

Zermelo-Markov-Dubins with two trailers

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Abstract: We study the minimum time problem for a simplified model of a ship towing a long spread of cables. Constraints are on the curvature of the trajectory as well as on the shape of what represent the spread of cables here. This model turns out to be the same as a cart towing two trailers and rolling without sleeping on a plane in uniform translation. We analyse the Hamiltonian system describing the extremal flow given by Pontrjagin maximum principle. We detail the equilibria of the system and prove that, contrary to the case of one trailer studied in Caillau et al. (2019), it is not solvable by quadratures. Preliminary numerical results are given.

Keywords: Zermelo navigation, minimum time, turnpike, integrability

1. INTRODUCTION

The present note is in a line of work motivated by the optimization of turns and maneuvers of marine vessels towing a set of long and fragile underwater cables. It is a follow-up to Caillau et al. (2019), where the interested reader can find many more details about the motivations in terms of marine seismic acquisition.

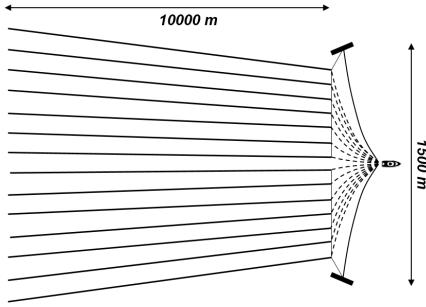


Fig. 1. dimensions of a seismic acquisition spread. Illustration from Caillau et al. (2019).

Each of these ships collects data from the ocean ground via sonic sources and sensors located in the spread of cables it is towing. This can be done only while the ship is sailing along a straight line. In a typical campaign, the ship runs on parallel pre-defined straight lines, and must perform a u-turn at the end of each straight line to position itself at the starting point of the next one, acquisition being stopped during this maneuver. This maneuver is not constrained to follow a specific path, and hence can be optimized. The objective is to perform it in minimum time with given starting and end points, while being gentle enough to preserve the integrity of this spread of

streamers. Since a typical U-turn can take more than an hour, minimizing time is important. Integrity of the spread of streamers, a few kilometers long, is primarily a matter of bounding the curvature of the trajectory. The starting point and end point are the end of a straight line and the beginning of the last one respectively; however if the state of the model takes into account the shape of the towed spreader, it should also be specified that it has to be in the right position at the end of the U-turn, i.e. in the relative equilibrium that is asymptotically attained during a straight line. There are also motivations in terms of traffic near airports of unmanned aerial vehicles Techy and Woolsey (2009).

After describing the model in Section 2, we apply Pontrjagin maximum principle: the minimum time extremals of the problem are solution of a Hamiltonian system and can be regular or singular, as detailed in Section 3. Then, equilibria of the singular extremals are studied in Section 4; among these, hyperbolic points play an important role and are related to the turnpike phenomenon. In Section 5, we give obstructions to solvability by quadratures (integrability) both for the singular and regular extremals thanks to differential Galois theory. Preliminary numerical simulations of the system with two trailers are provided in the final section.

2. MODEL

If one takes into account only the position and orientation of the ship and the constraint is a bound on the curvature of the trajectory, one gets the so called Dubin’s problem Dubins (1957): the magnitude of the speed being fixed, one seeks the shortest path from a point to another (including direction of the tangent) with a bound on cur-

vature. The maximum curvature has to be small enough in order to preserve the integrity of the towed equipment during the turn. See Dubins (1957); Sussmann and Tang (1991); Boissonnat et al. (1994), and textbooks for Dubin’s problem. There are two drawbacks to this approach: it does not take into account possible sea currents, and it does not contain any description of the hydrodynamic behavior of the towed cables (the dynamic equations only contain a kinematic of the ship itself).

- Adding the sea current into the problem, still without modelling the cables behavior, leads to a so-called Zermelo-Markov-Dubins problem (the term was apparently coined in Bakolas and Tsiotras (2013)) where it is well documented, see also Tych and Woolsey (2009).
- In Caillau et al. (2019), we introduced a possible model for the towed cables, consisting in replacing them with a finite number of rigid links or “trailers”, their dynamic (in fact kinematic) equations coming either from a simple punctual drag force applied by the ocean to the spread at each “joint”, or from mimicking the equations of rolling without slipping, in the frame that moves with the fluid, as if each link was a trailer on wheels. That are two types of models, that we call for short *rolling without slipping* models and *drag* models. Models also differ by the number of links, or number of trailers. For a single trailer, drag coincides with rolling without slipping. The state variables may be chosen as follows: two cartesian coordinates (x, y) and an angle θ for the position and orientation of the ship, or towing vehicle in the vehicle with trailers point of view, plus as many angles as trailers, α_i being the angles between the $(i - 1)^{\text{th}}$ and the i^{th} trailer (where the towing vehicle is counted as the 0^{th} trailer).

The case of a single trailer (drag or rolling without slipping: we just mentionned that they coincide) was examined in Caillau et al. (2019) where we explain among other things that this optimal control problem is Liouville int egable, which Here we investigate only the case of *two* trailers and “rolling without slipping”; the conclusions do not differ for the “drag” model but we do not present them due to space limitation. They enjoy interesting properties but we prove that they are not integrable, prohibiting an almost explicit resolution.

Since the model is a heuristic approached model for a ship towing a spread of streamers but an *exact* kinematic model for a cart towing two trailers all rolling without slipping for instance on a conveyor belt in translation, and despite the motivation for navigation, we now consider the latter, illustrated in figure 2, rather than the ship. Via a rescaling of space and time, we assume that the longitudinal speed of the cart and the maximum curvature are equal to 1, there remains only one parameter $\ell > 0$, that has to be larger than 1 (**explanation**). The state variables are $q = (x, y, \theta, \alpha_1, \alpha_2)$, we could take $\mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{S}^1$ as state space, we prefer the open subset $\mathbb{R}^2 \times \mathbb{S}^1 \times (-\pi/2, \pi/2) \times (-\pi/2, \pi/2)$, that is invariant in positive time since $\ell > 1$. The equations read

$$\dot{q} = F_0(q) + uF_1(q), \quad |u| \leq 1, \quad (1)$$

with

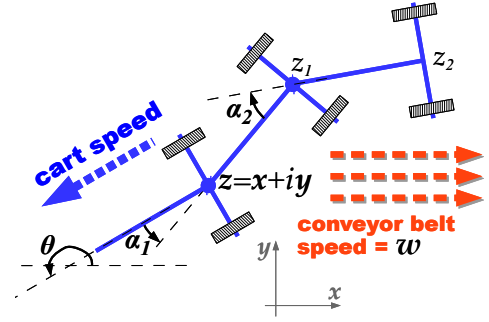


Fig. 2. **A cart with two trailers rolling without slipping on a conveyor belt.** The state variables $x, y, \theta, \alpha_1, \alpha_2$ have the following meaning, using complex notations in the plane.

$z = x + iy$ is the middle of the axle of the cart, the speed of the cart *with respect to the conveyor belt* is directed along the axis of the cart, since its magnitude is normalized to 1, it is equal to $e^{i\theta}$. The control u is the angular velocity $\dot{\theta}$ of the cart.

The first trailer is attached to the cart at point z , and its angle with respect to the axis of the cart is α_1 , so that the other end of the cart is at point $z_1 = z + \ell e^{i(\theta + \alpha_1)}$. Rolling without sleeping means that the velocity of point z_1 with respect to the conveyor belt is along the axis of the trailer, i.e. has polar angle $\theta + \alpha_1$.

Similarly, the second trailer is attached to the first one at z_1 , its angle with respect to first one is α_2 , the other end of that second cart is at $z_2 = z_1 + \ell e^{i(\theta + \alpha_1 + \alpha_2)}$, and the velocity of z_2 with respect to the conveyor belt has to have polar angle $\theta + \alpha_1 + \alpha_2$.

$$F_0 = (\cos \theta + w) \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y} - \frac{\sin \alpha_1}{\ell} \frac{\partial}{\partial \alpha_1} \quad (2)$$

$$- \left(\frac{\sin \alpha_1}{\ell} - \frac{\cos \alpha_1 \sin \alpha_2}{\ell} \right) \frac{\partial}{\partial \alpha_2} \quad (3)$$

and

$$F_1 = \frac{\partial}{\partial \theta} - \frac{\partial}{\partial \alpha_1}.$$

In ODE form, this yields:

$$\begin{cases} \dot{x} = \cos \theta + w \\ \dot{y} = \sin \theta \\ \dot{\theta} = u \\ \dot{\alpha}_1 = -u - \frac{\sin \alpha_1}{\ell} \\ \dot{\alpha}_2 = \frac{\sin \alpha_1}{\ell} - \frac{\cos \alpha_1 \sin \alpha_2}{\ell} \end{cases}$$

Here we normalise to 1 the longitudinal speed of the cart and the bound on the curvature. This is obtained by a rescaling of time and space. Then there is only one parameter remaining, ℓ . In terms of the original cart, ℓ would be its length normalized by the longitudinal speed. It can be seen that the realistic range for the parameter ℓ is $\ell \in (0, 1)$. Then for any value of the control with $|u| \leq 1$, the domain $\mathbb{R}^2 \times \mathbb{S}^1 \times (-\frac{\pi}{2}, \frac{\pi}{2}) \times (-\frac{\pi}{2}, \frac{\pi}{2})$ where

$$-\frac{\pi}{2} < \alpha_1 < \frac{\pi}{2} \text{ and } -\frac{\pi}{2} < \alpha_2 < \frac{\pi}{2} \quad (4)$$

is invariant in forward time.

3. EXTREMALS OF MINIMUM TIME PROBLEM

Hamiltonian:

$$H = p^0 + p_x(\cos \theta + w) + p_y \sin \theta p_y + (p_\theta - p_{\alpha_1})u - \frac{p_{\alpha_1}}{\ell} \sin \alpha_1 + \frac{p_{\alpha_2}}{\ell}(\sin \alpha_1 - \cos \alpha_1 \sin \alpha_2) \quad (5)$$

Adjoint system:

$$\begin{cases} \dot{p}_x = 0 \\ \dot{p}_y = 0 \\ \dot{p}_\theta = p_x \sin \theta - p_y \cos \theta \\ \dot{p}_{\alpha_1} = \frac{p_{\alpha_1}}{\ell} \cos \alpha_1 - \frac{p_{\alpha_2}}{\ell}(\sin \alpha_1 \sin \alpha_2 + \cos \alpha_1) \\ \dot{p}_{\alpha_2} = \frac{p_{\alpha_2}}{\ell} \cos \alpha_1 \cos \alpha_2 \end{cases}$$

Here $p_{\alpha_2} = 0$ is a defining value: either p_{α_2} never vanishes or p_{α_2} is identically zero. Assuming $p_{\alpha_2} = 0$, the same analysis holds for $p_{\alpha_1} = 0$. In general, it is a linear differential system in $(p_{\alpha_1}, p_{\alpha_2})$, meaning that either $(p_{\alpha_1}, p_{\alpha_2})$ is identically zero or never vanishes.

On the other hand, (p_x, p_y) is completely static, so we may use the normalisation $p_x^2 + p_y^2 = 1$, and the occasional

$$(p_x, p_y) = (\cos \phi, \sin \phi). \quad (6)$$

3.0.0.1. Along singular arcs. First, $H_1 = 0$ implies $p_\theta = p_{\alpha_1}$, and $H_{01} = 0$ implies

$$\frac{p_{\alpha_2}}{\ell}(\sin \alpha_1 \sin \alpha_2 + \cos \alpha_1) - \frac{p_{\alpha_1}}{\ell} \cos \alpha_1 + \sin(\theta - \phi) = 0 \quad (7)$$

At the third order, we have

$$H_{001} = \frac{-p_{\alpha_1} + p_{\alpha_2}(1 + \cos \alpha_2)}{\ell^2}$$

and

$$H_{101} = \cos(\theta - \phi) - \frac{p_{\alpha_1}}{\ell} \sin \alpha_1 + \frac{p_{\alpha_2}}{\ell}(\sin \alpha_1 - \cos \alpha_1 \sin \alpha_2).$$

Since $H = 0$, we obtain that $H_{101} = -p^0 - p_x w = \gamma$ is constant.

Hence

$$0 = \dot{H}_{01} = \frac{-p_{\alpha_1}}{\ell^2} + \frac{p_{\alpha_2}}{\ell^2}(1 + \cos \alpha_2) + \gamma u \quad (8)$$

It may be that $\gamma = 0$, but this case is actually made impossible by the bounds of the domain. Indeed, in that case $\dot{p}_{\alpha_1} = p_{\alpha_2}(1 + \cos \alpha_2)$. Then

$$H_{0001} = -\frac{p_{\alpha_1}}{\ell^3} \cos \alpha_1 - \frac{p_{\alpha_2}}{\ell^3}(2 + \cos \alpha_2) = \frac{p_{\alpha_2}}{\ell^3} \cos \alpha_1.$$

Since we assumed that $p_{\alpha_2} \neq 0$, this implies $\alpha_2 = \pi/2 + k\pi$, $k \in \mathbb{Z}$, which is excluded, see (4).

4. ANALYSIS OF EQUILIBRIA

We study the singular Hamiltonian. Let $u_s = -\frac{H_{001}}{\gamma}$ be the singular control. Then taking $p^0 = -\gamma - p_x w$, we consider the Hamiltonian in terms of $(\alpha_1, \alpha_2, p_{\alpha_1}, p_{\alpha_2})$:

$$H_s = -\gamma + \cos(\theta - \phi) + u_s p_\theta - \frac{p_{\alpha_1}}{\ell}(\sin \alpha_1 + u_s) + \frac{p_{\alpha_2}}{\ell}(\sin \alpha_1 - \cos \alpha_1 \sin \alpha_2) \quad (9)$$

The corresponding Hamiltonian system, with singular control is

$$\begin{cases} \dot{\alpha}_1 = -\frac{\sin \alpha_1}{\ell} - \frac{p_{\alpha_1}}{\gamma \ell^2} + \frac{p_{\alpha_2}}{\gamma \ell^2}(1 + \cos \alpha_2) \\ \dot{\alpha}_2 = \frac{1}{\ell}(\sin \alpha_1 - \cos \alpha_1 \sin \alpha_2) \\ \dot{p}_{\alpha_1} = \frac{p_{\alpha_1}}{\ell} \cos \alpha_1 - \frac{p_{\alpha_2}}{\ell}(\cos \alpha_1 + \sin \alpha_1 \sin \alpha_2) \\ \dot{p}_{\alpha_2} = \frac{p_{\alpha_2}}{\ell} \cos \alpha_1 \cos \alpha_2 \end{cases} \quad (10)$$

The normalisation (6) restricts the flow to the invariant surface

$$\left(\gamma + \frac{p_{\alpha_1}}{\ell} \sin \alpha_1 + \frac{p_{\alpha_2}}{\ell}(\sin \alpha_1 - \cos \alpha_1 \sin \alpha_2)\right)^2 + \left(\frac{p_{\alpha_2}}{\ell}(\sin \alpha_1 \sin \alpha_2 + \cos \alpha_1) - \frac{p_{\alpha_1}}{\ell} \cos \alpha_1\right)^2 = 1. \quad (11)$$

Lemma 1. System (10) has 8 equilibria (with angles taken modulo 2π) given by the 4-tuple

$$A = \{(0, 0, 0, 0), (\pi, 0, 0, 0), (0, \pi, 0, 0), (\pi, \pi, 0, 0)\}$$

and the 4-tuple

$$B = \left\{ \left(\frac{\pi}{4}, \frac{\pi}{2}, -\sqrt{2}\ell\gamma, -\frac{\ell\gamma}{\sqrt{2}} \right), \left(\frac{3\pi}{4}, -\frac{\pi}{2}, -\sqrt{2}\ell\gamma, -\frac{\ell\gamma}{\sqrt{2}} \right), \left(-\frac{\pi}{4}, \frac{\pi}{2}, \sqrt{2}\ell\gamma, \frac{\ell\gamma}{\sqrt{2}} \right), \left(-\frac{3\pi}{4}, -\frac{\pi}{2}, \sqrt{2}\ell\gamma, \frac{\ell\gamma}{\sqrt{2}} \right) \right\}.$$

Proof. Equilibria of the system are deduced by direct analysis as follows.

- From $\dot{p}_{\alpha_2} = 0$, either $p_{\alpha_2} = 0$ or $\cos \alpha_2 = 0$. Having $\cos \alpha_1 = 0$ is forbidden by $\dot{\alpha}_2 = 0$, however.
- If $p_{\alpha_2} = 0$, then $p_{\alpha_1} = 0$ follows from $\dot{p}_{\alpha_1} = 0$. $\dot{\alpha}_1 = 0$ and $\dot{\alpha}_2 = 0$ then yield $\sin \alpha_1 = \sin \alpha_2 = 0$. This implies equilibria in family A
- On the other hand, if $\cos \alpha_2 = 0$, then $\alpha_2 = \pm\pi/2 + 2k\pi$, $k \in \mathbb{Z}$. We now show these equilibria correspond to family B.
- If $\alpha_2 = \pi/2 + 2k\pi$, $k \in \mathbb{Z}$, then $\dot{\alpha}_2 = 0$ implies that $\cos \alpha_1 = \sin \alpha_1$, and $\dot{p}_{\alpha_1} = 0$ implies that $p_{\alpha_1} = 2p_{\alpha_2}$. Finally, $\dot{\alpha}_1 = 0$ implies that $p_{\alpha_2} = -\ell\gamma \cos \alpha_1 = \pm\ell\gamma/\sqrt{2}$.
- If $\alpha_2 = -\pi/2 + 2k\pi$, $k \in \mathbb{Z}$, then $\dot{\alpha}_2 = 0$ now implies that $\cos \alpha_1 = -\sin \alpha_1$, and $\dot{p}_{\alpha_1} = 0$ implies again that $p_{\alpha_1} = 2p_{\alpha_2}$. Finally, $\dot{\alpha}_1 = 0$ implies that $p_{\alpha_2} = \ell\gamma \cos \alpha_1 = \pm\ell\gamma/\sqrt{2}$.

This covers all possible cases. \square

By direct evaluation, we can obtain more information on the vector field near these equilibria.

Lemma 2. Equilibria in family A have a linear part with eigenvalues (with multiplicity)

$$\left\{ -\frac{1}{\ell}, -\frac{1}{\ell}, \frac{1}{\ell}, \frac{1}{\ell} \right\}$$

Equilibria in family B have a linear part with eigenvalues

$$\left\{ -\frac{1}{\ell}, \frac{1}{\ell}, -\frac{i}{\sqrt{2}\ell}, \frac{i}{\sqrt{2}\ell} \right\}$$

(with i denoting the imaginary unit $i^2 = -1$).

5. INTEGRABILITY PROPERTIES

In this section, we will study integrability of system (10). We see that the hyperplane $p_{\alpha_2} = 0$ is invariant, and the

system reduces

$$\begin{cases} \dot{\alpha}_1 = -\frac{\sin \alpha_1}{\ell} - \frac{p_{\alpha_1}}{\gamma \ell^2} \\ \dot{\alpha}_2 = \frac{1}{\ell}(\sin \alpha_1 - \cos \alpha_1 \sin \alpha_2) \\ \dot{p}_{\alpha_1} = \frac{p_{\alpha_1}}{\ell} \cos \alpha_1 \end{cases} \quad (12)$$

We remark that

$$H_r = \left(\sin(\alpha_1) + \frac{p_{\alpha_1}}{\gamma \ell} \right)^2 + \cos(\alpha_1)^2$$

is a first integral of (12). On the level $H_r = 1$, we can solve easily in p_{α_1}

$$p_{\alpha_1} = -2 \sin(\alpha_1) \gamma \ell \text{ or } p_{\alpha_1} = 0.$$

This first solution gives the equation $\dot{\alpha}_1 = \frac{\sin(\alpha_1)}{\ell}$ and we deduce the following solution

$$\alpha_1(t) = 2 \arctan \left(e^{t/\ell} \right), \quad p_{\alpha_1}(t) = -\frac{4\gamma \ell}{e^{t/\ell} + e^{-t/\ell}}. \quad (13)$$

We will now try to obtain the corresponding solution α_2 of system (12). We introduce a variable change

$$\alpha_2(t) = i \ln z(\tau), \quad t = \ell \ln \tau$$

and the equation in $z(\tau)$ becomes

$$z'(\tau) = \frac{(\tau^2 - 1)z(\tau)^2}{2\tau(\tau^2 + 1)} - \frac{2iz(\tau)}{\tau^2 + 1} - \frac{\tau^2 - 1}{2\tau(\tau^2 + 1)}. \quad (14)$$

This is a Riccati equation, the parameter ℓ disappears, and noting

$$z(\tau) = -\frac{2\tau(\tau^2 + 1)}{\tau^2 - 1} \frac{\phi'(\tau)}{\phi(\tau)}$$

this equation reduces to a second order linear ODE

$$\phi''(\tau) + \frac{\tau^3 + i\tau^2 - 3\tau + i}{\tau(\tau^2 - 1)(\tau - i)} \phi'(\tau) - \frac{(\tau^2 - 1)^2}{4(\tau^2 + 1)^2 \tau^2} \phi(\tau) = 0$$

Using the Kovacic algorithm, we find that this equation has Galois group $SL_2(\mathbb{C})$, and thus is not solvable by quadrature. Thus $\alpha_2(t)$ cannot be obtained by quadrature in system (12) when α_1, p_{α_1} are given by (13).

Theorem 3. The singular flow is not solvable by quadrature.

Proof. As it is impossible to obtain $\alpha_2(t)$ by quadrature in the particular case (13), then it is not possible in general, and thus system (12) and then system (10) are not solvable by quadrature. \square

Let us now look at the case of regular flow, which reduces on the invariant hyperplane $p_{\alpha_2} = 0$ to

$$\begin{cases} \dot{\alpha}_1 = \ell - \sin \alpha_1 \\ \dot{\alpha}_2 = \sin \alpha_1 - \cos \alpha_1 \sin \alpha_2 \\ \dot{p}_{\alpha_1} = p_{\alpha_1} \cos \alpha_1 \end{cases} \quad (15)$$

This system depends on a parameter $\ell \in]0, 1[$. Again this system admits a first integral $p_{\alpha_1}(\ell - \sin \alpha_1)$, which allows to recover p_{α_1} once α_1 is known. Solving the first equation in α_1 , we find the solution

$$\alpha_1(t) = -2 \arctan \left(\frac{\tanh \left(\frac{1}{2} t \sqrt{1 - \ell^2} \right) \sqrt{1 - \ell^2} - 1}{\ell} \right).$$

We substitute in the second equation, and making the variable change

$$\alpha_2(t) = i \ln(z(\tau)), \quad \tanh \left(\frac{1}{2} t \sqrt{1 - \ell^2} \right) \sqrt{1 - \ell^2} = \tau$$

we have a Riccati equation $z'(\tau) =$

$$\frac{(\ell^2 - (\tau - 1)^2)z(\tau)^2 - 4i\ell(\tau - 1)z(\tau) - \ell^2 + \tau^2 - 2\tau + 1}{((\ell^2 + \tau^2 - 1)(\ell^2 + (\tau - 1)^2))}$$

We can now transform this Riccati equation to a second order linear equation by the variable change

$$z(\tau) = -\frac{(\ell^2 + \tau^2 - 1)(\ell^2 + \tau^2 - 2\tau + 1)}{(\ell - 1 + \tau)(\ell + 1 - \tau)} \frac{\phi'(\tau)}{\phi(\tau)}$$

which gives the equation

$$\begin{aligned} & \frac{(i\tau^4 + (\ell - 3i)\tau^3 - (3i\ell^2 + 4\ell - 3i)\tau^2 - (3\ell^3 - 3i\ell^2 - 5\ell + i)\tau - 2\ell^3 + 2\ell)}{(i\tau + \ell - i)(\ell - 1 + \tau)(\ell + 1 - \tau)(\ell^2 + \tau^2 - 1)} \\ & \times 2\phi'(\tau) - \frac{(\ell - 1 + \tau)^2(\ell + 1 - \tau)^2\phi(\tau)}{(\ell^2 + \tau^2 - 1)^2(\ell^2 + \tau^2 - 2\tau + 1)^2} \end{aligned} \quad (16)$$

For a generic ℓ , the Kovacic algorithm finds no solutions, and thus generically the system (15) is not integrable by quadrature. The system could nevertheless be integrated for specific values of ℓ ; computing the possible confluences between singularities, we find $\ell = 0, \pm 1$ that are outside the interval studied. Thus for any $\ell \in (0, 1)$, there are exactly 6 singularities (the point at infinity is regular)

$$1 - \ell, 1 + \ell, 1 - i\ell, 1 + i\ell, \sqrt{1 - \ell^2}, -\sqrt{1 - \ell^2},$$

with respectively the local exponents

$$(0, 2), (0, 2), \frac{1}{2} \pm \frac{1}{2}\sqrt{2}, -\frac{1}{2} \pm \frac{1}{2}\sqrt{2},$$

$$\frac{i\ell\sqrt{1 - \ell^2} \pm \sqrt{2\ell^4 - 3\ell^2 + 1}}{2(1 - \ell^2)}, \frac{i\ell\sqrt{1 - \ell^2} \pm \sqrt{2\ell^4 - 3\ell^2 + 1}}{2(\ell^2 - 1)}$$

We will now follow the Kovacic algorithm case by case. We see that not all exponents are rational, and thus not all solutions of (16) can be algebraic. Thus case III is not possible. For case I, we need a hyperexponential solution, which requires that a sum of exponents to be a non positive integer. We obtain the following equation

$$\epsilon \frac{\sqrt{2\ell^4 - 3\ell^2 + 1}}{1 - \ell^2} + \kappa \sqrt{2} = -n, \quad n \in \mathbb{N} \quad (17)$$

where $\epsilon, \kappa \in \{-1, 0, 1\}$ depend on the choice of the exponents. This equation will give constraints to the possible ℓ . For case II, we need to consider the symmetric square of equation (16), which is a linear differential equation of order 3 with the same singularities. The computed exponents (3 for each singularity) are the following

$$\begin{aligned} & (0, 2, 4), (0, 2, 4), (1, \pm\sqrt{2}), (-1, \pm\sqrt{2}), \\ & \left(\frac{i\ell\sqrt{1 - \ell^2}}{1 - \ell^2}, \frac{i\ell\sqrt{1 - \ell^2} \pm \sqrt{2\ell^4 - 3\ell^2 + 1}}{1 - \ell^2} \right), \\ & \left(\frac{i\ell\sqrt{1 - \ell^2}}{\ell^2 - 1}, \frac{i\ell\sqrt{1 - \ell^2} \pm \sqrt{2\ell^4 - 3\ell^2 + 1}}{\ell^2 - 1} \right) \end{aligned}$$

We again need a hyperexponential solution, and thus a sum of exponents equal to a non positive integer. This gives again relation (17), but with $\epsilon, \kappa \in \{-2, -1, 0, 1, 2\}$.

For $\ell \in (0, 1)$, the possible non positive integer values are $-5, -4, -3, -2, -1, 0$, and the corresponding values of ℓ in $]0, 1[$ are

$$\begin{aligned} & \frac{1}{2}\sqrt{2}, \frac{1}{3}\sqrt{3}, \frac{1}{7}\sqrt{21}, \frac{1}{17}\sqrt{-51 + 136\sqrt{2}}, \\ & \frac{1}{2}\sqrt{3 - \sqrt{2}}, \frac{1}{7}\sqrt{42 - 14\sqrt{2}}, \frac{2}{21}\sqrt{84 - 21\sqrt{2}}, \\ & \frac{1}{21}\sqrt{357 - 168\sqrt{2}}, \frac{1}{31}\sqrt{837 - 496\sqrt{2}}, \\ & \frac{1}{69}\sqrt{3933 - 1104\sqrt{2}}, \frac{2}{7}\sqrt{7} - \frac{1}{14}\sqrt{14}, \frac{2}{7}\sqrt{14} - \frac{1}{7}\sqrt{7} \end{aligned}$$

For each of these distinguished values of ℓ , we apply the Kovacic algorithm and we find that none have a solvable Galois group. Summarising, we have proved the following result:

Theorem 4. For $\ell \in (0, 1)$, the regular flow is not solvable by quadrature.

Remark. The fact that systems (12) and (15), when reduced to $p_{\alpha_2} = 0$, can be solved through second order differential equations is not a generic situation, and thus further calculations could be possible through the use of special functions.

6. NUMERICAL SIMULATIONS

We use the BOCOP software (python version from the ct project¹) to provide preliminary numerical computations on the problem with two trailers. The algorithm uses the midpoint scheme to discretise then optimise the problem. Three simulations are presented Figure 3; they all share the same value of the current ($w = 0.8$), of ℓ ($\ell = 0.99 < 1$), the same initial conditions $(x_0, y_0, \theta_0) = (0, 0, \pi/7)$ and final conditions $(x_f, y_f) = (4, 7)$ (normalised values as explained in Section 2) while the final velocity angles differ (remember that θ is the argument of the velocity in a fixed frame, attached to the bottom of the sea): (i-a) $\theta = -\pi/2$, (i-b) $\theta = -\pi/2$ (other solution), (ii) $\theta = \pi/2$. The numerical simulations presented are reproducible online² on the web site of the ct project. In the two first cases, one can observe that the structure of the computed trajectories is similar: bang-singular-bang-bang-bang, namely $B_+SB_+B_-B_+$ in case (i-a) (B_+ = positive bang, that is left turn, while S = singular) and $B_+SB_-B_+B_-$ in case (i-b). As in the one trailer case Caillaud et al. (2019), the main part is the singular arc, which illustrates the so-called turnpike phenomenon; for a distant enough target in the (x, y) plane, the requested minimum time is long enough to allow the singular part of the extremal to come close to the hyperbolic equilibrium $(0, 0, 0, 0)$ of family A described in Section 4. The longer the minimum time, the closer this singular part is to the straight line encountered in the Zermelo-Markov-Dubins problem without trailer. In case (ii), the structure is $B_+B_-B_+B_-$ and reminiscent of the bang-bang-bang case of the Zermelo-Markov-Dubins problem without trailer. In contrast with the one trailer case, the trajectories are terminated not by two but three bang arcs that accommodate the alignment requirement of the two trailers at the end. These short bang arcs are much more time efficient than the "run-in" procedure (a final straight line) performed until now by ships at the end of the maneuver during real exploration campaigns. Such comparisons will be the topic of future work.

¹ ct: Control Toolbox, see ct.gitlabpages.inria.fr/gallery

² See ct.gitlabpages.inria.fr/gallery/nav/nav.html

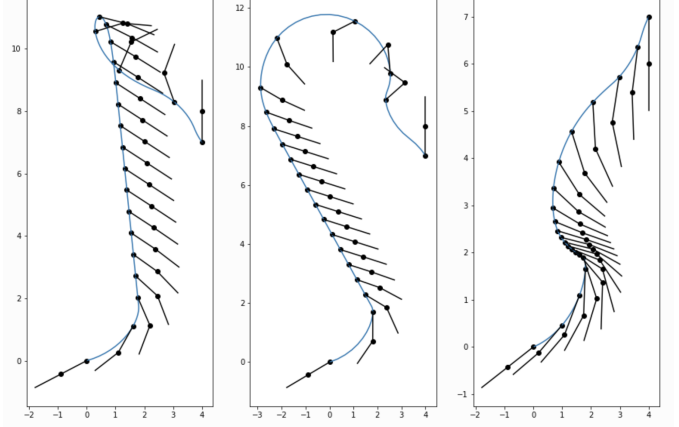


Fig. 3. Trajectories of a ship with two trailers in a constant current: cases (i-a), (i-b) and (ii) from left to right, all exhibiting a bang-singular-bang-bang structure. Respective minimum times are (i-a) $t_f = 26.7$, (i-b) $t_f = 41.0$ and (ii) $t_f = 13.2$. In particular, the extremal obtained case (i-b) for the same problem as (i-a) is a strict local minimum. Note that for case (i-1) the curvature constraint is indeed fulfilled; as the trajectories are portrayed in a fixed frame (attached to the bottom of the sea) and not in moving frame (moving at the speed of the current, directed along the (Ox) axis), the turns in cases (i-1) and (i-2) appear to have different curvatures although these curvatures are indeed equal but turns in opposite directions.

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