Zermelo-Markov-Dubins with two trailers
Preliminary draft

Ludovic Sacchelli * **Jean-Baptiste Caillau** **
Thierry Combot *** **Jean-Baptiste Pomet** ****

* *Univ. Lyon, Université Claude Bernard Lyon 1, CNRS, LAGEPP UMR 5007, 43 bd du 11 novembre 1918, F-69100 Villeurbanne, France;* `ludovic.sacchelli@univ-lyon1.fr`
** *Inria, Université Côte d'Azur, France.* `jean-baptiste@inria.ft`
*** *IMB, Université de Bourgogne, 9 avenue Alain Savary, 21078 Dijon, France.* `Thierry.Combot@u-bourgogne.fr`
**** *Inria, Université Côte d'Azur, France.* `jean-baptiste.pomet@inria.ft`

Abstract: We study the minimum time problem for a simplified model of a ship towing a long spread of cables. Constraints are on the curvature of the trajectory as well as on the shape of what represent the spread of cables here. This model turns out to be the same as a cart towing two trailers and rolling without sleeping on a plane in uniform translation. The Hamiltonian system resulting from Pontryagin Maximum principle is not integrable.

1. INTRODUCTION

The present note is in a line of work motivated by the optimization of turns and maneuvers of marine vessels towing a set of long and fragile underwater cables. It is a follow-up to Caillau et al. (2019), where the interested reader can find many more details about the motivations in terms of marine seismic acquisition.

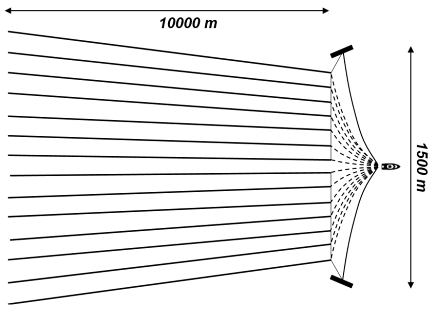

Fig. 1. dimensions of a seismic acquisition spread. Illustration from Caillau et al. (2019).

Each of these ships collects data from the ocean ground via sonic sources and sensors located in the spread of cables it is towing. This can be done only while the ship is sailing along a striaght line. In a typical campaign, the ship runs on parallel pre-defined straight lines, and must perform a u-turn at the end of each straight line to position itself at the starting point of the next one, acquisition being stopped during this maneuver. This maneuver is not constrained to follow a specific path, and hence can be optimized. The objective is to perform it in minimum time with given starting and end points, while being gentle enough to preserve the integrity of this spread of streamers. Since a typical U-turn can take more than an hour, minimizing time is important. Integrity of the spread of streamend, a few kilometers long, is primarily a matter of bounding the curvature of the trajectory. The starting point and end point are the end od a straight line and the beginning of the last one respectively; however if the state of the model takes into account the shape of the towed spreader, it should also be specified that it has to be in the right position at the end of the u-turn, i.e. in the relative equilibrium that is asymptotically attained during a straight line.

There are also motivations on terms of traffic near airports of unmanned aerial vehicles Techy and Woolsey (2009).

2. MODEL

If one takes into account only the position and orientation of the ship and the constraint is a bound on the curvature of the trajectory, one gets the so called Dubin's problem Dubins (1957): the magnitude of the speed being fixed, one seeks the shortest path from a point to another (including direction of the tangent) with a bound on curvature. The maximum curvature has to be small enough in order to preserve the integrity of the towed equipment during the turn. See Dubins (1957); Sussmann and Tang (1991); Boissonnat et al. (1994), and textbooks for Dubin's problem. There are two drawbacks to this approach: it does not take into account possible sea currents, and it does not contain any description of the hydrodynamic behavior of the towed cables (the dynamic equations only contain a kinematic of the ship itself).

- Adding the sea current into the problem, still without modelling the cable behavior, leads to a so-called Zermelo-Markov-Dubins problem (the term was apparently coined in Bakolas and Tsiotras (2013)) where

⋆ Sponsor and financial support acknowledgment goes here. Paper titles should be written in uppercase and lowercase letters, not all uppercase.

it is well documented, see also Techy and Woolsey (2009).

- In Caillau et al. (2019), we introduced a possible model for the towed cables, consisting in replacing them with a finite number of rigid links or “trailers”, theit dynamic (in fact kinematic equations coming either from a simple punctual drag force applie by the ocean to the stremers at each joint of the collection of trailers or of mimicking the equations of rolling without slipping, but in the frame that moves with the fluid.

The case of a single trailer (where “rolling without slipping” is the same model as the elementary drag force) was examined in Caillau et al. (2019) where we explain among other things that this optimal comtrol problem is Liouville intégable, which Here we investigate only the case of *two* trailers and “rolling without slipping”. They enjoy interesting properties but we prove that they are not integrable, prohibiting an almost expicit resolution.

Despite the motivation for navigation, and since the model is an approached heuristic model for a ship towing stremers and an exact kinematic model for a cart towing two trailers all rolling without slipping for instance on a conveyor belt in translation, we now talk af the latter,illustrated in figure 2, rather than the ship.

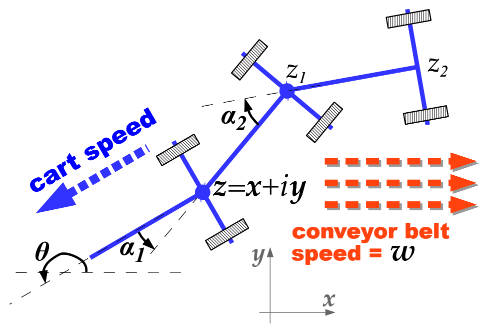

Fig. 2. **A cart with two trailers rolling without slipping on a conveyor belt.** The state variables $x, y, \theta, \alpha_1, \alpha_2$ have the following meaning, using complex notations in the plane.
$z = x + i\,y$ is the middle of the axle of the cart, the speed of the cart *with respect to the conveyor belt* is directed along the axis of the cart, since its magnitude is normalized to 1, it is equal to $e^{i\theta}$. The control u is the angular velocity $\dot\theta$ of the cart.
The first trailer is attached to the cart at point z, and its angle with respect to the axis of the cart is α_1, so that the other end of the cart is at point $z_1 = z + \ell e^{i\theta+\alpha_1}$, Rolling without sleeping means that the velocity of point z_1 with respect to the conveyor belt is along the axis of the trailer, i.e. has polar angle $\theta + \alpha_1$.
Similarly, the second trailer is attached to the first one at z_1, its angle with respect to first one is α_2, the other end of that second cart is at $z_2 = z_1 + \ell e^{i(\theta+\alpha_1+\alpha_2)}$, and the velocity of z_2 with respect to the conveyor belt has to have polar angle $\theta + \alpha_1 + \alpha_2$.

The equations take place in the following state space: $q = (x, y, \theta, \alpha_1, \alpha_2) \in \mathbb{R}^2 \times \mathbb{S}^1 \times (-\pi/2, \pi/2) \times (-\pi/2, \pi/2)$ and read

$$\dot q = F_0(q) + uF_1(q) \tag{1}$$

with

$$F_0 = (\cos\theta + w)\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y} - \frac{\sin\alpha_1}{\ell}\frac{\partial}{\partial\alpha_1} \tag{2}$$

$$- \left(\frac{\sin\alpha_1}{\ell} - \frac{\cos\alpha_1\sin\alpha_2}{\ell}\right)\frac{\partial}{\partial\alpha_2} \tag{3}$$

and

$$F_1 = \frac{\partial}{\partial\theta} - \frac{\partial}{\partial\alpha_1}.$$

In ODE form, this yields:

$$\begin{cases} \dot x = \cos\theta + w \\ \dot y = \sin\theta \\ \dot\theta = u \\ \dot\alpha_1 = -\,u - \dfrac{\sin\alpha_1}{\ell} \\ \dot\alpha_2 = \dfrac{\sin\alpha_1}{\ell} - \dfrac{\cos\alpha_1\sin\alpha_2}{\ell} \end{cases}$$

Here we normalise to 1 the longitudinal speed of the cart and the bound on the curvature. This is obtained by a rescaling of time ans space. Then there is only one parameter remaining, ℓ. In terms of the original cart, ℓ would be its length normalized by the longitudinal speed.

It can be seem that the realistic range for the parameter ℓ is

$$\ell > 1\,, \tag{4}$$

then for any value of the control with $|u| \le 1$, the domain $\mathbb{R}^2 \times \mathbb{S}^1 \times (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2})$ where

$$-\frac{\pi}{2} < \alpha_1 < \frac{\pi}{2} \text{ and } -\frac{\pi}{2} < \alpha_1 < \frac{\pi}{2} \tag{5}$$

is invariant in forward time; we only consider values of the state in this domain.

3. EXTREMALS OF MINIMUM TIME PROBLEM

Hamiltonian:

$$H = p^0 + p_x(\cos\theta + w) + p_y\sin\theta p_y + (p_\theta - p_{\alpha_1})u$$
$$- \frac{p_{\alpha_1}}{\ell}\sin\alpha_1 + \frac{p_{\alpha_2}}{\ell}(\sin\alpha_1 - \cos\alpha_1\sin\alpha_2) \tag{6}$$

Adjoint system:

$$\begin{cases} \dot p_x = 0 \\ \dot p_y = 0 \\ \dot p_\theta = p_x\sin\theta - p_y\cos\theta \\ \dot p_{\alpha_1} = \dfrac{p_{\alpha_1}}{\ell}\cos\alpha_1 - \dfrac{p_{\alpha_2}}{\ell}(\sin\alpha_1\sin\alpha_2 + \cos\alpha_1) \\ \dot p_{\alpha_2} = \dfrac{p_{\alpha_2}}{\ell}\cos\alpha_1\cos\alpha_2 \end{cases}$$

Here $p_{\alpha_2} = 0$ is a defining value: either p_{α_2} never vanishes or p_{α_2} is identically zero. Assuming $p_{\alpha_2} = 0$, the same analysis holds for $p_{\alpha_1} = 0$. In general, it is a linear differential system in $(p_{\alpha_1}, p_{\alpha_2})$, meaning that either $(p_{\alpha_1}, p_{\alpha_2})$ is identically zero or never vanishes.

On the other hand, (p_x, p_y) is completely static, so we may use the normalisation $p_x^2 + p_y^2 = 1$, and the occasional

$$(p_x, p_y) = (\cos\phi, \sin\phi)\,. \tag{7}$$

Along singular arcs. First, $H_1 = 0$ implies $p_\theta = p_{\alpha_1}$, and $H_{01} = 0$ implies

$$\frac{p_{\alpha_2}}{\ell}(\sin\alpha_1 \sin\alpha_2 + \cos\alpha_1) - \frac{p_{\alpha_1}}{\ell}\cos\alpha_1 + \sin(\theta-\phi) = 0 \quad (8)$$

At the third order, we have

$$H_{001} = \frac{-p_{\alpha_1} + p_{\alpha_2}(1+\cos\alpha_2)}{\ell^2}$$

and

$$H_{101} = \cos(\theta-\phi) - \frac{p_{\alpha_1}}{\ell}\sin\alpha_1 + \frac{p_{\alpha_2}}{\ell}(\sin\alpha_1 - \cos\alpha_1\sin\alpha_2).$$

Since $H = 0$, we obtain that $H_{101} = -p^0 - p_x w = \gamma$ is constant.

Hence

$$0 = \dot{H}_{01} = \frac{-p_{\alpha_1}}{\ell^2} + \frac{p_{\alpha_2}}{\ell^2}(1+\cos\alpha_2) + \gamma u \quad (9)$$

It may be that $\gamma = 0$, but this case is actually made impossible by the bounds of the domain. Indeed, in that case $\dot{p}_{\alpha_1} = p_{\alpha_2}(1+\cos\alpha_2)$. Then

$$H_{0001} = -\frac{p_{\alpha_1}}{\ell^3}\cos\alpha_1 - \frac{p_{\alpha_2}}{\ell^3}(2+\cos\alpha_2) = \frac{p_{\alpha_2}}{\ell^3}\cos\alpha_1.$$

Since we assumed that $p_{\alpha_2} \neq 0$, this implies $\alpha_2 = \pi/2 + k\pi$, $k \in \mathbb{Z}$, which is excluded, see (5).

4. ANALYSIS OF EQUILIBRIA

We study the singular Hamiltonian. Let $u_s = -\frac{H_{001}}{\gamma}$ be the singular control. Then taking $p^0 = -\gamma - p_x w$, we consider the Hamiltonian in terms of $(\alpha_1, \alpha_2, p_{\alpha_1}, p_{\alpha_2})$:

$$\begin{aligned} H_s =& -\gamma + \cos(\theta-\phi) + u_s p_\theta \\ &- \frac{p_{\alpha_1}}{\ell}(\sin\alpha_1 + u_s) + \frac{p_{\alpha_2}}{\ell}(\sin\alpha_1 - \cos\alpha_1\sin\alpha_2) \end{aligned} \quad (10)$$

The corresponding Hamiltonian system, with singular control is

$$\begin{cases} \dot{\alpha}_1 = -\dfrac{\sin\alpha_1}{\ell} - \dfrac{p_{\alpha_1}}{\gamma\ell^2} + \dfrac{p_{\alpha_2}}{\gamma\ell^2}(1+\cos\alpha_2) \\ \dot{\alpha}_2 = \dfrac{1}{\ell}(\sin\alpha_1 - \cos\alpha_1\sin\alpha_2) \\ \dot{p}_{\alpha_1} = \dfrac{p_{\alpha_1}}{\ell}\cos\alpha_1 - \dfrac{p_{\alpha_2}}{\ell}(\cos\alpha_1 + \sin\alpha_1\sin\alpha_2) \\ \dot{p}_{\alpha_2} = \dfrac{p_{\alpha_2}}{\ell}\cos\alpha_1\cos\alpha_2 \end{cases} \quad (11)$$

The normalisation (7) restricts the flow to the invariant surface

$$\begin{aligned} &\left(\gamma + \frac{p_{\alpha_1}}{\ell}\sin\alpha_1 + \frac{p_{\alpha_2}}{\ell}(\sin\alpha_1 - \cos\alpha_1\sin\alpha_2\right)^2 \\ &+ \left(\frac{p_{\alpha_2}}{\ell}(\sin\alpha_1\sin\alpha_2 + \cos\alpha_1) - \frac{p_{\alpha_1}}{\ell}\cos\alpha_1\right)^2 = 1. \end{aligned} \quad (12)$$

Lemma 1. System (11) has 8 equilibria (with angles taken modulo 2π) given by the 4-tuple

$$A = \{(0,0,0,0), (\pi,0,0,0), (0,\pi,0,0), (\pi,\pi,0,0)\}$$

and the 4-tuple

$$B = \left\{ \left(\frac{\pi}{4}, \frac{\pi}{2}, -\sqrt{2}\ell\gamma, -\frac{\ell\gamma}{\sqrt{2}}\right), \left(\frac{3\pi}{4}, -\frac{\pi}{2}, -\sqrt{2}\ell\gamma, -\frac{\ell\gamma}{\sqrt{2}}\right), \left(-\frac{\pi}{4}, \frac{\pi}{2}, \sqrt{2}\ell\gamma, \frac{\ell\gamma}{\sqrt{2}}\right), \left(-\frac{3\pi}{4}, -\frac{\pi}{2}, \sqrt{2}\ell\gamma, \frac{\ell\gamma}{\sqrt{2}}\right) \right\}.$$

Proof. Equilibria of the system are deduced by direct analysis as follows.

- From $\dot{p}_{\alpha_2} = 0$, either $p_{\alpha_2} = 0$ or $\cos\alpha_2 = 0$. Having $\cos\alpha_1 = 0$ is forbidden by $\dot{\alpha}_2 = 0$, however.
- If $p_{\alpha_2} = 0$, then $p_{\alpha_1} = 0$ follows from $\dot{p}_{\alpha_1} = 0$. $\dot{\alpha}_1 = 0$ and $\dot{\alpha}_2 = 0$ then yield $\sin\alpha_1 = \sin\alpha_2 = 0$. This implies equilibria in family A
- On the other hand, if $\cos\alpha_2 = 0$, then $\alpha_2 = \pm\pi/2 + 2k\pi$, $k \in \mathbb{Z}$. We now show these equilibria correspond to family B.
- If $\alpha_2 = \pi/2 + 2k\pi$, $k \in \mathbb{Z}$, then $\dot{\alpha}_2 = 0$ implies that $\cos\alpha_1 = \sin\alpha_1$, and $\dot{p}_{\alpha_1} = 0$ implies that $p_{\alpha_1} = 2p_{\alpha_2}$. Finally, $\dot{\alpha}_1 = 0$ implies that $p_{\alpha_2} = -\ell\gamma\cos\alpha_1 = \pm\ell\gamma/\sqrt{2}$.
- If $\alpha_2 = -\pi/2 + 2k\pi$, $k \in \mathbb{Z}$, then $\dot{\alpha}_2 = 0$ now implies that $\cos\alpha_1 = -\sin\alpha_1$, and $\dot{p}_{\alpha_1} = 0$ implies again that $p_{\alpha_1} = 2p_{\alpha_2}$. Finally, $\dot{\alpha}_1 = 0$ implies that $p_{\alpha_2} = \ell\gamma\cos\alpha_1 = \pm\ell\gamma/\sqrt{2}$.

This covers all possible cases.

By direct evaluation, we can obtain more information on the vector field near these equilibria.

Lemma 2. Equilibria in family A have a linear part with eigenvalues (with multiplicity)

$$\left\{-\frac{1}{\ell}, -\frac{1}{\ell}, \frac{1}{\ell}, \frac{1}{\ell}.\right\}$$

Equilibria in family B have a linear part with eigenvalues

$$\left\{-\frac{1}{\ell}, \frac{1}{\ell}, -\frac{i}{\sqrt{2}\ell}, \frac{i}{\sqrt{2}\ell}\right\}$$

(with i denoting the imaginary unit $i^2 = -1$).

5. INTEGRABILITY PROPERTIES

In this section, we will study integrability of system (11). We see that the hyperplane $p_{\alpha_2} = 0$ is invariant, and the system reduces

$$\begin{cases} \dot{\alpha}_1 = -\dfrac{\sin\alpha_1}{\ell} - \dfrac{p_{\alpha_1}}{\gamma\ell^2} \\ \dot{\alpha}_2 = \dfrac{1}{\ell}(\sin\alpha_1 - \cos\alpha_1\sin\alpha_2) \\ \dot{p}_{\alpha_1} = \dfrac{p_{\alpha_1}}{\ell}\cos\alpha_1 \end{cases} \quad (13)$$

We remark that

$$H_r = \left(\sin(\alpha_1) + \frac{p_{\alpha_1}}{\gamma\ell}\right)^2 + \cos(\alpha_1)^2$$

is a first integral of (13). On the level $H_r = 1$, we can solve easily in p_{α_1}

$$p_{\alpha_1} = -2\sin(\alpha_1)\gamma\ell \text{ or } p_{\alpha_1} = 0.$$

This first solution gives the equation $\dot{\alpha}_1 = \frac{\sin(\alpha_1)}{\ell}$ and we deduce the following solution

$$\alpha_1(t) = 2\mathrm{arctan}\left(e^{t/\ell}\right), \quad p_{\alpha_1}(t) = -\frac{4\gamma\ell}{e^{t/\ell} + e^{-t/\ell}}. \quad (14)$$

We will now try to obtain the corresponding solution α_2 of system (13). We introduce a variable change

$$\alpha_2(t) = i\ln z(\tau),\ t = \ell\ln\tau$$

and the equation in $z(\tau)$ becomes

$$z'(\tau) = \frac{(\tau^2-1)z(\tau)^2}{2\tau(\tau^2+1)} - \frac{2iz(\tau)}{\tau^2+1} - \frac{\tau^2-1}{2\tau(\tau^2+1)}. \quad (15)$$

This is a Riccati equation, the parameter ℓ disappeared, and noting

$$z(\tau)=-\frac{2\tau(\tau^2+1)}{\tau^2-1}\frac{\phi'(\tau)}{\phi(\tau)}$$

this equation reduces to a second order linear ODE

$$\phi''(\tau)+\frac{\tau^3+i\tau^2-3\tau+i}{\tau(\tau^2-1)(\tau-i)}\phi'(\tau)-\frac{(\tau^2-1)^2}{4(\tau^2+1)^2\tau^2}\phi(\tau)=0$$

Using the Kovacic algorithm , we find that this equation has Galois group $SL_2(\mathbb{C})$, and thus is not solvable by quadrature. Thus $\alpha_2(t)$ cannot be obtained by quadrature in system (13) when α_1, p_{α_1} are given by (14).

As it is impossible to obtain $\alpha_2(t)$ by quadrature in the particular case (14), then it is not possible in general, and thus system (13) and then system (11) are not solvable by quadrature.

Let us now look at the case of regular flow, which reduces on the invariant hyperplane $p_{\alpha_2}=0$ to

$$\begin{cases} \dot\alpha_1 = \ell-\sin\alpha_1\\ \dot\alpha_2 = \sin\alpha_1-\cos\alpha_1\sin\alpha_2\\ \dot p_{\alpha_1} = p_{\alpha_1}\cos\alpha_1 \end{cases}\tag{16}$$

This system depends on a parameter $l\in]0,1[$. Again this system admits a first integral $p_{\alpha_1}(l-\sin\alpha_1)$, which allows to recover p_{α_1} once α_1 is known. Solving the first equation in α_1, we find the solution

$$\alpha_1(t)=-2\arctan\left(\frac{\tanh\left(\frac{1}{2}t\sqrt{1-\ell^2}\right)\sqrt{1-\ell^2}-1}{\ell}\right).$$

We substitute in the second equation, and making the variable change

$$\alpha_2(t)=i\ln(z(\tau)),\quad \tanh\left(\frac{1}{2}t\sqrt{1-\ell^2}\right)\sqrt{1-\ell^2}=\tau$$

we have a Riccati equation $z'(\tau)=$

$$\frac{(\ell^2-(\tau-1)^2)z(\tau)^2-4i\ell(\tau-1)z(\tau)-\ell^2+\tau^2-2\tau+1)}{((\ell^2+\tau^2-1)(\ell^2+(\tau-1)^2))}$$

We can now transform this Ricatti equation to a second order linear equation by the variable change

$$z(\tau)=-\frac{(\ell^2+\tau^2-1)(\ell^2+\tau^2-2\tau+1)}{(\ell-1+\tau)(\ell+1-\tau)}\frac{\phi'(\tau)}{\phi(\tau)}$$

which gives the equation

$$\phi''(\tau)-\frac{(i\tau^4+(\ell-3i)\tau^3-(3i\ell^2+4\ell-3i)\tau^2-(3\ell^3-3i\ell^2-5\ell+i)\tau-2\ell^3+2\ell))}{(i\tau+\ell-i)(\ell-1+\tau)(\ell+1-\tau)(\ell^2+\tau^2-1)}\times 2\phi'(\tau)-\frac{(\ell-1+\tau)^2(\ell+1-\tau)^2\phi(\tau)}{(\ell^2+\tau^2-1)^2(\ell^2+\tau^2-2\tau+1)^2}\tag{17}$$

For a generic ℓ, the Kovacic algorithm finds no solutions, and thus generically the system (16) is not integrable by quadrature. But for specific values of ℓ, the system could maybe be integrated. Computing the possible confluences between singularities, we find $\ell=0,\pm1$ but are outside the interval studied. Thus for any $\ell\in]0,1[$, there are exactly 6 singularities (infinity is regular)

$$1-\ell, 1+\ell, 1-i\ell, 1+i\ell, \sqrt{1-\ell^2}, -\sqrt{1-\ell^2},$$

with respectively the local exponents

$$(0,2),(0,2),\frac{1}{2}\pm\frac{1}{2}\sqrt{2},-\frac{1}{2}\pm\frac{1}{2}\sqrt{2},$$

$$\frac{i\ell\sqrt{1-\ell^2}\pm\sqrt{2\ell^4-3\ell^2+1}}{2(1-\ell^2)},\frac{i\ell\sqrt{1-\ell^2}\pm\sqrt{2\ell^4-3\ell^2+1}}{2(\ell^2-1)}$$

We will now follow the Kovacic algorithm cite case by case. We see that not all exponents are rational, and thus not all solutions of (17) can be algebraic. Thus case III is not possible. For case I, we need a hyperexponential solution, which requires that a sum of exponents to be a non positive integer. We obtain the following equation

$$\epsilon\frac{\sqrt{2\ell^4-3\ell^2+1}}{1-\ell^2}+\kappa\sqrt{2}=-n,\ n\in\mathbb{N}\tag{18}$$

where $\epsilon,\kappa\in\{-1,0,1\}$ depend on the choice of the exponents. This equation will give constraints to the possible ℓ. For case II, we need to consider the symmetric square of equation (17), which is a linear differential equation of order 3 with the same singularities. The computed exponents (3 for each singularity) are the following

$$(0,2,4),(0,2,4),(1,\pm\sqrt{2}),(-1,\pm\sqrt{2}),$$

$$\left(\frac{i\ell\sqrt{1-\ell^2}}{1-\ell^2},\frac{i\ell\sqrt{1-\ell^2}\pm\sqrt{2\ell^4-3\ell^2+1}}{1-\ell^2}\right),$$

$$\left(\frac{i\ell\sqrt{1-\ell^2}}{\ell^2-1},\frac{i\ell\sqrt{1-\ell^2}\pm\sqrt{2\ell^4-3\ell^2+1}}{\ell^2-1}\right)$$

We again need a hyperexponential solution, and thus a sum of exponents equal to a non positive integer. This gives again relation (18), but with $\epsilon,\kappa\in\{-2,-1,0,1,2\}$.

For $\ell\in]0,1[$, the possible non positive integer values are $-5,-4,-3,-2,-1,0$, and the corresponding values of ℓ in $]0,1[$ are

$$\frac{1}{2}\sqrt{2},\frac{1}{3}\sqrt{3},\frac{1}{7}\sqrt{21},\frac{1}{17}\sqrt{-51+136\sqrt{2}},$$

$$\frac{1}{2}\sqrt{3-\sqrt{2}},\frac{1}{7}\sqrt{42-14\sqrt{2}},\frac{2}{21}\sqrt{84-21\sqrt{2}},$$

$$\frac{1}{21}\sqrt{357-168\sqrt{2}},\frac{1}{31}\sqrt{837-496\sqrt{2}},$$

$$\frac{1}{69}\sqrt{3933-1104\sqrt{2}},\frac{2}{7}\sqrt{7}-\frac{1}{14}\sqrt{14},\frac{2}{7}\sqrt{14}-\frac{1}{7}\sqrt{7}$$

For each of these distinguished values of ℓ, we apply the Kovacic algorithm and we find that none have a solvable Galois group.

Thus, for any $\ell\in]0,1[$, system (16) is not solvable by quadrature.

Remark however that the fact that systems (13) and (16), when reduced to $p_{\alpha_2}=0$, can be solved through second order differential equations is not a generic situation, and thus further calculations could be possible through the use of special functions.

6. NUMERICAL SIMULATIONS

ACKNOWLEDGEMENTS

The authors no longer thank T. Mensch and T. Moulinier (no longer with CGG) for introducing them to the problem, and Y. Du for her help with the numerical simulations.

REFERENCES

Bakolas, E. and Tsiotras, P. (2013). Optimal synthesis of the Zermelo-Markov-Dubins problem in a constant drift field. *J. Optim. Theory Appl.*, 156(2), 469–492. doi: 10.1007/s10957-012-0128-0.

Boissonnat, J.D., Cérézo, A., and Leblond, J. (1994). Shortest paths of bounded curvature in the plane. *J. Intell. Robot. Syst.*, 11, 5–20. doi:10.1007/BF01258291.

Caillau, J.B., Maslovskaya, S., Mensch, T., Moulinier, T., and Pomet, J.B. (2019). Zermelo-Markov-Dubins problem and extensions in marine navigation. In *58th IEEE Conf. Decision and Control (CDC 2019).* doi: 10.1109/CDC40024.2019.9029293. URL `https://hal.archives-ouvertes.fr/hal-02375015`.

Dubins, L.E. (1957). On curves of minimal length with a constraint on average curvature, and with prescribed initial and terminal positions and tangents. *American Journal of Mathematics*, 79, 497–516. doi:10.2307/2372560. URL `http://www.jstor.org/stable/2372560`.

Sussmann, H.J. and Tang, G. (1991). Shortest paths for the Reeds-Shepp car: a worked out example of the use of geometric techniques in nonlinear optimal control. Technical Report SYCON-91-10, Rutgers University, NJ, USA. URL `url`.

Techy, L. and Woolsey, C.A. (2009). Minimum-time path planning for unmanned aerial vehicles in steady uniform winds. *J. Guidance, Control, and Dynamics*, 32, 1736–1746. doi:10.2514/1.44580.