. This warrants the following definition, where the Whitney Theorem is seen as a metric property of a sub-Riemannian manifold.

Definition 2.2.3. The C^1_H *extension property* holds on (M, Δ, g) if for any closed subset $K \subset \mathbb{R}$ and any pair (f, L) satisfying the C^1_H Whitney condition on K, there exists a curve $\gamma : \mathbb{R} \to M$, horizontal and C^1, such that

$$\gamma_{|K} = f, \qquad \dot{\gamma}_{|K} = L.$$

In this problem we benefit from two aspects that may not be immediately apparent: the local nature of Whitney extensions and the topology of $\mathbb{R}$. Indeed, if $t_0 < t_1$ are points of $\mathbb{R}$ and $(f_0, L_0), (f_1, L_1) \in M \times \Delta$ are two jets of horizontal curves, finding a C^1 horizontal curve $\gamma : \mathbb{R} \to M$ such that $(\gamma(t_0), \dot{\gamma}(t_0)) = (f_0, L_0)$ and $(\gamma(t_1), \dot{\gamma}(t_1)) = (f_1, L_1)$ is a simple controlability matter.

Since the complementary of a closed subset of $\mathbb{R}$ is a countable union of open intervals, the existence of a continuous and almost everywhere horizontal extension is obtained by extending the curve on every interval. The difficulty lies in proving that there exists such an extension that is also C^1, because the continuity of the derivative can be lost at accumulation points of the border of K, where the surgery between the curve and the extension happens.

This is where nilpotent approximation comes into play. At accumulation points of $f(K)$, the pair (f, L) can be transposed to the metric tangent, and there we look for sufficiently robust controlability conditions to ensure that the curve admits a global C^1 extension.

In order to apply estimates of the sub-Riemannian distance, we are compelled to focus on the equiregular case for now.

2.2.2 The equiregular case

Crossing to the nilpotent approximation. We assume now that (M, Δ, g) is an equiregular sub-Riemannian manifold (see Section 2.1.2). Recall that in this case, the growth of Lie brackets does not depend on the position and that the metric tangent to the manifold has a Carnot group structure at any point.

Since the Whitney condition must be expressed in a given orthonormal frame of the distribution $(X_1, \dots, X_m)$ but does not depend on this choice, we will assume the frame to be fixed and we will indifferently consider $L(t) \in \Delta_{f(t)}$ and $u(t) \in \mathbb{R}^m$ such that $X_{u(t)} = L(t)$, for all $t \in K$.

We want to use uniform estimates of the sub-Riemannian distance, hence we introduce on an open set $\Omega \subset M$ a *continuously varying system of privileged coordinates*

$$\Phi : (p, q) \longmapsto \Phi_p(q) \in \mathbb{R}^d,$$

a continuous map such that Φ_p is a system of privileged coordinates at p, for all $p \in \Omega$.

With such coordinates, we can introduce the usual tools from nilpotent approximation, except we retain continuity with respect to the position. For $p \in \Omega$, recall δ^p_λ denotes the dilation centered at p. We use the locally uniform convergence of

$$\varepsilon \delta^p_{\frac{1}{\varepsilon}*} X_u \to \widehat{X}^{(p)}_u, \qquad \varepsilon \to 0, \tag{2.5}$$

and uniform distance estimates to give an alternative characterization of the C^1_H Whitney condition. It links the Whitney condition with accumulation points in K and the nilpotent approximation

Proposition 2.2.4. *The pair $(f, u) : K \to M \times \mathbb{R}^m$ satisfies the C^1_H Whitney condition on K if and only if for all $l \in K$, for all converging sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ in K such that $a_n < b_n$ and $a_n, b_n \to l \in K$, one has*

$$\lim_{n\to\infty} \delta^{f(a_n)}_{\frac{1}{b_n - a_n}}(f(b_n)) = \mathrm{e}^{\widehat{X}_{u(l)}}(0).$$

Despite the center of the dilation depending on n, thanks to (2.5) we easily get the locally uniform convergence

$$(b_n - a_n)\delta^{f(a_n)}_{\frac{1}{b_n - a_n}*} X_{u(a_n)} \xrightarrow[n\to\infty]{} \widehat{X}^{(f(l))}_{u(l)}.$$

The Whitney condition ensures (by means of distance estimates) that the distance between

$$\delta^{f(a_n)}_{\frac{1}{b_n - a_n}}(f(b_n)) \quad \text{and} \quad \delta^{f(a_n)}_{\frac{1}{b_n - a_n}}\left(\mathrm{e}^{(b_n - a_n)X_{u(a_n)}}(f(a_n))\right)$$

tends to 0 when n tends to infinity (see Figure 2.1). Since

$$\delta^{f(a_n)}_{\frac{1}{b_n - a_n}}\left(\mathrm{e}^{(b_n - a_n)X_{u(a_n)}}(f(a_n))\right) = \mathrm{e}^{(b_n - a_n)\delta^{f(a_n)}_{\frac{1}{b_n - a_n}*} X_{u(a_n)}}$$

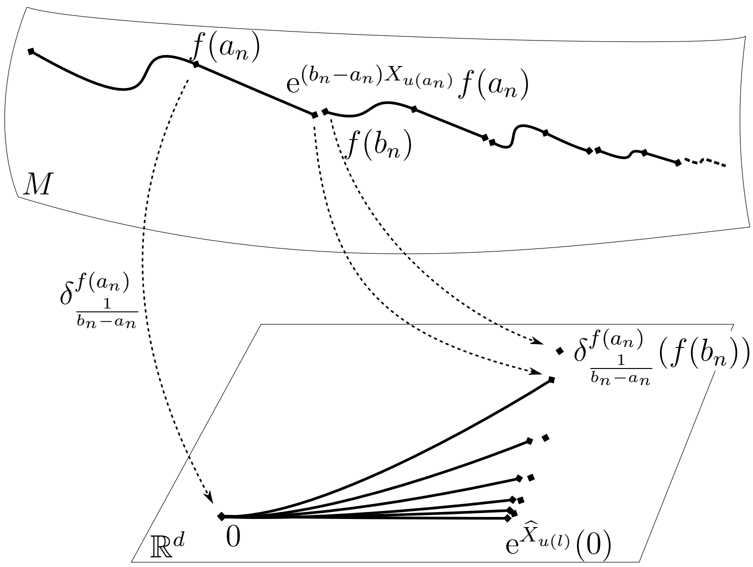


Figure 2.1 – Representation of the effect of dilations on a curve satisfying the Whitney condition. Once in $\mathbb{R}^d$, the dilated flow curves converge toward the straight curve $e^{\widehat{X}_{u(l)}}(0)$. The Whitney condition ensures that the distance between the final point of the flow curves $e^{(b_n-a_n)X_{u(a_n)}} f(a_n)$ and their target $f(b_n)$ converges sufficiently quickly towards 0 to maintain this convergence after blowup.

we get the stated result.

One can draw an (expected) interpretation of this result: the Whitney condition dictates that points of the curve should align with the derivative at the infinitesimal scale, a behavior that can be observed in the metric tangent (again, see Figure 2.1).

Endpoint maps. In light of these observations, it is useful to consider the endpoint map of the nilpotent approximation at accumulation points of the curve. In particular its openness: in the case of Carnot groups, if the endpoint map lacks openness, it is possible to construct a curve satisfying the Whitney condition but that cannot be extended (see [JS17]).

Let $(\widehat{X}_1^{(q)}, \dots, \widehat{X}_m^{(q)})$ be the nilpotent approximation at q of the orthonormal frame and let $\mathbb{G}_q$ be the Carnot group endowed with the associated left invariant sub-Riemannian structure.

For all $u \in L^1([0,1], \mathbb{R}^m)$ we denote by γ_u the curve such that

$$\gamma_u(0) = 0_{\mathbb{G}_q}, \qquad \dot{\gamma} = \widehat{X}_u(\gamma) \text{ p.p.}$$

The endpoint map $\widehat{E}_q : L^1([0,1], \mathbb{R}^m) \to \mathbb{G}_q$ is then defined by the relation

$$\widehat{E}_q(u) = \gamma_u(1).$$

(In this case, $\widehat{E}_q$ is the endpoint map of $\mathbb{G}_q$ at the point $0_{\mathbb{G}_q}$, as it was defined in Section 2.1.1.)

Therefore, Proposition 2.2.4 implies that for all $l \in K$, for all converging sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ in K such that $a_n < b_n$ and $a_n, b_n \to l \in K$, we have the convergence

$$\delta^{f(a_n)}_{\frac{1}{b_n-a_n}}(f(b_n)) \to \widehat{E}_{f(l)}(u(l)).$$

For all $n \in \mathbb{N}$, consider the sub-Riemannian structure (Δ_n, g_n) on $\mathbb{R}^d$ given by the orthonormal frame

$$\left((b_n - a_n)\delta^{f(a_n)}_{\frac{1}{b_n-a_n}*}X_i\right)_{1\leq i\leq m}.$$

Denoting E^n its endpoint map at 0, we have the locally uniform convergence $E^n \to \widehat{E}_{f(l)}$. Hence, it now appears that extending (f, L) requires finding for all n a continuous control v_n on $[0, 1]$ such that

$$E^n(v_n) = \delta^{f(a_n)}_{\frac{1}{b_n-a_n}}(f(b_n)), \qquad v_n(0) = u(a_n) \quad \text{and} \quad v_n(1) = u(b_n), \quad \forall n \in \mathbb{N}.$$

Since $u(a_n), u(b_n) \to u(l)$, $E^n(v_n) \to \widehat{E}_{f(l)}(u(l))$, a such sequence of controls $(v_n)_{n\in\mathbb{N}}$ must converge towards $u(l)$ for the corresponding extension to be C^1.

Hence we get the following intuition on the problem: an obstruction to the existence of an extension would be the map

$$\begin{array}{ccc} \{v \in C^0([0,1], \mathbb{R}^m) \mid v(0) = 0\} & \longrightarrow & \mathbb{G}_q \times \mathbb{R}^m \\ v & \longmapsto & \left(\widehat{E}_q(u(l) + v), v(1)\right) \end{array}$$

lacking openness at 0.

2.2.3 Strong pliability at regular points

Pliability, strong or not. In the case of a Carnot group $\mathbb{G}$, observe that the previous construction is tautological by homogeneity of the horizontal vector fields by dilation. Indeed, for all $1 \leq i \leq m$, elements of the orthonormal frame satisfy

$$X_i = (b_n - a_n)\delta^{f(a_n)}_{\frac{1}{b_n-a_n}*}X_i = \widehat{X}_i, \qquad \forall n \in \mathbb{N},$$

and as a result, the endpoint maps satisfy $E_{0_{\mathbb{G}}} = E^n = \widehat{E}$. These observation lead to the introduction of the notion of pliability of a horizontal left invariant vector field.

Definition 2.2.5 (Pliability [JS17])**.** Let $\mathbb{G}$ be a Carnot group and $(X_1, \ldots, X_m)$ be an orthonormal frame of a left invariant sub-Riemannian structure on $\mathbb{G}$. The left invariant vector field X_{u_0}, $u_0 \in \mathbb{R}^m$, is *pliable* if the map

$$\begin{array}{ccc} \mathcal{C}_0 & \longrightarrow & \mathbb{G} \times \mathbb{R}^m \\ v & \longmapsto & (E_0(u_0 + v), v(1)) \end{array}$$

is open at 0, where $\mathcal{C}_0 = \{v \in C^0([0,1], \mathbb{R}^m) \mid v(0) = 0\}$.

Notably, if there exists u_0 such that X_{u_0} is not pliable, that is, the map $E_{0_{\mathbb{G}}}$ not open at u_0, then it is possible to construct a curve that cannot be continuously differentiably extended but that still satisfies the C^1_H Whitney condition.

Theorem 2.2.6 (Juillet–Sigalotti [JS17])**.** *The C^1_H extension property holds on the Carnot group $\mathbb{G}$ if and only if every horizontal left invariant vector field is pliable.*

Remark 2.2.7. The name "pliability" references the non-rigidity of pliable curves. A C^1 horizontal curve $\gamma : [a, b] \to \mathbb{G}$ is rigid in the sense of Bryant–Hsu [BH93] if there exists a neighborhood $\mathcal{V}$ of γ in $C^1_H([a, b], \mathbb{G})$ such that if $\tilde{\gamma} \in \mathcal{V}$, $\tilde{\gamma}(a) = \gamma(a)$ and $\tilde{\gamma}(b) = \gamma(b)$, then $\tilde{\gamma}$ is a reparametrization of γ.

If X is a horizontal left invariant vector field and the curve $t \mapsto \mathrm{e}^{tX}$ is rigid then X is not pliable. Since there exist Carnot groups admitting rigid curves of this form [GK95], there exist Carnot groups where the extension property does not hold.

Taking inspiration from Carnot groups, we can propose a condition on $\widehat{E}_q$ that is sufficient to ensure that the extension property holds on (M, Δ, g) when it is equiregular. Now the maps $(E^n)_{n\in\mathbb{N}}$ and $\widehat{E}_q$ are different but we have the locally uniform convergence of $(E^n)_{n\in\mathbb{N}}$ towards $\widehat{E}_q$. What we expect is that the sufficient condition it should be sufficiently robust to ensure that E^n is open for n large enough.

If $v \mapsto \Big(\widehat{E}_q(u_0 + v), v(1)\Big)$ is a submersion at 0, one immediately has the openness of E^n, but there exists numerous examples where this is not true. Drawing from techniques on the local openness of the endpoint map ($F : \mathcal{U} \to M$ is locally open at u if $F(O_u)$ is a neighborhood of $F(u)$ for any open set $O_u \subset \mathcal{U}$ of u, see [AS04, Chapter 20]) it is possible to give a sufficient condition that also holds in a wide variety of cases where u_0 is a critical point of $\widehat{E}_q$.

Definition 2.2.8 (Strong Pliability)**.** The pair $(q, u_0) \in M \times \mathbb{R}^m$ is *strongly pliable* if for all $\eta > 0$ there exists $v \in \mathcal{C}_0$, $\|v\|_\infty < \eta$, such that the map

$$\begin{array}{rccc} \mathcal{F}^{u_0}_q : & \mathcal{C}_0 & \longrightarrow & \mathbb{G}_q \times \mathbb{R}^m \\ & v & \longmapsto & \Big(\widehat{E}_q(u_0 + v), v(1)\Big) \end{array}$$

is a submersion at v and $\mathcal{F}^{u_0}_q(v) = \mathcal{F}^{u_0}_q(0)$.

To study the strong pliability of a pair (q, u_0), we can now turn to more classical methods from geometric control theory. We set

$$\begin{array}{rccc} G^{u_0}_q : & L^\infty([0, 1], \mathbb{R}^m) & \longrightarrow & \mathbb{G}_q \\ & v & \longmapsto & \widehat{E}_q(u_0 + v). \end{array}$$

We can already limit the study of strong pliability to the study of the map $G^{u_0}_q$.

Lemma 2.2.9. *The pair (q, u_0) is strongly pliable if and only if for all $\eta > 0$ there exists $v \in L^\infty([0,1], \mathbb{R}^m)$ such that $\|v\|_{L^\infty} < \eta$, $G_q^{u_0}(v) = G_q^{u_0}(0)$ and $G_q^{u_0}$ is a submersion at v.*

As stated earlier, the most straightforward case is when $G_q^{u_0}$ is a submersion at 0.

Proposition 2.2.10. *If 0 is a regular point of $G_q^{u_0}$ then (q, u_0) is strongly pliable.*

However, if 0 is not a regular point of $G_q^{u_0}$, there exists classical methods to study singular values of an endpoint map. We now turn to second order conditions on the Hessian of $G_q^{u_0}$ at 0 (see [AS04, Chapter 20]).

The point of view of controlability allows to interpret these techniques. If $D_0 G_q^{u_0}$ is not surjective, we cannot assert that a neighborhood of $G_q^{u_0}(0)$ can be covered with controls of small L^∞ norm. We then look at the Hessian of $G_q^{u_0}$ at 0 to check if such a covering is possible (as some sort of second order approximation of the endpoint map). Specifically, if we can ensure that for all nonzero λ in $\mathrm{im} D_0 G_q^{u_0 \perp}$, the (negative) index of the quadratic form $\lambda \mathrm{Hess}_0 G_q^{u_0}$ on $\ker D_0 G_q^{u_0}$, is larger than the corank of $G_q^{u_0}$ at 0, then we have the proof of the existence of regular points of G arbitrarily close to 0.

We can find rather general criteria to check if this condition holds. It is a matter of checking if the singular curve corresponding to $G_q^{u_0}(0)$, that is to say $t \mapsto \mathrm{e}^{\widehat{X}_{u_0}}(0)$, is not a projection of a Goh or a (weak) Legendre curve (see [AS96]). If that is not the case, then $\lambda \mathrm{Hess}_0 G_q^{u_0}$ is $+\infty$ for all non-zero $\lambda \in \mathrm{im} D_0 G_q^{u_0 \perp}$. These observations lead to the inclusion diagram illustrated in Figure 2.2.

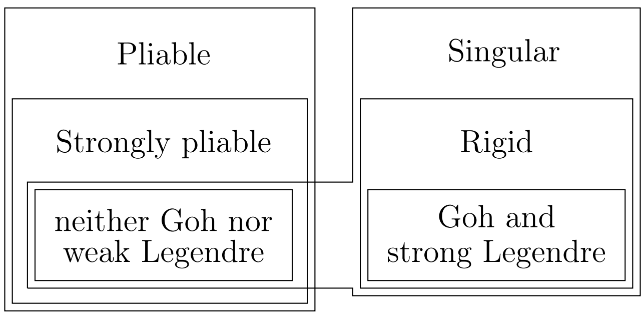


Figure 2.2 – Inclusion diagram of different families of horizontal curves associated with singularities of the endpoint map and pliability.

From this analysis, one can draw several criteria to apply on the manifold (M, Δ, g) to prove strong pliability of every pair (q, u). In particular, we obtain the following result on step 2 distributions.

Proposition 2.2.11. *If (M, Δ, g) is an equiregular sub-Riemannian manifold of step 2, then every pair (q, u_0) is strongly pliable.*

A Whitney extension theorem. As stated previously, strong pliability allows the construction of horizontal C^1 extensions.

Theorem 2.2.12. *Let (M, Δ, g) be an equiregular sub-Riemannian manifold. If every pair $(q, u) \in M \times \mathbb{R}^m$ is strongly pliable, then the C^1_H extension property holds on (M, Δ, g).*

Let (f, L) satisfying the Whitney condition on K. By controlability arguments, we can assert that for all $(a, b) \subset \mathbb{R} \setminus K$, $a, b \in K$, there exists a continuous control $v : [a, b] \to \mathbb{R}^m$ such that

$$X_{v(a)} = L(a), \qquad X_{v(b)} = L(b) \quad \text{and} \quad E_{f(a)}\left(v\left[a + (b-a)\cdot\right]\right) = f(b). \tag{2.6}$$

We use strong pliability to ensure that there exist extensions v on each connected component of K^c such that the global extension of L is continuous.

For $(a, b) \subset K^c$, $a, b \in K$, $\eta > 0$, let $\mathcal{P}_\eta([a, b]) \subset C^0([a, b], B_{\mathbb{R}^m}(0, \eta))$. It is the set of controls with L^∞ norm bounded by η such that $u(a) + v$ is an admissible extension on $[a, b]$, that is, satisfying (2.6), for every $v \in \mathcal{P}_\eta([a, b])$.

Controlability implies

$$\inf\left\{\eta > 0 \mid \mathcal{P}_\eta([a, b]) \neq \emptyset\right\} < +\infty.$$

for all $(a, b) \subset \mathbb{R} \setminus K$, $a, b \in K$. Strong pliability, on the other hand, implies the following.

Lemma 2.2.13. *Under the hypothesis of Theorem 2.2.12, let $(a_n)_n$ and $(b_n)_n$ be two sequences in ∂K such that $(a_n, b_n) \subset \mathbb{R} \setminus K$ for all $n \in \mathbb{N}$. If $\lim a_n = \lim b_n = l \in \mathbb{R}$ then*

$$\inf\left\{\eta > 0 \mid \mathcal{P}_\eta([a_n, b_n]) \neq \emptyset\right\} \to 0.$$

The method we use remains the same. We apply a dilation centered at $f(a_n)$ of coefficient $\frac{1}{b_n - a_n}$. This allows in the limit to transform the problem into a matter of controlabilty in the nilpotent approximation. The sequence of maps $(\mathcal{F}^n)_{n\in\mathbb{N}}$ defined on $\mathcal{C}_0 = \{v \in C^0([0, 1], \mathbb{R}^m) \mid v(0) = 0\}$ by

$$\mathcal{F}^n(v) = \left(\delta^{f(a_n)}_{\frac{1}{b_n - a_n}}\left(E_{f(a_n)}(u(a_n) + v\right), v(1)\right)$$

locally uniformly converges towards $\mathcal{F}^{u_0}_q$, similarly to what we have seen before.

By definition of $\mathcal{F}^n$, if $v_n \in C^0([a, b], B_{\mathbb{R}^m}(0, \eta))$, $v_n(0) = 0$, and

$$\mathcal{F}_n(v_n) = \left(\delta^{f(a_n)}_{\frac{1}{b_n - a_n}}(f(b_n)), u(b_n) - u(a_n)\right)$$

then $v_n \in \mathcal{P}_\eta([a_n, b_n])$. By hypothesis, the map $\mathcal{F}^{u_0}_q$ is regular at points arbitrarily close to 0, and $\mathcal{F}^n(v_n) \to \mathcal{F}^{u_0}_q(0)$. Using a topological degree argument relying on the uniform convergence of $(\mathcal{F}_n)_{n\in\mathbb{N}}$ (see Appendix 3.A), we get that for all $\eta > 0$, $\mathcal{P}_\eta([a_n, b_n])$ is not empty for n large enough. This proves Lemma 2.2.13 and Theorem 1.2.12.

2.2.4 Implications of the equiregular case

At singular points. The previous analysis may be extended to the case of sub-Riemannian manifolds that are not equiregular, however the sub-Riemannian distance estimates that we employed are dedicated to the equiregular case. Something can still be said in this case by using desingularization techniques.

The premise of desingularization stands on two observations. The first one is that the structure of the projection $\mathbb{G}_q \to \mathbb{G}_q/\mathbb{H}_q$ on the metric tangent to (M, Δ, g) at q can be replicated on M. Indeed for $d' = \dim \mathbb{G}_q$, there exists a projection $\pi : \widetilde{M} = M \times \mathbb{R}^{d'-d} \to M$ and a sub-Riemannian structure $(\widetilde{M}, \widetilde{\Delta}, \widetilde{g})$, of same rank m and same step r, such that

— For a given orthonormal frame $(\widetilde{X}_1, \dots, \widetilde{X}_m)$ of $(\widetilde{M}, \widetilde{\Delta}, \widetilde{g})$, the family

$$(\pi_* \widetilde{X}_1, \dots, \pi_* \widetilde{X}_m)$$

is an orthonormal frame of (M, Δ, g).

— The metric tangent to $(\widetilde{M}, \widetilde{\Delta}, \widetilde{g})$ at $(q, 0)$ is isometric to $\mathbb{G}_q$.

The second observation is that it is possible to choose $\mathbb{G}_q = N_{m,r}$, the free group of rank m and step r (see [Jea01, Jea14]). The main feature of this choice is that it implies that $(\widetilde{M}, \widetilde{\Delta}, \widetilde{g})$ is regular at q.

We call this (local) construction an equiregular lift of (M, Δ, g).

Proposition 2.2.14. *Let $(\widetilde{M}, \widetilde{\Delta}, \widetilde{g})$ be an equiregular lift of (M, Δ, g) on the open set Ω. If $(f, X_u) : K \to \Omega \times TM$ satisfies the C^1_H Whitney condition on K, there exists a continuous lift $(\widetilde{f}, \widetilde{X}_u) : K \to \widetilde{M} \times T\widetilde{M}$ of (f, X_u) satisfying (in the manifold $(\widetilde{M}, \widetilde{\Delta}, \widetilde{g})$) the C^1_H Whitney condition on K.*

Observe that if we have an equiregular lift of the manifold, a lift of an absolutely continuous curve can be obtained by integrating its derivative in the lifted manifold, after choosing a starting point for the integration. The idea of the proof is simply to choose a horizontal and continuous extension of f, arbitrarily picked so that the lift obtained by integration still satisfies the C^1_H Whitney condition.

As a consequence, since every step 2 sub-Riemannian manifold admits an equiregular step 2 lift, we get the following conclusion.

Corollary 2.2.15. *The C^1_H extension property holds on every step 2 sub-Riemannian manifold.*

Application to Lusin approximations and rectifiability. The Lusin approximation theorem is often associated with the Whitney extension theorem. Indeed, it is frequently observed that the Rademacher theorem, together with the Whitney extension theorem, usually implies Lusin approximation theorem [LDS16, Spe16, JS17, Zim18].

We then adapt a Rademacher theorem for curves in sub-Riemannian manifolds owed to Vodpyanov [Vod06] and we can prove a Lusin approximation theorem for horizontal curves.

Proposition 2.2.16 (Lusin approximation of a horizontal curve)**.** *Let (M, Δ, g) be a sub-Riemannian manifold satisfying the C^1_H extension property and let $\gamma : [a, b] \to M$ be an absolutely continuous horizontal curve. For all $\varepsilon > 0$ there exists a compact set $K \subset [a, b]$ of measure $\mathcal{L}([a, b] \setminus K) < \varepsilon$ and a C^1_H curve $\gamma_1 : [a, b] \to M$ such that γ and γ_1 coincide on K.*

This result finds an immediate corollary in the theory of 1-countable rectifiability in sub-Riemannian manifolds ([LDS16]). Recall that a set $E \subset M$ is said to be 1-countably rectifiable if there exists a countable family of Lipschitz-continuous curves $f_k : \mathbb{R} \to M$, $k \in M$, such that the residual set $E \setminus \cup_k f_k(\mathbb{R})$ has 0 Hausdorff measure.

Corollary 2.2.17 (1-countable rectifiability)**.** *Let (M, Δ, g) be a sub-Riemannian manifold satisfying the C^1_H extension property. If E is a 1-rectifiable subset of M, then there exists a countable family of C^1_H curves $f_k : \mathbb{R} \to M$ such that $\mathcal{H}^1 (E \setminus \cup_k f_k(\mathbb{R})) = 0$.*

2.3 The caustic of contact sub-Riemannian manifolds

Definition 2.3.1. A contact sub-Riemannian manifold is a smooth sub-Riemannian manifold (M, Δ, g) of dimension $2n + 1$, $n \geq 1$, rank $2n$ and step 2. The distribution being non-integrable, it coincides (locally) with the kernel of a contact 1-form $\omega \in \Lambda^1 M$ such that $\omega \wedge (\mathrm{d}\omega)^n \neq 0$.

Contact sub-Riemannian manifolds form a class of manifolds that benefit from more structure but still offer a wide variety of phenomena to study. In particular, because of its link to the geometry of sub-Riemannian manifolds, the study of singularities of the exponential map is a classical sub-Riemannian problem (see, for instance, [DV09, Hug95]). In the case of contact manifolds of dimension 3, the resolution of this problem relies on a fruitful perturbative approach [Agr96, ACGZ00, EAGK96].

Tools were implemented to tackle the case of higher dimensional contact manifolds, but the analysis has not been carried, except for some particular cases. One richness of this case comes from the fact that starting from dimension 5, one cannot ensure that the metric tangents at two distinct points of a contact sub-Riemannian manifolds are isometric. Hence we are interested in producing a first study of the small scale geometric features of these manifolds, via the construction of an approximation of the sub-Riemannian exponential map and its conjugate locus, which play a key role in this analysis.

2.3.1 Heisenberg groups

Since contact sub-Riemannian manifolds are equiregular by definition, we should start by the case of contact Carnot groups.

For any integer $n \geq 1$, we denote by $\mathbb{H}_{2n+1}$ the (real) Heisenberg group of dimension $2n+1$. Recall (see, for instance, [BBS16, Gav77, LR17]) that the Heisenberg group is a Lie group admitting the following the matrix representation in $\mathrm{GL}_{n+2}(\mathbb{R})$, $\mathbb{H}_{2n+1} = \{m(x,y,z) \mid (x,y,z) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}\}$, with

$$m(x,y,z) = \begin{pmatrix} 1 & x_1 & x_2 & \cdots & x_n & z - \frac{1}{2}\sum_{i=1}^n x_i y_i \\ & 1 & & (0) & & y_1 \\ & & \ddots & & & y_2 \\ & (0) & & 1 & & \vdots \\ & & & & & y_n \\ & & & & & 1 \end{pmatrix}.$$

Denoting by $E_{i,j}$ the $(n+2)\times(n+2)$ matrix whose only non-zero coefficient is a 1 in (i,j) position, we set $X_i = E_{1,i+1}$, $Y_i = E_{n+2,i+1}$, for all $1 \leq i \leq n$, and $Z = E_{1,n+2}$. The Lie algebra $\mathfrak{h}_{2n+1}$ of $\mathbb{H}_{2n+1}$ admits $\{X_i, Y_i, Z \mid, 1 \leq i \leq n\}$ as basis in this representation.

Since for all $1 \leq i,j \leq n$,

$$[X_i, X_j] = [Y_i, Y_j] = [X_i, Z] = [Y_i, Z] = 0 \qquad \text{et} \qquad [X_i, Y_j] = \delta_{ij},$$

we recover the fact that the Lie group $\mathbb{H}_{2n+1}$ is a step 2 Carnot group, with first stratum generated by $\{X_i, Y_i \mid 1 \leq i \leq n\}$. The distribution Δ is obtained by left translation of the vectors of the first stratum of $\mathfrak{h}_{2n+1}$.

By denoting $Z^* : \mathfrak{h}_{2n+1} \to \mathbb{R}$ the linear form dual to Z on $\mathfrak{h}_{2n+1}$, Z^* admits by left translation an extension $\omega \in \Lambda^1\mathbb{H}_{2n+1}$ that is a contact form of kernel Δ on $\mathbb{H}_{2n+1}$.

Let $Q : \mathrm{Vect}\{X_i, Y_i \mid 1 \leq i \leq n\} \to \mathbb{R}$ be a positive definite quadratic form. By left translation of Q, we set on $\mathbb{H}_{2n+1}$ a left invariant sub-Riemannian structure $(\mathbb{R}^{2n+1}, \Delta, g)$ such that $g_0 = Q$.

Proposition 2.3.2. *For every $n \geq 1$, $\mathbb{H}_{2n+1}$ is the only Carnot of dimension $2n+1$ that is a contact sub-Riemannian manifold once endowed with a left-invariant sub-Riemannian structure.*

From the differential point of view, for all $n \geq 1$ there exist only one Carnot group of dimension $2n+1$ that is also a contact manifold. However, there is in general no uniqueness from the metric point of view, that is, up to isometry, of $2n+1$-dimensional Carnot groups endowed with left-invariant sub-Riemannian structure.

We then introduce metric invariants that allow to classify these spaces. Let $(\mathbb{R}^{2n+1}, \Delta, g)$ be a sub-Riemannian manifold, left invariant for the group structure of $\mathbb{H}_{2n+1}$. Let ω be a contact 1-form such that $\ker\omega = \Delta$. At any given point $q \in \mathbb{R}^{2n+1}$, there exists a linear map $A(q) : \Delta_q \to \Delta_q$, skew-symmetric for g_q, such that for all $q \in \mathbb{R}^{2n+1}$ and all $X, Y \in \Delta$,

$$\mathrm{d}\omega(X,Y)(q) = g_q(A(q)X(q), Y(q)).$$

Notice that $A(q)$ does not depend on the choice of ω but only on g_q and Δ_q. Indeed, for any smooth non-vanishing real valued function α, $\alpha\omega$ is also a contact form such that $\ker\alpha\omega = \Delta$, but this implies for all $X, Y \in \Delta$,

$$\mathrm{d}\alpha\omega(X,Y)(q) = \mathrm{d}\omega(X,Y)(q) + \mathrm{d}\alpha \wedge \omega(X,Y)(q) = \mathrm{d}\omega(X,Y)(q).$$

Since $\omega \wedge (\mathrm{d}\omega)^n \neq 0$, there exist n positive reals $(b_1, \dots, b_n)$ such that the spectrum of A is $\{\pm ib_1, \cdots \pm ib_n\}$ (by left invariance, the eigenvalues of the matrix $A(q)$ do not depend on the position).

For any vector $(b_1, \dots, b_n)$ in $(\mathbb{R}_+^*)^n$ there exists a classical representative of this structure for which A admits $\{\pm ib_1, \cdots \pm ib_n\}$ as spectrum. It is given by the frame $(X_i)_{1\leq i\leq 2n}$ defined on $\mathbb{R}^{2n+1}$ endowed with coordinates $(x,z) \in \mathbb{R}^{2n} \times \mathbb{R}$, such that for all $1 \leq i \leq 2n$

$$X_{2i-1} = \partial_{x_{2i-1}} + \frac{b_i}{2} x_{2i}\partial_z, \qquad X_{2i} = \partial_{x_{2i}} - \frac{b_i}{2} x_{2i-1}\partial_z. \tag{2.7}$$

We denote by $\mathbb{H}_{2n+1}(b_1, \dots, b_n)$ the metric space $(\mathbb{R}^{2n+1}, \Delta, g)$ admitting $(X_i)_{1\leq i\leq 2n}$ as an orthonormal frame of the distribution. It is a left invariant structure for $\mathbb{H}_{2n+1}$ that brings to light the fact that $(b_1, \dots, b_n)$ defines a metric invariant for left invariant contact sub-Riemannian structures.

Theorem 2.3.3. *Every left invariant contact sub-Riemannian structure on $\mathbb{H}_{2n+1}$ is isometric to a unique $\mathbb{H}_{2n+1}(b_1, \dots, b_n)$ such that $1 = b_1 \geq b_2 \geq \cdots \geq b_n > 0$.*

Corollary 2.3.4. *There exists a unique left-invariant contact sub-Riemannian structure on $\mathbb{H}_3$ (up to isometry).*

Heisenberg groups are then the metric tangents to contact sub-Riemannian manifolds, and we are going to see the role of these observations in the study of the conjugate locus.

2.3.2 Sub-Riemannian geodesics

Symplectic considerations. Let (M, Δ, g) be a sub-Riemannian manifold of dimension d. We are interested in using tools arising from the symplectic structure of T^*M to draw from the maximum principle some information on minimizing curves in the contact case. We denote by σ the canonical symplectic structure on T^*M and recall that

$$\begin{array}{rccc} H: & T^*M & \longrightarrow & M \\ & (p,q) & \longmapsto & \dfrac{1}{2} \max\limits_{v\in\Delta_q\setminus\{0\}} \dfrac{\langle p,v\rangle^2}{g_q(v,v)} \end{array}$$

denotes the sub-Riemannian Hamiltonian.

Let N be a differential manifold and let $\Omega \in \Lambda^2 N$ be a 2-form. A Lipschitz-continuous curve $\gamma : [0,T] \to N$ is characteristic for Ω if for almost every $t \in [0,T]$, $\dot\gamma(t) \in \ker \Omega_\gamma(t)$. This definition allows to bring together all minimizing curves under the umbrella of characteristic curves of σ after restriction of its domain.

Proposition 2.3.5 ([ABB16, Propositions 4.31-4.34])**.** *Let H be the sub-Riemannian Hamiltonian and let $c \geq 0$ be a regular value of H. The curve γ is characteristic for $\sigma_{|H^{-1}(c)}$ if and only if γ is the reparametrization of an extremal curve taking its values in $H^{-1}(c)$.*

In particular, if $c > 0$, the extremal curve is normal, and abnormal if $c = 0$.

In the case of an equiregular sub-Riemannian manifold, 0 is always a regular value of H since $H^{-1}(0)$ is the annihilator of the distribution, the $(d+1)$-dimensional submanifold of T^*M orthogonal to Δ in the sense of duality between TM and T^*M.

If this sub-Riemannian manifold is contact, there exists an integer $n \geq 1$ such that $d = 2n+1$, and $\omega \in \Lambda^1 M$ such that $\omega(\Delta) = 0$ and $\omega \wedge (\mathrm{d}\omega)^n \neq 0$ (at least locally).

Elements of $H^{-1}(0)$ are then proportional to ω in the fibers of T^*M. This implies that the restriction to $H^{-1}(0)$ of the Liouville 1-form s is proportional to ω: there exists a real valued map $\alpha : M \to \mathbb{R} \setminus \{0\}$ such that $s_{|H^{-1}(0)} = \alpha\omega$. Since $\mathrm{d}s = \sigma$, we get

$$\sigma_{|H^{-1}(0)} = \mathrm{d}\alpha \wedge \omega + \alpha\, \mathrm{d}\omega.$$

These observations are crucial because one can check that, by non-integrability of the contact distribution,

$$\left(\sigma_{|H^{-1}(0)}\right)^{n+1} = (n+1)\, \mathrm{d}\alpha \wedge \omega \wedge (\mathrm{d}\omega)^n \neq 0.$$

Since the dimension of $H^{-1}(0)$ is $2n+2$, this equation implies that the kernel of $\sigma_{|H^{-1}(0)}$ is reduced to $\{0\}$ at any point.

As a consequence of Proposition 2.3.5, we deduce the following statement on abnormal extremals in contact sub-Riemannian manifolds.

Proposition 2.3.6. *Let (M, Δ, g) be a contact sub-Riemannian manifold. Abnormal extremals are projections of stationary curves in T^*M*

Hence the only non-trivial length minimizing curves remaining are normal extremals, which are projections of solutions of an autonomous Hamiltonian system.

Hamiltonian vector fields. Let M be a differential manifold, recall that for any function $a \in C^\infty(T^*M, \mathbb{R})$, we denote by $\vec{a}$ the Hamiltonian vector field associated to a on T^*M. It is defined by the relation

$$\sigma(\cdot, \vec{a}) = \mathrm{d}a.$$

In canonical coordinates (p, x) on T^*M, $\sigma = \sum_{i=1}^d \mathrm{d}p_i \wedge \mathrm{d}x_i$ and $\vec{a}$ takes the classical form

$$\vec{a} = \sum_{i=1}^{d} \frac{\partial a}{\partial p_i}\frac{\partial}{\partial x_i} - \frac{\partial a}{\partial x_i}\frac{\partial}{\partial p_i}.$$

In the case of sub-Riemannian manifolds, normal extremals are projections of flow curves of $\vec{H}$, but it is convenient to introduce a different form of the equation of integral curves of $\vec{H}$. Denoting $\{\cdot,\cdot\} : C^\infty(T^*M,\mathbb{R}) \times C^\infty(T^*M,\mathbb{R}) \to C^\infty(T^*M,\mathbb{R})$ the Poisson bracket on T^*M, recall that for all $a, b \in C^\infty(T^*M,\mathbb{R})$,

$$\{a, b\} = \vec{a} \cdot b.$$

Therefore, with $t \mapsto \lambda(t) = \mathrm{e}^{t\vec{a}}\lambda(0)$ an integral curve of the Hamiltonian vector field $\vec{a}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} b(\lambda(t)) = \{a, b\}(\lambda(t)).$$

Hence we can give an expression of the flow of $\vec{H}$ using coordinates that are well suited to the analysis of its trajectories in the case of a contact manifold (M, Δ, g). Let $(X_1, \dots, X_{2n})$ be an orthonormal frame of (Δ, g), and let X_0 be a vector field transverse to the distribution such that $(X_1(q), \dots, X_{2n}(q), X_0(q))$ is a basis of T_qM at any $q \in M$ (or on an open subset of M if there exist no global basis).

The family $(X_i)_{0\le i\le 2n}$ allows to define coordinates on the fibers of T^*M. For all $(p, q) \in T^*M$, we denote

$$h_i(p, q) = \langle p, X_i(q)\rangle, \qquad 0 \le i \le 2n.$$

By definition of H, we now have

$$H(p, q) = \frac{1}{2}\sum_{i=1}^{2n} h_i^2(p, q) \qquad \forall (p, q) \in T^*M,$$

and the Hamiltonian equation of normal extremals becomes in coordinates (x, h)

$$\begin{cases} \dot{x} = \displaystyle\sum_{i=1}^{2n} h_i X_i(x), \\ \dot{h}_i = \{H, h_i\} \qquad \forall\, 0 \le i \le 2n. \end{cases}$$

This equation can be refined by applying classical properties of Poisson brackets. The Poisson bracket being a derivation with respect of each of its arguments,

$$\dot{h}_i = \left\{\frac{1}{2}\sum_{j=1}^{2n} h_j^2, h_i\right\} = \sum_{j=1}^{2n} h_j \{h_j, h_i\}.$$

Furthermore, for two vector fields X_i and X_j, $0 \leq i, j \leq 2n$, we have the fundamental property of Poisson brackets,

$$\{h_i, h_j\}(p, q) = \langle p, [X_i, X_j]\,(q)\rangle\,, \qquad (p, q) \in T^*M.$$

Thus, denoting $\left(c_{ij}^k\right)_{0\leq i,j,k\leq 2n} : M \to \mathbb{R}$ the family of smooth functions such that

$$[X_i, X_j] = \sum_{k=0}^{2n} c_{ij}^k X_k,$$

we get the geodesic equation we study in this thesis,

$$\begin{cases} \dot{x} = \displaystyle\sum_{i=1}^{2n} h_i X_i(x), \\ \dot{h}_i = h_0 \displaystyle\sum_{j=1}^{2n} c_{ji}^0 h_j + \sum_{j=1}^{2n}\sum_{k=1}^{2n} c_{ji}^k h_j h_k, \qquad \forall\, 0 \leq i \leq 2n. \end{cases} \tag{2.8}$$

The Heisenberg case. Regarding Heisenberg groups, it is possible to give an exact expression of the solutions of (2.8) thanks to the normal form (2.7). We choose as complement to the frame $(X_i)_{1\leq i\leq 2n}$ the vector field $X_0 = \partial_z$.

We have immediately that the brackets $[X_i, X_j] = 0$ if there exists no $k \in \mathbb{N}$, $1 \leq k \leq n$, such that $\{i, j\} = \{2k-1, 2k\}$, and $[X_{2i-1}, X_{2i}] = -[X_{2i-1}, X_{2i}] = -b_i$ otherwise. Denoting the matrix $\bar{J} = \left(c_{ji}^0\right)_{1\leq i,j\leq 2n}$, we get the block-diagonal matrix

$$\bar{J} = \begin{pmatrix} 0 & b_1 & & & & & \\ -b_1 & 0 & & & & (0) & \\ & & 0 & b_2 & & & \\ & & -b_2 & 0 & & & \\ & & & & \ddots & & \\ & (0) & & & & 0 & b_n \\ & & & & & -b_n & 0 \end{pmatrix},$$

and Equation (2.8) becomes (by denoting h the vector $(h_1, \ldots, h_{2n})$)

$$\begin{cases} \dot{x} = h, \\ \dot{z} = \displaystyle\sum_{i=1}^{n} \frac{b_i}{2}\left(h_{2i-1}x_{2i} - h_{2i}x_{2i-1}\right), \\ \dot{h} = h_0 \bar{J} h, \\ \dot{h}_0 = 0. \end{cases}$$

Hence the normal extremal of initial covector $\left(\bar{h}_1, \ldots, \bar{h}_{2n}, \bar{h}_0\right)$, starting from 0 (one can get the other solutions by left translation in the Heisenberg group) admits the parametrized equation, for all $i \in \mathbb{N}$, $1 \leq i \leq n$, (with $\bar{h}_0 \neq 0$, otherwise it is stationary)

$$\begin{aligned}
x_{2i-1}(t) &= \frac{1}{b_1 \bar{h}_0} \left[\bar{h}_{2i-1} \sin(b_i \bar{h}_0 t) + \bar{h}_{2i}(1 - \cos(b_i \bar{h}_0 t))\right], \\
x_{2i}(t) &= \frac{1}{b_1 \bar{h}_0} \left[\bar{h}_{2i-1}(\cos(b_i \bar{h}_0 t) - 1) + \bar{h}_{2i} \sin(b_i \bar{h}_0 t)\right], \\
z(t) &= \sum_{i=1}^{n} \frac{\left(\bar{h}_{2i-1}^2 + \bar{h}_{2i}^2\right)}{2 b_i \bar{h}_0^2} (b_i \bar{h}_0 t - \sin(b_i \bar{h}_0 t)), \\
h_{2i-1}(t) &= \bar{h}_{2i-1} \cos(b_i \bar{h}_0 t) + \bar{h}_{2i} \sin(b_i \bar{h}_0 t), \\
h_{2i}(t) &= \bar{h}_{2i} \cos(b_i \bar{h}_0 t) - \bar{h}_{2i-1} \sin(b_i \bar{h}_0 t), \\
h_0(t) &= \bar{h}_0.
\end{aligned} \tag{2.9}$$

Notice that there exists infinitely many parametrizations of the curve minimizing the distance to $(\bar{x}_1, \ldots, \bar{x}_{2n}, \bar{z})$. One can limit the choice of initial covectors to the set of covectors such that $\sum_{i=1}^{2n} h_i^2 = 1$, or reduce to the case of geodesics evaluated at time 1, which makes the curve unique in most cases.

2.3.3 The sub-Riemannian caustic

Conjugate locus. Let (M, Δ, g) be a contact sub-Riemannian manifold of dimension $2n+1$, $n \geq 1$, and let $q_0 \in M$. In this section, $\pi : T^*M \to M$ denotes the canonical projection of the cotangent bundle. For all $q_0 \in M$, we denote

$$\begin{array}{rccc}
\mathcal{E}_{q_0}^1 : & T_{q_0}^* M & \longrightarrow & M \\
& p & \longmapsto & \pi\left(\mathrm{e}^{\vec{H}}(p, q_0)\right)
\end{array}$$

the sub-Riemannian exponential (at time 1), that maps the covector $p \in T_{q_0}^* M$ to the end-point at time 1 of the normal extremal starting from q_0 with initial covector p. *Conjugate points* are the singular points of $\mathcal{E}_{q_0}^1$. More can be said after observing this fundamental property of the Hamiltonian flow.

Proposition 2.3.7. *Let $(p, q) \in T^*M$. For all $\mu \in \mathbb{R}$,*

$$\mathrm{e}^{\mu \vec{H}}(p, q) = \mathrm{e}^{\vec{H}}(\mu\, p, q).$$

As a consequence, we can define the exponential at time t such that $\mathcal{E}_{q_0}^t(p) = \mathcal{E}_{q_0}^1(tp)$. The *conjugate time at* $p \in T_{q_0}^* M$ is then $t_c(p) = \inf\{t > 0 \mid \mathcal{E}_{q_0}^t \text{ singular at } p\}$ and the (first) *conjugate locus at* q_0 is the set of points

$$\left\{\mathcal{E}_{q_0}^{t_c(p)}(p) \mid p \in \mathcal{C}_{q_0}(1/2)\right\}.$$

We also call this set *sub-Riemannian caustic*. Notice that for all $p \in T_{q_0}^* M$, the conjugate time $t_c(p)$ is the date at which the geodesic of initial covector p loses local optimality.

Lastly, recall that the *conjugate length* $l_c(p)$, $p \in T^*_{q_0}M$, is the length of the curve $t \mapsto \mathcal{E}^t_{q_0}(p)$ defined on $[0, t_c(p)]$. The conjugate length has the benefit of having an intrinsic definition depending only on the point of the caustic. Indeed for all $p \in T^*_{q_0}M$, we have by the dilation property of the Hamiltonian flow that $t_c(p) = \mu t_c(\mu p)$ for all $\mu > 0$, whereas $\mathcal{E}^{t_c(\mu p)}_{q_0}(\mu p) = \mathcal{E}^{t_c(p)}_{q_0}(p)$. However $l_c(\mu p) = l_c(p)$ for all $\mu > 0$ and thus it only depends on the point $\mathcal{E}^{t_c(p)}_{q_0}(p)$. Notice that if $p \in T^*_{q_0}M$ is such that $H(p, q_0) = 1/2$, then the normal extremal is parametrized with constant speed 1 and $l_c(p) = t_c(p)$.

In the following of this section, we study the conjugate locus at a set point $q_0 \in M$. For all $c \geq 0$, $\mathcal{C}_{q_0}(c) \subset T^M_{q_0}$ denotes the set of covectors $p \in T^*_{q_0}M$ such that $H(p, q_0) = c$. Notice that in the contact case, $\mathcal{C}_{q_0}(0)$ is a line and if $c > 0$, the set $\mathcal{C}_{q_0}(c)$ has the topology of the cylinder $\mathbb{S}^{2n-1} \times \mathbb{R}$. We may also abuse notations by denoting, for $V \subset \mathbb{R}^+$, $\mathcal{C}_{q_0}(V) = \cup_{c \in V} \mathcal{C}_{q_0}(c)$.

Heisenberg case. We recalled in the previous section that normal extremals admit the parametrization (2.9). It is then possible to compute the conjugate locus is this case. Indeed, the normal extremal of initial covector $(\bar{h}, \bar{h}_0) \in \mathbb{R}^{2n} \times \mathbb{R}$ starting from 0 can be written

$$h(t) = \mathrm{e}^{\bar{h}_0 t \bar{J}} \bar{h} \qquad \text{and} \qquad x(t) = \frac{1}{\bar{h}_0} \bar{J}^{-1} \left(\mathrm{e}^{\bar{h}_0 t \bar{J}} - I_{2n} \right) \bar{h}.$$

Hence $D_h x(t) = \mathrm{e}^{\bar{h}_0 t \bar{J}} - I_{2n}$, which is an invertible matrix as long as $\bar{h}_0 t \neq \frac{2\pi}{b_i}$, for all $1 \leq i \leq n$. We then define the set $Z = \{2k\pi/b_i \mid k \in \mathbb{N}, 1 \leq i \leq n\}$. Observe the following fact on singular points of $\mathcal{E}^t_0 : (\bar{h}, \bar{h}_0) \mapsto (x(t), z(t))$ (see Section 4.3.2).

Proposition 2.3.8. *Assume $b_1 \geq b_2 \geq \cdots \geq b_n$. For all $p = (\bar{h}, \bar{h}_0) \in H^{-1}(1/2)$, for all $t \in \left(0, \frac{2\pi}{b_1 \bar{h}_0}\right)$, we have $\det(\mathrm{Jac}_p \mathcal{E}^t_0) \neq 0$. Hence for all $p = (\bar{h}, \bar{h}_0) \in \mathcal{C}_0(1/2)$,*

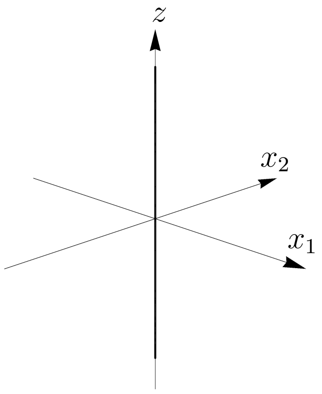
$$t_c(p) = \frac{2\pi}{b_1 \bar{h}_0},$$

and the conjugate locus at 0 of $\mathbb{H}_{2n+1}(b_1, \ldots, b_n)$ is given by the set

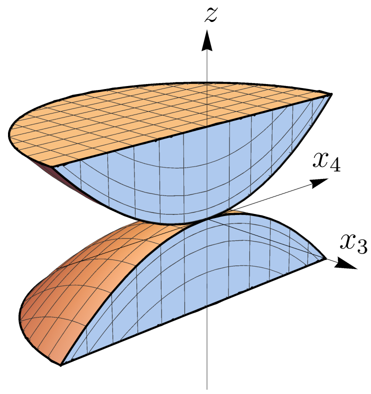
$$\left\{ \left(x\left(\frac{2\pi}{b_1 \bar{h}_0}\right), z\left(\frac{2\pi}{b_1 \bar{h}_0}\right) \right) \mid (\bar{h}, \bar{h}_0) \in \mathcal{C}_0(1/2) \right\}.$$

Notice that in the 3D case, one can check that the conjugate locus is the line $\{x_1 = x_2 = 0\}$. However in dimension $2n + 1 \geq 5$, the geometry of the sub-Riemannian caustic depends on the value of $(b_i)_{1 \leq i \leq n}$. If all b_i are equal, the conjugate locus is still the line $\{x_1 = \cdots = x_{2n} = 0\}$. On the other hand, if there exist two distinct i, j, $1 \leq i, j \leq n$, such that $b_i \neq b_j$, then the conjugate locus is reduced to 0 in the coordinates (x_{2i-1}, x_{2i}) such that $b_i = \max_j b_j$ only (see Figure 2.3).

We also recover the famous fact that the point 0 is in the closure of the conjugate locus.



(a) Section in the subspace (x_1, x_2, z)



(b) Section in the subspace $(x_3, x_4, z) \cap \{x_3 < 0\}$

Figure 2.3 – Representation of the conjugate locus of a 5D Heisenberg group (with eigenvalues $b_1 > b_2$).

The 3-dimensional case. In the case of a contact sub-Riemannian manifold (M, Δ, g) of dimension 3, one cannot in general expect global results similar to what is obtained for Heisenberg groups. We apply a different approach. Since the conjugate locus at q_0 contains q_0 in its closure, we focus on the geometry of the conjugate locus near its starting point. We base our method on the following observation (see Proposition 5.2.2 for a proof of this fact in the general case).

Proposition 2.3.9. *Let (M, Δ, g) be a contact sub-Riemannian manifold of dimension 3. For all $A > 0$, there exists $\varepsilon > 0$ such that for all initial covector $(h, h_0) \in \mathcal{C}_{q_0}(1/2)$, $t_c(h, h_0) < \varepsilon$ implies $|h_0| > A$.*

Since $t_c(p)$ is an upper bound of the distance between q_0 and $\mathcal{E}_{q_0}^{t_c(p)}(p)$ when $p \in \mathcal{C}_{q_0}(1/2)$, this implies that conjugate points near q_0 correspond to initial covectors $p = (h, h_0)$ such that $|h_0|$ tends to infinity. Hence we focus on the following problem: compute asymptotic expansions of $t_c(p)$ and $\mathcal{E}_{q_0}^{t_c(p)}(p)$ to obtain a description of the conjugate locus in a small neighborhood of q_0.

To this end, one has the following result.

Theorem 2.3.10 ([EAGK96, Agr96])**.** *Let (M, Δ, g) be a 3D contact sub-Riemannian manifold, and let $q_0 \in M$. On a neighborhood of q_0, there exist an orthonormal frame (X_1, X_2) of (Δ, g) and (x_1, x_2, z), coordinates centered at q_0 such that $X_i(q_0) = \partial_{x_i}$, such that the following asymptotic expansions hold. Denoting $p_0 = (\cos\theta, \sin\theta, h_0) \in \mathcal{C}_{q_0}(1/2)$, we have as $|h_0| \to \infty$*

$$t_c(p_0) = \frac{2\pi}{|h_0|} - \frac{\pi\kappa}{|h_0|^3} + O\left(\frac{1}{|h_0|^4}\right)$$

and

$$\mathcal{E}_{q_0}^{t_c(p_0)}(p_0) = \operatorname{sign}(h_0)\frac{\pi}{h_0^2}(0,0,1) + \frac{2\pi\chi}{h_0^3}\left(-\sin^3\theta, \cos^3\theta, 0\right) + O\left(\frac{1}{|h_0|^4}\right),$$

where κ and χ are structural invariants depending on q_0.

Moreover, at times such that $\chi \neq 0$, the cut locus admits the asymptotic expansion

$$t_{\text{cut}}(\theta, h_0) = \frac{2\pi}{|h_0|} - \frac{\pi\kappa + 2\chi\sin^2\theta}{|h_0|^3} + O\left(\frac{1}{|h_0|^4}\right)$$

and

$$\mathcal{E}_{q_0}^{t_{\text{cut}}(p_0)}(\theta, h_0) = \operatorname{sign}(h_0)\frac{\pi}{h_0^2}(0,0,1) + \frac{2\pi\chi}{h_0^3}(\cos\theta, 0, 0) + O\left(\frac{1}{|h_0|^4}\right).$$

(See Figure 2.4.)

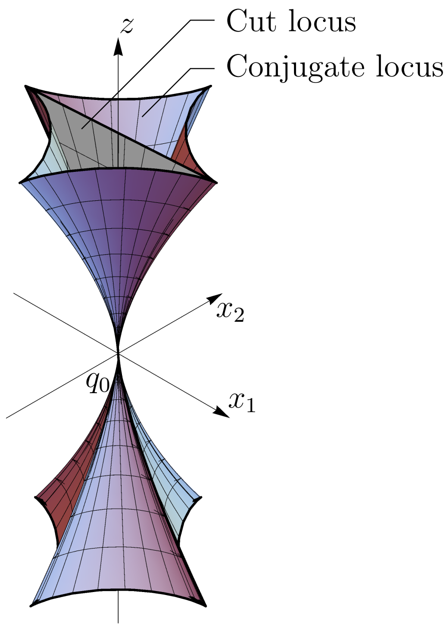


Figure 2.4 – The contact 3D conjugate locus and the cut locus of at points q_0 such that $\chi(q_0) \neq 0$.
Notice the astroidal shape of the sections of the caustic at constant z, characteristic of stable singular points of wave fronts.
Notice also the growth rate in $z^{3/2}$ of the radius of the astroid. It is a feature of the contact 3D sub-Riemannian caustic that makes special the ∂_z direction: it is transverse to the distribution (generated by $\partial_{x_1}, \partial_{x_2}$ at q_0) and tangent to the conjugate locus at q_0.

The proofs of this result in [EAGK96] and [Agr96] are different but the underlying principle is the same. When the distance to q_0 is very small, the metric can be written as a small perturbation of the metric obtained by nilpotent approximation at q_0. The sub-Riemannian exponential can then be computed by giving an approximation of the Hamiltonian flow modeled after the Hamiltonian flow of the Heisenberg group.

The method used in [EAGK96] rests in particular on the construction of coordinates and an orthonormal frame of the distribution that highlights this similarity

Theorem 2.3.11 ([EAGK96]). *Let (M, Δ, g) be a contact sub-Riemannian manifold of dimension 3. For all $q_0 \in M$ there exists (on a neighborhood of q_0) coordinates (x, y, z) centered at q_0 and (F, G) an orthonormal frame of Δ such that $F(0) = \partial_x$, $G(0) = \partial_y$,*

$[F,G](0) = \partial_z$ *and*

$$\begin{cases} F = (1+y^2\beta)\partial_x - xy\beta\partial_y + \dfrac{y}{2}(1+\gamma)\partial_z, \\ G = -xy\beta\partial_x + (1+x^2\beta)\partial_y - \dfrac{x}{2}(1+\gamma)\partial_z, \end{cases}$$

where β *and* γ *smooth functions from* $\mathbb{R}^3$ *to* $\mathbb{R}$ *such that*

$$\beta(0,0,z) = \gamma(0,0,z) = \partial_x\gamma(0,0,z) = \partial_y\gamma(0,0,z) = 0.$$

In such coordinates, (F,G) is already written as a perturbation of the normal form of a left-invariant sub-Riemannian structure on the Heisenberg group.

This study has an indirect but essential consequence for the study of the small scale geometry of contact sub-Riemannian manifolds. The front obtained by nilpotent approximation of the exponential is an extremely precise approximation.

Proposition 2.3.12. *Let* (M,Δ,g) *a contact sub-Riemannian manifold of dimension* 3*. Let* $q_0 \in M$*. There exists system of privileged coordinates centered at* q_0 *such that, denoting* $\widehat{\mathcal{E}}$ *the sub-Riemannian exponential at* 0 *of the contact structure obtained by nilpotent approximation, for all* $(h,h_0) \in \mathcal{C}_{q_0}(1/2)$*, as* $|h_0| \to \infty$

$$\mathcal{E}_{q_0}^{t/h_0}((h,h_0)) = \widehat{\mathcal{E}}^{t/h_0}((h,h_0)) + O\left(\frac{1}{h_0^3}\right).$$

Notice that $\left|\mathcal{E}_{q_0}^{t/h_0}((h,h_0))\right| \propto |h_0|^{-1}$, hence this result highlights that there exist coordinates for which we obtain two orders of approximation instead of one with the nilpotent approximation. As a consequence, the geometry of the nilpotent approximation is a better approximation of the geometry of 3D contact sub-Riemannian manifolds than what could have been expected.

2.3.4 The higher dimensional case

Novelties in higher dimension. The study of the sub-Riemannian exponential and its conjugate locus represents a first step towards understanding the small scale geometry of contact sub-Riemannian manifolds of dimension greater than 3. We wish to extend the perturbative approach used in dimension 3 to lay the groundwork of this analysis.

Let (M,Δ,g) be a contact sub-Riemannian manifold of dimension $2n+1$, $n \geq 2$. As explained in the previous sections, the geometry of (M,Δ,g) is deeply linked with the geometry of its nilpotent approximation. The nilpotent approximation of M at q_0 is given by a left-invariant structure on the Heisenberg group $\mathbb{H}_{2n+1}$. When $n \geq 2$, Theorem 2.3.3 implies that there exists infinitely many such structure that are not isometric. Thus, at any $q_0 \in M$ there exists a family $(b_1(q_0),\dots,b_n(q_0))$ of positive reals (unique up to multiplication by a positive constant and rearrangements) such that the nilpotent approximation at q_0 of (M, Δ, g) is given by $\mathbb{H}_{2n+1}(b_1(q_0), \dots b_n(q_0))$. We assume in the following that $b_1(q_0) \dots b_n(q_0) = \frac{1}{n!}$. (Observe that we could compute $(b_i)_{1 \leq i \leq n}$ without nilpotent approximation, in the same way we computed them in Section 2.3.1, see in particular Section 4.1.)

As seen in Section 2.3.3, the value of the family $(b_i)_{1 \leq i \leq n}$ greatly influences the geometry of the conjugate locus, hence we make the choice of limiting ourselves to the case of $(b_i)_{1 \leq i \leq n}$ such that $b_1 > b_2 \geq \cdots \geq b_n$. This choice can be justified by genericity arguments, since the subset of M on which there exist two invariants b_i, b_j, $i \neq j$, such that $b_i = b_j$, is a codimension 3 stratified subset of M in the generic case (see [Cha02]). Under this genericity hypothesis, we only consider points in the complementary of this stratified subset.

Similarly to what is done in dimension 3, we choose to apply perturbative methods to take advantage of our knowledge of the nilpotent approximation thanks to normal forms. We use a counterpart found in [AG01] of Theorem 1.3.11 in the case of contact sub-Riemannian manifold of arbitrary dimension (we recall the main aspects of the construction in Appendix 4.A). For all $q_0 \in M$, this theorem provides on a neighborhood of q_0 coordinates $(x_1, \dots, x_{2n}, z) : M \to \mathbb{R}^{2n+1}$ and an orthonormal frame $(X_1, \dots, X_{2n})$ of the distribution benefiting from the symmetries of the contact structure and its similarities to $\mathbb{H}_{2n+1}(b_1, \dots, b_n)$. We reference this framework as Agrachev–Gauthier normal form.

We are looking for an approximation of the sub-Riemannian conjugate locus in the neighborhood of the starting point of the caustic. Thanks to the Agrachev–Gauthier normal form, we can extend Proposition 2.3.9, which shows that points near q_0 that are conjugate to q_0 are associated with initial covectors $(h, h_0) \in \mathcal{C}_{q_0}(1/2)$ such that $|h_0|$ tends to infinity.

First and foremost, it is primordial to obtain an approximation of $t_c((h, h_0))$ as $h_0 \to \infty$, the approximation of the conjugate locus follows by evaluating the exponential at the conjugate time. Thanks to the dilation property of the Hamiltonian flow (see Proposition 2.3.7), we can compute the conjugate time at q_0 for the covector p_0 by finding the first vanishing time $t > 0$ of

$$\Phi(p_0, t) = \det\left(\frac{\partial \mathcal{E}^t_{q_0}}{\partial h_1}, \dots, \frac{\partial \mathcal{E}^t_{q_0}}{\partial h_{2n}}, \frac{\partial \mathcal{E}^t_{q_0}}{\partial h_0}\right).$$

From the identity $\Phi(p_0, t_c(p_0)) = 0$ we gather the information we need to compute an approximation of $t_c((h, h_0))$ as $h_0 \to \infty$.

Conjugate time. We introduce $\eta = 1/h_0$. A difficulty that appears is the (expected) fact that $t_c((h, \eta^{-1})) \to 0$ as $\eta \to 0$. In the case of the nilpotent approximation, recall we had precisely (with our choice $b_1 > b_2 \geq \cdots \geq b_n$)

$$t_c(h, \eta) = \frac{2\pi}{b_1}\eta.$$

Hence we make the choice of reparametrizing time with $t = \eta\tau$, so that τ_c, the conjugate time for the τ variable, is constant in the nilpotent approximation. Then we introduce $F : \mathbb{R} \times \mathbb{R}^{2n} \times \mathbb{R} \to M$ such that

$$F(\tau, h, \eta) = \mathcal{E}_{q_0}^{\eta\tau}((h, \eta^{-1}))$$

and

$$\Phi((h, \eta^{-1}), \eta\tau) = \det\left(\frac{\partial F}{\partial h_1}, \dots, \frac{\partial F}{\partial h_{2n}}, -\eta^2\frac{\partial F}{\partial \eta} + \eta\tau\frac{\partial F}{\partial \tau}\right).$$

One can check that F is smooth with respect to η at 0, which allows to express the function F as a power series in η. We formally write $F(\tau, h, \eta) = \sum_{k=0}^{\infty} \eta^k F^{(k)}(\tau, h)$ and we can compute the first non-zero terms $F^{(1)}$ and $F^{(2)}$. Denote $\widehat{\mathcal{E}}$ the exponential at 0 of the nilpotent approximation, as a result of this analysis we get

$$\begin{aligned}\widehat{\mathcal{E}}^{\eta\tau}((h, \eta^{-1}))_i &= \eta F_i^{(1)}(\tau, h), \qquad \forall\, 1 \leq i \leq 2n,\\ \widehat{\mathcal{E}}^{\eta\tau}((h, \eta^{-1}))_{2n+1} &= \eta^2 F_{2n+1}^{(2)}(\tau, h)\end{aligned}$$

This implies that a first order approximation of the conjugate time is the nilpotent conjugate time. However observe the following property of conjugate times in the nilpotent approximation that plays an important role here.

Proposition 2.3.13. *Let $b_1 > b_2 \geq \dots \geq b_n$ and let $(M, \Delta, g) = \mathbb{H}_{2n+1}(b_1, \dots, b_n)$. For a covector $p_0 \in \mathcal{C}_0(1/2)$, $t_c'(p_0)$ denotes the second conjugate time at 0 for the initial covector p_0, that is the infimum of times $t > t_c(p_0)$ such that $\mathcal{E}_0^t(p_0)$ is conjugated to 0. Then for $p_0 = (r_1\cos\theta_1, r_1\sin\theta_1, h_3, \dots, h_{2n}, h_0) \in \mathcal{C}_0(1/2)$, there exists a function $f(h_3, \dots, h_{2n})$ such that $t_c'(p_0)$ admits the following asymptotic expansion as $r_1 \to 0^+$:*

$$t_c'(p_0) = \frac{2\pi}{b_1 h_0} + r_1^2\frac{f(h_3, \dots, h_{2n})}{h_0} + o(r_1^2).$$

Hence it appears that in the nilpotent case, for initial covectors of the form

$$(0, 0, h_3, \dots, h_{2n}, h_0)$$

the nilpotent conjugate time $\frac{2\pi}{b_1 h_0}$ is a double zero of the function Φ. However double zeros are unstable by perturbation, and while it can remain double in the general case, it could also split into two simple zeros or vanish after perturbation.

Double conjugate times are not new in the context of equiregular step-2 sub-Riemannian structures, since this phenomenon can be found in dimension 4 quasi-contact structures [Cha02]. We choose a different approach to [Cha02] by blowing up the singularity at $h_1 = h_2 = 0$. Hence we split $\mathcal{C}_{q_0}(1/2)$ in two domains, depending on whether or not $h_1^2 + h_2^2 > \varepsilon$, for some arbitrary $\varepsilon \in (0, 1)$. In the case $h_1^2 + h_2^2 < \varepsilon$, since we are looking for perturbations of order $1/h_0^2$ of the nilpotent conjugate time $\frac{2\pi}{b_1 h_0}$, we consider initial covectors of the form, as $h_0 \to +\infty$,

$$\left(\frac{h_1}{\sqrt{h_0}}, \frac{h_2}{\sqrt{h_0}}, h_3, \dots, h_{2n}, h_0\right).$$

Remark 2.3.14. The principle of this method is to limit the introduction in the computations of elements resulting from higher orders of approximation of the exponential. We are able to restrict the computation to invariants of order 2 of the jets of the metric. The general method, mirroring what is done in [EAGK96, Cha02], would require to compute better approximations of the exponential by introducing elements resulting of jets of order 3. Given the growth of the number of invariants at each degree of approximation, this would have limited computations.

This approach leads to the following asymptotic expansion.

Theorem 2.3.15. *Let (M, Δ, g) be a contact sub-Riemannian manifold of dimension $2n+1$, $n \geq 2$, and let $q_0 \in M$ be such that $b_1(q_0) > b_2(q_0) \geq \cdots \geq b_n(q_0)$.*

For all $\varepsilon \in (0,1)$, let $S(\varepsilon) = \{(h_1, \dots, h_{2n}, h_0) \in \mathcal{C}_{q_0}(1/2) \mid h_1^2 + h_2^2 < \varepsilon^2\}$. There exists four functions $c_1, c_2, c_3, c_4 : \mathbb{R}^{2n} \to \mathbb{R}$ such that for all $\varepsilon \in (0,1)$

— *If $p_0 = (h, h_0) \in \mathcal{C}_{q_0}(1/2) \setminus S(\varepsilon)$, as $h_0 \to +\infty$*

$$t_c\,(h, h_0) = \frac{2\pi}{b_1 h_0} + \frac{1}{h_0^2} t_c^{(2)}(h) + O\left(\frac{1}{h_0^3}\right)$$

where $t_c^{(2)}$ is solution of

$$X(h_1^2 + h_2^2) + c_4(h) = 0.$$

— *If $h_0 \to +\infty$, the expansion*

$$t_c\left(\frac{h_1}{\sqrt{h_0}}, \frac{h_2}{\sqrt{h_0}}, h_3, \dots, h_{2n}, h_0\right) = \frac{2\pi}{b_1 h_0} + O\left(\frac{1}{h_0^2}\right)$$

holds if and only if the quadratic polynomial equation

$$X^2 c_1(h) + X\left[(h_1^2 + h_2^2) + c_2(h)\right] + [c_3(h) + c_4(h)] = 0.$$

admits real valued solutions. In this case, $\tilde{t}_c^{(2)}$ denotes the smallest of its two real solutions, and for $p_0 = \left(\frac{h_1}{\sqrt{h_0}}, \frac{h_2}{\sqrt{h_0}}, h_3, \dots, h_{2n}, h_0\right) \in S(\varepsilon)$, as $h_0 \to +\infty$,

$$t_c\left(\frac{h_1}{\sqrt{h_0}}, \frac{h_2}{\sqrt{h_0}}, h_3, \dots, h_{2n}, h_0\right) = \frac{2\pi}{b_1 h_0} + \frac{1}{h_0^2}\tilde{t}_c^{(2)}(h_1, \dots, h_{2n}) + O\left(\frac{1}{h_0^3}\right).$$

Sub-Riemannian caustic. These computations allow to determine an approximation of the conjugate locus represented in the following figures. Figure 2.5 gives an indication of the geometry of the conjugate locus in the dimensions (x_1, x_2, z) where it is considerably altered from the nilpotent case (see Figure2.3), contrarily to coordinates $(x_3, \dots x_{2n})$. We also present a schematic representation of the nilpotent (Figure 2.6) and general (Figure 2.7) conjugate locus to compare their global geometry.

Remark 2.3.16. As it can be seen in Theorem 2.3.15, the conjugate time may not always be a perturbation of the nilpotent conjugate time $2\pi/(b_1 h_0)$. We will not cover the case of conjugate points associated with such initial covectors. In the following, we then restrict this study to singular points $\mathcal{E}_{q_0}^{t_c(p)}(p)$ such that $t_c(p) \sim 2\pi/(b_1 h_0)$.

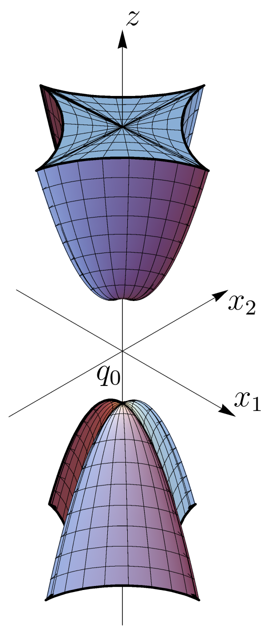


Figure 2.5 – Section of the 5D conjugate locus by the subspace $(x_3, x_4) \neq (0,0)$ and $z \in [-1,1]$ (represented without blowup of the singularity at $x_1 = x_2 = 0$.

Notice that some geometric properties of the caustic are lost in this case. For initial covectors such that $(h_3, h_4) \neq (0,0)$, the conjugate locus now appears as two separate connected components.
We may have an astroidal section at z constant, but is depends of the value of the pair (h_3, h_4) (see Figure 5.1 in Chapter 5 for more precision). The radius of a section does not grow as $z^{3/2}$ anymore. Indeed, denoting $r_1 = (h_1^2 + h_2^2)^{1/2}$ and $r_2 = (h_3^2 + h_4^2)^{1/2}$, there exists constants $c_1, c_2 > 0$ such that $F_5^{(2)}(\tau_c) = \eta^2(c_1 r_1^2 + c_2 r_2^2)$, whereas $F_i^{(2)}(\tau_c) \propto \eta^2 r_1^2 r_2$ for $i \in \{1,2\}$.

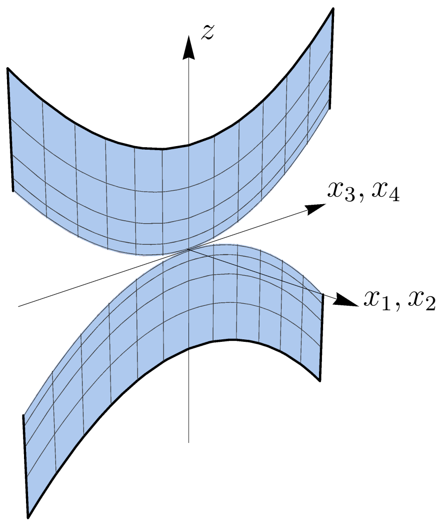


Figure 2.6 – Schematic representation of the conjugate locus of a 5D Heisenberg group (with distinct eigenvalues)

To discuss the geometry of the conjugate locus, we use in the following a schematic representation gathering x_1, x_2 in one dimension and x_3, x_4 in the other. The representation of the conjugate locus for a left-invariant structure on the Heisenberg group can be seen in Figure 1.6.

Sub-Riemannian stability in dimension 5. Let (M, Δ, g) and $(M, \widetilde{\Delta}, \widetilde{g})$ be two contact sub-Riemannian structures on M, of which H and $\widetilde{H}$ denote their respective Hamiltonians. Let $(p_0, q_0) \in T^*M$. The respective sub-Riemannian exponentials $\mathcal{E}_{q_0}^1$ and $\widetilde{\mathcal{E}}_{q_0}^1$ are *sub-Riemannian equivalent at* p_0 if thee exists two neighborhoods $\Omega, \widetilde{\Omega} \subset TM$ of (p_0, q_0) and two maps $\phi : \pi(\Omega) \to \pi(\Omega')$, $\Phi : \Omega \to \Omega'$ such that, for σ the canonical symplectic form, π the canonical projection of the cotangent bundle T^*M

$$\Phi^*\sigma = \sigma, \qquad \pi \circ \Phi = \phi \circ \pi \quad \text{and} \quad \Phi \circ \mathrm{e}^{\vec{H}} = \mathrm{e}^{\vec{\widetilde{H}}}.$$

The main interest for us is the following: if two exponential maps are sub-Riemannian equivalent at a point, their caustics are diffeomorphic (in a neighborhood of the image

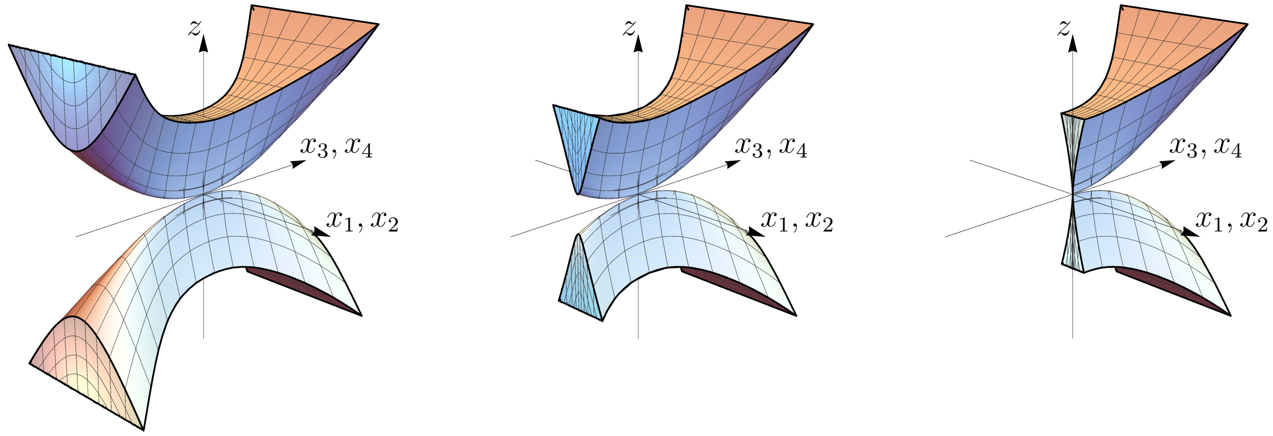


Figure 2.7 – Schematic representation of the caustic of a 5D contact structure (with distinct eigenvalues). Notice that the conjugate locus is pinched at q_0.

of this point). The sub-Riemannian exponential $\mathcal{E}^1_{q_0}$ is *sub-Riemannian-stable at* p_0 if there exists a neighborhood V of p_0 and a neighborhood W of $\mathcal{E}^1_{q_0|V}$ in the class of sub-Riemannian exponentials at q_0 for the C^∞ Whitney topology such that all elements of W are sub-Riemannian-equivalent to $\mathcal{E}^1_{q_0}$ at p_0.

Notice that sub-Riemannian stability is the restriction of Lagrangian stability to the class of sub-Riemannian exponentials (see [AGnZV85, Ben85, IFR+16]). Indeed, as the projection of the Hamiltonian flow, the sub-Riemannian exponential has a natural structure of Lagrangian map. A well known fact on the stability of Lagrangian maps is that in dimension up to 5 there exists only finitely many equivalence classes for Lagrangian stability (see, for instance, the summary in [BBCN16, Theorem 2]).

Theorem 2.3.17. *A generic Lagrangian map from $\mathbb{R}^5$ to $\mathbb{R}^5$ has only stable singular points, of type $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$, $\mathcal{A}_5$, $\mathcal{A}_6$, $\mathcal{D}_4^\pm$, $\mathcal{D}_5^\pm$, $\mathcal{D}_6^\pm$ and $\mathcal{E}_6^\pm$.*

We give a counterpart to this result in the case of contact sub-Riemannian manifolds of dimension 5, which allows to state the stability of the sub-Riemannian caustic outside of its starting point q_0.

Theorem 2.3.18. *Let (M, Δ, g) be a generic contact sub-Riemannian manifold of dimension 5. For all q_0 in the complementary of a stratified codimension 1 subset of M there exists a neighborhood V_{q_0} of q_0 such that for all open set U containing q_0, the intersection of the caustic with $V_{q_0} \setminus U$ is (sub-Riemannian-)stable and has only Lagrangian singularities of type $\mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{D}_4^+$ and $\mathcal{A}_5$.*

Remark 2.3.19. We extend previous results on the stability of the conjugate locus for singularities of 3D contact structures and 4D quasi-contact structures. Under the same assumption of restriction to the complementary of a codimension 1 stratified subset of the manifold, we get the stability of the sub-Riemannian caustic on an open neighborhood of q_0 (minus a small neighborhood of q_0), and the types of singularities are $\mathcal{A}_2, \mathcal{A}_3$ in contact 3D [Agr96, EAGK96] and $\mathcal{A}_2, \mathcal{A}_3, \mathcal{D}_4^+$ in quasi-contact 4D [Cha02].

The proof of this fact relies on a classification of the singularities of the sub-Riemannian exponential. We rely on the fact that the exponential is also a Lagrangian map.

Proposition 2.3.20. *Let (M, Δ, g) be a contact sub-Riemannian manifold of dimension and let $(p_0, q_0) \in T^*M$. If $\mathcal{E}_{q_0}$ is Lagrange-stable at p_0 then it is sub-Riemannian-stable at p_0.*

The classification is carried on three domains, each corresponding to one of three cases: (h_1, h_2) small, (h_3, h_4) small and their complementary. The introduction of a third domain ((h_3, h_4) small) is a result of the geometry of the conjugate locus. Indeed, the pinching of the conjugate locus at q_0 (see Figure 2.7) implies the annihilation of an order of approximation, similarly to what happens in 3D. We choose to treat this case via blowup, similarly to what is done with (h_1, h_2) small.

The classification domains and the corresponding singularities are represented in Figure 2.8.

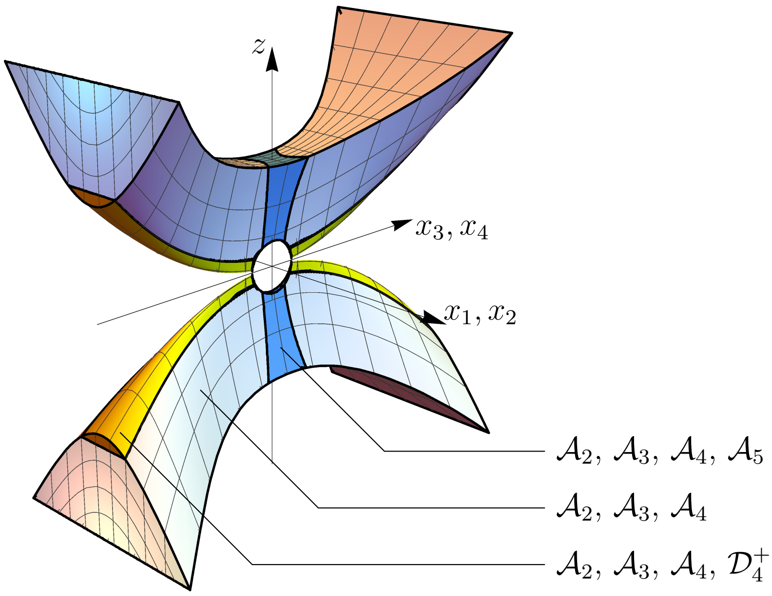


Figure 2.8 – Schematic representation of the three domains of the studied caustic, resulting in the classification.

2.3.5 The influence of invariants

One motivation of the study we outlined is to understand the similarities that exist between the similarities between the well understood 3D contact case and higher dimensional contact sub-Riemannian manifolds This study reveals that the geometry of sub-Riemannian manifolds of higher dimension can be quite different as a consequence of the presence of invariants of the metric tangent that break the symmetry if the 3D structure. We present this fact with two final discussions.

Approximations of the exponential. As we highlighted in the commentary of Proposition 2.3.12, in the 3D case, the sub-Riemannian exponential is very well approximated by the exponential at 0 of the nilpotent approximation, assuming the coordinates are chosen well enough to capture symmetries of the nilpotent approximation. The conjugate time asymptotic in dimension greater than 3 seems to contradict this observation.

In 3D, one can link the existence of coordinates such that

$$\mathcal{E}_{q_0}^{t/h_0}(h, h_0) = \widehat{\mathcal{E}}^{t/h_0}(h, h_0) + O(1/h_0^3) \tag{2.10}$$

to the fact that the conjugate time admits in such coordinates an asymptotic expansion of the form

$$t_c(h, h_0) = \frac{2\pi}{h_0} + O(1/h_0^3).$$

Indeed if we denote $t_c^{(2)}$ such that $t_c(h, h_0) = \frac{2\pi}{h_0} + \frac{t_c^{(2)}}{h_0^2} + O(1/h_0^3)$,

$$\mathcal{E}_{q_0}^{t_c}(h, h_0) = \widehat{\mathcal{E}}^{2\pi/h_0}(h, h_0) + \frac{t_c^{(2)}}{h_0^2}\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\mathcal{E}}^{2\pi/h_0} + O(1/h_0^3).$$

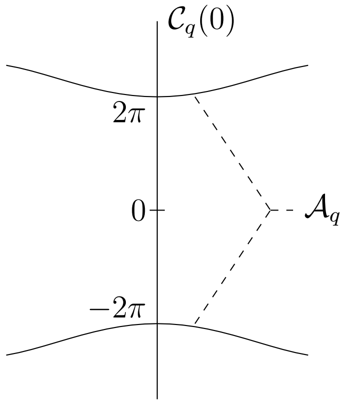
The conjugate locus being tangent to a line transverse to the distribution, $t_c^{(2)}\frac{\mathrm{d}}{\mathrm{d}t}\widehat{\mathcal{E}}^{2\pi/h_0}$ must be 0 for coordinates (x, z) such that the z-axis carries the tangent at q_0 of the conjugate locus. Hence $t_c^{(2)} = 0$.

In other words, from the geometry of the conjugate locus we can gather information on the asymptotic behavior of the exponential. We take inspiration from this observation to show that expansion (2.10) does not generalize.

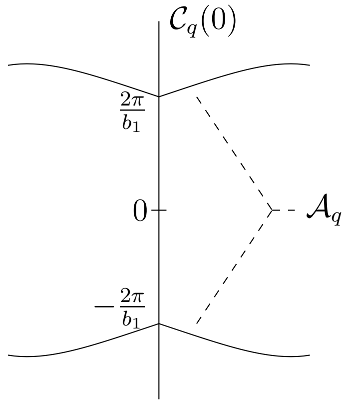
In dimension greater than 3, we cannot expect the tangent to the conjugate locus to carry as much information because of the shape of the nilpotent conjugate locus. We consider then a geometric object that holds a dual role. Let (M, Δ, g) be a contact sub-Riemannian manifold and let $q \in M$. We set $\mathcal{A}_q \subset T_q^*M$ such that

$$\mathcal{A}_q = \overline{\left\{t_c(p)\, p \mid p \in T_q^*M, H(p, q) > 0\right\}}.$$

It is the set of singular points of the exponential at time 1 that do not correspond to abnormal extremals. The set $\mathcal{A}_q$ is an immersed hypersurface, such that for $r > 0$ small enough, $\mathcal{A}_q \cap \mathcal{C}_q([0, r])$ has only two connected components (see Figure 2.9). In particular, the intersection of $\mathcal{A}_q$ with $\mathcal{C}_q(0)$ is reduced to two points, symmetric with respect to the 0 of T_q^*M, denoted by p_+ and p_-, of coordinates $(0, \pm\frac{2\pi}{\max b_i})$ (with $\max b_i = 1$ in the 3D case). In the nilpotent case for instance, $\mathcal{A}_q$ is the union of the two hyperplanes $\{h_0 = \pm\frac{2\pi}{\max b_i}\}$.



(a) 3D case



(b) 5D case

Figure 2.9 – Schematic representation of the set $\mathcal{A}_q$, always smooth at its intersection with $\mathcal{C}_q(0)$ in the 3D case only.

The counterpart of the reasoning on the geometry at q_0 of the conjugate locus is to consider the geometry of $\mathcal{A}_q$ at p_+ (or p_- equivalently). The existence of asymptotic expansions similar to (2.10) is then reduced to proving that the immersed surface $\mathcal{A}_q$ admits a planar tangent at p_+, which is always true in dimension 3, and false in most case in higher dimension.

We can interpret this difference as follows. In the 3D nilpotent case, the natural symmetry of the geometry of the wave front of the Heisenberg group is SO(2). For all $\theta \in \mathrm{SO}(2)$, the action of θ commutes with the exponential $\theta \cdot \widehat{\mathcal{E}}_0(p) = \widehat{\mathcal{E}}_0(\theta \cdot p)$, provided that it stabilizes ∂_z on the left side and $\mathcal{C}_q(0)$ on the right. Hence at the conjugate locus, the action of SO(2) stabilizes the conjugate points.

In higher dimension, this symmetry is generalized by the action of the group $\mathrm{SO}(2)^n$, to each b_i corresponds a rotation in the coordinates (x_{2i-1}, x_{2i}). At conjugate points, a trace of the initial covector is maintained in the coordinates (x_{2i-1}, x_{2i}) such that b_i is not maximal. This last point then implies the non-flatness of $\mathcal{A}_q$ at p^+ and p^-.

If $\mathcal{A}_q$ were to admit a tangent plane at its intersection with $\mathcal{C}_q(0)$, the natural symmetries would be given by the group $\mathrm{SO}(2n)$ acting on T_q^*M and stabilizing $C_q(0)$. It is clear that if $n = 1$, $\mathrm{SO}(2n) = \mathrm{SO}(2)^n$, however in higher dimension, this amounts to requiring $b_1 = \cdots = b_n$.

Intrinsic complements. All these considerations find resonates with the problem of constructing intrinsic vector fields transverse to the distribution.

Let (M, Δ, g) be a contact sub-Riemannian manifold of dimension $2n+1$. There exists (locally) a contact 1-form ω such that $\omega(\Delta) = 0$ and $\omega \wedge (\mathrm{d}\omega)^n \neq 0$. For any smooth non vanishing function $f : M \to \mathbb{R} \setminus \{0\}$, the 1-form $f\omega$ is also a contact form. The contact form ω is characterized by the smooth non-vanishing function $\alpha : M \to \mathbb{R} \setminus \{0\}$ such that $(\mathrm{d}\omega_{|\Delta})^n = \alpha \, \mathrm{vol}_g$, where vol_g is the volume form induce by g on Δ.

The kernel of the two-form $\mathrm{d}\omega$ defines at any point of M a 1-dimensional subspace of the tangent space and any non vanishing vector field X_0 such that $\mathrm{d}\omega(X_0, \cdot) = 0$ is characterized by the smooth non vanishing function $\beta : M \to \mathbb{R} \setminus \{0\}$ such that $\omega(X_0) = \beta$ (the Reeb vector field corresponds to $\beta = 1$).

We can consider X_0 to be intrinsic if α and β are functions of invariants of the metric tangent. In dimension 3, there is no invariant depending on the position, and constant α, β are the only possible choice.

In higher dimension, invariants of the metric tangent (that is, $(b_i)_{1 \leq i \leq n}$) depend on the position. Then the vector X_0 can be arbitrarily set at a point by choosing the functions α and β. Geometrically, we indeed notice that in dimension larger than 3, no intrinsic direction appears canonical by arguments of geometry of the conjugate locus. Neither the conjugate locus nor its preimage allow to definitively discriminate such a direction. This observation can be found in other aspects of the geometry of contact sub-Riemannian manifolds.

In [BNR17], the authors compare two distinct constructions of sub-Riemannian Laplacians. One is obtained via the choice of a volume form vol on the manifold and the other via the choice of a complement $\mathbf{c}$ of the distribution. Then L_{vol} and $L^{\mathbf{c}}$ denote the Laplacians obtained with these respective constructions. Regarding contact manifolds, one can find the following observation.

Let ω be a contact 1-form for (M, Δ, g) and let X_0 be a non-vanishing vector field transverse to the distribution. Set $\omega' = \frac{\mathrm{d}\omega(X_0, \cdot)}{\omega(X_0)}$ and $g = \frac{\mathrm{d}\omega' \wedge \omega \wedge (\mathrm{d}\omega)^{n-1}}{\omega \wedge (\mathrm{d}\omega)^n}$. If $\mathrm{d}\omega' = \mathrm{d}g \wedge \omega + g \, \mathrm{d}\omega$, then ω' is a contact form with Reeb X_0, and there exists a unique volume vol (up to multiplication by a constant) such that $L_{\mathrm{vol}} = L^{\mathbf{c}}$ with $\mathbf{c}$ generated by $\{X_0\}$. The arbitrary choice of a complement of the distribution is not limited either by the existence of a volume such that $L_{\mathrm{vol}} = L^{\mathbf{c}}$.

2.4 Singularities of line fields

Sub-Riemannian geometry finds an application in the modeling of the sensitivity of neurons of the visual cortex V1, represented as a 3D contact sub-Riemannian structure on $\mathbb{R}^2 \times \mathbb{P}^1$. The spatial organization of neurons in the visual cortex does not respect this geometry, and rather admits a mathematical modeling as a line field with many singularities. Some singularity patterns are occurring among multiple domains that can be modeled by line fields, which encourages to study the stability of these patterns. The natural definition of line fields is not suitable for such an analysis however, hence why we propose an alternative construction to model and study these singularities

2.4.1 Line fileds and their singularities

Sub-Riemannian geometry, vision and line fields. We owe to Hubel and Wiesel [HW62] the observation that the visual cortex V1 of some mammals is endowed with neurons sensible to both positions and orientations. This implies the existence of an underlying structure to neurons, whose sensitivity can be represented as points of the space $PT\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{P}^1$, the orientation bundle of the plane. Preferential connections between neurons sensitive to similar positions and orientations is then modeled by a contact sub-Riemannian structure on $PT\mathbb{R}^2$ [CS06, Pet08], the classic structure generated by the orthonormal frame (with $\beta > 0$ a parameter of the distance)

$$X_1 = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 \\ 0 \\ \beta \end{pmatrix}.$$

Regarding the spatial layout of neurons in the visual cortex, we observe they are not distributed according to this structure but rather in columns, whereby two neurons of same position but different depth in the cortex share the same sensitivity. Hence the distribution of orientation sensitivities can be modeled as a line field, a section of the orientation bundle of $\mathbb{R}^2$, which associates to each point in $\mathbb{R}^2$ a direction in $\mathbb{P}^1$ (see Figure 2.10).

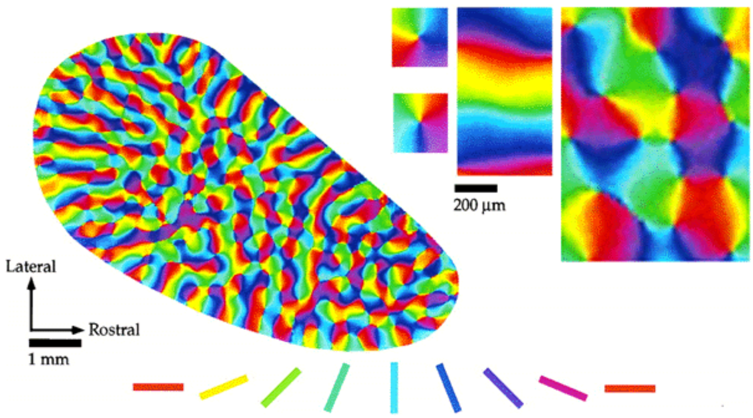


Figure 2.10 – Map of preferential orientations in the cortex visuel V1 obtained by medical imaging [BZSF97].

The medical imaging works on mapping orientation sensitivity reveal a particular structure. We observe a high density of singular points where all orientations sensitivities accumulate at a point (pinwheels) . Furthermore, two types of singularities are noticed, stopping points and triple points, corresponding to the two possible chiralities for the organization of orientations around a singular point.

Line fields. One of the first studies of line fields and their singularities removed from its historical applications is due to Hopf in [Hop03]. There, line fields are defined as maps that associate to each points of a manifold M a tangent line to M at the point. A line field singularity is then a point of M at which the map cannot be continuously extended. This early definition allows to extend the definition of index of vector field singularities to the case of line fields. More precisely, for a line field L continuous on a neighborhood of a point $p \in M$, minus p, the *index of L at p* is a real number quantifying the angle variation of L winding around p. For X a continuous vector field, not singular at p and C a closed simple curve winding around p in a small enough neighborhood, the index is given by

$$\mathrm{ind}_p L = \frac{1}{2\pi}\delta_C \angle[X, L],$$

where $\angle[X, L]$ denotes the oriented angle between X and L (in $[0, \pi)$), and $\delta_C \angle[X, L]$ the total variation of angle between X and L along C. In the case of a continuous vector field, the index is always an integer, whereas in the case of a line field continuous on a neighborhood of p, except at p, the index is a half-integer.

Hopf analysis allows to extend his theorem linking the total index of a line field with the Euler characteristic of a surface.

Theorem 2.4.1 (Poincaré-Hopf)**.** *Let (M, g) be a 2D Riemannian manifold, orientable and compact, of Euler characteristic $\chi(M)$, and let L be a line field on M with isolated singularities. Let z_L be the set of singularities of L, we have*

$$\sum_{p \in z_L} \mathrm{ind}_p L = \chi(M).$$

By analogy with vector field, we choose an alternative more precise definition of line fields. Let M be a manifld of dimension 2. The *orientation bundle PTM* is the fiber bundle of tangent lines to M, that can be obtained by projectivization $PT_pM$of the vector space T_pM at any $p \in M$. We call *line field* on M a section of the bundle PTM. A smooth line field is then a smooth section of the fiber bundle PTM.

A consequence of Theorem 1.4.1 on vector fields is that a continuous vector field must vanish at least once on a compact manifold with non-zero Euler characteristic. The counterpart of this observation for line fields is that there exists no continuous section of PTM if $\chi(M) \neq 0$. In other words, the definition of a regular line fields, as a section of the orientation bundle, requires to remove the singularities form its domain *a priori*. It is this fundamental observation that motivates the introduction of elements presented in the next section.

Modeling tool. Even though line fields and their singularities are more delicate to define than vector field, their usage as a modeling tool can be justified by their emergence in numerous fields beside the pinwheel structures in the visual cortex.

Similarly to integral lines of vector fields, we define *integral manifolds* of a line field L on M to be the set of curves foliating the manifold M and tangent to L at any point. Integral manifolds and their singularities can be seen in multiple classical applicative domains, such as fingerprints [Pen79, KW87], liquid crystals [Cha92, Pro95] and more recently structure optimization [PT08] (see also the thesis of P. Geoffroy). (See Figure 2.11.)

(a) Liquid crystals in nematic phase.

(b) Fingerprints.

(c) Structure optimization with lattice materials.

Figure 2.11 – Singularities as observed in liquid crystals[1], fingerprints and structure optimization[2].

A noticeable feature throughout these applications is the prevalence of singularities of index $\pm 1/2$ (see Figure 2.12). From the point of view of geometric modeling of singularities of line fields, this echoes the concepts of stability and genericity of Thom's school [Tho72], only singularities stable through small perturbations can be observed in nature.

Principal directions of curvature. The field of Gaussian geometry of surfaces has been one of the major domains where line fields and their singularities received an interest [Cay63, Por01]. At any point of a surface S immersed in $\mathbb{R}^3$, the principal directions of curvature define two line fields (orthogonal to each other). Umbilics of a surface S are the points at which both principal curvature are equal, and they correspond to singularities of the line fields associated with principal directions of curvature. (see Figure 2.13).

Line fields associated with directions of principal curvature are endowed with a natural topology given by the immersion of the surface $\alpha : M \to S \subset \mathbb{R}^3$. Three singularities, *Lemon*, *Monstar*, *Star* (see [BH77], and Figure 2.14), where identified by Darboux in [Dar96]. The structural stability of these singularities (in the sense of homeomorphisms between integral manifolds) has been proved by Gutierrez and Sotomayor [SG82] (see also [SG08]).

1. images of singularities of liquid crystals in nematic phase kindly provided by S.J. DeCamp.
2. images of singularities in the lattice structure of a post kindly provided by P. Geoffroy.

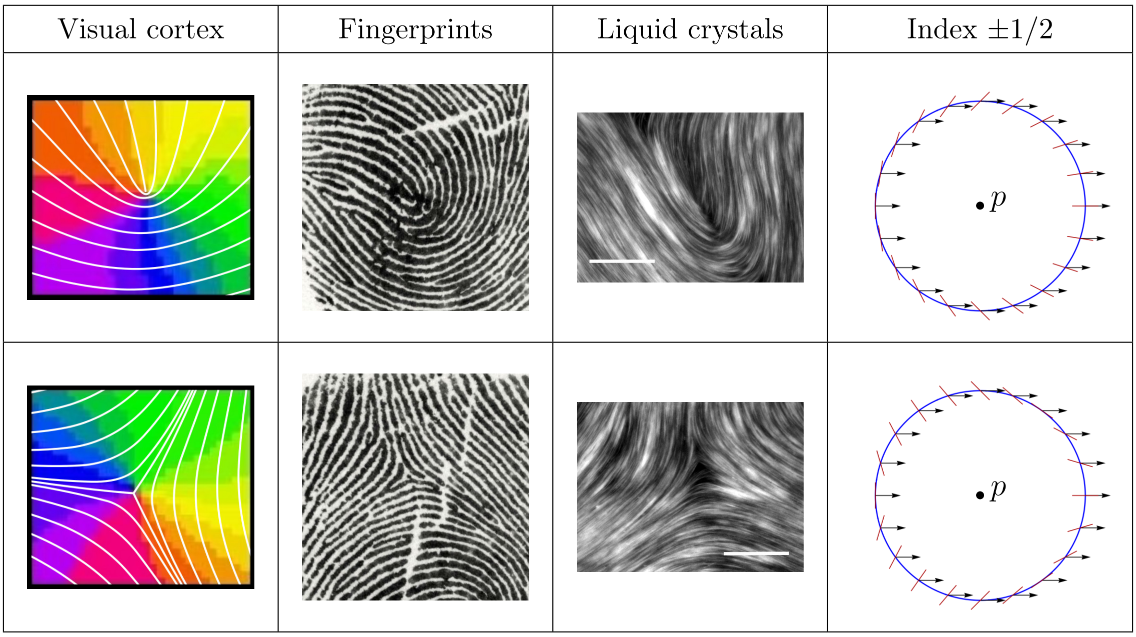
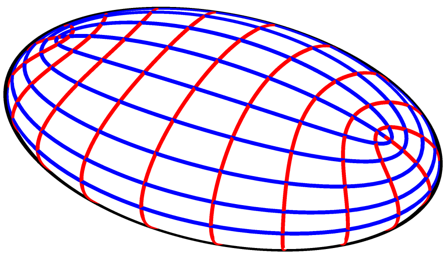
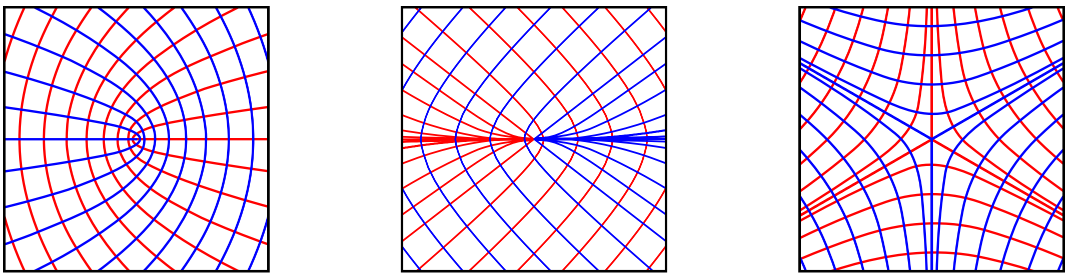


Figure 2.12 – Singularities of line fields found in nature predominantly bear singularities of index $\pm 1/2$.

Figure 2.13 – Integral manifolds of the directions principal curvature on a triaxial ellipsoid. This surface has four umbilics, all corresponding to line field singularities of index $1/2$.





(a) Lemon (b) Monstar (c) Star

Figure 2.14 – Representation of integral manifolds of the directions of principal curvature in the neighborhood of the three stable singularities.

2.4.2 Singularities of bisector line fields

Bisection of vector fields. As stated in the previous section, we would like to have tools better suited to the study of stability of singularities of line fields. The definition of a section of the orientation bundle requires, unlike vector fields, to remove the singular set of a line field from its domain. As a consequence the natural topology of line fields does not allow to perturb the position of the singularities of a field. We introduce a construction inspired by vector fields that allows to more easily define and study line fields, their singularities and their topology.

We consider in the following a smooth differential manifold M of dimension 2, endowed with a Riemannian structure g. Notice that only the conformal structure provided by g is necessary here, since it allows to measure angles. A local orientation is also necessary to consider angles modulo 2π, but it is not necessary to assume that M is orientable.

Definition 2.4.2. Let (M, g) be a 2D Riemannian manifold. A *proto line field* is a pair of vector fields (X, Y) on M.

If z_X and z_Y denote the zeros of X and Y. The *line field associated with* (X, Y), denoted $B(X, Y)$, is the section of $PT(M \backslash (z_X \cup z_Y))$ defined at $p \in M \backslash (z_X \cup z_Y)$ as the line $B(X(p), Y(p))$ of T_pM that bisects $(X(p), Y(p))$ for the metric $g(p)$.

Hence singularities of X and Y coincide with singularities of $B(X, Y)$. We also endowed the set of line fields obtained by bisection with a natural topology, the C^k Whitney topology on pairs of vector fields. We can justify the merit of this definition with the following observations.

Proposition 2.4.3. *Let (M, g) be a 2D Riemannian manifold K be a closed subset of M and L be a smooth section of $PT(M \backslash K)$. There exist smooth vector fields X and Y such that $L = B(X, Y)$.*

Proposition 2.4.4. *Let (X, Y) a proto line field on (M, g). Given an isolated point p of $z_X \cup z_Y$, we have $\mathrm{ind}_p(B(X, Y)) = \frac{1}{2}(\mathrm{ind}_p(X) + \mathrm{ind}_p(Y))$. (see Figure 2.15)*

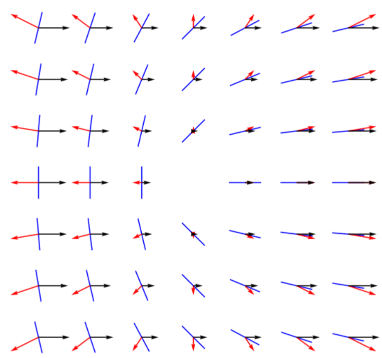


Figure 2.15 – Bisection of a vector field with an index 1 singularity and a vector field without singularity forms a line field with a 1/2 index singularity.

Hence, line fields obtained by bisection show expected behaviors of classical line fields, but are now endowed with a definition and a topology which allows to study the stability of singularities.

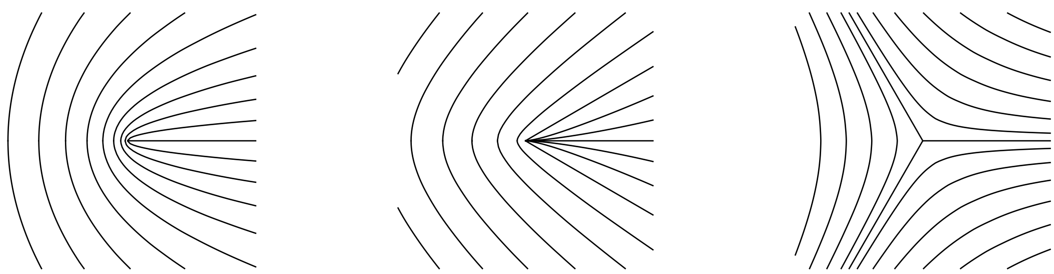
Stability and genericity. From the C^1 Whitney topology on pairs of vector fields, we can immediately obtain a first encouraging result in the classification of stable singularities of line fields.

Proposition 2.4.5. *The bisector line field of a generic pair* (X, Y) *has only isolated singularities of index* $\pm 1/2$.

By analogy with vector fields, we choose to take interest in structural stability of bisector line fields at their singularities. A proto line field is *locally structurally stable at* $p \in M$ if for all neighborhood V_p of p there exist a neighborhood $V_{(X,Y)}$ of (X, Y) for the C^1 Whitney topology such that for all $(X', Y') \in V_{(X,Y)}$, $B(X, Y)$ at p is topologically equivalent to $B(X', Y')$ at $p' \in V_p$ (in the sense of homemorphisms mapping integral manifolds onto integral manifolds). We are compelled to define these topological notions on the pairs (X, Y) because of the non-injectivity of the bisection operation.

Most of the work in the chapter is dedicated to proving the following theorem.

Theorem 2.4.6. *Generically with respect to Whitney topology on the pairs of vector fields, the proto line field* (X, Y) *is locally structurally stable and has only singularities of type Lemon, Monstar and Star.*



(a) Lemon (b) Monstar (c) Star

The methods used to prove this theorem are mostly classical methods from the theory of structural stability of vector fields in dimension 2 (see, for instance, [DLA06]). The approach is similar, we first prove that a linearized proto line field is generically of one of the three types listed in Theorem 2.4.6. Then we prove that in the case of a generic pair, every element in a small neighborhood of must have the same type of linearized. This property allows to construct by hand a homeomorphism which maps integral manifolds of two singularities of the same type onto each others.

Regarding the linear step, assuming that (in coordinates) $X(0) = 0$ and $Y(0) \neq 0$ (it is the generic case), we are interested in the bisector of the proto line field $L(p) = (D_0X \cdot p, Y(0))$ and the function

$$\varphi(\theta) = \angle\left[e_1, B(L)(\cos\theta, \sin\theta)\right]$$

(with $e_1 = (1, 0)$). The function φ measures the angle between the bisector line field of L at $(\cos\theta, \sin\theta)$ and the (arbitrary) horizontal $(1, 0)$. Considering fixed points of φ and their attractiveness, we are able to discriminates between the cases of Theorem 2.4.6.

Regarding the step of construction of homeomorphisms, notice that the bisection at singularities can be compared to the square-root operation $z \mapsto z^{1/2}$ in $\mathbb{C}$. An idea is then to apply the inverse operation by constructing a squaring operation on line fields, allowing to obtain vector fields, whose stability is easier to prove. We couple this operation with a blowup of the singularity, which allows to classically compute the homeomorphisms necessary for the topological equivalence.

Remark 2.4.7 (On the stability of Monstar singularities). One can be surprised by the fact that we discriminate three stable singularities when most applications discriminate only two. We can point out that structural stability may not be the most natural choice in regards of applications since there exists a limit in resolution of observations which limits our ability to differentiate a Monstar singularity from a Lemon singularity.

Notice however that this difficulty can be circumvented by considering the analog of the function φ defined earlier. This function allows to discern the distribution of orientations in the immediate neighborhood of the singularity, but also to notice that there is no essential difference between the two stable singularities of positive index. Their difference is mostly the homogeneity of the distribution of orientations around the singularity (see Figure 6.5 in Chapter 6).

Chapter 3

On the Whitney extension property for continuously differentiable horizontal curves in sub-Riemannian manifolds

In this chapter we study the validity of the Whitney C^1 extension property for horizontal curves in sub-Riemannian manifolds that satisfy a first-order Taylor expansion compatibility condition. We first consider the equiregular case, where we show that the extension property holds true whenever a suitable non-singularity property holds for the endpoint map on the Carnot groups obtained by nilpotent approximation. We then discuss the case of sub-Riemannian manifolds with singular points and we show that all step-2 manifolds satisfy the C^1 extension property. We conclude by showing that the C^1 extension property implies a Lusin-like approximation theorem for horizontal curves on sub-Riemannian manifolds.

Contents

3.1 Introduction

The success of sub-Riemannian geometry in geometric measure theory and nonlinear control is due to its simplicity and flexibility as a modeling tool and to the richness of the phenomena that it exhibits and allows to study. Other branches of mathematics, such as hypoelliptic operator theory and rough path theory use sub-Riemannian geometry as a natural underlying structure [BBS16].

A sub-Riemannian structure on a manifold M is characterized by a distribution $\Delta \subset TM$ endowed with a point-dependent norm which can be used for measuring the length of *horizontal curves*, i.e., absolutely continuous curves which are tangent to Δ. Horizontal curves play a fundamental role in sub-Riemannian geometry, since the *sub-Riemannian distance* is defined as the minimal length of an horizontal curve connecting two points.

A natural metric property that it makes sense to test on a sub-Riemannian structure is the extendability of regular horizontal curves. The Euclidean counterpart of this property is the well-known Whitney extension theorem for a map γ from a closed subset K of $\mathbb{R}$ into $\mathbb{R}^d$, $d \in \mathbb{N}$. In this case, the extendability holds under the sole assumption that the variation of the jets of γ on K is compatible with Taylor's expansions.

We study in this chapter the counterpart of the Whitney extension theorem for C^1 horizontal (C^1_H in the following) curves in sub-Riemannian manifolds. A useful intermediate ground where this problem can be set is provided by Carnot groups. Whitney extension theorems for maps between Carnot groups have been the object of research in the past: in particular the case of real-valued C^1 functions on the Heisenberg group has been considered in [FSSC01] and extended to the general case of real-valued C^m functions on Carnot groups in [VP06]. The problem which we consider here is different, since the domain of the map which we seek to extend is contained in $\mathbb{R}$ and the sub-Riemannian structure is taken on the codomain. The latter formulation of the problem has been proposed for Carnot groups by F. Serra Cassano in [SC16].

The question of extendability of C^1_H curves in Carnot groups has been answered positively in the case of the Heisenberg group by S. Zimmerman in [Zim18]. For general Carnot groups, it has been proved in [JS17] that the C^1_H extension property holds if and only if the group is *pliable*, that is, for every horizontal vector v the endpoint map $\mathcal{C}_0 \ni u \mapsto E(v+u)$ is locally open at 0, where $\mathcal{C}_0$ is the space of continuous functions u

from $[0,1]$ to the horizontal distribution of the Carnot group such that $u(0)=0$ and $E(w)$ is the endpoint at time 1 of the trajectory starting at the identity and tangent to w. This characterization in terms of pliability, together with some tools from geometric control theory, are used in [JS17] to prove that step-2 Carnot groups satisfy the C^1_H extension property. Several examples of non-pliable Carnot groups are also presented.

A closely related subject is the one of Lusin-like approximations in Carnot groups, namely the property that an absolutely continuous horizontal curve coincides, out of a set of arbitrarily small measure, with a C^1_H curve. The validity of such a Lusin approximation theorem in the Heisenberg group has been proved in [Spe16, Zim18] and extended to the case of step-2 Carnot groups in [LDS16]. In [JS17] it is shown that if a Carnot group is pliable then it satisfies the Lusin approximation property.

In this chapter we enlarge the analysis from Carnot groups to general sub-Riemannian manifolds (not necessarily equiregular). The first step of this program is to provide a suitable definition of the C^1_H extension property in sub-Riemannian manifolds. This is done by showing that C^1_H curves admit an intrinsic first-order Taylor expansion with uniform reminder, evaluated with respect to the sub-Riemannian distance. This can be seen as a form of uniform Pansu-differentiability for C^1_H curves (see also the results in [VP06], partially recalled in Section 3.5). The key for investigating this property is the use of nilpotent approximations, which characterize the infinitesimal metric structure at a point of the sub-Riemannian manifold. In the equiregular case, nilpotent approximations have a Carnot group structure and one can take advantage of the metric estimates given by the celebrated Ball-Box theorem [Bel96]. The non-equiregular case can be tackled by desingularization.

The second step of our analysis consists in providing sufficient conditions for the C^1_H extension property in sub-Riemannian manifolds to hold true. In the equiregular case, the conditions are expressed in terms of the pliability properties of the nilpotent approximations. More precisely, we prove that the C^1_H extension property holds if every Carnot group corresponding to a nilpotent approximation of the sub-Riemannian manifold is *strongly pliable*, that is, for every horizontal vector v, not only the endpoint map $\mathcal{C}_0 \ni u \mapsto E(v+u)$ is locally open at 0, but also there exists a sequence of points u_n converging to 0 in $\mathcal{C}_0$ such that $E(v+u_n)=E(v)$ and $u \mapsto E(v+u)$ is a submersion at u_n for every $n \in \mathbb{N}$. Strong pliability introduces a guarantee of structural stability in the inversibility of the endpoint map, allowing to deduce extendability properties on the sub-Riemannian manifold from those of the nilpotent approximations. As a consequence of this condition, we deduce that step-2 sub-Riemannian manifolds satisfy the C^1_H extension property. More generally, second order conditions for the local openness of the endpoint map can be used to express sufficient conditions for strong pliability in terms of the Goh and the generalized Legendre conditions.

Beyond its own metric interest, the C^1_H extension property can be used to characterize rectifiability in sub-Riemannian manifolds: we show that, if the C^1_H extension property holds true, then rectifiability by Lipschitz curves is equivalent to rectifiability by C^1_H curves. This equivalence is based on a generalization of Lusin approximation theorem for sub-Riemannian manifolds that satisfy the C^1_H extension property.

3.1.1 Chapter walkthrough

In Section 3.2 we recall some basic definitions and properties in sub-Riemannian geometry. In particular, we discuss two important tools used in the chapter, namely the nilpotent approximation of a sub-Riemannian structure and the distance estimates given by the Ball-Box theorem. We also recollect some basic facts about the chronological exponential notation and we use the variation of constant formula to prove a useful distance estimate (Lemma 3.2.8).

Section 3.3 is dedicated to the study of the C^1_H-Whitney condition. Taking inspiration from the uniform Pansu-differentiability of C^1_H-curves (Proposition 3.3.1), we propose a first definition of the C^1_H-Whitney condition on equiregular manifolds (Definition 3.3.2) that is independent of the choice of frame (Proposition 3.3.4). In order to extend this result to general sub-Riemannian manifolds, we introduce in Section 3.3.2 forward and backward C^1_H-Whitney conditions. These asymmetric conditions turn out to be equivalent to the original C^1_H-Whitney condition on equiregular manifolds (Proposition 3.3.9). Forward and backward C^1_H-Whitney conditions can be recast in terms of the nilpotent approximations of the sub-Riemannian structure (Propositions 3.3.7 and 3.3.8). This proves useful in Section 3.3.3, where the C^1_H-Whitney condition is defined on singular sub-Riemannian manifolds (Definition 3.3.2) and where it is proved that the C^1_H extension property is inherited by the projection of an equiregular lift (Corollary 3.3.12).

In Section 3.4, we propose a sufficient condition for the C^1_H extension property to hold in terms of strong pliability (Definition 3.4.1). Strong pliability is always satisfied at regular values of the endpoint map and can be investigated through second order conditions at critical points (Section 3.4.1.2). In Section 3.4.2, we use uniform estimates for the nilpotent approximations to prove that strong pliability implies the C^1_H extension property for equiregular sub-Riemannian manifolds (Theorem 3.4.11). As a consequence of this result, together with the previous desingularization analysis, we are able to prove that all step-2 sub-Riemannian manifolds have the C^1_H extension property (Corollary 3.4.13).

As a conclusion, we give in Section 3.5 an application of the C^1_H extension property, proving that it implies the Lusin approximation of horizontal curves (Proposition 3.5.3), which in turns can be used to characterize 1-rectifiability (Corollary 3.5.4).

3.2 Sub-Riemannian Geometry

In this section we introduce some classical notions from the fields of sub-Riemannian geometry and control theory that we use in the following sections. For more details, we refer to the publications cited below, in particular the books [AS04], [Jea14] and [ABB16].

3.2.1 Sub-Riemannian manifolds

We give the definition of a sub-Riemannian manifold as it can be found in [ABB16, BCGS13].

Definition 3.2.1. Let M be a smooth connected manifold. A *sub-Riemannian structure* on M is a pair (U, f) where

i. U is a Euclidean bundle with base M and Euclidean fiber U_q, *i.e.*, for every $q \in M$, U_q is a vector space endowed with a scalar product $(\cdot \mid \cdot)_q$ smooth with respect to q. In particular, the dimension of U_q is constant with respect to $q \in M$.

ii. $f : U \to TM$ is a smooth map that is a morphism of vector bundles, *i.e.*, f is linear on fibers and the diagram

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[dr, "\pi_U"'] & TM \arrow[d, "\pi"] \\ & M \end{tikzcd}$$

is commutative (with $\pi_U : U \to M$ and $\pi : TM \to M$ the canonical projections).

iii. The set of *horizontal vector fields* $\Delta = \{f(\sigma) \mid \sigma : M \to U \text{ smooth section}\}$ is a Lie bracket-generating family of vector fields.

A *sub-Riemannian manifold* is then a triple (M, U, f) where M is a smooth manifold endowed with a sub-Riemannian structure (U, f). The *distribution* of this manifold is the family of subspaces

$$(\Delta_q)_{q \in M} \text{ where } \Delta_q = f(U_q) \subset T_qM,$$

and $\dim \Delta_q$ is called the *rank of the sub-Riemannian structure at* q.

With an abuse of notation we will sometimes denote the sub-Riemannian manifold (M, U, f) by (M, Δ, g), with g a quadratic form on Δ obtained by projection of the Euclidean structure, as explained in the next definition.

Example 3.2.2. Recall that the Grushin plane [Gru70] is a rank-varying sub-Riemannian structure on $\mathbb{R}^2$ having as moving orthonormal frame

$$X_1 = \partial_x, \qquad X_2 = x\partial_y.$$

In terms of Definition 3.2.1, such a sub-Riemannian structure is identified with the triple $(\mathbb{R}^2, U, f)$ where $U \simeq \mathbb{R}^2 \times \mathbb{R}^2$ is the standard 2-dimensional Euclidean fiber over $\mathbb{R}^2$, and

$$\begin{array}{rccc} f: & U & \longrightarrow & T\mathbb{R}^2 \simeq \mathbb{R}^2 \times \mathbb{R}^2 \\ & ((x,y),(u,v)) & \longmapsto & ((x,y),(u,vx)). \end{array}$$

Definition 3.2.3. An absolutely continuous curve $\gamma : I \to M$ is said to be *horizontal* if there exist a control $u : I \to U$ measurable and essentially bounded such that $\gamma = \pi_U(u)$ and $\dot{\gamma}(t) = f(\gamma(t), u(t))$ for almost every $t \in I$. Moreover, if there exists such a control u that is continuous then the curve γ is said to be C^1_H.

For $v \in \Delta_q$,

$$g(v,v) = \inf\{(u \mid u)_q \mid f(q,u) = (q,v), (q,u) \in U\}.$$

We then define the *length of the horizontal curve* γ as

$$l(\gamma) = \int_I g(\dot{\gamma}(t), \dot{\gamma}(t))^{1/2}\, \mathrm{d}t.$$

With this length we are able to define the *Carnot-Caratheodory distance between two points* $p, q \in M$ as

$$\mathrm{d_{SR}}(p,q) = \inf\left\{l(\gamma) \mid \gamma : (a,b) \to M \text{ horizontal}, \gamma(a) = p, \gamma(b) = q\right\}.$$

Definition 3.2.4. Let (M, U, f) be a sub-Riemannian manifold and Ω be an open subset of M. We call *frame of the distribution on* Ω a family of horizontal vector fields $(X_1, \ldots, X_m)$ such that there exists a smooth Euclidean frame $(e_1, \ldots, e_m)$ of the Euclidean bundle on $\pi_U^{-1}(\Omega)$ that satisfies

$$X_j = f(e_j), \quad 1 \le j \le m.$$

As a direct consequence of this definition, we have that for any two frames $(X_1, \ldots, X_m)$ and $(Y_1, \ldots, Y_m)$ on Ω, there exists a smooth map c from Ω to the orthogonal group $\mathrm{O}(m)$ such that

$$X_i(q) = \sum_{j=1}^m c_{ij}(q) Y_j(q), \quad 1 \le i \le m,\ q \in \Omega.$$

Definition 3.2.5. Let $(\widetilde{M}, \widetilde{U}, \widetilde{f})$ and (M, U, f) be two sub-Riemannian manifolds. We say that $(\widetilde{M}, \widetilde{U}, \widetilde{f})$ is a *lift* of (M, U, f) if there exists $\phi : \widetilde{U} \to U$ a fiberwise isometry and $\psi : \widetilde{M} \to M$ a submersion such that the diagram

$$\begin{tikzcd} \widetilde{U} \arrow[r, "\widetilde{f}"] \arrow[d, "\phi"'] & T\widetilde{M} \arrow[d, "\psi_*"] \\ U \arrow[r, "f"] & TM \end{tikzcd}$$

is commutative.

A consequence of the isometry condition is that for $\widetilde{\mathrm{d}}_{\mathrm{SR}}$ and d_{SR} the respective sub-Riemannian distances of $(\widetilde{M}, \widetilde{U}, \widetilde{f})$ and (M, U, f), we have

$$\mathrm{d}_{\mathrm{SR}}(\psi(p), \psi(q)) \leq \widetilde{\mathrm{d}}_{\mathrm{SR}}(p, q), \qquad \forall p, q \in \widetilde{M}. \tag{3.1}$$

Moreover, any frame $(X_1, \dots, X_m)$ of (M, U, f) admits a lift $(\widetilde{X}_1, \dots, \widetilde{X}_m)$ (*i.e.*, $\psi_* \widetilde{X}_i = X_i$ for all $1 \leq i \leq m$) that is a frame of $(\widetilde{M}, \widetilde{U}, \widetilde{f})$.

Example 3.2.6. Consider the standard Heisenberg group structure on $\mathbb{R}^3$, endowed with canonical coordinates (x, y, z) and frame (X_1, X_2) such that

$$X_1 = \partial_x - \frac{y}{2}\partial_z \qquad X_2 = \partial_y + \frac{x}{2}\partial_z.$$

Its expression in terms of sub-Riemannian manifold is the following. We set $\widetilde{U} \simeq \mathbb{R}^3 \times \mathbb{R}^2$ to be the standard 2-dimensional Euclidean fiber over $\mathbb{R}^3$, and $T\mathbb{R}^3 \simeq \mathbb{R}^3 \times \mathbb{R}^3$. By setting

$$\begin{array}{cccc} \widetilde{f}: & \widetilde{U} & \longrightarrow & T\mathbb{R}^3 \\ & ((x,y,z),(u,v)) & \longmapsto & ((x,y,z),(u,v,(vx-uy)/2)) \end{array}$$

we have that the diagram

$$\begin{tikzcd} \mathbb{R}^3 \times \mathbb{R}^2 \arrow[r, "\widetilde{f}"] \arrow[dr, "\widetilde{\pi}_U"'] & T\mathbb{R}^3 \arrow[d, "\widetilde{\pi}"] \\ & \mathbb{R}^3 \end{tikzcd}$$

is commutative. The submersion

$$\begin{array}{cccc} \psi: & \mathbb{R}^3 & \longrightarrow & \mathbb{R}^2 \\ & (x,y,z) & \longmapsto & (x, z + xy/2), \end{array}$$

induces the differential map

$$\begin{array}{cccc} \psi_*: & T\mathbb{R}^3 & \longrightarrow & T\mathbb{R}^2 \\ & ((x,y,z),(u,v,w)) & \longmapsto & ((x, z+xy/2),(u, uy/2 + vx/2 + w)). \end{array}$$

As a consequence,

$$\psi_* \circ \widetilde{f}((x,y,z),(u,v)) = ((x, z+xy/2),(u, vx)).$$

Denoting by ϕ the fiberwise isometry

$$\begin{array}{cccc} \phi: & \widetilde{U} & \longrightarrow & U \simeq \mathbb{R}^2 \times \mathbb{R}^2 \\ & ((x,y,z),(u,v)) & \longmapsto & ((x, z+xy/2),(u,v)) \end{array}$$

and by f the smooth bundle morphism of the Grushin plane

$$\begin{array}{cccc} f: & U & \longrightarrow & T\mathbb{R}^2 \\ & ((x,y),(u,v)) & \longmapsto & ((x,y),(u,vx)) \end{array}$$

(see Example 3.2.2), one easily checks that $\psi_* \circ \widetilde{f} = f \circ \phi$. Hence the Heisenberg group $(\mathbb{R}^3, \widetilde{U}, \widetilde{f})$ is a lift of the Grushin plane $(\mathbb{R}^2, U, f)$.

3.2.2 Nilpotent approximation

Set $\Delta^1 = \Delta$ and $\Delta^{k+1} = \Delta^k + [\Delta^k, \Delta]$ for every integer $k \geq 1$. At any point $p \in M$, the Lie bracket generating condition ensures that there exists an integer $r \geq 1$, that we call *step of the sub-Riemannian structure at* p, such that

$$\Delta_p^1 \subseteq \Delta_p^2 \subseteq \dots \Delta_p^{r-1} \subsetneq \Delta_p^r = T_pM. \tag{3.2}$$

The finite sequence of integers $(\dim \Delta_p^1, \dots, \dim \Delta_p^r) = (n_1, \dots, n_r)$ is called *growth vector at* p. If the growth vector is constant on a neighborhood of p, p is said to be *regular*, and *singular* otherwise. The manifold itself is said to be *equiregular* if each of its points is regular and we say that it is singular if it contains singular points.

We call *desingularization of* (M, Δ, g) a lift $(\widetilde{M}, \widetilde{\Delta}, \widetilde{g})$ of (M, Δ, g) that is equiregular. As shown in [Jea14, Lemma 2.5], at any given point $p \in M$, there exists a desingularization on some open neighborhood of p that has the same step as Δ at p.

The relation between the flag (3.2) and the distance $\mathrm{d_{SR}}$ is characterized by the so-called *weights at* p, that is the sequence of integers $w = (w_1, \dots, w_d)$ such that $w_j = s$ if $n_s < j \leq n_{s+1}$ (with $n_0 = 0$). Written in full, that is

$$w = (\underbrace{1, \dots, 1}_{n_1 \text{times}}, \underbrace{2, \dots, 2}_{(n_2 - n_1)\text{times}}, \dots, \underbrace{r, \dots, r}_{(n_r - n_{r-1})\text{times}}).$$

A (smooth) system of coordinates $(x_1, \dots, x_d) : \Omega \to \mathbb{R}^d$ is said to be a *system of privileged coordinates at* p if Ω is a neighborhood of p and

$$\sup \{s \in \mathbb{R} \mid x_j(q) = O(\mathrm{d_{SR}}(p, q)^s)\} = w_j, \quad 1 \leq j \leq d.$$

This definition implies that privileged coordinates belong to the class of linearly adapted coordinates at p, *i.e.*, coordinates $(x_1, \dots, x_d)$ that satisfy

$$\mathrm{d}x_i(\Delta_p^{w_i}) \neq 0, \mathrm{d}x_i(\Delta_p^{w_i - 1}) = 0, \quad 1 \leq i \leq d,$$

with $\Delta_p^0 = \{0\}$. Existence of privileged coordinates has been proved in [AS87, BS90, Bel96].

A *continuously varying system of privileged coordinates* on an open $\Omega \subset M$ is a continuous map

$$\Phi : (p, q) \longmapsto \Phi_p(q) \in \mathbb{R}^d$$

defined on a neighborhood of the set $\{(p, p) \mid p \in \Omega\}$ in $M \times M$ such that for each $p \in \Omega$, the mapping Φ_p is a system of privileged coordinates at p.

The system of coordinates Φ can be used to define a pseudo-norm and a dilation, as follows: the *pseudo-norm at* $p \in \Omega$ is the map $\| \cdot \|_p : \mathbb{R}^d \to \mathbb{R}$ defined by

$$\|(x_1, \dots, x_d)\|_p = \sum_{i=1}^{d} |x_i|^{1/w_i}, \tag{3.3}$$

with $(w_i)_{1\le i\le d}$ the weights at p. Let $\lambda > 0$ and $\mathfrak{d}_\lambda : \mathbb{R}^d \to \mathbb{R}^d$ be defined by

$$(\mathfrak{d}_\lambda(y))_i = \lambda^{w_i} y_i, \quad 1 \le i \le d.$$

For $p \in \Omega$, let δ^p_λ be the *quasi-homogeneous dilation centered at* p,

$$\begin{array}{rccc} \delta^p_\lambda : & \Omega & \longrightarrow & \mathbb{R}^d \\ & q & \longmapsto & \mathfrak{d}_\lambda \circ \Phi_p(q). \end{array}$$

Then by construction,

$$\|\delta^p_\lambda(q)\|_p = \lambda \|\Phi_p(q)\|_p.$$

For every horizontal vector field X and every $p \in M$, we call *nilpotent approximation of* X *at* p the uniform limit $\widehat{X}$ on compact sets of $\mathbb{R}^d$ of the vector field $\frac{1}{\lambda}\delta^p_{\lambda *}X$ as $\lambda \to +\infty$ (see for instance [ABB16, Proposition 10.48]). Given a frame $(X_1, \dots, X_m)$ of the distribution, the vector bundle $(\widehat{X}_1, \dots, \widehat{X}_m)$ endows $\mathbb{R}^d$ with a structure of homogeneous space that depends on the point p but neither on the frame nor the system of privileged coordinates. This object is referred to as *nilpotent approximation of* (M, Δ, g) *at* p, and is denoted by $(\mathbb{R}^d, (\widehat{X}_1, \dots, \widehat{X}_m))$ when referring to a specific choice of frame. In the equiregular case, the nilpotent approximation is not only a homogeneous space but actually has a Carnot group structure ([Bel96]).

3.2.3 Uniform distance estimates

Privileged coordinates and nilpotent approximations play a fundamental role in distance estimates which compare the pseudo norm (3.3) with the sub-Riemannian distance. The result below will be applied repeatedly in the rest of the chapter.

Theorem 3.2.7 ([Jea14, Theorem 2.3])**.** *Let $\bar{p} \in M$ be a regular point. There exist an open neighborhood Ω of $\bar{p}$, a continuously varying system of privileged coordinates Φ on Ω, and two positive constants ε, C, such that for every pair $(p, q) \in \Omega \times \Omega$ with $\mathrm{d}_{\mathrm{SR}}(p, q) \le \varepsilon$,*

$$\frac{1}{C}\|\Phi_p(q)\|_p \le \mathrm{d}_{\mathrm{SR}}(p, q) \le C\|\Phi_p(q)\|_p.$$

An application of this theorem is the following technical lemma (Lemma 3.2.8) that will be useful in later results.

In order to prove the lemma, we introduce a useful notation for the flow of time-dependent vector fields, the so-called *chronological exponential* [AG78, AS04]. Let $\mathbb{R} \times M \ni (t, q) \mapsto X_t(q)$ be a complete time-dependent vector field, measurable and locally bounded with respect to t and smooth with respect to q. For $a, b \in \mathbb{R}$, $a \le b$, we denote by

$$\overrightarrow{\exp} \int_a^b X_t \, \mathrm{d}t : M \longrightarrow M$$

the map from M onto itself such that the curve $\gamma : [a,b] \to M$ defined by $\gamma(t) = \overrightarrow{\exp} \int_a^t X_\tau \, \mathrm{d}\tau(q_0)$ is absolutely continuous, satisfies $\gamma(a) = q_0$ and $\dot\gamma(t) = X_t(\gamma(t))$ for almost every t. For $a \geq b$ we set $\overrightarrow{\exp} \int_a^b X_\tau \, \mathrm{d}\tau = \left(\overrightarrow{\exp} \int_b^a X_\tau \, \mathrm{d}\tau\right)^{-1}$. Let us recall the variation of constant formula. (For a reference, see Equation (2.28) in [AS04]; notice that here we use the standard notational rule for the composition of maps, which explains the difference between the two expressions.) If X_τ, Y_τ are two time-dependent vector fields then

$$\overrightarrow{\exp} \int_0^t (X_\tau + Y_\tau) \, \mathrm{d}\tau = \overrightarrow{\exp} \int_0^t \left(\overrightarrow{\exp} \int_\tau^t X_\sigma \, \mathrm{d}\sigma\right)_* Y_\tau \, \mathrm{d}\tau \circ \overrightarrow{\exp} \int_0^t X_\tau \, \mathrm{d}\tau, \tag{3.4}$$

where $P_* X$ is used to denote the pushforward of the vector field X along the diffeomorphism P. (In order to justify the writing in (3.4), all the vector fields should be complete. In the following, variations formula are used for local reasonnings around a point, so that completeness can be guaranteed by multiplying all vector fields by a suitable cut-off function.) In particular if X is a time-independent vector field, Equation (3.4) takes the form

$$\overrightarrow{\exp} \int_0^t (X + Y_\tau) \, \mathrm{d}\tau = \overrightarrow{\exp} \int_0^t \mathrm{e}^{(\tau - t)\mathrm{ad}X} Y_\tau \, \mathrm{d}\tau \circ \mathrm{e}^{tX}, \tag{3.5}$$

where for each smooth vector field V on M we denote by $\mathrm{ad}V$ the endomorphism of the space of smooth vector fields on M defined by

$$\mathrm{ad}V(W) = [V, W].$$

Then $\mathrm{e}^{\sigma \mathrm{ad}X} Y_\tau$ admits the series expansion

$$\mathrm{e}^{\sigma \mathrm{ad}X} Y_\tau = Y_\tau + \sum_{k=1}^{N-1} \frac{\sigma^k}{k!} [X, \dots, [X, Y_\tau] \cdots] + R_N(\tau, \sigma), \quad N \in \mathbb{N}, \tag{3.6}$$

and there exists $C > 0$ such that for all compact K contained in a given coordinate neighborhood of M, all integer $j \geq 0$,

$$\|R_N(\tau, \sigma)\|_{j,K} \leq \frac{C}{N!} \mathrm{e}^{C\sigma \|X\|_{j+1,K}} \sigma^N \|X\|_{j+N,K}^N \|Y_\tau\|_{j+N,K}, \tag{3.7}$$

where $\| \cdot \|_{j,K}$ denotes the semi-norm on the space of smooth vector fields

$$\|f\|_{j,K} = \sup \left\{ \left| \frac{\partial^\alpha f}{\partial x^\alpha}(x) \right| \mid x \in K, \alpha \in \mathbb{N}^d, |\alpha| \leq j \right\}.$$

(See [AS04, Equation 2.24].) As a consequence, if $\tau \mapsto \|Y_\tau\|_{j+N,K}$ is bounded in $L^\infty([0,\tau])$, then for σ near 0,

$$\|R_N(\tau, \sigma)\|_{j,K} = O(\sigma^N).$$

In the following, for a given frame $(X_1, \dots, X_m)$ and a given $u \in \mathbb{R}^m$, we denote by X_u the horizontal vector field $\sum_{i=1}^m u_i X_i$.

Lemma 3.2.8. *Let (M, Δ, g) be an equiregular sub-Riemannian manifold of dimension d, rank m and step r. Let $(X_1, \dots, X_m)$ be a frame of Δ defined on an open subset Ω of M. Pick $\bar{p} \in \Omega$, $\bar{u} \in \mathbb{R}^m$, and $\phi : [0, +\infty) \to [0, +\infty)$, continuous at 0 and such that $\phi(0) = 0$.*

Then there exist $T > 0$, $V_{\bar{p}} \subset \Omega$, $V_{\bar{u}} \subset \mathbb{R}^m$ neighborhoods of $\bar{p}$, $\bar{u}$ respectively, a function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ with $\omega(t) = o(t)$ at 0^+, such that if $t \in [0, T]$, $p \in V_{\bar{p}}$, $u, v : [0, T] \to V_{\bar{u}}$ continuous at 0 and

$$|u - u(0)| \leq \phi, \quad |v - v(0)| \leq \phi, \quad u(0) = v(0),$$

then

$$\mathrm{d_{SR}} \left(\overrightarrow{\exp} \int_0^t X_{u(s)} \, \mathrm{d}s(p), \overrightarrow{\exp} \int_0^t X_{v(s)} \, \mathrm{d}s(p) \right) \leq \omega(t).$$

Proof. Without loss of generality we can assume u to be constant and the general result follows by triangular inequality.

We apply Theorem 3.2.7 to endow a compact neighborhood $\Omega' \subset \Omega$ of $\bar{p}$ with a continuously varying system of privileged coordinates Φ.

We fix $T > 0$, $V_{\bar{p}} \subset \Omega'$ and $V_{\bar{u}} \subset \mathbb{R}^m$ neighborhoods of $\bar{p}$ and $\bar{u}$, respectively, such that $\overrightarrow{\exp} \int_0^t X_{v(s)} \, \mathrm{d}s(p)$ is in Ω' for every $t \in [0, T]$, $p \in V_{\bar{p}}$, $v : [0, T] \to V_{\bar{u}}$ continuous at 0.

Let $p \in V_{\bar{p}}$, $u \in V_{\bar{u}}$ and $v : [0, T] \to V_{\bar{u}}$, continuous at 0, be such that $|u - v| \leq \phi$. For all $t \in [0, T]$ let

$$\gamma(t) = \mathrm{e}^{tX_u}(p) \quad \text{and} \quad \xi(t) = \overrightarrow{\exp} \int_0^t X_{v(s)} \, \mathrm{d}s(p).$$

Step 1: rewriting ξ as a perturbation of γ.
Let us set $X = X_u$ and $Z_t = X_{v(t)} - X_u$, so that

$$\dot{\xi}(t) = X(\xi(t)) + Z_t(\xi(t)).$$

By the variation of constants formula,

$$\xi(t) = \overrightarrow{\exp} \int_0^t (X + Z_s) \, \mathrm{d}s(p) = \overrightarrow{\exp} \int_0^t \mathrm{e}^{(s-t)\mathrm{ad}X} Z_s \, \mathrm{d}s \left(\mathrm{e}^{tX}(p) \right).$$

Let $W^t_s = \mathrm{e}^{(s-t)\mathrm{ad}X} Z_s$ and denote its integral curve by

$$\eta_t(\tau) = \overrightarrow{\exp} \int_0^\tau W^t_s \, \mathrm{d}s(\gamma(t)), \qquad \forall \tau \in (0, t).$$

Hence the problem consists in proving that the distance $\mathrm{d_{SR}}(\xi(t), \eta_t(0))$ is a $o(t)$.

Let us first establish a broader bound on $\mathrm{d_{SR}}(\xi(t), \eta_t(\tau))$. For every $\tau \in (0, t)$, by applying the variation of constants formula at $\xi(\tau)$ we get

$$\begin{aligned} \xi(t) &= \overrightarrow{\exp} \int_\tau^t (X + Z_s) \, \mathrm{d}s(\xi(\tau)) = \overrightarrow{\exp} \int_\tau^t \mathrm{e}^{(s-t)\mathrm{ad}X} Z_s \, \mathrm{d}s \left(\mathrm{e}^{(t-\tau)X}(\xi(\tau)) \right) \\ &= \overrightarrow{\exp} \int_\tau^t W^t_s \, \mathrm{d}s \left(\mathrm{e}^{(t-\tau)X}(\xi(\tau)) \right). \end{aligned}$$

On the other hand,

$$\xi(t) = \eta_t(t) = \overrightarrow{\exp} \int_\tau^t W_s^t \, \mathrm{d}s \, (\eta_t(\tau)) \,,$$

and therefore $\eta_t(\tau) = \mathrm{e}^{(t-\tau)X}(\xi(\tau))$ for all $\tau \in [0,t]$. In particular there exists $C > 0$ such that

$$\mathrm{d}_{\mathrm{SR}}(\xi(t), \eta_t(\tau)) \leq C(t-\tau). \tag{3.8}$$

(See Figure 3.1.)

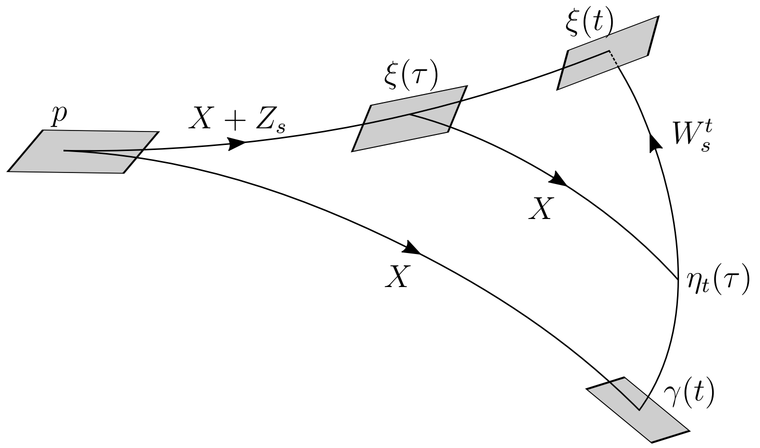


Figure 3.1 – Representation of the curve η_t, where $\eta_t(\tau)$ can be seen both as the evaluation at time τ of an integral curve of the non-horizontal vector field W_s^t and as the endpoint of the concatenation of an integral curve of $X + Z_s$ over $[0,\tau]$ and an integral curve of X of duration $t - \tau$.

Step 2: bounding the pseudo-norm centered at $\xi(t)$ of $\gamma(t)$.

By possibly reducing T, we can assume that $\eta_t(\tau) \in \Omega'$ for every $0 \leq \tau \leq t \leq T$. We then use the privileged coordinates $\Phi : \Omega \times \Omega \to \mathbb{R}^d$ at $\xi(t)$ to compute the pseudo-norm $\|\Phi_{\xi(t)}(\eta_t(\tau))\|_{\xi(t)}$.

Denote by $(n_1, \dots, n_r)$ the growth vector of the sub-Riemannian structure, by $(w_1, \dots, w_d)$ the corresponding weights, and by $x = (x_1, \dots, x_d)$ the coordinates $\Phi_{\xi(t)}$. We want to evaluate for all $1 \leq i \leq d$ the absolute value of

$$x_i(\gamma(t)) = -\int_0^t \left(W_s^t(\eta_t(s))\right)_i \mathrm{d}s.$$

Taking $N = r$ in the expansion (3.6) we have

$$W_s^t = Z_s + \sum_{k=1}^{r-1} \frac{(s-t)^k}{k!} \, [X, \dots, [X, Z_s] \cdots] + R_r(s, s-t). \tag{3.9}$$

The remainder R_r can be bounded using (3.7). For each coordinate $(R_r(s,s-t))_i$, $1 \leq i \leq d$, we have that

$$\|(R_r(s,s-t))_i\|_{0,\Omega'} \leq \frac{C}{r!} \mathrm{e}^{C(s-t)\|X\|_{1,\Omega'}} (s-t)^r \|X\|^r_{r,\Omega'} \|Z_s\|_{r,\Omega'}.$$

Since $|v-u| \leq \phi$, the compactness of Ω' and $V_{\bar{u}}$ implies the existence of C uniform such that

$$|(R_r(s,s-t))_i| \leq C(s-t)^r \phi(s).$$

We now use non-holonomic order arguments (see [Jea14, Section 2.1]) to bound the other terms in the expansion in (3.9).

Let $V_s^k = (\mathrm{ad} X)^{k-1} Z_s$ for $k \geq 1$. The vector fields X and Z_s being horizontal, $V_s^k \in \Delta^k$. As a consequence, the vector V_s^k has a non-holonomic order greater than or equal to $-k$ at any $p \in \Omega'$. Coordinate-wise, at $\xi(t)$, $\mathrm{ord}_{\xi(t)}(\partial_{x_i}) = -w_i$, so that $(V_s^k)_i$ then has non-holonomic order

$$\mathrm{ord}_{\xi(t)}((V_s^k)_i) \geq \max(w_i - k, 0).$$

Since V_s^k depends linearly on $v-u$, there exists $C_k > 0$ such that for $q \in \Omega'$ sufficiently close to $\xi(t)$,

$$\left|(V_s^k)_i(q)\right| \leq C_k \phi(s) \, \mathrm{d}_{\mathrm{SR}}(\xi(t), q)^{\max(w_i - k, 0)}.$$

Using estimate (3.8),

$$\left|(V_s^k)_i(\eta_t(s))\right| \leq C_k \phi(s) t^{\max(w_i - k, 0)}.$$

Thus

$$\int_0^t |(Z_s)_i(\eta_t(s))| \, \mathrm{d}s \leq C_1 t^{\max(w_i - 1, 0)} \int_0^t \phi(s) \, \mathrm{d}s = t^{w_i} \psi^1(t),$$

and

$$\int_0^t \left|\left((t-s)^{k-1} V_s^k\right)_i (\eta_t(s))\right| \mathrm{d}s \leq C_k t^{\max(w_i - k, 0)} \int_0^t (t-s)^{k-1} \phi(s) \, \mathrm{d}s = t^{w_i} \psi^k(t),$$

where for all positive integer k we denote by ψ^k the positive bounded function $\psi^k : [0,T] \ni t \mapsto C_k t^{-k} \int_0^t (t-s)^{k-1} \phi(s) \, \mathrm{d}s$, which is continuous and such that $\psi^k(0) = 0$ (by continuity at 0 of ϕ). Thus, for all $1 \leq i \leq d$,

$$|x_i(\gamma(t))| \leq t^{w_i} \Psi_i(t),$$

with $\Psi_i : [0,T] \to \mathbb{R}^+$ a bounded function continuous at 0 such that $\Psi_i(0) = 0$. Hence we have the uniform bound

$$\|\Phi_{\xi(t)}(\gamma(t))\|_{\xi(t)} \leq \sum_{i=1}^d t (\Psi_i(t))^{1/w_i} = \psi(t) \, t, \tag{3.10}$$

with $\psi : [0,T] \to \mathbb{R}^+$ a function continuous at 0 such that $\psi(0) = 0$.

Step 3: Uniform estimates.

Let $\varepsilon, C > 0$ be the constants associated with the neighborhood Ω in Theorem 3.2.7. Take $T > 0$ such that if $0 \leq t \leq T$,

$$\mathrm{d_{SR}}\left(\gamma(t), \xi(t)\right) \leq \varepsilon.$$

Therefore, by Theorem 3.2.7,

$$\mathrm{d_{SR}}\left(\gamma(t), \xi(t)\right) \leq C\|\Phi_{\xi(t)}(\gamma(t))\|_{\xi(t)},$$

and, plugging in (3.10), we get

$$\mathrm{d_{SR}}\left(\gamma(t), \xi(t)\right) \leq C\psi(t)\, t.$$

Thus, letting $\omega(t) = C\psi(t)t$, we have the uniform bound

$$\mathrm{d_{SR}}\left(\mathrm{e}^{tX_u}(p), \overrightarrow{\exp}\int_0^t X_{v(s)}\,\mathrm{d}s(p)\right) \leq \omega(t).$$

□

3.3 C^1_H-Whitney condition on sub-Riemannian manifolds

We begin this section by proposing a definition of C^1_H-Whitney condition for curves in sub-Riemannian structures that requires the choice of a frame of the sub-Riemannian structure. We then show that such a definition is intrinsic and we explore a few consequences.

Recall that for a given frame $(X_1, \ldots, X_m)$ and a given $u \in \mathbb{R}^m$, we denote by X_u the horizontal vector field $\sum_{i=1}^m u_i X_i$.

3.3.1 Whitney frame-wise condition

We denote by (M, Δ, g) a sub-Riemannian manifold of dimension d. Let $(X_1, \ldots, X_m)$ be a frame of Δ defined on an open subset Ω of M.

We say that (f, L) satisfies property $\mathcal{P}_X$ on K if the following is true.

$\mathcal{P}_X$: K is a compact in $\mathbb{R}$, $f : K \to \Omega$ is continuous, and $L : K \to TM$ is continuous such that $L(t) \in \Delta_{f(t)}$ for all $t \in K$. Moreover there exist $u : K \to \mathbb{R}^m$ continuous such that $X_{u(s)}(f(s)) = L(s)$ and $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\omega(t) = o(t)$ at 0^+ and for all $t, s \in K$

$$\mathrm{d_{SR}}\left(f(t), \mathrm{e}^{(t-s)X_{u(s)}} f(s)\right) \leq \omega(|t-s|).$$

This definition is motivated by the following proposition.

Proposition 3.3.1. *Let (M, Δ, g) be a sub-Riemannian manifold and $(X_1, \ldots, X_m)$ be a frame of Δ defined on an open subset Ω of M. Let $f : \mathbb{R} \to \Omega$ be a C^1_H curve. Then $(f|_K, \dot{f}|_K)$ satisfies $\mathcal{P}_X$ on K for any compact subset K of $\mathbb{R}$.*

Proof. Let $u \in C(\mathbb{R}, \mathbb{R}^m)$ be such that $X_{u(\tau)}(f(\tau)) = \dot{f}(\tau)$ for every $\tau \in \mathbb{R}$. Then

$$f(t) = \overrightarrow{\exp} \int_s^t X_{u(\tau)}\, \mathrm{d}\tau(f(s))$$

for every $s, t \in \mathbb{R}$.

Let K be a compact subset of $\mathbb{R}$ and K' a connected compact set of $\mathbb{R}$ containing K. For every $s \in [0, +\infty)$, set $\phi(s) = \sup_{t \in K'} |u(t+s) - u(t)|$. Notice that $\lim_{s \to 0^+} \phi(s) = 0$ by uniform continuity of u on compact subsets of $\mathbb{R}$.

Let us first consider the equiregular case. Let $s \in K$. We set $v = u(s) \in \mathbb{R}^m$ and we apply Lemma 3.2.8 at $f(s)$ with ϕ as above. We get that there exist a neighborhood $V_s \subset \mathbb{R}$ of s and a function $\omega_s : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\omega_s(t) = o(t)$ at 0^+ and

$$\mathrm{d}_{\mathrm{SR}}\left(f(t), \mathrm{e}^{(t-t')X_{u(t')}} f(t')\right) \leq \omega_s(|t - t'|) \tag{3.11}$$

for all $t, t' \in V_s \cap K$. Taking a finite cover $V_{s_1}, \ldots, V_{s_N}$ of K we have that $\omega(t) = \max_{1 \leq i \leq N} \omega_{s_i}(t) = o(t)$ at 0^+ and we deduce that $(f|_K, \dot{f}|_K)$ satisfies $\mathcal{P}_X$ on K.

Assume now that the manifold is singular. For every $s \in K$ there exists a neighborhood Ω_s of $f(s)$ contained in Ω and a desingularization $\psi : \widetilde{\Omega}_s \to \Omega_s$ such that $\widetilde{\Omega}_s$ is an open set in an equiregular sub-Riemannian manifold $\widetilde{M}$. Let $(\widetilde{X}_1, \ldots, \widetilde{X}_m)$ be the lifted frame of $(X_1, \ldots, X_m)$ on $\widetilde{\Omega}_s$.

Then we fix $\widetilde{f}(s) \in \psi^{-1}(f(s))$ and we set for all t in a neighborhood of s

$$\widetilde{f}(t) = \overrightarrow{\exp} \int_s^t \widetilde{X}_{u(\tau)}\, \mathrm{d}\tau(\widetilde{f}(s)).$$

By construction, $\psi(\widetilde{f}) = f$. We apply the equiregular reasoning on $\widetilde{f}$ at s, and we get the existence of $\omega_s : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\omega_s(t) = o(t)$ at 0^+ and

$$\widetilde{\mathrm{d}}_{\mathrm{SR}}\left(\widetilde{f}(t), \mathrm{e}^{(t-t')\widetilde{X}_{u(t')}} \widetilde{f}(t')\right) \leq \omega_s(|t - t'|)$$

for all t and t' close enough to s. By projecting this inequality (see Inequality (3.1)), we get

$$\mathrm{d}_{\mathrm{SR}}\left(f(t), \mathrm{e}^{(t-t')X_{u(t')}} f(t')\right) \leq \omega_s(|t - t'|).$$

We conclude with the same compactness argument as in the equiregular case. $\square$

We then define the C^1_H-Whitney condition and the C^1_H-extension property as follows.

Definition 3.3.2 (C^1_H-Whitney condition)**.** Let K be a closed subset of $\mathbb{R}$, $f : K \to M$ continuous, and $L : K \to TM$ continuous. We say that the *$\mathcal{C}^1_H$-Whitney condition holds for (f, L) on K* if for every $t \in K$, there exist a compact neighborhood K' of t in K, an open set $\Omega \subset M$ and a local frame X of Δ on Ω such that $(f|_{K'}, L|_{K'})$ satisfies $\mathcal{P}_X$ on K'.

Definition 3.3.3 (C^1_H-extension property)**.** We say that a sub-Riemannian manifold M has the C^1_H *extension property* if for all closed subset K of $\mathbb{R}$, all pair $(f, L) : K \to M \times TM$ continuous satisfying the C^1_H-Whitney condition, there exists a C^1_H-curve $\gamma : \mathbb{R} \to M$ such that

$$\gamma_{|K} = f, \quad \dot{\gamma}_{|K} = L.$$

In the case where (M, Δ, g) is equiregular, we are able to show that the property of satisfying $\mathcal{P}_X$ is intrinsic to the curve and does not depend on the choice of the frame.

Proposition 3.3.4. *Assume (M, Δ, g) to be an equiregular sub-Riemannian manifold. Let $(X_1, \dots, X_m)$ and $(Y_1, \dots, Y_m)$ be two frames of Δ defined on an open subset Ω of M. Let $K \subset \mathbb{R}$ be compact and $(f, L) : K \to \Omega \times TM$ be continuous. Then (f, L) satisfies $\mathcal{P}_X$ on K if and only if it satisfies $\mathcal{P}_Y$ on K.*

Proof. Let us assume that (f, L) satisfies $\mathcal{P}_X$ on K. In particular, there exists $\omega_X : \mathbb{R}^+ \to \mathbb{R}^+$, such that $\omega_X(t) = o(t)$ at 0^+ and, for all $t, s \in K$,

$$d_{SR}\left(f(s), e^{(s-t)X_u} f(t)\right) \leq \omega_X(|s - t|),$$

with $u_1, \dots, u_m$ defined by the relation $L(t) = \sum_{i=1}^m u_i X_i(f(t))$. Let us prove that (f, L) satisfies $\mathcal{P}_Y$.

Let us define $u, v : K \to \mathbb{R}^m$ continuous such that $X_{u(t)}(f(t)) = Y_{v(t)}(f(t)) = L(t)$ for every $t \in K$. Since $(X_1, \dots, X_m)$ and $(Y_1, \dots, Y_m)$ are both frames of Δ, there exist smooth functions $(c_{ij})_{1 \leq i,j \leq m}$ such that for all $q \in \Omega$,

$$Y_j(q) = \sum_{i=1}^m c_{ij}(q) X_i(q).$$

Then

$$\sum_{j=1}^m v_j(t) Y_j(q) = \sum_{i=1}^m \left(\sum_{j=1}^m v_j(t) c_{ij}(q) \right) X_i(q), \quad t \in K, \ q \in M.$$

As a consequence of Lemma 3.2.8, for all $t \in K$ there exist $T_t > 0$, $V_{f(t)} \subset \Omega$, $V_{u(t)} \subset \mathbb{R}^m$ neighborhoods of $f(t)$, $u(t)$ respectively, there exists $\omega_t : \mathbb{R}^+ \to \mathbb{R}^+$ with $\omega_t(s) = o(s)$ at 0^+, such that if $p \in V_{f(t)}$, $u \in V_{u(t)}$ and $v \in \mathbb{R}^m$ satisfy $X_u(p) = Y_v(p)$ then

$$d_{SR}\left(e^{sX_u} p, e^{sY_v} p\right) \leq \omega_t(s), \quad s \in [0, T_t].$$

By compactness of $f(K)$, there exists a finite cover $V_{f(t_1)}, \dots, V_{f(t_N)}$ of $f(K)$. Then $\omega(t) = \max_{1\le i\le N} \omega_{t_i}(s) = o(s)$ at 0^+ and we deduce that

$$\mathrm{d_{SR}}\left(\mathrm{e}^{(s-t)X_{u(t)}} f(t), \mathrm{e}^{(s-t)Y_{v(t)}} f(t)\right) \le \omega(|s-t|)$$

for all $s, t \in K$ close enough. Then, for $s, t \in K$ close enough,

$$\begin{aligned}\mathrm{d_{SR}}\left(f(s), \mathrm{e}^{(s-t)Y_{v(t)}} f(t)\right) &\le \mathrm{d_{SR}}\left(f(s), \mathrm{e}^{(s-t)X_{u(t)}} f(t)\right) + \mathrm{d_{SR}}\left(\mathrm{e}^{(s-t)X_{u(t)}} f(t), \mathrm{e}^{(s-t)Y_{v(t)}} f(t)\right)\\ &\le \omega_X(|s-t|) + \omega(|s-t|).\end{aligned}$$

Thus (f, L) satisfies $\mathcal{P}_Y$ on K. □

3.3.2 Forward and backward Whitney condition

To expand the notion of $\mathcal{C}^1_H$-Whitney condition to singular sub-Riemannian manifolds, we first have to break the symmetry in the definition of $\mathcal{P}_X$ by comparing only flows going forward or backward in time (that is, by requiring either $s < t$ or $s > t$ in the statement of $\mathcal{P}_X$). This new definition has two virtues. First, the asymmetric definition turns out to be equivalent to the symmetric one and it is easier to lift on a desingularized manifold. Second, the asymmetric definition lends itself well to the use of dilations, which will be useful in Section 3.4.

Consider an equiregular sub-Riemannian manifold (M, Δ, g) of dimension d, rank m and step r. Let $(X_1, \dots, X_m)$ be a frame of Δ defined on an open subset Ω of M.

We say that (f, L) satisfies the property $\mathcal{P}_X$-forward, denoted by $\mathcal{P}^F_X$, or $\mathcal{P}_X$-backward, denoted by $\mathcal{P}^B_X$, on K if the following is true.

$\mathcal{P}^F_X$: K is a compact subset of $\mathbb{R}$, $f : K \to \Omega$ is continuous, and $L : K \to TM$ is continuous such that $L(t) \in \Delta_{f(t)}$ for all $t \in K$. Moreover there exists $\omega : \mathbb{R}^+ \to \mathbb{R}^+$, such that $\omega(t) = o(t)$ at 0^+ and

$$\mathrm{d_{SR}}\left(f(t), \mathrm{e}^{(t-s)X_{u(s)}} f(s)\right) \le \omega(t-s) \qquad \forall t > s \in K,$$

with $u(s) \in \mathbb{R}^m$ pointwise defined by the relation $X_{u(s)}(f(s)) = L(s)$.

$\mathcal{P}^B_X$: K is a compact subset of $\mathbb{R}$, $f : K \to \Omega$ is continuous, and $L : K \to TM$ is continuous such that $L(t) \in \Delta_{f(t)}$ for all $t \in K$. Moreover there exists $\omega : \mathbb{R}^+ \to \mathbb{R}^+$, such that $\omega(t) = o(t)$ at 0^+ and

$$\mathrm{d_{SR}}\left(f(t), \mathrm{e}^{(t-s)X_{u(s)}} f(s)\right) \le \omega(s-t) \qquad \forall t < s \in K,$$

with $u(s) \in \mathbb{R}^m$ pointwise defined by the relation $X_{u(s)}(f(s)) = L(s)$.

Again, we emphasize that the difference between $\mathcal{P}^B_X$ and $\mathcal{P}^F_X$ is in the requirement that either $t > s$ (for $\mathcal{P}^F_X$) or $s < t$ (for $\mathcal{P}^B_X$). Furthermore, one can observe that satisfying $\mathcal{P}_X$ is strictly equivalent to satisfying both $\mathcal{P}^B_X$ and $\mathcal{P}^F_X$. In analogy with Definition 3.3.2, we introduce the following notion.

Definition 3.3.5 (Backward and forward C^1_H-Whitney condition)**.** Let K be a closed subset of $\mathbb{R}$, $f : K \to M$ continuous, and $L : K \to TM$ continuous be such that $L(t) \in \Delta_{f(t)}$ for all $t \in K$. We say that the *backward (respectively, forward) $\mathcal{C}^1_H$-Whitney condition holds for* (f, L) *on* K if for every $t \in K$ there exist a compact neighborhood K' of t in K, an open set $\Omega \subset M$ and a local frame X of Δ on Ω such that $(f|_{K'}, L|_{K'})$ satisfies $\mathcal{P}^B_X$ (respectively, $\mathcal{P}^F_X$).

The reasoning in Section 3.3.1 still holds when we consider $\mathcal{P}_X$-backward and $\mathcal{P}_X$-forward. Hence the following result.

Proposition 3.3.6. *Assume (M, Δ, g) to be an equiregular sub-Riemannian manifold. Let $(X_1, \dots, X_m)$ and $(Y_1, \dots, Y_m)$ be two frames of Δ defined on an open subset Ω of M. Let $K \subset \mathbb{R}$ be compact and $(f, L) : K \to \Omega \times TM$ be continuous. Then (f, L) satisfies $\mathcal{P}^F_X$ (respectively, $\mathcal{P}^B_X$) on K if and only if it satisfies $\mathcal{P}^F_Y$ (respectively, $\mathcal{P}^B_Y$) on K.*

Proposition 3.3.7 below reformulates the forward C^1_H-Whitney condition using dilations in privileged coordinates.

Proposition 3.3.7. *Let (M, Δ, g) be an equiregular sub-Riemannian manifold. Let $K \subset \mathbb{R}$ be compact set, Ω be an open subset of M and $(f, L) : K \to \Omega \times TM$ be continuous. Assume that there exists a continuously varying system of privileged coordinates $\Phi : \Omega \times \Omega \to \mathbb{R}^d$ as in Theorem 3.2.7. Then the pair (f, L) satisfies $\mathcal{P}^F_X$ if and only if for all $l \in K$, for all sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ in K such that $a_n < b_n$ and $a_n, b_n \to l \in K$, we have*

$$\lim_{n\to\infty} \delta^{f(b_n)}_{\frac{1}{b_n - a_n}}(f(a_n)) = \mathrm{e}^{-\widehat{X}_{u(l)}}(0).$$

Proof. Let $l \in K$, $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ in K be such that $a_n < b_n$ and $a_n, b_n \to l \in K$. By assumption there exists two positive constants ε, C, such that for every pair $(p, q) \in \Omega \times \Omega$ with $\mathrm{d_{SR}}(p, q) \le \varepsilon$, it holds

$$\frac{1}{C}\|\Phi_p(q)\|_p \le \mathrm{d_{SR}}(p, q) \le C\|\Phi_p(q)\|_p.$$

Then, for n large enough,

$$\begin{aligned}\frac{1}{C}\left\|\Phi_{f(b_n)}\left(\mathrm{e}^{(b_n-a_n)X_{u(a_n)}} f(a_n)\right)\right\|_{f(b_n)} \le \\ \mathrm{d_{SR}}\left(f(b_n), \mathrm{e}^{(b_n-a_n)X_{u(a_n)}} f(a_n)\right) \\ \le C\left\|\Phi_{f(b_n)}\left(\mathrm{e}^{(b_n-a_n)X_{u(a_n)}} f(a_n)\right)\right\|_{f(b_n)}. \qquad (3.12)\end{aligned}$$

By introducing a dilation in the pseudo norm, we get

$$\begin{aligned}\left\|\Phi_{f(b_n)}\left(\mathrm{e}^{(b_n-a_n)X_{u(a_n)}} f(a_n)\right)\right\|_{f(b_n)} &= (b_n - a_n)\left\|\mathfrak{d}_{\frac{1}{b_n-a_n}} \circ \Phi_{f(b_n)}\left(\mathrm{e}^{(b_n-a_n)X_{u(a_n)}} f(a_n)\right)\right\|_{f(b_n)} \\ &= (b_n - a_n)\left\|\delta^{f(b_n)}_{\frac{1}{b_n-a_n}}\left(\mathrm{e}^{(b_n-a_n)X_{u(a_n)}} f(a_n)\right)\right\|_{f(b_n)}.\end{aligned}$$

Denoting $t_n = b_n - a_n$, we get

$$\delta^{f(b_n)}_{1/t_n}\left(\mathrm{e}^{t_n X_{u(a_n)}}(f(a_n))\right) = \mathrm{e}^{t_n \delta^{f(b_n)}_{1/t_n}{}_* X_{u(a_n)}}\left(\delta^{f(b_n)}_{1/t_n}(f(a_n))\right).$$

Hence

$$\delta^{f(b_n)}_{1/t_n}(f(a_n)) = \mathrm{e}^{-t_n \delta^{f(b_n)}_{1/t_n}{}_* X_{u(a_n)}}\left(\delta^{f(b_n)}_{1/t_n}\left(\mathrm{e}^{t_n X_{u(a_n)}}(f(a_n))\right)\right). \tag{3.13}$$

Since u and f are continuous on K and $a_n, b_n \to l \in K$, $t_n \delta^{f(b_n)}_{1/t_n}{}_* X_{u(a_n)}$ locally uniformly converges towards $\widehat{X}_{u(l)}$. This is a consequence of the local uniform convergence of $\lambda\delta_{1/\lambda *}X_u$ towards $\widehat{X}_u$ as $\lambda \to 0$ (see [ABB16, Proposition 10.48]). Thus $\mathrm{e}^{-t_n \delta^{f(b_n)}_{1/t_n}{}_* X_{u(a_n)}}$ locally uniformly converges towards $e^{-\widehat{X}_{u(l)}}$.

If (f, L) satisfies $\mathcal{P}^F_X$ then Equation (3.12) implies that

$$\lim_{n\to\infty} \left\| \delta^{f(b_n)}_{1/t_n}\left(\mathrm{e}^{t_n X_{u(a_n)}} f(a_n)\right)\right\|_{f(b_n)} = 0.$$

It follows from (3.13) and the local uniform convergence of $\mathrm{e}^{-t_n \delta^{f(b_n)}_{1/t_n}{}_* X_{u(a_n)}}$ towards $e^{-\widehat{X}_{u(l)}}$ that

$$\lim_{n\to\infty} \delta^{f(b_n)}_{1/t_n}(f(a_n)) = \mathrm{e}^{-\widehat{X}_{u(l)}}(0).$$

Conversely, assume now that for all $l \in K$, for all sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ in K such that $a_n < b_n$ and $a_n, b_n \to l \in K$, we have

$$\lim_{n\to\infty} \delta^{f(b_n)}_{\frac{1}{b_n - a_n}}(f(a_n)) = \mathrm{e}^{-\widehat{X}_{u(l)}}(0).$$

To prove that (f, L) satisfies $\mathcal{P}^F_X$, we prove that

$$\lim_{t\to 0} \sup_{\substack{a,b\in K\\ 0<b-a<t}} \frac{1}{b-a} \mathrm{d}_{\mathrm{SR}}\left(f(b), \mathrm{e}^{(b-a)X_{u(a)}} f(a)\right) = 0.$$

Thanks to estimate (3.12), we are left to prove that

$$\lim_{t\to 0} \sup_{\substack{a,b\in K\\ 0<b-a<t}} \left\| \mathrm{e}^{(b-a)\delta^{f(b)}_{1/(b-a)*} X_{u(a)}}\left(\delta^{f(b)}_{1/(b-a)}(f(a))\right)\right\|_{f(b)} = 0.$$

Assume that there exist $\eta > 0$, $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ in K such that, for all $n \in \mathbb{N}$, $0 < b_n - a_n < 1/n$ and

$$\left\| \mathrm{e}^{(b_n-a_n)\delta^{f(b_n)}_{1/(b_n-a_n)*} X_{u(a_n)}}\left(\delta^{f(b_n)}_{1/(b_n-a_n)}(f(a_n))\right)\right\|_{f(b_n)} > \eta.$$

Up to extraction, the sequences $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ converge to some $l \in K$, so that

$$\delta^{f(b_n)}_{\frac{1}{b_n-a_n}}(f(a_n)) \to \mathrm{e}^{-\widehat{X}_{u(l)}}(0).$$

Moreover

$$\mathrm{e}^{(b_n-a_n)\delta^{f(b_n)}_{1/(b_n-a_n)_*}X_{u(a)}} \to \mathrm{e}^{\widehat{X}_{u(l)}}$$

locally uniformly on $\mathbb{R}^d$. Thus

$$\mathrm{e}^{(b_n-a_n)\delta^{f(b_n)}_{1/(b_n-a_n)_*}X_{u(a_n)}}\left(\delta^{f(b_n)}_{1/(b_n-a_n)}(f(a_n))\right) \longrightarrow 0$$

in $\mathbb{R}^d$, concluding the contradiction argument. □

With an analogous proof we obtain the similar backward result.

Proposition 3.3.8. *Let (M,Δ,g) be an equiregular sub-Riemannian manifold. Let $K\subset\mathbb{R}$ be compact and $(f,L): K\to\Omega\times TM$ be continuous. Assume that there exists a continuously varying system of privileged coordinates $\Phi:\Omega\times\Omega\to\mathbb{R}^d$ as in Theorem 3.2.7. Then the pair (f,L) satisfies $\mathcal{P}^B_X$ on a compact $K\subset\mathbb{R}$ if and only if for all $l\in K$, for all sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ in K such that $a_n<b_n$ and $a_n,b_n\to l\in K$, we have that*

$$\lim_{n\to\infty}\delta^{f(a_n)}_{\frac{1}{b_n-a_n}}(f(b_n)) = \mathrm{e}^{\widehat{X}_{u(l)}}(0).$$

We are now ready to prove that forward and backward C^1_H Whitney conditions are equivalent.

Proposition 3.3.9. *Assume (M,Δ,g) to be an equiregular sub-Riemannian manifold. Let $(X_1,\dots,X_m)$ be a frame of Δ defined on an open subset Ω of M. Let $K\subset\mathbb{R}$ be compact and $(f,L):K\to\Omega\times TM$ be continuous. The pair (f,L) satisfies $\mathcal{P}^F_X$ on K if and only if (f,L) satisfies $\mathcal{P}^B_X$ on K. Both are equivalent to (f,L) satisfying $\mathcal{P}_X$ on K.*

Proof. By symmetry of the definitions, we only prove that $\mathcal{P}^F_X \Rightarrow \mathcal{P}^B_X$ using Proposition 3.3.8. The converse would use Proposition 3.3.7.

By applying Theorem 3.2.7 at $f(l)$, we select a compact neighborhood Ω of $f(l)$, a continuously varying system of privileged coordinates Φ on Ω and two constants ε and C such that for all $(p,q)\in\Omega\times\Omega$ with $\mathrm{d}_{\mathrm{SR}}(p,q)\le\varepsilon$,

$$\frac{1}{C}\left\|\Phi_p(q)\right\|_p \le \mathrm{d}_{\mathrm{SR}}(p,q)\le C\left\|\Phi_p(q)\right\|_p.$$

Let $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ be two sequences in K such that $a_n<b_n$ and $\lim a_n=\lim b_n = l\in K$. Let us denote by $t_n=b_n-a_n$.

By assumption $\frac{1}{t_n}\mathrm{d}_{\mathrm{SR}}\left(\mathrm{e}^{t_nX_{u(a_n)}}f(a_n),f(b_n)\right)\to 0$. Applying Proposition 3.3.8, let us prove that

$$\delta^{f(a_n)}_{1/t_n}(f(b_n)) \xrightarrow[n\to+\infty]{} \mathrm{e}^{\widehat{X}_{u(l)}}(0).$$

Since $\mathrm{d}_{\mathrm{SR}}\left(\mathrm{e}^{t_nX_{u(a_n)}}f(a_n),f(b_n)\right)\to 0$, there exists sequence $(v_n)_{n\in\mathbb{N}}$ of controls, $v_n:[0,1]\to\mathbb{R}^m$, such that

$$|v_n|\le 2\mathrm{d}_{\mathrm{SR}}\left(f(b_n),\mathrm{e}^{t_nX_{u(a_n)}}f(a_n)\right)$$

almost everywhere on $[0,1]$ and

$$f(b_n) = \overrightarrow{\exp} \int_0^1 X_{v_n(s)}\, \mathrm{d}s \left(\mathrm{e}^{t_n X_{u(a_n)}} f(a_n)\right).$$

Then

$$\delta_{1/t_n}^{f(a_n)}(f(b_n)) = \overrightarrow{\exp} \int_0^1 {\delta_{1/t_n}^{f(a_n)}}_* X_{v_n(s)}\, \mathrm{d}s \left(\delta_{1/t_n}^{f(a_n)} \left(\mathrm{e}^{t_n X_{u(a_n)}} f(a_n)\right)\right). \tag{3.14}$$

Now, notice that

$${\delta_{1/t_n}^{f(a_n)}}_* X_{v_n(s)} = \sum_{i=1}^m \frac{v_n^i(s)}{t_n} \left(t_n {\delta_{1/t_n}^{f(a_n)}}_* X_i\right)$$

and that $\left(t_n {\delta_{1/t_n}^{f(a_n)}}_* X_i\right)_{n\in\mathbb{N}}$ locally uniformly converges towards $\widehat{X}_i$ on $\mathbb{R}^d$, while the L^∞-norm of $\frac{v_n}{t_n}$ is upper bounded by

$$\frac{2}{t_n} \mathrm{d}_{\mathrm{SR}}\left(f(b_n), \mathrm{e}^{t_n X_{u(a_n)}} f(a_n)\right),$$

which converges toward 0 almost everywhere on $[0,1]$.

Hence, by continuity of the endpoint map with respect to the control, $\overrightarrow{\exp} \int_0^1 {\delta_{1/t_n}^{f(a_n)}}_* X_{v_n(s)}\, \mathrm{d}s$ locally uniformly converges towards the identity. On the other hand,

$$\delta_{1/t_n}^{f(a_n)} \left(\mathrm{e}^{t_n X_{u(a_n)}} f(a_n)\right) = \mathrm{e}^{t_n {\delta_{1/t_n}^{f(a_n)}}_* X_{u(a_n)}}(0).$$

Again by local uniform convergence of $t_n {\delta_{1/t_n}^{f(a_n)}}_* X_i$ towards $\widehat{X}_i$ for all $1 \le i \le m$ and by the convergence of $(u(a_n))_{n\in\mathbb{N}}$ towards $u(l)$,

$$\delta_{1/t_n}^{f(a_n)} \left(\mathrm{e}^{t_n X_{u(a_n)}} f(a_n)\right) \to \mathrm{e}^{\widehat{X}_{u(l)}}(0),$$

so that (3.14) implies

$$\delta_{1/t_n}^{f(a_n)}(f(b_n)) \xrightarrow[n\to+\infty]{} \mathrm{e}^{\widehat{X}_{u(l)}}(0).$$

□

3.3.3 Whitney condition on singular sub-Riemannian manifolds

The aim of this section is to extend what we know about the C^1_H-Whitney condition to the case of sub-Riemannian manifolds with singular points. This extension stands on the following result.

Proposition 3.3.10. *Let $(\widetilde{M}, \widetilde{\Delta}, \widetilde{g})$ be an equiregular lift of (M, Δ, g) on the open set Ω. Let $(X_1, \ldots, X_m)$ be a frame of (M, Δ, g) on Ω and $(\widetilde{X}_1, \ldots, \widetilde{X}_m)$ be a frame of $(\widetilde{M}, \widetilde{\Delta}, \widetilde{g})$ on $\widetilde{\Omega}$, the lift of $(X_1, \ldots, X_m)$.*

Let $K \subset \mathbb{R}$ be compact and $(f, L) : K \to \Omega \times TM$ be continuous. If (f, L) satisfies $\mathcal{P}_X$ on K, then there exists a continuous lift $(\widetilde{f}, \widetilde{L}) : K \to \widetilde{\Omega} \times T\widetilde{M}$ of (f, L) that satisfies $\mathcal{P}_{\widetilde{X}}$ on K.

Proof. Let $(f, L) : K \to \Omega \times TM$ be continuous and satisfying $\mathcal{P}_X$ on K. The proof works by constructing $(\widetilde{f}, \widetilde{L})$ as the lift of a suitable absolutely continuous extension of f. More precisely we will select $u : \mathbb{R} \to \mathbb{R}^m$ such that the non-autonomous vector field X_u steers $f(t_0)$ to $f(t)$ for every $t_0, t \in K$ and we will integrate $\widetilde{X}_u$ to obtain $\widetilde{f}$. Moreover, u can be chosen in such a way that the Whitney condition for $(\widetilde{f}, \widetilde{L})$ is deduced from the one of (f, L) and the equivalences obtained in Propositions 3.3.8 and 3.3.9.

Let $u : \mathbb{R} \to \mathbb{R}^m$ be defined as follows. We set $u|_K$ to be continuous and such that $X_{u(t)} = L(t)$ for all $t \in K$. For every $(a, b) \subset K^c$ such that $a, b \in K$, let $d(a, b) = \mathrm{d}_{\mathrm{SR}}\left(\mathrm{e}^{(b-a)X_{u(a)}} f(a), f(b)\right)$ and let

$$M = 2 \sup \left\{ \frac{d(a,b)}{b-a} \mid (a,b) \subset K^c, a, b \in K \right\} < \infty.$$

Let $(a, b) \subset K^c$, $a, b \in K$. Since $b - \frac{d(a,b)}{M} > a$, we can define u on $\mathbb{R}$ in the following way:

— if $t \in \left(a, b - \frac{d(a,b)}{M}\right)$, we set

$$u(t) = \frac{b-a}{b-a-d(a,b)/M} u(a),$$

— on $\left[b - \frac{d(a,b)}{M}, b\right)$ we take u measurable such that

$$\overrightarrow{\exp} \int_{b-d(a,b)/M}^{b} X_{u(s)} \, \mathrm{d}s \left(\mathrm{e}^{(b-a)X_{u(a)}} f(a)\right) = f(b).$$

By definition of M, we can further assume that

$$|u(t)| \le M \quad \text{for all } t \in \left[b - \frac{d(a,b)}{M}, b\right).$$

On the non-compact components of K^c, we set u to be such that $X_{u(t)} = L(b)$ if the component is of the form $(-\infty, b)$, and $X_{u(t)} = L(a)$ if the component is of the form (a, ∞).

Let us prove that for any $t_0, t \in K$, we have

$$\overrightarrow{\exp} \int_{t_0}^{t} X_{u(s)} \, \mathrm{d}s \, (f(t_0)) = f(t). \tag{3.15}$$

For all $t_0 \in K$, there exists an open neighborhood O of $f(t_0)$, $\Phi : \bar{O} \to \mathbb{R}^d$ a smooth system of coordinates and an open interval $I \subset \mathbb{R}$ such that $t_0 \in I$, $f(K \cap I) \subset O$ and for all $t \in I$, $\overrightarrow{\exp} \int_{t_0}^{t} X_{u(s)} \, \mathrm{d}s \, (f(t_0)) \in O$.

Let

$$f^*(t) = \Phi\left(\overrightarrow{\exp} \int_{t_0}^{t} X_{u(s)} \, \mathrm{d}s \, (f(t_0))\right).$$

In order to prove (3.15), let us show that $\Phi(f(t)) = f^*(t)$ for all $t \in K \cap I$.

We denote by $\|\cdot\|$ the Euclidean norm with respect to the coordinates Φ. By continuity of Φ, there exists $C > 0$ such that

$$\|X_i\| \leq C, \qquad \forall q \in O, \forall 1 \leq i \leq m$$

and

$$\left\|f(t) - \mathrm{e}^{(t-s)X_{u(s)}} f(s)\right\| \leq C \mathrm{d}_{\mathrm{SR}}\left(f(t), \mathrm{e}^{(t-s)X_{u(s)}} f(s)\right) \leq C\omega(|t-s|), \qquad \forall t, s \in K \cap I.$$

Hence for all $t, s \in I \cap K$,

$$\begin{aligned} \|f(t) - f(s)\| &\leq \left\|f(t) - \mathrm{e}^{(t-s)X_{u(s)}} f(s)\right\| + \left\|\mathrm{e}^{(t-s)X_{u(s)}} f(s) - f(s)\right\| \\ &\leq C\left(\omega(|t-s|) + \|u\|_\infty |t-s|\right) \\ &\leq C'|t-s|, \end{aligned}$$

that is, $\Phi(f)$ is C'-Lipschitz continuous on $K \cap I$. Then $\Phi(f)$ admits a C'-Lipschitz continuous extension $\hat{f}$ on I.

To show (3.15), we then show that f^* and $\hat{f}$ coincide on $K \cap I$. Both are absolutely continuous and satisfy $f^*(t_0) = \hat{f}(t_0)$. Furthermore, the derivatives of f^* and $\hat{f}$ are almost everywhere equal on K, and for all $(a, b) \subset K^c$, $a, b \in K$,

$$\int_a^b f^{*\prime}(s)\,\mathrm{d}s = \Phi(f(b)) - \Phi(f(a)) = \hat{f}(b) - \hat{f}(a) = \int_a^b \hat{f}'(s)\,\mathrm{d}s \tag{3.16}$$

since, by construction of u,

$$\overrightarrow{\exp} \int_a^b X_{u(s)}\,\mathrm{d}s\,(f(a)) = \overrightarrow{\exp} \int_{b-d(a,b)/M}^b X_{u(s)}\,\mathrm{d}s\left(\mathrm{e}^{(b-a)X_{u(a)}}\,(f(a))\right) = f(b).$$

Hence for all $t \in K \cap I$,

$$f^*(t) = \int_{t_0}^t f^{*\prime}(s)\,\mathrm{d}s = \int_{(t_0,t)\cap K} f^{*\prime}(s)\,\mathrm{d}s + \int_{(t_0,t)\cap K^c} f^{*\prime}(s)\,\mathrm{d}s = \int_{t_0}^t \hat{f}'(s)\,\mathrm{d}s = \hat{f}(t),$$

where we used (3.16) and the fact that $\int_{(t_0,t)\cap K^c} f^{*\prime}(s)\,\mathrm{d}s$ is the (possibly infinite) sum of all integrals $\int_a^b f^{*\prime}(s)\,\mathrm{d}s$ such that $(a, b) \subset K^c$, $a, b \in K$.

Let $t_0 \in K$ and let $\widetilde{f}(t_0)$ be such that $\psi\left(\widetilde{f}(t_0)\right) = f(t_0)$, where ψ is as in Definition 3.2.5. We define the curve

$$\gamma : \mathbb{R} \to \widetilde{M}$$

such that for all $t \in \mathbb{R}$,

$$\gamma(t) = \overrightarrow{\exp} \int_{t_0}^t \widetilde{X}_{u(s)}\,\mathrm{d}s\left(\widetilde{f}(t_0)\right).$$

Then for all $t \in K$, we set $\widetilde{f}(t) = \gamma(t)$ and $\widetilde{L}(t) = \dot{\gamma}(t) = \widetilde{X}_{u(t)}(\gamma(t))$.

We claim that $(\widetilde{f}, \widetilde{L})$ is a lift of (f, L). By construction of u and γ,

$$\psi(\gamma(t)) = \overrightarrow{\exp} \int_{t_0}^{t} \psi_* \widetilde{X}_{u(s)}\, \mathrm{d}s \left(\psi(\widetilde{f}(t_0))\right) = \overrightarrow{\exp} \int_{t_0}^{t} X_{u(s)}\, \mathrm{d}s \left(f(t_0)\right) = f(t)$$

for all $t \in K$ and, since $\dot{\gamma}(t) = \widetilde{X}_{u(t)}$, we have $\psi_* \widetilde{L}(t) = X_{u(t)} = L(t)$.

Let us now prove that such a lift satisfies the Whitney condition. To alleviate notations in the following, we set $g = \widetilde{f}$ and $Y_i = \widetilde{X}_i$, $1 \le i \le m$. Moreover, up to restricting $\widetilde{\Omega}$, we assume that there exists a continuously varying system of privileged coordinates

$$\Phi : (p, q) \longmapsto \Phi_p(q) \in \mathbb{R}^d,$$

on $\widetilde{\Omega}$. As a consequence of Propositions 3.3.8 and 3.3.9, it is enough to show that for all $l \in K$, for all sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ in K such that $a_n < b_n$ and $a_n, b_n \to l \in K$, we have

$$\lim_{n\to\infty} \delta^{g(a_n)}_{\frac{1}{b_n - a_n}}(g(b_n)) = \mathrm{e}^{\widehat{Y}_{u(l)}}(0). \tag{3.17}$$

For any interval (a, b) in $\mathbb{R}$, by construction

$$g(b) = \overrightarrow{\exp} \int_a^b Y_{u(s)}\, \mathrm{d}s(g(a)),$$

thus, by reparametrizing,

$$\delta^{g(a)}_{\frac{1}{b-a}}(g(b)) = \overrightarrow{\exp} \int_a^b \delta^{g(a)}_{\frac{1}{b-a}\,*} Y_{u(s)}\, \mathrm{d}s(0) = \overrightarrow{\exp} \int_0^1 (b-a)\delta^{g(a)}_{\frac{1}{b-a}\,*} Y_{u(a+t(b-a))}\, \mathrm{d}t(0).$$

For any sequences $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ in K such that $a_n < b_n$ and $a_n, b_n \to l \in K$, $(b_n - a_n)\delta^{g(a_n)}_{\frac{1}{b_n-a_n}\,*} Y_i \to \widehat{Y}_i$ locally uniformly on $\mathbb{R}^d$, for all $1 \le i \le m$. Hence to prove (3.17), we now show that for $v_n(t) = u(a_n + t(b_n - a_n))$, $t \in [0, 1]$,

$$v_n \xrightarrow[n\to+\infty]{L^1((0,1),\mathbb{R}^m)} u(l).$$

For all $n \in \mathbb{N}$, let $K_n = (b_n - a_n)^{-1}(K - a_n) \cap [0, 1]$. By uniform continuity of $u|_K$ on compact subsets of K, $(v_n|_{K_n})_{n\in\mathbb{N}}$ uniformly converges to $u(l)$. Regarding $(v_n|_{K_n^c})_{n\in\mathbb{N}}$, as a first step, let us compare u to $u(l)$ on an interval $(a, b) \subset K^c$, $a, b \in K$.

For $t \in (a, b - d(a, b)/M)$, we have set $u(t) = \frac{b-a}{b-a-d(a,b)/M} u(a)$, and for $t \in [b - d(a, b)/M, b)$, we have imposed $|u(t)| \le M$. Then

$$\begin{aligned} \int_a^b |u(s) - u(l)|\, \mathrm{d}s &\le \left(b - a - \frac{d(a,b)}{M}\right) \left|u(a)\frac{b-a}{b-a-d(a,b)/M} - u(l)\right| + \frac{d(a,b)}{M}(M + |u(l)|), \\ &\le (b-a)\,|u(a) - u(l)| + \omega(b-a)\left(1 + \frac{2|u(l)|}{M}\right), \end{aligned} \tag{3.18}$$

where we used $\mathcal{P}_X$ for the inequality $d\,(a, b) \le \omega\,(b - a)$.

Since K is a closed subset of $\mathbb{R}$, K^c is a countable union of open intervals, notably for all $n \in \mathbb{N}$ there exist $I(n) \subset \mathbb{N}$ and two countable (or finite) families of reals $(c_n^k)_{k\in I(n)}$ and $(d_n^k)_{k\in I(n)}$ such that

$$K^c \cap (a_n, b_n) = \bigcup_{k\in I(n)} (c_n^k, d_n^k).$$

Then

$$K_n^c \cap (0,1) = \bigcup_{k\in I(n)} \left(\frac{c_n^k - a_n}{b_n - a_n}, \frac{d_n^k - a_n}{b_n - a_n} \right),$$

and for all $n \in \mathbb{N}$, $k \in I(n)$, (c_n^k, d_n^k) is a connected component of K^c, hence a bound of type (3.18) holds. Thus

$$\begin{aligned}\int_{\frac{c_n^k - a_n}{b_n - a_n}}^{\frac{d_n^k - a_n}{b_n - a_n}} |v_n(t) - u(l)|\,\mathrm{d}t &= \frac{1}{b_n - a_n} \int_{c_n^k}^{d_n^k} |u(s) - u(l)|\,\mathrm{d}s \\ &\leq \frac{(c_n^k - d_n^k)}{b_n - a_n} \left|u(c_n^k) - u(l)\right| + \frac{\omega(d_n^k - c_n^k)}{(b_n - a_n)} \left(1 + \frac{2|u(l)|}{M}\right),\end{aligned}$$

and

$$\begin{aligned}\int_{K_n^c} |v_n(t) - u(l)|\,\mathrm{d}t &= \sum_{k\in I(n)} \int_{\frac{c_n^k - a_n}{b_n - a_n}}^{\frac{d_n^k - a_n}{b_n - a_n}} |v_n(t) - u(l)|\,\mathrm{d}t \\ &\leq \|v_n - u(l)\|_{L^\infty(K_n)} + \left(1 + \frac{2|u(l)|}{M}\right) \sum_{k\in I(n)} \frac{\omega\left(d_n^k - c_n^k\right)}{b_n - a_n}.\end{aligned}$$

As shown previously, $\|v_n - u(l)\|_{L^\infty(K_n)} \to 0$. Regarding $\sum_{k\in I(n)} \frac{\omega(d_n^k - c_n^k)}{b_n - a_n}$, recall that $\omega(t) = t\phi(t)$ with $\phi(t) \to 0$ as $t \to 0^+$. Then

$$\frac{1}{b_n - a_n} \sum_{k\in I(n)} \omega(d_n^k - c_n^k) < \frac{1}{b_n - a_n} \sum_{k\in I(n)} (d_n^k - c_n^k)\phi(d_n^k - c_n^k) < \sup_{[0, b_n - a_n]} \phi.$$

In other terms,

$$\sum_{k\in I(n)} \frac{\omega(d_n^k - c_n^k)}{b_n - a_n} \xrightarrow[n\to+\infty]{} 0,$$

and

$$\int_0^1 |v_n(t) - u(l)|\,\mathrm{d}t = \int_{K_n \cap (0,1)} |v_n(t) - u(l)|\,\mathrm{d}t + \int_{K_n^c \cap (0,1)} |v_n(t) - u(l)|\,\mathrm{d}t \xrightarrow[n\to+\infty]{} 0.$$

□

Corollary 3.3.11. *Let $(X_1, \ldots, X_m)$ and $(Y_1, \ldots, Y_m)$ be two frames of the singular sub-Riemannian structure (M, Δ, g) on the open subset $\Omega \subset M$. Let $K \subset \mathbb{R}$ be compact and $(f, L) : K \to \Omega \times TM$ be continuous. Then, as in the equiregular case, (f, L) satisfies $\mathcal{P}_X$ if and only if (f, L) on K satisfies $\mathcal{P}_Y$ on K.*

Proof. Let $c : \Omega \to \mathrm{O}(m)$ be a smooth map onto the orthogonal group such that

$$Y_i = \sum_{j=1}^{m} c_{ij} X_j, \quad 1 \leq i \leq m.$$

(See Definition 3.2.4.) Without loss of generality, there exists an equiregular lift $(\widetilde{M}, \widetilde{X}_1, \dots, \widetilde{X}_m)$ of the sub-Riemannian structure $(M, X_1, \dots, X_m)$ on Ω. We denote by $\psi : \widetilde{M} \to M$ the associated submersion. Then let us define for all $q \in \widetilde{\Omega}$

$$\widetilde{Y}_i(q) = \sum_{j=1}^{m} c_{ij}(\psi(q)) \widetilde{X}_j(q), \quad 1 \leq i \leq m.$$

Then $(\widetilde{Y}_1, \dots, \widetilde{Y}_m)$ is a smooth frame for the sub-Riemannian manifold $(\widetilde{M}, \widetilde{\Delta}, \widetilde{g})$, and

$$\psi_* \widetilde{Y}_i(\psi(q)) = \sum_{j=1}^{m} c_{ij}(\psi(q)) \psi_* \widetilde{X}_j(\psi(q)) = Y_i, \quad 1 \leq i \leq m.$$

Let (f, L) be a curve in Ω satisfying $\mathcal{P}_X$ on K. By applying Proposition 3.3.10, it can be lifted to a curve $(\widetilde{f}, \widetilde{L})$ satisfying $\mathcal{P}_{\widetilde{X}}$ on K. Since the structure is regular on $\widetilde{\Omega}$, we have that $(\widetilde{f}, \widetilde{L})$ satisfies $\mathcal{P}_{\widetilde{Y}}$ on K by Proposition 3.3.4. Let us conclude by proving that (f, L) must then satisfy $\mathcal{P}_Y$ on K.

As consequence of (3.1), for all $u \in \mathbb{R}^m$,

$$\mathrm{d}_{\mathrm{SR}} \left(\widetilde{f}(t), \mathrm{e}^{(t-s)\widetilde{Y}_u} \widetilde{f}(s) \right) \geq \mathrm{d}_{\mathrm{SR}} \left(\psi \left(\widetilde{f}(t) \right), \psi \left(\mathrm{e}^{(t-s)\widetilde{Y}_u} \widetilde{f}(s) \right) \right) = \mathrm{d}_{\mathrm{SR}} \left(f(t), \mathrm{e}^{(t-s)Y_u} f(s) \right).$$

Thus $\mathcal{P}_X \Rightarrow \mathcal{P}_Y$. □

As a consequence, Definitions 3.3.2 and 3.3.3 for $\mathcal{C}^1_H$-Whitney condition and extension property are independent of the choice of the frame, and we have the following immediate corollary.

Corollary 3.3.12. *Let (M, Δ, g) be a possibly singular sub-Riemannian manifold and let $(\widetilde{M}, \widetilde{\Delta}, \widetilde{g})$ be an equiregular lift of (M, Δ, g). If $(\widetilde{M}, \widetilde{\Delta}, \widetilde{g})$ has the C^1_H extension property, then so does (M, Δ, g).*

3.4 A sufficient condition for the C^1_H extension property

3.4.1 Strong pliability

Definition 3.4.1 (Strong pliability)**.** Let $(q, u) \in M \times \mathbb{R}^m$ and let $\mathbb{G} = (\mathbb{R}^d, (\widehat{X}_1, \dots, \widehat{X}_m))$ be a nilpotent approximation of (M, Δ, g) at q. Define the space $\mathcal{C}_0 = \{v \in C^0([0,1], \mathbb{R}^m) \mid v(0) = 0\}$ and the map

$$\begin{array}{rccc} \mathcal{F}^u : & \mathcal{C}_0 & \longrightarrow & \mathbb{G} \times \mathbb{R}^m \\ & v & \longmapsto & \left(\overrightarrow{\exp} \int_0^1 \widehat{X}_{u+v(s)} \mathrm{d}s(0_{\mathbb{G}}), v(1) \right). \end{array}$$

The pair (q, u) is said to be *strongly pliable* if for all $\eta > 0$ there exists $v \in \mathcal{C}_0$ such that $\|v\|_\infty < \eta$, $\mathcal{F}^u(v) = \mathcal{F}^u(0)$ and $\mathcal{F}^u$ is a submersion at v.

A pair $(q, u) \in M \times \mathbb{R}^m$ is strongly pliable in particular when $\mathcal{F}^u$ is submersion at 0. This definition relates to what has been called pliability of the vector $\widehat{X}_u$ in [JS17], *i.e.*, the property that $\mathcal{F}^u$ is locally open at 0. Naturally, if (q, u) is strongly pliable then $\widehat{X}_u$ is pliable. Recall also that if (q, u) is pliable then the curve $[0, 1] \ni t \mapsto e^{tX_u}$ cannot be rigid in the sense of [BH93].

3.4.1.1 Regular points of the endpoint map

For every $v \in \mathcal{C}_0$, set $F(v) = \overrightarrow{\exp} \int_0^1 \widehat{X}_{u+v(s)}\, \mathrm{d}s(0_{\mathbb{G}})$ and $j(v) = v(1)$ so that $\mathcal{F}^u(v) = (F(v), j(v))$. We have $D_v j = j$ by linearity, so that $D_v\mathcal{F}^u = (D_vF, j)$. With $P_t = \overrightarrow{\exp} \int_0^t \widehat{X}_{u+v(s)}\, \mathrm{d}s$, $0 \leq t \leq 1$, we have $D_vF(w) = \int_0^1 P_{1*}{P_s^{-1}}_* \widehat{X}_{w(s)}\, \mathrm{d}s(0_{\mathbb{G}})$. We use this characterization to evaluate the corank of $\mathcal{F}^u$ at v.

Let $(\lambda, \mu) \in \mathbb{R}^{d+m*}$. If $(\lambda, \mu) \in \mathrm{Im} D_v\mathcal{F}^{u\perp}$ then

$$\lambda \cdot \int_0^1 P_{1*}{P_s^{-1}}_* \widehat{X}_{w(s)}\, \mathrm{d}s(0_{\mathbb{G}}) + \mu \cdot w(1) = 0 \qquad \forall w \in \mathcal{C}_0.$$

Let us rewrite

$$\lambda \cdot \int_0^1 P_{1*}{P_s^{-1}}_* \widehat{X}_{w(s)}\, \mathrm{d}s(0_{\mathbb{G}}) = \sum_{i=1}^m \int_0^1 w_i(s)\psi_i(s)\, \mathrm{d}s = \langle w, \psi \rangle_{L^2((0,1),\mathbb{R}^m)}$$

with $\psi_i(s) = \lambda \cdot P_{1*}{P_s^{-1}}_* \widehat{X}_i$. Then for all $w \in \langle \psi \rangle^\perp$, the codimension 0 or 1 subspace of $\mathcal{C}_0$ orthogonal to ψ, we have that

$$(\lambda, \mu) \cdot D_v\mathcal{F}^u(w) = \mu \cdot w(1) = 0.$$

Having $\mu \cdot w(1) = 0$ for all $w \in \langle \psi \rangle^\perp$ implies that $\mu = 0$, since $\left\{ w(1) \mid w \in \langle \psi \rangle^\perp \right\} = \mathbb{R}^m$. Hence elements of $\mathrm{Im} D_v\mathcal{F}^{u\perp}$ are of the form $(\lambda, 0)$ with $\lambda \in \mathrm{Im} D_vF^\perp$, and regular values of $\mathcal{F}^u$ need only be regular values of F.

To study the regularity points of F, we introduce a more classical endpoint map, that is, the extension of F to $L^\infty([0, 1], \mathbb{R}^m)$:

$$\begin{array}{rccc} G: & L^\infty([0,1],\mathbb{R}^m) & \longrightarrow & \mathbb{G} \\ & v & \longmapsto & \overrightarrow{\exp} \int_0^1 \widehat{X}_{u+v(s)} \mathrm{d}s(0_{\mathbb{G}}). \end{array}$$

Lemma 3.4.2. *The pair (q, u) is strongly pliable if and only if for all $\eta > 0$ there exists $v \in L^\infty([0, 1], \mathbb{R}^m)$ such that $\|v\|_{L^\infty} < \eta$, $G(v) = G(0)$ and G is a submersion at v.*

Proof. If (q, u) is strongly pliable, then for all $\eta > 0$ there exists $v \in \mathcal{C}_0$ such that $\|v\|_\infty < \eta$, $F(v) = F(0)$ and F is a submersion at v. Since G is an extension of F, the same conclusion follows by replacing F by G.

Let now $\eta > 0$ and pick $v \in L^\infty([0,1],\mathbb{R}^m)$ such that $\|v\|_{L^\infty} < \eta$, $G(v) = G(0)$ and G is a submersion at v, *i.e.*, $D_vG : L^\infty([0,1],\mathbb{R}^m) \to \mathbb{R}^d$ is surjective. By the remarks above, we are left to prove that there exists $w \in \mathcal{C}_0$ such that $\|w\|_\infty \leq 2\eta$ and $F(w) = F(0)$ and F is a submersion at w.

We have that $G \in C^1(L^\infty([0,1],\mathbb{R}^m),\mathbb{R}^d)$ where $L^\infty([0,1],\mathbb{R}^m)$ is endowed with the L^2-topology (see for instance [Tré05, Proposition 5.1.2], [Tré00, Section 3]). Moreover, $\mathcal{C}_0$ is dense in $L^\infty([0,1],\mathbb{R}^m)$ for the L^2 topology. The conclusion then follows from Lemma 3.A.1 in the Appendix, taking $V = L^\infty([0,1],\mathbb{R}^m)$ endowed with the L^2-topology, $W = \mathcal{C}_0$, $\mathcal{F} = G$, and $\mathcal{F}_n = F$, $v_n = G(0)$ for every $n \in \mathbb{N}$. □

Corollary 3.4.3. *If* 0 *is a regular value of* G *then the pair* (q,u) *is strongly pliable.*

Remark 3.4.4. An equivalent formulation of Corollary 3.4.3, extending [JS17, Section 6] is that if (q,u) is not strongly pliable, then $[0,1] \ni t \mapsto \mathrm{e}^{t\widehat{X}_u}(0_{\mathbb{G}})$ is an abnormal curve. We recall that a curve

$$[0,T] \ni t \longmapsto \overrightarrow{\exp}\int_0^t \widehat{X}_{v(s)}\,\mathrm{d}s(0_{\mathbb{G}})$$

is *abnormal* if the map

$$\begin{array}{ccc} L^\infty([0,T],\mathbb{R}^m) & \longrightarrow & \mathbb{G} \\ w & \longmapsto & \overrightarrow{\exp}\int_0^T \widehat{X}_{w(s)}\mathrm{d}s(0_{\mathbb{G}}) \end{array}$$

is singular at v.

Remark 3.4.5. For every sub-Riemannian manifold (M,Δ,g), for all $q \in M$, the pair $(q,0)$ is strongly pliable. Indeed, the regularity of 0 for G in the case $u = 0$ is a consequence of Chow's theorem (see for instance [ABB16]). The proof straightforwardly extends to pairs (q,u) such that $X_u(q) = 0$.

3.4.1.2 Second order conditions

When 0 is a singular point for G, we can still give conditions ensuring strong pliability of (q,u) in terms of classical optimality and rigidity conditions (see for instance [AS96]). The result of this discussion is summarized in Figure 3.2.

Let us recall some classical conditions for G to have regular values $v \in L^\infty([0,1],\mathbb{R}^m)$ arbitrarily close to 0 such that $G(v) = G(0)$. Namely, it is sufficient for $[0,1] \ni t \mapsto \mathrm{e}^{tX_u}0$ not to be the projection of a Goh or a weak Legendre singular extremal (see [AS96], [AS04, Section 20.4], [ABB16, Section 12.3]). These conditions summarize as follows.

Proposition 3.4.6. *For* $\lambda \in \mathrm{Im}D_0F^\perp \subset T^*_{\mathrm{e}^{\widehat{X}_u}(0_{\mathbb{G}})}\mathbb{G}$, $\lambda \neq 0$, *and for all* $t \in [0,1]$, *let*

$$\lambda_t = \mathrm{e}^{(1-t)\widehat{X}_u{}^*}\lambda. \tag{3.19}$$

(In particular $\lambda_1 = \lambda$.) Let $B_G(\lambda, t)$ and $B_L(\lambda, t)$ be two bilinear forms on $\mathbb{R}^m$ defined by

$$B_G(\lambda, t; v_1, v_2) = \lambda_t \cdot \left[\widehat{X}_{v_1}, \widehat{X}_{v_2}\right], \qquad \forall v_1, v_2 \in \mathbb{R}^m,$$

and

$$B_L(\lambda, t; v_1, v_2) = \lambda_t \cdot \left[\left[\widehat{X}_u, \widehat{X}_{v_1}\right], \widehat{X}_{v_2}\right], \qquad \forall v_1, v_2 \in \mathbb{R}^m.$$

If for all $\lambda \in \mathrm{Im} D_0 F^\perp$, $\lambda \neq 0$, there exist some $t \in [0, 1]$, $v_1, v_2 \in \mathbb{R}^m$ such that either

$$B_G(\lambda, t; v_1, v_2) \neq 0 \tag{\textbf{G}}$$

or

$$B_L(\lambda, t; v_1, v_1) < 0 \tag{\textbf{L}}$$

then (q, u) is strongly pliable.

Proof. It follows from Lemmas 20.7 and 20.8 of [AS04] that, as soon as

$$\mathrm{ind}_{-}\lambda\, \mathrm{Hess}_0\, G \geq k, \qquad \forall \lambda \in \mathrm{Im} D_0 G^\perp, \lambda \neq 0,$$

with k the corank of G at 0, G has regular values $v \in L^\infty([0, 1], \mathbb{R}^m)$ arbitrarily close to 0 such that $G(v) = G(0)$.

Moreover, for $\lambda \in \mathrm{Im} D_0 G^\perp$, $\lambda \neq 0$, if there exist some $t \in [0, 1]$, $v_1, v_2 \in \mathbb{R}^m$ such that either $B_G(\lambda, t; v_1, v_2) \neq 0$ or $B_L(\lambda, t; v_1, v_1) < 0$ then $\mathrm{ind}_{-}\lambda\, \mathrm{Hess}_0\, G = +\infty$ ([AS04, Section 20.4]).

By smoothness of G with respect to the L^2 topology and L^2-density of $\mathcal{C}_0$ in $L^\infty([0, 1], \mathbb{R}^m)$, we have $\mathrm{Im} D_0 G^\perp = \mathrm{Im} D_0 F^\perp$. The conclusion then follows from Lemma 3.4.2. $\square$

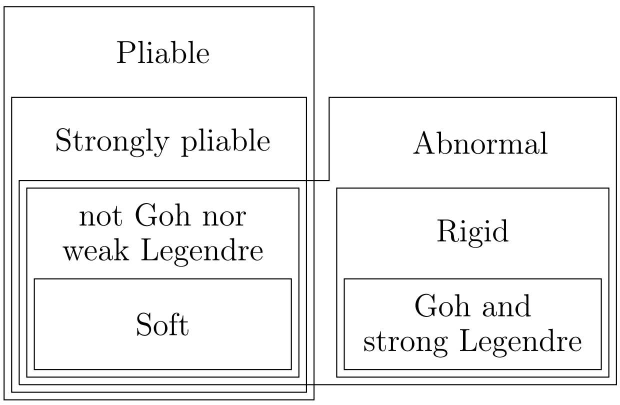


Figure 3.2 – Inclusion diagram of different classes of horizontal curves.

In Figure 3.2, we represent the inclusion diagram of horizontal curves having properties related to strong pliability. By Goh, we intend curves that are the projection of some λ_t (as in (3.19)) such that $\lambda_1 \in \mathrm{Im} D_0 F^\perp$, $\lambda_1 \neq 0$, and for all $t \in [0, 1]$, $v_1, v_2 \in \mathbb{R}^m$

, $B_G(\lambda_1, t; v_1, v_2) = 0$. A Goh curve is strong (respectively, weak) Legendre if for all $t \in [0,1]$, $v \in \mathbb{R}^m$, $B_L(\lambda_1, t; v, v) > 0$ (respectively, $B_L(\lambda_1, t; v, v) \geq 0$) (see [AS04]). An example of non-Goh curves are soft abnormals, introduced in [ABL17, Definition 1].

Let us present some consequences of Proposition 3.4.6. The following result is based on the fact that a non-Goh curve is strongly pliable.

Proposition 3.4.7. *Let $q \in M$ and $\mathbb{G} = (\mathbb{R}^d, (\widehat{X}_1, \dots, \widehat{X}_m))$ be the nilpotent approximation of (M, Δ, g) at q. Let $\widehat{\Delta}$ be the distribution generated by $\widehat{X}_1, \dots, \widehat{X}_m$. Let $u \in \mathbb{R}^m$ be such that*

$$\sum_{k=0}^{+\infty} \left((\mathrm{ad}\widehat{X}_u)^k \widehat{\Delta}^2 \right)_0 = T_0\mathbb{G}. \tag{3.20}$$

Then $(q, u) \in M \times \mathbb{R}^m$ is strongly pliable.

Proof. The proof works by proving that, under condition (3.20), the curve $t \mapsto \mathrm{e}^{t\widehat{X}_u}(0)$ is not Goh. The conclusion then follows from Proposition 3.4.6.

Assume by contradiction that λ_t is a lift of the integral curve of $\widehat{X}_u$ such that $\lambda_1 \in \mathrm{Im}D_0F^\perp$, $\lambda_1 \neq 0$, $B_G(\lambda, t; v_1, v_2) = 0$ for all $t \in [0,1]$ and all $v_1, v_2 \in \mathbb{R}^m$. Thus by differentiating $t \mapsto B_G(\lambda, t; v_1, v_2)$ k times and passing to the limit as $t \to 1^-$, we get

$$\lambda \cdot (\mathrm{ad}\widehat{X}_u)^k [\widehat{X}_{v_1}, \widehat{X}_{v_2}] = 0 \qquad \forall k \in \mathbb{N}, \forall v_1, v_2 \in \mathbb{R}^m.$$

If (3.20) is satisfied, then $\lambda \in \bigcap_{k=0}^{+\infty} ((\mathrm{ad}\widehat{X}_u)^k \widehat{\Delta}^2)_0^\perp = T_0\mathbb{G}^\perp = \{0\}$, hence the statement. □

We show below how Proposition 3.4.7 can be used to assess strong pliability for step-2 distributions and more generally to medium-fat distributions (see [Rif14]). This result extends [JS17, Theorem 6.4] where it was proved, as an application of [BS90, Corollary 1.2], that for a step-2 Carnot group $\mathbb{G}$ every vector $\widehat{X}_u$, $u \in \mathbb{R}^m$, is pliable.

Corollary 3.4.8. *Let (M, Δ, g) be a sub-Riemannian manifold and assume that Δ is medium-fat, i.e., for every $q \in M$ and every $u \in \mathbb{R}^m$ such that $X_u(q) \neq 0$,*

$$\Delta_q^2 + \left[X_u, \Delta^2\right]_q = T_qM. \tag{3.21}$$

Then every pair $(q, u) \in M \times \mathbb{R}^m$ is strongly pliable.

Proof. The case where $X_u(q) = 0$ follows from Remark 3.4.5. Let now $q \in M$ and $u \in \mathbb{R}^m$ be such that $X_u(q) \neq 0$. Let Φ be a system of privileged coordinates at q. Recall that for all positive integer $k > 0$ $\widehat{\Delta}_0^k = \Phi_* \Delta_q^k$ and that

$$\left[\widehat{X}_u, \left[\widehat{X}_i, \widehat{X}_j\right]\right](0) \in \Phi_* \left[X_u, [X_i, X_j]\right](q) + \widehat{\Delta}_0^2, \quad \text{for every } 1 \leq i, j \leq m. \tag{3.22}$$

Assumption (3.21) then implies that $\widehat{\Delta}_0^2 + \left[\widehat{X}_u, \widehat{\Delta}^2\right]_0 = T_0\mathbb{G}$. The conclusion follows from Proposition 3.4.7. □

Another consequence of Proposition 3.4.6 (in particular, of the property that a curve that is not weak Legendre is strongly pliable) is the following.

Corollary 3.4.9. *Let (M, Δ, g) be a step-3 sub-Riemannian manifold. Let $q \in M$ and $u \in \mathbb{R}^m$ be such that the convex positive cone*

$$C_u = \operatorname{conv}\left\{W(q) + [[X_u, V], V](q) \mid V \in \Delta, W \in \Delta^2\right\} \tag{3.23}$$

is equal to T_qM. Then (q, u) is strongly pliable.

Proof. Let $\mathbb{G} = (\mathbb{R}^d, (\widehat{X}_1, \dots, \widehat{X}_m))$ be the nilpotent approximation of (M, Δ, g) at q and $\widehat{\Delta}$ be the distribution generated by $\widehat{X}_1, \dots, \widehat{X}_m$. Then $\mathbb{G}$ is of step 3 and

$$\widehat{C}_u = \operatorname{conv}\left\{W(0) + [[\widehat{X}_u, V], V](0) \mid V \in \widehat{\Delta}, W \in \widehat{\Delta}^2\right\} \tag{3.24}$$

is equal to $T_0\mathbb{G}$ (see (3.22)).

Following Proposition 3.4.6, we assume by contradiction that there exists $\lambda \in \operatorname{Im} D_0F^{\perp} \setminus \{0\}$ such that $\lambda \in \left(\widehat{\Delta}^2_0\right)^{\perp}$ and

$$\lambda \cdot [[\widehat{X}_u, V], V](0) \geq 0$$

for every $V \in \widehat{\Delta}$. It then follows from the equality $\widehat{C}_u = T_0\mathbb{G}$ that $\lambda \cdot Z \geq 0$ for every $Z \in T_0\mathbb{G}$. This leads to a contradiction since $\lambda \neq 0$. □

Example 3.4.10. Let $(x_1, x_2, x_3, y_1, y_2, w)$ be the canonical coordinates on $\mathbb{R}^6$ and define

$$\begin{cases} X_1 = \partial_{x_1}, \\ X_2 = \partial_{x_2}, \\ X_3 = \partial_{x_3} + x_1\partial_{y_1} + x_2\partial_{y_2} + \frac{1}{2}\left(x_1^2 + \alpha x_2^2\right)\partial_w, \end{cases}$$

with $\alpha < 0$. We set

$$\begin{aligned} &W_1 = [X_1, X_3] = \partial_{y_1} + x_1\partial_w, \\ &W_2 = [X_2, X_3] = \partial_{y_2} + \alpha x_2\partial_w, \\ &Z = \partial_w, \end{aligned}$$

and we notice that

$$\begin{aligned} &[X_1, X_2] = [X_1, W_2] = [X_2, W_1] = 0, \\ &[X_1, W_1] = Z, \\ &[X_2, W_2] = \alpha Z. \end{aligned}$$

By [BLU07, Theorem 4.2.10], the Lie algebra generated by $\{X_1, X_2, X_3\}$ is a Carnot algebra.

Take $u \in \mathbb{R}^3$ and let us prove the strong pliability of $(0, u)$. If $u = 0$, this is a consequence of Remark 3.4.5. If either u_1 or u_2 is non-zero, the strong pliability of $(0, u)$ is a consequence of Proposition 3.4.7 since

$$[X_u, [X_1, X_3]] = u_1 Z, \quad [X_u, [X_2, X_3]] = \alpha u_2 Z.$$

Finally, if $u_1 = 0$, $u_2 = 0$ and $u_3 \neq 0$,

$$[[X_u, X_1], X_1] = u_3 Z, \quad [[X_u, X_2], X_2] = \alpha u_3 Z,$$

and the strong pliability of $(0, u)$ follows from Corollary 3.4.9.

3.4.2 Strong pliability implies the C^1_H extension property

Theorem 3.4.11. *Let (M, Δ, g) be an equiregular sub-Riemannian manifold. If every pair $(q, u) \in M \times \mathbb{R}^m$ is strongly pliable then the C^1_H extension property holds for (M, Δ, g).*

Proof. Let (f, L) satisfy the Whitney condition on a closed set K. Without loss of generality, K is compact and there exists a global frame $(X_1, \dots, X_m)$ of Δ on M. In particular, we can rewrite $L = X_u$ with $u : K \to \mathbb{R}^m$ continuous. Let us define $\bar{f}$ on K^c, which is a countable and disjoint union of open intervals.

Let $(a, b) \subset K^c$ be such that $a, b \in K$. For any $\eta > 0$, we define the set $\mathcal{P}_\eta([a, b]) \subset C^0([a, b], \mathbb{R}^m)$ of controls $v \in C^0([a, b], B_\eta(0))$ such that the integral curve of $X_{u(a)+v}$ is a $C^1_H([a, b])$ extension of f on $[a, b]$. In other words for $v \in \mathcal{P}_\eta([a, b])$ we have

$$\begin{aligned}
&v(a) = 0,\\
&v(b) = u(b) - u(a),\\
&\|v\|_\infty < \eta,\\
&f(b) = \left(\overrightarrow{\exp} \int_a^b X_{u(a)+v(s)} \,\mathrm{d}s\right) f(a).
\end{aligned}$$

Notice that $\mathcal{P}_\eta([a, b]) \subset \mathcal{P}_{\eta'}([a, b])$ if $0 < \eta \leq \eta'$ and define

$$\eta([a, b]) = |b - a| + \inf \{\eta > 0 \mid \mathcal{P}_\eta([a, b]) \neq \emptyset\}.$$

We claim that $\inf \{\eta > 0 \mid \mathcal{P}_\eta([a, b]) \neq \emptyset\} < +\infty$, that is, there exists $w \in C^0([a, b], \mathbb{R}^m)$ such that $w(a) = u(a)$, $w(b) = u(b)$ and

$$f(b) = \left(\overrightarrow{\exp} \int_a^b X_{w(s)} \,\mathrm{d}s\right) f(a).$$

This can be deduced, for instance, from Lemma 3.A.1 in the Appendix taking as V the space of piecewise continuous controls on $[a, b]$ endowed with the L^2 topology, $W = \{w \in C^0([a, b], \mathbb{R}^m) \mid w(a) = u(a),\, w(b) = u(b)\}$, $\mathcal{F}(v) = \left(\overrightarrow{\exp} \int_a^b X_{v(s)} \,\mathrm{d}s\right) f(a)$, and $\mathcal{F}_n = \mathcal{F}|_W$, $z_n = f(b)$ for every $n \in \mathbb{N}$. The existence of $v \in V$ such that $\mathcal{F}$ is a submersion at v is a standard consequence of the Lie bracket-generating condition (see, e.g., [Sus76]).

Denoting by $(-\infty, \bar{b})$ and $(\bar{a}, +\infty)$ the two unbounded components of $\mathbb{R} \setminus K$, we set $\bar{f}$ to be such that

$$\bar{f}(t) = \begin{cases} \exp\left((t - \bar{b})\, X_{u(\bar{b})}\right)\left(f(\bar{b})\right) & \text{if } t < \bar{b},\\ \exp\left((t - \bar{a})\, X_{u(\bar{a})}\right)\left(f(\bar{a})\right) & \text{if } t > \bar{a}.\end{cases}$$

We complete the extension $(\bar f, \bar u)$ of (f,u) on $\mathbb{R}$ by taking for each $(a,b) \subset K^c$ with $a, b \in K$ some $v \in \mathcal{P}_{\eta([a,b])}([a,b])$ and setting

$$\bar u(t) = u(a) + v(t), \quad \bar f(t) = \left(\overrightarrow{\exp} \int_a^t X_{u(a)+v(s)}\, \mathrm{d}s \right) f(a), \quad \forall t \in [a,b].$$

By construction, $\bar f : \mathbb{R} \to M$ is an extension of f and, for every $t \in \mathbb{R}$ such that u is continuous at t, the derivative $\dot{\bar f}(t)$ exists and is equal to $X_{u(t)}(f(t))$. We are left to prove that u is continuous on $\mathbb{R}$. Notice that by construction $u|_K$ and $u|_{K^c}$ are continuous. We then focus on the continuity of u at points of ∂K.

Take $\tau_\infty \in \partial K$ and a sequence $(\tau_n)_n \subset \mathbb{R} \setminus K$ such that $\tau_n \to \tau_\infty$. For every n, let $(a_n, b_n) \subset K^c$ be such that $a_n, b_n \in K$ and $\tau_n \in (a_n, b_n)$. If there exists a constant subsequence $((a_{n_k}, b_{n_k}))_{k \in \mathbb{N}}$ of $((a_n, b_n))_{n \in \mathbb{N}}$ then $\lim_{k \to \infty} u(\tau_{n_k}) = u(\tau_\infty)$ by the continuity of $u|_{[a_{n_k}, b_{n_k}]}$. Assume then, without loss of generality, that $\lim a_n = \lim b_n = \tau_\infty$.

Since

$$|u(\tau_n) - u(a_n)| \leq \eta([a_n, b_n]) \text{ and } u(a_n) \to u(\tau_\infty),$$

the proof of the theorem is concluded by Lemma 3.4.12 below. □

Lemma 3.4.12. *Let $(a_n)_n$ and $(b_n)_n$ be two sequences in ∂K such that $(a_n, b_n) \subset \mathbb{R} \setminus K$. If $\lim a_n = \lim b_n = \tau_\infty \in \mathbb{R}$ then $\eta([a_n, b_n]) \to 0$.*

Proof. To prove the lemma, we show that if $\lim a_n = \lim b_n < \infty$ then

$$\inf \{\eta > 0 \mid \mathcal{P}_\eta([a_n, b_n]) \neq \emptyset\} \to 0, \quad \text{as } n \to \infty.$$

Equivalently, given $\eta > 0$, we prove that there exists $N(\eta)$ such that if $n > N(\eta)$ then there exists $v_n \in C^0([a_n, b_n], \mathbb{R}^d)$ such that

$$\begin{cases} v_n(a_n) = 0, \\ v_n(b_n) = u(b_n) - u(a_n), \\ \|v_n\|_\infty < \eta, \\ f(b_n) = \overrightarrow{\exp} \displaystyle\int_{a_n}^{b_n} X_{u(a_n)+v_n(s)}\, \mathrm{d}s\, (f(a_n)). \end{cases} \tag{3.25}$$

By applying Theorem 3.2.7 at $f(\tau_\infty)$, we pick a neighborhood Ω of $f(\tau_\infty)$ and a continuously varying system of privileged coordinates

$$\begin{aligned} \Phi : \ \Omega \times \Omega &\longrightarrow \mathbb{R}^d \\ (x,y) &\longmapsto \Phi_x(y). \end{aligned}$$

For $x \in \Omega$ and $\lambda > 0$, we associate with Φ the quasi-homogeneous dilation δ^x_λ.

There exists a neighborhood $\mathcal{V}$ of 0 in $\mathcal{C}_0 = \{w \in C^0([0,1], \mathbb{R}^m) \mid w(0) = 0\}$ such that

$$\overrightarrow{\exp} \int_0^t (b_n - a_n) X_{u(a_n)+w(s)} \mathrm{d}s \, (f(a_n)) \in \Omega$$

for all $t \in [0,1]$, $w \in \mathcal{V}$ and n large enough. Hence, setting $t_n = b_n - a_n$, we define the endpoint map

$$\begin{array}{rccl} \mathcal{F}_n : & \mathcal{V} & \longrightarrow & \mathbb{R}^d \times \mathbb{R}^m \\ & w & \longmapsto & \left(\delta_{1/t_n}^{f(b_n)} \left(\overrightarrow{\exp} \int_0^1 t_n X_{u(a_n)+w(s)} \mathrm{d}s \, (f(a_n)) \right), u(a_n) + w(1) \right). \end{array}$$

If $\mathcal{F}_n(w) = (0, u(b_n))$ and $\|w\|_\infty < \eta$ then $[a_n, b_n] \ni s \mapsto w\left(\frac{s-a_n}{b_n-a_n}\right)$ satisfies (3.25) (recall that $\delta_{1/t_n}^{f(b_n)}(f(b_n)) = 0$). Then we are left to prove that there exists $N(\eta)$ such that if $n > N(\eta)$, there exists $w_n \in \mathcal{V}$ such that $\|w_n\|_\infty < \eta$ and

$$\mathcal{F}_n(w_n) = (0, u(b_n)).$$

Distributing the dilation we get

$$\mathcal{F}_n(w) = \left(\overrightarrow{\exp} \int_0^1 \left(t_n {\delta_{1/t_n}^{f(b_n)}}_* X_{u(a_n)+w(s)} \right) \mathrm{d}s \, \left(\delta_{1/t_n}^{f(b_n)}(f(a_n)) \right), u(a_n) + w(1) \right).$$

The Whitney condition ensures that $\delta_{1/t_n}^{f(b_n)}(f(a_n)) \to \mathrm{e}^{-\widehat{X}_{u(\tau_\infty)}}(0)$ (see Propositions 3.3.7 and 3.3.9), and $t_n {\delta_{1/t_n}^{f(b_n)}}_* X_{u(a_n)+w(s)}$ is bounded and locally uniformly converges towards $\widehat{X}_{u(\tau_\infty)+w(s)}$. Hence $\mathcal{F}_n$ locally uniformly converges towards

$$\begin{array}{rccl} \mathcal{F}_\infty : & \mathcal{V} & \longrightarrow & \mathbb{R}^d \times \mathbb{R}^m \\ & w & \longmapsto & \left(\overrightarrow{\exp} \int_0^1 \widehat{X}_{u(\tau_\infty)+w(s)} \mathrm{d}s \left(\mathrm{e}^{-\widehat{X}_{u(\tau_\infty)}}(0) \right), u(\tau_\infty) + w(1) \right). \end{array}$$

Let $\mathbb{G} = (\mathbb{R}^d, (\widehat{X}_1, \ldots, \widehat{X}_m))$ be the Carnot group structure of the nilpotent approximation of (M, Δ, g) at $f(\tau_\infty)$. Denote by $*$ its group operation, and recall that horizontal vector fields on $\mathbb{G}$ are left-invariant with respect to $*$. Then

$$\mathcal{F}_\infty(w) = \left(\left(\mathrm{e}^{-\widehat{X}_{u(\tau_\infty)}}(0) \right) * \overrightarrow{\exp} \int_0^1 \widehat{X}_{u(\tau_\infty)+w(s)} \mathrm{d}s(0), u(\tau_\infty) + w(1) \right).$$

With $\psi(g, u) = \left(\left(\mathrm{e}^{-\widehat{X}_{u(\tau_\infty)}}(0) \right) * g, u(\tau_\infty) + u \right)$, which is a diffeomorphism from $\mathbb{R}^d \times \mathbb{R}^m$ onto itself, we have that

$$\mathcal{F}_\infty = \psi \circ \mathcal{F}^{u(\tau_\infty)},$$

where $\mathcal{F}^{u(\tau_\infty)}$ stands for the map introduced in Definition 3.4.1.

Therefore, by the strong pliability hypothesis, there exists w_η in $\mathcal{V}$ such that $\|w_\eta\| < \eta/2$, $\mathcal{F}_\infty(w_\eta) = \mathcal{F}_\infty(0)$, and $\mathcal{F}_\infty$ is a submersion at w_η.

Notice that

$$\mathcal{F}_\infty(0) = \psi \circ \mathcal{F}^{u(\tau_\infty)}(0) = \psi\left(\mathrm{e}^{\widehat{X}_{u(\tau_\infty)}}(0), 0\right) = \left(\mathrm{e}^{-\widehat{X}_{u(\tau_\infty)}}(0) * \mathrm{e}^{\widehat{X}_{u(\tau_\infty)}}(0), u(\tau_\infty)\right) = (0, u(\tau_\infty)),$$

again by applying the $*$-left-invariance of $\widehat{X}_{u(\tau_\infty)}$.

It follows from Lemma 3.A.1 in the Appendix, with $V = W = \mathcal{V}$, $\mathcal{F} = \mathcal{F}_\infty$ and $z_n = (0, u(b_n))$ for $n \in \mathbb{N}$, that given $\eta > 0$, there exists $N(\eta) > 0$ such that for all $n > N(\eta)$ the equation $\mathcal{F}_n(w_n) = (0, u(b_n))$ has a solution w_n with $\|w_n\|_\infty < \eta$. This concludes the proof of the lemma. □

Corollary 3.4.13. *All step-2 sub-Riemannian manifolds have the C^1_H extension property.*

Proof. If the manifold (M, Δ, g) is equiregular, all pairs $(q, u) \in M \times \mathbb{R}^m$ are strongly pliable (see section 3.4.1.1, Corollary 3.4.8), hence the result by Theorem 3.4.11.

If the manifold (M, Δ, g) is not equiregular, it can be locally lifted to a step-2 equiregular manifold (see *e.g.* [Jea14, Section 2.4]). By Corollary 3.3.12, since equiregular step-2 sub-Riemannian manifolds have the C^1_H extension property, so does (M, Δ, g). □

3.5 Lusin approximation of horizontal curves

Let (M, Δ, g) be a sub-Riemannian manifold, and let $(X_1, \dots, X_m)$ be a frame of the distribution. As a consequence of [Vod06, Theorem 2], we have the following Rademacher-type theorem.

Theorem 3.5.1. *Let (M, Δ, g) be an equiregular sub-Riemannian manifold and $(X_1, \dots, X_m)$ be a frame of the distribution. Let $\gamma : [a, b] \to M$ be an absolutely continuous and horizontal curve on M.*

Let Φ be a continuously varying system of privileged coordinates. For almost every $t \in [a, b]$ there exists $u \in \mathbb{R}^m$, such that

$$\lim_{h \to 0} \frac{1}{h} \widehat{\mathrm{d}}_{\mathrm{SR}} \left(\Phi_{\gamma(t)}(\gamma(t+h)), \mathrm{e}^{h\widehat{X}_u}(0)\right) = 0,$$

where $\widehat{\mathrm{d}}_{\mathrm{SR}}$ is the Carnot-Caratheodory distance for the sub-Riemannian structure on $\mathbb{R}^d$ having $(\widehat{X}_1, \dots, \widehat{X}_d)$ as a frame, with $\mathbb{G} = (\mathbb{R}^d, (\widehat{X}_1, \dots, \widehat{X}_d))$ the nilpotent approximation of (M, Δ, g) at $\gamma(t)$.

We will use the following corollary.

Corollary 3.5.2. *Let (M, Δ, g) be a sub-Riemannian manifold. Let $[a, b] \subset \mathbb{R}$ and $\gamma : [a, b] \to M$ be an horizontal absolutely continuous curve. Then for almost every $t \in [a, b]$ there exists $u \in \mathbb{R}^m$ such that*

$$\lim_{h \to 0} \frac{1}{h} \mathrm{d}_{\mathrm{SR}} \left(\gamma(t+h), \mathrm{e}^{hX_u}\gamma(t)\right) = 0. \tag{3.26}$$

Proof. Let us first consider the equiregular case. Let $t \in [a,b]$ be such that there exists $u(t) \in \mathbb{R}^m$ such that

$$\lim_{h\to 0} \frac{1}{h}\widehat{\mathrm{d}}_{\mathrm{SR}}\left(\Phi_{\gamma(t)}(\gamma(t+h)), \mathrm{e}^{h\widehat{X}_{u(t)}}(0)\right) = 0.$$

Applying [Bel96, Theorem 7.32] at $\gamma(t)$, there exist $\varepsilon > 0$, $C > 0$ such that, as soon as

$$\max(\mathrm{d}_{\mathrm{SR}}(\gamma(t), q), \mathrm{d}_{\mathrm{SR}}(\gamma(t), q')) \leq \varepsilon,$$

we have

$$\mathrm{d}_{\mathrm{SR}}(q,q') \leq \widehat{\mathrm{d}}_{\mathrm{SR}}\left(\Phi_{\gamma(t)}(q), \Phi_{\gamma(t)}(q')\right) + C\widehat{\mathrm{d}}_{\mathrm{SR}}\left(0, \Phi_{\gamma(t)}(q')\right)\widehat{\mathrm{d}}_{\mathrm{SR}}\left(\Phi_{\gamma(t)}(q), \Phi_{\gamma(t)}(q')\right)^{1/r} \tag{3.27}$$

where r is the step of (M, Δ, g).

For $|h|$ sufficiently small, $\Phi^{-1}_{\gamma(t)}\left(\mathrm{e}^{h\widehat{X}_{u(t)}}(0)\right)$ is well defined and the triangular inequality yields

$$\begin{aligned}\mathrm{d}_{\mathrm{SR}}\left(\gamma(t+h), \mathrm{e}^{hX_{u(t)}}\gamma(t)\right) \leq & \mathrm{d}_{\mathrm{SR}}\left(\gamma(t+h), \Phi^{-1}_{\gamma(t)}\left(\mathrm{e}^{h\widehat{X}_{u(t)}}(0)\right)\right) \\ & + \mathrm{d}_{\mathrm{SR}}\left(\mathrm{e}^{hX_{u(t)}}\gamma(t), \Phi^{-1}_{\gamma(t)}\left(\mathrm{e}^{h\widehat{X}_{u(t)}}(0)\right)\right).\end{aligned}$$

As a consequence of (3.27), since $\widehat{\mathrm{d}}_{\mathrm{SR}}\left(0, \mathrm{e}^{h\widehat{X}_{u(t)}}(0)\right) \leq |h||u(t)|$, in order to prove (3.26) it is sufficient to have both

$$\lim_{h\to 0} \frac{1}{h}\widehat{\mathrm{d}}_{\mathrm{SR}}\left(\Phi_{\gamma(t)}(\gamma(t+h)), \mathrm{e}^{h\widehat{X}_{u(t)}}(0)\right) = 0$$

and

$$\lim_{h\to 0} \frac{1}{h}\widehat{\mathrm{d}}_{\mathrm{SR}}\left(\Phi_{\gamma(t)}\left(\mathrm{e}^{hX_{u(t)}}\gamma(t)\right), \mathrm{e}^{h\widehat{X}_{u(t)}}(0)\right) = 0.$$

The first limit coincides with our assumption on t, and the second one is a consequence of two distance estimates for h small enough. First, from [Bel96, Propsition 7.26, Equation (50)],

$$\begin{aligned}\frac{1}{C}\widehat{\mathrm{d}}_{\mathrm{SR}}\left(\Phi_{\gamma(t)}\left(\mathrm{e}^{hX_{u(t)}}\gamma(t)\right), \mathrm{e}^{h\widehat{X}_{u(t)}}(0)\right) \leq & \left\|\Phi_{\gamma(t)}\left(\mathrm{e}^{hX_{u(t)}}\gamma(t)\right) - \mathrm{e}^{h\widehat{X}_{u(t)}}(0)\right\|_{\gamma(t)} \\ & + \left\|\mathrm{e}^{h\widehat{X}_{u(t)}}(0)\right\|^{1-1/r}_{\gamma(t)}\left\|\Phi_{\gamma(t)}\left(\mathrm{e}^{hX_{u(t)}}\gamma(t)\right) - \mathrm{e}^{h\widehat{X}_{u(t)}}(0)\right\|^{1/r}_{\gamma(t)}.\end{aligned}$$

Second, from [Jea14, Theorem 2.3, Equation (2.14)],

$$\left\|\Phi_{\gamma(t)}\left(\mathrm{e}^{hX_{u(t)}}\gamma(t)\right) - \mathrm{e}^{h\widehat{X}_{u(t)}}(0)\right\|_{\gamma(t)} \leq C|u(t)|^{1+1/r}|h|^{1+1/r}.$$

Combining the two,

$$\widehat{\mathrm{d}}_{\mathrm{SR}}\left(\Phi_{\gamma(t)}\left(\mathrm{e}^{hX_{u(t)}}\gamma(t)\right), \mathrm{e}^{h\widehat{X}_{u(t)}}(0)\right) \leq C|u(t)||h|\left(|u(t)|^{1/r}|h|^{1/r} + |u(t)|^{1/r^2}|h|^{1/r^2}\right),$$

hence the result in the equiregular case.

If the manifold is not equiregular, as in the proofs of Propositions 3.3.1 and Corollary 3.4.13, we exploit the existence of local lifts of the sub-Riemannian structure that are equiregular. Consider an horizontal lift $\widetilde{\gamma}$ of γ. By the first part of the proof, we deduce that for almost every $t \in [a,b]$ there exists $u \in \mathbb{R}^m$ such that

$$\lim_{h \to 0} \frac{1}{h} \widetilde{\mathrm{d}}_{\mathrm{SR}} \left(\widetilde{\gamma}(t+h), \mathrm{e}^{h\widetilde{X}_u} \widetilde{\gamma}(t) \right) = 0,$$

where $\widetilde{X}_i$ is the lift of X_i for every $1 \leq i \leq m$. Using (3.1), we deduce (3.26). □

Following the classical scheme of proof for Lusin approximation theorems (see [LDS16, Spe16, JS17] for the case of Carnot groups), we give a version for general sub-Riemannian manifolds.

In the following we denote by $\mathcal{L}$ the Lebesgue measure on $\mathbb{R}$.

Proposition 3.5.3 (Lusin approximation of an horizontal curve)**.** *Let (M, Δ, g) be a sub-Riemannian manifold having the C^1_H-extension property and let $\gamma : [a,b] \to M$ absolutely continuous be an horizontal curve. Then for any $\varepsilon > 0$ there exists $K \subset [a,b]$ compact with $\mathcal{L}([a,b] \setminus K) < \varepsilon$ and a curve $\gamma_1 : [a,b] \to M$ of class C^1_H such that γ and γ_1 coincide on K.*

Proof. Let $\varepsilon > 0$. We want to prove that there exists a compact set $K \subset [a,b]$ with $\mathcal{L}([a,b] \setminus K) < \varepsilon$ such that the C^1_H-Whitney condition holds for $(\gamma, \dot{\gamma})$ on K. The proposition then follows from the C^1_H-extension property.

By Corollary 3.5.2, there exists $A \subset [a,b]$ of full measure such that, for any $t \in A$, the curve γ admits an horizontal derivative at t, denoted by $X_{u(t)}(\gamma(t))$ using the local frame $(X_1, \dots, X_m)$. Moreover, the family of functions over A, $(f_h)_{h \in (0,1)}$, defined by

$$f_h(t) = \frac{1}{h} \mathrm{d}_{\mathrm{SR}} \left(\gamma(t+h), \mathrm{e}^{hX_{u(t)}} \gamma(t) \right)$$

pointwise converges to 0 as $h \to 0$. Applying the classical Lusin Theorem to the map $u : A \to \mathbb{R}^m$, there exists a compact set $K \subset A$ such that u is uniformly continuous on K and $\mathcal{L}(A \setminus K) < \varepsilon/2$. Furthermore, by Egorov's Theorem, we have the uniform convergence of $(f_h)_h$ towards 0 on a compact subset $K' \subset K$ such that $\mathcal{L}(K \setminus K') < \varepsilon/2$.

This implies that there exist $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ and K' such that $\omega(t) = o(t)$ at 0^+, $\mathcal{L}([a,b] \setminus K') < \varepsilon$, and

$$\mathrm{d}_{\mathrm{SR}} \left(\gamma(t), \mathrm{e}^{(t-s)X_{u(s)}} \gamma(s) \right) \leq \omega(|t-s|), \quad \forall t, s \in K'.$$

□

The above result finds its application in the study of 1-countably rectifiable sets (see [LDS16]). A set $E \subset M$ is said to be 1-*countably rectifiable* if there exists a countable family of Lipschitz curves $f_k : \mathbb{R} \to M$ such that $\mathcal{H}^1 \left(E \setminus \cup_k f_k(\mathbb{R}) \right) = 0$, where $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure.

Corollary 3.5.4. *Let (M, Δ, g) be a sub-Riemannian manifold having the C^1_H-Whitney extension property and let E be a 1-countably rectifiable subset of M. Then there exists a countable family of C^1_H-curves $f_k : \mathbb{R} \to M$ such that $\mathcal{H}^1 \left(E \setminus \cup_k f_k(\mathbb{R})\right) = 0$.*

Appendices

3.A A topological degree lemma

Let us present here a useful technical result based on standard topological degree considerations.

Lemma 3.A.1. *Let V be a normed space and W an affine and dense subspace of V. Fix $v \in V$ and let $\mathcal{F} \in C^1(V, \mathbb{R}^d)$ be a submersion at v. Consider a sequence of functions $\mathcal{F}_n : W \to \mathbb{R}^d$ with the property that $\mathcal{F}_n$ locally uniformly converges to $\mathcal{F}|_W$. Let, moreover, z_n be a sequence in $\mathbb{R}^d$ converging to $\mathcal{F}(v)$. Then there exists a sequence w_n in W converging to v in V and such that, for n large enough, $\mathcal{F}_n(w_n) = z_n$.*

Proof. By assumption there exist $\phi_1, \dots, \phi_d \in V$ such that the map

$$\begin{array}{rccl} \mathfrak{F}: & \mathbb{R}^d & \longrightarrow & \mathbb{R}^d \\ & (x_1, \dots, x_d) & \longmapsto & \mathcal{F}(v + x_1\phi_1 + \dots + x_d\phi_d) \end{array}$$

is a local diffeomorphism at 0.

Let v_n be a sequence in W converging to v in V. Denote by W_L the linear space $\{w - w' \mid w, w' \in W\}$ and consider, for each $i = 1, \dots, d$, a sequence φ_i^n in W_L converging to ϕ_i in V. Then the sequence of maps

$$\begin{array}{rccl} \mathfrak{G}_n: & \mathbb{R}^d & \longrightarrow & \mathbb{R}^d \\ & (x_1, \dots, x_d) & \longmapsto & \mathcal{F}_n(v_n + x_1\phi_1^n + \dots + x_d\phi_d^n) \end{array}$$

locally uniformly converges to $\mathfrak{F}$.

Let $r > 0$ be small enough so that the restriction of $\mathfrak{F}$ to the ball $B_r(0)$ of center the origin and radius r is a diffeomorphism between $B_r(0)$ and $\mathfrak{F}(B_r(0))$. Then $\mathfrak{G}_n|_{\overline{B_r(0)}}$ uniformly converges to $\mathfrak{F}|_{\overline{B_r(0)}}$. Hence, for any K compactly contained in $\mathfrak{F}(B_r(0))$ and for n large enough, the topological degree $d(\mathfrak{G}_n, B_r(0), z)$ is equal to 1 or -1 for every $z \in K$. In particular, choosing $K = \mathfrak{F}(\overline{B_{\rho r}(0)})$ with $\rho \in (0, 1/2)$ and replacing r by $2\rho r$ in the above argument, we have that for n large enough, there exists $x^n \in B_{2\rho r}(0)$ such that

$$\mathcal{F}_n(v_n + x_1^n\phi_1^n + \dots + x_d^n\phi_d^n) = \mathfrak{G}_n(x^n) = z_n.$$

In order to recover the convergence to v of the sequence $v_n + x_1^n \phi_1^n + \cdots + x_d^n \phi_d^n$ it suffices to notice that if z_n is in $\mathfrak{F}(\overline{B_{\rho r}(0)})$, then x^n can be chosen of norm smaller than $2\rho r$. The conclusion then follows from the convergence of v_n to v and the uniform boundedness of $\{\phi_j^n \mid j = 1, \ldots, d,\, n \in \mathbb{N}\}$. $\square$

Chapter 4

Short geodesics losing optimality in contact sub-Riemannian manifolds

We study the sub-Riemannian exponential for contact distributions on manifolds of dimension greater or equal to 5. We compute an approximation of the sub-Riemannian Hamiltonian flow and show that the conjugate time can be double in this case. We obtain an approximation of the first conjugate locus for small radii and introduce a geometric invariant to show that the metric for contact distributions typically exhibits an original behavior, different from the classical 3-dimensional case.

Contents

4.1 Introduction

Context. Let M be a smooth (C^∞) manifold of dimension $2n+1$, with $n \geq 1$ integer. The *contact distribution* is a $2n$-dimensional vector sub-bundle $\Delta \subset TM$ that locally coincides with the kernel of a smooth 1-form ω on M such that $\omega \wedge (\mathrm{d}\omega)^n \neq 0$. The sub-Riemannian structure on M is given by a smooth scalar product g on Δ, and we call (M, Δ, g) a *contact sub-Riemannian manifold* (see, for instance, [ABB16, ABR16]).

The small scale geometry of general 3-dimensional contact sub-Riemannian manifolds is well understood but not much can be said for dimension 5 and beyond, apart from the particular case of Carnot groups. We are interested in a qualitative description of the local geometry of contact sub-Riemannian manifolds by describing the family of short locally-length-minimizing curves (or geodesics) originating at a given point. In the case of contact sub-Riemannian manifolds, all length-minimizing curves are projections of integral curves of an intrinsic Hamiltonian vector field on T^*M, and as such, geodesics are characterized by their initial point and initial covector.

By analogy with the Riemannian case, for all $q \in M$, we denote by $\mathcal{E}_q$ the *sub-Riemannian exponential*, that maps a covector $p \in T^*_qM$ to the evaluation at time 1 of the geodesic curve starting at q of initial covector p. An essential observation on length minimizing curves in sub-Riemannian geometry is that there exist locally-length-minimizing curves that lose local optimality arbitrarily close to their starting point. Hence the geometry of sub-Riemannian balls of small radii is inherently linked with the geometry of the conjugate locus, that is, at q, the set of points $\mathcal{E}_q(p)$ such that p is a critical point of $p \mapsto \mathcal{E}_q(p)$.

The sub-Riemannian exponential has a natural structure of Lagrangian map, since it is the projection of a Hamiltonian flow over T^*M, and its conjugate locus of Lagrangian caustic. In small dimension, this observation permits the study of the stability of the caustic and the classification of singular points of the exponential from the point of view of Lagrangian singularities (see, for instance, [AGnZV85]).

In the 3-dimensional case, this analysis has been conducted, with different approaches, in [Agr96] and [EAGK96]. These works describe asymptotics of the sub-Riemannian exponential, the conjugate and cut loci near the starting point. The aim of the present work is to extend this study to higher dimensional contact sub-Riemannian manifolds, following the methodology developed in [EAGK96] and [Cha02] (the latter focusing on a similar study of quasi-contact sub-Riemannian manifolds). More precisely, we use a perturbative approach to compute approximations of the Hamiltonian flow. This is made possible by using a general well-suited normal form for contact sub-Riemannian structures. The normal form that we use has been obtained in [AG01] and we recall its properties in Appendix 4.A.

Finally, it can be noted that classical behaviors displayed by 3-dimensional sub-Riemannian structures may not be typical in larger dimension. The 3-dimensional case is very rigid in the class of sub-Riemannian manifolds and it appears to be so even in regard of contact sub-Riemannian manifolds of arbitrary dimension. Therefore, part of our focus is dedicated to highlighting the differences between this classical case and those of larger dimension.

Notation. In the following, for any two integers $m, n \in \mathbb{N}$, $m \leq n$, we denote by $[\![m, n]\!]$ the set of integers $k \in \mathbb{N}$ such that $m \leq k \leq n$.

Let (M, Δ, g) be a contact sub-Riemannian manifold of dimension $2n + 1$, $n \geq 1$ integer.

Invariants of the nilpotent approximation. Consider a 1-form ω such that $\ker \omega = \Delta$ and $\omega \wedge (\mathrm{d}\omega)^n \neq 0$ (unique up to multiplication by a smooth non-vanishing function). For all $q \in M$, there exists a linear map $A(q) : \Delta_q \to \Delta_q$, skew-symmetric with respect to g_q, such that for all $X, Y \in \Delta$, $\mathrm{d}\omega(X, Y)(q) = g_q(A(q)X(q), Y(q))$. Then the eigenvalues of $A(q)$, $\{\pm i b_1, \dots, \pm i b_n\}$, are invariants of the sub-Riemannian structure at q (up to a multiplicative constant). In the following, we will assume that the invariants $\{b_1, \dots, b_n\} \in \mathbb{R}^+$ are rescaled so that $b_1 \cdots b_n = \frac{1}{n!}$.

These invariants are parameters of the metric tangent to the sub-Riemannian structure at q, or nilpotent approximation at q (see [Bel96]), which admits a structure of Carnot group. Notice in particular that the nilpotent approximations of a contact sub-Riemannian structure at two points $q_1, q_2 \in M$ may not be isometric if the dimension $2n + 1$ is larger than 3.

For a given $q \in M$, there always exists a set of coordinates $(x_1, \dots, x_{2n}, z) : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ such that a frame $\left(\widehat{X}_1, \dots, \widehat{X}_{2n}\right)$ of the nilpotent approximation at q can be written in the normal form

$$\widehat{X}_{2i-1} = \partial_{x_{2i-1}} + \frac{b_i}{2} x_{2i} \partial_z, \quad \widehat{X}_{2i} = \partial_{x_{2i}} - \frac{b_i}{2} x_{2i-1} \partial_z, \qquad \forall i \in [\![1, n]\!].$$

The central idea we follow is that the sub-Riemannian structure at a point $q \in M$ can be expressed as a small perturbation of the nilpotent structure at q_0 for points q close to q_0. An important tool we use is the Agrachev–Gauthier normal form, introduced in [AG01], which asserts, for any given $q_0 \in M$, the existence of coordinates at q_0, $(x_1, \dots x_{2n}, z)$:

$M \to \mathbb{R}^{2n+1}$, and a frame of (Δ, g), $(X_1, \ldots, X_{2n})$, such that

$$X_i(x,z) = \widehat{X}_i(x,z) + O\left(|x|^2\right).$$

Asymptotics and covectors. Let

$$H(p,q) = \frac{1}{2} \sup_{v \in \Delta_q \setminus \{0\}} \frac{\langle p, v\rangle^2}{g_q(v,v)}$$

be the sub-Riemannian Hamiltonian. For all $q \in M$, $H(q,\cdot)$ is a positive quadratic form on T_q^*M of rank $2n$. Then for all $r > 0$, the set $\{H(p,q) = r \mid p \in T_q^*M\}$ is an unbounded subset of T_q^*M with the topology of the cylinder $\mathbb{S}^{2n-1} \times \mathbb{R}$ (see for instance [ABB16, ABR16]). In the following, for all $q \in M$ and $r \geq 0$, we denote this set by

$$\mathcal{C}_q(r) = \{H(p,q) = r \mid p \in T_q^*M\}.$$

Abusing notations, for $V \subset \mathbb{R}^+$, we denote $\mathcal{C}_q(V) = \cup_{r \in V} \mathcal{C}_q(r)$. We choose coordinates $p = (h, h_0)$ on T_q^*M where for a given $r > 0$, h_0 denotes the unbounded component of $p \in \mathcal{C}_q(r)$.

An important observation is that in the nilpotent case, geodesics losing optimality near their starting point correspond to initial covectors in $\mathcal{C}_q(r)$ such that $|h_0|/r$ tends to infinity (see, for instance, [BBN12, BN16]). The expansions obtained in this paper rely on the same type of asymptotics.

Section 4.2 is dedicated to the computation of an approximation of the flow of the Hamiltonian vector field $\vec{H}$ as $h_0 \to \infty$. Since $\vec{H}$ is a quadratic Hamiltonian vector field, its integral curves satisfy the symmetry

$$\mathrm{e}^{t\vec{H}}(p_0, q_0) = \mathrm{e}^{\vec{H}}(tp_0, q_0), \qquad \forall q_0 \in M, p_0 \in T_{q_0}^*M, t \in \mathbb{R}.$$

Hence it is useful for us to consider the time-dependent exponential that maps the pair $(t,p) \in \mathbb{R} \times \mathcal{C}_q(1/2)$ to the geodesic of initial covector p evaluated at time t. Using the approximation of the Hamiltonian flow as $h_0 \to \infty$, Section 4.3 is dedicated to the computation of the conjugate time. For a given $q \in M$, the conjugate time $t_c(p)$ is the smallest positive time such that $\mathcal{E}_q(t_c(p),\cdot)$ is critical at p. The computation of the conjugate locus follows once the conjugate time is known.

Notice in particular that for a given initial covector $p \in \mathcal{C}_q(1/2)$, $t_c(p)$ is then an upper bound of the sub-Riemannian distance between q and $\mathcal{E}_q(t_c(p),p)$ (and we have equality if $\mathcal{E}_q(t_c(p),p)$ is also in the cut locus).

In the 3D case it is proven in [Agr96, EAGK96] that for an initial covector $(\cos\theta, \sin\theta, h_0) \in \mathcal{C}_q(1/2)$, the conjugate time at q satisfies as $h_0 \to \pm\infty$

$$t_c(\cos\theta, \sin\theta, h_0) = \frac{2\pi}{|h_0|} - \frac{\pi\kappa}{|h_0|^3} + O\left(\frac{1}{|h_0|^4}\right) \tag{4.1}$$

and the first conjugate point satisfies (in well chosen adapted coordinates at q)

$$\mathcal{E}_q(t_c(\cos\theta,\sin\theta,h_0),(\cos\theta,\sin\theta,h_0)) = \\ \pm\frac{\pi}{|h_0|^2}(0,0,1)\pm\frac{2\pi\chi}{|h_0|^3}(-\sin^3\theta,\cos^3\theta,0)+O\left(\frac{1}{|h_0|^4}\right).$$

The analysis we carry in Sections 4.2 and 4.3 aims at generalizing such expansions. (Notice that we focus only on the case $h_0\to+\infty$ but the case $h_0\to-\infty$ is the same.) Our results, however, provide an important distinction between the classical 3D contact case and higher dimensional ones. Indeed, a very useful fact in the analysis of the geometry of the 3D case is that a 3D sub-Riemannian contact structure is very well approximated by its nilpotent approximation (as exemplified in [Bar13], for instance).

This can be illustrated by using the 3D version of the Agrachev–Gauthier normal form, as introduced in [EAGK96]. Let us denote by $\widehat{\mathcal{E}}_q$ the exponential of the nilpotent approximation of the sub-Riemannian structure at q_0 in normal form. Then as $h_0\to+\infty$, we have the expansion

$$\mathcal{E}_q(\tau/h_0,(h_1,h_2,h_0))=\widehat{\mathcal{E}}_q(\tau/h_0,(h_1,h_2,h_0))+O\left(\frac{1}{h_0^3}\right). \tag{4.2}$$

As a result, one immediately obtains a rudimentary version of expansion (4.1),

$$t_c(\cos\theta,\sin\theta,h_0)=\frac{2\pi}{|h_0|}+O\left(\frac{1}{|h_0|^3}\right). \tag{4.3}$$

However, expansion (4.3) is not true in general when we consider contact manifolds of dimension larger than 3 (that is, the conjugate time is not a third order perturbation of the nilpotent conjugate time $2\pi/|h_0|$). As an application of Theorem 4.3.6, which gives a general second order approximation of the conjugate time in dimension greater or equal to 5, we are able to prove that the expansion (4.2) cannot be generalized.

Theorem 4.1.1. *Let (M,Δ,g) be a generic contact sub-Riemannian manifold. There exists a codimension 1 stratified subset $\mathfrak{S}$ of M such that for all $q\in M\setminus\mathfrak{S}$, for all linearly adapted coordinates at q and for all $T>0$,*

$$\limsup_{h_0\to+\infty}\left(h_0^2\sup_{\tau\in(0,T)}\left|\mathcal{E}_q\left(\frac{\tau}{h_0},(h_1,\dots,h_{2n},h_0)\right)-\widehat{\mathcal{E}}_q\left(\frac{\tau}{h_0},(h_1,\dots,h_{2n},h_0)\right)\right|\right)>0. \tag{4.4}$$

This observation needs to be put in perspective with some already observed differences between 3D contact sub-Riemannian manifolds and those of greater dimension. For a given 1-form ω such that $\ker\omega=\Delta$ and $\omega\wedge(\mathrm{d}\omega)^n\neq0$, the Reeb vector field is the unique vector field X_0 such that $\omega(X_0)=1$ and $\iota_{X_0}\mathrm{d}\omega=0$. The form ω is not unique (only up to multiplication by a smooth non-vanishing function), and so X_0 is neither. In 3D however, the conjugate locus lies tangent to a single line that carries a Reeb vector field that is deemed canonical. In larger dimension, this uniqueness property is not true in general. For this reason, we introduce a geometric invariant that plays a similar role in measuring how the conjugate locus lies with respect to the nilpotent conjugate locus and use it to prove Theorem 4.4.

The main difference seems to be a lack of symmetry in greater dimensions. Indeed the existence of a unique Reeb vector field (up to rescaling) points toward the idea of a natural $\mathrm{SO}(2n)$ symmetry of the nilpotent structure. However the actual symmetry of a contact sub-Riemannian manifold (or rather its nilpotent approximation) is $\mathrm{SO}(2)^n$ (on the subject, see, for instance, [AG01]). Of course, when $n = 1$, $\mathrm{SO}(2)^n = \mathrm{SO}(2n)$.

4.1.1 Content

In Section 4.2, we compute an approximation of the exponential map for small time and large h_0 (Proposition 4.2.1). Using the Agrachev–Gauthier normal form (recalled in Appendix 4.A), the exponential appears to be a small perturbation of the standard n-Heisenberg exponential.

Section 4.3-4.4 are dedicated to the approximation of the conjugate time (as summarized in Theorem 4.3.6), from which an approximation of the conjugate locus can be obtained. A careful analysis of the conjugate time for the nilpotent approximation shows that, under some conditions, the second conjugate time accumulates on the first (Section 4.3.2). We rely on this observation to compute a second order approximation of the conjugate time (Section 4.4) and treat the problem of a double conjugate time via blow-up (Section 4.4.2). With the aim of proving stability of the caustic, we conclude the section by computing a third order approximation of the conjugate time for a small subset of initial covectors (Section 4.5).

4.2 Normal extremals

4.2.1 Geodesic equation in perturbed form

In this section we establish the dynamical system satisfied by geodesics in terms of small perturbations of the Heisenberg case.

Let (M, Δ, g) be a $(2n+1)$-dimensional contact sub-Riemannian manifold. Let V be an open subset of M and $(X_1, \ldots, X_{2n})$ be a frame of (Δ, g) on V, that is, a family of vector fields such that $g_q(X_i(q), X_j(q)) = \delta_i^j$ for all $i, j \in [\![1, 2n]\!]$ and all $q \in V$ (such a family always exists for V sufficiently small). The sub-Riemannian Hamiltonian can be written

$$H(p, q) = \frac{1}{2}\sum_{i=1}^{2n} \langle p, X_i(q)\rangle^2.$$

In the case of contact distributions, locally-length-minimizing curves are projections of normal extremals, the integral curves of the Hamiltonian vector field $\vec{H}$ on T^*M (see for instance [ABB16, ABR16]). In other words, a normal extremal $t \mapsto (p(t), q(t))$ satisfies in coordinates the Hamiltonian ordinary differential equation

$$\begin{cases} \dfrac{\mathrm{d}q}{\mathrm{d}t} = \displaystyle\sum_{i=1}^{2n} \langle p, X_i(q) \rangle X_i(q), \\ \dfrac{\mathrm{d}p}{\mathrm{d}t} = -\displaystyle\sum_{i=1}^{2n} \langle p, X_i(q) \rangle \, {}^t p \, D_q X_i(q). \end{cases} \tag{4.5}$$

For V sufficiently small, we can arbitrarily choose a non-vanishing vector field X_0 transverse to Δ in order to complete $(X_1(q), \ldots, X_{2n}(q))$ into a basis $(X_1(q), \ldots, X_{2n}(q), X_0(q))$ of T_qM at any point q of V. We use the family $(X_1, \ldots, X_{2n}, X_0)$ to endow T^*M with dual coordinates $(h_1, \ldots, h_{2n}, h_0)$ such that

$$h_i(p, q) = \langle p, X_i(q) \rangle \qquad \forall i \in [\![0, 2n]\!], \forall q \in V, \forall p \in T_q^*M.$$

We also introduce the structural constants $(c_{ij}^k)_{i,j,k \in [\![0,2n]\!]}$ on V, defined by the relations

$$[X_i, X_j](q) = \sum_{k=0}^{2n} c_{ij}^k(q) X_k(q), \qquad \forall i, j \in [\![0, 2n]\!], \forall q \in V.$$

In terms of the coordinates $(h_i)_{i \in [\![0,2n]\!]}$, along a normal extremal, Equation (4.5) yields (see [ABB16, Chapter 4])

$$\frac{\mathrm{d}h_i}{\mathrm{d}t} = \{H, h_i\} = \sum_{j=0}^{2n} \sum_{k=0}^{2n} c_{ji}^k h_j h_k, \quad \forall i \in [\![0, 2n]\!].$$

We set $J : V \to \mathcal{M}_{2n}(\mathbb{R})$ to be the matrix such that

$$J_{ij} = c_{ji}^0, \quad \forall i, j \in [\![1, 2n]\!],$$

and $Q : V \longrightarrow (\mathbb{R}^{2n} \to \mathbb{R}^{2n})$ to be the map such that for all $i \in [\![1, 2n]\!]$,

$$Q_i(h_1, \ldots h_{2n}) = \sum_{j=1}^{2n} \sum_{k=1}^{2n} c_{ji}^k h_j h_k.$$

By denoting $h = (h_1, \ldots, h_{2n})$ we then have

$$\frac{\mathrm{d}h}{\mathrm{d}t} = h_0 J h + Q(h). \tag{4.6}$$

As stated in Section 4.1, we want an approximation of the geodesics for small time when $h_0(0) \to +\infty$, thus we introduce $w = \frac{h_0(0)}{h_0}$ and $\eta = h_0(0)^{-1}$. Then

$$\frac{\mathrm{d}w}{\mathrm{d}t} = -\eta w^2 \frac{\mathrm{d}h_0}{\mathrm{d}t}.$$

We separate the terms containing h_0 in the derivative of w to obtain an equation in the form of (4.6). We set $L : V \to \mathcal{M}_{1\times 2n}(\mathbb{R})$ to be the line matrix such that

$$L_i = c_{i0}^0, \quad \forall i \in [\![1, 2n]\!],$$

and $Q_0 : V \longrightarrow (\mathbb{R}^{2n} \to \mathbb{R})$ to be the map such that

$$Q_0(h_1, \dots h_{2n}) = \sum_{j=1}^{2n} \sum_{k=1}^{2n} c_{j0}^k h_j h_k,$$

so that

$$\frac{\mathrm{d}w}{\mathrm{d}t} = -wLh - \eta w^2 Q_0(h).$$

Finally, rescaling time with $\tau = t/\eta$, we obtain

$$\begin{cases} \dfrac{\mathrm{d}q}{\mathrm{d}\tau} = \eta \displaystyle\sum_{i=1}^{2n} h_i X_i(q), \\ \dfrac{\mathrm{d}h}{\mathrm{d}\tau} = \dfrac{1}{w} Jh + \eta\, Q(h), \\ \dfrac{\mathrm{d}w}{\mathrm{d}\tau} = -\eta w L h - \eta^2 w^2 Q_0(h). \end{cases} \tag{4.7}$$

Hence to the solution of (4.5) with initial condition $(q_0, (h(0), \eta^{-1}))$ corresponds the solution of the parameter depending differential equation (4.7) of initial condition $(q_0, h(0), w(0))$ and parameter η. Since $w(0) = 1$, the flow of this ODE is well defined (at least for τ small enough), and smooth with respect to $\eta \in (-\varepsilon, \varepsilon)$, for some $\varepsilon > 0$.

This warrants a study of its solutions as $\eta \to 0$.

4.2.2 Approximation of the Hamiltonian flow

Let $q_0 \in M$. In the rest of the paper, except when explicitly stated otherwise, we assume that the structure at q_0 has been put in the Agrachev–Gauthier normal form introduced in [AG01]. That is, we have an open neighborhood $V \subset M$ of q_0, linearly adapted coordinates at q_0 $(x_1, \dots x_{2n}, z) : V \to \mathbb{R}^{2n+1}$ and a frame of (Δ, g), $(X_1, \dots, X_{2n})$, satisfying Theorems 4.A.1-4.A.2 (see Appendix 4.A). The family is locally completed as a basis of TM with $X_0 = \frac{\partial}{\partial z}$.

Let us introduce a few notations. Let $\bar{J} = J(q_0)$. As a consequence of the choice of frame, in particular Equations (4.23) and (4.24), $\bar{J}$ is already in reduced form $\mathrm{diag}(\bar{J}_1, \dots, \bar{J}_n)$, that is, block diagonal with 2×2 blocks

$$\bar{J}_i = \begin{pmatrix} 0 & b_i \\ -b_i & 0 \end{pmatrix}, \qquad \forall i \in [\![1, n]\!],$$

where $(b_i)_{i\in[\![1,n]\!]}$, are the nilpotent invariants of the contact structure at q_0. Then define $\widehat{h}:\mathbb{R}\times\mathbb{R}^{2n}\to\mathbb{R}^{2n}$, $\widehat{x}:\mathbb{R}\times\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ and $\widehat{z}:\mathbb{R}\times\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ by

$$\begin{aligned}
\widehat{h}(t,h) &= \mathrm{e}^{t\bar{J}}h,\\
\widehat{x}(t,h) &= \bar{J}^{-1}(\mathrm{e}^{t\bar{J}}-I_{2n})h,\\
\widehat{z}(t,h) &= \sum_{i=1}^{n}\left(h_{2i-1}^2+h_{2i}^2\right)\frac{b_it-\sin(b_it)}{2b_i},
\end{aligned}$$

for all $t\in\mathbb{R}$ and all $h\in\mathbb{R}^{2n}$.

We also set $J^{(1)}:\mathbb{R}^{2n}\to\mathcal{M}_{2n}(\mathbb{R})$ such that

$$J^{(1)}_{i,j}(y)=\sum_{k=1}^{2n}\left(\frac{\partial^2(X_i)_{2n+1}}{\partial x_j\partial x_k}-\frac{\partial^2(X_j)_{2n+1}}{\partial x_i\partial x_k}\right)y_k,\qquad \forall i,j\in[\![1,2n]\!],$$

where for any vector field Y, we denote by $(Y)_i$, $1\leq i\leq 2n+1$, the i-th coordinate of Y, written in the basis $(\partial_{x_1},\dots,\partial_{x_{2n}},\partial_z)$.

Let us denote $B_R=\{h\in\mathbb{R}^{2n}\mid\sum_{i=1}^{2n}h_i^2\leq R\}$.

Proposition 4.2.1. *For all $T,R>0$, normal extremals with initial covector $(h(0),\eta^{-1})$ have the following order 2 expansion at time $\eta\tau$, as $\eta\to0^+$, uniformly with respect to $\tau\in[0,T]$ and $h(0)\in B_R$. In normal form coordinates, we denote*

$$\mathrm{e}^{\eta\tau\vec{H}}\left((0,0),\left(h(0),\eta^{-1}\right)\right)=\left((x(\tau),z(\tau)),\left(h(\tau),\eta w(\tau)^{-1}\right)\right).$$

Then

$$\begin{aligned}
x(\tau)&=\eta\widehat{x}(\tau,h(0))+\eta^2\int_0^\tau\int_0^\sigma \mathrm{e}^{(\sigma-\rho)\bar{J}}J^{(1)}\left(\widehat{x}(\rho,h(0))\right)\widehat{h}(\rho,h(0))\,\mathrm{d}\rho\,\mathrm{d}\sigma+O(\eta^3),\\
z(\tau)&=\eta^2\widehat{z}(\tau,h(0))+O(\eta^3),
\end{aligned}$$

and

$$\begin{aligned}
h(\tau)&=\widehat{h}(\tau,h(0))+\eta\int_0^\tau \mathrm{e}^{(\tau-\sigma)\bar{J}}J^{(1)}\left(\widehat{x}(\sigma,h(0))\right)\widehat{h}(\sigma,h(0))\,\mathrm{d}\sigma+O(\eta^2),\\
w(\tau)&=1+O(\eta^2).
\end{aligned}$$

Proof. This is a consequence of the integration of the time-rescaled system (4.7). Since the system smoothly depends on η near 0, we prove this result by successive integration of the terms of the power series in η of $x=\sum\eta^kx^{(k)}$, $z=\sum\eta^kz^{(k)}$, $h=\sum\eta^kh^{(k)}$, and $w=\sum\eta^kw^{(k)}$.

Let $T,R>0$. All asymptotic expressions are to be understood uniform with respect to $\tau\in[0,T)$ and $h(0)\in B_R$. Solutions of (4.7) are integral curves of a Hamiltonian vector field $\vec{H}$, hence H is preserved along the trajectory, that is, for all $\tau\in[0,T)$,

$$\sum_{i=1}^{2n}h_i(\tau)^2=\sum_{i=1}^{2n}h_i(0)^2.$$

Furthermore, we have by (4.7)

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = O(\eta), \quad \frac{\mathrm{d}z}{\mathrm{d}\tau} = O(\eta),$$

and since $x(0) = 0$ and $z(0) = 0$, we have $x(\tau) = O(\eta)$ and $z(\tau) = O(\eta)$.

As a consequence of the choice of frame, in particular conditions (4.23),(4.24), $c_{ij}^k(q_0) \neq 0$ if and only if $k = 0$ and there exists $l \in [\![1, n]\!]$ such that $\{i, j\} = \{2l-1, 2l\}$.

Hence for all $j \in [\![1, 2n]\!]$, $c_{j0}^0(q(\tau)) = O(\eta)$ and $Lh = O(\eta)$. Similarly, $Q_i(h) = O(\eta)$ for all $i \in [\![0, 2n]\!]$, and since $w(0) = 1$, we have that $\frac{\mathrm{d}w}{\mathrm{d}\tau} = O(\eta^2)$ and $w(\tau) = 1 + O(\eta^2)$.

Since $J(q_0) = \bar{J}$, we have $J(q) = \bar{J} + O(\eta)$ and thus

$$\frac{\mathrm{d}h}{\mathrm{d}\tau} = \bar{J}h + O(\eta).$$

Hence h is a small perturbation of the solution of $\frac{\mathrm{d}h}{\mathrm{d}\tau} = \bar{J}h$ with initial condition $h(0)$, that is, $h(\tau) = \widehat{h}(\tau, h(0)) + O(\eta)$.

Since $X_i(q_0) = \frac{\partial}{\partial x_i}$ for all $i \in [\![1, 2n]\!]$ (as a consequence of (4.21)),

$$\frac{\mathrm{d}x^{(1)}}{\mathrm{d}\tau} = h^{(0)}(\tau) = \widehat{h}(\tau, h(0)), \quad \frac{\mathrm{d}z^{(1)}}{\mathrm{d}\tau} = 0,$$

and since $x(0) = 0$, $z(0) = 0$, $x(\tau) = \eta\widehat{x}(\tau, h(0)) + O(\eta^2)$ and $z(\tau) = O(\eta^2)$.

The definition of $J^{(1)}$ implies

$$J^{(1)}\left(x^{(1)}\right) = \left.\frac{\partial J(q)}{\partial \eta}\right|_{\eta=0}.$$

Then, since $Q(h) = O(\eta)$, $h^{(1)}$ is solution of

$$\frac{\mathrm{d}h^{(1)}}{\mathrm{d}\tau} = \bar{J}h^{(1)} + J^{(1)}\left(x^{(1)}\right)$$

with initial condition $h^{(1)}(0) = 0$. Hence

$$h^{(1)}(\tau) = \int_0^\tau \mathrm{e}^{(\tau-\sigma)\bar{J}} J^{(1)}\left(\widehat{x}(\sigma, h(0))\right) \widehat{h}(\sigma, h(0))\, \mathrm{d}\sigma.$$

Since $\frac{\partial (X_i)_j}{\partial x_k} = 0$ for all $i, j, k \in [\![1, 2n]\!]$ (as stated in (4.22)),

$$X_{2i-1}(q(\tau)) = \partial_{x_{2i-1}} + \eta\, \widehat{x}_{2i}(\tau, h(0))\frac{b_i}{2}\partial_z + O(\eta^2),$$
$$X_{2i}(q(\tau)) = \partial_{x_{2i}} - \eta\, \widehat{x}_{2i-1}(\tau, h(0))\frac{b_i}{2}\partial_z + O(\eta^2).$$

Thus

$$\frac{\mathrm{d}x^{(2)}}{\mathrm{d}\tau} = h^{(1)}, \quad \frac{\mathrm{d}z^{(2)}}{\mathrm{d}\tau} = \sum_{i=1}^n \frac{b_i}{2}\left(\widehat{h}_{2i-1}\widehat{x}_{2i} - \widehat{h}_{2i}\widehat{x}_{2i-1}\right).$$

Hence the result by integration. □

4.3 Conjugate time

4.3.1 Singularities of the sub-Riemannian exponential

Definition 4.3.1. Let $q_0 \in M$. We call *sub-Riemannian exponential at* q_0 the map

$$\begin{array}{rccc} \mathcal{E}_{q_0}: & \mathbb{R}^+ \times T^*_{q_0}M & \longrightarrow & M \\ & (t, p_0) & \longmapsto & \mathcal{E}_{q_0}(t, p_0) = \pi \circ \mathrm{e}^{t\vec{H}}(p_0, q_0) \end{array}$$

where $\pi : T^*M \to M$ is the canonical fiber projection.

Recall that the flow of the Hamiltonian vector field $\vec{H}$ satisfies the equality

$$\mathrm{e}^{t\vec{H}}(p_0, q_0) = \mathrm{e}^{\vec{H}}(tp_0, q_0), \qquad \forall q_0 \in M, p_0 \in T^*_{q_0}M, t \in \mathbb{R}.$$

We use this property to our advantage to compute the sub-Riemannian caustic. Indeed, the caustic at q_0 is defined as the set of critical values of $\mathcal{E}_{q_0}(1, \cdot)$. But for any time $t > 0$, the caustic is also the set of critical values of $\mathcal{E}_{q_0}(t, \cdot)$. Hence instead of classifying the covectors p_0 such that $\mathcal{E}_{q_0}(1, \cdot)$ is critical at p_0, we compute for a given p_0 the conjugate time $t_c(p_0)$ such that $\mathcal{E}_{q_0}(t_c(p_0), \cdot)$ is critical at p_0.

Definition 4.3.2. Let $q_0 \in M$, and $p_0 \in T^*_{q_0}M$. A *conjugate time for* p_0 is a positive time $t > 0$ such that the map $\mathcal{E}_{q_0}(t, \cdot)$ is critical at p_0. The *conjugate locus of* q_0 is the subset of M

$$\left\{\mathcal{E}_{q_0}(t, p_0) \mid t \text{ is a conjugate time for } p_0 \in T_{q_0}M\right\}.$$

The *first conjugate time for* p_0, denoted $t_c(p_0)$, is the infimum of conjugate times for p_0. The *first conjugate locus of* q_0 is the subset of M

$$\left\{\mathcal{E}_{q_0}(t, p_0) \mid t \text{ is the first conjugate time for } p_0 \in T_{q_0}M\right\}.$$

In the following, we restrict our study of the sub-Riemannian caustic to the first conjugate locus.

From now on, let us index the nilpotent invariants in descending order $b_1 \geq b_2 \geq \cdots \geq b_n > 0$. Let $\mathfrak{S}_1 \subset M$ be the set of points of M such that $b_1 = b_2 = \max_{1 \leq k \leq n} b_k$. Assuming the sub-Riemannian structure on M is generic, $\mathfrak{S}_1$ is a stratified subset of M of codimension 3.

Remark 4.3.3. This is a consequence of Thom's transversality theorem. The sub-Riemannian structure can be seen as a smooth map on M. Under the assumption that the sub-Riemannian structure on M is generic, a submersion $\phi : \mathrm{Jet}^k\left(M, \mathbb{R}^l\right) \to \mathbb{R}^d$ on the jets of order k of the sub-Riemannian structure to $\mathbb{R}^d$ vanishes on a stratified subset of codimension d (that is, on the disjoint union of finitely many submanifolds of codimension at least d).

Furthermore, for a given $q_0 \in M$, if the sub-Riemannian structure at q_0 is in Agrachev–Gauthier normal form (see Appendix 4.A) then the jets of order k at q_0 of the sub-Riemannian structure are given by the jets at 0 of the vector fields $X_1, \dots, X_{2n}$.

In coordinates, conjugate points satisfy the following equality

$$\det\left(\frac{\partial \mathcal{E}_{q_0}}{\partial h_1}, \dots, \frac{\partial \mathcal{E}_{q_0}}{\partial h_{2n}}, \frac{\partial \mathcal{E}_{q_0}}{\partial h_0}\right)\Bigg|_{(t,p_0)} = 0. \tag{4.8}$$

To use this equation in relation with the results of Proposition 4.2.1, we introduce

$$F(\tau, h, \eta) = \mathcal{E}_{q_0}(\eta\tau; (h, \eta^{-1})), \quad \forall \tau > 0, h \in \mathbb{R}^{2n}, \eta > 0.$$

Then

$$\frac{\partial \mathcal{E}_{q_0}}{\partial h_i}(\eta\tau; (h, \eta^{-1})) = \frac{\partial F}{\partial h_i}(\tau, h, \eta), \quad \forall i \in [\![1, 2n]\!]$$

and

$$\frac{\partial \mathcal{E}_{q_0}}{\partial h_0}(\eta\tau; (h, \eta^{-1})) = -\eta\left(\eta\frac{\partial F}{\partial \eta}(\tau, h, \eta) - \tau\frac{\partial F}{\partial \tau}(\tau, h, \eta)\right).$$

Hence (4.8) equates to

$$\det\left(\frac{\partial F}{\partial h_1}, \dots, \frac{\partial F}{\partial h_{2n}}, \eta\frac{\partial F}{\partial \eta} - \tau\frac{\partial F}{\partial \tau}\right)\Bigg|_{(\tau,h,\eta)} = 0. \tag{4.9}$$

We have shown in Proposition 4.2.1 that the map F is a perturbation of the simpler map $(\widehat{x}, \widehat{z})$, the nilpotent exponential map. Hence the conjugate time is expected to be a perturbation of the conjugate time for $(\widehat{x}, \widehat{z})$. To get an approximation of the conjugate time for a covector (h, η^{-1}) as $\eta \to 0$, we use expansions from Proposition 4.2.1 to derive equations on a power series expansion of the conjugate time.

4.3.2 Nilpotent order and doubling of the conjugate time

Let us define

$$\Phi(\tau, h, \eta) = \det\left(\frac{\partial F}{\partial h_1}, \dots, \frac{\partial F}{\partial h_{2n}}, \eta\frac{\partial F}{\partial \eta} - \tau\frac{\partial F}{\partial \tau}\right)\Bigg|_{(\tau,h,\eta)} \tag{4.10}$$

and its power series expansion

$$\Phi(\tau, h, \eta) = \sum_{k \geq 0} \eta^k \Phi^{(k)}(\tau, h).$$

As a first application of Proposition 4.2.1, notice that $F_i = O(\eta)$ for all $i \in [\![1, 2n]\!]$, while $F_{2n+1} = O(\eta^2)$. Hence, one gets $\Phi^{(k)} = 0$ for all $k \in [\![0, 2n+1]\!]$, and $\Phi^{(2n+2)}$ is the first non-trivial term in the power series.

To study $\Phi^{(2n+2)}$, let us introduce the set $Z = \{2k\pi/b_i \mid i \in [\![1,n]\!], k \in \mathbb{N}\}$ and the map $\psi : (\mathbb{R}^+ \setminus Z) \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$\psi(\tau, r) = \sum_{i=1}^{n} \frac{r_i^2}{2} \left(3\tau - b_i\tau^2 \frac{\cos(b_i\tau/2)}{\sin(b_i\tau/2)} - \frac{\sin(b_i\tau)}{b_i} \right), \quad \forall (\tau, r) \in (\mathbb{R}^+ \setminus Z) \times \mathbb{R}^n.$$

We first need the following result on the zeros of ψ.

Lemma 4.3.4. *Assume $b_1 > b_2 \geq \cdots \geq b_n$. For all $r \in (\mathbb{R}^+)^n$, let $\tau_1(r)$ be the first positive time in $\mathbb{R}^+ \setminus Z$ such that $\psi(\tau_1, r) = 0$. Then $\tau_1(r_1, \ldots, r_n) > 2\pi/b_1$ and there exists $f(r_2, \ldots, r_n) > 0$ such that, as $r_1 \to 0^+$,*

$$\tau_1(r_1, \ldots, r_n) = 2\pi/b_1 + f(r_2, \ldots, r_n)r_1^2 + o(r_1^2). \tag{4.11}$$

Proof. Let $T = \min(2\pi/b_2, 4\pi/b_1)$, so that $(2\pi/b_1, T)$ is a connected component of $\mathbb{R}^+ \setminus Z$. For all $i \in [\![1,n]\!]$, let

$$\begin{array}{rccc} \psi_i : & \mathbb{R} \setminus \cup_{k\in\mathbb{N}}\{2k\pi/b_i\} & \longrightarrow & \mathbb{R} \\ & \tau & \longmapsto & 3\tau - b_i\tau^2 \frac{\cos(b_i\tau/2)}{\sin(b_i\tau/2)} - \frac{\sin(b_i\tau)}{b_i}. \end{array}$$

For all $i \in [\![1,n]\!]$, ψ_i is smooth and has a positive derivative over $\mathbb{R} \setminus \cup_{k\in\mathbb{N}}\{2k\pi/b_i\}$. Moreover $\psi_i(0) = 0$, and for all $k \in \mathbb{N}$, $k > 0$,

$$\lim_{t \to 2k\pi/b_i^+} \psi_i(t) = -\infty \quad \text{and} \quad \lim_{t \to 2k\pi/b_i^-} \psi_i(t) = +\infty.$$

This immediately implies that $\tau_1(r) > 2\pi/b_1$. Furthermore, since $\psi(\tau, r) = \sum_{i=1}^n r_i^2\psi_i(\tau)$, for all $r \in (\mathbb{R}^+)^n$

$$\lim_{t \to 2\pi/b_1^+} \psi(\tau, r) = -\infty \quad \text{and} \quad \lim_{t \to T^-} \psi(\tau, r) = +\infty$$

and $\psi(\cdot, r)$ vanishes exactly once on $(2\pi/b_1, T)$ at time $\tau_1(r)$. Since for all $i \in [\![2,n]\!]$, $\psi_i > 0$ on $(2\pi/b_1, T)$, we have that

$$\psi_1(\tau_1(r)) = -\frac{1}{r_1^2} \sum_{i=2}^{n} r_i^2 \psi_i(\tau_1(r)) < 0.$$

This equality implies that $r_1 \mapsto \tau_1(r)$ is an increasing function. Indeed let $r, r' \in (\mathbb{R}^+)^n$ be such that $r_1 < r_1'$ and $r_i = r_i'$ for all $i \in [\![2,n]\!]$, then for all $\tau \in (2\pi/b_1, T)$,

$$-\frac{1}{r_1^2} \sum_{i=2}^{n} r_i^2 \psi_i(\tau) < -\frac{1}{{r_1'}^2} \sum_{i=2}^{n} {r_i'}^2 \psi_i(\tau) < 0.$$

Since $\tau \mapsto -\frac{1}{r_1^2} \sum_{i=2}^n r_i^2 \psi_i$ and $\tau \mapsto -\frac{1}{{r_1'}^2} \sum_{i=2}^n {r_i'}^2 \psi_i$ are both decreasing functions over $(2\pi/b_1, T)$, since ψ_1 is an increasing function over $(2\pi/b_1, T)$, this implies $\tau_1(r) < \tau_1(r')$.

In particular, τ_1 being continuous, it converges towards a limit $l(r_2,\dots r_n) \in [2\pi/b_1, T)$ as $r_1 \to 0^+$, and

$$\lim_{r_1\to 0^+} \sum_{i=2}^n {r_i}^2 \psi_i(\tau_1(r)) = \sum_{i=2}^n {r_i}^2 \psi_i(l(r_2,\dots r_n)) > 0.$$

Hence $\lim_{r_1\to 0^+} \psi_1(\tau_1(r_1,\dots,r_n)) = -\infty$, and by inverting ψ_1 we obtain

$$\lim_{r_1\to 0^+} \tau_1(r_1,\dots,r_n) = 2\pi/b_1.$$

Notice in particular that as $\delta t \to 0^+$, $\psi_1(2\pi/b_1+\delta t) \sim -\frac{8\pi^2}{b_1^2 \delta t}$. Hence we get by inverting ψ_1

$$\psi_1^{-1}\left(-\frac{1}{r_1^2}\sum_{i=2}^n r_i^2\psi_i(\tau_1(r))\right) - 2\pi/b_1 \sim \frac{8\pi^2}{b_1^2 \sum_{i=2}^n r_i^2 \psi_i(2\pi/b_1))} r_1^2,$$

hence expansion (4.11). □

The zeros of $\Phi^{(2n+2)}$ can be deduced from the zeros of ψ, as shown in the following proposition (see also Figure 4.1).

Proposition 4.3.5. *Assume $b_1 > b_2 \geq \dots \geq b_n$. Let $h \in \mathbb{R}^{2n}\setminus\{0\}$ and $r \in \mathbb{R}^n$ be such that $r_i = \sqrt{h_{2i-1}^2 + h_{2i}^2}$ for all $i \in [\![1,n]\!]$. Then $\Phi^{(2n+2)}(\tau,h) = 0$ if and only if $\tau \in Z$ or $\psi(\tau,r) = 0$. In particular*

$$\Phi^{(2n+2)}(\tau,h) \neq 0 \quad \forall \tau \in (0, 2\pi/b_1), \forall h \in \mathbb{R}^{2n}\setminus\{0\}.$$

Proof. By factorizing powers of η in Φ, we obtain that $\Phi^{(2n+2)}$ is given by the determinant of the matrix

$$M = \left(\begin{array}{c|c} D_h\widehat{x}(\tau) & \widehat{x}(\tau) - \tau\widehat{h}(\tau) \\ \hline D_h\widehat{z}(\tau) & \widehat{z}(\tau) - \tau\frac{\mathrm{d}}{\mathrm{d}\tau}\widehat{z}(\tau) \end{array}\right).$$

The Jacobian matrix $D_h\widehat{x} = \bar{J}^{-1}(\mathrm{e}^{\tau\bar{J}} - I_{2n})$ is invertible for $\tau \in \mathbb{R}^+\setminus Z$ and of rank $2n-2$ for $\tau \in Z$. Hence, the matrix M is not invertible for $\tau \in \mathbb{R}^+\setminus Z$ if and only of we have the linear dependance of the family

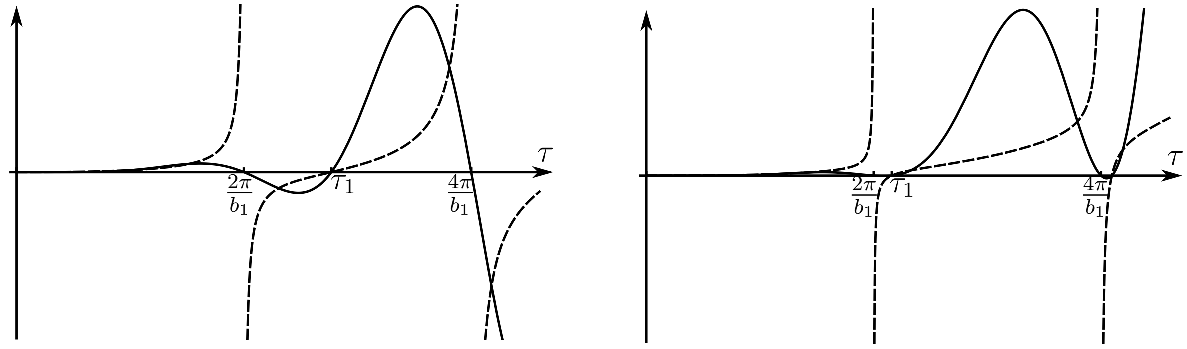
$$\left\{\frac{\partial}{\partial h_1}\left(\frac{\widehat{x}(\tau)}{\widehat{z}(\tau)}\right), \dots, \frac{\partial}{\partial h_{2n}}\left(\frac{\widehat{x}(\tau)}{\widehat{z}(\tau)}\right), \left(\frac{\widehat{x}(\tau) - \tau\widehat{h}(\tau)}{\widehat{z}(\tau) - \tau\frac{\mathrm{d}}{\mathrm{d}\tau}\widehat{z}(\tau)}\right)\right\}.$$

This implies the existence of $\mu \in \mathbb{R}^{2n}$ such that both $D_h\widehat{x}(\tau)\mu = \widehat{x}(\tau) - \tau\widehat{h}(\tau)$ and $D_h\widehat{z}(\tau)\mu = \widehat{z}(\tau) - \tau\frac{\mathrm{d}}{\mathrm{d}\tau}\widehat{z}(\tau)$. That is

$$D_h\widehat{z}(\tau)\left(D_h\widehat{x}(\tau)\right)^{-1}\left(\widehat{x}(\tau) - \tau\widehat{h}(\tau)\right) = \widehat{z}(\tau) - \tau\frac{\mathrm{d}}{\mathrm{d}\tau}\widehat{z}(\tau).$$

We explicitly have $\widehat{z}(\tau) - \tau\frac{\mathrm{d}}{\mathrm{d}\tau}\widehat{z}(\tau) = \sum_{i=1}^n \frac{r_i^2}{2}\left(\tau\cos b_i\tau - \frac{\sin b_i\tau}{b_i}\right)$ and

$$D_h\widehat{z}(\tau)\left(D_h\widehat{x}(\tau)\right)^{-1}\left(\widehat{x}(\tau) - \tau\widehat{h}(\tau)\right) = \sum_{i=1}^n r_i^2\left(\sin b_i\tau - b_i\tau\right)\frac{b_i\tau\cos(b_i\tau/2) - 2\sin(b_i\tau/2)}{2b_i\sin(b_i\tau/2)}.$$



(a) $r_1 = r_2$ (b) $r_1 = \dfrac{r_2}{10}$.

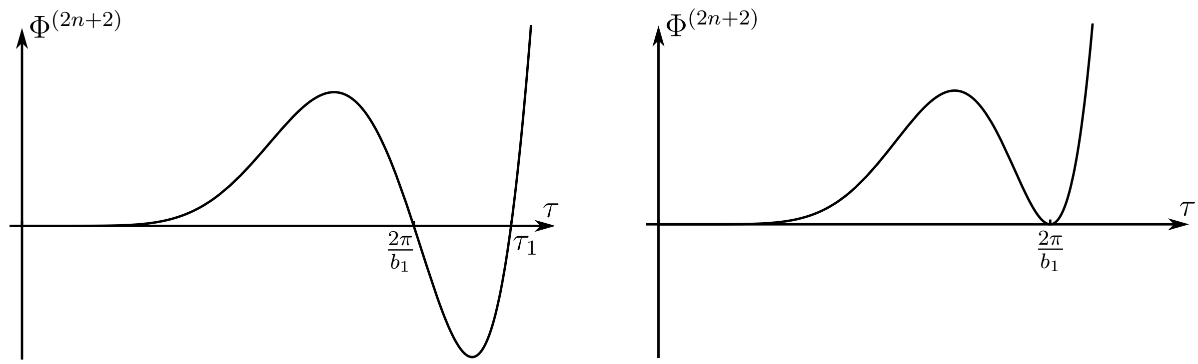
Figure 4.1 – Representation of $\Phi^{(2n+2)}$ (plain) and ψ (dashed) as functions of τ in the case $n = 2$ for two values of r_1/r_2 (with $b_1 = 2$, $b_2 = 1/4$ and $r_2 = 1$).

Hence

$$D_h\widehat{z}\,(D_h\widehat{x})^{-1}\left(\widehat{x} - \tau\widehat{h}\right) - \left(\widehat{z} - \tau\frac{\mathrm{d}\widehat{z}}{\mathrm{d}\tau}\right) = \sum_{i=1}^{n}\frac{r_i^2}{2}\left(3\tau - b_i\tau^2\frac{\cos(b_i\tau/2)}{\sin(b_i\tau/2)} - \frac{\sin(b_i\tau)}{b_i}\right) = \psi(\tau, r).$$

Thus times $\tau \in \mathbb{R}^+$ such that $\Phi_k^{(2n+2)}(\tau, h) = 0$ are either multiples of $2\pi b_i$, $i \in [\![1, n]\!]$, or zeros of ψ. Under the assumption that $h \in \mathbb{R}^{2n} \setminus \{0\}$ and $\tau \in (0, 2b_i\pi)$, we have $\psi(\tau, r) > 0$, hence the result. $\square$

We can draw some conclusions regarding the coming stability analysis of the conjugate locus. From Proposition 4.3.5, we have that $2\pi/b_1$ is the first zero of $\Phi^{(2n+2)}(\cdot, h)$ for all $h \in \mathbb{R}^{2n} \setminus \{0\}$. From Lemma 4.3.4 we also know that $2\pi/b_1$ is a simple zero if $r_1 > 0$ and a double zero otherwise (see Figure 4.2). Zeros of order larger than 1 can be unstable under perturbation and this case requires a separate analysis, either by high order approximation or by blowup. We choose the latter for computational reasons.



(a) $\Phi^{(2n+2)}$ as $r_1 = \dfrac{r_2}{4}$. (b) $\Phi^{(2n+2)}$ as $r_1 = 0$.

Figure 4.2 – Representation of $\Phi^{(2n+2)}$ as a function of τ in the case $n = 2$, as $r_1 \neq 0$ and $r_1 = 0$ (with $b_1 = 2$, $b_2 = 1/4$ and $r_2 = 1$).

From Equation (4.11) in Lemma 4.3.4, we have that the blowup $r_1 \leftarrow \eta^\alpha r_1$ corresponds to

$$\tau_1(\eta^\alpha r_1, r_2, \ldots, r_n) = 2\pi/b_1 + \eta^{2\alpha} f(r_2, \ldots, r_n) r_1^2 + o(\eta^{2\alpha}).$$

Since we have an approximation of the exponential that is a perturbation of order η of the nilpotent exponential, we expect the conjugate time to be a perturbation of order η of the nilpotent conjugate time. Hence it is natural to chose $\alpha = 1/2$ to obtain a perturbation of comparable order in η.

We separate the cases in the following way.

— We can compute the conjugate time assuming $r_1 > \varepsilon$ for some arbitrary ε (in Section 4.4.1);

— we use the blowup $r_1 \leftarrow \sqrt{\eta} r_1$ to get the conjugate time near $r_1 = 0$ (in Section 4.4.2).

4.3.3 Statement of the conjugate time asymptotics

The focus of this chapter is now devoted to the proof of the following asymptotic expansion theorem for the conjugate time on $M \setminus \mathfrak{S}_1$. Let S_1 be the subspace of $T_{q_0}^*M$ defined by

$$S_1 = \left\{(h_1, \ldots, h_{2n}, h_0) \in T_{q_0}^*M \setminus \mathcal{C}_{q_0}(0) \mid h_1 = h_2 = 0, H \neq 0\right\},$$

and for all $\varepsilon > 0$, let us denote by S_1^ε the subset of $T_{q_0}^*M$ containing S_1:

$$S_1^\varepsilon = \left\{(h_1, \ldots, h_{2n}, h_0) \in T_{q_0}^*M \setminus \mathcal{C}_{q_0}(0) \mid h_1^2 + h_2^2 < \varepsilon H(h_1, \ldots, h_{2n}, h_0)\right\}.$$

Theorem 4.3.6. *Let $q_0 \in M \setminus \mathfrak{S}_1$. There exist real valued structural invariants α,β, $(\kappa_{i,j}^k)_{\substack{i,k\in[\![1,2]\!],\\ j\in[\![3,2n]\!]}}$, such that we have the following asymptotic behavior for initial covectors $p_0 \in T_{q_0}^*M$ with $h_0 \to +\infty$.*

1. *(Away from S_1.) For all $R > 0, \varepsilon \in (0,1)$, uniformly with respect to $p_0 = (h_1, \ldots, h_{2n}, h_0) \in \mathcal{C}_{q_0}((0,R)) \setminus S_1^\varepsilon$, we have as $h_0 \to +\infty$*

$$t_c(h_1, \ldots, h_{2n}, h_0) = \frac{2\pi}{b_1 h_0} + \frac{1}{h_0^2} t_c^{(2)}(h_1, \ldots, h_{2n}) + O\left(\frac{1}{h_0^3}\right)$$

where $t_c^{(2)}$ satisfies

$$(h_1^2 + h_2^2) t_c^{(2)}(h) = -2(\alpha h_1 + \beta h_2)\left(h_1^2 + h_2^2\right) + (\gamma_{12} + \gamma_{21}) h_1 h_2 - \gamma_{22} h_1^2 - \gamma_{11} h_2^2, \quad (4.12)$$

denoting

$$\gamma_{ij} = \sum_{k=3}^{2n} \kappa_{ik}^j h_k, \qquad \forall i,j \in [\![1,2n]\!].$$

2. *(Near S_1.) The asymptotic expansion*

$$t_c\left(\frac{h_1}{\sqrt{h_0}}, \frac{h_2}{\sqrt{h_0}}, h_3, \ldots, h_{2n}, h_0\right) = \frac{2\pi}{b_1 h_0} + O\left(\frac{1}{h_0^2}\right)$$

holds if and only if the quadratic polynomial equation in X

$$X^2 K' - X\left[\frac{2\pi}{b_1}(h_1^2+h_2^2) - K'(\gamma_{11}+\gamma_{22})\right] + \frac{2\pi}{b_1}\left[(\gamma_{12}+\gamma_{21})h_1h_2 - \gamma_{22}h_1^2 - \gamma_{11}h_2^2\right] + K'(\gamma_{11}\gamma_{22}-\gamma_{12}\gamma_{21}) = 0 \quad (4.13)$$

admits a real solution, where

$$K' = \sum_{i=2}^{n}(h_{2i-1}^2+h_{2i}^2)\left(1 - \frac{b_i}{b_1}\pi\cot\frac{b_i\pi}{b_1}\right) > 0.$$

If that is the case, denote by $\tilde{t}_c^{(2)}(h_1,\ldots,h_{2n})$ the smallest of its two (possibly double) solutions. Then, for all $R>0, \varepsilon\in(0,1)$, uniformly with respect to $p_0 = \left(\frac{h_1}{\sqrt{h_0}}, \frac{h_2}{\sqrt{h_0}}, h_3, \ldots, h_{2n}, h_0\right) \in \mathcal{C}_{q_0}((0,R)) \cap S_1^\varepsilon$, we have

$$t_c\left(\frac{h_1}{\sqrt{h_0}}, \frac{h_2}{\sqrt{h_0}}, h_3, \ldots, h_{2n}, h_0\right) = \frac{2\pi}{b_1 h_0} + \frac{1}{h_0^2}\tilde{t}_c^{(2)}(h_1,\ldots,h_{2n}) + O\left(\frac{1}{h_0^3}\right).$$

Corollary 4.3.7. *Let $q_0 \in M \setminus \mathfrak{S}_1$. There exist real valued structural invariants α,β, $(\kappa_{i,j}^k)_{\substack{k\in[\![1,2]\!],\\ i,j\in[\![3,2n]\!]}}$, such that we have the following asymptotic expansion for the conjugate locus as $\eta\to 0^+$, uniformly with respect to $p_0 = (h_1,\ldots,h_{2n},\eta^{-1}) \in \mathcal{C}_q((0,R))\setminus S_1^\varepsilon$ for all $R>0, \varepsilon\in(0,1)$. We have for all $i\in[\![2,n]\!]$ (in normal form coordinates)*

$$(\mathcal{E}_{q_0}(t_c(p_0),p_0))_1 = \frac{\eta^2}{h_1^2+h_2^2}\left((\gamma_{11}-\gamma_{22})h_1^3 + \gamma_{21}h_2^3 + (\gamma_{12}+2\gamma_{21})h_1^2h_2 + d_1\right) + O(\eta^3),$$

$$(\mathcal{E}_{q_0}(t_c(p_0),p_0))_2 = \frac{\eta^2}{h_1^2+h_2^2}\left(\gamma_{12}h_1^3 - (\gamma_{11}-\gamma_{22})h_2^3 + (\gamma_{21}+2\gamma_{12})h_1h_2^2 + d_2\right) + O(\eta^3),$$

$$(\mathcal{E}_{q_0}(t_c(p_0),p_0))_{2i-1} = \frac{\eta}{b_i}\left[\sin\left(\frac{2b_i\pi}{b_1}\right)h_{2i-1} + \left(1-\cos\left(\frac{2b_i\pi}{b_1}\right)\right)h_{2i}\right] + O(\eta^2),$$

$$(\mathcal{E}_{q_0}(t_c(p_0),p_0))_{2i} = \frac{\eta}{b_i}\left[\left(\cos\left(\frac{2b_i\pi}{b_1}\right)-1\right)h_{2i-1} + \sin\left(\frac{2b_i\pi}{b_1}\right)h_{2i}\right] + O(\eta^2),$$

$$(\mathcal{E}_{q_0}(t_c(p_0),p_0))_{2n+1} = \eta^2\sum_{i=1}^{n}\left(h_{2i-1}^2+h_{2i}^2\right)\left(\frac{\pi}{b_1} - \frac{\sin(2\pi b_i/b_1)}{2b_i}\right) + O(\eta^3),$$

where we denoted

$$d_1 = \alpha(h_1^2+h_2^2)^2 + \sum_{3\le i<j\le 2n}^{2n}\kappa_{ij}^1 h_ih_j, \qquad d_2 = \beta(h_1^2+h_2^2)^2 + \sum_{3\le i<j\le 2n}^{2n}\kappa_{ij}^1 h_ih_j.$$

4.4 Perturbations of the conjugate time

Thanks to the previous section, we have a sufficiently precise picture of the behavior of the conjugate time for the nilpotent approximation. We now introduce small perturbations of the exponential map in accordance with Proposition 4.2.1. As stated previously, we treat separately the case of initial covectors away from S_1 and near S_1 since S_1 corresponds to the set of covectors such that $r_1 = \sqrt{h_1^2 + h_2^2} = 0$. Recall also that we assumed $q_0 \in M \setminus \mathfrak{S}_1$.

However, rather than computing t_c, we compute $\tau_c = t_c/\eta$, the rescaled conjugate time, since we use asymptotics in rescaled time from Proposition 4.2.1.

4.4.1 Asymptotics for covectors in $T^*_{q_0}M \setminus S_1$

In this section we assume that $(h_1, h_2) \neq (0,0)$. Recall that

$$F(\tau, h, \eta) = \mathcal{E}(\eta\tau; (h, \eta^{-1})), \quad \forall \tau > 0, h \in \mathbb{R}^{2n}, \eta > 0.$$

The function F admits a power series expansion

$$F(\tau, h, \eta) = \sum_{k \geq 0} \eta^k F^{(k)}(\tau, h),$$

and for $\delta t \in \mathbb{R}$, $h \in \mathbb{R}^{2n}$, evaluating F at the perturbed conjugate time $\frac{2\pi}{b_1} + \eta\delta t$ yields

$$F\left(\frac{2\pi}{b_1} + \eta\delta t, h, \eta\right) = \eta\, F^{(1)}\Big|_{\tau=\frac{2\pi}{b_1}} + \eta^2 \left[F^{(2)} + \delta t \frac{\partial F^{(1)}}{\partial \tau}\right]\Bigg|_{\tau=\frac{2\pi}{b_1}} + O(\eta^3). \tag{4.14}$$

In the previous section, we highlighted the role of the function Φ defined by (4.10). We have the following statement on the first nontrivial term of the expansion of Φ at times near $2\pi/b_1$.

Lemma 4.4.1. *We have*

$$\Phi\left(2\pi/b_1 + \eta\delta t, h, \eta\right) = \eta^{2n+3} K d + O(\eta^{2n+4}),$$

where

$$K = 2^{2n-2} \prod_{i=2}^{n} \frac{1}{b_i^2} \sin^2\left(\frac{\pi b_i}{b_1}\right) > 0$$

and

$$d = \begin{vmatrix} \frac{\partial}{\partial h_1}\left(F^{(2)} + \delta t \frac{\partial F^{(1)}}{\partial \tau}\right)_1 & \frac{\partial}{\partial h_2}\left(F^{(2)} + \delta t \frac{\partial F^{(1)}}{\partial \tau}\right)_1 & -\tau\left(\frac{\partial F^{(1)}}{\partial \tau}\right)_1 \\ \frac{\partial}{\partial h_1}\left(F^{(2)} + \delta t \frac{\partial F^{(1)}}{\partial \tau}\right)_2 & \frac{\partial}{\partial h_2}\left(F^{(2)} + \delta t \frac{\partial F^{(1)}}{\partial \tau}\right)_2 & -\tau\left(\frac{\partial F^{(1)}}{\partial \tau}\right)_2 \\ \frac{\partial}{\partial h_1}\left(F^{(2)}\right)_{2n+1} & \frac{\partial}{\partial h_2}\left(F^{(2)}\right)_{2n+1} & 0 \end{vmatrix}_{\tau=2\pi/b_1}.$$

Proof. From Proposition 4.2.1, we have that $F^{(1)}\left(\frac{2\pi}{b_1}, h\right) = \left(\widehat{x}\left(\frac{2\pi}{b_1}, h\right), 0\right)$, with $\widehat{x}_1\left(\frac{2\pi}{b_1}, h\right) = \widehat{x}_2\left(\frac{2\pi}{b_1}, h\right) = 0$. Furthermore, observe that

$$\begin{pmatrix} \widehat{x}_3(2\pi/b_1, h) \\ \vdots \\ \widehat{x}_{2n}(2\pi/b_1, h) \end{pmatrix} = A \begin{pmatrix} h_3 \\ \vdots \\ h_{2n} \end{pmatrix}$$

where $A \in \mathcal{M}_{2n-2}(\mathbb{R})$ is the block-diagonal matrix $\operatorname{diag}(A_2, \dots, A_n)$ of 2×2 blocks

$$A_i = \frac{1}{b_i} \begin{pmatrix} \sin(\frac{2b_i\pi}{b_1}) & 1 - \cos(\frac{2b_i\pi}{b_1}) \\ \cos(\frac{2b_i\pi}{b_1}) - & \sin(\frac{2b_i\pi}{b_1}) \end{pmatrix}, \qquad \forall i \in [\![2, n]\!].$$

Thus Equation (4.14) entails, by factorizing η,

$$\Phi\left(2\pi/b_1 + \eta\delta t, h, \eta\right) = \\ \eta^{2n+3} \left|\begin{array}{cc|c|c} \frac{\partial}{\partial h_1}\left(F^{(2)} + \delta t \frac{\partial F^{(1)}}{\partial \tau}\right)_1 & \frac{\partial}{\partial h_2}\left(F^{(2)} + \delta t \frac{\partial F^{(1)}}{\partial \tau}\right)_1 & 0 \cdots 0 & -\tau\left(\frac{\partial F^{(1)}}{\partial \tau}\right)_1 \\ \frac{\partial}{\partial h_1}\left(F^{(2)} + \delta t \frac{\partial F^{(1)}}{\partial \tau}\right)_2 & \frac{\partial}{\partial h_2}\left(F^{(2)} + \delta t \frac{\partial F^{(1)}}{\partial \tau}\right)_2 & 0 \cdots 0 & -\tau\left(\frac{\partial F^{(1)}}{\partial \tau}\right)_2 \\ \hline 0 & 0 & & 0 \\ \vdots & \vdots & A & \vdots \\ 0 & 0 & & 0 \\ \hline \frac{\partial}{\partial h_1}\left(F^{(2)}\right)_{2n+1} & \frac{\partial}{\partial h_2}\left(F^{(2)}\right)_{2n+1} & 0 \cdots 0 & 0 \end{array}\right|_{\tau = 2\pi/b_1} \\ + O(\eta^{2n+4}).$$

From Lemma 4.C.1 in Appendix 4.C, $\det(A) = K$ and we have the stated result. $\square$

We now apply this lemma to solve Equation (4.9), that is, $\Phi = 0$, after observing that τ_c must annihilate every term in the expansion of $\Phi(\tau_c)$.

Proposition 4.4.2. *Let $\tau_c(h, \eta) = \sum_{k=0}^{+\infty} \eta^k \tau_c^{(k)}(h)$ be the formal power series expansion of τ_c, for all $(h, \eta^{-1}) \in T_{q_0}^* M$. Then $\tau_c^{(0)} = 2\pi/b_1$ and $\tau_c^{(1)}$ must satisfy*

$$(h_1^2 + h_2^2)\tau_c^{(1)}(h) = -h_1^2 \frac{\partial\left(F^{(2)}\right)_2}{\partial h_2} - h_2^2 \frac{\partial\left(F^{(2)}\right)_1}{\partial h_1} + h_1 h_2 \left(\frac{\partial\left(F^{(2)}\right)_1}{\partial h_2} + \frac{\partial\left(F^{(2)}\right)_2}{\partial h_1}\right). \quad (4.15)$$

Proof. This is a consequence of Proposition 4.2.1 and Lemma 4.4.1. By noticing that $\left(F^{(2)}\right)_{2n+1} = \widehat{z}$, $\frac{\partial F^{(1)}}{\partial \tau} = \widehat{h}$, and that $\partial_{h_1}\widehat{z} = 2\pi h_1/b_1$, $\partial_{h_2}\widehat{z} = 2\pi h_2/b_1$, we get the stated result by solving for δt

$$\left|\begin{array}{ccc} \frac{\partial}{\partial h_1}\left(F^{(2)}\right)_1 + \delta t & \frac{\partial}{\partial h_2}\left(F^{(2)}\right)_1 & h_1 \\ \frac{\partial}{\partial h_1}\left(F^{(2)}\right)_2 & \frac{\partial}{\partial h_2}\left(F^{(2)}\right)_2 + \delta t & h_2 \\ h_1 & h_2 & 0 \end{array}\right|_{\tau = 2\pi/b_1} = 0.$$

$\square$

Remark 4.4.3. The relation (4.15) is degenerate at $r_1 = 0$. This is another illustration of the behavior we highlighted in the previous section, that is, $\tau_c^{(1)}$ is a zero of order 2 at $r_1 = 0$.

As a consequence of Proposition 4.2.1, it appears that for all $k \in [\![1, 2n]\!]$ and all $\tau > 0$, each function $h \mapsto x_k^{(2)}(\tau)$ can be seen as a quadratic form on $(h_1, \dots, h_{2n})$. Hence we introduce the structural invariants $(\kappa_{i,j}^k)_{i,j,k \in [\![1,2n]\!]}$ such that

$$F_k^{(2)}\left(\frac{2\pi}{b_1}, h\right) = \sum_{1 \le i \le j \le 2n} \kappa_{i,j}^k h_i h_j \qquad \forall k \in [\![1, 2n]\!].$$

As a consequence of Lemmas 4.B.1 through 4.B.4 in Appendix 4.B, we have the following result.

Proposition 4.4.4. *The structural invariants $(\kappa_{i,j}^k)_{i,j,k \in [\![1,2n]\!]}$ linearly depend on the family*

$$\left(\frac{\partial^2 (X_i)_{2n+1}}{\partial x_j \partial x_k}(q_0)\right)_{i,j,k \in [\![1,2n]\!]}.$$

There exist $\alpha, \beta \in \mathbb{R}$ such that we have the symmetries

$$\begin{aligned}
\kappa_{1,1}^1 &= 3\alpha, & \kappa_{1,1}^2 &= \beta, \\
\kappa_{2,2}^1 &= \alpha, & \kappa_{2,2}^2 &= 3\beta, \\
\kappa_{1,2}^1 &= 2\beta, & \kappa_{1,2}^2 &= 2\alpha,
\end{aligned}$$

and for all $i \in [\![2, n]\!]$, $\left(\kappa_{k,l}^m\right)_{\substack{k,m \in [\![1,2]\!] \\ l \in [\![2i-1,2i]\!]}}$ only depend on the family

$$\left\{\left(\frac{\partial^2 (X_k)_{2n+1}}{\partial x_l \partial x_m}(q_0)\right) \mid (k, l, m) \in [\![2i-1, 2i]\!] \times [\![1, 2]\!]^2 \cup [\![1, 2]\!]^2 \times [\![2i-1, 2i]\!]\right\}.$$

Furthermore, the corresponding linear map $\zeta_i : \mathbb{R}^{15} \to \mathbb{R}^8$ such that

$$\zeta_i\left(\left(\frac{\partial^2 (X_k)_{2n+1}}{\partial x_l \partial x_m}(q_0)\right)_{k,l,m \in \{1,2\} \cup \{2i-1,2i\}}\right) = \left(\kappa_{k,l}^m\right)_{\substack{k,m \in \{1,2\} \\ l \in \{2i-1,2i\}}}$$

is of rank at least 7 (and of rank 8 on the complementary of a codimension 1 subset $\mathfrak{S}_3$ of M).

Remark 4.4.5. A consequence of the rank of ζ_i being 7, for all $2 \le i \le n$, is that a single condition of codimension $k \ge 2$ on $\left(\kappa_{k,l}^m\right)_{\substack{k,m \in [\![1,2]\!] \\ l \in [\![2i-1,2i]\!]}}$ is then a condition of codimension at least $k - 1$ on the jets of order 2 of the sub-Riemannian structure at q_0.

Using this notation, we can give a first approximation of the conjugate locus.

Proposition 4.4.6. *Let $q_0 \in M \setminus \mathfrak{S}_1$. As $\eta \to 0^+$, uniformly with respect to $p_0 = (h_1, \dots, h_{2n}, \eta^{-1}) \in \mathcal{C}_q((0,R)) \setminus S_1^\varepsilon$ for all $R > 0, \varepsilon \in (0,1)$, we have (in normal form coordinates)*

$$(F(\tau_c(h,\eta)), h, \eta))_1 = \frac{\eta^2}{h_1^2 + h_2^2}\big((\gamma_{11} - \gamma_{22})h_1^3 + \gamma_{21}h_2^3 + (\gamma_{12} + 2\gamma_{21})h_1^2 h_2 + d_1\big) + O(\eta^3)$$
$$(F(\tau_c(h,\eta)), h, \eta))_2 = \frac{\eta^2}{h_1^2 + h_2^2}\big(\gamma_{12}h_1^3 - (\gamma_{11} - \gamma_{22})h_2^3 + (\gamma_{21} + 2\gamma_{12})h_1 h_2^2 + d_2\big) + O(\eta^3)$$

with

$$\gamma_{ij} = \sum_{k=3}^{2n} \kappa_{ik}^j h_k, \qquad \forall i, j \in [\![1, 2n]\!],$$

and

$$d_1 = \alpha(h_1^2 + h_2^2)^2 + \sum_{3 \le i < j \le 2n}^{2n} \kappa_{ij}^1 h_i h_j, \qquad d_2 = \beta(h_1^2 + h_2^2)^2 + \sum_{3 \le i < j \le 2n}^{2n} \kappa_{ij}^1 h_i h_j.$$

If there exists a covector such that $\gamma_{11} - \gamma_{22} = \gamma_{12} = \gamma_{21} = 0$ then this first order approximation of the conjugate locus is not sufficient to prove stability and more orders of approximation are necessary. This occurs for instance when $h_3 = \dots = h_{2n} = 0$, and

$$(F(\tau_c(h,\eta)), h, \eta))_1 = \eta^2 \alpha(h_1^2 + h_2^2) + O(\eta^3),$$
$$(F(\tau_c(h,\eta)), h, \eta))_2 = \eta^2 \beta(h_1^2 + h_2^2) + O(\eta^3).$$

Proposition 4.4.7. *Let M be a generic contact sub-Riemannian manifold of dimension $2n+1 \ge 5$. Let $\mathfrak{S}_2 \subset M$ be the set of points at which the linear system in $(h_3, \dots h_{2n})$*

$$\begin{cases} \sum_{i=3}^{2n} (\kappa_{1,i}^1 - \kappa_{2,i}^2) h_i = 0, \\ \sum_{i=3}^{2n} \kappa_{1,i}^2 h_i = 0, \\ \sum_{i=3}^{2n} \kappa_{2,i}^1 h_i = 0, \end{cases}$$

admits non-trivial solutions. If $\dim M \ge 7$, then $M = \mathfrak{S}_2$. However if $\dim M = 5$, the set $\mathfrak{S}_2$ is codimension 1 stratified subset of M.

Proof. If we assume $(r_2, \dots, r_n) \neq 0$ then $\gamma_{11} - \gamma_{22} = \gamma_{12} = \gamma_{21} = 0$ reduces to the existence of a non-zero vector of $\mathbb{R}^{2n-2}$ in the intersection

$$\mathrm{Span}\{(\kappa_{1,3}^1 - \kappa_{2,3}^2, \dots, \kappa_{1,2n}^1 - \kappa_{2,2n}^2)\}^\perp \cap \mathrm{Span}\{(\kappa_{1,3}^2, \dots, \kappa_{1,2n}^2)\}^\perp \cap \mathrm{Span}\{(\kappa_{2,3}^1, \dots, \kappa_{2,2n}^1)\}^\perp.$$

This space is never reduced to a single point for $n > 2$, hence $M = \mathfrak{S}_2$. However for $n = 2$, this requires the three vectors

$$(\kappa_{1,3}^1 - \kappa_{2,3}^2, \kappa_{1,4}^1 - \kappa_{2,4}^2), \quad (\kappa_{1,3}^2, \kappa_{1,4}^2), \quad (\kappa_{2,3}^1, \kappa_{2,4}^1), \tag{4.16}$$

to be co-linear, which is a constraint of codimension 2 on the family $\left(\kappa_{k,l}^m\right)_{\substack{k,m \in \{1,2\} \\ l \in \{3,4\}}}$. By Remark 4.4.5, this is a codimension 1 (at least) constraint on the jets of order 2 of the sub-Riemannian structure at q_0, hence the result. $\square$

4.4.2 Asymptotics for covectors near S_1

We reiterate the previous construction for a special class of initial covector in the vicinity of $S_1 = \{(h_1, \dots, h_{2n}, h_0) \in T^*_{q_0}M \mid h_1 = h_2 = 0\}$, in accordance with the discussion in Section 4.3.2.

Let $\bar{h} \in \mathbb{R}^{2n}$ be such that $(\bar{h}_1, \bar{h}_2) \neq (0,0)$ and $(\bar{h}_3, \dots, \bar{h}_{2n}) \neq (0, \dots, 0)$. We blowup the singularity at $h_1 = h_2 = 0$ by computing an approximation of the conjugate locus for

$$h(0) = (\sqrt{\eta}\bar{h}_1, \sqrt{\eta}\bar{h}_2, \bar{h}_3, \dots, \bar{h}_{2n}). \tag{4.17}$$

Let Λ be the square $2n \times 2n$ matrix such that

$$\Lambda_{i,j} = \begin{cases} 1 & \text{if } i = j = 1 \text{ or } i = j = 2, \\ 0 & \text{otherwise,} \end{cases}$$

so that $h(0) = \sqrt{\eta}\Lambda\bar{h} + (I_{2n} - \Lambda)\bar{h}$.

Recall the power series notation $f(\eta\tau, h(0)) = \sum \eta^k f^{(k)}(\tau, h(0))$. As a consequence of Proposition 4.2.1, we can give a new expansion of the Hamiltonian flow for the special class of initial covectors of type (4.17) in terms of coefficients of the power series of x, z, h, w. (Recall for all $R > 0$, B_R denotes the set $\{h \in \mathbb{R}^{2n} \mid \sum_{i=1}^{2n} h_i^2 \leq R\}$.)

Proposition 4.4.8. *For all $T, R > 0$, normal extremals with initial covector*

$$(\sqrt{\eta}\Lambda\bar{h} + (I_{2n} - \Lambda)\bar{h}, \eta^{-1})$$

have the following order 3 expansion at time $\eta\tau$, as $\eta \to 0^+$, uniformly with respect to $\tau \in [0, T]$ and $h(0) \in B_R$:

$$\begin{aligned} x(\eta\tau) = \eta\widehat{x}(\tau, (I_{2n} - \Lambda)\bar{h}) + \eta^{3/2}\left[\widehat{x}\left(\tau, \Lambda\bar{h}\right)\right] + \eta^2\left[x^{(2)}\left(\tau, (I_{2n} - \Lambda)\bar{h}\right)\right] \\ + \eta^{5/2}\left[x^{(2)}\left(\tau, \bar{h}\right) - x^{(2)}\left(\tau, (I_{2n} - \Lambda)\bar{h}\right) - x^{(2)}\left(\tau, \Lambda\bar{h}\right)\right] + O(\eta^3), \end{aligned}$$

$$z(\eta\tau) = \eta^2\widehat{z}(\tau, (I_{2n} - \Lambda)\bar{h}) + \eta^3\left[z^{(3)}(\tau, (I_{2n} - \Lambda)\bar{h}) + \widehat{z}(\tau, \Lambda\bar{h})\right] + O(\eta^4).$$

Likewise, the associated covector has the expansion

$$\begin{aligned} h(\eta\tau) = \widehat{h}(\tau, (I_{2n} - \Lambda)\bar{h}) + \sqrt{\eta}\left[\widehat{h}(\tau, \Lambda\bar{h})\right] + \eta\left[h^{(1)}(\tau, (I_{2n} - \Lambda)\bar{h})\right] \\ + \eta^{3/2}\left[h^{(1)}(\tau, \bar{h}) - h^{(1)}(\tau, \Lambda\bar{h}) - h^{(1)}(\tau, (I_{2n} - \Lambda)\bar{h})\right] + O(\eta^2), \end{aligned}$$

$$w(\eta\tau) = 1 + O(\eta^2).$$

Proof. For a linear map $\phi : \mathbb{R}^{2n} \to \mathbb{R}^m$ we have

$$\phi(\Lambda\bar{h} + \sqrt{\eta}(I_{2n} - \Lambda)\bar{h}) = \phi(\Lambda\bar{h}) + \sqrt{\eta}\phi((I_{2n} - \Lambda)\bar{h}),$$

and for a quadratic form $\psi : \mathbb{R}^{2n} \to \mathbb{R}$ we have by polarization identity

$$\psi\left(\Lambda\bar{h} + \sqrt{\eta}(I_{2n} - \Lambda)\bar{h}\right) = \psi(\Lambda\bar{h}) + \sqrt{\eta}\left[\psi(\bar{h}) - \psi(\Lambda\bar{h}) - \psi((I_{2n} - \Lambda)\bar{h})\right] \\ + \eta\psi((I_{2n} - \Lambda)\bar{h}).$$

Hence the result since we proved in Proposition 4.2.1 that $x^{(1)}(\eta\tau, \cdot)$, $h^{(0)}(\eta\tau, \cdot)$ are linear and $x^{(2)}(\eta\tau, \cdot)$, $h^{(1)}(\eta\tau, \cdot)$, $z^{(2)}(\eta\tau, \cdot)$ are quadratic, coordinate-wise. The case of w comes from the fact that $w^{(1)} = 0$. □

We set

$$G(\tau, \bar{h}, \eta) = F\left(\tau, \sqrt{\eta}\Lambda\bar{h} + (I_{2n} - \Lambda)\bar{h}, \eta\right), \quad \forall \tau > 0, \bar{h} \in \mathbb{R}^{2n}, \eta > 0.$$

The function G admits a power series expansion in $\sqrt{\eta}$

$$G(\tau, \bar{h}, \eta) = \sum_{k \geq 0} \eta^{k/2} G^{(k/2)}(\tau, \bar{h}).$$

We first check that the conjugate time is not a perturbation of order $\sqrt{\eta}$ of the nilpotent conjugate time $2\pi/b_1$, that is, $\tau_c(\sqrt{\eta}\Lambda\bar{h} + (I_{2n} - \Lambda)\bar{h}) = 2\pi/b_1 + O(\eta)$. With $\delta t \in \mathbb{R}$, $\bar{h} \in \mathbb{R}^{2n}$, we have

$$G\left(\frac{2\pi}{b_1} + \sqrt{\eta}\delta t, \bar{h}, \eta\right) = \eta\, G^{(1)}\big|_{\tau = \frac{2\pi}{b_1}} + \eta^{3/2}\left(G^{(3/2)} + \delta t \frac{\partial G^{(1)}}{\partial \tau}\right)\bigg|_{\tau = \frac{2\pi}{b_1}} + O(\eta^{5/2}). \quad (4.18)$$

Recall that we set

$$\Phi(\tau, h, \eta) = \det\left(\frac{\partial F}{\partial h_1}, \ldots, \frac{\partial F}{\partial h_{2n}}, \eta\frac{\partial F}{\partial \eta} - \tau\frac{\partial F}{\partial \tau}\right)\bigg|_{(\tau, h, \eta)}.$$

To evaluate $\Phi\left(2\pi/b_1 + \sqrt{\eta}\delta t, \sqrt{\eta}\Lambda\bar{h} + (I_{2n} - \Lambda)\bar{h}, \eta\right)$, $\delta t \in \mathbb{R}$, $\bar{h} \in \mathbb{R}^{2n}$, notice that

$$\frac{\partial F}{\partial h_i} = \frac{1}{\sqrt{\eta}}\frac{\partial G}{\partial \bar{h}_i}, \quad \forall i \in [\![1, 2]\!], \quad \text{and} \quad \frac{\partial F}{\partial h_i} = \frac{\partial G}{\partial \bar{h}_i} \quad \forall i \in [\![3, 2n]\!].$$

Then for all $i \in [\![1, 2n]\!]$, we set $C_i = \dfrac{\partial G}{\partial \bar{h}_i}$ and $C_{2n+1} = \eta\dfrac{\partial G}{\partial \eta} - \tau\dfrac{\partial G}{\partial \tau}$, evaluated at time $\tau = 2\pi/b_1 + \sqrt{\eta}\delta t$. For all $i \in [\![1, 2n+1]\!]$, the vector $C_i \in \mathbb{R}^{2n+1}$ also admits a power series expansion in $\sqrt{\eta}$,

$$C_i = \sum_{k=0}^{\infty} \eta^{k/2} C_i^{(k/2)}.$$

Notice that by definition of $(C_i)_{i \in [\![1, 2n+1]\!]}$ we have $C_i^{(0)} = C_i^{(1/2)} = C_i^{(1)} = 0$ for all $i \in [\![1, 2n]\!]$. As a consequence we can obtain an equation satisfied by a potential perturbation of order $1/2$ of the nilpotent conjugate time.

Lemma 4.4.9. *Let*

$$K = 2^{2n-2}\prod_{i=2}^{n}\frac{1}{b_i^2}\sin^2\left(\frac{b_i\pi}{b_1}\right) > 0, \qquad K' = \sum_{i=2}^{n}(h_{2i-1}^2 + h_{2i}^2)\left(1 - \frac{b_i}{b_1}\pi\cot\frac{b_i\pi}{b_1}\right) > 0.$$

We have

$$\Phi\left(2\pi/b_1 + \sqrt{\eta}\delta t, \sqrt{\eta}\Lambda\bar{h} + (I_{2n} - \Lambda)\bar{h}, \eta\right) = -\frac{2\pi}{b_1^2}\eta^{2n+4}KK'\delta t^2 + o(\eta^{2n+4}).$$

As a consequence, $\tau_c^{(1/2)} = 0$.

Proof. From Proposition 4.4.8, we get that neither $G^{(1)}$ nor $G^{(2)}$ depend on (h_1, h_2), hence from expression (4.18) we deduce

$$C_1 = \eta^2\delta t_{1/2}\partial_{\bar{h}_1}\partial_\tau G^{(3/2)} + O(\eta^{5/2}), \qquad C_2 = \delta t_{1/2}\partial_{\bar{h}_2}\partial_\tau G^{(3/2)} + O(\eta^{5/2}).$$

Hence, evaluating Φ at $\left(2\pi/b_1 + \sqrt{\eta}\delta t, \sqrt{\eta}\Lambda\bar{h} + (I_{2n} - \Lambda)\bar{h}, \eta\right)$ and eliminating higher order terms, there exist $V, W \in \mathcal{M}_{1\times 2n-2}(\mathbb{R})$, $v \in \mathbb{R}$ such that

$$\Phi = \eta^{2n+4}\left|\begin{array}{cc|c|c} \delta t & 0 & 0\cdots 0 & 0 \\ 0 & \delta t & 0\cdots 0 & 0 \\ \hline 0 & 0 & & \\ \vdots & \vdots & A & W \\ 0 & 0 & & \\ \hline 0 & 0 & {}^tV & v \end{array}\right|_{\tau=2\pi/b_1} + o(\eta^{2n+4}).$$

Recall that $\det(A) = K$ (see Lemma 4.C.1). To get the statement, let us show that

$$\left|\begin{array}{c|c} A & W \\ \hline {}^tV & v \end{array}\right| = -\frac{2\pi}{b_1^2}KK'.$$

From Lemma 4.C.2 in Appendix 4.C, we have

$$\frac{1}{\det(A)}\left|\begin{array}{c|c} A & W \\ \hline {}^tV & v \end{array}\right| = v + \frac{1}{2}\sum_{i=2}^{n} b_i\left(V_{2i-1}W_{2i} - V_{2i}W_{2i-1} - (V_{2i-1}W_{2i-1} + V_{2i}W_{2i})\cot\frac{b_i\pi}{b_1}\right).$$

In our case, for all $i \in [\![2, n]\!]$, $(V_{2i-3}, V_{2i-2}) = (\bar{h}_{2i-1}, \bar{h}_{2i})\left(\frac{2\pi}{b_1} - \frac{1}{b_i}\sin\left(\frac{2b_i\pi}{b_1}\right)\right)$ and

$$\begin{pmatrix} W_{2i-3} \\ W_{2i-2} \end{pmatrix} = \begin{pmatrix} \frac{1}{b_i}\sin\frac{2\pi b_i}{b_1} - \frac{2\pi}{b_1}\cos\frac{2\pi b_i}{b_1} & \frac{1}{b_i} - \frac{2\pi}{b_1}\sin\frac{2\pi b_i}{b_1} - \frac{1}{b_i}\cos\frac{2\pi b_i}{b_1} \\ \frac{2\pi}{b_1}\sin\frac{2\pi b_i}{b_1} + \frac{1}{b_i}\cos\frac{2\pi b_i}{b_1} - \frac{1}{b_i} & \frac{1}{b_i}\sin\frac{2\pi b_i}{b_1} - \frac{2\pi}{b_1}\cos\frac{2\pi b_i}{b_1} \end{pmatrix}\begin{pmatrix} \bar{h}_{2i-1} \\ \bar{h}_{2i} \end{pmatrix}.$$

Finally, $v = \sum_{i=2}^{n}\left(\bar{h}_{2i-1}^2 + \bar{h}_{2i}^2\right)\left(\frac{2\pi}{b_1}\cos^2\frac{\pi b_i}{b_1} - \frac{1}{b_i}\sin\frac{2\pi b_i}{b_1}\right)$, hence the result by summation. $\square$

Computing the perturbation of the conjugate time is then a matter of computing

$$\Phi\left(2\pi/b_1 + \eta\delta t, \sqrt{\eta}\Lambda\bar{h} + (I_{2n} - \Lambda)\bar{h}, \eta\right).$$

Regarding G, we have

$$\begin{aligned} G\left(\frac{2\pi}{b_1} + \eta\delta t, \bar{h}, \eta\right) =& \eta\, G^{(1)}\big|_{\tau=\frac{2\pi}{b_1}} + \eta^{3/2}\, G^{(3/2)}\big|_{\tau=\frac{2\pi}{b_1}} + \eta^2 \left[G^{(2)} + \delta t \frac{\partial G^{(1)}}{\partial \tau}\right]\Bigg|_{\tau=\frac{2\pi}{b_1}} \\ &+ \eta^{5/2}\left[G^{(5/2)} + \delta t \frac{\partial G^{(3/2)}}{\partial \tau}\right]\Bigg|_{\tau=\frac{2\pi}{b_1}} + O(\eta^3). \end{aligned} \tag{4.19}$$

Proposition 4.4.10. *We have*

$$\Phi\left(2\pi/b_1 + \eta\delta t, h, \eta\right) = \eta^{2n+5} K\left(d - \frac{2\pi}{b_1^2} K' d'\right) + o(\eta^{2n+5}),$$

where

$$d = \left|\begin{matrix} \frac{\partial}{\partial \bar{h}_1}\left(G^{(5/2)} + \delta t \frac{\partial G^{(3/2)}}{\partial \tau}\right)_1 & \frac{\partial}{\partial \bar{h}_2}\left(G^{(5/2)} + \delta t \frac{\partial G^{(3/2)}}{\partial \tau}\right)_1 & -\tau\left(\frac{\partial G^{(3/2)}}{\partial \tau}\right)_1 \\ \frac{\partial}{\partial \bar{h}_1}\left(G^{(5/2)} + \delta t \frac{\partial G^{(3/2)}}{\partial \tau}\right)_2 & \frac{\partial}{\partial \bar{h}_2}\left(G^{(5/2)} + \delta t \frac{\partial G^{(3/2)}}{\partial \tau}\right)_2 & -\tau\left(\frac{\partial G^{(3/2)}}{\partial \tau}\right)_2 \\ \frac{\partial}{\partial \bar{h}_1}\left(G^{(3)}\right)_{2n+1} & \frac{\partial}{\partial \bar{h}_2}\left(G^{(3)}\right)_{2n+1} & 0 \end{matrix}\right|_{\tau=2\pi/b_1}$$

and

$$d' = \left|\begin{matrix} \frac{\partial}{\partial \bar{h}_1}\left(G^{(2)} + \delta t \frac{\partial G^{(1)}}{\partial \tau}\right)_1 & \frac{\partial}{\partial \bar{h}_2}\left(G^{(2)} + \delta t \frac{\partial G^{(1)}}{\partial \tau}\right)_1 \\ \frac{\partial}{\partial \bar{h}_1}\left(G^{(2)} + \delta t \frac{\partial G^{(1)}}{\partial \tau}\right)_2 & \frac{\partial}{\partial \bar{h}_2}\left(G^{(2)} + \delta t \frac{\partial G^{(1)}}{\partial \tau}\right)_2 \end{matrix}\right|_{\tau=2\pi/b_1}.$$

Proof. The proof is similar to that of Lemma 4.4.9. From Proposition 4.4.8 and expression (4.19) we deduce

$$C_1 = \eta^{5/2}\partial_{\bar{h}_1}\left(G^{(5/2)} + \delta t \partial_\tau G^{(3/2)}\right) + O(\eta^3), \qquad C_2 = \eta^{5/2}\partial_{\bar{h}_2}\left(G^{(5/2)} + \delta t \partial_\tau G^{(3/2)}\right) + O(\eta^3).$$

Again, similarly to Lemma 4.4.9, evaluating Φ at $\left(2\pi/b_1 + \eta\delta t, \sqrt{\eta}\Lambda\bar{h} + (I_{2n} - \Lambda)\bar{h}, \eta\right)$ and eliminating higher order terms, there exist $V, W \in \mathcal{M}_{1\times 2n-2}(\mathbb{R})$, $v \in \mathbb{R}$ such that at

$\tau = 2\pi/b_1$,

$$\Phi = \eta^{2n+5}\left|\begin{array}{cc|c|c}
\frac{\partial}{\partial \bar{h}_1}\left(G^{(5/2)} + \delta t\frac{\partial G^{(3/2)}}{\partial \tau}\right)_1 & \frac{\partial}{\partial \bar{h}_2}\left(G^{(5/2)} + \delta t\frac{\partial G^{(3/2)}}{\partial \tau}\right)_1 & 0\cdots 0 & 0 \\
\frac{\partial}{\partial \bar{h}_1}\left(G^{(5/2)} + \delta t\frac{\partial G^{(3/2)}}{\partial \tau}\right)_2 & \frac{\partial}{\partial \bar{h}_2}\left(G^{(5/2)} + \delta t\frac{\partial G^{(3/2)}}{\partial \tau}\right)_2 & 0\cdots 0 & 0 \\
\hline
0 & 0 & & \\
\vdots & \vdots & A & W \\
0 & 0 & & \\
\hline
0 & 0 & {}^tV & v
\end{array}\right|$$

$$+\, \eta^{2n+5}\left|\begin{array}{cc|c|c}
\frac{\partial}{\partial \bar{h}_1}\left(G^{(5/2)} + \delta t\frac{\partial G^{(3/2)}}{\partial \tau}\right)_1 & \frac{\partial}{\partial \bar{h}_2}\left(G^{(5/2)} + \delta t\frac{\partial G^{(3/2)}}{\partial \tau}\right)_1 & 0\cdots 0 & -\tau\left(\frac{\partial G^{(3/2)}}{\partial \tau}\right)_1 \\
\frac{\partial}{\partial \bar{h}_1}\left(G^{(5/2)} + \delta t\frac{\partial G^{(3/2)}}{\partial \tau}\right)_2 & \frac{\partial}{\partial \bar{h}_2}\left(G^{(5/2)} + \delta t\frac{\partial G^{(3/2)}}{\partial \tau}\right)_2 & 0\cdots 0 & -\tau\left(\frac{\partial G^{(3/2)}}{\partial \tau}\right)_2 \\
\hline
0 & 0 & & 0 \\
\vdots & \vdots & A & \vdots \\
0 & 0 & & 0 \\
\hline
\frac{\partial}{\partial \bar{h}_1}\left(G^{(3)}\right)_{2n+1} & \frac{\partial}{\partial \bar{h}_2}\left(G^{(3)}\right)_{2n+1} & 0\cdots 0 & 0
\end{array}\right| + o(\eta^{2n+5}).$$

Hence the result since $\det(A) = K$ and $\left|\begin{array}{c|c} A & W \\ \hline {}^tV & v \end{array}\right| = -\frac{2\pi}{b_1^2}KK'$ (as showed in the proof of Lemma 4.4.9). $\square$

Now we have enough information to state the equation satisfied by the perturbation of the conjugate time in terms of the power expansion of G, similarly to Proposition 4.4.2. As a consequence of the accumulation of the second nilpotent conjugate time to $2\pi/b_1$ for initial covectors near S_1, the conjugate time is now solution of a quadratic equation, as explained in Section 4.3.2.

Proposition 4.4.11. *$\tau_c^{(1)}\left(\sqrt{\eta}\Lambda\bar{h} + (I_{2n} - \Lambda)\bar{h}\right)$ is solution of the following quadratic equation in δt:*

$$\begin{aligned}
-\,\delta t^2 K' + \delta t\left(\frac{2\pi}{b_1}\left(\bar{h}_1^2 + \bar{h}_2^2\right) - K'\left(\frac{\partial G_1^{(5/2)}}{\partial \bar{h}_1} + \frac{\partial G_2^{(5/2)}}{\partial \bar{h}_2}\right)\right) \\
+ \frac{2\pi}{b_1}\left(\bar{h}_2^2\frac{\partial G_1^{(5/2)}}{\partial \bar{h}_1} + \bar{h}_1^2\frac{\partial G_2^{(5/2)}}{\partial \bar{h}_2} - \bar{h}_1\bar{h}_2\left(\frac{\partial G_2^{(5/2)}}{\partial \bar{h}_1} + \frac{\partial G_1^{(5/2)}}{\partial \bar{h}_2}\right)\right) \\
+ K'\left(\frac{\partial G_2^{(5/2)}}{\partial \bar{h}_1}\frac{\partial G_1^{(5/2)}}{\partial \bar{h}_2} - \frac{\partial G_1^{(5/2)}}{\partial \bar{h}_1}\frac{\partial G_2^{(5/2)}}{\partial \bar{h}_2}\right) = 0. \quad (4.20)
\end{aligned}$$

Let

$$\Delta(\bar h) = \left(\frac{2\pi}{b_1}\left(\bar h_1^2+\bar h_2^2\right) - K'\left(\frac{\partial G_1^{(5/2)}}{\partial \bar h_1} + \frac{\partial G_2^{(5/2)}}{\partial \bar h_2}\right)\right)^2 + \\ 4K'\left(\frac{2\pi}{b_1}\left(\bar h_2^2\frac{\partial G_1^{(5/2)}}{\partial \bar h_1} + \bar h_1^2\frac{\partial G_2^{(5/2)}}{\partial \bar h_2} - \bar h_1\bar h_2\left(\frac{\partial G_2^{(5/2)}}{\partial \bar h_1} + \frac{\partial G_1^{(5/2)}}{\partial \bar h_2}\right)\right) + \right. \\ \left. K'\left(\frac{\partial G_2^{(5/2)}}{\partial \bar h_1}\frac{\partial G_1^{(5/2)}}{\partial \bar h_2} - \frac{\partial G_1^{(5/2)}}{\partial \bar h_1}\frac{\partial G_2^{(5/2)}}{\partial \bar h_2}\right)\right).$$

We have the following cases:

- *if* $\Delta(\bar h) > 0$, $\tau_c^{(1)}$ *is the smallest of the two solutions* (4.20)*;*
- *if* $\Delta(\bar h) = 0$, $\tau_c^{(1)}$ *is the double solution of* (4.20)*;*
- *if* $\Delta(\bar h) < 0$, *the first conjugate time is not a perturbation of* $2\pi/b_1$*.*

Proof. This is a direct application Proposition 4.4.10. By computing explicitly $d\frac{b_1^2}{2\pi} - K'd' = 0$, we obtain (4.20). The discriminant analysis of (4.20) implies that either there exist conjugate times that are perturbations of $2\pi/b_1$, and the smallest of the two is the first conjugate time, or the system does not admit a perturbation of $2\pi/b_1$ as a first conjugate time after being perturbed. □

Remark 4.4.12. Contrarily to (4.15), Equation (4.20) not degenerate at $\bar h_1 = \bar h_2 = 0$.

4.4.3 Proof of Theorems 4.1.1 and 4.3.6

It appears now that proving Theorem 4.3.6 is a matter of summarizing what we know about the conjugate time from the previous results of Section 4.4.

Proof of Theorem 4.3.6. In the previous section we computed the rescaled conjugate time τ_c. We have for all covector $p_0 = (\bar h_1, \dots, \bar h_{2n}, \eta^{-1}) \in T^*_{q_0}M$,

$$t_c(\bar h, \eta^{-1}) = \eta\tau_c(\bar h, \eta^{-1})$$

From Proposition 4.3.5, we deduce that under the assumption $(\bar h_1, \bar h_2) \neq (0,0)$, we have as $\eta \to 0^+$ that $\tau_c(\bar h, \eta^{-1}) = 2\pi/b_1 + O(\eta)$. From Proposition 4.4.2, we deduce the existence of $t_c^{(2)} = \eta\tau_c^{(1)}$ that satisfies the given equation, using the structural invariants introduced in Proposition 4.4.4.

On the other hand, by performing the blow up at $(0, 0, \bar h_3 \dots, \bar h_{2n})$, we compute an approximation of

$$t_c(\sqrt{\eta}\bar h_1, \sqrt{\eta}\bar h_2, \bar h_3, \dots, \bar h_{2n}, \eta^{-1}) = \eta\tau_c(\sqrt{\eta}\bar h_1, \sqrt{\eta}\bar h_2, \bar h_3, \dots, \bar h_{2n}, \eta^{-1}).$$

Again, from Proposition 4.3.5, we deduce that under the assumption $(\bar{h}_1, \bar{h}_2) \neq (0,0)$, a possible approximation is $\tau_c(\sqrt{\eta}\Lambda\bar{h} + (I_{2n} - \Lambda)\bar{h}, \eta^{-1}) = 2\pi/b_1 + O(\eta)$. However from Lemma 4.3.4, we now know that in the nilpotent case, $2\pi/b_1$ is a zero of order two at $(\bar{h}_1, \bar{h}_2) = (0,0)$. Thus computing a perturbation of the conjugate time, one gets the statement for $\tilde{t}_c^{(2)}$ from Proposition 4.4.11 and the expression in terms of invariants from Proposition 4.4.8. $\square$

Likewise, the proof of Corollary 4.3.7 is straightforward.

Proof of Corollary 4.3.7. By applying Proposition 4.2.1 at time $\tau = 2\pi/b_1$, one gets the statement for $(\mathcal{E}_{q_0}(t_c(p_0); p_0))_i$ for $3 \leq i \leq 2n$. The statement for $(\mathcal{E}_{q_0}(t_c(p_0); p_0))_1$, $(\mathcal{E}_{q_0}(t_c(p_0); p_0))_2$, is then given by Proposition 4.4.6. $\square$

Having proved Theorem 4.3.6, we can introduce a geometrical invariant that will help us prove Theorem 4.1.1. For all $q \in M \setminus \mathfrak{S}_1$, let

$$\mathcal{A}_q = \overline{\{t_c(p)p \mid H(p,q) = 1/2\}}.$$

By the usual property of the Hamiltonian flow, the first conjugate locus at q is given by $\mathcal{E}_q(1, \mathcal{A}_q)$. Furthermore, the set $\mathcal{A}_q$ is an immersed hypersurface of T_q^*M and $\mathcal{A}_q \cap \mathcal{C}_q(0)$ is reduced to the two points $p^+ = (0, \ldots, 0, 2\pi/b_1)$, $p^- = (0, \ldots, 0, -2\pi/b_1)$. Then let $\bar{\mathcal{A}}_q$ be the tangent cone to $\mathcal{A}_q$ at p^+.

Observe that $\bar{\mathcal{A}}_q$ is a geometrical invariant independent of the choice of coordinates on M. It can be computed once the asymptotics of the conjugate time are known.

Proof of Theorem 4.1.1. We prove the theorem by contradiction. Assume there exists a set of coordinates for which (4.4) does not hold, *i.e.*

$$\lim_{h_0 \to +\infty} \left(h_0^2 \sup_{\tau \in (0,T)} \left| \mathcal{E}_q\left(\frac{\tau}{h_0}, (h_1, \ldots, h_{2n}, h_0)\right) - \widehat{\mathcal{E}}_q\left(\frac{\tau}{h_0}, (h_1, \ldots, h_{2n}, h_0)\right) \right| \right) = 0.$$

Then we have that uniformly with respect to $\tau \in (0,T)$,

$$\mathcal{E}_q\left(\eta\tau, (h_1, \ldots, \bar{h}_{2n}, \eta^{-1})\right) = \widehat{\mathcal{E}}_q\left(\eta\tau, (h_1, \ldots, \bar{h}_{2n}, \eta^{-1})\right) + o(\eta^2).$$

That is, the exponential is a second order perturbation of the nilpotent exponential. If that is the case, as a consequence of Section 4.4, and in particular Proposition 4.4.2, we have that for $p_0 = (h_1, \ldots, h_{2n}, \eta^{-1}) \in T_q^*M$,

$$t_c(p_0) = \frac{2\pi}{b_1}\eta + o(\eta^2).$$

Then

$$t_c(p_0)p_0 = \left(0, \ldots, 0, \frac{2\pi}{b_1}\right) + \eta\left(\frac{2\pi}{b_1}h_1, \ldots, \frac{2\pi}{b_1}h_{2n}, 0\right) + o(\eta)$$

and the cone $\bar{\mathcal{A}}_q$ is the affine plane $(0,\dots,0,2\pi/b_1)+\{h_0=0\}$.

However, as a consequence of Theorem 4.3.6, the cone $\bar{\mathcal{A}}_q$ can be computed using the Agrachev–Gauthier frame, where we have for $p_0=(h_1,\dots,h_{2n},\eta^{-1})\in T_q^*M\setminus S$,

$$t_c(p_0)p_0=\left(0,\dots,0,\frac{2\pi}{b_1}\right)+\eta\left(\frac{2\pi}{b_1}h_1,\dots,\frac{2\pi}{b_1}h_{2n},t_c^{(2)}(h_1,\dots,h_{2n})\right)+o(\eta).$$

For $\bar{\mathcal{A}}_q$ to be planar, the following symmetry for $t_c^{(2)}$ is needed:

$$\lim_{r_1\to0^+}t_c^{(2)}(r_1\cos\theta_1,r_1\sin\theta_1,h_3,\dots,h_{2n})=-\lim_{r_1\to0^+}t_c^{(2)}(-r_1\cos\theta_1,-r_1\sin\theta_1,h_3,\dots,h_{2n})$$

for all $(h_3,\dots,h_{2n})\in\mathbb{R}^{2n-2}$. Given the expression (4.12), we have rather

$$\lim_{r_1\to0^+}t_c^{(2)}(r_1\cos\theta_1,r_1\sin\theta_1,h_3,\dots,h_{2n})=\lim_{r_1\to0^+}t_c^{(2)}(-r_1\cos\theta_1,-r_1\sin\theta_1,h_3,\dots,h_{2n})\neq0,$$

unless $\gamma_1^1=\gamma_2^2=\gamma_1^2+\gamma_2^1=0$ for all $(h_3,\dots,h_{2n})\in\mathbb{R}^{2n-2}$. That is $\kappa_{1i}^1=\kappa_{2i}^2=\kappa_{1i}^2+\kappa_{2i}^1=0$ for all $i\in[\![3,2n]\!]$, which is not generic with respect to the sub-Riemannian structure at $q\in M\setminus(\mathfrak{S}_1\cup\mathfrak{S}_3)$ (see Proposition 4.4.4 and Appendix 4.B).

In consequence, we have proven that generically with respect to the sub-Riemannian structure at $q\in M\setminus\mathfrak{S}$, there does not exist a set of privileged coordinates at q and $T>0$ such that the limit (4.4) holds. □

Remark 4.4.13. Regarding the non-genericity of $\kappa_{1i}^1=\kappa_{2i}^2=\kappa_{1i}^2+\kappa_{2i}^1=0$, notice that it constitutes $6(n-1)$ independent conditions on the family $\left(\kappa_{i,j}^k\right)_{\substack{i,k\in[\![1,2]\!],\\ j\in[\![3,2n]\!]}}$ and thus a codimension $5(n-1)$ condition (at least) on the 2-jets of the sub-Riemannian structure at q.

Notice that $5n-5>2n+1$ if $n>2$ and $5n-5=2n+1$ when $n=2$. Hence in the $n=2$ case we can impose $\mathfrak{S}_3\subset\mathfrak{S}$ (see Proposition 4.4.4) so that in this particular case, we have a codimension 6 condition on the 2-jets of the sub-Riemannian structure.

4.5 Next order perturbations

As observed in Section 4.4.1, there exists a subset of initial covectors in $T_{q_0}^*\setminus S_1$ for which the conjugate locus is degenerate (which makes the second order approximation unstable as Lagrangian map). In particular, for all $q_0\in M$, this set contains $S_2=\{(h_1,h_2,0,\dots,0,\eta^{-1})\in T_{q_0}^*M\}$. As proved in Proposition 4.4.7, this set is reduced to S_2 at points q_0 in the complement of a startified codimension 1 subset $\mathfrak{S}_2$ of M if $n=2$.

Hence in preparation of the stability analysis of Chapter 5, we compute here a third order approximation of the conjugate time in the case of covectors near S_2. When $n=2$, we get a complete description of the sub-Riemannian caustic at points of $M\setminus\mathfrak{S}_2$ as a result.

We use a blowup technique similar to Section 4.4.2. Let $\bar{h} \in \mathbb{R}^{2n}$ be such that $(\bar{h}_1, \bar{h}_2) \neq (0,0)$. We blowup the singularity at $(\bar{h}_1, \bar{h}_2, 0, \dots, 0)$ by computing an approximation of the conjugate locus with

$$h(0) = (\bar{h}_1, \bar{h}_2, \eta\bar{h}_3, \dots, \eta\bar{h}_{2n}).$$

Let Λ be the square $2n \times 2n$ matrix such that

$$\Lambda_{i,j} = \begin{cases} 1 & \text{if } i = j = 1 \text{ or } i = j = 2, \\ 0 & \text{otherwise.} \end{cases}$$

so that $h(0) = \Lambda\bar{h} + \eta(I_{2n} - \Lambda)\bar{h}$.

We give an equivalent of Proposition 4.4.8 for this case.

Proposition 4.5.1. *For all $T, R > 0$, normal extremals with initial covector $(\Lambda\bar{h} + \eta(I_{2n} - \Lambda)\bar{h}, \eta^{-1})$ have the following order 3 expansion at time $\eta\tau$, as $\eta \to 0^+$, uniformly with respect to $\tau \in [0, T]$ and $h(0) \in B_R$:*

$$\begin{aligned} x(\eta\tau, \Lambda\bar{h} + \eta(I_{2n} - \Lambda)\bar{h}) &= \eta\widehat{x}(\tau, \Lambda\bar{h}) + \eta^2 \left[x^{(2)}\left(\tau, \Lambda\bar{h}\right) + \widehat{x}\left(\tau, (I_{2n} - \Lambda)\bar{h}\right)\right] \\ &+ \eta^3 \left[x^{(3)}\left(\tau, \Lambda\bar{h}\right) + x^{(2)}\left(\tau, \bar{h}\right) - x^{(2)}\left(\tau, \Lambda\bar{h}\right) - x^{(2)}\left(\tau, (I_{2n} - \Lambda)\bar{h}\right)\right] + O(\eta^4), \end{aligned}$$

$$z(\eta\tau) = \eta^2\widehat{z}(\tau, \Lambda\bar{h}) + \eta^3 z^{(3)}(\tau, \Lambda\bar{h}) + O(\eta^4).$$

Likewise, the associated covector has the following expansion:

$$\begin{aligned} h(\eta\tau, \Lambda\bar{h} + \eta(I_{2n} - \Lambda)\bar{h}) &= \widehat{h}(\tau, \Lambda\bar{h}) + \eta\left[h^{(1)}(\tau, \Lambda\bar{h}) + \widehat{h}(\tau, (I_{2n} - \Lambda)\bar{h})\right] \\ &+ \eta^2 \left[h^{(2)}(\tau, \Lambda\bar{h}) + h^{(1)}(\tau, \bar{h}) - h^{(1)}(\tau, \Lambda\bar{h}) - h^{(1)}(\tau, (I_{2n} - \Lambda)\bar{h})\right] + O(\eta^3), \end{aligned}$$

$$w(\eta\tau) = 1 + \eta^2 w^{(2)}(\tau, \Lambda\bar{h}) + O(\eta^4).$$

Proof. The proof relies on the same arguments as that of Proposition 4.4.8. □

We aim to obtain a second order approximation of τ_c in the case of an initial covector of the form $(\Lambda\bar{h} + \eta(I_{2n} - \Lambda)\bar{h}, \eta^{-1})$, for $\bar{h} \in \mathbb{R}^{2n}$. The previous section, together with Proposition 4.5.1, applies to give us

$$\tau_c^{(1)}(\Lambda\bar{h} + \eta(I_{2n} - \Lambda)\bar{h}) = \tau_c^{(1)}(\Lambda\bar{h}), \qquad \forall\bar{h} \in \mathbb{R}^{2n}.$$

Similarly to Section 4.4.2, for all $\tau > 0$, $h \in \mathbb{R}^{2n}$ and $\eta > 0$, we denote $F(\tau, h, \eta) = \mathcal{E}(\eta\tau; (h, \eta^{-1}))$, and we set

$$G(\tau, \bar{h}, \eta) = F\left(\tau, \Lambda\bar{h} + \eta(I_{2n} - \Lambda)\bar{h}, \eta\right), \quad \forall\tau > 0, \bar{h} \in \mathbb{R}^{2n}, \eta > 0.$$

The function G admits a formal power series expansion in η $G(\tau,\bar h,\eta)=\sum_{k\geq 0}\eta^k G^{(k)}(\tau,\bar h)$. With $\delta t_1,\delta t_2\in\mathbb{R}$, $\bar h\in\mathbb{R}^{2n}$, we have

$$G\left(\frac{2\pi}{b_1}+\eta\delta t_1+\eta^2\delta t_2,\bar h,\eta\right)=\eta\, G^{(1)}\big|_{\tau=\frac{2\pi}{b_1}}+\eta^2\left(G^{(2)}+\delta t_1\frac{\partial G^{(1)}}{\partial\tau}\right)\Big|_{\tau=\frac{2\pi}{b_1}}$$
$$+\eta^3\left[G^{(3)}+\delta t_2\frac{\partial G^{(1)}}{\partial\tau}+\frac{\delta t_1^2}{2}\frac{\partial^2 G^{(1)}}{\partial\tau^2}+\delta t_1\frac{\partial G^{(2)}}{\partial\tau}\right]\Big|_{\tau=\frac{2\pi}{b_1}}+O(\eta^3).$$

Recall that

$$\Phi(\tau,h,\eta)=\det\left(\frac{\partial F}{\partial h_1},\dots,\frac{\partial F}{\partial h_{2n}},\eta\frac{\partial F}{\partial\eta}-\tau\frac{\partial F}{\partial\tau}\right)\Big|_{(\tau,h,\eta)}.$$

To evaluate $\Phi\left(2\pi/b_1+\eta\delta t_1+\eta^2\delta t_2,\Lambda\bar h+\eta(I_{2n}-\Lambda)\bar h,\eta\right)$, $\delta t_1,\delta t_2\in\mathbb{R}$, $\bar h\in\mathbb{R}^{2n}$, notice that

$$\frac{\partial F}{\partial h_i}=\frac{\partial G}{\partial\bar h_i},\quad\forall i\in[\![1,2]\!]\quad\text{and}\quad\frac{\partial F}{\partial h_i}=\frac{1}{\eta}\frac{\partial G}{\partial\bar h_i},\quad\forall i\in[\![3,2n]\!].$$

Then for all $i\in[\![1,2n]\!]$, let C_i and C_{2n+1} be the respective evaluations at time $\tau=2\pi/b_1+\eta\delta t_1+\eta^2\delta t_2$ of the vectors $\dfrac{\partial G}{\partial\bar h_i}$ and $\eta\dfrac{\partial G}{\partial\eta}-\tau\dfrac{\partial G}{\partial\tau}$. For all $i\in[\![1,2n+1]\!]$, the vector $C_i\in\mathbb{R}^{2n+1}$ also admits a formal power series in η, $C_i=\sum_{k=1}^{\infty}\eta^k C_i^{(k)}$. Notice that by definition of $(C_i)_{i\in[\![1,2n+1]\!]}$ we have

$$C_i^{(0)}=C_i^{(1)}=0\qquad\forall i\in[\![1,2n]\!].$$

Hence we have a priori $\Phi\left(2\pi/b_1,\Lambda\bar h+\eta(I_{2n}-\Lambda)\bar h,\eta\right)=O(\eta^{4n+1})$. We can use these elements to give the following refinement on Lemma 4.4.1.

Proposition 4.5.2. *For all $\bar h\in\mathbb{R}^{2n}$, $\delta t_1=\tau_c^{(1)}(\Lambda\bar h)$ is the only solution to*

$$\Phi\left(2\pi/b_1+\eta\delta t_1+\eta^2\delta t_2,\Lambda\bar h+\eta(I_{2n}-\Lambda)\bar h,\eta\right)=O(\eta^{4n+2}).$$

Furthermore

$$\Phi\left(2\pi/b_1+\eta\tau_c^{(1)}(\Lambda\bar h)+\eta^2\delta t_2,\Lambda\bar h+\eta(I_{2n}-\Lambda)\bar h,\eta\right)=\eta^{4n+2}K\left(\sum_{i=1}^{2n+1}d_i\right)+O(\eta^{4n+3}),$$

where $K=2^{2n-2}\prod_{i=2}^{n}\frac{1}{b_i^2}\sin^2\left(\frac{\pi b_i}{b_1}\right)>0$ and

$$d_1=\begin{vmatrix}\left(C_1^{(3)}\right)_1 & \left(C_2^{(3)}\right)_1 & \left(C_{2n+1}^{(1)}\right)_1\\ \left(C_1^{(3)}\right)_2 & \left(C_2^{(3)}\right)_2 & \left(C_{2n+1}^{(1)}\right)_2\\ \left(C_1^{(2)}\right)_{2n+1} & \left(C_2^{(2)}\right)_{2n+1} & 0\end{vmatrix},$$

$$d_2 = \begin{vmatrix} \left(C_1^{(2)}\right)_1 & \left(C_2^{(2)}\right)_1 & \left(C_{2n+1}^{(1)}\right)_1 \\ \left(C_1^{(2)}\right)_2 & \left(C_2^{(2)}\right)_2 & \left(C_{2n+1}^{(1)}\right)_2 \\ \left(C_1^{(3)}\right)_{2n+1} & \left(C_2^{(3)}\right)_{2n+1} & 0 \end{vmatrix},$$

$$d_{2n+1} = \begin{vmatrix} \left(C_1^{(2)}\right)_1 & \left(C_2^{(2)}\right)_1 & \left(C_{2n+1}^{(2)}\right)_1 \\ \left(C_1^{(2)}\right)_2 & \left(C_2^{(2)}\right)_2 & \left(C_{2n+1}^{(2)}\right)_2 \\ \left(C_1^{(2)}\right)_{2n+1} & \left(C_2^{(2)}\right)_{2n+1} & \left(C_{2n+1}^{(2)}\right)_{2n+1} \end{vmatrix}$$

and

$$d_k = \frac{2\pi^2}{b_1^2} e_k \left(-h_2 \partial_{h_k} \left(G^{(3)}\right)_1 + h_1 \partial_{h_k} \left(G^{(3)}\right)_2\right) \qquad \forall k \in [\![3, 2n]\!],$$

with $e \in \mathbb{R}^{2n-2}$ *the vector such that* $Ae = \left(h_2 C_1^{(2)} - h_1 C_2^{(2)}\right)_{3,\dots,2n}$, *with* $A \in \mathcal{M}_{2n-2}(\mathbb{R})$ *the matrix introduced in Lemma 4.4.1 and where we denote* $(v)_{3,\dots,2n} = (v_3, \dots, v_{2n}) \in \mathbb{R}^{2n-2}$ *for all* $v \in \mathbb{R}^{2n+1}$.

Proof. The first part of the statement is an application of Lemma 4.4.1 in the case of an initial covector of the form $h(0) = \Lambda\bar{h} + \eta(I_{2n} - \Lambda)\bar{h}$. Indeed

$$\Phi\left(2\pi/b_1 + \eta\tau_c^{(1)} + \eta^2 \delta t_2, \Lambda\bar{h} + \eta(I_{2n} - \Lambda)\bar{h}, \eta\right) = \eta^{4n+1} \det\left(C_1^{(2)}, \dots, C_{2n}^{(2)}, C_{2n+1}^{(1)}\right) + O(\eta^{4n+2}).$$

The equation satisfied by $\tau_c^{(1)}$ comes down to $\det\left(C_1^{(2)}, \dots, C_{2n}^{(2)}, C_{2n+1}^{(1)}\right)\Big|_{\tau = 2\pi/b_1 + \eta\tau_c^{(1)}} = 0$, hence

$$\Phi\left(2\pi/b_1 + \eta\tau_c^{(1)} + \eta^2 \delta t_2, \Lambda\bar{h} + \eta(I_{2n} - \Lambda)\bar{h}, \eta\right) = \eta^{4n+2}\left[\det\left(C_1^{(2)}, \dots, C_{2n+1}^{(2)}\right) + \sum_{k=1}^{2n} \det\left(C_1^{(2)}, \dots, C_{k-1}^{(2)}, C_k^{(3)}, C_{k+1}^{(2)}, \dots, C_{2n}^{(2)}, C_{2n+1}^{(1)}\right)\right] + O(\eta^{4n+3}).$$

Setting $d'_k = \det\left(C_1^{(2)}, \dots, C_{k-1}^{(2)}, C_k^{(3)}, C_{k+1}^{(2)}, \dots, C_{2n+1}^{(2)}\right)$, for all $k \in [\![3, 2n]\!]$, we first prove $d'_k = K d_k$, for all $k \in [\![3, 2n]\!]$.

We proceed to the following transformation on the columns $(C_i)_{i \in [\![1, 2n+1]\!]}$ of the Jacobian matrix. First, $C_1 \leftarrow h_2 C_1 - h_1 C_2$ and $C_2 \leftarrow h_1 C_1 + h_2 C_2$, then we transpose $C_k \leftrightarrow C_1$ and finally we cycle $C_{i+1} \leftarrow C_i$ for $i \in [\![3, 2n]\!]$ and $C_3 \leftarrow C_{2n+1}$. This yields

$$d'_k = \frac{1}{h_1^2 + h_2^2} \det\left(C_k^{(3)}, h_1 C_1^{(2)} + h_2 C_2^{(2)}, C_{2n+1}^{(1)}, C_3^{(2)}, \dots, C_{k-1}^{(2)}, h_2 C_1^{(2)} - h_1 C_2^{(2)}, C_{k+1}^{(2)}, \dots, C_{2n}^{(2)}\right).$$

Using Proposition 4.5.1, $\left(C_i^{(2)}\right)_1 = \left(C_i^{(2)}\right)_2 = \left(C_i^{(2)}\right)_{2n+1} = 0$, $i \in [\![3, 2n]\!]$. All columns of this new matrix have zero $2n+1$ component except for $h_1 C_1^{(2)} + h_2 C_2^{(2)}$, and zero 1 and 2 component except for $C_k^{(3)}$, $h_1 C_1^{(2)} + h_2 C_2^{(2)}$ and $C_{2n+1}^{(1)}$. One can apply the Cramer rule for computing the k-th coefficient of $e = A^{-1}(h_2 C_1^{(2)} - h_1 C_2^{(2)})$ when computing the determinant of the square submatrix of lines and columns 3 through $2n$.

Hence we have

$$d_k' = \frac{K e_k}{h_1^2 + h_2^2} \det\left(\widetilde{C}_k^{(3)}, h_1 \widetilde{C}_1^{(2)} + h_2 \widetilde{C}_2^{(2)}, \widetilde{C}_{2n+1}^{(1)}\right)$$

with $\widetilde{C}_i = \left((C_i)_1, (C_i)_2, (C_i)_{2n+1}\right)$, and we get the value of d_k by computing the remaining determinant.

Similarly, we obtain the stated relation for d_1, d_2 and d_{2n+1} by noticing that $C_{2n+1}^{(1)} = 0$ and isolating the three 3×3 matrices given by lines and columns 1, 2 and $2n+1$. $\square$

The value of determinants d_1 through d_{2n+1} can be explicitly stated in terms of second order invariants thanks to the computations in Appendix 4.B.2.

Lemma 4.5.3. *We have* $d_{2n+1} = 0$, $d_2 = -\frac{2\pi}{b_1}(\bar{h}_1^2 + \bar{h}_2^2)(\beta \bar{h}_1 - \alpha \bar{h}_2)^2$ *and*

$$\begin{aligned} d_1 =& \frac{4\pi^2}{b_1^2}(\bar{h}_1^2 + \bar{h}_2^2)\left(\delta t_2 + 4 b_1 (\beta \bar{h}_1 - \alpha \bar{h}_2)(\alpha \bar{h}_1 + \beta \bar{h}_2)\right) \\ & \frac{4\pi^2}{b_1^2}\left(\bar{h}_1^2 \frac{\partial \left(G^{(3)}\right)_2}{\partial \bar{h}_2} + \bar{h}_2^2 \frac{\partial \left(G^{(3)}\right)_1}{\partial \bar{h}_1} - \bar{h}_1 \bar{h}_2 \left(\frac{\partial \left(G^{(3)}\right)_1}{\partial \bar{h}_2} + \frac{\partial \left(G^{(3)}\right)_2}{\partial \bar{h}_1}\right)\right). \end{aligned}$$

Furthermore, for all $i \in [\![2, n]\!]$, *we have*

$$\begin{aligned} d_{2i-1} =& \frac{2\pi^2 b_i}{b_1^2}\left(h_1(h_1 \kappa_{1,2i-1}^2 + h_2 \kappa_{2,2i-1}^2) - h_2(h_1 \kappa_{1,2i-1}^1 + h_2 \kappa_{2,2i-1}^1)\right) \\ & \left[\cot\left(\frac{\pi b_i}{b_1}\right)\left(\kappa_{12}^{2i-1}(h_2^2 - h_1^2) + 2 h_1 h_2 (\kappa_{11}^{2i-1} - \kappa_{22}^{2i-1})\right)\right. \\ & \left. - \left(\kappa_{12}^{2i}(h_2^2 - h_1^2) + 2 h_1 h_2 (\kappa_{11}^{2i} - \kappa_{22}^{2i})\right)\right], \end{aligned}$$

$$\begin{aligned} d_{2i} =& \frac{2\pi^2 b_i}{b_1^2}\left(h_1(h_1 \kappa_{1,2i}^2 + h_2 \kappa_{2,2i}^2) - h_2(h_1 \kappa_{1,2i}^1 + h_2 \kappa_{2,2i}^1)\right) \\ & \left[\cot\left(\frac{\pi b_i}{b_1}\right)\left(\kappa_{12}^{2i}(h_2^2 - h_1^2) + 2 h_1 h_2 (\kappa_{11}^{2i} - \kappa_{22}^{2i})\right)\right. \\ & \left. + \left(\kappa_{12}^{2i-1}(h_2^2 - h_1^2) + 2 h_1 h_2 (\kappa_{11}^{2i-1} - \kappa_{22}^{2i-1})\right)\right]. \end{aligned}$$

Proof. First, recall that

$$\begin{aligned} x_1^{(2)}(2\pi/b_1, \Lambda \bar{h}) &= \alpha(3\bar{h}_1^2 + \bar{h}_2^2) + 2\beta \bar{h}_1 \bar{h}_2, \\ x_2^{(2)}(2\pi/b_1, \Lambda \bar{h}) &= 2\alpha \bar{h}_1 \bar{h}_2 + \beta(\bar{h}_1^2 + 3\bar{h}_2^2) \end{aligned}$$

and $\tau_c^{(1)}(\Lambda\bar{h}) = -2(\alpha\bar{h}_1 + \beta\bar{h}_2)$. Using Lemma 4.B.6 from the appendix, we have the value of $z^{(3)}\left(\frac{2\pi}{b_1}, \Lambda\bar{h}\right)$ and we can compute the 3×3 determinant d_2. (Remark that $F_{2n+1}^{(3)} = z^{(3)} + \tau_c^{(1)}\partial_\tau z^{(2)}(2\pi/b_1)$ and that $\partial_\tau z^{(2)}(2\pi/b_1) = 0$.) Similarly we can compute d_{2n+1} by noticing, for $i \in [\![1, 2]\!]$,

$$(\eta\partial_\eta F_i - \tau\partial_\tau F_i)_{|\tau=2\pi/b_1+\eta\tau_c^{(1)}} = 2x_i^{(2)}\left(\frac{2\pi}{b_1}, \Lambda\bar{h}\right) - \frac{2\pi}{b_1}\left(h_i^{(1)}\left(\frac{2\pi}{b_1}, \Lambda\bar{h}\right) + \tau_c^{(1)}(\Lambda\bar{h})(J\bar{h})_i\right).$$

Regarding d_k, $k \in [\![3, 2n]\!]$, we obtain the result by explicitly computing the vector $e \in \mathbb{R}^{2n-2}$. First, since $\partial_{\bar{h}_k} z^{(3)} = 0$

$$\det\left(\widetilde{C}_k^{(3)}, h_1\widetilde{C}_1^{(2)} + h_2\widetilde{C}_1^{(2)}, \widetilde{C}_{2n+1}^{(1)}\right) = -\frac{4\pi^2}{b_1^2}(h_1^2+h_2^2)\left[h_1h_2(\kappa_{1,k}^1 - \kappa_{2,k}^2) + h_2^2\kappa_{2,k}^1 - h_1^2\kappa_{1,k}^2)\right].$$

On the other hand, we have $Ae = h_2C_1^{(2)} - h_1C_2^{(2)}$ and for all $3 \leq i \leq 2n$

$$\left(h_2C_1^{(2)} - h_1C_2^{(2)}\right)_i = \kappa_{12}^i(h_2^2 - h_1^2) + 2h_1h_2(\kappa_{11}^i - \kappa_{22}^i).$$

We then get the stated result since A^{-1} is block diagonal with blocks in position $i-1$ being, for all $i \in [\![2, n]\!]$,

$$\frac{b_i}{2}\begin{pmatrix} \cot \pi b_i/b_1 & -1 \\ 1 & \cot \pi b_i/b_1 \end{pmatrix}.$$

□

Up to the computation of $G^{(3)}$, which is carried out in Appendix 4.B.2, we have enough information to compute the conjugate time, similarly to Proposition 4.4.2.

Proposition 4.5.4. *The second order perturbation of τ_c with initial covector $h(0) = \Lambda\bar{h} + \eta(I_{2n} - \Lambda)\bar{h}$ satisfies the equation*

$$(\bar{h}_1^2 + \bar{h}_2^2)\tau_c^{(2)}(h(0)) = -\bar{h}_1^2\frac{\partial\left(G^{(3)}\right)_2}{\partial\bar{h}_2} - \bar{h}_2^2\frac{\partial\left(G^{(3)}\right)_1}{\partial\bar{h}_1} + \bar{h}_1\bar{h}_2\left(\frac{\partial\left(G^{(3)}\right)_1}{\partial\bar{h}_2} + \frac{\partial\left(G^{(3)}\right)_2}{\partial\bar{h}_1}\right)$$
$$+ (\bar{h}_1^2 + \bar{h}_2^2)(\alpha\bar{h}_2 - \beta\bar{h}_1)\left(\frac{b_1}{2\pi}(\beta\bar{h}_1 - \alpha\bar{h}_2) + 4b_1(\alpha\bar{h}_1 + \beta\bar{h}_2)\right) + \sum_{i=3}^{2n} d_i.$$

Remark 4.5.5. By definition of the invariants $\chi_{11}, \chi_{12}, \chi_{22}$ introduced in Appendix 4.B, the third dimensional case would correspond to the case $\kappa_{ij}^k = 0$ if $3 \leq i, j, k \leq 2n$, $\alpha = \beta = 0$. Under these conditions, one has

$$\tau_c^{(1)}(\bar{h}) = 0, \qquad \tau_c^{(2)}(\bar{h}) = -3(\chi_{11} + \chi_{22})(\bar{h}_1^2 + \bar{h}_2^2)$$

and

$$\left[\mathcal{E}(\eta\tau_c; (h, \eta^{-1}))\right]_1 = \eta^3\left(2\bar{h}_1^3(\chi_{22} - \chi_{11}) + 3\bar{h}_1^2\bar{h}_2\chi_{12} + \bar{h}_2^3\chi_{12}\right) + O(\eta^4),$$
$$\left[\mathcal{E}(\eta\tau_c; (h, \eta^{-1}))\right]_2 = \eta^3\left(2\bar{h}_2^3(\chi_{11} - \chi_{22}) + 3\bar{h}_1\bar{h}_2^2\chi_{12} + \bar{h}_1^3\chi_{12}\right) + O(\eta^4).$$

This expression corresponds to the classical astroidal caustic expansion observed in the 3-dimensional contact case.

Appendices

4.A Agrachev–Gauthier normal form

Let (M, Δ, g) be a contact sub-Riemannian manifold of dimension $2n+1$. In [AG01], the authors prove the existence at any $q_0 \in M$ of a set of coordinates and vector fields for which the contact sub-Riemannian structure satisfies interesting symmetries. Here we recall the properties of this normal form, that we call Agrachev–Gauthier normal form.

On a contact manifold, there exists a 1-form ω such that $\omega \wedge (\mathrm{d}\omega)^n$ never vanishes and $\ker \omega = \Delta$. Notice that for any smooth non-vanishing function $f : M \to \mathbb{R}$, $\ker f\omega = \Delta$. Hence ω can be chosen so that

$$(\mathrm{d}\omega)^n_{|\Delta} = \mathrm{vol}_g$$

where vol_g is the volume form induced by g on Δ. Then there exists a unique vector field X_0, the Reeb vector field, such that

$$\omega(X_0) = 1 \quad \text{and} \quad \iota_{X_0} \mathrm{d}\omega = 0.$$

In the following, for any vector field Y, for all $i \in [\![1, 2n+1]\!]$, we denote by $(Y)_i$ the i-th coordinate of Y written in the basis $(\partial_{x_1}, \dots, \partial_{x_{2n}}, \partial_z)$.

Theorem 4.A.1 ([AG01, Section 6]). *Let (M, Δ, g) be a contact sub-Riemannian manifold of dimension $2n+1$ and $q_0 \in M$. There exist privileged coordinates at q_0, $(x_1, \dots x_{2n}, z) : M \to \mathbb{R}^{2n+1}$, and a frame of (Δ, g), $(X_1, \dots, X_{2n})$, that satisfy the following properties on a small neighborhood of $q_0 = (0, \dots, 0)$.*

(1) *The horizontal components of the vector fields $X_1, \dots, X_{2n}$ satisfy the following two symmetries: for all $1 \le i, j \le 2n$, we have*

$$(X_i)_j = (X_j)_i$$

and

$$\sum_{j=1}^{2n} (X_j)_i \, x_j = x_i.$$

(2) *The vertical components of* $X_1, \ldots, X_{2n}$ *satisfy the symmetry*

$$\sum_{j=1}^{2n} (X_j)_{2n+1} x_j = 0.$$

(3) $X_0 = \frac{\partial}{\partial z}$, $\omega(X_0) = 1$ *and* $\iota_{X_0} \mathrm{d}\omega = 0$.

This is further detailed by evaluating the elements $(X_i)_j$ at some well chosen positions. Let us denote by $V_1, \ldots, V_n$ the 3-dimensional subspaces of M defined by

$$V_i = \cap_{j \neq i} \{x_{2j-1} = 0\} \cap \{x_{2j} = 0\} \qquad \forall i \in [\![1, n]\!].$$

Theorem 4.A.2 ([AG01, Theorem 6.6])**.** *Let* (M, Δ, g) *be a contact sub-Riemannian manifold of dimension* $2n+1$ *and* $q_0 \in M$. *Let* $(x_1, \ldots x_{2n}, z) : M \to \mathbb{R}^{2n+1}$ *be privileged coordinates at* q_0, *and* $(X_1, \ldots, X_{2n})$ *be a frame of* (Δ, g), *both as in statement of Theorem 4.A.1. Then*

(i) *For all* $i, j \in [\![1, 2n]\!]$,

$$(X_i)_j (0, z) = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ otherwise} \end{cases} \tag{4.21}$$

and for all $k \in [\![1, 2n]\!]$

$$\partial_{x_k} (X_i)_j (0, z) = 0. \tag{4.22}$$

Furthermore, there exist $\beta_1, \ldots, \beta_n : \mathbb{R}^3 \to \mathbb{R}$ *such that for all* $i \in [\![1, n]\!]$, $\beta_i(0, 0, z) = 0$ *and*

$$\begin{aligned} &\begin{cases} (X_{2i-1})_{2i-1}\big|_{V_i} &= 1 + x_{2i}^2 \beta_i(x_{2i-1}, x_{2i}, z), \\ (X_{2i-1})_{2i}\big|_{V_i} &= -\, x_{2i-1} x_{2i} \beta_i(x_{2i-1}, x_{2i}, z), \end{cases} \\ &\begin{cases} (X_{2i})_{2i-1}\big|_{V_i} &= -\, x_{2i-1} x_{2i} \beta_i(x_{2i-1}, x_{2i}, z), \\ (X_{2i})_{2i}\big|_{V_i} &= 1 + x_{2i-1}^2 \beta_i(x_{2i-1}, x_{2i}, z). \end{cases} \end{aligned} \tag{4.23}$$

(ii) *There exist* $\alpha_1, \ldots, \alpha_n : \mathbb{R}^3 \to \mathbb{R}$ *such that for all* $i \in [\![1, n]\!]$,

$$(X_{2i-1})_{2n+1}\big|_{V_i} = x_{2i} \alpha_i(x_{2i-1}, x_{2i}, z)/2, \quad (X_{2i})_{2n+1}\big|_{V_i} = -x_{2i-1} \alpha_i(x_{2i-1}, x_{2i}, z)/2. \tag{4.24}$$

(iii) *We have*

$$\prod_{i=1}^{n} \alpha_i(0, 0, z) = \frac{1}{n!},$$

and for all $i \in [\![1, n]\!]$, *we denote*

$$\widetilde{L}_i = \frac{\partial (X_{2i})_{2n+1}}{\partial x_{2i-1}} - \frac{\partial (X_{2i-1})_{2n+1}}{\partial x_{2i}}.$$

Then for all $i \in [\![1, n]\!]$,

$$\widetilde{L}_i\Big|_{V_i} = \alpha_i, \qquad \forall i \in [\![1, n]\!],$$

and

$$\sum_{j=1}^{n} \partial_{x_{2k-1}} \widetilde{L}_j(0, z) \prod_{i \neq j} \alpha_i(0, z) = \sum_{j=1}^{n} \partial_{x_{2k}} \widetilde{L}_j(0, z) \prod_{i \neq j} \alpha_i(0, z) = 0.$$

Remark 4.A.3. A few observations on Theorem 4.A.2.

— Notice that points (i), (ii), (iii) are respectively consequences of points **(1)**, **(2)**, **(3)** of Theorem 4.A.1.

— The eigenvalues $b_1, \dots, b_n$ of the skew-symmetric matrix $J = (c^0_{ij})_{i,j \in [\![1,2n]\!]}$ at q_0 satisfy (up to reordering)

$$b_i = \alpha_i(0, 0, 0), \qquad \forall i \in [\![1, n]\!].$$

— In the Agrachev–Gauthier normal form, the frame $(X_1, \dots, X_{2n})$ naturally appears as a perturbation of the frame of a nilpotent contact structure over $\mathbb{R}^{2n+1}$, $\left(\widehat{X}_1, \dots, \widehat{X}_{2n}\right)$, written in the normal form

$$\widehat{X}_{2i-1} = \partial_{x_{2i-1}} + \frac{b_i}{2} x_{2i} \partial_z, \quad \widehat{X}_{2i} = \partial_{x_{2i}} - \frac{b_i}{2} x_{2i-1} \partial_z, \qquad \forall i \in [\![1, n]\!].$$

— We can deduce from (i) the following equalities. For all $r, s \in \mathbb{N}$,

$$\begin{aligned} 2 \left(\partial_{x_{2i-1}}\right)^r \left(\partial_{x_{2i}}\right)^s \beta_i(0, z) &= \left(\partial_{x_{2i-1}}\right)^r \left(\partial_{x_{2i}}\right)^{s+2} \left(X_{2i-1}\right)_{2i-1}(0, z) \\ &= \left(\partial_{x_{2i-1}}\right)^{r+2} \left(\partial_{x_{2i}}\right)^s \left(X_{2i}\right)_{2i}(0, z) \\ &= -2 \left(\partial_{x_{2i-1}}\right)^{r+1} \left(\partial_{x_{2i}}\right)^{s+1} \left(X_{2i-1}\right)_{2i}(0, z) \\ &= -2 \left(\partial_{x_{2i-1}}\right)^{r+1} \left(\partial_{x_{2i}}\right)^{s+1} \left(X_{2i}\right)_{2i-1}(0, z). \end{aligned} \tag{4.25}$$

In particular,

$$\begin{aligned} 0 = \beta_i(0, 0, z) &= \left(\partial_{x_{2i}}\right)^2 \left(X_{2i-1}\right)_{2i-1}(0, z) \\ &= \left(\partial_{x_{2i-1}}\right)^2 \left(X_{2i}\right)_{2i}(0, z) \\ &= -2 \left(\partial_{x_{2i-1}}\right) \left(\partial_{x_{2i}}\right) \left(X_{2i-1}\right)_{2i}(0, z) \\ &= -2 \left(\partial_{x_{2i-1}}\right) \left(\partial_{x_{2i}}\right) \left(X_{2i}\right)_{2i-1}(0, z). \end{aligned} \tag{4.26}$$

4.B Computation of structural invariants

4.B.1 Second order invariants

For all $l \in [\![1, 2n]\!]$, let $J_l \in \mathcal{M}_{2n}(\mathbb{R})$ be the matrix such that

$$(J_l)_{k,m} = \frac{\partial^2 (X_l)_{2n+1}}{\partial x_k \partial x_m}(q_0) - \frac{\partial^2 (X_k)_{2n+1}}{\partial x_l \partial x_m}(q_0), \qquad \forall k, l, m \in [\![1, 2n]\!],$$

so that for all $x, y \in \mathbb{R}^{2n}$, the vector $J^{(1)}(x)\, y$ satisfies $(J^{(1)}(x)\, y)_l = J_l x \cdot y$.

Let $V_{i,j}(\sigma) \in \mathbb{R}^{2n}$ be the vector such that

$$(V_{i,j}(\sigma))_l = \left(\left(\mathrm{e}^{-\sigma \bar{J}} - I_{2n}\right) \bar{J}^{-1}\, {}^t J_l\, \mathrm{e}^{\sigma \bar{J}}\right)_{i,j} + \left(\left(\mathrm{e}^{-\sigma \bar{J}} - I_{2n}\right) \bar{J}^{-1}\, {}^t J_l\, \mathrm{e}^{\sigma \bar{J}}\right)_{j,i}.$$

Lemma 4.B.1. *For all $i, j, k \in [\![1, 2n]\!]$*

$$\kappa^k_{i,j} = \varepsilon(i,j) \int_0^{\frac{2\pi}{b_1}} \int_0^{\tau} \left[\mathrm{e}^{(\tau-\sigma)\bar{J}} V_{i,j}(\sigma)\right]_k \mathrm{d}\sigma\, \mathrm{d}\tau,$$

where

$$\varepsilon(i,j) = \begin{cases} 1 & \text{if } i \neq j, \\ 1/2 & \text{if } i = j. \end{cases}$$

Proof. From Proposition 4.2.1, we have to compute for all $i, j, k \in [\![1, 2n]\!]$,

$$\varepsilon(i,j) \frac{\partial^2 x^{(2)}_k}{\partial h_i \partial h_j} \left(\frac{2\pi}{b_1}, h\right) = \kappa^k_{i,j}$$

Observe that for all $i, j \in [\![1, 2n]\!]$,

$$\frac{\partial^2 x^{(2)}}{\partial h_i \partial h_j} \left(\frac{2\pi}{b_1}, h\right) = \int_0^{\frac{2\pi}{b_1}} \int_0^{\tau} \mathrm{e}^{(\tau-\sigma)\bar{J}} \left(J^{(1)}\left(\widehat{x}(\sigma, e_i)\right) \widehat{h}(\sigma, e_j) + J^{(1)}\left(\widehat{x}(\sigma, e_j)\right) \widehat{h}(\sigma, e_i)\right) \mathrm{d}\sigma\, \mathrm{d}\tau,$$

where, for all

$$m \in [\![1, 2n]\!]$$

, $e_m \in \mathbb{R}^{2n}$ is the vector such that $(e_m)_l = 1$ if $l = m$ and $(e_m)_l = 0$ otherwise. Using the fact that $(J^{(1)}(x)y)_l = (J_l x) \cdot y$, we have

$$\begin{aligned}
\left[J^{(1)}\left(\widehat{x}(\sigma, e_i)\right) \widehat{h}(\sigma, e_j)\right]_l &= \left(J_l \bar{J}^{-1} \left(\mathrm{e}^{\sigma \bar{J}} - I_{2n}\right) e_i\right) \cdot \mathrm{e}^{\sigma \bar{J}} e_j \\
&= e_i \cdot \left({}^t\left(\mathrm{e}^{\sigma \bar{J}} - I_{2n}\right) {}^t\bar{J}^{-1}\, {}^t J_l\right) \mathrm{e}^{\sigma \bar{J}} e_j \\
&= e_i \cdot \left(I_{2n} - \mathrm{e}^{-\sigma \bar{J}}\right) \bar{J}^{-1}\, {}^t J_l\, \mathrm{e}^{\sigma \bar{J}} e_j \\
&= \left(\left(I_{2n} - \mathrm{e}^{-\sigma \bar{J}}\right) \bar{J}^{-1}\, {}^t J_l\, \mathrm{e}^{\sigma \bar{J}}\right)_{i,j}.
\end{aligned}$$

Hence the result. ☐

To compute κ^k_{ij} we use the following lemma.

Lemma 4.B.2. *For all $r, s \in [\![1, n]\!]$, for all $M \in \mathcal{M}_{2n}(\mathbb{R})$, let us define the (r, s) 2×2 sub-block of M, $\mathsf{B}_{rs}[M] \in \mathcal{M}_2(\mathbb{R})$ by*

$$\mathsf{B}_{rs}[M] = \begin{pmatrix} M_{2r-1,2s-1} & M_{2r,2s-1} \\ M_{2r-1,2s} & M_{2r,2s} \end{pmatrix}.$$

For all $\theta \in \mathbb{R}$, *let*

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad S(\theta) = \begin{pmatrix} \sin\theta & 1-\cos\theta \\ \cos\theta - 1 & \sin\theta \end{pmatrix}.$$

Then

$$\mathsf{B}_{rs}\left[(V(\sigma))_l\right] = \frac{1}{b_r} S(b_r\sigma) \mathsf{B}_{rs}\left[{}^t J_l\right] R(b_s\sigma) + \frac{1}{b_s} S(b_s\sigma) \mathsf{B}_{sr}\left[{}^t J_l\right] R(b_r\sigma).$$

Proof. Since the matrices $\bar{J}$ and $\mathrm{e}^{\sigma\bar{J}}$ are block-diagonal,

$$\mathsf{B}_{rs}\left[\left(\mathrm{e}^{-\sigma\bar{J}} - I_{2n}\right)\bar{J}^{-1}\,{}^t J_l\,\mathrm{e}^{\sigma\bar{J}}\right] = \mathsf{B}_{rr}\left[\left(I_{2n} - \mathrm{e}^{-\sigma\bar{J}}\right)\bar{J}^{-1}\right]\mathsf{B}_{rs}\left[{}^t J_l\right]\,\mathsf{B}_{ss}\left[\mathrm{e}^{\sigma\bar{J}}\right].$$

Hence the result since

$$\mathsf{B}_{rr}\left[\left(I_{2n} - \mathrm{e}^{-\sigma\bar{J}}\right)\bar{J}^{-1}\right] = \frac{1}{b_r} S(b_r\sigma), \qquad \mathsf{B}_{rr}\left[\left(\mathrm{e}^{\sigma\bar{J}}\right)\bar{J}^{-1}\right] = R(b_r\sigma), \qquad \forall r \in [\![1,n]\!].$$

$\square$

Some interesting computational properties can be deduced from this result.

Lemma 4.B.3. *Let*

$$\alpha = \frac{\pi}{b_1^3}\left(\frac{\partial^2 (X_2)_{2n+1}}{\partial x_1 \partial x_2}(q_0) - \frac{\partial^2 (X_1)_{2n+1}}{\partial x_2^2}(q_0)\right),$$
$$\beta = -\frac{\pi}{b_1^3}\left(\frac{\partial^2 (X_2)_{2n+1}}{\partial x_1^2}(q_0) - \frac{\partial^2 (X_1)_{2n+1}}{\partial x_1 \partial x_2}(q_0)\right).$$

Then

$$\kappa_{1,1}^1 = 3\alpha, \quad \kappa_{1,1}^2 = \beta,$$
$$\kappa_{2,2}^1 = \alpha, \quad \kappa_{2,2}^2 = 3\beta,$$
$$\kappa_{1,2}^1 = 2\beta, \quad \kappa_{1,2}^2 = 2\alpha.$$

Lemma 4.B.4. *For all* $i \in [\![2,n]\!]$, $\left(\kappa_{k,l}^m\right)_{\substack{k,m\in\{1,2\} \\ l\in\{2i-1,2i\}}}$ *only depend on the family*

$$\left\{\left(\frac{\partial^2 (X_k)_{2n+1}}{\partial x_l \partial x_m}(q_0)\right) \mid (k,l,m) \in \{2i-1,2i\}\times\{1,2\}^2 \cup \{1,2\}^2 \times \{2i-1,2i\}\right\}.$$

Let $\zeta_i : \mathbb{R}^{15} \to \mathbb{R}^8$ *be the linear map such that*

$$\zeta_i\left(\left(\frac{\partial^2 (X_k)_{2n+1}}{\partial x_l \partial x_m}(q_0)\right)_{k,l,m\in\{1,2\}\cup\{2i-1,2i\}}\right) = \left(\kappa_{k,l}^m\right)_{\substack{k,m\in\{1,2\} \\ l\in\{2i-1,2i\}}}$$

is of rank 8 *on the complementary of codimension* 1 *subset* $\mathfrak{S}_3 \subset M$, *and rank* 7 *on* $\mathfrak{S}$.

Proof. The first part of the result is a direct application of Lemma 4.B.2. Let $\bar{\zeta}_i$ be the restriction of ζ_i to

$$\left(\frac{\partial^2 (X_k)_{2n+1}}{\partial x_l \partial x_m}(q_0)\right)_{\substack{k,l\in\{1,2\}\\ m\in\{2i-1,2i\}}}.$$

Explicit computation of ζ_i yields that the rank of $\bar{\zeta}_i$ is 8, except for when

$$\begin{aligned} 0 =& 2\pi^2\rho^5 + 2\pi^2\rho^4 - 2\pi^2\rho^3 - 2\pi^2\rho^2 - 2\rho + 1 \\ &+ \left(-4\pi\rho^3 + 10\pi\rho^2 + 2\pi\rho\right)\sin(2\pi\rho) + \left(2\pi\rho^3 - 6\pi\rho^2 + 4\pi\rho\right)\sin(4\pi\rho) \\ &+ \left(-4\pi^2\rho^5 + 8\pi^2\rho^4 + 4\pi^2\rho^3 - 8\pi^2\rho^2 + 3\rho - 3\right)\cos(2\pi\rho) + (2-\rho)\cos(4\pi\rho) \end{aligned} \tag{4.27}$$

where $\rho = b_i/b_1 < 1$. Furthermore, if ρ satisfies (4.27), then the rank of $\bar{\zeta}_i$ is 7. Hence the existence of $\mathfrak{S}_3 \subset M$, by the existence of a codimension 1 constraint on the 1-jet of the sub-Riemannian structure at q_0. □

4.B.2 Third order invariants

In this section we compute a more precise approximation of the exponential map in the case of initial covectors of the form $(h_1, h_2, 0, \dots, 0, \eta^{-1}) \in T^*_{q_0}M$.

Lemma 4.B.5. *For all $T, R > 0$, normal extremals with initial covector $(\Lambda\bar{h}, \eta^{-1})$ have the following order 3 terms at time $\eta\tau$, uniformly with respect to $h(0) \in B_R$ and $\tau \in [0,T]$, as $\eta \to 0^+$:*

$$\begin{aligned} x^{(3)}(\tau, \Lambda\bar{h}) &= \int_0^\tau h^{(2)}(\sigma, \Lambda\bar{h})\,\mathrm{d}\sigma, \\ z^{(3)}(\tau, \Lambda\bar{h}) &= \int_0^\tau \left(h_2^{(1)}\widehat{x}_1 - h_1^{(1)}\widehat{x}_2 + \widehat{h}_1\left(X_1^{(2)}\right)_{2n+1} + \widehat{h}_2\left(X_2^{(2)}\right)_{2n+1}\right)(\sigma, \Lambda\bar{h})\,\mathrm{d}\sigma, \end{aligned}$$

with

$$\begin{aligned} h^{(2)}(\tau, \Lambda\bar{h}) = \int_0^\tau \mathrm{e}^{(\tau-\sigma)\bar{J}} \Big[& J^{(1)}(x^{(2)})\widehat{h} + J^{(1)}(\widehat{x})\,h^{(1)} + J^{(2)}(\widehat{x})\,\widehat{h} + J_z(\widehat{z})\,\widehat{h} \\ & + Q^{(1)}\left(\widehat{x}, \widehat{h}\right) - w^{(2)}\bar{J}\widehat{h}\Big](\sigma, \Lambda\bar{h})\,\mathrm{d}\sigma \end{aligned}$$

and

$$Q^{(1)}(x,h) = \sum_{i=1}^{2n} \frac{\partial Q(h)}{\partial x_i} x_i,$$

$$J^{(1)}(x) = \sum_{i=1}^{2n} \frac{\partial J}{\partial x_i} x_i \qquad J^{(2)}(x) = \sum_{i=1}^{2n}\sum_{j=1}^{2n} \frac{\partial^2 J}{\partial x_i \partial x_j} x_i x_j, \qquad J_z(z) = \frac{\partial J}{\partial z} z.$$

Proof. We have

$$\frac{\mathrm{d}q^{(3)}}{\mathrm{d}\tau} = \sum_{i=1}^{2n} \widehat{h}_i(\tau, \Lambda\bar{h}) X_i^{(2)}(\widehat{x}(\tau, \Lambda\bar{h})) + h_i^{(1)}(\tau, \Lambda\bar{h}) X_i^{(1)}(\widehat{x}(\tau, \Lambda\bar{h})) + h_i^{(2)}(\tau, \Lambda\bar{h}) X_i^{(0)}.$$

Since $\widehat{x}(\tau,\Lambda\bar{h})_i = 0$ and $\widehat{h}(\tau,\Lambda\bar{h}) = 0$ for $3 \leq i \leq 2n$, the horizontal part of

$$h_i^{(0)}(\tau,\Lambda\bar{h})X_i^{(2)}(\widehat{x}(\tau,\Lambda\bar{h}))$$

vanishes in the Agrachev–Gauthier frame. The same goes for the horizontal part of $X_i^{(1)}$, $1 \leq i \leq 2n$. Thus

$$\begin{aligned}\frac{\mathrm{d}x^{(3)}}{\mathrm{d}\tau} &= h^{(2)}(\tau,\Lambda\bar{h})\\ \frac{\mathrm{d}z^{(3)}}{\mathrm{d}\tau} &= \sum_{i=1}^{2n}\left[h^{(1)}\left(X_i^{(1)}\right)_{2n+1} + \widehat{h}\left(X_i^{(2)}\right)_{2n+1}\right](\tau,\Lambda\bar{h}).\end{aligned}$$

Regarding $h^{(2)}$, we get the result by computing the order 2 in η of $\frac{\mathrm{d}h}{\mathrm{d}\tau}$. We have

$$\frac{\mathrm{d}h}{\mathrm{d}\tau} = \frac{\eta}{w}Jh + \eta Q(h)$$

with

$$\begin{gathered}\frac{1}{w} = (1+\eta^2 w^{(2)} + O(\eta^3))^{-1} = 1 - \eta^2 w^{(2)} + O(\eta^3),\\ Q(h) = \eta Q^{(1)}(x^{(1)},h^{(0)}) + O(\eta^2),\\ J = \bar{J} + \eta J^{(1)}\left(x^{(1)}\right) + \eta^2\left(J^{(1)}(x^{(2)}) + J^{(2)}(x^{(1)}) + J_z(z^{(2)})\right) + O(\eta^3).\end{gathered}$$

Then evaluated at $(\tau,\Lambda\bar{h})$, we have

$$\frac{\mathrm{d}h^{(2)}}{\mathrm{d}\tau} = \bar{J}h^{(2)} + J^{(1)}\left(x^{(2)}\right)\widehat{h} + J^{(1)}\left(\widehat{x}\right)h^{(1)} + J^{(2)}\left(\widehat{x}\right)\widehat{h} + J_z\left(\widehat{z}\right)\widehat{h} + Q^{(1)}\left(\widehat{x},\widehat{h}\right)\widehat{h} - w^{(2)}\bar{J}\widehat{h}.$$

Hence the result. □

We can immediately apply this result to give an expression of $z^{(3)}$, using only the second order invariants introduced in the previous sections.

Lemma 4.B.6. *Using the prior notations, we have*

$$z^{(3)}\left(\frac{2\pi}{b_1},\Lambda\bar{h}\right) = \frac{1}{2}\left(\bar{h}_1^2+\bar{h}_2^2\right)(\alpha\bar{h}_1 + \beta\bar{h}_2).$$

Proof. As stated before, it is a matter of evaluating the terms for the Agrachev–Gauthier frame. We have

$$\frac{\mathrm{d}z^{(3)}}{\mathrm{d}\tau} = \sum_{i=1}^{2n}\left[h_i^{(1)}\left(X_i^{(1)}\right)_{2n+1} + \widehat{h}_i\left(X_i^{(2)}\right)_{2n+1}\right](\tau,\Lambda\bar{h}).$$

For $3 \leq i \leq 2n$, $\left(X_i^{(1)}\right)_{2n+1}\left(\widehat{x}\left(\tau,\Lambda\bar{h}\right)\right) = 0$,

$$\left(X_1^{(1)}\right)_{2n+1}\left(\widehat{x}\left(\tau,\Lambda\bar{h}\right)\right) = \frac{b_1}{2}\widehat{x}_2 \qquad \text{and} \qquad \left(X_2^{(1)}\right)_{2n+1}\left(\widehat{x}\left(\tau,\Lambda\bar{h}\right)\right) = -\frac{b_1}{2}\widehat{x}_1.$$

We have

$$h^{(1)}(\tau,\Lambda\bar{h}) = \int_0^\tau \mathrm{e}^{(\tau-\sigma)\bar{J}} J^{(1)}\left(\widehat{x}(\sigma,\Lambda\bar{h})\right)\widehat{h}(\sigma,\Lambda\bar{h})\,\mathrm{d}\sigma,$$

with

$$\left[J^{(1)}\left(\widehat{x}(\tau,\Lambda\bar{h})\right)\right]_{12} = \widehat{x}_1\frac{\partial c_{21}^0}{\partial x_1} + \widehat{x}_2\frac{\partial c_{21}^0}{\partial x_2}.$$

Since $\frac{\partial c_{21}^0}{\partial x_1} = \frac{b_1^3}{\pi}\beta$, and $\frac{\partial c_{21}^0}{\partial x_2} = -\frac{b_1^3}{\pi}\alpha$, with $h(0)=\Lambda\bar{h}$,

$$J^{(1)}\left(\widehat{x}\right)\widehat{h} = \frac{b_1^3}{\pi}\begin{pmatrix} \widehat{h}_2(\beta\widehat{x}_1-\alpha\widehat{x}_2) \\ -\widehat{h}_1(\beta\widehat{x}_1-\alpha\widehat{x}_2) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Similarly, we have

$$\left(X_1^{(2)}\right)_{2n+1} = \frac{b_1^3}{2\pi}(-\beta\widehat{x}_1\widehat{x}_2 + \alpha\widehat{x}_2^2/2),$$
$$\left(X_2^{(2)}\right)_{2n+1} = \frac{b_1^3}{2\pi}(\beta\widehat{x}_1^2/2 - \alpha\widehat{x}_1\widehat{x}_2).$$

We then obtain obtain the result by integration. □

Since we are only interested in the first two coordinates of the exponential map, we state the following result.

Lemma 4.B.7. *For all τ, for all $\bar{h}\in\mathbb{R}^{2n}$,*

$$Q_1^{(1)}\left(\widehat{x}(\tau,\Lambda\bar{h}),\widehat{h}(\tau,\Lambda\bar{h})\right) = Q_2^{(1)}\left(\widehat{x}(\tau,\Lambda\bar{h}),\widehat{h}(\tau,\Lambda\bar{h})\right) = 0.$$

Proof. Recall that $Q:\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ is the map such that

$$Q_i(h_1,\dots h_{2n}) = \sum_{j=1}^{2n}\sum_{k=1}^{2n} c_{ji}^k h_j h_k, \qquad i\in[\![1,2n]\!].$$

Since $h(\tau)=\widehat{h}(\tau,\Lambda\bar{h})+O(\eta)$ and $x(\tau)=\eta\widehat{x}(\tau,\Lambda\bar{h})+O(\eta^2)$,

$$Q_1(h) = c_{21}^1(\widehat{x})\widehat{h}_1\widehat{h}_2 + c_{21}^2(\widehat{x})\widehat{h}_2^2 + O(\eta^2),$$
$$Q_2(h) = c_{12}^1(\widehat{x})\widehat{h}_1^2 + c_{12}^2(\widehat{x})\widehat{h}_1\widehat{h}_2 + O(\eta^2).$$

Recall that for all $i,j\in[\![1,2n]\!]$

$$\frac{\partial c_{12}^j}{\partial x_i} = \frac{\partial(X_2)_j}{\partial x_i\partial x_1} - \frac{\partial(X_1)_j}{\partial x_i\partial x_2},$$

and thus in the Agrachev–Gauthier frame, evaluated at q_0,

$$\frac{\partial c_{12}^1}{\partial x_1} = \frac{\partial c_{12}^1}{\partial x_2} = \frac{\partial c_{12}^2}{\partial x_1} = \frac{\partial c_{12}^2}{\partial x_2} = 0.$$

Hence

$$Q_1^{(1)}\left(\widehat{x}(\tau,\Lambda\bar{h}),\widehat{h}(\tau,\Lambda\bar{h})\right) = Q_2^{(1)}\left(\widehat{x}(\tau,\Lambda\bar{h}),\widehat{h}(\tau,\Lambda\bar{h})\right) = 0.$$

□

Let us introduce the invariant $\xi \in \mathbb{R}$, given in the Agrachev–Gauthier frame by the formula

$$\xi = \frac{\pi}{b_1^3}\frac{\partial^2 X_1}{\partial z \partial x_2}(q_0).$$

This invariant, which is 0 in the 3-dimensional contact case, naturally appears in some terms of the third order expansion of the exponential map.

Lemma 4.B.8. *We have*

$$w^{(2)}(\tau,\Lambda\bar{h}) = -\frac{2b_1^2\xi}{\pi}\widehat{z}(\tau,\Lambda\bar{h})$$

and

$$J_z(\widehat{z}(\tau,\Lambda\bar{h}))\widehat{h}(\tau,\Lambda\bar{h}) = -\frac{2b_1^2\xi}{\pi}\widehat{z}(\tau,\Lambda\bar{h})\bar{J}\Lambda\widehat{h}(\tau).$$

Proof. As seen in the proof of Proposition 4.2.1, $\frac{\mathrm{d}w}{\mathrm{d}\tau} = -\eta w L h - \eta^2 w^2 Q_0(h) = O(\eta^2)$. Then

$$\frac{\mathrm{d}w^{(2)}}{\mathrm{d}\tau} = -w^{(1)}L^{(0)}h^{(0)} - w^{(0)}L^{(1)}h^{(0)} - w^{(0)}L^{(0)}h^{(1)} - Q_0^{(0)}\left(h^{(0)}\right).$$

In the Agrachev–Gauthier frame, $c_{i0}^j(q_0) = -\partial_z(X_i)_j$, for all $i,j \in [\![1,2n]\!]$. Hence $c_{i0}^j(q_0) = 0$, which implies $Q_0^{(0)} = 0$. Likewise, $c_{i0}^0(q_0) = -\partial_z(X_i)_{2n+1}$ for all $i \in [\![1,2n]\!]$, hence $c_{i0}^0(q_0) = 0$ and $L^{(0)} = 0$.

With $h(\tau) = \widehat{h}(\tau,\Lambda\bar{h}) + O(\eta)$ and $x(\tau) = \eta\widehat{x}(\tau,\Lambda h) + O(\eta^2)$, we then have

$$\frac{\mathrm{d}w^{(2)}}{\mathrm{d}\tau} = \left(\frac{\partial c_{10}^0}{\partial x_1}\widehat{x}_1 + \frac{\partial c_{10}^0}{\partial x_2}\widehat{x}_2\right)\widehat{h}_1 + \left(\frac{\partial c_{20}^0}{\partial x_1}\widehat{x}_1 + \frac{\partial c_{20}^0}{\partial x_2}\widehat{x}_2\right)\widehat{h}_2.$$

Again in the Agrachev–Gauthier frame, at q_0,

$$\frac{\partial c_{10}^0}{\partial x_1} = \frac{\partial c_{20}^0}{\partial x_2} = 0, \qquad \text{and} \qquad \frac{\partial c_{10}^0}{\partial x_2} = -\frac{\partial c_{20}^0}{\partial x_1} = -\frac{1}{2}\frac{\partial b_1}{\partial z} = -\frac{b_1^3\xi}{\pi}.$$

As a result

$$\frac{\mathrm{d}w^{(2)}}{\mathrm{d}\tau} = -\frac{b_1^3\xi}{\pi}\left(\widehat{x}_2\widehat{h}_1 - \widehat{x}_1\widehat{h}_2\right),$$

hence the result by recognizing $\frac{\mathrm{d}\widehat{z}}{\mathrm{d}\tau}$ and $w^{(2)}(0) = 0$.

The same reasoning applies for J_z, where $(J_z)_{1,2} = -\frac{\partial c_{21}^0}{\partial z} = -\frac{2b_1^3}{\pi}\xi$. □

We now know enough to compute $x^{(3)}(2\pi/b_1,\Lambda\bar{h})$ (or at least its first two coordinates). By direct integration we have the following expression for the terms of the expansion that depend on ξ.

Lemma 4.B.9. *Let*

$$x_{w^{(2)}} = \int_0^{2\pi/b_1} \int_0^{\tau} \mathrm{e}^{(\tau-\sigma)\bar{J}} \left[-w^{(2)} \bar{J}\widehat{h}\right] (\sigma, \Lambda\bar{h})\,\mathrm{d}\sigma\,\mathrm{d}\tau$$

and

$$x_{J_z} = \int_0^{2\pi/b_1} \int_0^{\tau} \mathrm{e}^{(\tau-\sigma)\bar{J}} \left[J_z\left(\widehat{z}\right)\widehat{h}\right] (\sigma, \Lambda\bar{h})\,\mathrm{d}\sigma\,\mathrm{d}\tau.$$

Then $x_{w^{(2)}} = -x_{J_z}$.

We use the same method to compute the other terms of the expansion. Let

$$\chi_{11} = -\frac{b_1^4}{\pi}\frac{\partial^3 X_1}{\partial x_1^2 \partial x_2}, \qquad \chi_{12} = \frac{2b_1^4}{\pi}\frac{\partial^3 X_1}{\partial x_1 \partial x_2^2}, \qquad \chi_{22} = -\frac{b_1^4}{\pi}\frac{\partial^3 X_1}{\partial x_2^3}.$$

Lemma 4.B.10. *Let*

$$x_{J^{(2)}} = \int_0^{2\pi/b_1} \int_0^{\tau} \mathrm{e}^{(\tau-\sigma)\bar{J}} \left[J^{(2)}\left(\widehat{x}\right)\widehat{h}\right] (\sigma, \Lambda\bar{h})\,\mathrm{d}\sigma\,\mathrm{d}\tau.$$

We have

$$\begin{aligned}
\left(x_{J^{(2)}}\right)_1 &= \left(\chi_{11} + 5\chi_{22}\right)\bar{h}_1^3 + 3\chi_{12}\bar{h}_2\bar{h}_1^2 + 3\left(\chi_{11} + \chi_{22}\right)\bar{h}_2^2\bar{h}_1 + \chi_{12}\bar{h}_2^3,\\
\left(x_{J^{(2)}}\right)_2 &= \left(5\chi_{11} + \chi_{22}\right)\bar{h}_2^3 + 3\chi_{12}\bar{h}_2^2\bar{h}_1 + 3\left(\chi_{11} + \chi_{22}\right)\bar{h}_1^2\bar{h}_2 + \chi_{12}\bar{h}_1^3.
\end{aligned}$$

Proof. First notice that

$$J^{(2)}\left(\widehat{x}(\tau, \Lambda\bar{h})\right)_{1,2} = -J^{(2)}\left(\widehat{x}(\tau, \Lambda\bar{h})\right)_{2,1} = \frac{\pi}{b_1^4}\left(-\chi_{11}\,\widehat{x}_1^2 + \chi_{12}\,\widehat{x}_1\widehat{x}_2 - \chi_{22}\,\widehat{x}_2^2\right)(\tau, \Lambda\bar{h}).$$

The stated result is obtained by direct integration. □

Lemma 4.B.11. *Let*

$$x_{J^{(1)}} = \int_0^{2\pi/b_1} \int_0^{\tau} \mathrm{e}^{(\tau-\sigma)\bar{J}} \left[J^{(1)}\left(x^{(2)}\right)\widehat{h} + J^{(1)}\left(\widehat{x}\right)h^{(1)}\right] (\sigma, \Lambda\bar{h})\,\mathrm{d}\sigma\,\mathrm{d}\tau.$$

We have

$$\begin{aligned}
\left(x_{J^{(1)}}(\tau, \Lambda\bar{h})\right)_1 =& \frac{1}{2b_1^2}\left[-\bar{h}_1^3\left(15\alpha^2 + 3\beta^2\right) + \bar{h}_1^2\bar{h}_2\left(4\pi\alpha^2 - 18\alpha\beta\right)\right.\\
&\left. -\bar{h}_1\bar{h}_2^2\left(9\alpha^2 - 8\pi\alpha\beta + 9\beta^2\right) + \bar{h}_2^3\left(4\pi\beta^2 - 6\alpha\beta\right)\right],\\
\left(x_{J^{(1)}}(\tau, \Lambda\bar{h})\right)_2 =& -\frac{1}{2b_1^2}\left[\bar{h}_1^3\left(4\pi\alpha^2 + 6\alpha\beta\right) + \bar{h}_1^2\bar{h}_2\left(9\alpha^2 + 8\pi\alpha\beta + 9\beta^2\right)\right.\\
&\left. + \bar{h}_1\bar{h}_2^2\left(4\pi\beta^2 + 18\alpha\beta\right) + \bar{h}_2^3\left(3\alpha^2 + 15\beta^2\right)\right].
\end{aligned}$$

Proof. Let $\tau \in \mathbb{R}$, $h \in \mathbb{R}^{2n}$. Evaluated at $(\tau, \Lambda\bar{h})$, we have

$$\left(J^{(1)}\left(x^{(2)}\right)\widehat{h}\right)_1 = \widehat{h}_2(\beta x_1^{(2)} - \alpha x_2^{(2)}), \qquad \left(J^{(1)}\left(x^{(2)}\right)\widehat{h}\right)_2 = -\widehat{h}_1(\beta x_1^{(2)} - \alpha x_2^{(2)})$$

and

$$\left(J^{(1)}\left(\widehat{x}\right)h^{(1)}\right)_1 = h_2^{(1)}(\beta\widehat{x}_1 - \alpha\widehat{x}_2), \qquad \left(J^{(1)}\left(\widehat{x}\right)h^{(1)}\right)_2 = -h_1^{(1)}(\beta\widehat{x}_1 - \alpha\widehat{x}_2).$$

Both $h^{(1)}$ and $x^{(2)}$ have been computed before and we have the stated result by integration. □

Summing up, we have proven the following.

Proposition 4.B.12. *We have* $\left[x^{(3)}\left(\frac{2\pi}{b_1}, \Lambda\bar{h}\right)\right]_{1,2} = [x_{J^{(1)}} + x_{J^{(2)}}]_{1,2}$. *Explicitly, this yields*

$$\begin{aligned}
\left[x^{(3)}\left(\frac{2\pi}{b_1}, \Lambda\bar{h}\right)\right]_1 =& \bar{h}_1^3\left(\frac{3}{2b_1^2}\left(5\alpha^2+\beta^2\right)+\chi_{11}+5\chi_{22}\right)\\
&+\bar{h}_1^2\bar{h}_2\left(\frac{\alpha}{b_1^2}(2\pi\alpha-9\beta)+3\chi_{12}\right)\\
&+\bar{h}_1\bar{h}_2^2\left(-\frac{1}{2b_1^2}\left(9\alpha^2-8\pi\alpha\beta+9\beta^2\right)+3(\chi_{11}+\chi_{22})\right)\\
&+\bar{h}_2^3\left(-\frac{\beta}{b_1^2}(2\pi\beta-3\alpha)+\chi_{12}\right),\\
\left[x^{(3)}\left(\frac{2\pi}{b_1}, \Lambda\bar{h}\right)\right]_2 =& \bar{h}_1^3\left(-\frac{\alpha}{b_1^2}(2\pi\alpha+3\beta)+\chi_{12}\right)\\
&+\bar{h}_1^2\bar{h}_2\left(-\frac{1}{2b_1^2}\left(9\alpha^2+8\pi\alpha\beta+9\beta^2\right)+3(\chi_{11}+\chi_{22})\right)\\
&+\bar{h}_1\bar{h}_2^2\left(-\frac{\beta}{b_1^2}(2\pi\beta+9\alpha)+3\chi_{12}\right)\\
&+\bar{h}_2^3\left(-\frac{3}{2b_1^2}\left(\alpha^2+5\beta^2\right)+5\chi_{11}+\chi_{22}\right).
\end{aligned}$$

4.C Computational lemmas

In this section we prove some computational results useful in multiple proofs. Let $n \in \mathbb{N}$, $n > 1$, and $b_1, \dots, b_n \in \mathbb{R}$ be such that $0 < b_i < b_1$ for all $i \in [\![2, n]\!]$.

Let $A \in \mathcal{M}_{2n-2}(\mathbb{R})$ be the block-diagonal square matrix

$$A = \begin{pmatrix} \frac{1}{b_2}\begin{pmatrix} \sin(\frac{2b_2\pi}{b_1}) & 1-\cos(\frac{2b_2\pi}{b_1}) \\ \cos(\frac{2b_2\pi}{b_1})-1 & \sin(\frac{2b_2\pi}{b_1}) \end{pmatrix} & & (0) \\ & \ddots & \\ (0) & & \frac{1}{b_n}\begin{pmatrix} \sin(\frac{2b_n\pi}{b_1}) & 1-\cos(\frac{2b_n\pi}{b_1}) \\ \cos(\frac{2b_n\pi}{b_1})-1 & \sin(\frac{2b_n\pi}{b_1}) \end{pmatrix} \end{pmatrix}.$$

Lemma 4.C.1. *We have*

$$\det(A) = 2^{2n-2}\prod_{i=2}^{n}\frac{1}{b_i^2}\sin^2\left(\frac{\pi b_i}{b_1}\right) > 0.$$

Proof. This is a consequence of

$$\begin{vmatrix} \sin(\frac{2b_i\pi}{b_1}) & 1-\cos(\frac{2b_i\pi}{b_1}) \\ \cos(\frac{2b_i\pi}{b_1})-1 & \sin(\frac{2b_i\pi}{b_1}) \end{vmatrix} = 4\sin^2\left(\frac{\pi b_i}{b_1}\right) \qquad \forall i \in [\![2, n]\!].$$

Since $0 < b_i < b_1$ for all $i \in [\![2, n]\!]$, we have the stated sign. □

Lemma 4.C.2. *Let $V, W \in \mathcal{M}_{1\times 2n-2}(\mathbb{R})$, $v \in \mathbb{R}$. Then*

$$\frac{1}{\det(A)}\left|\begin{array}{c|c} A & W \\ \hline {}^tV & v \end{array}\right| = v + \frac{1}{2}\sum_{i=2}^{n} b_i\left(V_{2i-1}W_{2i} - V_{2i}W_{2i-1} - (V_{2i-1}W_{2i-1} + V_{2i}W_{2i})\cot\frac{b_i\pi}{b_1}\right).$$

Proof. To prove this result, we develop along the last column the determinant of

$$\left(\begin{array}{c|c} A & W \\ \hline {}^tV & v \end{array}\right).$$

We get

$$\left|\begin{array}{c|c} A & W \\ \hline {}^tV & v \end{array}\right| = v\det(A) + \det(A)\sum_{i=2}^{n}\frac{b_i^2}{4\sin^2\left(\frac{\pi b_i}{b_1}\right)}\left(\frac{W_{2i-1}}{b_i}\begin{vmatrix}\cos(\frac{2b_i\pi}{b_1}) - 1 & \sin(\frac{2b_i\pi}{b_1}) \\ V_{2i-1} & V_{2i}\end{vmatrix} - \frac{W_{2i}}{b_i}\begin{vmatrix}\sin(\frac{2b_i\pi}{b_1}) & 1 - \cos(\frac{2b_i\pi}{b_1}) \\ V_{2i-1} & V_{2i}\end{vmatrix}\right).$$

Hence the result by trigonometric identification. □

Chapter 5

Stability of the caustics of five-dimensional contact sub-Riemannian manifolds

We apply the methods introduced in Chapter 4 to the case of 5-dimensional contact manifolds. We provide a stability analysis of the sub-Riemannian caustic under the assumptions previously discussed. We first discuss our method of approximation of the exponential map in regard of stability, then we classify the singular points of the exponential map on each of the three domains we exhibited in the previous chapter.

Contents

5.1 Introduction

We wish to apply the results of Chapter 4 to the study of stability of the caustic in the 5-dimensional case. This study has been carried for 3-dimensional contact sub-Riemannian manifolds in [EAGK96] and for 4-dimensional quasi-contact sub-Riemannian manifolds in [Cha02]. To understand the interest of stability in the sense of sub-Riemannian geometry in small dimension, we must first understand stability from the point of view of Lagrangian manifolds.

Lagrangian singularities. (See [AGnZV85, Chapters 18, 21] and also [Ben85, IFR$^+$16].) Let (E, σ) be a $2d$-dimensional symplectic manifold. A smooth submanifold L of M is said to be a *Lagrangian submanifold* if L is d-dimensional and $\sigma_{|L} = 0$. The fiber bundle $\pi : E \to N$ is said to be a *Lagrangian fibration* if its fibers are Lagrangian submanifolds. For L a Lagrangian submanifold of E and $i : L \to E$ an immersion of L into E such that $i^*\sigma = 0$, the map $\pi \circ i : L \to N$ is called a *Lagrangian map*.

Let (E, σ), (E', σ') be two symplectic structures, let $\pi : E \to N$, $\pi' : E' \to N'$ be two Lagrangian fibrations. Two Lagrangian maps $\pi \circ i : L \to N$, $\pi' \circ i' : L' \to N'$ are said to be Lagrange equivalent if there exists two diffeomorphisms $\Phi : E \to E'$ and $\phi : N \to N'$ such that $\Phi^*\sigma' = \sigma$, $\pi' \circ \Phi = \phi \circ \pi$ (the two Lagrangian fibrations are Lagrange equivalent) and $\Phi \circ i(L) = i'(L')$.

The *caustic* of a Lagrangian map is the set of its critical values. A consequence of the definition of Lagrangian equivalence is that if two Lagrangian maps are Lagrange equivalent then their caustics are diffeomorphic.

A Lagrangian map $f : L \to N$ is said to be *(Lagrange-)stable at $q \in L$* if there exists a neighborhood V_q of q and a neighborhood V_f of $f_{|V_q}$ for the Whitney C^∞-topology such that any Lagrangian map $g \in V_f$ is Lagrange equivalent to f (see [Cha02]). In the following we may refer to points of a caustic as stable when they are critical values of a stable Lagrangian map.

For dimensions $d \leq 5$, there exists only a finite number of equivalence classes for stable singularities of Lagrangian maps (for instance, one can find a summary in [BBCN16, Theorem 2]).

Theorem 5.1.1 (Lagrangian stability in dimension 5). *A generic Lagrangian map $f : \mathbb{R}^5 \to \mathbb{R}^5$ has only stable singularities of type $\mathcal{A}_2, \dots, \mathcal{A}_6$, $\mathcal{D}_4^\pm, \mathcal{D}_5^\pm, \mathcal{D}_6^\pm$ and $\mathcal{E}_6^\pm$.*

Sub-Riemannian exponential maps form a subclass of Lagrangian maps and we can define sub-Riemannian stability as Lagrangian stability restricted to the class of sub-Riemannian exponential maps. Notouriously, the point q_0 is an unstable critical value of the sub-Riemannian exponential $\mathcal{E}_{q_0}$, as the starting point of the geodesics defining $\mathcal{E}_{q_0}$.

In the following, we refer to the set of critical values of the sub-Riemannian exponential at $q_0 \in M$ as the (sub-Riemannian) caustic at q_0. Our study of the stability of the caustic can be summed up in the following theorem (see also Figures 5.1, 5.2).

Theorem 5.1.2 (Sub-Riemannian stability in dimension 5)**.** *Let (M, Δ, g) be a generic 5-dimensional contact sub-Riemannian manifold. There exists a stratified set $\mathfrak{S} \subset M$ of codimension 1 for which all $q_0 \in M \setminus \mathfrak{S}$ admit an open neighborhood V_{q_0} such that for all U open neighborhood of q_0 small enough, the intersection of the caustic at q_0 with $V_{q_0} \setminus U$ is (sub-Riemannian) stable and has only Lagrangian singularities of type $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$, $\mathcal{D}_4^+$ and $\mathcal{A}_5$.*

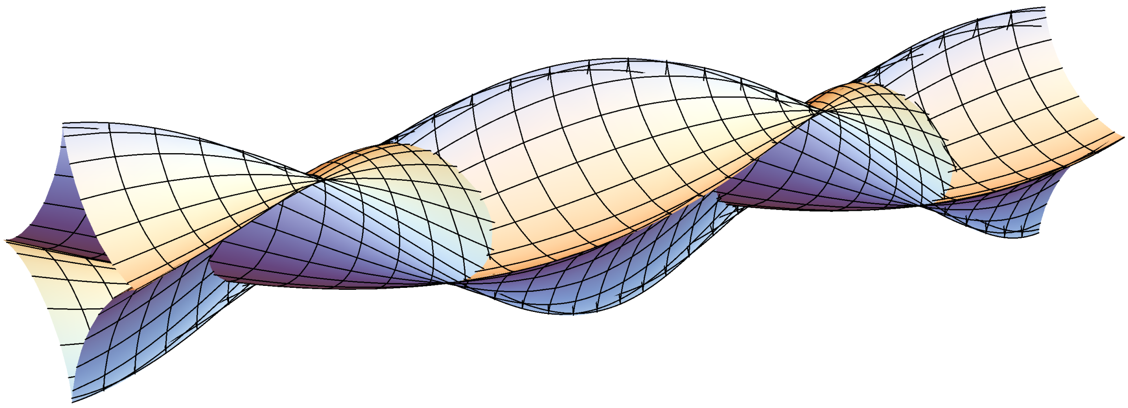


Figure 5.1 – Section of the caustic of a 5-dimensional sub-Riemannian manifold, at a point of the manifold chosen so that it exhibits $\mathcal{A}_4$ singularities. This representation is obtained after sectioning by the hyperplanes $\{z = z_0\}$, $\{x_3 = R_2 \cos\omega\}$, $\{x_4 = R_2 \sin\omega\}$ (all in Agrachev–Gauthier normal form coordinates), and plotting for all $\omega \in [0, 2\pi)$, with fixed $z_0, R_2 > 0$.

This result rests on two foundations. On the one hand, a careful study of the problem of conjugate points in contact sub-Riemannian manifolds, provided in Chapter 4, and on the other hand, a stability analysis from the point of view of Lagrangian singularities in small dimension.

5.2 Stability of the sub-Riemannian caustic

5.2.1 Sub-Riemannian to Lagrangian stability

The aim of the whole classification is to prove Theorem 5.1.2. Recall we denote by $\mathcal{E}^1_{q_0} : T^*_{q_0}M \to M$ the sub-Riemannian exponential at time 1, that is $\mathcal{E}^1_{q_0} = \mathcal{E}_{q_0}(1, \cdot)$. We first observe the following immediate fact.

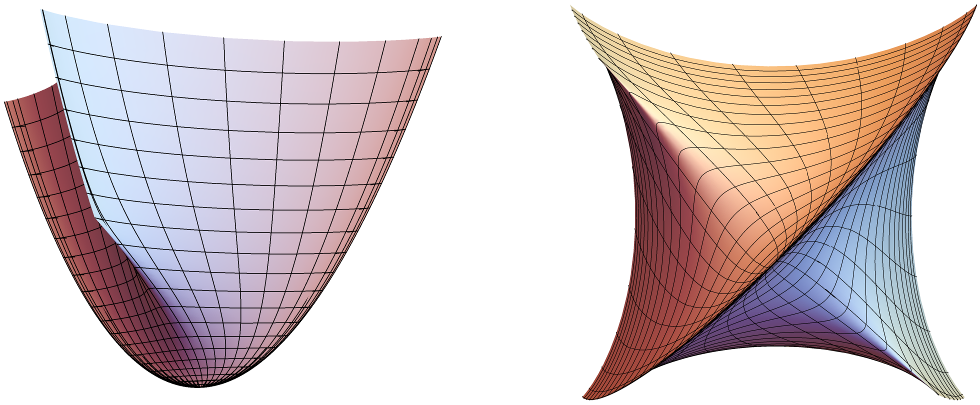


Figure 5.2 – Section of the caustic of a 5-dimensional sub-Riemannian manifold, at a point of the manifold chosen so that it exhibits $\mathcal{D}_4^+$ singularities. This representation is obtained after sectioning by the hyperplanes $\{z = z_0\}$, $\{x_3 = R_2 \cos\omega\}$, $\{x_4 = R_2 \sin\omega\}$, and plotting for all $z_0 \in [0, \bar{z}_0]$, with fixed $\bar{z}_0, R_2, \omega > 0$.

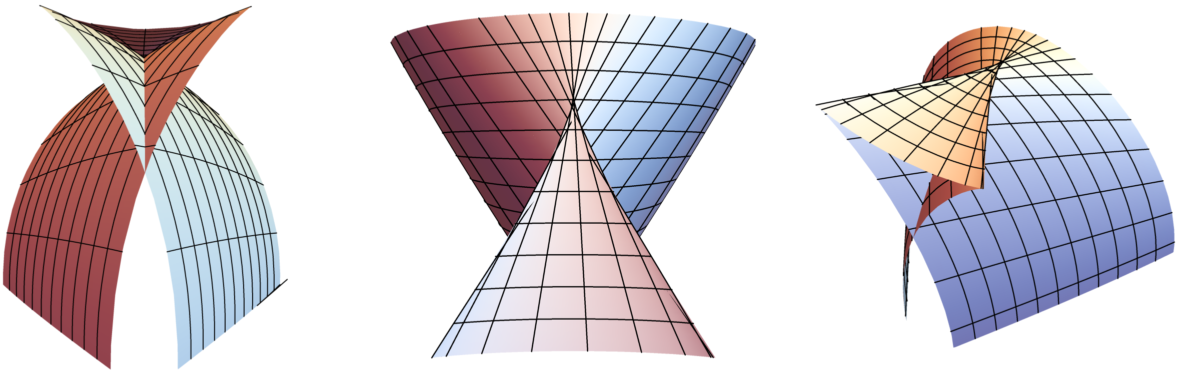


Figure 5.3 – Close-up of one $\mathcal{A}_4$ singularity, as appearing in Figure 5.1.

Proposition 5.2.1. *Let (M, Δ, g) be a sub-Riemannian manifold and let $q_0 \in M$. If the exponential map at time 1, $\mathcal{E}^1_{q_0} : T^*_{q_0}M \to M$, is Lagrange stable at $p \in T_{q_0}M$, then $\mathcal{E}^1_{q_0}$ is sub-Riemannian stable at p.*

Proof. By definition of Lagrangian stability of $\mathcal{E}^1_{q_0}$ at p, there exist a neighborhood $V_p \subset T^*_{q_0}M$, and a neighborhood V of $\mathcal{E}^1_{q_0|V_p}$ in the Whitney C^∞-topology such that any Lagrangian map $g \in V$ is Lagrange equivalent to $\mathcal{E}^1_{q_0|V_p}$.

Let $(\widetilde{\Delta}, \widetilde{g})$ be a sub-Riemannian structure on M, and let $\widetilde{\mathcal{E}}^1_{q_0} : T^*_{q_0}M \to M$ be its corresponding exponential map at time 1. Since $\widetilde{\mathcal{E}}^1_{q_0}$ is also a Lagrangian map, if $\left(\widetilde{\mathcal{E}}^1_{q_0}\right)_{|V_p} \in V$, then $\left(\widetilde{\mathcal{E}}^1_{q_0}\right)_{|V_p}$ is Lagrange equivalent to $\mathcal{E}^1_{q_0|V_p}$. Hence the restriction of V to the set of exponential maps is sufficient to have the sub-Riemannian stability of $\mathcal{E}^1_{q_0}$ at p. $\square$

We have stated before that the study of the sub-Riemannian caustic near its starting point requires to consider initial covectors in $\mathcal{C}_q(1/2)$ such that h_0 is near infinity. We have motivated this choice by analogy with the nilpotent case, but we can make this argument more precise. Observe that for covectors in $\mathcal{C}_q(1/2)$, the shorter the conjugate time, the closer the conjugate point is to the starting point of the caustic. Then what has to be proved is that a short conjugate time corresponds only to covectors with large h_0.

From the point of view of the exponential at time 1, this means that singular points close to the origin of the caustic must belong to a sufficiently narrow cone containing $\mathcal{C}_{q_0}(0)$ (this can be understood by considering that $\mathcal{E}_{q_0}(t,p) = \mathcal{E}^1_{q_0}(tp)$). This can be stated in the following way, considering that $\mathcal{E}^1_{q_0}(\mathcal{C}_{q_0}([0,R)))$ is a neighborhood of q_0.

Proposition 5.2.2. *Let (M,Δ,g) be a contact sub-Riemannian manifold and $q_0 \in M$. For all $\alpha > 0$, there exists $R > 0$ such that the set of singular points of the exponential at time 1 in $\mathcal{C}_{q_0}((0,R))$ is a subset of $\{h_0^2 > \alpha H\}$.*

Equivalently, for all $\bar{h}_0 > 0$, there exists $\varepsilon > 0$ such that all $p \in \mathcal{C}_q(1/2)$ with $t_c(p) < \varepsilon$ have $|h_0(p)| > \bar{h}_0$.

Proof. Notice that both statements are equivalent since any $p \in \mathcal{C}_q(1/2)$ satisfies $t_c(p) = \sqrt{2H(t_c(p)p, q_0)}$.

We prove this statement by contradiction. Assume there exist $\alpha > 0$ and a sequence of singular points for $\mathcal{E}^1_{q_0}$, $(p_k)_{k\in\mathbb{N}} \in \{H > 0\}$, such that $H(p_k, q_0) = \frac{1}{2k^2}$ and $h_0(p_k)^2 \leq \alpha H(p_k, q_0)$.

Then $kp_k = \frac{p_k}{\sqrt{2H(p_k,q_0)}} \in \mathcal{C}_q(1/2) \cap \{h_0^2 \leq \alpha/2\}$. The sequence $(kp_k)_{k\in\mathbb{N}}$ converges up to extraction and there exist $(k_n)_{n\in\mathbb{N}} \in \mathbb{N}$, $p'_\infty \in \mathcal{C}_q(1/2) \cap \{h_0^2 \leq \alpha/2\}$ such that $k_n p_{k_n} \to p'_\infty$.

Hence there exists a converging sequence $(p_{k_n})_{n\in\mathbb{N}} \in \mathcal{C}_q(1/2) \cap \{|h_0| \leq \alpha'\}$ that admits as conjugate time $t_c(p_{k_n}) = 1/k_n$. Let us prove that this is contradictory with the assumptions on the contact sub-Riemannian structure.

Since the sequence $(p_{k_n})_{n\in\mathbb{N}}$ converges towards p'_∞, we can chose an arbitrarily small neighborhood of p'_∞, $V \subset T^*_{q_0}M$, and assume the sequence $(p_{k_n})_{n\in\mathbb{N}}$ stays in V. Then we use the expansion of $q(t) = \mathcal{E}_{q_0}(t, h_1, \dots, h_{2n}, h_0)$, uniform with respect to $p \in V$,

$$q(1/k) = \sum_{l=1}^{3} \frac{q^{(l)}(0)}{k^l l!} + o(1/k^4).$$

We use the Agrachev–Gauthier normal form to prove that this map cannot be singular for $p \in V$ and k large enough.

Indeed, notice first that the Jacobian of $\dot{q}(0) = \sum_{i=1}^{2n} h_i(0) X_i(q_0)$ is just the diagonal matrix $\mathrm{diag}(1,\dots,1,0)$. Furthermore, for all $i \in [\![1,n]\!]$, as a consequence of (4.21)-(4.24),

$$h_{2i-1} D_{q_0} X_{2i-1} \dot{q}(0) = (0,\dots,0, 2b_i h_{2i} h_{2i-1}) \text{ and } h_{2i} D_{q_0} X_{2i} \dot{q}(0) = (0,\dots,0,-2b_i h_{2i} h_{2i-1}),$$

hence the last line of the Jacobian of $\ddot{q}(0)$ is empty. Thus the Jacobian matrix has the form

$$\mathrm{Jac}_p q(1/k) = \frac{1}{k}\mathrm{diag}(1,\dots,1,0) + \frac{1}{k^2}\begin{pmatrix} * & \cdots & * \\ \vdots & * & \vdots \\ * & \cdots & * \\ 0 & \cdots & 0 \end{pmatrix} + O\left(\frac{1}{k^3}\right).$$

Hence if the $(2n+1, 2n+1)$-coefficient is not a $o(1/k^3)$, the Jacobian matrix has a non-zero determinant for k large enough.

Then for $i \in [\![1, 2n]\!]$, $\partial_{h_0}\partial_t^2(h_i(t)X_i(q(t))_{|t=0} = \partial_{h_0}\dot{h}_i(0)D_{q_0}X_i{\cdot}h(0) = \left(\bar{J}h(0)\right)_i\left(2\bar{J}h(0)\right)_i$ and the $(2n+1, 2n+1)$-coefficient is $2|Jh(0)|_2^2 > 0$, hence the result. □

This proposition then implies that we have to classify Lagrangian stable singularities of the sub-Riemannian exponential with inital covectors in $\mathcal{C}_q(1/2)$ such that h_0 is very large. In Section 5.4, we prove the following theorem, of which Theorem 5.1.2 is a corollary.

Theorem 5.2.3. *Let (M, Δ, g) be a generic 5-dimensional contact sub-Riemannian manifold and let $q_0 \in M \setminus \mathfrak{S}$. There exist $\bar{\eta} > 0$ such that for all $(h_1, h_2, h_3, h_4, h_0) \in \mathcal{C}_q(1/2) \cap \{|h_0| > \bar{\eta}^{-1}\}$, the first conjugate point of $\mathcal{E}_{q_0}$ with initial covector $(h_1, h_2, h_3, h_4, h_0)$ is a Lagrange stable singular point of type $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$, $\mathcal{D}_4^+$ or $\mathcal{A}_5$.*

Assuming Theorem 5.2.3 holds, we can now prove Theorem 5.1.2.

Proof of Theorem 5.1.2. As a consequence of Proposition 5.2.1, we prove the Lagrange stability of the singular points of $\mathcal{E}_{q_0}^1$. For all $t > 0$, $p_0 \in T_{q_0}^*M$, $\mathcal{E}_{q_0}^1(tp_0) = \mathcal{E}_{q_0}(t, p_0)$. Hence for a given covector $p_0 \in \{H \neq 0\}$, $t_c(p_0)p_0$ is a critical point of $\mathcal{E}_{q_0}^1$.

Recall that for all $q \in M$, we have set $\mathcal{A}_{q_0} = \overline{\{t_c(p_0)p_0 \mid H(p_0, q_0) = 1/2\}}$, and the caustic is the set $\mathcal{E}_{q_0}^1(\mathcal{A}_{q_0})$.

Since $\mathcal{E}_{q_0}^1(\mathcal{C}_q(0)) = q_0$, to prove the statement it is sufficient to show the existence of V_{q_0} neighborhood of q_0 such that $\mathcal{E}_{q_0}^1$ is Lagrange stable at every point of $\mathcal{A}_{q_0} \cap \left(\mathcal{E}_{q_0}^1\right)^{-1}(V_{q_0}) \cap \{H > 0\}$ (and satisfies the stated classification). As a result of Theorem 5.2.3, what remains to prove is that there exists $R > 0$ such that for all covectors $p \in \mathcal{A}_{q_0} \cap \mathcal{C}_{q_0}((0, R))$,

$$\frac{p}{\sqrt{2H(p, q_0)}} \in \mathcal{C}_q(1/2) \cap \{|h_0| > \bar{\eta}^{-1}\}$$

with $\bar{\eta} > 0$ as in the statement of Theorem 5.2.3, but this is Proposition 5.2.2. □

5.2.2 Classification methodology

We first recall normal forms for the stable singularities that appear in Theorem 5.2.3.

Definition 5.2.4. Let $f : \mathbb{R}^5 \to \mathbb{R}^5$ be a smooth map singular at $q \in \mathbb{R}^5$. Assume there exist variables x centered at q and and variables centered at $f(q)$ such that

— $f(x_1, \dots, x_5) = (x_1^2,\ x_2,\ x_3,\ x_4,\ x_5)$, then the singularity is of type $\mathcal{A}_2$;
— $f(x_1, \dots, x_5) = (x_1^3 + x_1x_2,\ x_2,\ x_3,\ x_4,\ x_5)$, then the singularity is of type $\mathcal{A}_3$;
— $f(x_1, \dots, x_5) = (x_1^4 + x_1^2x_2 + x_1x_3,\ x_2,\ x_3,\ x_4,\ x_5)$, then the singularity is of type $\mathcal{A}_4$;
— $f(x_1, \dots, x_5) = (x_1^5 + x_1^3x_2 + x_1^2x_3 + x_1x_4,\ x_2,\ x_3,\ x_4,\ x_5)$, then the singularity is of type $\mathcal{A}_5$;
— $f(x_1, \dots, x_5) = (x_1^2 + x_2^2 + x_1x_3,\ x_1x_2,\ x_3,\ x_4,\ x_5)$, then the singularity is of type $\mathcal{D}_4^+$.

We use these normal forms to characterize the singularities in terms of jets.

Proposition 5.2.5. *Let M be a 5-dimensional manifold, let $q_0 \in M$ and let $g : T^*_{q_0}M \to M$ be a Lagrangian map. Let p_0 be a critical point of g. If the kernel $\ker \mathrm{Jac}_{p_0} g$ is 1-dimensional, then the singularity will be of type*

— $\mathcal{A}_2$ *if there exists a set of coordinates $(x_1, \dots, x_5)$ such that*

$$\partial_{x_1} g = 0,\ \partial^2_{x_1} g \notin \mathrm{imJac}_{p_0} g;$$

— $\mathcal{A}_3$ *if there exists a set of coordinates $(x_1, \dots, x_5)$ such that*

$$\partial_{x_1} g = 0,\ \partial^2_{x_1} g \in \mathrm{imJac}_{p_0} g$$

whereas

$$\partial_{x_1}\partial_{x_2} g,\ \partial^3_{x_1} g \notin \mathrm{imJac}_{p_0} g;$$

— $\mathcal{A}_4$ *if there exists a set of coordinates $(x_1, \dots, x_5)$ such that*

$$\partial_{x_1} g = 0,\ \partial^2_{x_1} g,\ \partial^3_{x_1} g,\ \partial_{x_1}\partial_{x_2} g \in \mathrm{imJac}_{p_0} g$$

whereas

$$\partial_{x_1}\partial_{x_3} g,\ \partial^2_{x_1}\partial_{x_2} g,\ \partial^4_{x_1} g \notin \mathrm{imJac}_{p_0} g;$$

— $\mathcal{A}_5$ *if there exists a set of coordinates $(x_1, \dots, x_5)$ such that*

$$\partial_{x_1} g = 0,\ \partial^2_{x_1} g,\ \partial^3_{x_1} g,\ \partial^4_{x_1} g,\ \partial_{x_1}\partial_{x_2} g,\ \partial^2_{x_1}\partial_{x_2} g,\ \partial_{x_1}\partial_{x_3} g\ \in \mathrm{imJac}_{p_0} g$$

whereas

$$\partial_{x_1}\partial_{x_4} g,\ \partial^2_{x_1}\partial_{x_3} g,\ \partial^3_{x_1}\partial_{x_2} g,\ \partial^5_{x_1} g \notin \mathrm{imJac}_{p_0} g.$$

If $\ker \mathrm{Jac}_{p_0} g$ is 2-dimensional, something similar can be said for $\mathcal{D}_4^+$ singularities. Indeed the singularity is of type $\mathcal{D}_4^+$ if there exist coordinates $(x_1, \dots, x_5)$ such that $\partial_{x_1} g = \partial_{x_2} g = 0$, $\partial_{x_1}\partial_{x_2} g \notin \mathrm{imJac}_{p_0} g$,

$$\partial^2_{x_1} g, \partial^2_{x_2} g, \partial_{x_1}\partial_{x_3} g \notin \mathrm{imJac}_{p_0} g \oplus \mathrm{Span}(\partial_{x_1}\partial_{x_2} g),$$

and finally

$$\det\left(\partial^2_{x_1} g, \partial_{x_1}\partial_{x_2} g, \partial_{x_3} g, \partial_{x_4} g, \partial_{x_5} g\right) \cdot \det\left(\partial^2_{x_2} g, \partial_{x_1}\partial_{x_2} g, \partial_{x_3} g, \partial_{x_4} g, \partial_{x_5} g\right) > 0. \tag{5.1}$$

Proof. This is a matter of proving that g has the same k-jets as the normal form for $\mathcal{A}_k$ singularities, $k \in [\![2,5]\!]$, and 2-jet for $\mathcal{D}_4^+$. For each of the stated cases, the existence of changes of coordinates at p_0 and $g(p_0)$ such that it is the case is then warranted by the stated conditions. $\square$

Remark 5.2.6. Equation (5.1) corresponds to the distinction between $\mathcal{D}_4^+$ and $\mathcal{D}_4^-$ singularities, the latter corresponding to the opposite sign.

Let (M, Δ, g) be a contact sub-Riemannian manifold of dimension 5 and let $q_0 \in M$. A covector $p_0 = (h_1, h_2, h_3, h_4, h_0) \in T^*M$ is a critical point of the exponential at t if the pair (t, p_0) satisfies

$$\det\left(\frac{\partial \mathcal{E}_{q_0}}{\partial h_1}, \frac{\partial \mathcal{E}_{q_0}}{\partial h_2}, \frac{\partial \mathcal{E}_{q_0}}{\partial h_3}, \frac{\partial \mathcal{E}_{q_0}}{\partial h_4}, \frac{\partial \mathcal{E}_{q_0}}{\partial h_0}\right)\Bigg|_{(t,p_0)} = 0.$$

With $F(\tau, h, \eta) = \mathcal{E}_{q_0}(\eta\tau; (h, \eta^{-1}))$, for all $\tau > 0$, $h \in \mathbb{R}^4$, $\eta > 0$, recall we have

$$\frac{\partial \mathcal{E}_{q_0}}{\partial h_i}(\eta\tau; (h, \eta^{-1})) = \frac{\partial F}{\partial h_i}(\tau, h, \eta), \quad \forall i \in [\![1,4]\!]$$

and

$$\frac{\partial \mathcal{E}_{q_0}}{\partial h_0}(\eta\tau; (h, \eta^{-1})) = -\eta\left(\eta\frac{\partial F}{\partial \eta}(\tau, h, \eta) - \tau\frac{\partial F}{\partial \tau}(\tau, h, \eta)\right).$$

Hence for a given $p_0 = (h, \eta^{-1})) \in T^*_{q_0}M$ and a given $t > 0$, the Jacobian matrix $\operatorname{Jac}\mathcal{E}_{q_0}(t) : \mathbb{R}^5 \to \mathcal{M}_5(\mathbb{R})$ is the matrix

$$\left(\frac{\partial F}{\partial h_1}, \frac{\partial F}{\partial h_2}, \frac{\partial F}{\partial h_3}, \frac{\partial F}{\partial h_4}, -\eta^2\frac{\partial F}{\partial \eta} + \eta\tau\frac{\partial F}{\partial \tau}\right)\Bigg|_{\tau = t/\eta}.$$

To study the sub-Riemannian caustic, we study for a given p_0 the stability at $p_0 \in \mathcal{C}_q(1/2)$ of $\mathcal{E}_{q_0}(t_c(p_0), \cdot)$. To apply Proposition 5.2.5, we first compute an approximation the linear spaces $\ker \operatorname{Jac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0))$ and $\operatorname{imJac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0))$. Then we compute approximations of derivatives of order larger than 1 of $\mathcal{E}_{q_0}(t_c(p_0))$ and check the conditions from Proposition 5.2.5 by computing

$$v \mapsto \det\left(v, \operatorname{imJac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0))\right),$$

after choosing a suitable representation of $\operatorname{imJac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0))$.

Let S_1, S_2 be the subspaces of $T^*_{q_0}M$ such that

$$S_1 = \left\{(h_1, h_2, h_3, h_4, h_0) \in T^*_{q_0}M \mid h_1 = h_2 = 0\right\},$$
$$S_2 = \left\{(h_1, h_2, h_3, h_4, h_0) \in T^*_{q_0}M \mid h_3 = h_4 = 0\right\}.$$

As a consequence of Section 4.3, this stability analysis is carried independently on the three domains of initial covectors : $T^*_{q_0}M \setminus (S_1 \cup S_2)$, near S_1 and near S_2.

Remark 5.2.7. Let $\tau \in \mathbb{R}^+$ and $(h, \eta) \in \mathbb{R}^5$. The map $\mathcal{E}_{q_0}(\eta\tau)$ is critical at (h, η^{-1}) if there exists $v \in \mathbb{R}^5$ such that $\mathrm{Jac}_{p_0}\mathcal{E}_{q_0}(\eta\tau) \cdot v = 0$.

Denoting $\partial_i = \partial_{h_i}$, for all $i \in [\![1, 4]\!]$, and $\partial_5 = -\eta^2\partial_\eta + \eta\tau\partial_\tau$, we have

$$\mathrm{Jac}_{p_0}\mathcal{E}_{q_0}(\eta\tau) = (\partial_1 F, \partial_2 F, \partial_3 F, \partial_4 F, \partial_5 F)\,.$$

Then for a coordinate $\mathrm{x}_1 : M \to \mathbb{R}$ such that $\partial_{\mathrm{x}_1} = \sum_{i=1}^5 v_i\partial_i$, we have $\partial_{\mathrm{x}_1}\mathcal{E}_{q_0}(\eta\tau, p_0) = 0$.

Hence, rather than giving a description of the coordinates $(\mathrm{x}_1, \dots, \mathrm{x}_5)$ at each critical point, we use an implicit characterization by instead focusing on computing the vectors $v_1, \dots, v_5 \in \mathbb{R}^5$ such that

$$\partial_{\mathrm{x}_k} = \sum_{i=1}^{5} (v_k)_i \partial_i, \qquad \forall i \in [\![1, 5]\!].$$

This way we can compute partial derivatives of $\mathcal{E}_{q_0}$ using only the usual coordinates $(h_1, h_2, h_3, h_4, h_0)$. For instance, if we denote by id the identity map in coordinates $(h_1, h_2, h_3, h_4, h_0)$, then $v_i = \partial_{\mathrm{x}_i}\mathrm{id}$ and we have the following implicit formulations. For $f : \mathbb{R}^5 \to \mathbb{R}$,

$$\begin{aligned}
\partial_{\mathrm{x}_1} f &= Df(\partial_{\mathrm{x}_1}\mathrm{id}) = Df(v_1),\\
\partial_{\mathrm{x}_2}\partial_{\mathrm{x}_1} f &= D^2 f(\partial_{\mathrm{x}_1}\mathrm{id}, \partial_{\mathrm{x}_2}\mathrm{id}) + Df(\partial_{\mathrm{x}_2}\partial_{\mathrm{x}_1}\mathrm{id})\\
&= D^2 f(v_1, v_2) + Df(Dv_1(v_2)),
\end{aligned}$$

$$\begin{aligned}
\partial_{\mathrm{x}_3}\partial_{\mathrm{x}_2}\partial_{\mathrm{x}_1} f =& D^3 f(\partial_{\mathrm{x}_1}\mathrm{id}, \partial_{\mathrm{x}_2}\mathrm{id}, \partial_{\mathrm{x}_3}\mathrm{id}) + D^2 f(\partial_{\mathrm{x}_3}\mathrm{id}, \partial_{\mathrm{x}_2}\partial_{\mathrm{x}_1}\mathrm{id}) + D^2 f(\partial_{\mathrm{x}_3}\partial_{\mathrm{x}_1}\mathrm{id}, \partial_{\mathrm{x}_2}\mathrm{id})\\
&+ D^2 f(\partial_{\mathrm{x}_1}\mathrm{id}, \partial_{\mathrm{x}_3}\partial_{\mathrm{x}_2}\mathrm{id}) + Df(\partial_{\mathrm{x}_3}\partial_{\mathrm{x}_2}\partial_{\mathrm{x}_1}\mathrm{id})\\
=& D^3 f(v_1, v_2, v_3) + D^2 f(v_3, Dv_1(v_2)) + D^2 f(Dv_1(v_3), v_2)\\
&+ D^2 f(v_1, Dv_2(v_3)) + Df\left(D^2 v_1(v_2, v_3) + Dv_1(Dv_2(v_3))\right),
\end{aligned}$$

and so on.

5.3 Description of the Jacobian matrix

In this section we compute an approximation of the kernel of the Jacobian matrix of the exponential. Recall that the rank is lower semi-continuous as a map from $\mathcal{M}_5(\mathbb{R})$ to $\mathbb{N}$. This implies, as a consequence of the previous sections, that the Jacobian matrix can have a kernel of dimension at most 2 at times near $2\pi/b_1$, as it is the case for the first order approximation $\widehat{\mathcal{E}}$.

We decompose the matrix $\mathrm{Jac}_{p_0}\mathcal{E}_{q_0}$ into the following sub matrices:

$$\left(\begin{array}{c|c|c} A_1 & A_2 & C_1 \\ \hline A_3 & A_4 & C_2 \\ \hline L_1 & L_2 & E \end{array}\right)$$

with $A_1, A_2, A_3, A_4 \in \mathcal{M}_{2\times 2}(\mathbb{R})$, $L_1, L_2 \in \mathcal{M}_{1\times 2}(\mathbb{R})$, $C_1, C_2 \in \mathcal{M}_{2\times 1}(\mathbb{R})$ and $E \in \mathcal{M}_{1\times 1}(\mathbb{R})$.

A vector v in the kernel of $\operatorname{Jac}_{p_0}\mathcal{E}_{q_0}$ must the satisfy the equations

$$A_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + A_2 \begin{pmatrix} v_3 \\ v_4 \end{pmatrix} + C_1 v_5 = 0, \tag{5.2}$$

$$A_3 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + A_4 \begin{pmatrix} v_3 \\ v_4 \end{pmatrix} + C_2 v_5 = 0, \tag{5.3}$$

$$L_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + L_2 \begin{pmatrix} v_3 \\ v_4 \end{pmatrix} + E v_5 = 0. \tag{5.4}$$

In the following three sections, we compute approximations of elements of the kernel with initial covectors of the form

$$\left(h_1, h_2, h_3, h_4, \eta^{-1}\right), \qquad \left(\sqrt{\eta}h_1, \sqrt{\eta}h_2, h_3, h_4, \eta^{-1}\right) \qquad \text{and} \qquad \left(h_1, h_2, \eta h_3, \eta h_4, \eta^{-1}\right).$$

All expansions as $\eta \to 0$ are assumed uniform under the condition $h_1^2 + h_2^2 + h_3^3 + h_4^4 < R$ for some arbitrary $R > 0$.

Remark 5.3.1. The following computations make abundant use of explicit expressions of the approximations of the exponential map obtained in Section 4.3. Readers wishing to precisely follow the computations are referred to Propositions 4.2.1, 4.4.8 and 4.5.1 for a general expression of the approximation of the exponential map, and the results of Section 4.3 and Appendix 4.B for expressions in terms of structural invariants.

5.3.1 Initial covectors in $T_{q_0}^*M \setminus (S_1 \cup S_2)$

From computations of the conjugate time, we know that $\ker \operatorname{Jac}_{p_0}\mathcal{E}_{q_0} \neq \{0\}$ at $t = t_c(p_0)$. Let us compute a first approximation of the set of solutions of the equation $\operatorname{Jac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0)) \cdot v = 0$ (thanks to our approximation of $F(\tau) = \mathcal{E}(\eta\tau)$).

Proposition 5.3.2. *The kernel of* $\operatorname{Jac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0))$ *is* 1*-dimensional and there exists* $\nu(h_1, h_2, h_3, h_4)$ *such that it is generated by the vector*

$$(-h_2, h_1, 0, 0, \nu) + O(\eta).$$

Proof. According to the computations carried in Section 4.4.1, we have

$$A_i = O(\eta^2), \quad A_4 \neq O(\eta^2), \qquad i \in [\![1, 3]\!],$$

$$C_i = O(\eta^2), \quad L_i = O(\eta^2), \quad E = O(\eta^3), \qquad i \in [\![1, 2]\!].$$

Regarding C_1, C_2, E, this is in particular a consequence of $\partial_5 F = -\eta^2\partial_\eta F + \eta\tau\partial_\tau F$. Then (5.3) implies $v_3 = O(\eta)$ and $v_4 = O(\eta)$ since $A_4^{(1)}$ is invertible, and from (5.4) we obtain

$$L_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = O(\eta^3).$$

That is $h_1v_1 + h_2v_2 = O(\eta)$, hence there exists $\lambda \in \mathbb{R}$ such that $v_1 = -\lambda h_2 + O(\eta)$, $v_2 = \lambda h_1 + O(\eta)$. Similarly, (5.2) yields

$$A_1^{(2)} \begin{pmatrix} v_1^{(0)} \\ v_2^{(0)} \end{pmatrix} + C_1^{(2)} v_5^{(0)} = 0.$$

Since $\tau_c^{(1)}$ corresponds to the fact that $A_1^{(2)} \begin{pmatrix} -h_2 \\ h_1 \end{pmatrix}$ is colinear to $C_1^{(2)} = \dfrac{2\pi}{b_1} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$, with $(v_1^{(0)}, v_2^{(0)}) = \lambda(-h_2, h_1)$, $v_5^{(0)}$ is uniquely defined, linearly dependent on λ. Similarly, we compute

$$\begin{pmatrix} v_3^{(1)} \\ v_4^{(1)} \end{pmatrix} = -\left(A_4^{(1)}\right)^{-1} \left(v_5^{(0)} C_2^{(2)} + A_3^{(2)} \begin{pmatrix} v_1^{(0)} \\ v_2^{(0)} \end{pmatrix} \right).$$

Hence the statement. The kernel of $\mathrm{Jac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0))$ is in particular 1-dimensional as a consequence of the lower semi-continuity of the rank. $\square$

Regarding the image space, we have can give a description as a consequence of Lemma 5.3.2.

Lemma 5.3.3. *Let $p_0 \in T^*_{q_0}M \setminus (S_1 \cup S_2)$. The image of the Jacobian at p_0 of the exponential at the conjugate time admits the representation*

$$\mathrm{im}\mathrm{Jac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0)) = \mathrm{Span}\left\{ h_1\partial_1 F + h_2\partial_2 F, \partial_3 F, \partial_4 F, \partial_5 F \right\}.$$

Proof. Let v_1 be such that $\ker \mathrm{Jac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0)) = \mathrm{Span}(v_1)$. For any 4 vectors v_2, v_3, v_4, v_5 such that $\mathrm{rk}(v_1, v_2, v_3, v_4, v_5) = 5$, we have the property that

$$\mathrm{im}\mathrm{Jac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0)) = \mathrm{Span}\left(\sum_{k=1}^{5} (v_i)_k \partial_k F \right)_{i \in [\![2,5]\!]}.$$

One possible choice is then $v_2 = (h_1, h_2, 0, 0, 0)$, $v_3 = (0, 0, 1, 0, 0)$, $v_4 = (0, 0, 0, 1, 0)$, and $v_5 = (0, 0, 0, 0, 1)$. $\square$

5.3.2 Initial covectors near S_1

The idea is the same as before, now we consider initial covectors of the form

$$p_0 = \left(\sqrt{\eta}h_1, \sqrt{\eta}h_2, h_3, h_4, \eta^{-1}\right).$$

Proposition 5.3.4. *If there exist a time near $2\pi\eta/b_1$ that is conjugate for p_0 then the kernel of $\mathrm{Jac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0))$ is either 1 or 2-dimensional. If $(h_1, h_2) \neq (0, 0)$ then there exist two vectors*

$$v_{\theta_1} = (-h_2, h_1, 0, 0, 0) + O(\eta) \quad \textit{and} \quad v_{r_1} = \left(h_1, h_2, 0, 0, -\frac{2\pi(h_1^2 + h_2^2)}{b_1 K'} \right) + O(\eta)$$

such that the kernel of $\mathrm{Jac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0))$ is either $\mathrm{Span}\left(\lambda_{\theta_1} v_{\theta_1} + \lambda_{r_1} v_{r_1}\right)$ or $\mathrm{Span}\left(v_{\theta_1}, v_{r_1}\right)$.

Proof. From the computations in Section 4.4.2, we have

$$A_i = O(\eta^{5/2}), \quad A_4 \neq O(\eta^2), \qquad i \in [\![1,3]\!],$$

$$C_1 = O(\eta^{5/2}), \quad L_1 = O(\eta^3), \quad C_2 = O(\eta^2), \quad L_2 = O(\eta^2), \quad E = O(\eta^3).$$

As previously, (5.3) implies $v_3 = O(\eta)$ and $v_4 = O(\eta)$ and similarly to Section 5.3.1, $\left(v_3^{(1)}, v_4^{(1)}\right)$ can be computed as

$$\begin{pmatrix} v_3^{(1)} \\ v_4^{(1)} \end{pmatrix} = -v_5^{(0)} \left(A_4^{(1)}\right)^{-1} C_2^{(2)}.$$

Hence the smallest non-vanishing order of the system (5.2)-(5.3)-(5.4) reduces to the 3×3 system

$$A_1^{(5/2)} \begin{pmatrix} v_1^{(0)} \\ v_2^{(0)} \end{pmatrix} + C_1^{(5/2)} v_5^{(0)} = 0, \tag{5.5}$$

$$L_1^{(3)} \begin{pmatrix} v_1^{(0)} \\ v_2^{(0)} \end{pmatrix} + \left(E^{(3)} - L_2^{(2)} \left(A_4^{(1)}\right)^{-1} C_2^{(2)}\right) v_5^{(0)} = 0. \tag{5.6}$$

Now observe that $E^{(3)} - L_2^{(2)} \left(A_4^{(1)}\right)^{-1} C_2^{(2)} = -\frac{2\pi}{b_1} K'$, where K' is the constant introduced in Lemma 4.4.9. Furthermore from Propositions 4.4.10 and 4.4.11, we know that the first conjugate time is a perturbation of $2\pi\eta/b_1$ if

$$\det \left(\begin{array}{c|c} A_1^{(5/2)} & C_1^{(5/2)} \\ \hline L_1^{(3)} & \frac{2\pi}{b_1} K' \end{array}\right) = 0. \tag{5.7}$$

When that is the case, the set of solutions of (5.5)-(5.6) is at least 1-dimensional, otherwise it is reduced to $\{0\}$.

Assume (5.7) holds and that $(h_1, h_2) \neq (0,0)$. Let us denote $e_{r_1} = (h_1, h_2)$ and $e_{\theta_1} = (-h_2, h_1)$. There exist unique $\lambda_{r_1}, \lambda_{\theta_1} \in \mathbb{R}$ such that $\left(v_1^{(0)}, v_2^{(0)}\right) = \lambda_{r_1} e_{r_1} + \lambda_{\theta_1} e_{\theta_1}$. Since $\left(\partial_{h_1} F_5^{(3)}, \partial_{h_2} F_5^{(3)}\right) \in \mathrm{Span}(e_{r_1})$, we have from (5.6) that

$$v_5^{(0)} = -\lambda_{r_1} \frac{b_1 L_1^{(3)} e_{r_1}}{2\pi K'},$$

and from (5.5) we get

$$\lambda_{r_1} \left(A_1^{(5/2)} e_{r_1} - \frac{b_1 L_1^{(3)} e_{r_1}}{2\pi K'} C_1\right) + \lambda_{\theta_1} A_1^{(5/2)} e_{\theta_1} = 0.$$

Recall that $L_1^{(3)} = \frac{2\pi}{b_1} \begin{pmatrix} h_1 & h_2 \end{pmatrix}$, thus $\frac{b_1 L_1^{(3)} e_{r_1}}{2\pi K'} = \frac{h_1^2 + h_2^2}{K'}$. Elements of the kernel must be linear combination of the vectors

$$v_{\theta_1} = (-h_2, h_1, 0, 0, 0) + O(\eta) \quad \text{and} \quad v_{r_1} = (h_1, h_2, 0, 0, -(h_1^2 + h_2^2)/K') + O(\eta).$$

Assuming (5.7) holds, there are two cases:

1. Either $A_1^{(5/2)} e_{r_1} + \frac{h_1^2+h_2^2}{K'} C_1 \neq 0$ or $A_1^{(5/2)} e_{\theta_1} \neq 0$, and the kernel is a 1-dimensional space generated by a linear combination of v_{θ_1} and v_{r_1}.
2. Both $A_1^{(5/2)} e_{r_1} + \frac{h_1^2+h_2^2}{K'} C_1 = 0$ and $A_1^{(5/2)} e_{\theta_1} = 0$, and the kernel is the 2-dimensional space $\mathrm{Span}(v_{\theta_1}, v_{r_1})$.

If $h_1 = h_2 = 0$, assuming (5.7) holds implies that $v_5^{(0)} = 0$ and the kernel is of the dimension of $\ker A_1^{(5/2)}$. □

Remark 5.3.5. Notice that a 2-dimensional kernel implies that the conjugate time is a zero of order 2, that is, $\Delta = 0$. (The converse may not be true however.) Indeed, if $(h_1, h_2) \neq (0,0)$, $A_1^{(5/2)} e_{\theta_1} = 0$ implies we must have for some $a, b \in \mathbb{R}$

$$A_1^{(5/2)} = \begin{pmatrix} ah_1 & ah_2 \\ bh_1 & bh_2 \end{pmatrix}.$$

Then $A_1^{(5/2)} e_{r_1} = -\frac{h_1^2+h_2^2}{K'} C_1$ implies $a = -h_1 \frac{2\pi}{b_1 K'}$, $b = -h_2 \frac{2\pi}{b_1 K'}$. Under these conditions, one can check that the zero is of order 2.

If $(h_1, h_2) = (0,0)$ however, having a 2-dimensional kernel corresponds to $A_1^{(5/2)} = 0$. However, in that case, using notations from Theorem 4.3.6, this implies that $\gamma_2^1 = \gamma_2^1 = \gamma_1^1 - \gamma_2^2 = 0$. From Proposition 4.4.7, this is exactly stating that $q_0 \in \mathfrak{S}_2$, hence the kernel of $\mathrm{Jac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0))$ for an initial covector p_0 in S_1 is of dimension at most 1 at points of $M \setminus \mathfrak{S}_2$.

Finally, let us give a useful description of the image set of the Jacobian matrix of $\mathcal{E}_{q_0}(t_c(p_0))$ in the case of 1D kernel with initial covector such that $(h_1, h_2) \neq (0,0)$.

Let $\lambda_{r_1}, \lambda_{\theta_1}$ be such that $\mathrm{Span}(\lambda_{r_1} v_{r_1} + \lambda_{\theta_1} v_{\theta_1}) = \ker \mathrm{Jac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0))$, and let V, W be two vectors in the image set of $\mathrm{Jac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0))$ such that

$$W = \partial_5 \bar{F} - \eta w_3 \partial_3 \bar{F} - \eta w_4 \partial_4 \bar{F}, \text{ with } \begin{pmatrix} w_3 \\ w_4 \end{pmatrix} = -\left(A_4^{(1)}\right)^{-1} C_2^{(2)}$$

and

$$V = -\lambda_{\theta_1}\left(h_1 \partial_1 \bar{F} + h_2 \partial_2 \bar{F} + \frac{(h_1^2+h_2^2)}{K'} W\right) + \lambda_r \left(-h_2 \partial_1 \bar{F} + h_1 \partial_2 \bar{F}\right).$$

They have been chosen to simplify low order terms in their expansions as $\eta \to 0$. Indeed

$$W_1 = \eta^{5/2} 2\pi / b_1 h_1 + o(\eta^{5/2}), \quad W_2 = \eta^{5/2} 2\pi / b_1 h_2 + o(\eta^{5/2}),$$

$$(W_3, W_4) = o(\eta^{5/2}) \quad \text{and} \quad W_5 = -\eta^3 \frac{2\pi}{b_1} K' + o(\eta^3).$$

Likewise, $(V_1, V_2) \neq o(\eta^{5/2})$ but $V_3, V_4 = O(\eta^{5/2})$ and $V_5 = o(\eta^3)$. (This observation is useful for the next section in particular.)

Lemma 5.3.6. *Assume* $p_0 = \left(\sqrt{\eta}h_1, \sqrt{\eta}h_2, h_3, h_4, \eta^{-1}\right)$ *is an initial covector such that* $(h_1, h_2) \neq (0,0)$ *and the kernel of* $\mathrm{Jac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0))$ *is of dimension* 1. *Then*

$$\mathrm{imJac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0)) = \mathrm{Span}\left\{V, W, \partial_3 F, \partial_4 F\right\}.$$

Proof. The proof is analogous to the proof of Lemma 5.3.3. The kernel is spanned by $\lambda_{\theta_1} v_{\theta_1} + \lambda_{r_1} v_{r_1}$.

Let $v_3 = (0,0,1,0,0)$, $v_4 = (0,0,0,1,0)$, $w = (0,0,-\eta w_3, -\eta w_4, 1)$ and

$$v = -\lambda_{\theta_1}(v_{r_1} - \eta w_3 v_3 - \eta w_4 v_4) + \lambda_{r_1} v_{\theta_1}.$$

By construction,

$$\mathrm{rk}(\lambda_{\theta_1} v_{\theta_1} + \lambda_{r_1} v_{r_1}, v, w, v_3, v_4) = 5,$$

Hence the result since $V = \mathrm{Jac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0)) \cdot v$ and $W = \mathrm{Jac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0)) \cdot w$. □

5.3.3 Initial covectors near S_2

We now consider initial covectors of the form

$$p_0 = \left(h_1, h_2, \eta h_3, \eta h_4, \eta^{-1}\right).$$

The approach here is similar to Section 5.3.1, however we need two orders of approximation. For two matrices $A, B \in \mathcal{M}_n(\mathbb{R})$, and two vectors $u, v \in \mathbb{R}^n$, having $(A + \eta B)(u + \eta v) = 0$ yields $Au = 0$ and $Av + Bu = 0$. This relates to the computation of the conjugate time in Section 4.5, but we only proved $\det(A + \eta B) = o(\eta)$, hence the existence *a priori* of $u \in \mathbb{R}^n$ such that $Au = 0$ but not of $v \in \mathbb{R}^n$ such that $Av + Bu = 0$.

Lemma 5.3.7. *Let* $A, B \in \mathcal{M}_n(\mathbb{R})$. *If* $\mathrm{rank}(A) = n - 1$ *and* $\det(A + \eta B) = o(\eta)$ *as* $\eta \to 0$ *then* $B \cdot \ker A \subset \mathrm{im}A$.

Proof. Since $\mathrm{rank}(A) = n - 1$, there exists $P, Q \in \mathrm{GL}_n(\mathbb{R})$ such that $A = PA'Q$, with A' the diagonal matrix with diagonal $(0, 1, \ldots, 1)$. Let $u \in \ker A \setminus \{0\}$. Then Qu is colinear to $e_1 = (1, 0, \ldots, 0)$. Without loss of generality, we can assume $Qu = e_1$. Then, denoting $B' = P^{-1}BQ^{-1}$, $Bu \in \mathrm{im}A$ is equivalent to $B'e_1 \in \mathrm{im}A'$, that is $B'_{11} = 0$.

On the other hand $\det(A + \eta B) = o(\eta)$ implies $\det(A' + \eta B') = o(\eta)$, and developing the determinant with respect to η yields $\det(A' + \eta B') = \eta B'_{11} + o(\eta)$. Hence the result. □

Proposition 5.3.8. *The kernel of* $\mathrm{Jac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0))$ *is* 1*-dimensional and there exists* $\nu(h_1, h_2) \in \mathbb{R}$, $\mu_3(h_1, h_2, h_3, h_4) \in \mathbb{R}$, *such that* $\ker \mathrm{Jac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0))$ *is generated by the vector*

$$(-h_2, h_1, *, *, \nu) + \eta\left[\frac{3\nu}{4}(h_1, h_2, *, *, 0) + \mu_3\left(-\nu h_2, \nu h_1, *, *, -\left(h_1^2 + h_2^2\right)\right)\right] + O(\eta^2).$$

Proof. From computations in Section 4.5, we have

$$A_1 = O(\eta^2), \quad A_3 = O(\eta^2), \quad A_4 = O(\eta^2), \quad \text{and} \quad A_3 = O(\eta^3),$$

$$C_1 = O(\eta^2), \quad L_1 = O(\eta^2), \quad C_2 = O(\eta^3), \quad L_2 = O(\eta^3), \quad E = O(\eta^3).$$

Equation (5.4) then implies

$$L_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = O(\eta^3).$$

Hence, as previously, there exists $\lambda \in \mathbb{R}$ such that $(v_1, v_2) = \lambda(-h_2, h_1) + O(\eta)$. Now however, since $A_4 = O(\eta^2)$ and $A_2 = O(\eta^3)$,

$$A_3 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + A_4 \begin{pmatrix} v_3 \\ v_4 \end{pmatrix} = O(\eta^3)$$

and

$$A_1 \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + C_1 v_5 = O(\eta^3).$$

Hence we have

$$v_5^{(0)} C_1^{(2)} = -A_1^{(2)} \begin{pmatrix} v_1^{(0)} \\ v_2^{(0)} \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} v_3^{(0)} \\ v_4^{(0)} \end{pmatrix} = -\left(A_4^{(2)}\right)^{-1} A_3^{(2)} \begin{pmatrix} v_1^{(0)} \\ v_2^{(0)} \end{pmatrix}.$$

The lower semi-continuity of the rank implies that the kernel is indeed 1-dimensional. We can apply Lemma 5.3.7 and compute $v^{(1)} \in \ker A^{\perp}$ such that (focusing on $v_1^{(1)}, v_2^{(1)}, v_5^{(1)}$)

$$A_1^{(2)} \begin{pmatrix} v_1^{(1)} \\ v_2^{(1)} \end{pmatrix} + \left(A_1^{(3)} - A_2^{(3)} \left(A_4^{(2)}\right)^{-1} A_3^{(2)}\right) \begin{pmatrix} v_1^{(0)} \\ v_2^{(0)} \end{pmatrix} + v_5^{(0)} C_1^{(3)} + v_5^{(1)} C_1^{(2)} = 0 \tag{5.8}$$

$$L_1^{(2)} \begin{pmatrix} v_1^{(1)} \\ v_2^{(1)} \end{pmatrix} + L_1^{(3)} \begin{pmatrix} v_1^{(0)} \\ v_2^{(0)} \end{pmatrix} + v_5^{(0)} E^{(3)} = 0. \tag{5.9}$$

We can assume $(h_1, h_2) \neq (0, 0)$, since we are considering covectors near S_2 but not S_1. Still focusing on $v_1^{(1)}, v_2^{(1)}, v_5^{(1)}$ and looking for solutions in $\ker A^{\perp}$, we use a more suited basis of $\mathbb{R}^3$. We have ν such that $\nu C_1^{(2)} = -A_1^{(2)} \begin{pmatrix} -h_2 \\ h_1 \end{pmatrix}$, so that with $f_1 = (-h_2, h_1, \nu)$, $\left(v_1^{(0)}, v_2^{(0)}, v_5^{(0)}\right) = \lambda f_1$. Then we set $f_2 = (h_1, h_2, 0)$ and $f_3 = (-\nu h_2, \nu h_1, -(h_1^2 + h_2^2))$, and $\left(v_1^{(1)}, v_2^{(1)}, v_5^{(1)}\right) = \mu_2 f_2 + \mu_3 f_3$.

Then Equations (5.8)-(5.9) yield

$$\mu_2 A_1^{(2)} e_{r_1} + \mu_3 \nu A_1^{(2)} e_{\theta_1} + \lambda \left(A_1^{(3)} - A_2^{(3)} \left(A_4^{(2)}\right)^{-1} A_3^{(2)}\right) e_{\theta_1} + \lambda \nu C_1^{(3)} - \mu_3 (h_1^2 + h_2^2) C_1^{(2)} = 0,$$

$$\mu_2 L_1^{(2)} e_{r_1} + \lambda L_1^{(3)} e_{\theta_1} + \lambda \nu E^{(3)} = 0.$$

Then $\mu_2 = -\frac{\lambda}{L_1^{(2)} e_{r_1}} \left(L_1^{(3)} e_{\theta_1} + \nu E^{(3)} \right) = 3\lambda\nu/4$ (see the proof of Lemma 4.5.3 to find an explicit expression of $L_1^{(3)}$ and $E^{(3)}$) and

$$\lambda \frac{3}{4} \nu A_1^{(2)} e_{r_1} + \lambda \left(A_1^{(3)} - A_2^{(3)} \left(A_4^{(2)} \right)^{-1} A_3^{(2)} \right) e_{\theta_1} + \lambda\nu C_1^{(3)} = \mu_3 (h_1^2 + h_2^2 + \nu^2) C_1^{(2)}.$$

Hence the result with $\mu = \mu_3/\lambda$. □

Again, we end the section with a handy description of the image of $\mathrm{Jac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0))$. Let

$$V' = h_1 \partial_1 \bar{F} + h_2 \partial_2 \bar{F} - w_3 \partial_3 \bar{F} - w_4 \partial_4 \bar{F}, \text{ where } \begin{pmatrix} w_3 \\ w_4 \end{pmatrix} = \left(A_4^{(2)} \right)^{-1} A_3^{(2)} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix},$$

so that $(V_3', V_4') = O(\eta^3)$.

Lemma 5.3.9. *Let $p_0 = (h_1, h_2, \eta h_3, \eta h_4, \eta^{-1}) \in T_{q_0}^* M$. The image of the Jacobian matrix at p_0 of the exponential at the conjugate time admits the representation*

$$\mathrm{imJac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0)) = \mathrm{Span}\left\{ V', \partial_3 F, \partial_4 F, \partial_5 F \right\}.$$

Proof. The proof is again straightforward. With v such that $\ker \mathrm{Jac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0)) = \mathrm{Span}(v)$, as given by Proposition 5.3.8, $v' = (h_1, h_2, w_3, w_4, 0)$, $v_3 = (0,0,1,0,0)$, $v_4 = (0,0,0,1,0)$, $v_5 = (0,0,0,0,1)$, it is immediate that

$$\mathrm{rk}(v, v', v_3, v_4, v_5) = 5.$$

Hence the result, similarly to Lemma 5.3.3. □

5.4 Singularity classification

For a given $p_0 \in T_{q_0}^* M$ we compute explicit conditions for the jets to be as discussed in Proposition 5.2.5. For some $k \in \mathbb{N}$ and some $i_1, \ldots, i_k \in [\![1, 5]\!]$, we compute approximations as $\eta \to 0$ of

$$\partial_{\mathrm{x}_{i_1}} \ldots \partial_{\mathrm{x}_{i_k}} F \pmod{\mathrm{imJac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0))}$$

and (depending on either the kernel of the Jacobian matrix is of dimension 1 or 2)

$$\phi_{i_1,\ldots,i_k}(p_0) = \det \left(\partial_{\mathrm{x}_{i_1}} \ldots \partial_{\mathrm{x}_{i_k}} F, \mathrm{imJac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0)) \right), \quad \text{if } \dim\ker \mathrm{Jac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0)) = 1,$$

$$\phi'_{i_1,\ldots,i_k}(p_0) = \det \left(\partial_{\mathrm{x}_{i_1}} \ldots \partial_{\mathrm{x}_{i_k}} F, \partial_{x_1}\partial_{x_2} F, \mathrm{imJac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0)) \right), \quad \text{if } \partial_{x_1} F = \partial_{x_2} F = 0.$$

In terms of $\phi_{i_1,\ldots i_k}$, we have the following immediate corollary of Proposition 5.2.5.

Corollary 5.4.1. *Let M be a 5-dimensional manifold, let $q_0 \in M$ and $p_0 \in T^*_{q_0}M$. Assume the kernel $\ker \mathrm{Jac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0))$ is 1-dimensional, if there exists coordinates $(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4, \mathrm{x}_5)$ such that $\partial_{x_1}F = 0$ and one of the following holds*

- *$\phi_{11}(p_0) \neq 0$ then p_0 is a singular point of type $\mathcal{A}_2$;*
- *$\phi_{11}(p_0) = 0$, $\phi_{111} \cdot \phi_{12}(p_0) \neq 0$ then p_0 is a singular point of type $\mathcal{A}_3$;*
- *$\phi_{11}(p_0) = \phi_{111} \cdot \phi_{12}(p_0) = 0$, $\phi_{1111} \cdot \phi_{112} \cdot \phi_{13}(p_0) \neq 0$ then p_0 is a singular point of type $\mathcal{A}_4$;*
- *$\phi_{11}(p_0) = \phi_{111} \cdot \phi_{12}(p_0) = \phi_{1111} \cdot \phi_{112} \cdot \phi_{13}(p_0) = 0$, $\phi_{11111} \cdot \phi_{1112} \cdot \phi_{113}\phi_{14}(p_0) \neq 0$ then p_0 is a singular point of type $\mathcal{A}_5$.*

Assume the kernel $\ker \mathrm{Jac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0))$ is 2-dimensional, if there exists coordinates $(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4, \mathrm{x}_5)$ such that $\partial_{x_1}F = \partial_{x_2}F = 0$ and $\phi'_{11} \cdot \phi'_{22}(p_0) > 0$, $\phi'_{13}(p_0) \neq 0$ then p_0 is a singular point of type $\mathcal{D}_4^+$.

Recall that we are considering points $q_0 \in M \setminus (\mathfrak{S}_1 \cup \mathfrak{S}_2)$, where $\mathfrak{S}_1$ (introduced at the beginning of Section 4.3) and $\mathfrak{S}_2$ (introduced in Proposition 4.4.7) are both stratified subsets of M of codimension 1 at most.

Remark 5.4.2. Precisely checking the conditions of Corollary 5.4.1 will require explicit computations executed in the computer algebra system Mathematica. An example of this procedure is laid out in Appendix 5.A.

5.4.1 Classification on $T^*_{q_0}M \setminus (S_1 \cup S_2)$

We first introduce a computational lemma approximate the functions ϕ.

Lemma 5.4.3. *For all $i \in [\![1, 5]\!]$, let $U_i : \mathbb{R}^4 \to \mathbb{R}$ and let*

$$\Psi(u_1, u_2, u_5) = -u_5 \left({}^t e_{\theta_1} A_1^{(2)} e_{r_1}\right) + \frac{2\pi}{b_1}(h_1^2 + h_2^2)\,(h_1 u_2 - h_2 u_1)\,.$$

Then we have for $p_0 = (h, \eta^{-1})$, uniformly with respect to $h \in B_R$ as $\eta \to 0$,

$$\det\left(U(h), h_1\partial_1 F + h_2\partial_2 F, \partial_3 F, \partial_4 F, \partial_5 F\right) = \eta^6 \frac{8\pi}{b_1 b_2^2} \sin^2\left(\frac{\pi b_2}{b_1}\right) \Psi(U_1(h), U_2(h), U_5(h)) + o(\eta^6).$$

Proof. We compute the dominant term of $\det(h_1\partial_1 F + h_2\partial_2 F, \partial_3 F, \partial_4 F, \partial_5 F, U(h))$. Using notations from Section 5.3.1 and a similar reasoning to what can be found in Section 4.4, we obtain

$$\det(h_1\partial_1 F + h_2\partial_2 F, \partial_3 F, \partial_4 F, \partial_5 F, U(h)) = \left|\begin{array}{c|cc|c|c} \multicolumn{1}{c|}{\multirow{2}{*}{$\eta^2 A_1^{(2)} e_{r_1}$}} & 0 & 0 & \multirow{2}{*}{$\eta^2 \frac{2\pi}{b_1} e_{r_1}$} & U_1(h) \\ & 0 & 0 & & U_2(h) \\ \hline 0 & \multicolumn{2}{c|}{\multirow{2}{*}{$\eta A_4^{(1)}$}} & 0 & 0 \\ 0 & & & 0 & 0 \\ \hline \eta^2 \frac{2\pi}{b_1}(h_1^2 + h_2^2) & 0 & 0 & 0 & U_5(h) \end{array}\right| + o(\eta^6).$$

We have the result once observed that $\det A_4^{(1)} = \frac{4}{b_2^2}\sin^2\left(\frac{\pi b_2}{b_1}\right)$ and

$$\det\left(\begin{array}{c|c|c} A_1^{(2)}e_{r_1} & e_{r_1} & \begin{matrix} U_1 \\ U_2 \end{matrix} \\ \hline \frac{2\pi}{b_1}(h_1^2+h_2^2) & 0 & U_5 \end{array}\right) = -U_5\left({}^t e_{\theta_1} A_1^{(2)} e_{r_1}\right) + \frac{2\pi}{b_1}(h_1^2+h_2^2)\left(h_1U_2 - h_2U_1\right)$$
$$= \Psi(U_1, U_2, U_5).$$

□

Let $p_0 \in T^*_{q_0}M \setminus (S_1 \cup S_2)$ and v be as in the statement of Proposition 5.3.2 so that $\ker \mathrm{Jac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0)) = \mathrm{Span}(v)$. As explained in Remark 5.2.7, we choose the first coordinate $\mathrm{x}_1 : M \to \mathbb{R}$ such that $\partial_{\mathrm{x}_1} = \sum_{i=1}^5 v_i\partial_i$. Since $v_3, v_4 = O(\eta)$, we have that $\partial^k_{x_1}F = O(\eta^2)$ for all integer $k \geq 2$.

If we denote $V' : \mathbb{R}^4 \to \mathbb{R}^5$ such that $\partial^2_{x_1}F = \eta^2 V'(h) + o(\eta^2)$ then let $\Psi_2(h) = \Psi(V'_1, V'_2, V'_5)$. Similarly, define $V'' : \mathbb{R}^4 \to \mathbb{R}^5$ such that $\partial^3_{x_1}F = \eta^2 V''(h) + o(\eta^2)$ and $V''' : \mathbb{R}^4 \to \mathbb{R}^5$ such that $\partial^4_{x_1}F = \eta^2 V'''(h) + o(\eta^2)$; and define $\Psi_3(h) = \Psi(V''_1, V''_2, V''_5)$, $\Psi_4(h) = \Psi(V'''_1, V'''_2, V'''_5)$.

Since the length of expressions is still manageable in this case, we can give the explicit form of Ψ_2, Ψ_3 and Ψ_4 (up to multiplication by $2\pi(h_1^2+h_2^2)/b_1$):

$$\begin{aligned}
\Psi_2(h_1,h_2,h_3,h_4) = &- (h_3(\kappa^2_{13} + 2\kappa^1_{23}) + h_4(\kappa^2_{14} + 2\kappa^1_{24}))h_1^2 \\
&+ 3(h_3(\kappa^1_{13} - \kappa^2_{23}) + h_4(\kappa^1_{14} - \kappa^2_{24}))h_1h_2 \\
&+ (h_3(2\kappa^2_{13} + \kappa^1_{23}) + h_4(2\kappa^2_{14} + \kappa^1_{24}))h_2^2, \\
\Psi_3(h_1,h_2,h_3,h_4) = &+ (h_3(\kappa^1_{13} - \kappa^2_{23}) + h_4(\kappa^1_{14} - \kappa^2_{24}))h_1^2 \\
&+ 2(h_3(\kappa^2_{13} + \kappa^1_{23}) + h_4(\kappa^2_{14} - \kappa^1_{24}))h_1h_2 \\
&- (h_3(\kappa^1_{13} - \kappa^2_{23}) + h_4(\kappa^1_{14} - \kappa^2_{24}))h_2^2, \\
\Psi_4(h_1,h_2,h_3,h_4) = &- (h_3(3\kappa^2_{13} + 4\kappa^1_{23}) + h_4(3\kappa^2_{14} + 4\kappa^1_{24}))h_1^2 \\
&+ 7(h_3(\kappa^1_{13} - \kappa^2_{23}) + h_4(\kappa^1_{14} - \kappa^2_{24}))h_1h_2 \\
&+ (h_3(4\kappa^2_{13} + 3\kappa^1_{23}) + h_4(4\kappa^2_{14} + 3\kappa^1_{24}))h_2^2.
\end{aligned}$$

As an application of Lemma 5.4.3, and the analysis of the Jacobian matrix of $\mathcal{E}_{q_0}(t_c(p_0))$ of Section 5.3.1, we immediately obtain that for η small enough

$$\Psi_2(h) \neq 0 \Rightarrow \phi_{11}(p_0) \neq 0, \qquad \Psi_3(h) \neq 0 \Rightarrow \phi_{111}(p_0) \neq 0, \qquad \Psi_4(h) \neq 0 \Rightarrow \phi_{1111}(p_0) \neq 0.$$

We can further numerically check as an application of Corollary 5.4.1 that

- if $\Psi_2 \neq 0$ then the singularity is of type $\mathcal{A}_2$;
- if $\Psi_3 \neq 0$ and the singularity is not of type $\mathcal{A}_2$ then the singularity is of type $\mathcal{A}_3$;
- if $\Psi_4 \neq 0$ and the singularity is not of type $\mathcal{A}_2, \mathcal{A}_3$ then the singularity is of type $\mathcal{A}_4$.

Then we have the following conclusion.

Proposition 5.4.4. *Let (M, Δ, g) be a generic sub-Riemannian structure and let $q_0 \in M \backslash \mathfrak{S}$. There exists $\bar{\eta} > 0$ such that for all covectors p_0 in $(\mathcal{C}_q(1/2) \cap \{h_0 > \bar{\eta}^{-1}\}) \backslash (S_1 \cup S_2)$, the singularity at p_0 of $\mathcal{E}_{q_0}(t_c(p_0))$ is a Lagrange stable singular point of type $\mathcal{A}_2$, $\mathcal{A}_3$ or $\mathcal{A}_4$.*

Proof. As a consequence of our discussion, what remains to be proved is that generically with respect to the sub-Riemannian structure, there are no points $(h_1, h_2, h_3, h_4) \in (\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}^2 \setminus \{0\})$ such that

$$\Psi_2(h_1, h_2, h_3, h_4) = \Psi_3(h_1, h_2, h_3, h_4) = \Psi_4(h_1, h_2, h_3, h_4) = 0. \tag{5.10}$$

Since for all $i \in [\![2, 4]\!]$, for all $\lambda, \mu \in \mathbb{R}$,

$$\Psi_i(\lambda h_1, \lambda h_2, \mu h_3, \mu h_4) = \lambda^2 \mu \Psi_i(h_1, h_2, h_3, h_4),$$

the equality $\Psi_i(h_1, h_2, h_3, h_4) = 0$ holds if and only if

$$\Psi_i\left(\frac{h_1}{\sqrt{h_1^2 + h_2^2}}, \frac{h_2}{\sqrt{h_1^2 + h_2^2}}, \frac{h_3}{\sqrt{h_3^2 + h_4^2}}, \frac{h_4}{\sqrt{h_3^2 + h_4^2}}\right) = 0.$$

Hence, denoting by r the rank of the family $\{\Psi_2, \Psi_3, \Psi_4\}$, (5.10) is satisfied on a codimension r subspace of the torus $\mathbb{S}^1 \times \mathbb{S}^1$.

Since we assumed $q_0 \notin \mathfrak{S}_2$ however, having $r < 3$ is a condition of codimension 6 on the sub-Riemannian structure at q_0. Hence generically with respect to the sub-Riemannian structure at q_0 there is no solution to (5.10) on $(\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}^2 \setminus \{0\})$. □

5.4.2 Classification near S_1

Again, we introduce a lemma to help us approximate the functions ϕ.

Lemma 5.4.5. *Let V, W be as in the statement of Lemma 5.3.6. For all $i \in [\![1, 5]\!]$, let $U_i : \mathbb{R}^4 \to \mathbb{R}$ and let*

$$\Phi(u_1, u_2, u_5) = u_5\left(V_1^{(5/2)} h_2 - V_2^{(5/2)} h_1\right) + K'\left(V_2^{(5/2)} u_1 - V_1^{(5/2)} u_2\right).$$

Let also $\mathfrak{d}_\eta : \mathbb{R}^5 \to \mathbb{R}^5$ be such that $\mathfrak{d}_\eta(u) = (\eta^{5/2} u_1, \eta^{5/2} u_2, \eta^{5/2} u_3, \eta^{5/2} u_4, \eta^3 u_5)$.

With $p_0 = \left(\sqrt{\eta} h_1, \sqrt{\eta} h_2, h_3, h_4, \eta^{-1}\right)$, uniformly with respect to $h \in B_R$ as $\eta \to 0$, we have at p_0

$$\det\left(\mathfrak{d}_\eta(U(h)), V, W, \partial_3 F, \partial_4 F\right) = \eta^{7+1/2} \frac{8\pi}{b_1 b_2^2} \sin^2\left(\frac{\pi b_2}{b_1}\right) \Phi(U_1(h), U_2(h), U_5(h)) + o\left(\eta^{7+1/2}\right).$$

Proof. We compute the dominant term of $\det(\mathfrak{d}_\eta(U(h)), V, W, \partial_3 F, \partial_4 F)$. Similarly to what is done in the proof of Lemma 5.4.3, we get from the assumptions and the construction of V and W in Section 5.3.2

$$\det(\mathfrak{d}_\eta(U(h)), V, W, \partial_3 F, \partial_4 F) = \left|\begin{array}{ccc|cc} \eta^{5/2}U_1 & \eta^{5/2}V_1^{(5/2)} & \eta^{5/2}\frac{2\pi}{b_1}h_1 & 0 & 0 \\ \eta^{5/2}U_2 & \eta^{5/2}V_2^{(5/2)} & \eta^{5/2}\frac{2\pi}{b_1}h_2 & 0 & 0 \\ \hline 0 & 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\eta A_4^{(1)}$}} \\ 0 & 0 & 0 & & \\ \hline \eta^3 U_5 & 0 & \eta^3\frac{2\pi}{b_1}K' & 0 & 0 \end{array}\right| + o(\eta^{7+1/2}).$$

Hence the statement since $\Phi(U_1, U_2, U_5) = \begin{vmatrix} U_1 & V_1 & h_1 \\ U_2 & V_2 & h_2 \\ U_5 & 0 & K' \end{vmatrix}$ and $\det A_4^{(1)} = \frac{4}{b_2^2}\sin^2\left(\frac{\pi b_2}{b_1}\right)$. $\square$

Let $q_0 \in M \setminus \mathfrak{S}$ and $p_0 = \left(\sqrt{\eta}h_1, \sqrt{\eta}h_2, h_3, h_4, \eta^{-1}\right) \in T^*_{q_0}M$. We can separate cases depending on the dimension of $\ker \mathrm{Jac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0))$.

Let us first treat the case of a 2-dimensional kernel. Let S^+ be the subset of $T^*_{q_0}M$ on which $\dim\ker \mathrm{Jac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0)) = 2$. Following the analysis in Remark 5.3.5, singular points with dimension 2 kernel on $M \setminus \mathfrak{S}$ correspond to covectors such that $(h_1, h_2) \neq (0,0)$ and

$$\gamma_1^2 = -\frac{2\pi h_1 h_2}{b_1}, \quad \gamma_2^1 = -\frac{2\pi h_1 h_2}{b_1 K'}, \quad \gamma_2^2 - \gamma_1^1 = \frac{2\pi\,(h_1^2 - h_2^2)}{b_1 K'}.$$

Hence if $\left(\sqrt{\eta}h_1, \sqrt{\eta}h_2, h_3, h_4, \eta^{-1}\right) \in S^+$ then $\Phi_+(h) = 0$ with

$$\Phi_+(h) = \left(\gamma_1^2 + \frac{2\pi h_1 h_2}{b_1 K'}\right)^2 + \left(\gamma_2^1 + \frac{2\pi h_1 h_2}{b_1 K'}\right)^2 + \left(\gamma_2^2 - \gamma_1^1 - \frac{2\pi\,(h_1^2 - h_2^2)}{b_1 K'}\right)^2.$$

Furthermore, $\ker \mathrm{Jac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0))$ is generated by v_{θ_1}, v_{r_1}, hence we choose the coordinates $\mathrm{x}_1, \mathrm{x}_2$ such that $\partial_{\mathrm{x}_1}\mathrm{id} = v_{\theta_1}$, $\partial_{\mathrm{x}_2}\mathrm{id} = v_{r_1}$, and we can check that the singularity is always of type $\mathcal{D}_4^+$.

Assume now that the kernel of $\mathrm{Jac}\mathcal{E}_{q_0}(t_c(p_0))$ is 1-dimensional. As a consequence of Proposition 5.3.2, assuming $(h_1, h_2) \neq (0,0)$, the kernel is generatedby $v = \lambda_{\theta_1} v_{\theta_1} + \lambda_{r_1} v_{r_1}$. We choose the first coordinate $\mathrm{x}_1 : M \to \mathbb{R}$ such that $\partial_{\mathrm{x}_1} = \sum_{i=1}^5 v_i \partial_i$. Since $v_3, v_4 = O(\eta)$, we have that $\partial_{x_1}^k F = O(\eta^{5/2})$ and $\partial_{x_1}^k F_5 = O(\eta^3)$ for all integer $k \geq 2$.

If we denote $V' : \mathbb{R}^4 \to \mathbb{R}^5$ such that (coordinate-wise) $\partial_{x_1}^2 F = \mathfrak{d}_\eta(V'(h)) + o(\mathfrak{d}_\eta(1))$ then let $\Phi_2(h) = \Phi(V_1', V_2', V_5')$. Similarly, define $V'' : \mathbb{R}^4 \to \mathbb{R}^5$ such that $\partial_{x_1}^3 F = \mathfrak{d}_\eta(V''(h)) + o(\mathfrak{d}_\eta(1))$ and $V''' : \mathbb{R}^4 \to \mathbb{R}^5$ such that $\partial_{x_1}^4 F = \mathfrak{d}_\eta(V'''(h)) + o(\mathfrak{d}_\eta(1))$; and define $\Phi_3(h) = \Phi(V_1'', V_2'', V_5'')$, $\Phi_4(h) = \Phi(V_1''', V_2''', V_5''')$.

We can check that the singularity is of type $\mathcal{A}_3$ if $(h_1, h_2) = (0,0)$ as an application of Remark 5.3.5. As an application of Lemma 5.4.5, and the analysis of the Jacobian matrix of $\mathcal{E}_{q_0}(t_c(p_0))$ of Section 5.3.2, we immediately obtain that for η small enough

$$\Phi_2(h) \neq 0 \Rightarrow \phi_{11}(p_0) \neq 0, \qquad \Phi_3(h) \neq 0 \Rightarrow \phi_{111}(p_0) \neq 0, \qquad \Phi_4(h) \neq 0 \Rightarrow \phi_{1111}(p_0) \neq 0.$$

We can further numerically check as an application of Corollary 5.4.1 that

— if $\Phi_2 \neq 0$ then the singularity is of type $\mathcal{A}_2$;

— if $\Phi_3 \neq 0$ and the singularity is not of type $\mathcal{A}_2$ then the singularity is of type $\mathcal{A}_3$;

— if $\Phi_4 \neq 0$ and the singularity is not of type $\mathcal{A}_2, \mathcal{A}_3$ then the singularity is of type $\mathcal{A}_4$.

Then we have the following conclusion.

Proposition 5.4.6. *Let (M, Δ, g) be a generic sub-Riemannian structure and let $q_0 \in M \setminus \mathfrak{S}$. There exists $\bar{\eta} > 0$ such that for all covectors p_0 in $\mathcal{C}_q(1/2) \cap \{h_0 > \bar{\eta}^{-1}\} \cap \{h_1^2 + h_2^2 < \bar{\eta}\}$, the singularity at p_0 of $\mathcal{E}_{q_0}(t_c(p_0))$ is a Lagrange stable singular point of type $\mathcal{A}_2$, $\mathcal{A}_3$ or $\mathcal{A}_4$ or $\mathcal{D}_4^+$.*

Proof. As a consequence of our discussion, what remains to be proved is that generically with respect to the sub-Riemannian structure, there are no element $(h_1, h_2, h_3, h_4) \in (\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}^2 \setminus \{0\})$ such that $\Phi_+(h) = \Phi_2(h) = \Phi_3(h) = \Phi_4(h) = 0$.

Let $\lambda > 0$ and $\delta_\lambda : \mathbb{R}^4 \to \mathbb{R}^4$ be the linear map such that

$$\delta_\lambda(h_1, h_2, h_3, h_4) = (\lambda^3 h_1, \lambda^3 h_2, \lambda^2 h_3, \lambda^2 h_4).$$

First, notice that $\Phi_+ \circ \delta_\lambda = \lambda^4 \Phi_+$ for all $\lambda > 0$. Similarly, one can check that for all $i \in [\![2, 4]\!]$, there exits $l > 0$ such that $\Phi_i \circ \delta_\lambda = \lambda^{14} \Phi_i$. Indeed by choice of x_1, for a monomial $f(h)$ we have $\partial_{\mathrm{x}_1}(f \circ \delta_\lambda) = (\partial_{\mathrm{x}_1} f) \circ \delta_\lambda + o(\eta)$. Then we can check that $V_1'(\delta_\lambda(h)) = \lambda^5 V_1'(h)$ and $V_2'(\delta_\lambda(h)) = \lambda^5 V_2'(h)$, while $V_5'(\delta_\lambda(h)) = \lambda^6 V_5'(h)$. Hence the result by applying the function Φ (notice for instance that $V_i^{(5/2)}(\delta_\lambda(h)) = \lambda^5 V_i^{(5/2)}(h)$ for $i \in [\![1, 2]\!]$). The same goes for Φ_3 and Φ_4.

Denoting by $\sim$ the equivalence relation on $\mathbb{R}^4 \setminus \{0\}$ such that $h \sim h'$ if there exists $\lambda > 0$ satisfying $\delta_\lambda(h) = h'$, and denoting by r the rank of the family $\{\Phi_2, \Phi_3, \Phi_4, \Phi_+\}$, the equality

$$\Phi_2(h) = \Phi_3(h) = \Phi_4(h) = \Phi_+(h) = 0$$

is satisfied on a codimension r subset of the dimension 3 quotient set $((\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}^2 \setminus \{0\})) / \sim$. Generically with respect to the the sub-Riemannian structure at $q_0 \notin \mathfrak{S}$, $r = 4$, and the singularity is either of type $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$ or $\mathcal{D}_4^+$. □

5.4.3 Classification near S_2

We repeat the process a last time, except we now need two orders of approximation.

Lemma 5.4.7. *Let V' be as in the statement of Lemma 5.3.9. For all $i \in [\![1, 5]\!]$, let $U, U' : \mathbb{R}^4 \to \mathbb{R}^5$ and for $u, u' \in \mathbb{R}^5$, let*

$$\Psi'(u) = u_5 (\alpha h_2 - \beta h_1) + \frac{\pi}{b_1} (h_1 u_2 - h_2 u_1)$$

and

$$\begin{aligned}\Gamma(u,u') =& \frac{4\pi}{b_1^2}\left(b_1u_5'(\alpha h_2-\beta h_1)+\pi h_1u_2'-\pi h_2u_1'\right)\\ &+\frac{3\pi}{b_1}(\alpha h_1+\beta h_2)(h_1u_2-h_2u_1)+\frac{2\pi\, u_5}{b_1(h_1^2+h_2^2)}\left(h_2V_1'^{(3)}-h_1V_2'^{(3)}\right)\\ &+\frac{4}{b_1}(\alpha h_1+\beta h_2)\left(b_1u_5(\beta h_1-\alpha h_2)+\pi(h_2u_1-h_1u_2)\right)\\ &+\frac{2\pi^2b_2}{b_1^2}(h_1u_2+h_2u_1)\Big[h_1(\kappa_{14}^3-\kappa_{13}^4)+h_2(\kappa_{24}^3-\kappa_{23}^4)+\\ &\qquad\cot\left(\frac{\pi b_2}{b_1}\right)\left(h_1(\kappa_{13}^3+\kappa_{14}^4)+h_2(\kappa_{23}^3+\kappa_{24}^4)\right)\Big]\\ &+\frac{2\pi^2b_2}{b_1^2}\left(U_4-U_3\cot\left(\frac{\pi b_2}{b_1}\right)\right)\left(\kappa_{13}^2h_1^2-\kappa_{23}^1h_2^2+(\kappa_{23}^2-\kappa_{13}^1)h_1h_2\right)\\ &+\frac{2\pi^2b_2}{b_1^2}\left(U_3+U_4\cot\left(\frac{\pi b_2}{b_1}\right)\right)\left(\kappa_{24}^1h_2^2-\kappa_{14}^2h_1^2+(\kappa_{14}^1-\kappa_{24}^2)h_1h_2\right).\end{aligned}$$

With $p_0=(h_1,h_2,\eta h_3,\eta h_4,\eta^{-1})$*, uniformly with respect to* $h\in B_R$ *as* $\eta\to 0$*, we have at* p_0

$$\begin{aligned}\det\left(U(h)+\eta U'(h),V,\partial_3F,\partial_4F,\partial_5F\right)=&\eta^8\frac{8(h_1^2+h_2^2)}{b_2^2}\sin^2\left(\frac{\pi b_2}{b_1}\right)\Psi'(U(h))\\ &+\eta^9\frac{4(h_1^2+h_2^2)}{b_2^2}\sin^2\left(\frac{\pi b_2}{b_1}\right)\Gamma\left(U(h),U'(h)\right)+o(\eta^9).\end{aligned}$$

Proof. We compute the first two non-zero terms in the expansion of

$$\det\left(U(h)+\eta U'(h),V,\partial_3F,\partial_4F,\partial_5F\right).$$

Observe that

$$V=\eta^2V^{(2)}+\eta^3V^{(3)}+o(\eta^3)\quad\text{ and }\quad\partial_iF=\eta^2\partial_iF^{(2)}+\eta^3\partial_iF^{(3)}+o(\eta^3)\quad\forall i\in[\![3,5]\!].$$

Hence

$$\begin{aligned}\det\left(U(h),V,\partial_3F,\partial_4F,\partial_5F\right)=&\eta^8\det\left(U(h),V^{(2)},\partial_3F^{(2)},\partial_4F^{(2)},\partial_5F^{(2)}\right)\\ &+\eta^9K(d_1+d_2+d_3+d_4+d_5)+o(\eta^9)\end{aligned}$$

with (recall $K=\det\left(A_4^{(2)}\right)=\frac{4}{b_2^2}\sin^2\left(\frac{\pi b_2}{b_1}\right)$)

$$Kd_1=\det\left(U'(h),V^{(2)},\partial_3F^{(2)},\partial_4F^{(2)},\partial_5F^{(2)}\right)=K\begin{vmatrix}\begin{array}{c|c|c}U_1' & \multirow{2}{*}{$A_1^{(2)}e_{r_1}$} & \multirow{2}{*}{$C_1^{(2)}$}\\ U_2' & & \\ \hline U_5' & \frac{2\pi}{b_1}(h_1^2+h_2^2) & 0\end{array}\end{vmatrix}$$

$$Kd_2 = \det\left(U(h), V^{(3)}, \partial_3 F^{(2)}, \partial_4 F^{(2)}, \partial_5 F^{(2)}\right) =$$
$$K \left| \begin{array}{c|c|c} U_1 & & \\ U_2 & A_1^{(3)} e_{r_1} - A_2^{(3)} \left(A_4^{(2)}\right)^{-1} A_3^{(2)} e_{r_1} & C_1^{(2)} \\ \hline U_5 & L_1^{(3)} e_{r_1} & 0 \end{array} \right|$$

$$Kd_5 = \det\left(U(h), V^{(2)}, \partial_3 F^{(2)}, \partial_4 F^{(2)}, \partial_5 F^{(3)}\right) = K \left| \begin{array}{c|c|c} U_1 & A_1^{(2)} e_{r_1} & C_1^{(3)} \\ U_2 & & \\ \hline U_5 & \frac{2\pi}{b_1}(h_1^2+h_2^2) & E^{(3)} \end{array} \right|$$

and

$$Kd_3 = \det\left(U(h), V^{(2)}, \partial_3 F^{(3)}, \partial_4 F^{(2)}, \partial_5 F^{(2)}\right) = \frac{2\pi}{b_1}(h_1^2+h_2^2)$$
$$\left(\begin{vmatrix} U_3 & \left(A_4^{(2)}\right)_{1,2} \\ U_4 & \left(A_4^{(2)}\right)_{2,2} \end{vmatrix} \begin{vmatrix} \left(A_2^{(3)}\right)_{1,1} & \left(C_1^{(2)}\right)_1 \\ \left(A_2^{(3)}\right)_{2,1} & \left(C_1^{(2)}\right)_2 \end{vmatrix} - \begin{vmatrix} \left(A_4^{(3)}\right)_{1,1} & \left(A_4^{(2)}\right)_{1,2} \\ \left(A_4^{(3)}\right)_{2,1} & \left(A_4^{(2)}\right)_{2,2} \end{vmatrix} \begin{vmatrix} U_1 & \left(C_1^{(2)}\right)_1 \\ U_2 & \left(C_1^{(2)}\right)_2 \end{vmatrix} \right),$$

$$Kd_4 = \det\left(U(h), V^{(2)}, \partial_3 F^{(2)}, \partial_4 F^{(3)}, \partial_5 F^{(2)}\right) = -\frac{2\pi}{b_1}(h_1^2+h_2^2)$$
$$\left(\begin{vmatrix} U_3 & \left(A_4^{(2)}\right)_{1,1} \\ U_4 & \left(A_4^{(2)}\right)_{2,1} \end{vmatrix} \begin{vmatrix} \left(A_2^{(3)}\right)_{1,2} & \left(C_1^{(2)}\right)_1 \\ \left(A_2^{(3)}\right)_{2,2} & \left(C_1^{(2)}\right)_2 \end{vmatrix} - \begin{vmatrix} \left(A_4^{(3)}\right)_{1,2} & \left(A_4^{(2)}\right)_{1,1} \\ \left(A_4^{(3)}\right)_{2,2} & \left(A_4^{(2)}\right)_{2,1} \end{vmatrix} \begin{vmatrix} U_1 & \left(C_1^{(2)}\right)_1 \\ U_2 & \left(C_1^{(2)}\right)_2 \end{vmatrix} \right).$$

Notice that $\det\left(U(h), V^{(2)}, \partial_3 F^{(2)}, \partial_4 F^{(2)}, \partial_5 F^{(2)}\right) = \frac{4\pi K}{b_1^2}(h_1^2+h_2^2)\psi'(U(h))$. Furthermore,

$$d_1 = \frac{4\pi}{b_1^2}\left(h_1^2+h_2^2\right)\left(b_1 U_5'(\alpha h_2 - \beta h_1) + \pi h_1 U_2' - \pi h_2 U_1'\right),$$
$$d_2 = \frac{3\pi}{b_1}\left(h_1^2+h_2^2\right)(\alpha h_1 + \beta h_2)(h_1 U_2 - h_2 U_1) + \frac{2\pi}{b_1} U_5 \left(h_2 V_1'^{(3)} - h_1 V_2'^{(3)}\right),$$

$$\begin{aligned} d_3 = \frac{2\pi^2 b_2}{b_1^2}(h_1^2+h_2^2) \Bigg[&\left(U_4 - U_3 \cot\left(\frac{\pi b_2}{b_1}\right)\right)\left(\kappa_{13}^2 h_1^2 - \kappa_{23}^1 h_2^2 + (\kappa_{23}^2 - \kappa_{13}^1) h_1 h_2\right) \\ &+ (h_1 U_2 - h_2 U_1)\left(\cot\left(\frac{\pi b_2}{b_1}\right)(h_1 \kappa_{13}^3 + h_2 \kappa_{23}^3) - h_1 \kappa_{13}^4 - h_2 \kappa_{23}^4\right)\Bigg]; \end{aligned}$$

$$\begin{aligned} d_4 = \frac{2\pi^2 b_2}{b_1^2}(h_1^2+h_2^2) \Bigg[&\left(U_3 + U_4 \cot\left(\frac{\pi b_2}{b_1}\right)\right)\left(\kappa_{24}^1 h_2^2 - \kappa_{14}^2 h_1^2 + (\kappa_{14}^1 - \kappa_{24}^2) h_1 h_2\right) \\ &+ (h_1 U_2 - h_2 U_1)\left(\cot\left(\frac{\pi b_2}{b_1}\right)(h_1 \kappa_{14}^3 + h_2 \kappa_{24}^3) + h_1 \kappa_{14}^4 + h_2 \kappa_{24}^4\right)\Bigg]; \end{aligned}$$

and finally

$$d_5 = 4\left(h_1^2+h_2^2\right)(\alpha h_1 + \beta h_2)\left(U_5(\beta h_1 - \alpha h_2) + \frac{\pi}{b_1}(h_2 U_1 - h_1 U_2)\right).$$

Hence the statement. □

Let $q_0 \in M \setminus \mathfrak{S}$ and $p_0 = (h_1, h_2, \eta h_3, \eta h_4, \eta^{-1}) \in T^*_{q_0}M$. Let $p_0 \in T^*_{q_0}M$ and v be as in the statement of Proposition 5.3.8 so that $\ker \mathrm{Jac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0)) = \mathrm{Span}(v)$. As explained in Remark 5.2.7, we choose the first coordinate $\mathrm{x}_1 : M \to \mathbb{R}$ such that $\partial_{\mathrm{x}_1} = \sum_{i=1}^5 v_i \partial_i$ and we have that $\partial^k_{x_1} F = O(\eta^2)$ for all integer $k \geq 2$.

If we denote $V', W' : \mathbb{R}^4 \to \mathbb{R}^5$ such that $\partial^2_{x_1} F = \eta^2 V'(h) + \eta^3 W'(h) + o(\eta^3)$ then let $\Psi'_2(h) = \Psi'(V')$ and $\Gamma_2(h) = \Gamma(V', W')$. Similarly, define $V'', W'' : \mathbb{R}^4 \to \mathbb{R}^5$ such that $\partial^3_{x_1} F = \eta^2 V''(h) + \eta^3 W''(h) + o(\eta^2)$, $V''', W''' : \mathbb{R}^4 \to \mathbb{R}^5$ such that $\partial^4_{x_1} F = \eta^2 V'''(h) + \eta^3 W'''(h) + o(\eta^2)$ and $V'''', W'''' : \mathbb{R}^4 \to \mathbb{R}^5$ such that $\partial^5_{x_1} F = \eta^2 V''''(h) + \eta^3 W''''(h) + o(\eta^2)$; and define $\Psi'_3(h) = \Psi'(V'')$, $\Gamma_3(h) = \Gamma(V'', W'')$, $\Psi'_4(h) = \Psi'(V''')$, $\Gamma_4(h) = \Gamma(V''', W''')$, and $\Psi'_5(h) = \Psi'(V'''')$, $\Gamma_5(h) = \Gamma(V'''', W'''')$.

We would like to replicate what has been done in the previous two sections in regard of the functions Ψ'_i. However we can check that $\Psi'_i = 0$ for $i \in [\![2, 5]\!]$ and we should instead focus on the functions Γ_i. As an application of Lemma 5.4.7, and the analysis of the Jacobian matrix of $\mathcal{E}_{q_0}(t_c(p_0))$ of Section 5.3.3, we immediately obtain that for η small enough

$$\Psi_2(h) \neq 0 \Rightarrow \phi_{11}(p_0) \neq 0, \qquad \Psi_3(h) \neq 0 \Rightarrow \phi_{111}(p_0) \neq 0, \qquad \Psi_4(h) \neq 0 \Rightarrow \phi_{1111}(p_0) \neq 0.$$

We can further numerically check as an application of Corollary 5.4.1 that

- if $\Gamma_2 \neq 0$ then the singularity is of type $\mathcal{A}_2$;
- if $\Gamma_3 \neq 0$ and the singularity is not of type $\mathcal{A}_2$ then the singularity is of type $\mathcal{A}_3$;
- if $\Gamma_4 \neq 0$ and the singularity is not of type $\mathcal{A}_2, \mathcal{A}_3$ then the singularity is of type $\mathcal{A}_4$;
- if $\Gamma_5 \neq 0$ and the singularity is not of type $\mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ then the singularity is of type $\mathcal{A}_5$.

Then we have the following conclusion.

Proposition 5.4.8. *Let (M, Δ, g) be a generic sub-Riemannian structure and let $q_0 \in M \setminus \mathfrak{S}$. There exists $\bar{\eta} > 0$ such that for all covectors p_0 in $(\mathcal{C}_q(1/2) \cap \{h_0 > \bar{\eta}^{-1}\}) \cap \{h_3^2 + h_4^2 < \bar{\eta}^2\}$, the singularity at p_0 of $\mathcal{E}_{q_0}(t_c(p_0))$ is a Lagrange stable singular point of type $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$ or $\mathcal{A}_5$.*

Proof. The proof is almost identical to the other two cases. As a consequence of our discussion, what remains to be proved is that generically with respect to the sub-Riemannian structure, there are no points $(h_1, h_2, h_3, h_4) \in (\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}^2)$ such that $\Gamma_2(h_1, h_2, h_3, h_4) = \Gamma_3(h_1, h_2, h_3, h_4) = \Gamma_4(h_1, h_2, h_3, h_4) = \Gamma_5(h_1, h_2, h_3, h_4) = 0$.

Again, for $\lambda > 0$, denote by $\delta_\lambda : \mathbb{R}^4 \to \mathbb{R}^4$ the linear map such that

$$\delta_\lambda(h_1, h_2, h_3, h_4) = (\lambda h_1, \lambda h_2, \lambda^2 h_3, \lambda^2 h_4),$$

and by $\sim$ the equivalence relation on $\mathbb{R}^4 \setminus \{0\}$ such that $h \sim h'$ if there exists $\lambda > 0$ satisfying $\delta_\lambda(h) = h'$. Notice that for all $i \in [\![2, 5]\!]$, there exists $l \in \mathbb{N}$ such that for all

$\lambda > 0$, $\Gamma_i \circ \delta_\lambda = \lambda^l \Omega_i$. With r the rank of the family $\{\Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5\}$, the equality

$$\Gamma_2(h) = \Gamma_3(h) = \Gamma_4(h) = \Gamma_5(h) = 0$$

is satisfied on a codimension r subset of the dimension 3 quotient set

$$\left(\left(\mathbb{R}^2 \setminus \{0\}\right) \times \left(\mathbb{R}^2 \setminus \{0\}\right)\right) / \sim .$$

Generically with respect to the the sub-Riemannian structure at $q_0 \notin \mathfrak{S}$, $r = 4$ and the singularity is either of type $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$ or $\mathcal{A}_5$. $\square$

Appendices

5.A Numerical classification, a commented example

In this appendix, we provide an example of the numerical methods resulting form the analysis of Section 5.4. We use the notations introduced in Section 5.3.

Recall that Lemma 5.4.3 states that for a function $U : \mathbb{R}^4 \to \mathbb{R}^5$, for $p_0 = (h, \eta^{-1})$, uniformly with respect to $h \in B_R$ as $\eta \to 0$,

$$\det\left(U(h), h_1\partial_1 F + h_2\partial_2 F, \partial_3 F, \partial_4 F, \partial_5 F\right) = \eta^6 \frac{8\pi}{b_1 b_2^2} \sin^2\left(\frac{\pi b_2}{b_1}\right) \Psi(U_1(h), U_2(h), U_5(h)) + o(\eta^6).$$

with

$$\Psi(u_1, u_2, u_5) = u_5\left({}^t e_{\theta_1} A_1^{(2)} e_{r_1}\right) - \frac{2\pi}{b_1}(h_1^2 + h_2^2)\left(h_1 u_2 - h_2 u_1\right).$$

We are looking for a coordinate $\mathrm{x}_1 : M \to \mathbb{R}$ such that $\partial_{\mathrm{x}_1} F = 0$ and $\partial_{\mathrm{x}_1}^2 F \notin \mathrm{imJac}\bar{\mathcal{E}}$. Hence with $v \in \ker \mathrm{Jac}_{p_0}\bar{\mathcal{E}}_{q_0}(t_c(p_0))$ and x_1 such that $\partial_{\mathrm{x}_1} F = DF(v) = 0$, what remains to check is that

$$\det\left(D^2 F(v,v), h_1\partial_1 F + h_2\partial_2 F, \partial_3 F, \partial_4 F, \partial_5 F\right) \neq 0 \tag{5.11}$$

(recall Remark 5.2.7). Indeed

$$\partial_{\mathrm{x}_1}^2 F = \sum_{i,j=1}^{5} v_i v_j \partial_i\partial_j F + \sum_{i,j=1}^{5} v_i \partial_i v_j \partial_j F = \sum_{i,j=1}^{5} v_i v_j \partial_i\partial_j F \quad (\mathrm{mod\ imJac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0)))$$

Notice that $v_3 = O(\eta)$,$v_4 = O(\eta)$ and

$$\partial_i\partial_j F_l = O(\eta^3) \quad \forall i, j \in [\![1, 4]\!], l \in [\![3, 4]\!],$$

$$\partial_i F_l = O(\eta^3) \quad \forall i \in [\![1, 2]\!], l \in [\![3, 4]\!].$$

Hence

$$D^2 F(v,v)_l = \sum_{i,j=1}^{2} v_i v_j \partial_i\partial_j F_l + 2\sum_{i=1}^{2} v_i v_5 \partial_i\partial_5 F_l + O(\eta^3) \quad \forall l \in [\![1, 2]\!]$$

and

$$D^2F(v,v)_5 = v_1^2\partial_1^2 F_5 + v_2^2\partial_2^2 F_5 + O(\eta^3),$$

while

$$D^2F(v,v)_l = 2\sum_{i=3}^{4} v_i v_5 \partial_i \partial_5 F_l = O(\eta^3) \qquad l \in [\![3,4]\!].$$

Hence we can apply Lemma 5.4.3 to the coefficient of order 2 in η of $D^2F(v,v)$.

In the following we give an example of this particular method using the computer algebra system Mathematica. We start by introducing elements useful for the computation of the conjugate points.

```
In[1]:= J = {{0, -b1}, {b1, 0}};

In[2]:= h0 = { h1, h2};
hhat[t_] = MatrixExp[t J].h0;
xhat[t_] = Integrate[hhat[t], {t, 0, t}];
zhat[t_] =
  Integrate[
   b1 hhat[s][[1]] xhat[s][[2]]/2 -
    b1 hhat[s][[2]] xhat[s][[1]]/2, {s, 0, t}];

In[6]:= exp21 =
  3 a h1 h1 + a h2 h2 + 2 b h1 h2 + k131 h1 h3 + k141 h1 h4 +
   k231 h2 h3 + k241 h2 h4 + rest1[h3, h4];
exp22 = b h1 h1 + 3 b h2 h2 + 2 a h1 h2 + k132 h1 h3 + k142 h1 h4 +
   k232 h2 h3 + k242 h2 h4 + rest2[h3, h4];
```

Then we compute elements of $\mathrm{Jac}_{p_0}\mathcal{E}_{q_0}(2\pi\eta/b_1)$ and use them to compute $t_c^{(2)}$.

```
In[8]:= C10 = 2 Pi/b1 {h1, h2};
A0 = Simplify[{{D[exp21, h1], D[exp21, h2]}, {D[exp22, h1],
     D[exp22, h2]}}];

In[10]:= jacobian =
  Simplify[{{D[exp21, h1], D[exp21, h2], C10[[1]]}, {D[exp22, h1],
     D[exp22, h2], C10[[2]]}, {D[zhat[2 Pi/b1], h1],
     D[zhat[2 Pi/b1], h2], 0}}];

In[11]:= tc =
 dt /. Factor[
   Solve[Det[jacobian + {{dt, 0, 0}, {0, dt, 0}, {0, 0, 0}}] == 0,
     dt][[1]]]

Out[11]= -(1/(
 h1^2 + h2^2))(2 a h1^3 + 6 a h1^2 h2 - 4 b h1^2 h2 + 2 a h1 h2^2 +
   2 b h2^3 + h2^2 h3 k131 - h1 h2 h3 k132 + h2^2 h4 k141 -
```

```
  h1 h2 h4 k142 - h1 h2 h3 k231 + h1^2 h3 k232 - h1 h2 h4 k241 +
  h1^2 h4 k242)
```

We have sufficiently many elements to compute Ψ.

```
In[12]:= A12 = Simplify[A0 + {{tc, 0}, {0, tc}}];

In[13]:= Psi[u1_, u2_, u5_] =
 Simplify[u5 (A12.{h1, h2}).{h2, -h1} + 2 Pi/b1 (h1^2 + h2^2) (h1 u2 -
    h2 u1)]

Out[13]=(1/b1)(2 h1^3 (\[Pi] u2 - b b1 u5) -
  h1^2 (2 h2 \[Pi] u1 - 2 a b1 h2 u5 + b1 h3 k132 u5 +
     b1 h4 k142 u5) +
  h2^2 (-2 h2 \[Pi] u1 + 2 a b1 h2 u5 + b1 h3 k231 u5 +
     b1 h4 k241 u5) +
  h1 h2 (b1 (h3 k131 + h4 k141 - h3 k232 - h4 k242) u5 +
     2 h2 (\[Pi] u2 - b b1 u5)))
```

We enter $v \in \ker \mathrm{Jac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0))$ and the derivatives ∂_i.

```
In[14]:= nu =
  Simplify[nu0 /. Solve[A12.{-h2, h1} + nu0 C10 == 0, nu0][[1]]];

In[15]:= v[1] = -h2;
v[2] = h1;
v[0] = nu;

In[18]:= d[1][f_] := D[f, h1]
d[2][f_] := D[f, h2]
d[0][f_] := -eta^2 D[f, eta] + eta t D[f, t]
```

We compute $\partial_{x_1}^2 f \pmod{\mathrm{im}\mathrm{Jac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0))}$, that is, $\sum_{i,j=1}^{5} v_i v_j \partial_i \partial_j f$. (Recall $v_3, v_4 = O(\eta)$, hence they do not appear in this computation).

```
In[21]:= g = eta g1[t + eta dt1, h1, h2] + eta^2 g2[t, h1, h2];

In[22]:= d2f = Simplify[
   (
    D[Sum[v[i1] v[i2] d[i1][d[i2][g]], {i1, 0, 2}, {i2, 0, 2}], eta,
       eta]/2 /. eta -> 0
    )];

In[23]:= d2Exp = Simplify[(
     d2f /. {
       g1 -> Function[{t, h1, h2}, Evaluate[Append[xhat[t], 0]]],
```

```
    g2 ->  Function[{t, h1, h2},
      Evaluate[Append[{exp21, exp22}, zhat[t]]]]}
  ) /. t -> 2 Pi/b1];
```

We apply Ψ to $\partial^2_{x_1} f$.

```
In[24]:= Simplify[Apply[Psi, d2Exp]/(1/b1 2 (h1^2 + h2^2) \[Pi]) ]

Out[24]= -h2^2 (h3 (2 k132 + k231) + h4 (2 k142 + k241)) +
 h1^2 (h3 (k132 + 2 k231) + h4 (k142 + 2 k241)) -
 3 h1 h2 (h3 (k131 - k232) + h4 (k141 - k242))
```

We then recognize

$$\begin{aligned}\Psi_2(h_1, h_2, h_3, h_4) = &-(h_3(\kappa^2_{13} + 2\kappa^1_{23}) + h_4(\kappa^2_{14} + 2\kappa^1_{24}))h_1^2 \\ &+ 3(h_3(\kappa^1_{13} - \kappa^2_{23}) + h_4(\kappa^1_{14} - \kappa^2_{24}))h_1 h_2 \\ &+ (h_3(2\kappa^2_{13} + \kappa^1_{23}) + h_4(2\kappa^2_{14} + \kappa^1_{24}))h_2^2 \quad \neq 0.\end{aligned}$$

Chapter 6

Generic singularities of line fields on 2D manifolds

Generic singularities of line fields have been studied for lines of principal curvature of embedded surfaces. In this chapter we propose an approach to classify generic singularities of general line fields on 2D manifolds. The idea is to identify line fields as bisectors of pairs of vector fields on the manifold, with respect to a given conformal structure. The singularities correspond to the zeros of the vector fields and the genericity is considered with respect to a natural topology in the space of pairs of vector fields. Line fields at generic singularities turn out to be topologically equivalent to the Lemon, Star and Monstar singularities that one finds at umbilical points.

Contents

6.1 Introduction

A line field on a 2-dimensional manifold is a smooth map that associates with every $q \in M$ a line (*i.e.*, a 1-dimensional subspace) in T_qM. This is the definition used in [Hop03], where Hopf extends the classical Poincaré-Hopf Theorem to the case of line fields. Line fields appear often in nature as for instance in fingerprints [KW87, Pen79, WHP07], liquid crystals [Cha92, DRB$^+$15, Pro95] and in the pinwheel structure of the visual cortex V1 of mammals [BCGR14, BDGR12, CS06, HW62, Pet08]. Contrarily to what happens for vector fields, where the topology of the manifold forces the vector fields to have zeros, the topology of the manifold forces line fields to have singularities (*i.e.*, points where a line field is not defined).

Singularities of line fields are visible in nature as shown in Figure 6.1. Two types of singularities are usually observed, one of index $1/2$ and one of index $-1/2$, which have different names depending on the context.

Following Thom [Tho72], one expects that only singularities that do not disappear for small perturbations of the system are easily observed in nature (see also [Arn92]). For this reason, it is important to study which singularities are *structurally stable*. To define what structurally stable means, one needs two ingredients. First one needs a topology on the space of line fields. Second one needs a notion of local equivalence between line fields. The difficulty in studying this problem comes from the fact that there is no natural topology on the set of line fields, since the set of singular points depends on the line field itself.

This problem was completely solved in the case of lines of principal curvature on surfaces, since these line fields are given by the embedding of a surface in $\mathbb{R}^3$ and the natural topology is the one given by the embedding. Three types of singularities, called *Lemon*, *Monstar* and *Star* (see [BH77]), were identified by Darboux in [Dar96] (see Figure 6.2). It was proven in [SG82] that Lemon, Monstar and Star are the structurally stable singularities of lines of principal curvature with respect to the Whitney C^3-topology of immersions of a surface in $\mathbb{R}^3$. Related results have been obtained in the field of binary differential equations (see [BF89, BT95] and the survey paper [Rem06]).

The purpose of this chapter is to study the structurally stable singularities of line fields in a more general context than the one of lines of principal curvature. The starting point of the chapter is to give a definition of line field (that we call *proto-line-field*) that has a natural associated topology. For us a proto-line-field on a Riemannian surface is a pair of vector fields X and Y on M. The corresponding line field associated with the proto-line-field is the line field bisecting X and Y. The angle is computed using the Riemannian metric, actually a conformal would be sufficient. The zeros of X and Y become singularities of the associated line field. In Proposition 6.3.1, we prove that any line field with singularities can be realized in this way.

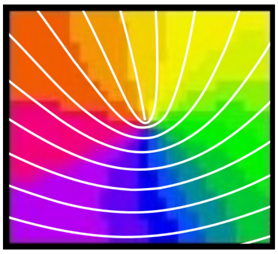
With this definition we naturally associate a topology on line fields, that is, the Whitney topology on pairs of vector fields on M. The main result of the chapter is that generically a proto-line-field has only structurally stable singularities, which are Lemon, Monstar or Star singularities. Hence the structurally stable singularities for lines of principal curvature are the same as for general proto-line-fields.

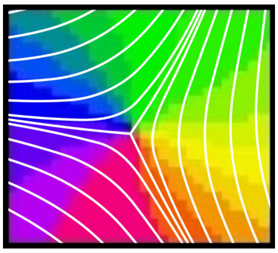
Notice that in nature it is not easy to distinguish between the Lemon and the Monstar singularity since they have the same index (see Section 6.3.2) and they look quite similar. This is why the observation of singularities of line fields in nature usually reports only two behaviors, characterized by the index of the singularity. One important issue for singularities of line fields (in particular for finger ridges) is their parameterization by a model with few parameters and capable to capture high curvature patterns ([WH07]). Our definition of proto-line-fields could be useful for such applications, since it could be used to detect fine properties, such as the difference between Lemon and Monstar singularities.

The structure of the chapter is the following. In Section 6.2 we give the definition of proto-line-field, of local structural stability and we state our main result (Theorem 6.2.7). In Section 6.3, we establish some basic properties of proto-line-fields and we prove that every line field (possibly with singularities) can be realized as a proto-line-field. Moreover, following Hopf we introduce the index of a proto-line-field and we show how to compute it starting from the indices of X and Y. We also deduce that the index of a singularity of a generic proto-line-field is $1/2$ or $-1/2$.

The main technical part of the chapter consists of Sections 6.4 and 6.5. In Section 6.4 we study the case of linear proto-line-fields in the Euclidean plane and we classify them into three categories corresponding to the three exhibited singularities. In Section 6.5 we study the general problem via a blow up and make use of the classification obtained in the linear case to prove the Theorem 6.2.7.

In Section 6.6 we study the role of the Riemannian metric on the identification between a proto-line-field and the corresponding line field. In particular we observe that bifurcations between Lemon and Monstar singularities can occur by changing the metric. Finally in Section 6.6.2 we show how to reduce the number of ingredients necessary to define proto-line-fields by constructing a Riemannian metric starting from two vector fields.


(a) Stopping point

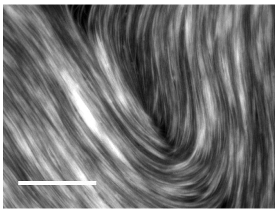

(b) Triple point

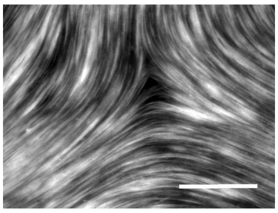
The pinwheel structure of the orientation columns of the visual cortex V1 can be modeled as a line field whose singularities are the pinwheels. Clockwise pinwheels, also called stopping points, have index $1/2$ and counter-clockwise pinwheels, also called triple points, have index $-1/2$.


(c) Core


(d) Delta

In an effort to classify fingerprints, the topology of the underlying line field in the ridge patterns can be used. Isolated singularities of index $1/2$ and $-1/2$ can be observed, and their total index is actually fixed by the number of fingers.


(e)


(f)

Singularities of index $\pm 1/2$ can be observed in nematic liquid crystals. Perpendicularly to a 1-dimensional dislocation in the material, the liquid crystal can be modeled as a line field with singularities called disclinations. (Images kindly provided by Stephen J. DeCamp.)

Figure 6.1 – Examples of singularities observed in nature with half-integer indices.

6.2 Basic definitions and statement of the main result

In this chapter, manifolds and vector fields are assumed to be smooth, *i.e.*, C^∞.

Definition 6.2.1. Let (M, g) be a 2-dimensional Riemannian manifold. A *proto-line-field* is a pair (X, Y) of vector fields on M. Denote by z_X and z_Y the sets of zeros of X and Y. The *line field associated with* (X, Y), denoted by $B(X, Y)$, is the section of $PT(M \backslash (z_X \cup z_Y))$ defined at a point $p \in M \backslash (z_X \cup z_Y)$ as the line $B(X(p), Y(p))$ of T_pM bisecting $(X(p), Y(p))$ for the metric $g(p)$.

In the definition above, the metric g is only used to measure angles. One could then replace g by a conformal structure. We are not assuming that (M, g) is orientable. When angles are measured, it is implicitly meant that we are choosing a local orientation. In the following we will denote the angle measured with respect to the metric g between the vectors V and W of T_pM by $\angle_g[V,W]$. This angle should be understood modulo 2π. We use the same notation to define the angle between two lines or between a vector and a line, in this case the angle should be understood modulo π.

Definition 6.2.2. A one-dimensional connected immersed submanifold N of $M\backslash(z_X \cup z_Y)$ is said to be an *integral manifold of the proto-line-field* (X,Y) if for any point p of N, the tangent line to N at p is given by $B(X,Y)$.

By involutivity of one-dimensional distributions, $M\backslash(z_X \cup z_Y)$ can be foliated by integral manifolds of (X,Y).

Example 6.2.3. We introduce here three proto-line-fields whose singularities correspond to the well-known Lemon, Monstar and Star singularities observed for lines of principal curvature. Their respective integral manifolds are represented in Figure 6.2.

The *Lemon proto-line-field* is the pair of vector fields on $(\mathbb{R}^2, \mathrm{Eucl})$ defined by

$$X_L(x,y) = \begin{pmatrix} x+y \\ y-x \end{pmatrix}, \ Y_L(x,y) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

The *Monstar proto-line-field* is the pair of vector fields on $(\mathbb{R}^2, \mathrm{Eucl})$ defined by

$$X_M(x,y) = \begin{pmatrix} x \\ 3y \end{pmatrix}, \ Y_M(x,y) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

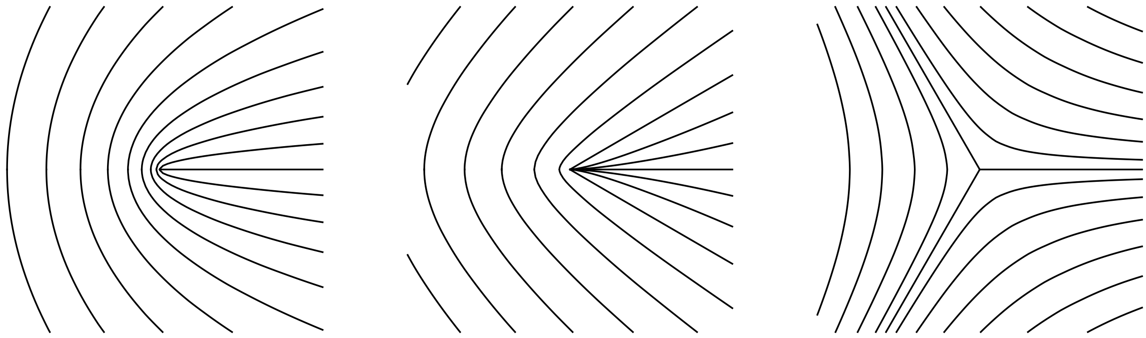
The *Star proto-line-field* is the pair of vector fields on $(\mathbb{R}^2, \mathrm{Eucl})$ defined by

$$X_S(x,y) = \begin{pmatrix} x \\ -y \end{pmatrix}, \ Y_S(x,y) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

For $k \in \mathbb{N}$, denote by $\mathcal{W}^k(M)$ the space of pairs of smooth vector fields of M endowed with the product Whitney C^k-topology.

Definition 6.2.4. Let (X,Y) be a proto-line-field on (M,g), and (X',Y') be a proto-line-field on (M',g'). Fix $p \in M$ and $p' \in M'$. Then (X,Y) and (X',Y') are said to be *topologically equivalent at p and p'* if there exist two neighborhoods V_p and $W_{p'}$ of p and p' respectively and a homeomorphism $h : V_p \to W_{p'}$, with $h(p) = p'$, which takes the integral manifolds of (X,Y) onto those of (X',Y').

Definition 6.2.5. Let (X,Y) be a proto-line-field on (M,g). We say that (X,Y) has a Lemon (respectively, Monstar, Star) singularity at $p \in M$, if it is topologically equivalent to (X_L,Y_L) (respectively, (X_M,Y_M), (X_S,Y_S)) at 0. We say that a singularity of a proto-line-field is *Darbouxian* if it is either a Lemon, a Monstar or a Star.



(a) Lemon (b) Monstar (c) Star

Figure 6.2 – Integral manifolds of the proto-line-fields of Example 6.2.3.

Definition 6.2.6. A proto-line-field (X, Y) on M is said to be *locally structurally stable at* $p \in M$ if for any neighborhood $U_p \subset M$ of p there exists a neighborhood $\mathcal{N}_{(X,Y)}$ of (X, Y) with respect to $\mathcal{W}^1(M)$ such that for any $(X', Y') \in \mathcal{N}_{(X,Y)}$, (X, Y) and (X', Y') are topologically equivalent at p and q, for some $q \in U_p$. Moreover, (X, Y) is said to be *locally structurally stable* if it is locally structurally stable at any point $p \in M$.

Recall that a residual set in a topological space is a countable intersection of open and dense subsets. We say that a property *holds generically for a proto-line-field in* $\mathcal{W}^1(M)$ if there exists a residual set $\mathcal{U}$ in $\mathcal{W}^1(M)$ such that the property is satisfied by every element of $\mathcal{U}$. In the case where M is compact, we could actually replace *residual* by *open and dense* in the definition of genericity, and all results stated in this chapter would still hold true.

Theorem 6.2.7 (Genericity theorem)**.** *Generically with respect to* $(X, Y) \in \mathcal{W}^1(M)$*, the proto-line-field* (X, Y) *is locally structurally stable and has only Darbouxian singularities.*

6.3 Basic properties of proto-line-fields

6.3.1 Every line field can be realized as a proto-line-field

Proposition 6.3.1. *Let* (M, g) *be a 2-dimensional Riemannian manifold,* K *be a closed subset of* M *and* L *be a section of* $PT(M\backslash K)$*. There exist two vector fields* X *and* Y *such that* $L = B(X, Y)$*.*

Proof. Let us first fix the vector field X. If K is empty, by Poincaré-Hopf Theorem for line fields (see [Hop03]), the Euler characteristic of M is 0. Hence we can take as X a never-vanishing smooth vector field on M. In the case where K is non-empty, we take instead as X any vector field on M vanishing at a single point q belonging to K.

In the case in which M is orientable, let $\alpha : M\backslash K \to \mathbb{R}/\pi\mathbb{Z}$ be the smooth function defined by $\angle[X, L]$ and $\varphi : M \to \mathbb{R}$ be a smooth function such that $\varphi(p) \neq 0$ for all $p \in M\backslash K$ and φ is equal to 0 with all its derivatives on K. Define the smooth vector field Y on M by $Y(p) = \varphi(p)R_{2\alpha(p)}(X(p))$, where $R_\theta : TM \to TM$ denotes the fiber-wise rotation by an angle θ. By construction $L = B(X, Y)$.

In the non-orientable case, even if α is not globally well defined on $M\backslash K$, the vector field $p \mapsto R_{2\alpha(p)}(X(p))$ is. Hence Y can be defined as above and the same conclusion follows. □

6.3.2 Index of line fields and hyperbolic singularities

Let U be an open subset of M and $p \in U$. Following Hopf, we define the index of a section $L : U \setminus \{p\} \to PTM$ at p as follows. Up to restricting U, we can assume it to be simply connected and we can consider a never vanishing vector field Z on U. Let $C : [0, 1] \to U$ be a simple closed curve encircling p counterclockwise. Then there exists a map $F : [0, 1] \to TM$ such that $L(C(t))$ is the span of $F(t)$ for every $t \in [0, 1]$. Let $\angle[Z, F]_{C(t)}$ be the angle between $F(t)$ and $Z(C(t))$ with respect to the Riemannian metric g, and let $\delta_C\angle[Z, F]$ be the total signed variation of this angle on the interval $[0, 1]$. We then define j by

$$2\pi j = \delta_C\angle[Z, F].$$

Since $\angle[Z, F]_{C(0)} = \angle[Z, F]_{C(1)} \pmod{\pi}$, $2j$ is an integer and it can be shown that j does not depend on Z, C nor on g. We say that j is the *index of L at p* and we write $\mathrm{ind}_p(L) = j$. We use the same symbol ind_p to denote the index of a vector field at the point p.

The following result holds (see [Hop03]).

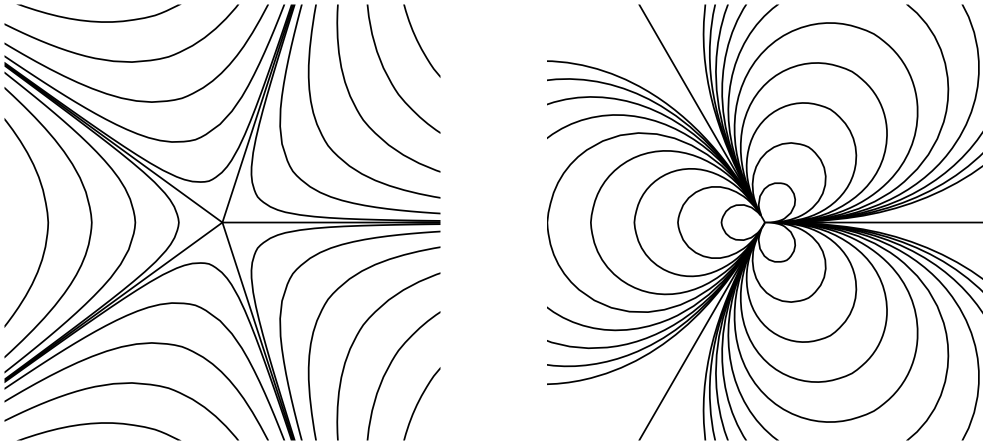
Theorem 6.3.2 (Poincaré-Hopf). *Let (M, g) be a compact, orientable 2-dimensional Riemannian manifold of Euler characteristic $\chi(M)$, and let L be a line field on M with isolated singularities. Let z_L be the set of singularities of L and j_p be the index of L at p for any $p \in z_L$. Then*

$$\sum_{p\in z_L} j_p = \chi(M).$$

Example 6.3.3. Let us construct some examples of sections with arbitrary index using complex numbers. We assume that the indetermination of the logaritheorem is set on $\mathbb{R}_-$. Then we can define for any half-integer $j \in \frac{1}{2}\mathbb{Z}$ the smooth function

$$\begin{array}{rccc} \Phi_j : & \mathbb{C}\backslash\mathbb{R}_- & \longrightarrow & \mathbb{C} \\ & z & \longmapsto & z^j. \end{array}$$

The logarithm is not defined everywhere on $\mathbb{C}$, but the section defined by $L_j(z) = \arg(\Phi_j(z)) \pmod{\pi}$ can be continuously extended on $\mathbb{C}\backslash\{0\}$ for any half-integer j. This section has a singularity of index j. Furthermore, notice that if j is not an integer, then the section cannot be induced from a continuous vector field that vanishes only at 0. (See Figure 6.3.)



(a) $j = -3/2$ (b) $j = 5/2$

Figure 6.3 – Examples of integral manifolds of sections of $PT(\mathbb{C} \setminus \{0\})$ near half-integer index singularities.

Proposition 6.3.4. *Let (X, Y) be a proto-line-field on (M, g). Given an isolated point p of $z_X \cup z_Y$, we have $\operatorname{ind}_p(B(X,Y)) = \frac{1}{2}(\operatorname{ind}_p(X) + \operatorname{ind}_p(Y))$.*

Proof. Fix $p \in M$. Take C, Z and $F : [0,1] \to TM$ as in the definition of the index of a singularity. Then for any $t \in [0,1]$

$$\begin{aligned}\angle[Z(C(t)), F(t)] &= \angle[Z(C(t)), X(C(t))] + \angle[X(C(t)), F(t)]\\ &= \angle[Z(C(t)), X(C(t))] + \frac{1}{2}\angle[X(C(t)), Y(C(t))]\\ &= \angle[Z(C(t)), X(C(t))] + \frac{1}{2}\left(\angle[Z(C(t)), Y(C(t))] + \angle[X(C(t)), Z(C(t))]\right)\\ &= \frac{1}{2}\left(\angle[Z(C(t)), X(C(t))] + \angle[Z(C(t)), Y(C(t))]\right),\end{aligned}$$

since $\angle[Z(C(t)), X(C(t))] = -\angle[X(C(t)), Z(C(t))]$. Hence

$$\operatorname{ind}_p(B(X,Y)) = \frac{1}{2}(\operatorname{ind}_p(X) + \operatorname{ind}_p(Y))$$

by definition of index. □

Definition 6.3.5. We say that a proto-line-field (X, Y) has a *hyperbolic singularity at a point $p \in M$* if one of the two vector fields has a hyperbolic singularity and the other is non-vanishing at p.

Proposition 6.3.6. *A generic proto-line-field has only hyperbolic singularities. In particular its singularities have indices either* $1/2$ *or* $-1/2$.

Proof. As a straightforward consequence of Thom's transversality theorem (see, for instance, [Hir12, p. 82]), for a generic $(X, Y) \in \mathcal{W}^1(M)$, both X and Y have only hyperbolic singularities, and they do not vanish at the same point. This proves the first part of the statement, while the second follows from Proposition 6.3.4. $\square$

6.3.3 Example: Lines of principal curvature on a triaxial ellipsoid

The study of lines of principal curvature on the triaxial ellipsoid is one of the most classical examples of this theory, that dates back to the work of Monge on the subject (see [Mon96, SG08]).

Consider the triaxial ellipsoid $\mathcal{E}$ of equation

$$\frac{x_1^2}{a} + \frac{x_2^2}{b} + \frac{x_3^2}{c} = 1$$

where we assume that $0 < a < b < c$. In order to introduce the coordinates on $\mathcal{E}$ used by Jacobi in [Jac41], consider the map from $\mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}/2\pi\mathbb{Z}$ onto $\mathcal{E}$ given by

$$\begin{aligned} x_1 &= \sqrt{\frac{a}{c-a}} \sin\varphi \sqrt{c \sin^2\psi + b\cos^2\psi - a}, \\ x_2 &= \sqrt{b}\cos\varphi \sin\psi, \\ x_3 &= \sqrt{\frac{c}{c-a}} \cos\psi \sqrt{c - b\sin^2\varphi - a\cos^2\varphi}, \end{aligned}$$

where $\varphi, \psi \in \mathbb{R}/2\pi\mathbb{Z}$. Although this map is a cover of $\mathcal{E}$ by the torus, the pair (φ, ψ) is referred to as *ellipsoidal coordinates*. Their main interest for us is that level sets of φ and ψ (*i.e.*, curves on which either φ or ψ is constant) are the two sets of lines of curvature on the ellipsoid, and the umbilical points of the surface are situated at the points of coordinates $(\varphi, \psi) = (\pm\pi/2, \pm\pi)$.

The ellipsoidal coordinates are used by Jacobi to express the first integral of motion along geodesics on the ellipsoid. Indeed any geodesics on $\mathcal{E}$ can be described by an equation of the type

$$\begin{aligned} \alpha = &\int \frac{\sqrt{b\sin^2\varphi + a\cos^2\varphi}}{\sqrt{c - b\sin^2\varphi - a\cos^2\varphi}\sqrt{(b-a)\cos^2\varphi - \beta}} \mathrm{d}\varphi \\ &- \int \frac{\sqrt{c\sin^2\psi + b\cos^2\psi}}{\sqrt{c\sin^2\psi + b\cos^2\psi - a}\sqrt{(c-b)\sin^2\psi + \beta}} \mathrm{d}\psi \end{aligned}$$

where

$$\beta = (b-a)\cos^2\varphi \sin^2\theta - (c-b)\sin^2\psi\cos^2\theta$$

is also constant along geodesics, with θ measuring the angle between the geodesic and the level set of ψ.

Among the geodesics, those which start at umbilical points of the ellipsoid satisfy very strong properties that we can use to characterize lines of principal curvature as integral manifolds of an explicitly identified proto-line-field. First, umbilics are the only points on the ellipsoid for which the cut locus is reduced to a single point, the antipodal umbilic, and all geodesics between them have the same length. Therefore, by any non-umbilical point of the ellipsoid pass exactly two minimizing geodesics originating from the two pair of antipodal umbilics. On these geodesics, the constants of motion α and β vanish.

Consider two non-antipodal umbilics Ω_1 and Ω_2 and two geodesics $t \mapsto (\varphi_1(t), \psi_1(t))$ and $t \mapsto (\varphi_2(t), \psi_2(t))$ starting at Ω_1 and Ω_2 respectively and meeting at $(\varphi_1(t_1), \psi_1(t_1)) = (\varphi_2(t_2), \psi_2(t_2))$. Since

$$\begin{aligned}
&(b-a)\cos^2\varphi_1(t_1)\sin^2\theta_1(t_1) - (c-b)\sin^2\psi_1(t_1)\cos^2\theta_1(t_1) = 0,\\
&(b-a)\cos^2\varphi_2(t_2)\sin^2\theta_2(t_2) - (c-b)\sin^2\psi_2(t_2)\cos^2\theta_2(t_2) = 0,\\
&\varphi_1(t_1) = \varphi_2(t_2),\\
&\psi_1(t_1) = \psi_2(t_2),
\end{aligned}$$

we have

$$\sin^2\theta_1(t_1) = \sin^2\theta_2(t_2)$$

and thus

$$\theta_1(t_1) = \pm\theta_2(t_2) \pmod{\pi}.$$

Since a geodesic is uniquely defined by its tangent line, $\theta_1 = \theta_2 \pmod{\pi}$ is excluded, and we have that $\theta_1(t_1) = -\theta_2(t_2) \pmod{\pi}$. By definition of θ_1 and θ_2, it follows that the level set of ψ bisects the angle between the tangent lines of the two geodesics. We can write this fact in terms of proto-line-fields.

Consider the Riemannian exponential exp on $\mathcal{E}$ and the length l of the geodesics connecting two umbilical points. Let $Y(x,y) = \sin\left(\frac{\pi}{l}\sqrt{x^2+y^2}\right)\begin{pmatrix} x \\ y \end{pmatrix}$ be a vector field on the closed disc $\bar{B}_{\mathbb{R}^2}(0,l)$ of radius l, vanishing at 0 and $\partial B_{\mathbb{R}^2}(0,l)$, and $X_1 = \exp^*_{\Omega_1} Y$, $X_2 = \exp^*_{\Omega_2} Y$ be two vector fields on $\mathcal{E}$. By construction, for each $i \in \{1,2\}$, X_i is tangent to the geodesics starting from Ω_i and vanishes at Ω_i and $-\Omega_i$.

Then at the point (φ_0, ψ_0), X_i forms an angle $\theta_i \pmod{\pi}$, $i \in \{1,2\}$, with the level set $\{\psi = \psi_0\}$. Since $\theta_1 = -\theta_2 \pmod{\pi}$, the line bisecting (X_1, X_2) for the metric on $\mathcal{E}$ induced from the Euclidean metric in $\mathbb{R}^3$ is either parallel or orthogonal to the level set $\{\psi = \psi_0\}$. In other words the line field bisecting the proto-line-field (X_1, X_2) is one of the line fields of principal curvature, and the other is bisecting $(-X_1, X_2)$. (See Figure 6.4.)

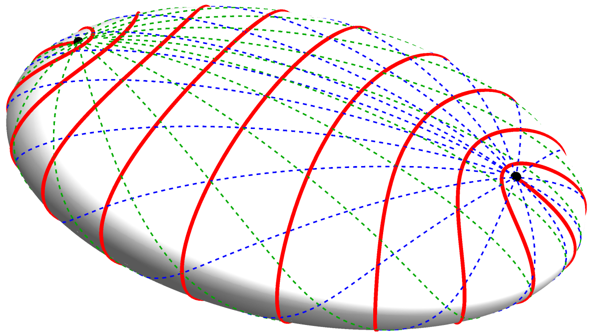


Figure 6.4 – Integral lines of X_1 and X_2 and integral manifolds of (X_1, X_2), shown respectively in blue, green and red.

6.4 Linear Euclidean case

Definition 6.4.1. Let (X, Y) be a proto-line-field on the Euclidean plane. We say that (X, Y) is a *linear proto-line-field* if one of the two vector fields is linear and the other one is constant. We say that (X, Y) is *linear hyperbolic* if it is linear and has a hyperbolic singularity at 0.

Example 6.4.2. The three proto-line-fields presented in Example 6.2.3 are linear hyperbolic.

From now on, when considering a linear proto-line-field (X, Y), we assume that X is linear and Y is constant, that is, we consider (X, Y) as an element of $\mathcal{M}_2(\mathbb{R}) \times \mathbb{R}^2$, where $\mathcal{M}_2(\mathbb{R})$ denotes the space of square 2-by-2 matrices.

Consider a linear proto-line-field $L = (X, Y)$. Along the rays $\{(r\cos\theta, r\sin\theta) \mid r > 0\}$, $\theta \in \mathbb{R}$, the direction of X and Y and thus of $B(X, Y)$ is constant. Hence we can define $\phi_L : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}/\pi\mathbb{Z}$ a parametrization of the direction of $B(X, Y)$ (after fixing an orthonormal basis basis) with

$$\phi_L(\theta) = \angle_{\text{Eucl}} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, B(L(\cos\theta, \sin\theta)) \right].$$

We call *fixed point of* L any point $\theta_0 \in \mathbb{R}/2\pi\mathbb{Z}$ such that $\phi_L(\theta_0) = \theta_0 \pmod{\pi}$. A fixed point θ_0 is said to be *attractive* if $\frac{d\phi_L}{d\theta}(\theta_0) > 1$, and *repulsive* if $\frac{d\phi_L}{d\theta}(\theta_0) < 1$.

Theorem 6.4.3. *Let X be a linear hyperbolic vector field on $\mathbb{R}^2$. Then there exists a set $\mathcal{E}_X \subset \mathbb{R}^2$ made of finitely many lines through the origin such that if $Y \in \mathbb{R}^2 \setminus \mathcal{E}_X$ then the linear proto-line-field $L = (X, Y)$ satisfies one of the following properties*

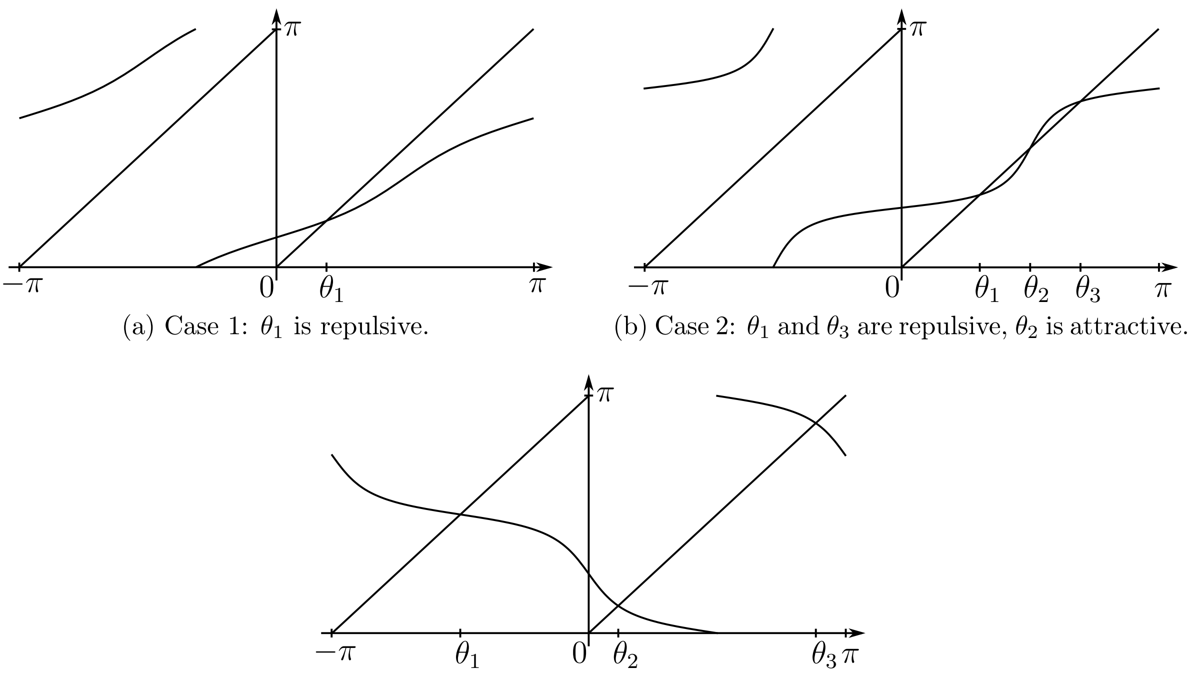
1. *ϕ_L has a unique fixed point, which is repulsive;*
2. *ϕ_L has three fixed points, all in the same half-plane. We can then identify two external fixed points, which are repulsive, and one internal, which is attractive;*

3. ϕ_L has three repulsive fixed points, which are not contained in a single half-plane.

Moreover, the set of linear hyperbolic proto-line-fields which do not fall in one of the stated cases is given by

$$\left\{ L \text{ linear hyperbolic} \mid \exists \theta_0 \in \mathbb{R}/2\pi\mathbb{Z} \text{ such that } \phi_L(\theta_0) = \theta_0 \pmod{\pi},\ \frac{d\phi_L}{d\theta}(\theta_0) = 1 \right\}$$

and has codimension 1 in $\mathcal{M}_2(\mathbb{R}) \times \mathbb{R}^2$.



(a) Case 1: θ_1 is repulsive.

(b) Case 2: θ_1 and θ_3 are repulsive, θ_2 is attractive.

(c) Case 3: θ_1, θ_2 and θ_3 are repulsive.

Figure 6.5 – Examples of ϕ_L in each of the three cases of Theorem 6.4.3. These specific examples correspond to $L = \left(\begin{pmatrix} 3x \\ 2y \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$, $L = \left(\begin{pmatrix} 4x \\ y \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$, and $L = \left(\begin{pmatrix} x \\ -3y \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$ respectively.

The behavior of ϕ_L in the three cases mentioned in the theorem is illustrated in Figure 6.5. In order to show that this is the case we split the proof of Theorem 6.4.3 in several steps. We start by proposing a suitable normal form for the vector field X.

Lemma 6.4.4. *Let X be a linear vector field with a hyperbolic singularity. Then there exist $E, C > 0$ and $\varphi \in [0, 2\pi)$ such that, in some orthonormal basis, for every $\theta \in \mathbb{R}$,*

C1 *If the singularity is a focus, then $X(E\cos(\theta), \sin(\theta)) = C \begin{pmatrix} E\cos(\theta + \varphi) \\ \sin(\theta + \varphi) \end{pmatrix}$, with $E > 1$.*

C2 *If the singularity is a node, then* $X(E\cos(\theta),\sin(\theta)) = C\begin{pmatrix} E\cos(\theta-\varphi) \\ \sin(\theta+\varphi)\end{pmatrix}$, *with* $\varphi \in (-\pi/4, \pi/4) \pmod{\pi}$.

C3 *If the singularity is a saddle, then* $X(E\cos(\theta),\sin(\theta)) = C\begin{pmatrix} E\cos(\theta-\varphi) \\ \sin(\theta+\varphi)\end{pmatrix}$, *with* $\varphi \in (\pi/4, 3\pi/4) \pmod{\pi}$.

Proof. In each of the three cases, it is possible to find an orthonormal basis such that the matrix of the linear vector field is of the form

$$A = \begin{pmatrix} a & b \\ c & a \end{pmatrix},$$

with $b, c \neq 0$. For $d = \sqrt{|bc|}$ and $E = \sqrt{|b/c|}$, we get for $\varepsilon = \pm 1$ and some $\varphi \in [0, 2\pi)$

$$A = \sqrt{a^2+d^2}\begin{pmatrix} \cos\varphi & \varepsilon E \sin\varphi \\ \frac{1}{E}\sin\varphi & \cos\varphi \end{pmatrix}.$$

In case **C1**, we know that the discriminant of A is negative, so that $4\varepsilon\sin^2\varphi < 0$, hence $\varepsilon = -1$. By further imposing $E > 1$, if necessary by changing the orientation of the basis and by replacing φ by $2\pi - \varphi$, we get the stated result.

In the two other cases, we know the discriminant of A to be positive, so that $4\varepsilon\sin^2\varphi > 0$, hence $\varepsilon = +1$. Since $\det A = (a^2+d^2)(\cos^2\varphi - \sin^2\varphi) = (a^2+d^2)\cos 2\varphi$, we know that if the singularity is a node then $\cos 2\varphi > 0$, hence $\varphi \in (-\pi/4, \pi/4) \pmod{\pi}$, and if the singularity is a saddle then $\cos 2\varphi < 0$, hence $\varphi \in (\pi/4, 3\pi/4) \pmod{\pi}$. □

In the following we assume an orthonormal basis of $\mathbb{R}^2$ as described in Lemma 6.4.4 has been fixed.

Lemma 6.4.5. *Let* (X, Y) *be a linear proto-line-field. Let* X *be in one of the three normal forms of Lemma 6.4.4 and* $\alpha = \angle_{\mathrm{Eucl}}[(\begin{smallmatrix}1\\0\end{smallmatrix}), Y] \in \mathbb{R}/2\pi\mathbb{Z}$. *Consider* $F, G : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}/2\pi\mathbb{Z}$ *smooth and such that*

$$F(\theta) = \angle_{\mathrm{Eucl}}\left[\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix} E\cos\theta \\ \sin\theta\end{pmatrix}\right], \quad G(\theta) = \angle_{\mathrm{Eucl}}\left[\begin{pmatrix}1\\0\end{pmatrix}, X(E\cos\theta, \sin\theta)\right]$$

for every $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.

Then $\theta_0 \in \mathbb{R}/2\pi\mathbb{Z}$ *is a fixed point of* ϕ_L *if and only if there exists* $\theta_1 \in \mathbb{R}/2\pi\mathbb{Z}$ *such that* $\theta_0 = F(\theta_1)$ *and* $2F(\theta_1) - G(\theta_1) = \alpha \pmod{2\pi}$. *Moreover* θ_0 *is attractive if* $(2F-G)'(\theta_1) < 0$, *and repulsive if* $(2F-G)'(\theta_1) > 0$.

Proof. By its definition, F is increasing and $F' > 0$. By definition of F, G and α, we have

$$\phi_L \circ F(\theta) = \frac{1}{2}G(\theta) + \frac{1}{2}\alpha \pmod{\pi}.$$

Fixed points of ϕ_L are then the images by F of the solutions θ of

$$\frac{1}{2}G(\theta) + \frac{1}{2}\alpha = F(\theta) \pmod{\pi}$$

that is,

$$2F(\theta) - G(\theta) = \alpha \pmod{2\pi}, \tag{6.1}$$

which proves the first part of the statement. Since $F' > 0$, the sign of $\phi_L' - 1 = (\phi_L - \mathrm{id})'$ is the sign of $(\phi_L \circ F - F)' = \frac{1}{2}(G - 2F)'$, which proves the second part of the statement. $\square$

The idea is now to study the variations of $2F - G$ to show that, depending on the values taken by α and the index of the singularity, we fall in one of the cases stated in Theorem 6.4.3.

Proposition 6.4.6. *Under the assumptions of Theorem 6.4.3, if the singularity of X has index 1, then there exist two constants $\Phi \in [-\pi, \pi)$ and $\kappa > 0$ such that the sign of $(2F - G)'(\theta)$ is the sign of $\cos(2\theta + \Phi) + \kappa$. If the singularity of X has index -1, then $(2F - G)' > 0$ everywhere on $[-\pi, \pi)$.*

The proof of the proposition can be found in Appendix 6.A.

Before concluding the proof of Theorem 6.4.3, let us emphasize the following two properties of F and G.

P1 $\forall \theta \in (-\pi, \pi)$, $F(-\theta) = -F(\theta)$ and $G(-\theta) = -G(\theta)$;

P2 $\forall \theta \in [-\pi, 0)$, $F(\theta + \pi) - F(\theta) = \pi$ and $G(\theta + \pi) - G(\theta) = \pi$.

Proof of Theorem 6.4.3 when X has a singularity of index 1. Let $\kappa > 0$ and $\Phi \in [-\pi, \pi)$ be as in Proposition 6.4.6.

First consider the case where $\kappa \geq 1$. The derivative $(2F - G)'$ is then always positive, except possibly at two points in the case $\kappa = 1$. Hence $2F - G$ is a bijection between $[-\pi, \pi)$ and its image. We claim that $2F - G \pmod{2\pi}$ is a bijection between $[-\pi, \pi)$ and $[0, 2\pi)$. Equivalently we have to show that the image of $2F - G$ is an interval of length 2π, which is immediate by application of **P2**.

Thus we get the uniqueness in $[-\pi, \pi)$ of the solution θ of $2F(\theta) - G(\theta) = \alpha \pmod{2\pi}$. When $(2F - G)'(\theta) > 0$ we deduce the repulsiveness from Lemma 6.4.5. In the case $\kappa = 1$ the set $\mathcal{E}_X$ is made of a single line through the origin, corresponding to the two values of α for which $(2F - G)'(\theta) = 0$ and $2F(\theta) - G(\theta) = \alpha \pmod{2\pi}$.

The case $\kappa \in (0, 1)$ requires a further study of $2F - G$. From Proposition 6.4.6, we know that $(2F - G)'(\theta)$ has the same sign as $\cos(2\theta + \Phi) + \kappa$. Let $\theta_0 \in [0, \pi)$ be such that $\cos(2\theta_0 + \Phi) + \kappa = 0$ and $-2\sin(2\theta_0 + \Phi) > 0$, and let $\theta_1 \in [\theta_0, \theta_0 + \pi)$ be such that $\cos(2\theta_1 + \Phi) + \kappa = 0$ and $-2\sin(2\theta_1 + \Phi) < 0$. Then $\cos(2\theta + \Phi) + \kappa > 0$ on (θ_0, θ_1) and $\cos(2\theta + \Phi) + \kappa < 0$ on $(\theta_1, \pi + \theta_0)$. Since $(2F - G)'$ is π-periodic, we let $x = \theta_1 - \theta_0 \in (0, \pi)$ and up to replacing θ by $\theta - \theta_0$ (which corresponds to an orthonormal

θ	$-\pi$	$y-\pi$	$x-\pi$		0	y	x		π
$(2F-G)'$	0	$+$	0	$-$	0	$+$	0	$-$	0
$2F-G$ (mod 2π)	0 ↗	π ↗	$< 2\pi$ ↘		π ↗	2π / 0 ↗	$< \pi$ ↘		0

Figure 6.6 – Qualitative behavior of $2F - G$.

change of coordinates), we can assume that $(2F-G)'$ is positive on $(0, x)$ and negative on (x, π). (See Figure 6.6.)

We are interested in characterizing the solutions of $2F - G = \alpha \pmod{2\pi}$ or, equivalently,

$$(2F-G) - (2F-G)(-\pi) = \alpha - (2F-G)(-\pi) \pmod{2\pi}.$$

Thus we can focus on the case $2F - G = \beta \pmod{2\pi}$, for some $\beta \in [-\pi, \pi)$ and $(2F - G)(-\pi) = 0$.

Since $(2F-G)(0) = \pi$, and since $(2F-G)'$ is negative on $(x-\pi, 0)$, there exists $y \in (0, x)$ such that $2F - G$ is increasing on $(-\pi, y-\pi)$ and $(2F-G)(y) = \pi$. Moreover

$$\max_{\theta\in[0,\pi]} (2F-G)(\theta) \leq 2 \max_{\theta\in[-\pi,0]} F(\theta) - \min_{\theta\in[-\pi,0]} G(\theta).$$

Since F and G are increasing, $\max_{\theta\in[-\pi,0]} F(\theta) = \pi$ and $\min_{\theta\in[-\pi,0]} G(\theta) = 0$, so that

$$\max_{\theta\in[0,\pi]} (2F-G)(\theta) \leq 2\pi.$$

Hence we know the behavior of the function $2F - G$, that is,

— $2F - G$ is increasing on $(-\pi, x-\pi)$, $(2F-G)(-\pi) = 0$, $(2F-G)(x-\pi) \leq 2\pi$;

— $2F - G$ is decreasing on $(x-\pi, 0)$, $(2F-G)(0) = \pi$;

— $2F - G$ is increasing on $(0, x)$, and $(2F-G)(y) = 2\pi$, then $(2F-G)(\theta) \leq 3\pi$ when $\theta \in [y, x]$;

— $2F - G$ is decreasing on $[x, 2\pi]$, $(2F-G)(2\pi) = 2\pi$.

We can see that if $\alpha \notin [0, (2F-G)(x)] \cup [\pi, (2F-G)(-\pi+x)] + 2k\pi$, $k \in \mathbb{Z}$, there is a unique repulsive solution. If, instead, $\alpha \in (0, (2F-G)(x)) \cup (\pi, (2F-G)(-\pi+x)) + 2k\pi$, $k \in \mathbb{Z}$, then there are three solutions, two repulsive and one attractive. Moreover, either the three solutions are all contained in $(y-\pi, y)$, with $(2F-G)' > 0$ on the first and third solutions, or they all are in $[0, 2\pi) \backslash [y-\pi, y]$ with $(2F-G)' > 0$ on the first and second solutions (see Figure 6.6), which corresponds to the case 2 of Theorem 6.4.3. Notice that the values of α which are not covered by this discussion are 0, $(2F-G)(x)$, π, $(2F-G)(-\pi+x)$, which correspond to an exceptional set $\mathcal{E}_X$ made of two lines. $\square$

Proof of Theorem 6.4.3 when X has a singularity of index -1. In this case we have that $G'<0$ and $F'>0$ on $[-\pi,\pi)$. So $2F-G$, as a function from $[-\pi,\pi)$ to $\mathbb{R}$ is increasing and has total variation 6π. Hence there exists $y\in(-\pi,0)$ such that $(2F-G)-(2F-G)(-\pi)$ is a bijection from $(-\pi,y)$ onto $(0,2\pi)$; it exists $x\in(y,\pi)$ such that $(2F-G)-(2F-G)(-\pi)$ is a bijection from (y,x) onto $(2\pi,4\pi)$; and again $(2F-G)-(2F-G)(-\pi)$ is a bijection from (x,π) onto $(4\pi,6\pi)$. Hence we have found that there are three solutions to (6.1).

Since $(2F-G)(\theta+\pi)=(2F-G)(\theta)+\pi \pmod{2\pi}$, we know that each half-line where the line field is orthogonal to the line of the position is in the opposite direction to one of the solutions of equation (6.1). By monotonicity of $2F-G$, we then conclude that the three fixed points cannot be in the same half-plane. $\square$

Theorem 6.4.3 motivates the following definition.

Definition 6.4.7. A linear hyperbolic proto-line-field L is said to be *hyper-hyperbolic* if for every $\theta_0\in\mathbb{R}$ such that $\phi_L(\theta_0)=\theta_0 \pmod{\pi}$, one has $\frac{d\phi_L}{d\theta}(\theta_0)\neq 1$.

6.5 Linearization, blow-up and proof of Theorem 6.2.7

The goal of this section is to prove the topological equivalence of a hyper-hyperbolic proto-line-field at a hyperbolic singularity and its linearization. As a direct consequence, we get a proof of Theorem 6.2.7.

6.5.1 Blow-up

Proposition 6.5.1 below is the main technical step in the construction of the topological equivalence. It provides a blow up of a hyperbolic singularity of a proto-line-field. The blow up sends the singularity into a line and allows to describe locally the line field by means of a vector field on a strip containing such line.

In what follows set Pol to be the Riemannian metric on $\mathbb{R}^+\times\mathbb{R}$ defined by $\mathrm{Pol}(r,\theta)=dr^2+r^2d\theta^2$ and recall that a Riemannian metric can always be diagonalized at a point by a suitable choice of coordinates.

Proposition 6.5.1. *Let $L=(X,Y)$ be a proto-line-field on (M,g) with a hyperbolic singularity at $p\in M$. Fix a system of coordinates (x,y) such that $p=(0,0)$, $g(0,0)=\mathrm{id}$ and assume that (x,y) defines a diffeomorphism between a neighborhood of p and the ball of center the origin and radius δ, for some $\delta>0$ such that p is the only singularity* *of L on the ball. Assume that $Y(0,0) \neq 0$ and consider the linear proto-line-field $\bar{L} = (DX(0,0), Y(0,0))$.*

For every $(r,\theta) \in (0,\delta) \times \mathbb{R}$ let $\phi_L(r,\theta)$ and $\phi_{\bar{L}}(\theta)$ in $\mathbb{R}/\pi\mathbb{Z}$ be defined by

$$\phi_L(r,\theta) = \angle_g \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, B(L(r\cos\theta, r\sin\theta)) \right] \text{ and } \phi_{\bar{L}}(\theta) = \angle_{\text{Eucl}} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, B(\bar{L}(r\cos\theta, r\sin\theta)) \right].$$

Then there exists a C^1 function $\widetilde{\phi}_L : (-\delta,\delta) \times \mathbb{R} \to \mathbb{R}$ such that for every $(r,\theta) \in (0,\delta) \times \mathbb{R}$

$$\begin{aligned} &\phi_L(r,\theta) = \widetilde{\phi}_L(r,\theta) \pmod{\pi}, \\ &\widetilde{\phi}_L(-r,\theta) = \widetilde{\phi}_L(r,\theta) \\ &\widetilde{\phi}_L(r,\theta + 4\pi) = \widetilde{\phi}_L(r,\theta) \pmod{2\pi}, \end{aligned}$$

and such that the vector field P on $(-\delta,\delta) \times (\mathbb{R}/4\pi\mathbb{Z})$ given by

$$P(r,\theta) = \begin{pmatrix} r\cos(\widetilde{\phi}_L(r,\theta) - \theta) \\ \sin(\widetilde{\phi}_L(r,\theta) - \theta) \end{pmatrix} \tag{6.2}$$

is C^1 and satisfies, for all $(r,\theta) \in (0,\delta) \times (\mathbb{R}/4\pi\mathbb{Z})$,

$$\angle_{\text{Pol}} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, P(r,\theta) \right] = \angle_g \left[\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, B(L(r\cos\theta, r\sin\theta)) \right] \pmod{\pi}. \tag{6.3}$$

The singularities of P in $(-\delta,\delta) \times (\mathbb{R}/4\pi\mathbb{Z})$ are the points $(0,\theta_0)$ such that $\phi_{\bar{L}}(\theta_0) = \theta_0$ $\pmod{\pi}$. Moreover, if θ_0 is a repulsive (respectively, attractive) fixed point of $\phi_{\bar{L}}$ then the singularity $(0,\theta_0)$ of P is a saddle (respectively, a node).

Proof. Since $\delta > 0$ has been chosen small enough so that the only singularity of L in $\{x^2 + y^2 \leq \delta^2\}$ is in $(0,0)$, then $\phi_L : (0,\delta) \times \mathbb{R} \to \mathbb{R}/\pi\mathbb{Z}$ can be lifted as a smooth function $\widetilde{\phi}_L : (0,\delta) \times \mathbb{R} \to \mathbb{R}$.

Lemma 6.B.1, found in appendix 6.B, states that the limits

$$\phi_L(r,\theta) \underset{r\to 0}{\longrightarrow} \phi_{\bar{L}}(\theta), \quad \frac{\partial \phi_L}{\partial \theta}(r,\theta) \underset{r\to 0}{\longrightarrow} \frac{d\phi_{\bar{L}}}{d\theta}(\theta), \text{ and } \quad \frac{\partial \phi_L}{\partial r}(r,\theta) \underset{r\to 0}{\longrightarrow} 0,$$

hold true, thus proving that $\widetilde{\phi}_L$ admits a C^1 extension on $[0,\delta) \times \mathbb{R}$. We can then symmetrically extend $\widetilde{\phi}_L$ onto $(-\delta,\delta) \times \mathbb{R}$ by setting $\widetilde{\phi}_L(r,\theta) = \widetilde{\phi}_L(-r,\theta)$ for any $r \in (-\delta, 0)$. By construction, $\widetilde{\phi}_L$ is C^1.

From this construction we deduce that P, defined as in (6.2), is C^1 on $(-\delta,\delta) \times (\mathbb{R}/4\pi\mathbb{Z})$ and that it vanishes exclusively at the points $(0,\theta_0)$, where $\phi_{\bar{L}}(\theta_0) = \theta_0 \pmod{\pi}$. Its definition further implies relation (6.3).

The differential of P at a point $(r,\theta) \in (-\delta,\delta) \times (\mathbb{R}/4\pi\mathbb{Z})$ is given by

$$DP(r,\theta) = \begin{pmatrix} \cos\left(\widetilde{\phi}_L - \theta\right) + r\frac{\partial \widetilde{\phi}_L}{\partial r} \sin\left(\widetilde{\phi}_L - \theta\right) & r\left(\frac{\partial \widetilde{\phi}_L}{\partial \theta} - 1\right) \sin\left(\widetilde{\phi}_L - \theta\right) \\ \frac{\partial \widetilde{\phi}_L}{\partial r} \cos\left(\widetilde{\phi}_L - \theta\right) & \left(\frac{\partial \widetilde{\phi}_L}{\partial \theta} - 1\right) \cos\left(\widetilde{\phi}_L - \theta\right) \end{pmatrix}.$$

Thus, at a singularity $(0, \theta_0)$, where $\phi_{\bar{L}}(\theta_0) = \theta_0 \pmod{\pi}$, the differential of P is given by

$$DP(0,\theta_0) = \cos\left(\widetilde{\phi}_L(0,\theta_0) - \theta_0\right)\begin{pmatrix} 1 & 0 \\ 0 & \frac{d\widetilde{\phi}_{\bar{L}}}{d\theta} - 1 \end{pmatrix}.$$

Since $\phi_{\bar{L}}(\theta_0) = \theta_0 \pmod{\pi}$, moreover, we have $\cos\left(\widetilde{\phi}_L(0,\theta_0) - \theta_0\right) = \pm 1$. Therefore, if θ_0 is attractive then $\frac{\partial \phi_{\bar{L}}}{\partial \theta} > 1$ and the singularity is a node, while if θ_0 is repulsive then $\frac{\partial \phi_{\bar{L}}}{\partial \theta} > 1$ and the singularity is a saddle. □

The proto-line-field $\bar{L}$ in Proposition 6.5.1 plays the role of the linearization of L. This motivates the following definition.

Definition 6.5.2. Let $L = (X, Y)$ be a proto-line-field on (M, g) with a singularity at $p \in M$ such that $Y(p) \neq 0$. Fix a system of coordinates (x, y) such that $p = (0, 0)$, $g(0, 0) = \mathrm{id}$. Then we call *linearization of L at p* the linear proto-line-field $\bar{L} = (DX(0, 0), Y(0, 0))$.

Notice that the condition $g(0, 0) = \mathrm{id}$ defines uniquely $\bar{L}$ up to an orthogonal transformation of $\mathbb{R}^2$. In particular, the fact that $\bar{L}$ satisfies one of the properties 1, 2 or 3 of Theorem 6.4.3 depends only on L.

6.5.2 Proof of Theorem 6.2.7

The proof of Theorem 6.2.7 is based on the following proposition.

Proposition 6.5.3. *Let L and L' be two proto-line-fields on (M, g) and (M', g') respectively. Let p and p' be two hyperbolic singularities of L and L' respectively. Let $\bar{L}$ and $\bar{L}'$ be the corresponding linearizations and assume that one of the properties 1, 2 or 3 of Theorem 6.4.3 is satisfied both by $\bar{L}$ and $\bar{L}'$. Then L and L' are topologically equivalent at p and p'.*

The proof of Proposition 6.5.3 is given in the next section. A first consequence of the proposition is the following corollary.

Corollary 6.5.4. *Let L be a proto-line-field on (M, g). Let p be a hyperbolic singularity of L. Let $\bar{L}$ be the corresponding linearization. If $\bar{L}$ is hyper-hyperbolic then L and $\bar{L}$ are topologically equivalent at p and 0.*

Theorem 6.2.7 can now be obtained by combining Theorem 6.4.3, Proposition 6.5.3 and Thom's transversality theorem.

Proof of Theorem 6.2.7. As a consequence of Thom's transversality theorem, generically with respect to (X, Y) in $\mathcal{W}^1(M)$, every singularity of the proto-line-field $L = (X, Y)$ is hyperbolic and its linearization is hyper-hyperbolic. Notice that the Lemon (respectively, Monstar, Star) proto-line-field satisfies property 1 (respectively, 2, 3) of Theorem 6.4.3. It then follows from Proposition 6.5.3 that all singularities of a generic proto-line-field are Darbouxian. □

6.5.3 Construction of the topological equivalence

In order to prove Proposition 6.5.3, let us first focus on the following two lemmas, which yield conditions for the existence of homeomorphisms preserving integral manifolds of proto-line-fields around singularities.

Lemma 6.5.5. *Let L and L' be two proto-line-fields on (M, g) and (M', g') respectively. Let p and p' be two hyperbolic singularities of L and L' respectively. Let $\bar{L}$ and $\bar{L}'$ be the corresponding linearizations and assume that one of the properties 1, 2 or 3 of Theorem 6.4.3 is satisfied both by $\bar{L}$ and $\bar{L}'$. Let $\delta > 0$ (respectively, $\delta' > 0$) and P (respectively, P') be the vector field on $(-\delta, \delta) \times (\mathbb{R}/4\pi\mathbb{Z})$ (respectively $(-\delta', \delta') \times (\mathbb{R}/4\pi\mathbb{Z})$) introduced in Proposition 6.5.1.*

Then there exist two neighborhoods V and V' of $\{0\} \times (\mathbb{R}/4\pi\mathbb{Z})$, invariant under the translation $T : (r, \theta) \mapsto (r, \theta + 2\pi)$, and an homeomorphism $h : V \to V'$ such that h maps the integral lines of P onto the integral lines of P' and $h \circ T = T \circ h$.

Proof. Choose $\eta > 0$ small enough so that P has no cycle nor integral curve with both ends at a saddle in $(-\eta, 0) \times (\mathbb{R}/4\pi\mathbb{Z})$. We are interested in studying the skeleton of P, *i.e.*, the union S of the set of zeros of P and integral curves in $(-\eta, \eta) \times (\mathbb{R}/4\pi\mathbb{Z})$ that reach a saddle singularity of P at one of its ends (or at both of them). The set $((-\eta, \eta) \times (\mathbb{R}/4\pi\mathbb{Z}))\backslash S$ has exactly twice as many connected components as P has saddles in $(-\eta, \eta) \times (\mathbb{R}/4\pi\mathbb{Z})$, and four times as many as $\phi_{\bar{L}}$ has repulsive fixed points in $\mathbb{R}/2\pi\mathbb{Z}$. (See Figure 6.7.)

Let $\{C_i \mid i \in I\}$ be the set of connected components of $((-\eta, \eta) \times (\mathbb{R}/4\pi\mathbb{Z}))\backslash S$.

The border ∂C of a cell C is the union of a segment of the type $\{0\} \times [\theta_1, \theta_2]$, of an arc of $\{\pm\eta\} \times (\mathbb{R}/4\pi\mathbb{Z})$, and of two integral curves γ_1, γ_2 of P that join $(0, \theta_1)$ and $(0, \theta_2)$.

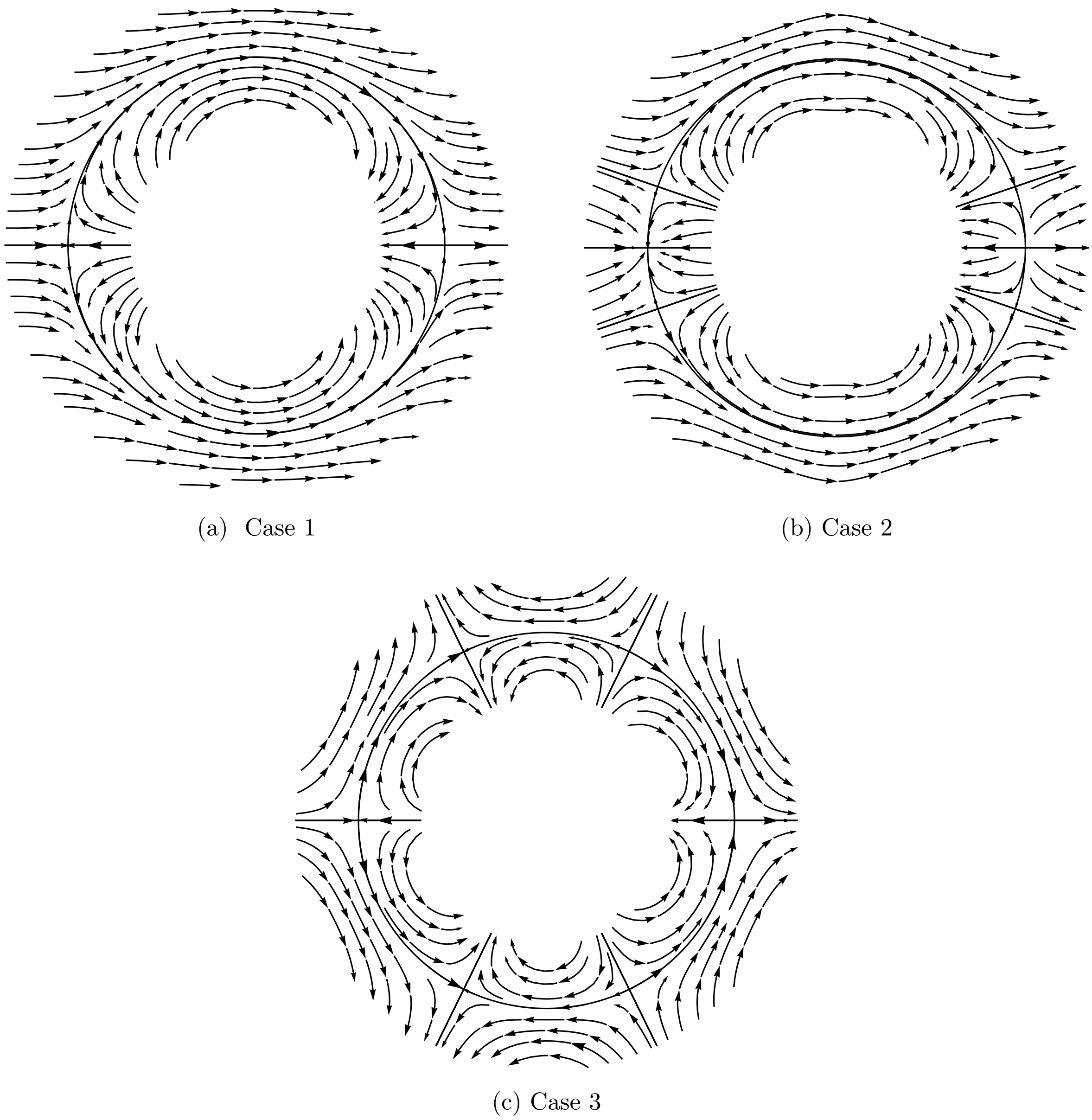
If there is no attractive fixed point of $\phi_{\bar{L}}$ in the interval (θ_1, θ_2) then we can find an integral line γ of P that is arbitrarily close to $\gamma_1 \cup (\{0\} \times [\theta_1, \theta_2]) \cup \gamma_2$ (see Figure 6.8a). Then we can assume that the vector field P is transverse to $\{\pm\eta\} \times (\mathbb{R}/4\pi\mathbb{Z})$ between γ and $\gamma_1 \cup (\{0\} \times [\theta_1, \theta_2]) \cup \gamma_2$, and that it is topologically equivalent on this subset to the parallel vector field

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ on } [0, 1] \times [0, 1].$$

If there is an attractive fixed point of $\phi_{\bar{L}}$ in the interval (θ_1, θ_2) then we can find $0 < \eta_1 < \eta$ so that P is transverse to $\{\pm\eta_1\} \times (\mathbb{R}/4\pi\mathbb{Z})$ (see Figure 6.8b) and P is topologically equivalent on the intersection of the cell with $(-\eta_1, \eta_1) \times (\mathbb{R}/4\pi\mathbb{Z})$ to

$$\begin{pmatrix} x \\ y \end{pmatrix} \text{ on } [0, 1] \times [0, 1].$$

Since L and L' satisfy the same property 1, 2 or 3 of Theorem 6.4.3, their skeletons are homeomorphic. Since the directions of P and P' are 2π-periodic, the construction above leads to a topological equivalence between P and P' as in the statement of the lemma. $\square$



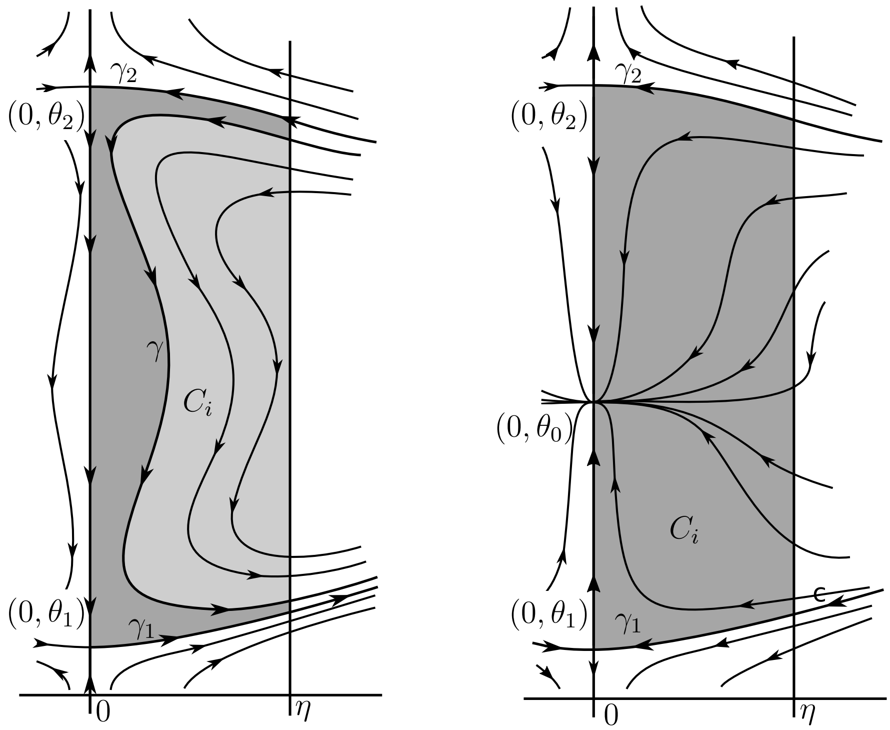
(a) Case 1

(b) Case 2

(c) Case 3

Figure 6.7 – Examples of P in each case of Theorem 6.4.3, where $(-\delta_0, \delta_0) \times (\mathbb{R}/4\pi\mathbb{Z})$ is represented as an annulus. The figures 6.7a, 6.7b and 6.7c where respectively computed with $Y = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $X = \begin{pmatrix} x \\ 10y/11 \end{pmatrix}$, $X = \begin{pmatrix} x \\ 5y \end{pmatrix}$, $X = \begin{pmatrix} x \\ -3y \end{pmatrix}$, respectively.

Lemma 6.5.6. *Let L be a proto-line-field on (M, g). Let p be a hyperbolic singularity of L. Assume that the linearization $\bar{L}$ of L at p is hyper-hyperbolic. Let delta > 0, the*



(a) First case: there is no attractive fixed point of $\phi_{\bar{L}}$ between θ_1 and θ_2. The topological equivalence is defined on a subset of C_i bounded by the curve γ.

(b) Second case: there is an attractive fixed point θ_0 of $\phi_{\bar{L}}$ between θ_1 and θ_2. For η small enough the topological equivalence is defined on the entire cell C_i.

Figure 6.8 – Representation of the two types of connected components of $((-\eta, \eta) \times (\mathbb{R}/4\pi\mathbb{Z}))\backslash S$.

system of coordinates (x, y), and the vector field P on $(-\delta, \delta) \times (\mathbb{R}/4\pi\mathbb{Z})$ be defined as in Proposition 6.5.1. Then the application

$$\begin{array}{rccl} \psi: & (0, \delta) \times (\mathbb{R}/4\pi\mathbb{Z}) & \longrightarrow & \{0 < x^2 + y^2 < \delta^2\} \\ & (r, \theta) & \longmapsto & (x, y) = (r\cos\theta, r\sin\theta) \end{array}$$

is a local diffeomorphism that maps the integral lines of P onto the integral manifolds of L.

Proof. The map ψ is a local diffeomorphism since the differential of ψ is given by

$$D\psi(r, \theta) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}.$$

Hence,

$$\psi_*(P(r,\theta)) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \begin{pmatrix} r\cos(\widetilde{\phi}_L(r,\theta)-\theta) \\ \sin(\widetilde{\phi}_L(r,\theta)-\theta) \end{pmatrix} = \begin{pmatrix} r\cos(\widetilde{\phi}_L(r,\theta)) \\ r\sin(\widetilde{\phi}_L(r,\theta)) \end{pmatrix}.$$

Thus, the locally defined vector field $\psi_* P$ is parallel to L, concluding the proof of the lemma. $\square$

We are now ready to prove Proposition 6.5.3.

Proof of Proposition 6.5.3. Let $\delta > 0$ (respectively, $\delta' > 0$) and P (respectively, P') be the vector field on $(-\delta,\delta)\times(\mathbb{R}/4\pi\mathbb{Z})$ (respectively $(-\delta',\delta')\times(\mathbb{R}/4\pi\mathbb{Z})$) introduced in Proposition 6.5.1. Consider V, V' and the homeomorphism $h : V \to V'$ introduced in Lemma 6.5.5. Finally, following Lemma 6.5.6, let ψ and ψ' be the local diffeomorphisms

$$\begin{array}{rccl} \psi: & (0,\delta)\times\mathbb{R}/4\pi\mathbb{Z} & \longrightarrow & D_p(\delta) = \{0 < x^2+y^2 < \delta^2\} \\ & (r,\theta) & \longmapsto & (x,y) = (r\cos\theta, r\sin\theta) \end{array}$$

and

$$\begin{array}{rccl} \psi': & (0,\delta')\times\mathbb{R}/4\pi\mathbb{Z} & \longrightarrow & D'_{p'}(\delta') = \{0 < x'^2+y'^2 < \delta'^2\} \\ & (r,\theta) & \longmapsto & (x',y') = (r\cos\theta, r\sin\theta). \end{array}$$

The translation $T : (r,\theta) \mapsto (r,\theta+2\pi)$ induces a natural fiber bundle structure $\pi : V \to V/_\sim$ (where $(r,\theta)\sim(r',\theta')$ if $r = r'$ and $\theta = \theta' \pmod{2\pi}$). Notice that $V/_\sim$ is a neighborhood of $\{0\}\times(\mathbb{R}/2\pi\mathbb{Z})$ in $(-\delta,\delta)\times(\mathbb{R}/2\pi\mathbb{Z})$. Likewise we define $\pi' : V' \to V'/_\sim \subset (-\delta',\delta')\times(\mathbb{R}/2\pi\mathbb{Z})$.

Since $h\circ T = T\circ h$, there exists a homeomorphism $\bar{h} : V/_\sim \to V'/_\sim$ such that

$$\begin{array}{ccc} V & \xrightarrow{h} & V' \\ \downarrow\pi & & \downarrow\pi' \\ V/_\sim & \xrightarrow{\bar{h}} & V'/_\sim \end{array} \tag{6.4}$$

commutes.

Since P and P' do not have singularities on $(0,\delta)\times(\mathbb{R}/4\pi\mathbb{Z})$ and $(0,\delta')\times(\mathbb{R}/4\pi\mathbb{Z})$ respectively, then the line field spanned by each of them has no singularity and is 2π-periodic with respect to the second variable. Therefore, one can identify $\pi_* P$ and $\pi'_* P'$ with two line fields without singularities on $(0,\delta)\times(\mathbb{R}/2\pi\mathbb{Z})$ and $(0,\delta')\times(\mathbb{R}/2\pi\mathbb{Z})$ respectively. The commutativity of diagram (6.4) and Lemma 6.5.5 then show that $\bar{h}$ is a homeomorphism between $V/_\sim$ and $V'/_\sim$ that maps the integral manifolds of $\pi_* P$ onto the integral manifolds of $\pi'_* P'$.

Since $\psi\circ T = \psi$ on $(0,\delta)\times(\mathbb{R}/4\pi\mathbb{Z})$, there exists a diffeomorphism $\widehat{\psi} : V_+/_\sim \to D_p(\delta)\backslash\{p\}$ such that $\widehat{\psi}\circ\pi = \psi$, where $V_+ = V\cap((0,\delta)\times(\mathbb{R}/4\pi\mathbb{Z}))$. In particular $\widehat{\psi}(V_+/_\sim)\cup\{p\}$ is a neighborhood of p. Lemma 6.5.6 implies that $\widehat{\psi}^{-1}$ is a homeomorphism from $\widehat{\psi}(V_+/_\sim)$ onto $V_+/_\sim$ that maps the integral manifolds of L onto the integral manifolds of $\pi_* P$. One can similarly define $\widehat{\psi}'$ on $V'_+/_\sim$, which satisfies analogous properties.

The topological equivalence of L and L' at p and p' can therefore be proven through the homeomorphism $H : \widehat{\psi}(V_+/_\sim) \cup \{p\} \to \widehat{\psi}'(V'_+/_\sim) \cup \{p\}$ defined by $H(p) = p'$ and

$$H : \widehat{\psi}(V_+/_\sim) \xrightarrow{\widehat{\psi}^{-1}} V_+/_\sim \xrightarrow{\bar{h}} V'_+/_\sim \xrightarrow{\widehat{\psi}'} \widehat{\psi}'(V'_+/_\sim).$$

By construction, H is indeed a homeomorphism between a neighborhood of p and a neighborhood of p' which takes the integral manifolds of L onto those of L'. $\square$

6.6 The role of the metric g

6.6.1 What changes if we change the metric g

In this chapter, the metric g is fixed from the beginning and the main results (Theorem 6.2.7, Propositions 6.3.1 and 6.3.4) are independent of its choice. It is natural to ask which properties are affected by the choice of g. The following example shows that a proto-line-field having a Lemon singularity at a point p for a certain metric can have a Monstar singularity for another metric. Notice however that the Star singularity cannot be transformed into a Lemon or Monstar singularity by changing g, since they have different indices.

Example 6.6.1. For every $\lambda > 0$ consider the Riemannian metric $g_\lambda = dx^2 + \lambda^2 dy^2$ on $\mathbb{R}^2$. Let $X(x,y) = \begin{pmatrix} x \\ y \end{pmatrix}$ and $Y(x,y) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then the singularity of $L = (X, Y)$ at $(0,0)$ can be a Lemon singularity or a Monstar singularity depending on λ. Indeed, let $\phi_X(\theta) = \angle_{g_\lambda}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, X(\cos\theta, \sin\theta)\right]$ and $\phi_L(\theta) = \angle_{g_\lambda}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, B_{g_\lambda}(X(\cos\theta, \sin\theta), Y(\cos\theta, \sin\theta))\right] = \frac{1}{2}\phi_X(\theta)$. Notice that $\theta = 0$ is a fixed point of ϕ_X and let us compute the derivative of ϕ_X at 0.

Since

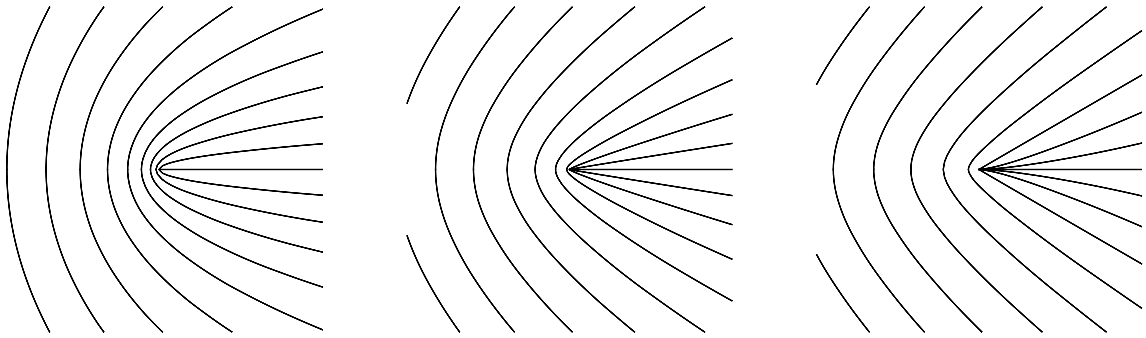
$$g_\lambda\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = 1, \quad g_\lambda\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \lambda^2,$$

one has

$$\|X(\cos\theta, \sin\theta)\|_{g_\lambda} \cos\phi_X(\theta) = \cos\theta, \quad \lambda\|X(\cos\theta, \sin\theta)\|_{g_\lambda} \sin\phi_X(\theta) = \sin\theta.$$

Therefore, $\tan\phi_X(\theta) = \lambda\tan\theta$ and thus $\frac{d}{d\theta}\phi_X(0) = \lambda$ and $\frac{d}{d\theta}\phi_L(0) = \lambda/2$.

If $0 < \lambda < 2$, then $\theta = 0$ is the only fixed point of ϕ_L and it is repulsive. If $\lambda > 2$, then $\theta = 0$ is an attractive fixed point of ϕ_L. (See Figure 6.9.)



(a) $\lambda = 1$ (b) $\lambda = 2$ (c) $\lambda = 3$

Figure 6.9 – Integral manifolds of the proto-line-fields of Example 6.6.1 for three values of λ.

Notice that the bifurcation value $\lambda = 2$ corresponds to a case which is not hyper-hyperbolic. Hence a proto-line-field that has a structurally stable singularity at a point p for a certain metric, can have non-structurally stable singularities at p for another metric.

The next proposition shows that if we take a smooth curve γ passing through a hyperbolic singularity of a proto-line-field L, then the angle between the line field associated with L and $\dot\gamma$, measured with respect to the metric g, makes a jump of $\pi/2$ at the singularity. Hence, changing the metric and keeping the same proto-line-field, produces a new line field for which the angle between $\dot\gamma$ and itself, measured with respect to the new metric, jumps again of $\pi/2$.

Proposition 6.6.2. *Let L be a proto-line-field on (M, g) with a hyperbolic singularity at $p \in M$. Let $\gamma : (-1, 1) \to M$ be a smooth curve on M such that $\gamma(0) = p$ and $\dot\gamma(0) \neq 0$. Then*

$$\lim_{t\to 0^+} \angle_g \left[\dot\gamma(t), B(L(\gamma(t)))\right] = \lim_{t\to 0^-} \angle_g \left[\dot\gamma(t), B(L(\gamma(t)))\right] + \pi/2 \pmod{\pi}.$$

Proof. Up to a change of parametrization, we can assume that γ is parametrized by arc length, so that we can fix a system of coordinates (x, y) on a neighborhood of $p = (0, 0)$ such that $g(0, 0) = \mathrm{id}$ and γ coincides with the curve $t \mapsto (t, 0)$.

Then using the notation of Section 6.5.1, for $t > 0$ we have $\angle_g \left[\dot\gamma(t), B(L(\gamma(t)))\right] = \phi_L(t, 0)$ and $\angle_g \left[\dot\gamma(-t), B(L(\gamma(-t)))\right] = \phi_L(t, \pi)$.

Since X has a hyperbolic singularity at p, we have that

$$\lim_{t\to 0^+} \frac{X(t, 0)}{\|X(t, 0)\|} = -\lim_{t\to 0^+} \frac{X(-t, 0)}{\|X(-t, 0)\|}.$$

By definition of L, we then have

$$\lim_{t\to 0^+} \phi_L(t, 0) = \lim_{t\to 0^+} \phi_L(t, \pi) + \pi/2 \pmod{\pi}.$$

□

Remark 6.6.3. We know from Proposition 6.3.1 that for every closed set $K \subset M$ and every section L of $PT(M \setminus K)$, for every Riemannian metric g on M, there exists a proto-line-field (X, Y) such that $B(X, Y) = L$ on $M \setminus K$.

Proposition 6.6.2 says that, even if K is made of isolated points, one cannot expect in addition that the singularities of X and Y are hyperbolic, unless some compatibility condition between L and g is satisfied at each point of K.

In particular, a line field associated with a proto-line-field with hyperbolic singularities for a certain Riemannian metric is not in general associated with any proto-line-field with hyperbolic singularities for a different Riemannian metric.

6.6.2 How to construct a Riemannian metric from a pair of vector fields

The procedure of defining a line field by using two vector fields and a Riemannian metric (or a conformal structure) may look greedy and one may wonder if some alternative definition involving less functional parameters can lead to similar characterizations of structurally stable singularities. In this section we propose a way to get rid of the requirement of fixing a Riemannian metric. This is done by constructing a Riemannian metric from two vector fields alone, at least in the generic case.

Proposition 6.6.4. *Let X, Y be two vector fields on M and set*

$$Z = [X, Y]\,,\ W^1 = [X, [X, Y]]\,,\ W^2 = [Y, [X, Y]]\,.$$

Generically with respect to $(X, Y) \in W^2(M)$, the vectors $X(p), Y(p), Z(p), W^1(p), W^2(p)$ span T_pM for every $p \in M$. In this case

$$\|V\| = \min\left\{\|u\|_2 \mid u \in \mathbb{R}^5, \left(u_1X + u_2Y + u_3Z + u_4W^1 + u_5W^2\right)(p) = V\right\}, \quad V \in T_pM, \tag{6.5}$$

is a norm on T_pM depending smoothly on p and

$$g_{X,Y}(V, V') = \frac{1}{2}\left(\|V + V'\|^2 - \|V\|^2 - \|V'\|^2\right), \quad V, V' \in T_pM,$$

defines a Riemannian metric on M.

Proof of Proposition 6.6.4. Up to reducing M to a coordinate chart, the set

$$\mathcal{F} = \left\{ j_2(X,Y)(p) \;\middle|\; \begin{array}{l} p \in M,\ X, Y \text{ vector fields on } M \\ X(p) \wedge Y(p) = 0 \\ [X, Y](p) \wedge X(p) = 0 \\ [X, [X, Y]]\,(p) \wedge X(p) = 0 \end{array} \right\}$$

can be identified with a submanifold of codimension 3 of $J^2(M, \mathbb{R}^4)$, the set of 2-jets of maps from M to $\mathbb{R}^4$. By Thom's transversality theorem, for a generic pair $(X, Y) \in W^2(M)$, there exist no $p \in M$ such that $j_2(X, Y)(p) \in \mathcal{F}$. In particular, $X(p), Y(p), Z(p), W^1(p), W^2(p)$ span T_pM for every $p \in M$.

Fix now $p \in M$, $V \in T_pM$ and let us give an explicit expression of the vector u realizing the minimum in (6.5), assuming that $X(p), Y(p), Z(p), W^1(p), W^2(p)$ span T_pM. Write in local coordinates $V = (V_1, V_2)$, $X = (X_1, X_2)$, $Y = (Y_1, Y_2)$, *etc.* Let

$$s_i = \left(X_i, Y_i, Z_i, W_i^1, W_i^2\right)(p) \in \mathbb{R}^5, \ i \in \{1, 2\},$$

and $r^4 = |s_1|^2|s_2|^2 - \langle s_1, s_2\rangle^2$, which is positive because s_1 and s_2 cannot be colinear. Since the two affine hyperplanes $H_1 = \{u \in \mathbb{R}^5 \mid \langle u, s_1\rangle = V_1\}$ and $H_2 = \{u \in \mathbb{R}^5 \mid \langle u, s_2\rangle = V_2\}$ are not parallel, minimizing $\|u\|_2$ in (6.5) comes down to finding the orthogonal projection u_V of $0_{\mathbb{R}^5}$ onto $H_1 \cap H_2$. Since the orthogonal subspace to $H_1 \cap H_2$ is the span of s_1 and s_2, then the point u_V is characterized by the conditions

$$\langle u_V, s_1\rangle = V_1, \quad \langle u_V, s_2\rangle = V_2, \quad u_V = \lambda_1 s_1 + \lambda_2 s_2, \ \lambda_1, \lambda_2 \in \mathbb{R}.$$

Hence (λ_1, λ_2) is the unique solution of the system

$$\begin{cases} \lambda_1|s_1|^2 + \lambda_2 \langle s_1, s_2\rangle &= V_1 \\ \lambda_1 \langle s_1, s_2\rangle + \lambda_2|s_2|^2 &= V_2, \end{cases} \tag{6.6}$$

which leads to the characterization of u_V as

$$u_V = \left(\frac{V_1|s_2|^2 - V_2\langle s_1, s_2\rangle}{r^4}\right) s_1 + \left(\frac{V_2|s_1|^2 - V_1\langle s_1, s_2\rangle}{r^4}\right) s_2.$$

Since u_V depends linearly on V, it is then easy to see that the norm $\|\cdot\| = \|u_\cdot\|_2$ depends smoothly on p. This norm derives from a scalar product since it verifies the parallelogram law, as we are now going to show. Let $V, V' \in T_pM$ and u_V, $u_{V'}$ be defined as above. Then $u_V = \lambda_1 s_1 + \lambda_2 s_2$ and $u_{V'} = \lambda_1' s_1 + \lambda_2' s_2$, where (λ_1, λ_2) and (λ_1', λ_2') are the respective unique solutions of system (6.6). Then, by linearity, we have that $u_{V+V'} = (\lambda_1 + \lambda_1')s_1 + (\lambda_2 + \lambda_2')s_2$ and $u_{V-V'} = (\lambda_1 - \lambda_1')s_1 + (\lambda_2 - \lambda_2')s_2$. Thus

$$\|V\|^2 + \|V'\|^2 = \|u_V\|_2^2 + \|u_{V'}\|_2^2 = \|u_V + u_{V'}\|_2^2 + \|u_V - u_{V'}\|_2^2 = \|V + V'\|^2 + \|V - V'\|^2.$$

□

Remark 6.6.5. Notice that for every compact $K \subset M$ the set of pairs (X, Y) such that the metric $g_{X,Y}$ introduced in Proposition 6.6.4 is well-defined on K is open in $\mathcal{W}^2(M)$ and $g_{X,Y}$ depends continuously on (X, Y) on it. Hence, because of the continuity of the linearization of a proto-line-field with respect to (X, Y) and g (see Definition 6.5.2), we deduce the local structural stability of Lemon, Monstar and Star singularities for proto-line-fields (X, Y) with respect to the metric $g_{X,Y}$, in the sense that if at a singular point p the linearized system is satisfies condition 1, 2, or 3 of Theorem 6.4.3, then the same property is satisfied by every small perturbation of (X, Y) in the $\mathcal{W}^2(M)$ topology.

In order to prove that generically with respect to (X, Y) the singularities of the proto-line-field (X, Y) with respect to the metric $g_{X,Y}$ are Darbouxian, one should further prove that non-Darbouxian singularities can be removed by small perturbations of (X, Y). Although we expect this result to be true (in the $\mathcal{W}^2(M)$ topology and not in the $\mathcal{W}^1(M)$ one as it was the case in Theorem 6.2.7), this does not follow directly from the results in this chapter.

Appendices

6.A Proof of Proposition 6.4.6

Let us first compute F' and G'. By definition of F we have $\sin\theta\cos F(\theta) = E\cos\theta\sin F(\theta)$ from which we obtain

$$E\cos^2\theta\sin F(\theta) - F'(\theta)\sin^2\theta\sin F(\theta) = E^2F'(\theta)\cos^2\theta\sin F(\theta) - E\sin^2\theta\sin F(\theta).$$

Then either $\sin F(\theta) = 0$, and then $\sin\theta = 0$, or

$$E\cos^2\theta - F'(\theta)\sin^2\theta = E^2F'(\theta)\cos^2\theta - E\sin^2\theta.$$

By smoothness of F,

$$F'(\theta) = \frac{E}{E^2\cos^2\theta + \sin^2\theta}, \quad \forall\theta \in [-\pi, \pi). \tag{6.7}$$

Concerning G, in the case of focuses, we immediately get that $G(\theta) = F(\theta + \varphi)$. In the other two cases, reasoning as for F, we get

$$G'(\theta) = \frac{E\cos(2\varphi)}{E^2\cos^2(\theta - \varphi) + \sin^2(\theta + \varphi)}, \quad \forall\theta \in [-\pi, \pi). \tag{6.8}$$

We prove Proposition 6.4.6 by considering separately the three cases corresponding to the type of singularity of X. The saddle case follows immediately from (6.7) and (6.8), since $\varphi \in (\pi/4, 3\pi/4) \pmod{\pi}$ implies that $\cos(2\varphi) < 0$. The following two lemmas consider the focus and node case, respectively.

Lemma 6.A.1. *Let the singularity of X be a focus and set $A = \sqrt{5 - 2\cos(2\varphi)}$. Then Φ and κ as in the statement of Proposition 6.4.6 exist and are characterized by*

$$\begin{cases} A\cos(\Phi) = & 2\cos(2\varphi) - 1 \\ A\sin(\Phi) = & 2\sin(2\varphi) \end{cases} \tag{6.9}$$

and

$$\kappa = \frac{E^2 + 1}{A(E^2 - 1)}.$$

Proof. Let Φ satisfy (6.9). By definition of Φ and A, the inequality

$$\cos(2\theta+\Phi) > -\frac{E^2+1}{A(E^2-1)}$$

is equivalent to

$$(E^2-1)\cos(2\theta)(2\cos(2\varphi)-1) - 2(E^2-1)\sin(2\theta)\sin(2\varphi) + E^2+1 > 0. \tag{6.10}$$

By elementary trigonometric identities, condition (6.10) is equivalent to

$$2(E^2\cos^2(\theta+\varphi)+\sin^2(\theta+\varphi)) - (E^2\cos^2\theta+\sin^2\theta) > 0,$$

which, in turns, is equivalent to

$$2F'(\theta) - G'(\theta) = \frac{2E}{E^2\cos^2\theta+\sin^2\theta} - \frac{E}{E^2\cos^2(\theta+\varphi)+\sin^2(\theta+\varphi)} > 0.$$

□

Lemma 6.A.2. *Let the singularity of X be a node and set*

$$A = \sqrt{\frac{1}{2}\left(5+6E^2+5E^4-(3+10E^2+3E^4)\cos 4\varphi\right)}.$$

Then Φ and κ as in the statement of Proposition 6.4.6 exist and are characterized by

$$\begin{cases} A\cos(\Phi) = & (E^2-1)\cos(2\varphi) \\ A\sin(\Phi) = & -2(E^2+1)\sin(2\varphi) \end{cases} \tag{6.11}$$

and

$$\kappa = \frac{(E^2+1)}{A}\left(2-\cos 2\varphi\right).$$

Proof. In this case we have

$$2F'(\theta) - G'(\theta) = \frac{2E}{E^2\cos^2\theta+\sin^2\theta} - \frac{E\cos(2\varphi)}{\sin^2(\theta+\varphi)+E^2\cos^2(\theta-\varphi)},$$

and it follows by elementary trigonometric identities that $2F'(\theta)-G'(\theta)>0$ if and only if

$$(E^2-1)\cos 2\varphi\cos 2\theta + 2(E^2+1)\sin 2\varphi\sin 2\theta + (1-E^2)(2-\cos 2\varphi) > 0.$$

By definition of A and letting Φ satisfy (6.11), this inequality is equivalent to

$$A\cos(2\theta+\Phi) > -\left(2(E^2+1)+(1-E^2)\cos 2\varphi\right).$$

□

6.B Extension of the direction at blown-up singularities

Lemma 6.B.1. *Let $L = (X, Y)$ be a proto-line-field on (M, g) with a hyperbolic singularity at $p \in M$. Fix a system of coordinates (x, y) such that $p = (0, 0)$, $g(0, 0) = \mathrm{id}$. Assume that $Y(0, 0) \neq 0$ and consider the linear proto-line-field $\bar{L} = (DX(0, 0), Y(0, 0))$.*

For every $r > 0$ small enough and $\theta \in \mathbb{R}$, let $\phi_L(r, \theta)$ and $\phi_{\bar{L}}(\theta)$ in $\mathbb{R}/\pi\mathbb{Z}$ be defined by

$$\phi_L(r,\theta) = \angle_g \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, B(L)(r\cos\theta, r\sin\theta) \right] \text{ and } \phi_{\bar{L}}(\theta) = \angle_{\mathrm{Eucl}} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, B(\bar{L})(r\cos\theta, r\sin\theta) \right].$$

Then

$$\phi_L(r,\theta) \underset{r\to 0}{\longrightarrow} \phi_{\bar{L}}(\theta), \quad \frac{\partial \phi_L}{\partial \theta}(r,\theta) \underset{r\to 0}{\longrightarrow} \frac{d\phi_{\bar{L}}}{d\theta}(\theta), \text{ and } \quad \frac{\partial \phi_L}{\partial r}(r,\theta) \underset{r\to 0}{\longrightarrow} 0.$$

Proof. Let $\bar{X}(x, y) = DX(0, 0) \begin{pmatrix} x \\ y \end{pmatrix}$ and define $\phi_X(r, \theta)$, $\phi_Y(r, \theta)$, $\phi_{\bar{X}}(\theta)$, and α in $\mathbb{R}/2\pi\mathbb{Z}$ by

$$\phi_X(r,\theta) = \angle_g \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, X(r\cos\theta, r\sin\theta) \right], \qquad \phi_{\bar{X}}(\theta) = \angle_{\mathrm{Eucl}} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \bar{X}(\cos\theta, \sin\theta) \right],$$
$$\phi_Y(r,\theta) = \angle_g \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, Y(r\cos\theta, r\sin\theta) \right], \qquad \alpha = \angle_{\mathrm{Eucl}} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, Y(0,0) \right].$$

Since $\phi_L = \frac{1}{2}\phi_X + \frac{1}{2}\phi_Y \pmod{\pi}$ and $\phi_{\bar{L}} = \frac{1}{2}\bar{\phi}_X + \frac{1}{2}\bar{\phi}_Y \pmod{\pi}$, we are left to prove that

$$\phi_X(r,\theta) \underset{r\to 0}{\longrightarrow} \phi_{\bar{X}}(\theta), \quad \frac{\partial \phi_X}{\partial \theta}(r,\theta) \underset{r\to 0}{\longrightarrow} \frac{d\phi_{\bar{X}}}{d\theta}(\theta), \quad \frac{\partial \phi_X}{\partial r}(r,\theta) \underset{r\to 0}{\longrightarrow} 0,$$

and

$$\phi_Y(r,\theta) \underset{r\to 0}{\longrightarrow} \alpha, \quad \frac{\partial \phi_Y}{\partial \theta}(r,\theta) \underset{r\to 0}{\longrightarrow} 0, \quad \frac{\partial \phi_Y}{\partial r}(r,\theta) \underset{r\to 0}{\longrightarrow} 0.$$

We are going to give a proof for ϕ_X only, the one for ϕ_Y being analogous.

Using the local coordinates (x, y), let us identify vector fields with their coordinate representation. Denote by (e_1, e_2) an oriented orthonormal frame for g in a neighborhood of p such that e_1 is a positive multiple of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Since $g(0, 0) = \mathrm{id}$, then $e_1(0, 0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \bar{e}_1$ and $e_2(0, 0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \bar{e}_2$. For $i = 1, 2$,

$$g(e_i, X) - \langle \bar{e}_i, \bar{X} \rangle = g(e_i, X) - \langle e_i, X \rangle + \langle e_i - \bar{e}_i, X \rangle + \langle \bar{e}_i, X - \bar{X} \rangle.$$

By an abuse of notation, we write in what follows $X(r, \theta)$ for $X(r\cos\theta, r\sin\theta)$ and similarly for e_1, e_2 and $\bar{X}$. By definition of $\bar{X}$, we have $\left\|(X - \bar{X})(r, \theta)\right\|_2 = O(r^2)$, so that $\langle \bar{e}_i, X - \bar{X} \rangle = O(r^2)$. Likewise $\|e_i(r, \theta) - \bar{e}_i\|_2 = O(r)$ and $\|X(r, \theta)\|_2 = O(r)$, so that

$\langle e_i - \bar{e}_i, X\rangle = O(r^2)$. Finally $|g(W,Z) - \langle W,Z\rangle| \leq \varepsilon(r)\|W\|_2\|Z\|_2$, with $\varepsilon(r) = O(r)$, for any pair of vector fields W,Z, so that $g(e_i,X) - \langle e_i, X\rangle = O(r^2)$.

In conclusion, $g(e_i,X) - \langle \bar{e}_i, \bar{X}\rangle = O(r^2)$. By definition of ϕ_X, we have

$$\cos(\phi_X)g(e_2,X) = \sin(\phi_X)g(e_1,X), \tag{6.12}$$

which can be rewritten as

$$\begin{aligned}\cos(\phi_X)\langle \bar{e}_2,\bar{X}\rangle + \cos(\phi_X)(g(e_2,X) - \langle \bar{e}_2,\bar{X}\rangle)& \\ = \sin(\phi_X)\langle \bar{e}_1,\bar{X}\rangle + \sin(\phi_X)(g(e_1,X) - \langle \bar{e}_1,\bar{X}\rangle).&\end{aligned} \tag{6.13}$$

Dividing equation (6.13) by r, we get

$$\phi_X(r,\theta) = \phi_{\bar{X}}(\theta) + O(r). \tag{6.14}$$

Regarding the partial derivatives of ϕ_X, by differentiating (6.12) we get

$$\partial_\theta \phi_X \left(\cos\phi_X g(e_1,X) + \sin\phi_X g(e_2,X)\right) = \cos\phi_X \partial_\theta(g(e_2,X)) - \sin\phi_X\partial_\theta(g(e_1,X)). \tag{6.15}$$

We have

$$\partial_\theta(g(e_i,X)) = (\partial_\theta g)(e_i,X) + g(\partial_\theta e_i, X) + g(e_i, \partial_\theta X).$$

By singularity of the polar parameterization, we have $\partial_\theta g = O(r)$ and $\partial_\theta e_i = O(r)$. Moreover, $g(e_i,\partial_\theta X)(r,\theta) = \langle \bar{e}_i, \partial_\theta \bar{X}(r,\theta)\rangle + O(r^2) = r\langle \bar{e}_i, \partial_\theta\bar{X}(1,\theta)\rangle + O(r^2)$.

By definition of ϕ_X, $\cos\phi_X g(e_1,X) + \sin\phi_X g(e_2,X) = \|X(r,\theta)\|_g = r\left\|r\bar{X}(1,\theta)\right\|_2 + r$. Hence we deduce from (6.14) and (6.15) that

$$\frac{\partial\phi_X}{\partial\theta}(r,\theta) = \frac{\cos\phi_{\bar{X}}(\theta)\langle \bar{e}_2, \partial_\theta\bar{X}(1,\theta)\rangle - \sin\phi_{\bar{X}}(\theta)\langle \bar{e}_2, \partial_\theta \bar{X}(1,\theta)\rangle + O(r)}{\left\|\bar{X}(1,\theta)\right\|_2 + O(r)} = \frac{d\bar{\phi}_X}{d\theta}(\theta) + O(r).$$

Similarly, we have

$$\partial_r\phi_X \left\|X\right\|_g = \cos\phi_X\partial_r(g(e_2,X)) - \sin\phi_X\partial_r(g(e_1,X)).$$

Since

$$\partial_r(g(e_i,X))(r,\theta) = r\langle \bar{e}_i, \bar{X}(1,\theta)\rangle + O(r^2)$$

and

$$\cos\phi_{\bar{X}}\langle \bar{e}_2, \bar{X}(1,\theta)\rangle - \sin\phi_{\bar{X}}\langle \bar{e}_1,\bar{X}(1,\theta)\rangle = 0,$$

we have

$$\frac{\partial\phi_X}{\partial r} = \frac{\cos\phi_X\partial_r(g(e_2,X)) - \sin\phi_X\partial_r(g(e_1,X))}{\|X\|_g} = O(r).$$

$\square$

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Titre : Singularités en géométrie sous-riemannienne

Mots clés : géométrie sous-riemannienne, singularités, stabilité, champs de direction, théorie géométrique de la mesure

Résumé : Nous étudions les relations qui existent entre des aspects de la géométrie sous-riemannienne et une diversité de singularités typiques dans ce contexte.
Avec les théorèmes de Whitney sous-riemanniens, nous conditionnons l'existence de prolongements globaux de courbes horizontales définies sur des fermés à des hypothèse de non-singularité de l'application point-final dans l'approximation nilpotente de la variété.
Nous appliquons des méthodes perturbatives pour obtenir des asymptotiques sur la longueur de courbes localement minimisantes perdant leur optimalité proche de leur point de départ dans le cas des variétés sous-riemanniennes de contact de dimension arbitraire. Nous décrivons la géométrie du lieu singulier et prouvons sa stabilité dans le cas des variétés de dimension 5.
Nous introduisons une construction permettant de définir des champs de directions à l'aide de couples de champs de vecteurs. Ceci fournit une topologie naturelle pour analyser la stabilité des singularités de champs de directions sur des surfaces.

Title : Singularities in sub-Riemannian geometry

Keywords : sub-Riemannian geometry, singularities, stability, line fields, geometric measure theory

Abstract : We investigate the relationship between features of sub-Riemannian geometry and an array of singularities that typically arise in this context.
With sub-Riemannian Whitney theorems, we ensure the existence of global extensions of horizontal curves defined on closed set by requiring a non-singularity hypothesis on the endpoint-map of the nilpotent approximation of the manifold to be satisfied.
We apply perturbative methods to obtain asymptotics on the length of short locally-length-minimizing curves losing optimality in contact sub-Riemannian manifolds of arbitrary dimension. We describe the geometry of the singular set and prove its stability in the case of manifolds of dimension 5.
We propose a construction to define line fields using pairs of vector fields. This provides a natural topology to study the stability of singularities of line fields on surfaces.

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