New perspectives on output feedback stabilization at an unobservable target

Lucas Brivadis, Jean-Paul Gauthier, Ludovic Sacchelli, Ulysse Serres

New perspectives on output feedback stabilization at an unobservable target

Lucas Brivadis[1], Jean-Paul Gauthier[2], Ludovic Sacchelli[1], and Ulysse Serres[1]

[1]Univ. Lyon, Université Claude Bernard Lyon 1, CNRS, LAGEPP UMR 5007, 43 bd du 11 novembre 1918, F-69100 Villeurbanne, France (e-mail: lucas.brivadis@univ-lyon1.fr, ludovic.sacchelli@univ-lyon1.fr, ulysse.serres@univ-lyon1.fr)
[2]Université de Toulon, Aix Marseille Univ, CNRS, LIS, France (e-mail: jean-paul.gauthier@univ-tln.fr)

Tuesday 24th November, 2020

Abstract

We address the problem of dynamic output feedback stabilization at an unobservable target point. The challenge lies in according the antagonistic nature of the objective and the properties of the system: the system tends to be less observable as it approaches the target. We illustrate two main ideas: well chosen perturbations of a state feedback law can yield new observability properties of the closed-loop system, and embedding systems into bilinear systems admitting observers with dissipative error systems allows to mitigate the observability issues. We apply them on a case of systems with linear dynamics and nonlinear observation map and make use of an *ad hoc* finite-dimensional embedding. More generally, we introduce a new strategy based on infinite-dimensional unitary embeddings. To do so, we extend the usual definition of dynamic output feedback stabilization in order to allow infinite-dimensional observers fed by the output. We show how this technique, based on representation theory, may be applied to achieve output feedback stabilization at an unobservable target.

1 Introduction

The problem of output feedback stabilization is one of deep interest in control theory. It consists of stabilizing the state of a dynamical system, that is only partially known, to a target point. Although a vast literature tackles this topic (see [4], and references therein), some fundamental issues remain mostly open. The case of completely uniformly observable systems has been solved in [20] and [36, 37]: a separation principle can be set up to achieve semi-global dynamic output feedback stabilization. However, as shown in [17], nonlinear systems are generically non-uniformly observable, *i.e.,* there exist inputs that make the system unobservable. Crucially, observability singularities prevent from applying classical tried-and-tested methods. However, such difficulties occur in practical engineering systems, where original strategies need to be explored [1, 2, 12, 16, 18, 29, 35], leading to a renewal of interest in the issue in recent years.

General methods, based on time-varying feedback laws have been developed to deal with singular inputs. Let us mention the seminal article [13] by J.-M. Coron, in which local stabilization is achieved by means of a periodic time-varying feedback, up to a Lie null-observability condition. A “sample-and-hold” strategy was developed in [32] for the practical semi-global stabilization as well. Furthermore, perturbations of the input, such as high-frequency excitation [12, 35], stochastic noise [16], or feedback law perturbation [9, 24], appear to be a key tool to enhance the observability properties of the system. In the later two papers, the closed-loop system remains autonomous, which is of interest for engineering applications.

Another important tool in stabilization theory is the use of weakly contractive systems, for which flows are non-expanding [10]. In [30], the authors proved that for non-uniformly observable state-affine systems, detectability at the target is a sufficient condition to set up a separation principle. In [24] a strategy of feedback perturbation is used in conjunction with the contraction property of a quantum control system to achieve stabilization at an unobservable target.

In the present paper, we coalesce the insights provided by these previous works in order to come up with solutions to attack the problem of dynamic output feedback stabilization at an unobservable target. The issue itself bears discussion, hence we first explore in Section 2.1 necessary conditions for dynamic output feedback stabilization. Furthermore, a simple example by J.-M. Coron in [13] highlights how some systems are not stabilizable by means of a finite-dimensional autonomous dynamic output feedback, even locally (Section 3.1). Similar examples, sharing the same unobservability issues, may nonetheless be stabilized, as we illustrate in Section 3.2.

In order to access the properties of contractive systems, we look into embedding techniques. In [11], the authors propose an observer design strategy based on infinite-dimensional unitary embeddings. We rely on their approach in the context of output feedback stabilization, which leads to a coupling of the finite-dimensional original system with an infinite-dimensional contractive observer system (Section 4.1). Interestingly, adding an infinite-dimensional virtual state fed by the output to the original system lifts the topological obstructions identified in [13]. We illustrate this strategy in Section 4.2 by focusing on examples with linear dynamics and nonlinear observation map, that we hope may pave the way to more general results in the future.

Notations: Denote by $\mathbb{R}$ (resp. $\mathbb{R}_+$) the set of real (resp. non-negative) numbers, by $\mathbb{C}$ the set of complex numbers and by $\mathbb{Z}$ (resp. $\mathbb{N}$) the set of integers (resp. non-negative integers). The Euclidean norm over $\mathbb{R}^n$ (or $\mathbb{C}^n$) is denoted by $|\cdot|$ for any $n \in \mathbb{N}$. The real and imaginary parts of $z \in \mathbb{C}$ are denoted by $\Re z$ and $\Im z$, respectively. For any normed vector space $(X, \|\cdot\|_X)$, we denote by $B_X(x, r)$ (resp. $\bar{B}_X(x, r)$) the open (resp. closed) ball of X centered at $x \in X$ of radius $r > 0$ for the norm $\|\cdot\|_X$. The identity operator over X is denoted by I_X. For all $k \in \mathbb{N} \cup \{\infty\}$ and all interval $U \subset \mathbb{R}$, the set $C^k(U, X)$ is the set of k-continuously differentiable functions from U to X.

2 Problem statement

Let n, m and p be positive integers, $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}^m$. For all $u \in C^0(\mathbb{R}_+, \mathbb{R}^p)$, consider the following observation-control system:

$$\begin{cases} \dot{x} = f(x, u) \\ y = h(x) \end{cases} \tag{1}$$

where x is the state of the system, u is the control (or input) and y is the observation (or output). This paper deals with the problem of dynamic output feedback stabilization of (1). Without loss of generality, we assume that the target point at which we aim to stabilize x is the origin $0 \in \mathbb{R}^n$, $h(0) = 0$ and $f(0, 0) = 0$.

Suppose that f is uniformly locally Lipschitz with respect to x and continuous. According to the Cauchy-Lipschitz theorem, for all input $u \in C^0(\mathbb{R}_+, \mathbb{R}^n)$, and all $x_0 \in \mathbb{R}^n$, there exists exactly one maximal solution $\varphi_t(x_0, u)$ defined for $t \in [0, T(x_0, u))$ such that $\varphi_0(x_0, u) = x_0$ and $\frac{\partial \varphi_t(x_0,u)}{\partial t} = f(\varphi_t(x_0, u), u(t))$ for all $t \in [0, T)$. The map φ is called the flow of (1).

Definition 2.1 (Dynamic output feedback stabilizability)**.** System (1) is said to be *locally* (resp. *globally*) *stabilizable by means of a dynamic output feedback* if and only if the following holds.

There exist two continuous map $\nu : \mathbb{R}^q \times \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^q$ and $\varpi : \mathbb{R}^q \times \mathbb{R}^m \to \mathbb{R}^p$ for some positive integer q such that $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^q$ is a locally (resp. globally) asymptotically stable[1] equilibrium point of the following system:

$$\begin{cases} \dot{x} = f(x, u) \\ y = h(x) \end{cases}, \qquad \begin{cases} \dot{w} = \nu(w, u, y) \\ u = \varpi(w, y). \end{cases} \tag{2}$$

Additionally, if for any compact set $\mathcal{K} \subset \mathbb{R}^n$, there exist two continuous map $\nu : \mathbb{R}^q \times \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^q$ and $\varpi : \mathbb{R}^q \times \mathbb{R}^m \to \mathbb{R}^p$ for some positive integer q, and a compact set $\widehat{\mathcal{K}} \subset \mathbb{R}^q$ such that $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^q$ is an asymptotically stable equilibrium point of (2) with basin of attraction containing $\mathcal{K} \times \widehat{\mathcal{K}}$, then (1) is said to be *semi-globally stabilizable by means of a dynamic output feedback*.

Remark 2.2. Clearly, we have following implications about the stabilizability by means of dynamic output feedback of (1):

$$\text{(Global stabilizability)} \implies \text{(Semi-global stabilizability)} \implies \text{(Local stabilizability)} \tag{3}$$

Remark 2.3. Since $h(0) = 0$ and $f(0, 0) = 0$, if (2) is locally asymptotically stable at $(0, 0)$, then $\varpi(0, 0) = 0 \in \mathbb{R}^p$ is the value of the control at the target point.

2.1 Necessary conditions

The problem of dynamic state feedback stabilization of (1) is equivalent to the dynamic output feedback stabilization in the case where $h(x) = x$. Therefore, dynamic state feedback stabilizability of (1) is a necessary condition for dynamic output feedback stabilizability. One may wonder if *static* state feedback stabilizability of (1) a necessary condition for dynamic output feedback stabilizability. In [4], the authors answer by the positive if a sufficiently regular selection function can be found. We recall their result below.

Definition 2.4 (State feedback stabilizability)**.** System (1) is said to be *locally* (resp. *globally*) *stabilizable by means of a (static) state feedback* if and only if there exists a continuous map $\phi : \mathbb{R}^n \to \mathbb{R}^p$ such that $0 \in \mathbb{R}^n$ is a locally (resp. globally) asymptotically stable equilibrium point of

$$\begin{cases} \dot{x} = f(x, u) \\ u = \phi(x). \end{cases} \tag{4}$$

Additionally, if for any compact set $\mathcal{K} \subset \mathbb{R}^n$, there exists a continuous map $\phi : \mathbb{R}^n \to \mathbb{R}^p$ such that $0 \in \mathbb{R}^n$ is an asymptotically stable equilibrium point of (4) with basin of attraction containing $\mathcal{K}$, then (1) is said to be *semi-globally stabilizable by means of a static state feedback*.

Remark 2.5. Clearly, the implications (3) hold for the stabilizability by means of static state feedback of (1).

[1]Recall that a dynamical system is said to be asymptotically stable at an equilibrium point with some basin of attraction if and only if each initial condition in the basin of attraction yields at least one solution to the corresponding Cauchy problem, each solution converges to the equilibrium point, and the equilibrium point is Lyapunov stable.

Theorem 2.6 ([4, Lemma 1, (1)])**.** *Assume that* (2) *is locally asymptotically stable at* $(0,0)$ *with basin of attraction*[2] $\mathcal{U}_x \times \mathcal{U}_w$. *Let* V *be a* $C^\infty(\mathcal{U}_x \times \mathcal{U}_w, \mathbb{R}_+)$ *strict proper Lyapunov function of* (2). *If there exists a selection map* $\mathcal{U}_x \ni x \mapsto \phi(x) \in \operatorname{argmin}_{\mathcal{U}_w} V(x,\cdot)$ *which is locally Hölder of order strictly larger than* $\frac{1}{2}$*, then* (4) *is locally asymptotically stable at* 0 *with basin of attraction containing* $\mathcal{U}_x$.

Therefore, up to the existence of a sufficiently regular selection map, this result implies that the following local (resp. semi-global, global) condition is necessary for the local (resp. semi-global, global) stabilizability of (1) by means of a dynamic output feedback.

Condition 2.7 (State feedback stabilizability — local, semi-global, global)**.** System (1) is locally (resp. semi-globally, globally) stabilizable by means of a static state feedback.

In [13], J.-M. Coron stated two additional conditions that he proved to be sufficient when local static state feedback stabilizability holds to ensure local dynamic output feedback stabilizability, provided that one allows the output feedback to depend on time (which we do *not* allow in this paper). The two following conditions are weaker versions of the ones of [13]. We prove that these two conditions are necessary to ensure dynamic output feedback stabilizability. The first one, known as 0-*detectability* is also used by E. Sontag in [34] in the context of abstract nonlinear regulation theory.

Condition 2.8 (0-detectability — local, global)**.** Let $\mathcal{X}_0 = \{x_0 \in \mathbb{R}^n \ : \ \forall t \in [0, T(x_0,0)), h(\varphi_t(x_0,0)) = 0\}$. Then $0 \in \mathcal{X}_0$ is a locally (resp. globally) asymptotically stable equilibrium point of the vector field $\mathcal{X}_0 \ni x \mapsto f(x,0)$.

Theorem 2.9. *If* (1) *is locally (resp. semi-globally, globally) stabilizable by means of a dynamic output feedback, then Condition 2.8 holds locally (resp. globally, globally).*

Proof. The set $\mathcal{X}_0$ is invariant for the vector field $x \mapsto f(x,0)$ and $0 \in \mathcal{X}_0$. Let $x_0 \in \mathcal{X}_0$. Assume that (1) is locally stabilizable by means of a dynamic output feedback, and that $(x_0,0)$ is in the basin of attraction of $(0,0)$ for (2).

Then $t \mapsto (\varphi_t(x_0,0),0)$ is a trajectory of (2) with initial condition $(x_0,0)$. Hence $\varphi_t(x_0,0)$ is well-defined for all $t \geqslant 0$ and tends towards 0 as t goes to infinity. Moreover, for all $R>0$, there exists $r>0$ such that, if $x_0 \in B_{\mathbb{R}^n}(x_0,r)$, then $\varphi_t(x_0,0) \in B_{\mathbb{R}^n}(x_0,R)$ for all $t \geqslant 0$.

If we assume that (1) is globally stabilizable by means of a dynamic output feedback, then the arguments still hold for any $x_0, \tilde{x}_0 \in \mathbb{R}^n$. If (1) is only semi-globally stabilizable by means of a dynamic output feedback, we first define $\mathcal{K}$ as in Definition 2.1 containing x_0. ∎

Condition 2.10 (Indistinguishability implies common stabilizability — local, global)**.** For all x_0, $\tilde{x}_0$ in some neighborhood of $0 \in \mathbb{R}^n$ (resp. for all x_0, $\tilde{x}_0$ in $\mathbb{R}^n$), if for all $u \in C^0(\mathbb{R}_+, \mathbb{R}^p)$ such that $T(x_0,u) = +\infty$ it holds that $h(\varphi_t(x_0,u)) = h(\varphi_t(\tilde{x}_0,u))$ for all $t \in [0, T(\tilde{x}_0,u))$, then there exists $v \in C^0(\mathbb{R}_+, \mathbb{R}^p)$ such that $\varphi_t(x_0,v)$ and $\varphi_t(\tilde{x}_0,v)$ are well-defined for all $t \in \mathbb{R}_+$ and tend towards 0 as t goes to infinity.

Theorem 2.11. *If* (1) *is locally (resp. semi-globally, globally) stabilizable by means of a dynamic output feedback, then Condition 2.10 holds locally (resp. globally, globally).*

Proof. Let $x_0, \tilde{x}_0 \in \mathbb{R}^n$ be such that for all $u \in C^0(\mathbb{R}_+, \mathbb{R}^p)$ such that $T(x_0,u) = +\infty$ it holds that $h(\varphi_t(x_0,u)) = h(\varphi_t(\tilde{x}_0,u))$ for all $t \in [0, T(\tilde{x}_0,u))$, Assume that (1) is locally stabilizable

[2] In [4, Lemma 1, (1)], the authors state only a global version of the result, that is, $\mathcal{U}_x = \mathbb{R}^n$ and $\mathcal{U}_w = \mathbb{R}^q$. However, the proof remains identical in the other cases.

by means of a dynamic output feedback, and that $(x_0, 0)$, $(\tilde{x}_0, 0)$ are in the basin of attraction of $(0, 0)$ for (2).

Let (x, w) be a solution of (2) starting from $(x_0, 0)$. Set $v = \varpi(w, h(x))$. Then $T(x_0, v) = +\infty$ and $\varphi_t(x_0, v) \to 0$ as $t \to +\infty$. Let $\tilde{x}(t) = \varphi_t(\tilde{x}_0, v)$ for all $t \in [0, T(\tilde{x}_0, v))$. Since $h(\varphi_t(x_0, v)) = h(\varphi_t(\tilde{x}_0, v))$ for all $t \in [0, T(\tilde{x}_0, v))$, $(\tilde{x}, w)$ is a solution of (2) starting from $(\tilde{x}_0, 0)$. Hence $T(\tilde{x}_0, v) = +\infty$ and $\varphi_t(\tilde{x}_0, v) \to 0$ as $t \to +\infty$.

If we assume that (1) is globally stabilizable by means of a dynamic output feedback, then the arguments still hold for any $x_0, \tilde{x}_0 \in \mathbb{R}^n$. If (1) is only semi-globally stabilizable by means of a dynamic output feedback, we first define $\mathcal{K}$ as in Definition 2.1 containing x_0 and $\tilde{x}_0$. ■

Remark 2.12. In [30], the authors consider the problem of dynamic output feedback stabilization of *dissipative* state-affine systems, that is, systems of the form

$$\begin{cases} \dot{x} = A(u)x + B(u) \\ y = Cx \end{cases} \tag{5}$$

where there exists some positive definite matrix $P \in \mathbb{R}^n \times \mathbb{R}^n$ such that, for all inputs u in some admissible set,

$$PA(u) + A(u)'P \leqslant 0. \tag{6}$$

For such systems, Conditions 2.7 (local) and 2.8 are proved to be sufficient to achieve the dynamic output feedback stabilization, which implies, by Theorem 2.11, that Condition 2.10 is also satisfied. In this paper, we therefore focus on systems that are not in the form of (5)-(6).

2.2 Separation principle

For linear systems, the problem of output feedback stabilization is solved by the so-called separation principle, which consists of designing "separately" a stabilizing state feedback law and a state observer. Then the coupled system, which consists of applying this feedback law to the state estimation given by the observer, provides a suitable dynamic output feedback. This strategy is known to fail in general for nonlinear systems.

In [20] and [36,37], the authors proved under a *complete uniform observability* assumption that any system semi-globally stabilizable by means of a static state feedback is also semi-globally stabilizable by means of a dynamic output feedback.

Definition 2.13 (Observability)**.** System (1) is said to be observable for some input $u \in C^0(\mathbb{R}_+, \mathbb{R}^p)$ in time $T > 0$ if and only if, for all initial conditions $(x_0, \tilde{x}_0) \in \mathbb{R}^n \times \mathbb{R}^n$, there exists $t \in [0, T]$ such that $h(\varphi_t(x_0, u) \neq h(\varphi_t(\tilde{x}_0, u))$.

The complete uniform observability assumption required in [36, 37] is stronger than the observability for all inputs and all times. However, as proved in [17], it is generic for nonlinear systems to be not completely uniformly observable when $m \geqslant p$. The problem of dynamic output feedback stabilization remains open when singular inputs (that are, inputs that make the system unobservable) exist. The main difficulties arise when the control input $u \equiv 0$, which is the input corresponding to the equilibrium point $(0, 0)$ of (2), is singular. Indeed, a contradiction may occur between the stabilization of the state at the target point, and the fact that the observer system may fail to properly estimate the state near the target. Various techniques have been introduced to remove this inconsistency.

Most of them rely on a perturbation of the input, that helps the observer system to estimate the state even near the target point. This strategy was employed in [13] (that achieved local stabilization by using a periodic time-dependent perturbation) and in [32] (that achieved practical stabilization by using a "sample and hold" time-dependent perturbation). Let us also mention the recent works of [12, 35], that also rely on a high-frequency excitation of the input. Adopting another point of view in line with [9, 24], we are interested in time-independent feedback laws. In other words, the perturbation acts directly on the feedback law instead of the control input.

In the context of output feedback stabilization of non-uniformly observable systems via a separation principle, the fact that the error between the state and the observer has contractive dynamics has proven to be a powerful tool in [24, 30], because it guarantees that the error is non-increasing, no matter the observability properties of the system. Therefore, embedding the original nonlinear system into a bilinear system and design an observer with dissipative error system is the second guideline that we aim to follow.

This paper is devoted to the illustration of these two guidelines for output feedback stabilization on enlightening examples. Essentially, we consider systems with linear conservative dynamics and nonlinear observation maps. Section 3 is devoted to the study of an illustrative example in which the specific form of the output allows us to find a finite-dimensional embedding of the system into a bilinear system, and to design a Luenberger observer with dissipative error system. In Section 4, we show how tools from representation theory may help to embed a system into a bilinear unitary infinite-dimensional one and stabilize a larger class of systems by means of dynamic output feedback. In both cases, the stabilizing state feedback law is modified with a perturbation that vanishes at the target point.

3 An illustrative example

3.1 An obstruction by J.-M. Coron

Consider the case where (1) is single-input single-output and f is a linear map, so that it can be written in the form of

$$\begin{cases} \dot{x} = Ax + bu, \\ y = h(x). \end{cases} \tag{7}$$

where $A \in \mathbb{R}^{n\times n}$ and $b \in \mathbb{R}^{n\times 1}$ and $h : \mathbb{R}^n \to \mathbb{R}$. If h is nonlinear and is not an invertible transformation of a linear map, then the usual theory of linear systems fails to be applied. Condition 2.7 reduces to the stabilizability of the pair (A, b). If it holds, then (7) is globally stabilizable by a linear static state feedback.

In [13], J.-M. Coron introduced the following illustrative one-dimensional example:

$$\dot{x} = u, \qquad y = x^2. \tag{8}$$

He proved that (8) is not locally stabilizable by means of a dynamic output feedback, unless introducing a time-dependent component in the feedback law. The difficulty with this system comes from the unobservability of the target point 0. Indeed, (8) is not observable for the constant input $u \equiv 0$ in any time $T > 0$, and. Indeed, the initial conditions x_0, $-x_0 \in \mathbb{R}$ are indistinguishable. In particular, the system is not uniformly observable, and consequently the results of [20, 36, 37] fail to be applied. To overcome this issue, [13] introduced time-dependent output feedback laws, and proved by this means the local stabilizability of (8). This system can also be stabilized by means of "dead-beat" or "sample-and-hold" techniques (see [26], [32], respectively).

A generalization of (8) in higher dimension is

$$\begin{cases} \dot{x} = Ax + bu, \\ y = h(x) \end{cases} \tag{9}$$

for a skew-symmetric matrix A and h radially symmetric[3]. Again, the constant input $u \equiv 0$ makes the system unobservable in any time $T > 0$ since for any initial conditions x_0, $\tilde{x}_0$ in $\mathbb{R}^n$ satisfying $|x_0| = |\tilde{x}_0|$, $h(\varphi_t(x_0)) = h(\tilde{x}_0) = h(x_0) = h(\varphi_t(x_0))$ for all $t \in \mathbb{R}_+$. Condition 2.7 (global) reduces to the stabilizability of (A, b) and Condition 2.8 (global) is always satisfied. Let us state a necessary condition for the stabilizability of (9) by means of a dynamic output feedback.

Theorem 3.1. *If* (9) *is locally stabilizable by means of a dynamic output feedback, then A is invertible.*

Proof. The proof is an adaptation of the one given in [13] in the one-dimensional context. Assume that $(0, 0)$ is a locally asymptotically stable equilibrium point of

$$\begin{cases} \dot{x} = Ax + bu, \\ y = h(x) \end{cases}, \qquad \begin{cases} \dot{w} = \nu(w, u, y) \\ u = \varpi(w, y) \end{cases} \tag{10}$$

for some positive integer q and two continuous map $\nu : \mathbb{R}^q \times \mathbb{R} \times \mathbb{R}$ and $\varpi : \mathbb{R}^q \times \mathbb{R}$. Set $F : \mathbb{R}^n \times \mathbb{R}^q \ni (x, w) \mapsto (Ax + b\varpi(w, h(x)), \nu(w, \varpi(w, h(x)), h(x)))$. Then, according to [21, Theorem 52.1] (see [14] when one does not have uniqueness of the solutions to the Cauchy problem), the index of $-F$ at $(0, 0)$ is 1. Assume, for the sake of contradiction, that A is not invertible. Let $\mathcal{N}$ be a one-dimensional subspace of $\ker A$. Denote by Σ the reflection through the hyperplane $\mathcal{N}^\perp$, that is, $\Sigma = \mathrm{I}_n - 2vv'$ for some unitary vector $v \in \mathcal{N}$. Then $\det \Sigma = -1$, $A\Sigma = A$ and $h(\Sigma x) = h(x)$. Hence $(x, w) \mapsto -F(\Sigma x, w)$ has index -1 at $(0, 0)$ and $F(\Sigma x, w) = F(x, w)$. Thus $1 = -1$ which is a contradiction. ∎

According to the spectral theorem, we have the following immediate corollary. If $n = 1$, we recover the result of J.-M. Coron in [13].

Corollary 3.2. *If n is odd and A is skew-symmetric, then* (9) *is not locally stabilizable by means of a dynamic output feedback.*

3.2 Converse theorem: a positive result of output feedback stabilization

One of the main results of this paper is the following theorem which is the converse of Theorem 3.1 in the case where $h(x) = \frac{1}{2}|x|^2$. The proof relies on the guidelines described in Section 2.2, that is, an embedding into a bilinear system, an observer design with dissipative error-system and a feedback perturbation.

Consider the special case for system (9):

$$\begin{cases} \dot{x} = Ax + bu, \\ y = h(x) = \dfrac{1}{2}|x|^2. \end{cases} \tag{9'}$$

Theorem 3.3. *If A is skew-symmetric and invertible and (A, b) is stabilizable, then* (9') *is semi-globally stabilizable by means of a dynamic output feedback.*

[3] Up to a change of scalar product, one may also consider the case where $PA + AP = 0$ for some positive definite matrix $P \in \mathbb{R}^{n\times n}$ and h such that $(x_1'Px_1 = x_2'Px_2) \Rightarrow (h(x_1) = h(x_2))$.

Remark 3.4. The dynamic output feedback is explicitly given in (17). It is easily implementable, and does not use time-dependent feedback laws.

The proof of Theorem 3.3 is the object of the section. We follow the same steps as in [2], with a very similar embedding strategy. The main difference is the observability analysis developped in Section 3.2.5: here the target is unobservable, while in [2] it was observable.

3.2.1 Embedding into a bilinear system of higher dimension

Consider the map

$$\begin{aligned} \tau : \mathbb{R}^n &\longrightarrow \mathbb{R}^{n+1} \\ x &\longmapsto \left(x, \tfrac{1}{2}|x|^2\right). \end{aligned}$$

For all $z = (z_1, \dots, z_{n+1}) \in \mathbb{R}^{n+1}$, define $\bar{z}_n = (z_1, \dots, z_n) \in \mathbb{R}^n$. If x is a solution of (9), then $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|x|^2 = x'Ax + x'bu = x'bu$ since A is skew-symmetric. Hence $z = \tau(x)$ defines an embedding of (9) into

$$\begin{cases} \dot{z} = \mathcal{A}(u)z + \mathcal{B}u \\ y = \mathcal{C}z. \end{cases} \tag{11}$$

where $\mathcal{A}(u) = \begin{pmatrix} A & 0 \\ ub' & 0 \end{pmatrix}$, $\mathcal{B} = \begin{pmatrix} b \\ 0 \end{pmatrix}$ and $\mathcal{C} = \begin{pmatrix} 0 & \cdots & 0 & 1 \end{pmatrix}$ and with initial conditions in $\mathcal{T} = \tau(\mathbb{R}^n)$. Moreover, the semi-trajectory z remains in $\mathcal{T}$.

3.2.2 Observer design with dissipative error system

Let us introduce a Luenberger observer with dynamic gain for (11). In order to make the error system dissipative, set $\mathcal{L}_\alpha(u) = \begin{pmatrix} bu \\ \alpha \end{pmatrix} \in \mathbb{R}^{n+1}$ for some positive constant α to be fixed later. The corresponding observer system is given by

$$\begin{cases} \dot{\varepsilon} = (\mathcal{A}(u) - \mathcal{L}_\alpha(u)\mathcal{C})\,\varepsilon \\ \dot{\hat{z}} = \mathcal{A}(u)\hat{z} + \mathcal{B}u - \mathcal{L}_\alpha(u)\mathcal{C}\varepsilon \end{cases} \tag{12}$$

where $z = \hat{z} - \varepsilon$ satisfies (11), $\hat{z}$ is the estimation of the state made by the observer system and ε is the error between the estimation of the state and the actual state of the system. Note that for all $u \in \mathbb{R}$,

$$\mathcal{A}(u) - \mathcal{L}_\alpha(u)\mathcal{C} = \begin{pmatrix} A & -bu \\ ub' & -\alpha \end{pmatrix} = \begin{pmatrix} A & -bu \\ ub' & 0 \end{pmatrix} - \alpha\mathcal{C}'\mathcal{C}. \tag{13}$$

It implies that the ε-subsystem of (12) is dissipative, that is, for all input $u \in C^0(\mathbb{R}_+, \mathbb{R})$, the solutions of (12) satisfy

$$\frac{\mathrm{d}|\varepsilon|^2}{\mathrm{d}t} = 2\varepsilon'\left(\mathcal{A}(u) - \mathcal{L}_\alpha(u)\mathcal{C}\right)\varepsilon - 2\alpha\varepsilon'\mathcal{C}'\mathcal{C}\varepsilon = -2\alpha|\mathcal{C}\varepsilon|^2 \leqslant 0. \tag{14}$$

This is the first key fact of the strategy applied below.

3.2.3 Feedback perturbation and closed-loop system

Because (A, b) is stabilizable, there exists $K \in \mathbb{R}^{1\times n}$ such that $A + bK$ is Hurwitz (in particular, (K, A) is detectable). Since A is skew-symmetric, its eigenvalues are purely imaginary. Hence, the Hautus lemmas for stabilizability (resp. detectability) and controllability (resp. observability) are equivalent. Therefore, (A, b) is controllable and (K, A) is observable.

With a separation principle in mind, a natural strategy for dynamic output feedback stabilization of (9) would be to combine the Luenberger observer (12) with the state feedback law $\phi : x \mapsto Kx$. However, it appears that this strategy fails to be applied due to the unobservability at the target. To overcome this difficulty, we rather consider a perturbed feedback law $\phi_\delta : x \mapsto Kx + \frac{\delta}{2}|x|^2$ for some positive constant δ to be fixed later. This is the second key fact of the strategy. For all $\delta > 0$, denote by $\mathcal{D}_\delta$ the basin of attraction of $0 \in \mathbb{R}^n$ of the vector field $\mathbb{R}^n \ni x \mapsto Ax + b\phi_\delta(x)$. Since the linearization of this vector field at 0 is $x \mapsto (A+bK)x$, it is locally asymptotically stable at 0 for all $\delta > 0$. As stated in the following lemma, the drawback of this perturbation is to pass from a globally stabilizing state feedback to a semi-globally stabilizing one.

Lemma 3.5. *For any compact set $\mathcal{K} \subset \mathbb{R}^n$, there exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$, $\mathcal{K} \subset \mathcal{D}_\delta$.*

Proof. Let $\rho > 0$ be such that $\mathcal{K} \subset B_{\mathbb{R}^n}(0, \rho)$. Since $A+bK$ is Hurwitz, there exists $P \in \mathbb{R}^{n\times n}$ positive definite such that $P(A+bK)+P'(A+bK)' < -2\mathrm{I}_{\mathbb{R}^n}$. Set $V : \mathbb{R}^n \ni x \mapsto x'Px$. Then, for all $x \in \mathcal{K}$,

$$\begin{aligned}\frac{\partial V}{\partial x}(x)(Ax + b\phi_\delta(x)) &= 2x'P(A + bK)x + \delta|x|^2 x'Pb\\ &\leqslant (-2 + \delta|x||Pb|)|x|^2\\ &\leqslant (-2 + \delta\rho|Pb|)|x|^2.\end{aligned}$$

Set $\delta_0 = \frac{1}{\rho|Pb|}$ and let $\delta \in (0, \delta_0)$. Then V is positive definite and

$$\frac{\partial V}{\partial x}(x)(Ax + b\phi_\delta(x)) < -|x|^2$$

for all $x \in \mathcal{K}$. Hence, $0 \in \mathbb{R}^n$ is a locally asymptotically (even exponentially) stable equilibrium point of the vector field $\mathbb{R}^n \ni x \mapsto Ax + b\phi_\delta(x)$ with basin of attraction containing $\mathcal{K}$. ■

Hence, for all compact set $\mathcal{K} \subset \mathbb{R}^n$ there exists $\delta_0 > 0$ such that if $\delta \in (0, \delta_0)$, then $\mathcal{K} \subset \mathcal{D}_\delta$. On system (11), we choose the feedback law

$$\lambda_\delta(z) = \begin{pmatrix} K & \delta \end{pmatrix} z, \tag{15}$$

which satisfies $\phi_\delta = \lambda_\delta \circ \tau$. The corresponding closed-loop system is given by

$$\begin{cases} \dot{\varepsilon} = (\mathcal{A}(\lambda_\delta(\hat{z})) - \mathcal{L}_\alpha(\lambda_\delta(\hat{z}))\mathcal{C})\,\varepsilon,\\ \dot{\hat{z}} = \mathcal{A}(\lambda_\delta(\hat{z}))\hat{z} + \mathcal{B}\lambda_\delta(\hat{z}) - \mathcal{L}_\alpha(\lambda_\delta(\hat{z}))\mathcal{C}\varepsilon. \end{cases} \tag{16}$$

By using $w = \hat{z}$ as the new dynamical system fed by the output y, we are now able to exhibit the coupled system that we intend to prove to give a solution to the semi-global dynamic output feedback stabilization problem of (9):

$$\begin{cases} \dot{x} = Ax + bu,\\ y = \dfrac{1}{2}|x|^2 \end{cases}, \qquad \begin{cases} \dot{\hat{z}} = \mathcal{A}(u)\hat{z} + \mathcal{B}u - \mathcal{L}_\alpha(u)\,(\mathcal{C}\hat{z} - y)\\ u = \lambda_\delta(\hat{z}). \end{cases} \tag{17}$$

It is now sufficient to prove the following theorem, which implies Theorem 3.3, in the next sections.

Theorem 3.6. *For all compact set $\mathcal{K} \times \widehat{\mathcal{K}} \subset \mathbb{R}^n \times \mathbb{R}^{n+1}$, there exist $\delta_0 > 0$ and $\alpha_0 > 0$ such that for all $\delta \in (0, \delta_0)$ and all $\alpha \in (\alpha_0, +\infty)$, $(0, 0) \in \mathbb{R}^n \times \mathbb{R}^{n+1}$ is a locally asymptotically stable equilibrium point of* (17) *with basin of attraction containing $\mathcal{K} \times \widehat{\mathcal{K}}$.*

3.2.4 Boundedness of trajectories

Since $\mathbb{R}^n \ni x \mapsto \frac{1}{2}|x|^2$ and ϕ_δ are locally Lipschitz continuous functions, according to the Cauchy-Lipschitz theorem, for any initial condition $(x_0, \hat{z}_0) \in \mathbb{R}^n \times \mathbb{R}^{n+1}$, there exists exactly one maximal solution $(x, \hat{z})$ of (17) such that $(x(0), \hat{z}(0)) = (x_0, \hat{z}_0)$. Before going into the proof of Theorem 3.6, we need to ensure the existence of global solutions.

Lemma 3.7. *For any compact set $\mathcal{K} \times \widehat{\mathcal{K}} \subset \mathbb{R}^n \times \mathbb{R}^{n+1}$, there exist $\delta_0 > 0$ and $\alpha_0 > 0$ such that for all $\delta \in (0, \delta_0)$ and all $\alpha \in (\alpha_0, +\infty)$,* (17) *has a unique global solution $(x, \hat{z})$ for each initial condition $(x_0, \hat{z}_0) \in \mathcal{K} \times \widehat{\mathcal{K}}$. Moreover, $(x, \hat{z})$ is bounded and $\bar{\hat{z}}_n$ remains in a compact subset of $\mathcal{D}_\delta$.*

Proof. Let $(x_0, \hat{z}_0) \in \mathcal{K} \times \widehat{\mathcal{K}}$ and $(x, \hat{z})$ be the corresponding maximal solution of (17). Set $z = \tau(x)$ and $\varepsilon = \hat{z} - z$, so that $(\varepsilon, \hat{z})$ is the maximal solution of (16) starting from $(\varepsilon_0, \hat{z}_0)$. Then, it is sufficient to prove that $(\varepsilon, \hat{z})$ is a global solution, $(\varepsilon, \hat{z})$ is bounded and $\bar{\hat{z}}_n$ remains in a compact subset of $\mathcal{D}_\delta$. According to (14), ε is bounded since $|\varepsilon|$ is non-increasing. Moreover, $\hat{z}_{n+1} = \varepsilon_{n+1} + \frac{1}{2}|\bar{z}_n|^2 = \varepsilon_{n+1} + \frac{1}{2}|\bar{\hat{z}}_n - \bar{\varepsilon}_n|^2$. Then, it remains to show that there exist $\delta_0 > 0$ and $\alpha_0 > 0$ such that for all $\delta \in (0, \delta_0)$ and all $\alpha \in (\alpha_0, +\infty)$, for all initial conditions $(\varepsilon_0, \hat{z}_0) \in \left(\widehat{\mathcal{K}} - \tau(\mathcal{K})\right) \times \widehat{\mathcal{K}}$, $\bar{\hat{z}}_n$ remains in a compact subset of $\mathcal{D}_\delta$.

Since $A + bK$ is Hurwitz, there exists $P \in \mathbb{R}^{n \times n}$ positive definite such that $P(A + bK) + P'(A + bK)' < -2\mathrm{I}_{\mathbb{R}^n}$. Then $V : \mathbb{R}^n \ni x \mapsto x'Px$ is a strict Lyapunov function for system (9) with feedback law ϕ. For all $r > 0$, set $D(r) = \{x \in \mathbb{R}^n \ : \ V(x) \leqslant r\}$. Let $\rho' > \rho > 0$ and $r' > r > 0$ be such that $B_{\mathbb{R}^{n+1}}(0, \rho)$ contains $\left(\widehat{\mathcal{K}} - \tau(\mathcal{K})\right)$ and $\widehat{\mathcal{K}}$ and $B_{\mathbb{R}^{n+1}}(0, \rho) \subset D(r) \subset D(r') \subset B_{\mathbb{R}^{n+1}}(0, \rho')$. According to Lemma 3.5, there exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$, $\mathcal{D}_\delta$ contains the closure of $B_{\mathbb{R}^n}(0, \rho')$. In the following, we show that there exists $\alpha_0 > 0$ such that, if $\alpha > \alpha_0$, then $\bar{\hat{z}}_n$ remains in $B_{\mathbb{R}^n}(0, \rho')$. Consider the vector fields over $\mathbb{R}^{n+1}$ given by

$$
\begin{aligned}
\mu_\delta^1(\hat{z}) &= \mathcal{A}(\phi_\delta(\bar{\hat{z}}_n))\hat{z} + \mathcal{B}\phi_\delta(\bar{\hat{z}}_n),\\
\mu_\delta^2(\hat{z}) &= (\mathcal{A}(\lambda_\delta(\hat{z})) - \mathcal{A}(\phi_\delta(\bar{\hat{z}}_n)))\hat{z} + \mathcal{B}(\lambda_\delta(\hat{z}) - \phi_\delta(\bar{\hat{z}}_n)),\\
\mu_{\delta,\alpha}^3(\varepsilon, \hat{z}) &= -\mathcal{L}_\alpha(\lambda_\delta(\hat{z}))\mathcal{C}\varepsilon,
\end{aligned}
$$

so that the solutions of (16) satisfy

$$
\dot{\hat{z}} = \mu_\delta^1(\hat{z}) + \mu_\delta^2(\hat{z}) + \mu_{\delta,\alpha}^3(\varepsilon, \hat{z}). \tag{18}
$$

In particular,

$$
\dot{\bar{\hat{z}}}_n = A\bar{\hat{z}}_n + \lambda_\delta(\hat{z})b - \lambda_\delta(\hat{z})\varepsilon_{n+1}b.
$$

By continuity of $\delta \mapsto \lambda_\delta$,

$$
\overline{M} := \sup_{\substack{\varepsilon, \hat{z} \in B_{\mathbb{R}^{n+1}}(0,\rho')\\ \delta \in [0,\delta_0]}} |A\bar{\hat{z}}_n + \lambda_\delta(\hat{z})b - \lambda_\delta(\hat{z})\varepsilon_{n+1}b| < \infty.
$$

Let $T_0 = \frac{\rho'-\rho}{M}$. Since $|\varepsilon|$ is non-increasing, any trajectory of (16) starting in $B_{\mathbb{R}^{n+1}}(0,\rho) \times B_{\mathbb{R}^{n+1}}(0,\rho)$ will be such that $\bar{\hat{z}}_n$ remains in $B_{\mathbb{R}^n}(0,\rho')$ over the time interval $[0,T_0]$. It remains to show that $\bar{\hat{z}}_n$ does not exit $B_{\mathbb{R}^n}(0,\rho')$ after time T_0.

Let $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ denotes the projection operator on the first n coordinates, that is, $\pi(\hat{z}) = \bar{\hat{z}}_n$. Then π is a left-inverse of τ: for all $x \in \mathbb{R}^n$, $\pi(\tau(x)) = x$. Note that $\mu_\delta^1(\hat{z}_1) = \mu_\delta^1(\hat{z}_2)$ if $\pi(\hat{z}_1) = \pi(\hat{z}_2)$. Then,

$$\underline{m} := - \max_{\substack{\bar{\hat{z}}_n \in \partial D(r') \\ \hat{z} \in B_{\mathbb{R}^{n+1}}(0,\rho')}} (L_{\mu_0^1} V \circ \pi)(\hat{z}) = - \max_{\substack{\pi(\hat{z}) \in \partial D(r') \\ \hat{z} \in B_{\mathbb{R}^{n+1}}(0,\rho')}} \frac{\partial V}{\partial x}(\pi(\hat{z}))\,(A+bK)\pi(\hat{z}) > 0.$$

Notice that $(\mu_\delta^1 - \mu_0^1 + \mu_\delta^2)(\hat{z}) = \delta \hat{z}_{n+1} \begin{pmatrix} b \\ b' \bar{\hat{z}}_n \end{pmatrix}$. Hence, without loss of generality, one can assume that $\delta_0 > 0$ is (small enough) such that for all $\delta \in (0,\delta_0)$,

$$\max_{B_{\mathbb{R}^{n+1}}(0,\rho')} |L_{\mu_\delta^1 - \mu_0^1 + \mu_\delta^2} V \circ \pi| \leqslant \frac{1}{3}\underline{m}.$$

Fix $\delta \in (0,\delta_0)$. Assume for the sake of contradiction that $\bar{\hat{z}}_n$ leaves $D(r')$ for the first time at $T_0' > T_0$. Then $\frac{\mathrm{d}}{\mathrm{d}t}_{|t=T_0'} V(\pi(\hat{z})) \geqslant 0$. We have

$$\begin{aligned}
0 &\leqslant \frac{\mathrm{d}}{\mathrm{d}t} V\left(\pi(\hat{z}(t))\right)\Big|_{t=T_0'} \\
&= (L_{\mu_0^1} V \circ \pi)(\hat{z}(T_0')) + (L_{\mu_\delta^1 - \mu_0^1 + \mu_\delta^2} V \circ \pi)(\hat{z}(T_0')) + \frac{\partial V \circ \pi}{\partial \hat{z}}(\hat{z}(T_0'))\mu_{\delta,\alpha}^3(\varepsilon(T_0'), \hat{z}(T_0')) \\
&\leqslant -\frac{2}{3}\underline{m} + \frac{\partial V \circ \pi}{\partial \hat{z}}(\hat{z}(T_0'))\mu_{\delta,\alpha}^3(\varepsilon(T_0'), \hat{z}(T_0'))
\end{aligned}$$

Now, we show that there exists $\alpha_0 > 0$ big enough such that for all $\alpha > \alpha_0$,

$$\frac{\partial V \circ \pi}{\partial \hat{z}}(\hat{z}(T_0'))\mu_{\delta,\alpha}^3(\varepsilon(T_0'), \hat{z}(T_0')) \leqslant \frac{1}{3}\underline{m}, \tag{19}$$

which contradicts $\underline{m} > 0$. By definition of $\mathcal{L}_\alpha$, π and $\mu_{\delta,\alpha}^3$,

$$\frac{\partial V \circ \pi}{\partial \hat{z}}(\hat{z})\mu_{\delta,\alpha}^3(\varepsilon, \hat{z}) = -\varepsilon_{n+1}\lambda_\delta(\hat{z})\frac{\partial V}{\partial x}(\pi(\hat{z}))b.$$

Let $Q = \max_{\substack{(\hat{z}_2,\hat{z}_3) \in \partial B_{\mathbb{R}^n}(0,\rho') \\ \varepsilon, \hat{z} \in B_{\mathbb{R}^{n+1}}(0,\rho')}} |\lambda_\delta(\hat{z})\frac{\partial V}{\partial x}(\pi(\hat{z}))b|$, so that $|\lambda_\delta(\hat{z}(T_0'))\frac{\partial V}{\partial x}(\pi(\hat{z}(T_0')))b| \leqslant Q$. Recall that

$$\dot{\varepsilon}_{n+1} = -\alpha\varepsilon_{n+1} + \lambda_\delta(\hat{z})b'\bar{\varepsilon}_n$$

and thus, for all $t \geqslant 0$,

$$\varepsilon_{n+1}(t) = e^{-\alpha t}\varepsilon_{n+1}(0) + \int_0^t e^{-\alpha(t-s)}\lambda_\delta(\hat{z}(s))b'\bar{\varepsilon}_n(s)\,\mathrm{d}s.$$

Moreover, $\varepsilon(t)$ and $\bar{\hat{z}}_n(t)$ are in $B_{\mathbb{R}^{n+1}}(0,\rho')$ for all $t \in [0, T_0']$ and

$$\lambda_\delta(\hat{z}) = \begin{pmatrix} K & \delta \end{pmatrix} \hat{z} = K\bar{\hat{z}}_n + \delta\left(\varepsilon_{n+1} + \frac{1}{2}|\bar{\hat{z}}_n - \bar{\varepsilon}_n|^2\right).$$

Hence,

$$|\lambda_\delta(\hat{z})| \leqslant \rho'\left(|K| + \delta(1+2\rho')\right).$$

As a consequence, for all $t \in [0, T_0']$,

$$|\varepsilon_{n+1}(t)| \leqslant \rho' \left(e^{-\alpha t} + \frac{\rho'^2 |b|}{\alpha} \left(|K| + \delta(1 + 2\rho') \right) \right).$$

Thus there exists $\alpha_0 > 0$ such that if $\alpha > \alpha_0$, then $|\varepsilon_{n+1}(T_0')| \leqslant \dfrac{m}{3Q}$. Fix $\alpha > \alpha_0$. Then (19) holds, which concludes the proof of the lemma. ■

3.2.5 Observability analysis

The following lemma is a crucial step of the proof of Theorem 3.3 that emphasizes the usefulness of the feedback perturbation described above. Indeed, one can easily see that its proof fails if $\delta = 0$ (since the matrix $\mathcal{Q}$ defined below is not invertible in this case).

Lemma 3.8. *Let $(z_0, \hat{z}_0) \in (\mathcal{T} \times \mathbb{R}^{n+1}) \setminus \{(0, 0)\}$. Let $(\varepsilon, \hat{z})$ be the semi-trajectory of (12) with initial condition $(\hat{z}_0 - z_0, \hat{z}_0)$. Then, for all $T > 0$, (11) is observable in time T for the input $u = \lambda_\delta(\hat{z})$.*

Proof. Let $\omega_0 \in \ker(\mathcal{C}) \setminus \{0\}$, and consider ω a solution of the dynamical system

$$\dot{\omega} = \mathcal{A}(\lambda_\delta(\hat{z}))\omega \tag{20}$$

with initial condition ω_0. To prove the result, it is sufficient to show that $\mathcal{C}\omega$ has a non-zero derivative of some order at $t = 0$ if $(\varepsilon_0, \hat{z}_0) \neq (0, 0)$. Indeed, it implies that for all initial conditions $z_0 \neq \tilde{z}_0$ in $\mathbb{R}^{n+1}$, if z (resp. $\tilde{z}$) is the solution of (11) with initial condition z_0 (resp. $\tilde{z}_0$), then $\omega = z - \tilde{z}$ is a solution to (20) starting at $\omega_0 \neq 0$ and $\mathcal{C}\omega$ is not constantly equal to zero on any time interval $[0, T] \subset \mathbb{R}_+$. We prove this fact by contradiction: assume that

$$\mathcal{C}\omega^{(k)}(0) = \omega_{n+1}^{(k)}(0) = 0 \qquad \forall k \in \mathbb{N}, \tag{21}$$

for some $\omega(0) \neq 0$, and prove that $(z_0, \hat{z}_0) = (0, 0)$. Let $u = \lambda_\delta(\hat{z})$. Then $\dot{\omega}_{n+1} = ub'\bar{\omega}_n$ and $\dot{\bar{\omega}}_n = A\bar{\omega}_n$. Hence

$$0 = \omega_{n+1}^{(k+1)}(0) = \sum_{i=0}^{k} \binom{k}{i} u^{(i)}(0) b' A^{k-i} \bar{\omega}_n(0) \tag{22}$$

for all $k \in \mathbb{N}$, where $\binom{k}{i}$ denote binomial coefficients. The proof goes through the following three steps.

Step 1: Show that $u^{(k)}(0) = 0$ for all $k \in \mathbb{N}$. Let $p \in \mathbb{N}$ be the smallest integer such that $u^{(p)}(0) \neq 0$ and look for a contradiction. Equation (22) yields

$$\sum_{i=0}^{k} \binom{p+k}{p+i} u^{(p+i)}(0) b' A^{k-i} \bar{\omega}_n(0) = 0 \tag{23}$$

for all $k \in \mathbb{N}$. Since (A, b) is controllable and $\bar{\omega}_n(0) \neq 0$, there exists $q \in \{0, \ldots, n\}$ such that $b'A^q\bar{\omega}_n(0) \neq 0$ and $b'A^i\bar{\omega}_n(0) = 0$ for all $i \in \{0, \ldots, q-1\}$. Then

$$0 = \sum_{i=0}^{q} \binom{p+q}{p+i} u^{(p+i)}(0) b' A^{q-i} \bar{\omega}_n(0) = \binom{p+q}{p} u^{(p)}(0) b' A^q \bar{\omega}_n(0). \tag{24}$$

which is a contradiction.

Step 2: Find $\mathcal{Q} \in \mathbb{R}^{(n+2)\times(n+2)}$ (invertible) such that $\mathcal{Q}\begin{pmatrix} \hat{z}(0) \\ \varepsilon_{n+1}(0) \end{pmatrix} = 0$. For all $k \in \mathbb{N}$,

$$0 = u^{(k)}(0) = \begin{pmatrix} K & \delta \end{pmatrix} \hat{z}^{(k)}(0).$$

Moreover,

$$\begin{pmatrix} \dot{\bar{\hat{z}}}_n \\ \dot{\hat{z}}_{n+1} \\ \dot{\varepsilon}_{n+1} \end{pmatrix} = \begin{pmatrix} A & -bu & 0 \\ b'u & 0 & -\alpha \\ 0 & 0 & -\alpha \end{pmatrix} \begin{pmatrix} \bar{\hat{z}}_n \\ \hat{z}_{n+1} \\ \varepsilon_{n+1} \end{pmatrix} + u \begin{pmatrix} b \\ 0 \\ b'\bar{\varepsilon}_n \end{pmatrix}.$$

Hence, for all $k \geqslant 1$, $\bar{\hat{z}}_n^{(k)}(0) = A^k \bar{\hat{z}}_n(0)$ and $\hat{z}_{n+1}^{(k)}(0) = \varepsilon_{n+1}^{(k)}(0) = (-\alpha)^k \varepsilon_{n+1}(0)$. Thus $\begin{pmatrix} KA^k & \delta(-\alpha)^k \end{pmatrix} \begin{pmatrix} \bar{\hat{z}}_n(0) \\ \varepsilon_{n+1}(0) \end{pmatrix} = 0$ for all $k \geqslant 1$. By setting

$$\mathcal{Q} = \begin{pmatrix} K & \delta & 0 \\ KA & 0 & -\delta\alpha \\ \vdots & \vdots & \vdots \\ KA^{n+1} & 0 & \delta(-\alpha)^{n+1} \end{pmatrix} \tag{25}$$

we get that $\mathcal{Q}\begin{pmatrix} \hat{z}(0) \\ \varepsilon_{n+1}(0) \end{pmatrix} = 0$.

Step 3: Conclusion. In Appendix A, we check that $\mathcal{Q}$ is invertible. Hence, $\hat{z}(0) = 0$ and $\varepsilon_{n+1}(0) = 0$. Thus, $\frac{1}{2}|\bar{z}_n(0)|^2 = z_{n+1}(0) = \hat{z}_{n+1}(0) - \varepsilon_{n+1}(0) = 0$ *i.e.* $(z_0, \hat{z}_0) = (0, 0)$ which is a contradiction. ■

On the basis of Lemmas 3.7 and 3.8, we are now in position to prove Theorem 3.6. Let $\mathcal{K} \times \widehat{\mathcal{K}} \subset \mathbb{R}^n \times \mathbb{R}^{n+1}$ be a compact set, and $\delta_0 > 0$ and $\alpha_0 > 0$ be as in Lemma 3.7. Fix $\delta \in (0, \delta_0)$ and $\alpha \in (\alpha_0, +\infty)$. Let $(x_0, \hat{z}_0) \in \mathcal{K} \times \widehat{\mathcal{K}}$ and $(x, \hat{z})$ be the corresponding solution of (17). Set $z = \tau(x)$, $\varepsilon = \hat{z} - z$ so that $(\varepsilon, \hat{z})$ is the solution of (16) starting from $(\varepsilon_0, \hat{z}_0)$, $\varepsilon_0 = \hat{z}_0 - \tau(x_0)$. We need to show the two following statements:

1. (Stability) $(0, 0)$ is a stable equilibrium point of (17),
2. (Attractivity) and its basin of attraction contains $\mathcal{K} \times \widehat{\mathcal{K}}$.

We prove the former in Section 3.2.6 and the latter in Section 3.2.7.

3.2.6 Stability

Let $R > 0$. We seek $r > 0$ such that, if $|x_0|, |\hat{z}_0| \leqslant r$, then $|x(t)|, |\hat{z}(t)| \leqslant R$ for all $t \in \mathbb{R}_+$. We have

$$\begin{aligned} \dot{x} &= Ax + b\lambda_\delta(\hat{z}) \\ &= Ax + b\lambda_\delta(\tau(x) + \varepsilon) \\ &= Ax + b\phi_\delta(x) + b\begin{pmatrix} K & \delta \end{pmatrix}\varepsilon. \end{aligned}$$

Fix $\eta > 0$ such that $R - \eta\sqrt{1 + \frac{\eta^2}{2}} > 0$. Since $x \mapsto Ax + b\phi_\delta(x)$ is locally asymptotically stable, there exists a positive constant $r_\varepsilon \leqslant R - \eta\sqrt{1 + \frac{\eta^2}{2}}$ such that, if $|\varepsilon(t)| \leqslant r_\varepsilon$ for all $t \in \mathbb{R}_+$, then

$|x(t)| \leqslant \eta$ for all $t \in \mathbb{R}_+$. Let $r > 0$ be such that $r + r\sqrt{1 + \frac{r^2}{2}} \leqslant r_\varepsilon$. Assume that $|x_0|, |\hat{z}_0| \leqslant r$. Then,

$$|\varepsilon_0| \leqslant |\hat{z}_0| + |\tau(x_0)| = |\hat{z}_0| + |x_0|\sqrt{1 + \frac{|x_0|^2}{2}} \leqslant r + r\sqrt{1 + \frac{r^2}{2}} \leqslant r_\varepsilon.$$

According to (14), $|\varepsilon|$ is non-increasing. Hence, $|x| \leqslant \eta \leqslant R$ and

$$|\hat{z}| \leqslant |\tau(x)| + |\varepsilon| \leqslant \eta\sqrt{1 + \frac{\eta^2}{2}} + r_\varepsilon \leqslant R.$$

3.2.7 Attractivity

According to (14), $\frac{\mathrm{d}|\varepsilon|^2}{\mathrm{d}t} = -2\alpha|\mathcal{C}\varepsilon|^2$. According to LaSalle's invariance principle, the ω-limit set of ε is the largest invariant subset of $\ker \mathcal{C}$. Since ε satisfies (20) on this set, Lemma 3.8 guarantees that either $\varepsilon \equiv 0$, or $(\varepsilon_0, \hat{z}_0) = (0, 0)$, which also implies $\varepsilon \equiv 0$. Therefore, the ω-limit set of ε reduces to $\{0\}$, *i.e.*, $\varepsilon \to 0$.

Since $\hat{z}_{n+1} = \varepsilon_{n+1} + \frac{1}{2}|\bar{\hat{z}}_n - \bar{\varepsilon}_n|^2$, it remains to prove that $\bar{\hat{z}}_n \to 0$. First, notice that

$$|\mu_\delta^2(\hat{z})| = |\lambda_\delta(\hat{z}) - \phi_\delta(\hat{z})|\sqrt{|b|^2 + |b'\bar{\hat{z}}_n|^2}$$

and

$$|\mu_{\delta,\alpha}^3(\varepsilon, \hat{z})| = \sqrt{\alpha^2 + |b|^2\lambda_\delta(\hat{z})^2}|\mathcal{C}\varepsilon|.$$

Since $C\varepsilon \to 0$ and $\hat{z}$ is bounded, $|\mu_{\delta,\alpha}^3(\varepsilon, \hat{z})| \to 0$. Likewise,

$$\begin{aligned}
\lambda_\delta(\hat{z}) - \phi_\delta(\hat{z}) &= \delta\left(\hat{z}_{n+1} - \frac{1}{2}|\bar{\hat{z}}_n|^2\right) \\
&= \delta\left(\varepsilon_{n+1} + z_{n+1} - \frac{1}{2}|\bar{\varepsilon}_n|^2 - \frac{1}{2}|\bar{z}_n|^2 + \bar{\varepsilon}_n'\bar{z}_n\right) \\
&= \delta\left(\varepsilon_{n+1} - \frac{1}{2}|\bar{\varepsilon}_n|^2 + \bar{\varepsilon}_n'\bar{z}_n\right).
\end{aligned}$$

Since $\varepsilon \to 0$ and z is bounded, $\mu_\delta^2(\hat{z}) \to 0$.

According to the converse Lyapunov theorem [38, Theorem 1], there exists a strict proper Lyapunov function V_δ for system (9) with feedback law $\phi_\delta : x \mapsto Kx + \frac{\delta}{2}|x|^2$ over the basin of attraction $\mathcal{D}_\delta$. For all $r > 0$, set $D(r) = \{x \in \mathcal{D}_\delta \ : \ V_\delta(x) \leqslant r\}$. In order to prove that $\bar{\hat{z}}_n \to 0$, we show that for all $r > 0$, there exists $T(r) \geqslant 0$ such that $\bar{\hat{z}}_n(t) \in D(r)$ for all $t \geqslant T(r)$. According to Lemma 3.7, there exists a compact set $\mathcal{K} \subset \mathcal{D}_\delta$ such that $\bar{\hat{z}}_n \in \mathcal{K}$. If $r > 0$ is such that $\mathcal{K} \subset D(r)$ then $T(r) = 0$ satisfies the statement. Let $0 < r < R$ be such that $\mathcal{K} \not\subset D(r)$ and $\mathcal{K} \subset D(R)$, then

$$\bar{m} := -\max_{D(R)\setminus D(r)} L_{\phi_\delta} V_\delta > 0.$$

Since $|g(\hat{z}(t))| \to 0$ and $|h(\varepsilon(t), \hat{z}(t))| \to 0$, there exists $T_1(r) > 0$ such that for all $t \geqslant T_1(r)$, if $\bar{\hat{z}}_n(t) \notin D(r)$, then

$$\frac{\mathrm{d}}{\mathrm{d}t} V_\delta(\bar{\hat{z}}_n) < -\frac{\bar{m}}{2}.$$

First, this implies that if $\pi(\hat{z}(t)) \in D(r)$ for some $t \geqslant T_1(r)$, then $\pi(\hat{z}(s)) \in D(r)$ for all $s \geqslant t$. Second, for all $t \geqslant 0$,

$$\begin{aligned}
V_\delta(\bar{\hat{z}}_n(T_1(r) + t)) &= V_\delta(\bar{\hat{z}}_n(T_1(r))) + \int_0^t \frac{\mathrm{d}}{\mathrm{d}t} V_\delta(\bar{\hat{z}}_n(T_1(r) + s))\,\mathrm{d}s \\
&\leqslant R - \frac{\bar{m}}{2}t \qquad \text{while } \bar{\hat{z}}_n(T_1(r) + t) \notin D(r).
\end{aligned}$$

Set $T_2(r) = \frac{2R-r}{\bar{m}}$ and $T(r) = T_1(r) + T_2(r)$. Then for all $t \geqslant T(r)$, $\bar{\hat{z}}_n(t) \in D(r)$, which concludes the proof of convergence, and therefore the proof of Theorem 3.6.

4 An infinite-dimensional perspective

Guided by the illustrative example of Section 3, we aim to provide more general results, based on the same two principles: embedding into a dissipative system, and feedback perturbation. The embedding strategy used in Section 3.2.1 appears to be specific to this example, and hardly generalizable, since it relies mostly on the form of the observation map. A different strategy must be found. In [11], the authors introduce a technique for the synthesis of observers for nonlinear systems. The method is based on representation theory, and embedding into bilinear unitary systems. It is far more general than the embedding found in Section 3.2.1. The price to pay is that the observer system can be infinite-dimensional. In this section, we show how to use this strategy in the context of dynamic output feedback stabilization. After exhibiting some general results when such an embedding exists, we investigate a case of systems with linear conservative dynamics and nonlinear observation maps.

Notations: For all Hilbert space X, denote by $\langle \cdot, \cdot \rangle_X$ the inner product over X and $\|\cdot\|_X$ the induced norm. If Y is also a Hilbert space, then $\mathscr{L}(X, Y)$ denotes the space of bounded linear maps from X to Y, and $\mathscr{L}(X) = \mathscr{L}(X, X)$. We identify the Hilbert spaces with their dual spaces via the canonical isometry, so that the adjoint of $\mathcal{C} \in \mathscr{L}(X, Y)$, denoted by $\mathcal{C}^*$, lies in $\mathscr{L}(Y, X)$. Let us recall the characterization of the strong and weak topologies on X. A sequence $(x_n)_{n\geqslant 0} \in X^{\mathbb{N}}$ is said to be strongly convergent to some $x^\star \in X$ if $\|x_n - x^\star\|_X \to 0$ as $n \to +\infty$, and we shall write $x_n \to x^\star$ as $n \to +\infty$. It is said to be weakly convergent to $x^\star$ if $\langle x_n - x^\star, \psi \rangle_X \to 0$ as $n \to +\infty$ for all $\psi \in X$, and we shall write $x_n \xrightharpoonup{w} x^\star$ as $n \to +\infty$. The strong topology on X is finer than the weak topology (see, *e.g.*, [7] for more properties on these usual topologies).

4.1 Embedding into infinite-dimensional unitary systems

4.1.1 Embedding into unitary systems and observer design

Let X be a Hilbert space and $\mathcal{D}$ be a dense subspace of X. For all $u \in \mathbb{R}^p$, let $\mathcal{A}(u) : \mathcal{D} \to X$ be the skew-adjoint generator of a strongly continuous unitary group on X and $\mathcal{C} \in \mathscr{L}(X, \mathbb{C}^{\mathfrak{m}})$ for some positive integer $\mathfrak{m}$. Let $u \in C^1(\mathbb{R}_+, \mathbb{R}^p)$ and $z_0 \in X$. Consider the non-autonomous linear abstract Cauchy problem with measured output

$$\begin{cases} \dot{z} = \mathcal{A}(u(t))z \\ z(0) = z_0 \end{cases} \qquad \mathfrak{y} = \mathcal{C}z. \tag{26}$$

According to [28, Chapter 5, Theorem 4.8], the family $(\mathcal{A}(u(t)))_{t\in\mathbb{R}_+}$ is the generator of a unique evolution system on X that we denote by $(\mathbb{T}_t(\cdot, u))_{t\in\mathbb{R}_+}$. For any $z_0 \in X$, (26) admits a unique solution $z \in C^0(\mathbb{R}_+, X)$ given by $z(t) = \mathbb{T}_t(z_0, u)$ for all $t \in \mathbb{R}_+$. Moreover, if $z_0 \in \mathcal{D}$, then $z \in C^0(\mathbb{R}_+, \mathcal{D}) \cap C^1(\mathbb{R}_+, X)$. The reader may refer to [28, Chapter 5], [15, Chapter VI.9] or [19] for more details on the evolution equations theory.

For such systems, a Luenberger observer with constant gain $\alpha > 0$ can be built as follows:

$$\begin{cases} \dot{\hat{z}} = \mathcal{A}(u(t))\hat{z} - \alpha\mathcal{C}^*(\mathcal{C}\hat{z} - \mathfrak{y}) \\ \hat{z}(0) = \hat{z}_0 \in X. \end{cases} \tag{27}$$

Set $\varepsilon = \hat{z} - z$ and $\varepsilon_0 = \hat{z}_0 - z_0$. From now on, $\hat{z}$ represents the state estimation made by the observer system and ε the error between this estimation and the actual state of the system. Then $\hat{z}$ satisfies (27) if and only if ε satisfies

$$\begin{cases} \dot{\varepsilon} = (\mathcal{A}(u(t)) - \alpha\mathcal{C}^*\mathcal{C})\varepsilon \\ \varepsilon(0) = \varepsilon_0. \end{cases} \tag{28}$$

Since $\mathcal{C} \in \mathscr{L}(X, \mathbb{C}^{\mathfrak{m}})$, [28, Chapter 5, Theorem 2.3] claims that $(\mathcal{A}(t) - \alpha\mathcal{C}^*\mathcal{C})_{t\geqslant 0}$ is also a stable family of generators of strongly continuous semigroups, and generates an evolution system on X denoted by $(\mathbb{S}_t(\cdot, u))_{t\in\mathbb{R}_+}$. Then, systems (27) and (28) have respectively a unique solution $\hat{z}$ and ε in $C^0(\mathbb{R}_+, X)$. Moreover, $\hat{z}(t) = \mathbb{T}_t(z_0, u) + \mathbb{S}_t(\varepsilon_0, u)$ and $\varepsilon(t) = \mathbb{S}_t(\varepsilon_0, u)$ for all $t \in \mathbb{R}_+$. If $(\hat{z}_0, \varepsilon_0) \in \mathcal{D}^2$, then $\hat{z}$, $\varepsilon \in C^0(\mathbb{R}_+, \mathcal{D}) \cap C^1(\mathbb{R}_+, X)$.

This infinite-dimensional Luenberger observer has been investigated in [11] (see also [8]), in which it is proved that $\varepsilon(t) \xrightarrow{w} 0$ as t goes to infinity if u is a *regularly persistent input*. Our goal is to embed the original system (1) into a unitary system, and to use this observer design in the context of dynamic output feedback stabilization.

Definition 4.1 (Embedding)**.** An injective map $\tau : \mathbb{R}^n \mapsto X$ is said to be an embedding[4] of (1) into the unitary system (26) if there exists $\mathfrak{h} : \mathbb{R}^m \to \mathbb{C}^{\mathfrak{m}}$ such that the following diagram is commutative for all $t \in \mathbb{R}_+$ and all $u \in C^1(\mathbb{R}_+, \mathbb{R}^p)$:

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "\varphi_t(\cdot,u)"] \arrow[d, "\tau"'] & \mathbb{R}^n \arrow[r, "h"] \arrow[d, "\tau"] & \mathbb{R}^m \arrow[r, "\mathfrak{h}"] & \mathbb{C}^{\mathfrak{m}} \\ X \arrow[r, "\mathbb{T}_t(\tau(\cdot),u)"'] & X \arrow[urr, "\mathcal{C}"'] & & \end{tikzcd} \tag{29}$$

i.e., for all $x_0 \in \mathbb{R}^n$, $\tau(\varphi_t(x_0, u)) = \mathbb{T}_t(\tau(x_0), u)$ and $\mathfrak{h}(h(x_0)) = \mathcal{C}\tau(x_0)$.

Here, the map $\mathfrak{h}$ is a degree of freedom that may be chosen to find an embedding of (1) into (26). Let $u \in C^1(\mathbb{R}_+, \mathbb{R}^p)$, z_0, $\varepsilon_0 \in \mathcal{D}$, $z(t) = \mathbb{T}_t(\hat{z}_0, u)$ and $\varepsilon(t) = \mathbb{S}_t(\varepsilon_0, u)$ for all $t \in \mathbb{R}_+$. For all $t \in \mathbb{R}_+$, $\mathcal{A}(u(t))$ is skew-adjoint, hence

$$\frac{\mathrm{d}\,\|z\|_X^2}{\mathrm{d}t}(t) = 2\Re\langle \mathcal{A}(u(t))z(t), z(t)\rangle_X = 0, \tag{30}$$

$$\frac{\mathrm{d}\,\|\varepsilon\|_X^2}{\mathrm{d}t}(t) = 2\Re\langle \mathcal{A}(u(t))\varepsilon(t), \varepsilon(t)\rangle_X - 2\alpha\Re\langle \mathcal{C}^*\mathcal{C}\varepsilon(t), \varepsilon(t)\rangle_X = -2\alpha\,\|\mathcal{C}\varepsilon(t)\|_X^2 \leqslant 0. \tag{31}$$

Thus $\|z\|_X$ is constant and $\|\varepsilon\|_X$ is non-increasing. Moreover, $t \mapsto \|\mathcal{C}\varepsilon(t)\|_X^2$ is non-negative, integrable over $\mathbb{R}_+$ (by (31)), and has bounded derivative (since $\mathcal{A}(u(t))$ is skew-adjoint). Hence, according to Barbalat's lemma, $\mathcal{C}\varepsilon(t) \to 0$ as $t \to +\infty$. Inequality (31) is similar to (14), and will be a key argument to achieve the dynamic output feedback stabilization.

4.1.2 Embedding inversion: from the embedded system's weak observer to the original system's observer

In Section 3, a crucial argument was the existence of a left-inverse π to the embedding τ. Now, X being infinite-dimensional, we must precise the notion of left-inverse, and, moreover, the convergence of the observer $\hat{z}$ to the embedded state z will hold only in the weak topology of X, namely, $\varepsilon \xrightarrow{w} 0$. This is an important issue, which causes difficulties in achieving output

[4]This definition does not coincides with the usual notion of embedding in differential topology.

feedback stabilization. However, in this section, we show that if the original state x remains bounded, and if the embedding τ is injective and analytic, then $\hat{x} = \pi(\hat{z})$ is actually an observer of x in the usual topology of $\mathbb{R}^n$, namely, $\hat{x} - x \to 0$. This is summarized in Corollary 4.6, which is an important result of the paper.

Definition 4.2 (Strong left-inverse)**.** Let $(X, \|\cdot\|_X)$ be a normed vector space, $\mathcal{K} \subset \mathbb{R}^n$ and $\tau : \mathbb{R}^n \to X$. A map $\pi : X \to \mathcal{K}$ is called a *strong left-inverse* of τ on $\mathcal{K}$ if and only if there exists a class $\mathcal{K}_\infty$ function[5] ρ^* and $Q \in \mathscr{L}(X, \mathbb{C}^q)$ for some a positive integer q such that, for all $(x, \xi) \in \mathcal{K} \times X$,

$$|\pi(\xi) - x| \leqslant \rho^*(|Q(\xi - \tau(x))|). \tag{32}$$

Remark 4.3. If π is a strong left-inverse of τ on $\mathcal{K}$, then (32) implies that π is also a left-inverse in the usual sense: for all $x \in \mathcal{K}$, $\pi(\tau(x)) = x$. In particular, τ is injective over $\mathcal{K}$.

The reason for which we look for a strong left-inverse of τ is the following lemma, which follows directly from (32) and the fact that $Q \in \mathscr{L}(X, \mathbb{C}^q)$.

Lemma 4.4. *Let $(X, \|\cdot\|_X)$ be a normed vector space, $\mathcal{K} \subset \mathbb{R}^n$ and $\tau : \mathbb{R}^n \to X$. Let $\pi : X \to \mathcal{K}$ be a strong left-inverse of τ on $\mathcal{K}$. Let $(x_n)_{n\in\mathbb{N}}$ and $(\xi_n)_{n\in\mathbb{N}}$ be two sequences in $\mathcal{K}$ and X, respectively. If $\xi_n - \tau(x_n) \xrightarrow{w} 0$ as n goes to infinity, then $|\pi(\xi_n) - x_n| \to 0$ as n goes to infinity.*

This justifies the denomination of *strong* left-inverse, in the sense that it allows to pass from weak convergence in the infinite-dimensional space X to (usual) convergence in the finite-dimensional space $\mathbb{R}^n$. The following theorem states sufficient conditions for the existence of a strong left-inverse.

Theorem 4.5. *Let X be a Hilbert space, $\tau : \mathbb{R}^n \to X$ be an analytic map and $\mathcal{K} \subset \mathbb{R}^n$ be a compact set . If $\tau|_{\mathcal{K}}$ is injective, then τ has a continuous strong left-inverse on $\mathcal{K}$.*

Proof. Let $(e_k)_{k\in\mathbb{N}}$ be a Hilbert basis of X. For all $i \in \mathbb{N}$, let

$$E_i = \{(x_a, x_b) \in \mathbb{R}^n \times \mathbb{R}^n \ : \ \forall k \in \{0, \dots, i-1\},\ \langle \tau(x_a) - \tau(x_b), e_k \rangle_X = 0\}.$$

Then $(E_i)_{i\in\mathbb{N}}$ is a non-increasing family of analytic sets. According to [25, Chapter 5, Corollary 1], $(E_i \cap \mathcal{K}^2)_{i\in\mathbb{N}}$ is stationary, *i.e.*, there exists $q \in \mathbb{N}$ such that $E_q \cap \mathcal{K}^2 = E_i \cap \mathcal{K}^2$ for all $i \geqslant q$. Hence,

$$\begin{aligned} E_q \cap \mathcal{K}^2 &= \bigcap_{k\in\mathbb{N}} E_k \cap \mathcal{K}^2 \\ &= \{(x_a, x_b) \in \mathcal{K}^2 \ : \ \tau(x_a) = \tau(x_b)\} && \text{(since } (e_k)_{k\in\mathbb{N}} \text{ is a Hilbert basis of } X\text{)} \\ &= \{(x_a, x_a) \ : \ x_a \in \mathcal{K}\}. && \text{(since } \tau \text{ is injective on } \mathcal{K}\text{)} \end{aligned}$$

Let $Q : X \ni \xi \mapsto (\langle \xi, e_k \rangle_X)_{k\in\{0,\dots,q-1\}} \in \mathbb{C}^q$ and $\tilde{\tau} = Q \circ \tau$. Then $\tilde{\tau}$ is continuous and injective on $\mathcal{K}$. Indeed, for all $(x_a, x_b) \in \mathcal{K}^2$, if $\tilde{\tau}(x_a) = \tilde{\tau}(x_b)$, then $(x_a, x_b) \in E_q \cap \mathcal{K}^2$ which yields $x_a = x_b$. Hence, combining [6, Lemma 6] and [3, Theorem 1], there exists a continuous map $\tilde{\pi} : \mathbb{C}^q \to \mathcal{K}$ and a class $\mathcal{K}_\infty$ function ρ^* such that for all $(x, \mathfrak{z}) \in \mathcal{K} \times \mathbb{C}^q$, $|\tilde{\pi}(\mathfrak{z}) - x| \leqslant \rho^*(|\mathfrak{z} - \tilde{\tau}(x)|)$. Set $\pi = \tilde{\pi} \circ Q$. Then π is continuous and for all $(x, \xi) \in \mathcal{K} \times X$,

$$|\pi(\xi) - x| \leqslant \rho^*(|Q(\xi) - \tilde{\tau}(x)|) = \rho^*(|Q(\xi - \tau(x))|).$$

■

[5] A class $\mathcal{K}_\infty$ function is a continuous function $\rho^* : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\rho^*(0) = 0$, ρ^* is strictly increasing and tends to infinity at infinity.

Applying Theorem 4.5, then Lemma 4.4, we get the following result in our context.

Corollary 4.6. *Let $\tau : \mathbb{R}^n \to X$ be an analytic embedding of* (1) *into the unitary system* (26) *and $\mathcal{K}$ be a compact subset of $\mathbb{R}^n$. Then τ has a continu ous strong left-inverse π on $\mathcal{K}$.*

Let $x_0 \in \mathcal{K}$, $\hat{z}_0 \in X$ and $u \in C^1(\mathbb{R}_+, \mathbb{R}^p)$. Denote by x and $\hat{z}$ the corresponding solutions of (1) *and* (27)*, respectively. Set $\hat{x} = \pi(\hat{z})$. Assume that $x(t) \in \mathcal{K}$ for all $t \in \mathbb{R}_+$. If $\hat{z} - \tau(x) \overset{w}{\rightharpoonup} 0$, then $\hat{x} - x \to 0$.*

Remark 4.7. Beyond the problem of output feedback stabilization, Corollary 4.6 may be used in the context of observer design. In [11], after embedding the original finite-dimensional system into an infinite-dimensional unitary system, the authors investigate only the convergence of the infinite-dimensional observer. Corollary 4.6 states that if the infinite-dimensional observer converges and if the original system's state trajectory remains bounded, then an observer can be built for the original system, by using a strong left-inverse of the embedding.

4.1.3 Feedback perturbation and closed-loop system

In order to set up a separation principle to solve the dynamic output feedback stabilization problem of (1), let us assume that Condition 2.7 (semi-global) and the following assumption are satisfied.

Assumption 4.8 (Existence of an embedding)**.** System (1) admits an analytic embedding into the unitary system (26).

Let $\mathcal{K}$ be a compact subset of $\mathbb{R}^n$. Denote by ϕ a locally asymptotically stabilizing state feedback of (1) with basin of attraction containing $\mathcal{K}$ and by τ an embedding of (1) into (26). According to Theorem 4.5, there exists $\pi : X \to \mathcal{K}$, a strong left-inverse of τ on $\mathcal{K}$. Then, a natural way to build a dynamic output feedback would be to combine (1)-(27) with the control input $u = \phi(\pi(\hat{z}))$, and to ensure that the state x of (1) remains in $\mathcal{K}$. However, due to the unobservability of the original system at the target, we propose, as in Section 3.2.3, to add a perturbation to this feedback law. In [11], the convergence of the error system (28) to 0, when it holds, is only in the weak topology of X. Therefore, the perturbation added to the feedback law must be chosen to vanish when the observer state $\hat{z}$ of (27) tends towards $\tau(0)$ in the weak topology. For this reason, let us define a weak norm on X.

Definition 4.9 (Weak norm)**.** Let $(e_k)_{k\in\mathbb{Z}}$ be a Hilbert basis of X. For all $\xi \in X$, set

$$\mathcal{N}(\xi) = \sqrt{\sum_{k\in\mathbb{Z}} \frac{|\langle \xi, e_k\rangle_X|^2}{k^2+1}}.$$

Then $\mathcal{N}$ defines a norm, we call the *weak norm*, on X.

Note that $\mathcal{N}$ is not equivalent to $\|\cdot\|_X$, but satisfies $\mathcal{N}(\cdot) \leqslant \nu \|\cdot\|_X$ with $\nu = \sqrt{\sum_{k\in\mathbb{Z}} \frac{1}{k^2+1}} < +\infty$. Moreover, $\mathcal{N}$ induces a metric on bounded sets of X endowed with the weak topology. More precisely, for any bounded sequence $(\xi_n)_{n\in\mathbb{N}}$ in X, $\mathcal{N}(\xi_n) \to 0$ as n goes to infinity if and only if $\xi_n \overset{w}{\rightharpoonup} 0$ as n goes to infinity. Now, for some positive constant δ to be fixed (small enough) later, we can add the perturbation $\hat{z} \mapsto \delta\mathcal{N}^2(\hat{z} - \tau(0))$ to the feedback law, and obtain the following full coupled system:

$$\begin{cases} \dot{x} = f(x, u) \\ y = h(x) \end{cases}, \qquad \begin{cases} \dot{\hat{z}} = \mathcal{A}(u)\hat{z} - \alpha\mathcal{C}^*(\mathcal{C}\hat{z} - \mathfrak{h}(y)) \\ u = \phi(\pi(\hat{z})) + \delta\mathcal{N}^2(\hat{z} - \tau(0)). \end{cases} \tag{33}$$

Since X is infinite-dimensional and $\hat{z}$ lies in X, Definition 2.1 of semi-global dynamic output feedback stabilization must be revised. Indeed, (33) does not exactly fit the form of (2).

Definition 4.10 (Infinite-dimensional embedding-based dynamic output feedback stabilizability)**.** Let $\mathcal{K} \subset \mathbb{R}^n$ be a compact set. System (1) is said to be *stabilizable over $\mathcal{K}$ by means of an infinite-dimensional embedding-based dynamic output feedback* if and only if the following holds.

There exists an embedding τ of (1) into (26), a strong left-inverse π of τ on $\mathcal{K}$, a map $\phi : \mathbb{R}^n \to \mathbb{R}^p$, two positive constants α and δ and a compact set $\widehat{\mathcal{K}} \subset \mathbb{R}^n$ such that:

(i) For all initial condition $(x_0, \hat{z}_0) \in \mathcal{K} \times \tau(\widehat{\mathcal{K}})$, (33) has at least one solution in X over $\mathbb{R}_+$.

(ii) For all $R_x, R_{\hat{z}} > 0$, there exist $r_x, r_{\hat{z}} > 0$ such that for all $(x_0, \hat{z}_0) \in \mathcal{K} \times \tau(\widehat{\mathcal{K}})$, if $|x_0| < r_x$ and $\|\hat{z}_0 - \tau(0)\|_X < r_{\hat{z}}$, then any solution $(x, \hat{z})$ of (33) starting from $(x_0, \hat{z}_0)$ satisfies $|x(t)| < R_x$ and $\|\hat{z}(t) - \tau(0)\|_X < R_{\hat{z}}$ for all $t \geqslant 0$.

(iii) Any solution $(x, \hat{z})$ of (33) with initial condition in $\mathcal{K} \times \tau(\widehat{\mathcal{K}})$ is such that $x(t) \to 0$ and $\hat{z}(t) \overset{w}{\rightharpoonup} \tau(0)$ as t goes to infinity.

If the previous conditions hold for any compact $\mathcal{K} \subset \mathbb{R}^n$, then system (1) is said to be *semi-globally stabilizable by means of an infinite-dimensional embedding-based dynamic output feedback.*

Remark 4.11. If X is finite-dimensional, then *(i)-(ii)-(iii)* is equivalent to the usual definition of asymptotic stability of (33) at $(0, \tau(0))$ with basin of attraction containing $\mathcal{K} \times \tau(\widehat{\mathcal{K}})$. However, when X is infinite-dimensional (the case of interest in this section), the convergence of trajectories towards the equilibrium point holds only in the weak topology. Hence, *(i)-(ii)-(iii)* is not equivalent to the usual definition of asymptotic stability of the infinite-dimensional system (33).

4.2 Back to the illustrative example

In this section, we illustrate the use of infinite-dimensional embeddings in the context of output feedback stabilization on a two-dimensional example with linear dynamics and nonlinear observation map. Let $h : \mathbb{R}^2 \to \mathbb{C}$. We consider the problem of stabilization by means of an infinite-dimensional embedding-based dynamic output feedback of the following system:

$$\begin{cases} \dot{x} = Ax + bu \\ y = h(x) \end{cases} \qquad \text{with } A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \text{ and } b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{34}$$

Since (A, b) is stabilizable, there exists $K \in \mathbb{R}^{1\times 2}$ such that $A + bK$ is Hurwitz. Moreover, A is skew-symmetric. Hence $\kappa = |K|$ can be chosen arbitrarily small. Then, the state feedback law $\phi : x \mapsto Kx$ is such that (34) with $u = \phi(x)$ is globally asymptotically stable at 0. Note that (34) does not exactly fit the form of (9) since h is not necessarily radially-symmetric. Of course, our analysis is of interest only if (34) is not uniformly observable. In Example 4.14, we give an example of non-radially symmetric h that makes the system non-uniformly observable, and on which our (infinite-dimensional) embedding-based strategy does apply. In the following we give some sufficient conditions on h allowing the design of a stabilizing infinite-dimensional dynamic output feedback. The main result of this section, Theorem 4.27 (stated in Section 4.2.4), relies on three main hypotheses: the existence of an embedding of (34) into (33), and two observability assumptions. For each of these assumptions, we provide examples of output maps h satisfying theses hypotheses.

4.2.1 Unitary representations and embeddings

In [11], the authors investigated the problem of observer design for (34) by using infinite-dimensional embeddings. We briefly recall their strategy, that relies on representation theory (see, *e.g.,* [5, 40]). The Lie group G of system (34) (the group of flows generated by the dynamical system (34) with constant inputs) is isomorphic to $\mathbb{R}^2 \rtimes_{\mathcal{R}} H$, where $H \simeq \{e^{tA}, t \in \mathbb{R}_+\} \simeq \mathcal{S}^1$ is the group of rotations (isomorphic to the unit circle), $\mathcal{R} : \mathcal{S}^1 \ni \theta \mapsto e^{\theta A}$ is an automorphism of $\mathbb{R}^2$ and $\rtimes_{\mathcal{R}}$ denotes the outer semi-direct product with respect to $\mathcal{R}$. Hence G is the group of motions of the plane. According to [40, Section IV.2], its unitary irreducible representations are given by a family $(\rho_\mu)_{\mu>0}$, where for each $\mu > 0$,

$$\begin{aligned} \rho_\mu : \quad & G \longrightarrow \mathscr{L}(L^2(\mathcal{S}^1, \mathbb{C})) \\ & (x, \vartheta) \longmapsto \left(\xi \in L^2(\mathcal{S}^1, \mathbb{C}) \mapsto \left(\mathcal{S}^1 \ni s \mapsto e^{i\mu(1,0)e^{sA'}x}\xi(s - \vartheta)\right)\right). \end{aligned}$$

Let $X = L^2(\mathcal{S}^1, \mathbb{C})$ be the set of real-valued square-integrable functions over $\mathcal{S}^1$. Then X is a Hilbert space endowed with the scalar product defined by $\langle \xi, \zeta \rangle_X = \frac{1}{2\pi}\int_0^{2\pi} \xi(s)\bar{\zeta}(s)\,\mathrm{d}s$ and the induced norm $\|\cdot\|_X$. Since $\mathcal{S}^1$ is compact, the constant function $\mathbb{1} : s \mapsto 1$ lies in X. Let $\mu > 0$ to be fixed later. Set

$$\begin{aligned} \tau_\mu : \mathbb{R}^2 &\longrightarrow X \\ x &\longmapsto \rho_\mu(x, 0)\mathbb{1}. \end{aligned}$$

Note that τ_μ depends on μ, but from now on we omit this dependence in the notation and write τ instead of τ_μ. Since ρ_μ is a unitary representation, $\|\tau(x)\|_X = 1$ for all $x \in \mathbb{R}^2$ and $\tau(0) = \mathbb{1}$. For all $x = (x_1, x_2) = (r\cos(\theta), r\sin(\theta))$ in $\mathbb{R}^2$, we have

$$\tau(x) : \mathcal{S}^1 \ni s \mapsto e^{i\mu(x_1\cos(s) + x_2\sin(s))} = e^{i\mu r\cos(s-\theta)}. \tag{35}$$

If $x, \tilde{x} \in \mathbb{R}^2$ are such that $\tau(x) = \tau(\tilde{x})$, then $(x_1 - \tilde{x}_1)\cos(s) + (x_2 - \tilde{x}_2)\sin(s) = 0$ for all $s \in \mathcal{S}^1$, hence $x = \tilde{x}$. Thus τ is injective. Let $u \in C^0(\mathbb{R}_+, \mathbb{R})$. Let x be a solution of (34) and set $z = \tau(x) \in C^0(\mathbb{R}_+, H^1(\mathcal{S}^1, \mathbb{C})) \cap C^1(\mathbb{R}_+, X)$. Then

$$\begin{aligned} \dot{z} &= i\mu\,(\dot{x}_1\cos(s) + \dot{x}_2\sin(s))\,z \\ &= i\mu\,(-x_2\cos(s) + x_1\sin(s) + u\sin(s))\,z \\ &= -\frac{\partial z}{\partial s} + iu\mu\sin(s)z \\ &= \mathcal{A}(u)z \end{aligned}$$

with $\mathcal{A}(u) = -\frac{\partial}{\partial s} + iu\sin(s)$ defined on the dense domain $\mathcal{D} = H^1(\mathcal{S}^1, \mathbb{C}) = \{f \in X \;:\; f' \in X\}$. The operator $\mathcal{A}(u)$ is the skew-adjoint generator of a strongly continuous unitary group on X for any $u \in \mathbb{R}$. In order to make τ an embedding of (34) into (26), we need the output map to be in the form $\mathfrak{y} = \mathcal{C}z$. This is where the freedom degree $\mathfrak{h}$ introduced in (33) may be employed. More specifically, we make the following first assumption on the observation map h.

Assumption 4.12 (Linearizable output map)**.** There exist $\mathfrak{h} : \mathbb{R}^m \to \mathbb{C}^\mathfrak{m}$ and $\mathcal{C} \in \mathscr{L}(X, \mathbb{C}^\mathfrak{m})$ such that $\mathfrak{h}(h(x)) = \mathcal{C}\tau(x)$ for all $x \in \mathbb{R}^2$.

Remark 4.13. If Assumption 4.12 is satisfied, then the embedding defined in (35) shows that Assumption 4.8 is satisfied.

Example 4.14. Denote by J_k the Bessel function of the first kind of order $k \in \mathbb{Z}$, that is,

$$J_k : \mathbb{R} \ni r \mapsto \frac{1}{2\pi}\int_0^{2\pi} e^{ir\sin(s) - iks}\,\mathrm{d}s \in \mathbb{R}. \tag{36}$$

For all $k \in \mathbb{Z}$, let

$$\begin{aligned} e_k : \mathcal{S}^1 &\longrightarrow \mathbb{C} \\ s &\longmapsto e^{iks}. \end{aligned}$$

The family $(e_k)_{k\in\mathbb{Z}}$ forms a Hilbert basis of X. In the rest of the paper, the weak norm $\mathcal{N}$ is always defined with respect to this Hilbert basis. Then, for all $x = (r\cos(\theta), r\sin(\theta)) \in \mathbb{R}^2$ and all $k \in \mathbb{Z}$,

$$\begin{aligned} \langle \tau(x), e_k\rangle_X &= \frac{1}{2\pi}\int_0^{2\pi} e^{i\mu r\cos(s-\theta)-iks}\,\mathrm{d}s \\ &= \frac{1}{2\pi}e^{-ik\theta+i\frac{\pi}{2}}\int_0^{2\pi} e^{i\mu r\sin(s)-iks}\,\mathrm{d}s \\ &= i^k J_k(\mu r)e^{-ik\theta}. \end{aligned} \tag{37}$$

Since $(e_k)_{k\in\mathbb{Z}}$ is a Hilbert basis of X, any function h such that $\mathfrak{h}(h(r\cos(\theta), r\sin(\theta))) = \sum_{k\in\mathbb{Z}} c_k J_k(\mu r)e^{-ik\theta}$ for some map $\mathfrak{h}$ and $(c_k)_{k\in\mathbb{Z}} \in l^2(\mathbb{Z}, \mathbb{C})$ satisfies Assumption 4.12. For example, $h(x) = J_0(\mu|x|) - 1$ (with $\mathfrak{h}(y) = y+1$), $h(x) = J_2(\mu|x|)\cos(2\theta)$ (with $\mathfrak{h}(y) = y$) and $h(x) = |x|$ (with $\mathfrak{h}(y) = J_0(\mu y)$) are suitable observations maps. In each of these cases, the constant input $u \equiv 0$ makes (34) unobservable. Moreover, $h(x) = J_0(\mu|x|) - 1$ and $h(x) = |x|$ are radially symmetric but $h(x) = J_2(\mu|x|)\cos(2\theta)$ is not. If $h(x) = |x|$, then (34) is a subcase of system (9).

Remark 4.15. According to the Gelfand–Raïkov theorem, the finite linear combinations of pure positive-type functions (*i.e.,* functions of the form $(x, \vartheta) \mapsto \langle \rho_\mu(x, \vartheta)\xi, \xi\rangle_X$, where $\mu > 0$ and $\xi \in X$) is dense for the uniform convergence on compact sets, in the continuous bounded complex-valued functions on G. Hence, the set of functions of the form $(r\cos(\theta), r\sin(\theta)) \mapsto \sum_{\ell\in I_1}\sum_{k\in I_2} c_k J_k(\mu_\ell r)e^{-ik\theta}$, where I_1 and I_2 are finite subsets of $\mathbb{Z}$, $\mu_\ell > 0$ and $c_k \in \mathbb{C}$, is dense for the uniform convergence on compact sets of $\mathbb{R}^2$, in the continuous bounded complex-valued functions on $\mathbb{R}^2$. In the examples of applications of our results, we will focus on output maps h of the form $\mathfrak{h}(h(x)) = \sum_{k\in I} c_k J_k(\mu r)e^{-ik\theta}$ for some $\mathfrak{h} : \mathbb{R}^m \to \mathbb{C}^{\mathfrak{m}}$ and some fixed $\mu > 0$.

4.2.2 Explicit strong left-inverse

Having in mind to use the strategy developed in the previous section, we now explicitly construct a strong left-inverse π of τ defined in (35) over some compact set. With Corollary 4.6, we already know that a strong left-inverse π exists. However, we would like to give an explicit expression. This can be done by employing the relationship between Bessel functions of the first kind given in (36) and the embedding τ, as shown in equation (37).

Indeed, let j_1 denote the first zero of J_1'. Then J_1 is increasing over $[0, j_1]$. Denote J_1^{-1} its inverse over $[0, j_1]$. Let $\Phi : \mathbb{C} \ni x_1 + ix_2 \mapsto (x_1, x_2) \in \mathbb{R}^2$ be the canonical bijection. Let $j \in (0, j_1)$. For all $\zeta \in \mathbb{C}$, let

$$\mathfrak{f}(\zeta) = \begin{cases} 0 & \text{if } \zeta = 0 \\ \Phi\left(\frac{i\bar{\zeta}}{\mu|\zeta|}J_1^{-1}(|\zeta|)\right) & \text{if } 0 < |\zeta| \leqslant J_1(j) \\ \Phi\left(\frac{i\bar{\zeta}}{\mu|\zeta|}j_1\right) & \text{if } |\zeta| \geqslant J_1(j_1) \end{cases} \tag{38}$$

If $J_1(j) < |\zeta| < J_1(j_1)$, define $\mathfrak{f}(\zeta)$ such that $\mathfrak{f}$ is continuously differentiable and globally Lipschitz over $\mathbb{C}$. Denote by $\ell_{\mathfrak{f}}$ its Lipschitz constant. Let $e_1 \in X$ be defined by $e_1(s) = e^{is}$ for all $s \in \mathcal{S}^1$. Let

$$\begin{aligned} \pi : X &\longrightarrow \mathbb{R}^2 \\ \xi &\longmapsto \mathfrak{f}(\langle \xi, e_1\rangle_X) \end{aligned} \tag{39}$$

Lemma 4.16. *The map* π *is a strong left-inverse of* τ *over* $\bar{B}_{\mathbb{R}^2}(0, \frac{j}{\mu})$.

Proof. Set $\mathcal{K} = \bar{B}_{\mathbb{R}^2}(0, \frac{j}{\mu})$. According to (38), $\phi(\xi) \in \mathcal{K}$ for all $x \in \mathcal{K}$.

Let $x = (r\cos(\theta), r\sin(\theta))$ in $\mathcal{K}$. Then, with (37),

$$\langle \tau(x), e_1 \rangle_X = ie^{-i\theta} J_1(\mu r) \in B_{\mathbb{C}}\left(0, J_1\left(j\right)\right).$$

Hence $\pi(\tau(x)) = \Phi(re^{i\theta}) = x$. Let $\xi \in X$. We have

$$|\pi(\xi) - x| = |\pi(\xi) - \pi(\tau(x))| = |\mathfrak{f}(\langle \xi, e_1 \rangle_X) - \mathfrak{f}(\langle \tau(x), e_1 \rangle_X)| \leqslant \ell_{\mathfrak{f}} |\langle \xi - \tau(x), e_1 \rangle_X|.$$

Hence π is a strong left-inverse of τ over $\mathcal{K}$. ■

Remark 4.17. Letting μ tends towards 0, the domain of the left-inverse tends towards $\mathbb{R}^2$, which will be of use to achieve semi-global stabilization.

4.2.3 Well-posedness and boundedness of trajectories

Since $\pi(\xi)$ is meaningful only if $|\langle \xi, e_1 \rangle_X| \leqslant J_1(j)$, we now check the well-posedness of the closed-loop system (33) and ensure that $\hat{z}$ remains in this domain. We now check the well-posedness of the closed-loop system (33). In a second step, since $\pi(\xi)$ is meaningful only if $|\langle \xi, e_1 \rangle_X| \leqslant J_1(j)$, we show that by selecting the (perturbation) parameter δ sufficiently small, $\hat{z}$ remains in this domain along the trajectories of the closed-loop system.

Lemma 4.18. *For all* $\mu, \alpha, \delta > 0$ *and all* $x_0, \hat{x}_0$ *in* $\bar{B}_{\mathbb{R}^2}(0, \frac{j}{\mu})$*, the system* (33) *(with* π *as in Lemma 4.16) admits a unique solution* $(x, \hat{z}) \in C^1(\mathbb{R}_+, \mathbb{R}^2) \times (C^0(\mathbb{R}_+, \mathcal{D}) \cap C^1(\mathbb{R}_+, X))$ *such that* $x(0) = x_0$ *and* $\hat{z}(0) = \tau(\hat{x}_0)$.

Proof. Let $\mathcal{K} = \bar{B}_{\mathbb{R}^2}(0, \frac{j}{\mu})$ and x_0, $\hat{x}_0$ in $\mathcal{K}$. Set $z_0 = \tau(x_0) \in \mathcal{D}$ and $\varepsilon_0 = \tau(\hat{x}_0) - \tau(x_0) \in \mathcal{D}$. The well-posedness of system (33) is equivalent to the well-posedness of the following system:

$$\begin{cases} \dot{z} = \mathcal{A}(u) z \\ \dot{\varepsilon} = (\mathcal{A}(u) - \alpha \mathcal{C}^* \mathcal{C}) \varepsilon \\ u = \phi(\pi(z + \varepsilon)) + \delta \mathcal{N}^2(z + \varepsilon - \mathbb{1}) \\ z(0) = z_0, \ \varepsilon(0) = \varepsilon_0 \end{cases} \tag{40}$$

where $\mathcal{A}(u) = -\frac{\partial}{\partial s} + i\mu u \sin$ and $\mathcal{C} \in \mathscr{L}(X, \mathbb{C}^{\mathfrak{m}})$. Set

$$\mathcal{A}_0 = \begin{pmatrix} -\frac{\partial}{\partial s} & 0 \\ 0 & -\frac{\partial}{\partial s} - \alpha \mathcal{C}^* \mathcal{C} \end{pmatrix}, \qquad \mathcal{F} : (z, \varepsilon) \mapsto \begin{pmatrix} i\mu \left(\phi(\pi(z + \varepsilon)) + \delta \mathcal{N}^2(z + \varepsilon - \mathbb{1}) \right) \sin(\cdot) z \\ i\mu \left(\phi(\pi(z + \varepsilon)) + \delta \mathcal{N}^2(z + \varepsilon - \mathbb{1}) \right) \sin(\cdot) \varepsilon \end{pmatrix}.$$

Since $\mathcal{C}$ is bounded and $\mathcal{A}_0$ is diagonal, $\mathcal{A}_0$ is the generator of a strongly continuous semigroup on X^2. Since π and $\mathcal{N}^2$ are locally Lipschitz, $\mathcal{F}$ is locally Lipschitz. Hence, according to [31, Theorem 1], system (40) admits a unique solution $(z, \varepsilon) \in C^0([0, T], X^2)$ for some $T \in \mathbb{R}_+^* \cup \{+\infty\}$. Moreover, since $\mathcal{A}(u)$ is skew-adjoint for all $u \in \mathbb{R}$, $\|z\|_X$ is constant and $\|\varepsilon\|_X$ is non-increasing. Hence, $T = +\infty$. Since π and $\mathcal{N}^2$ are continuously Fréchet differentiable, $\mathcal{F}$ is continuously Fréchet differentiable. Thus, $(z, \varepsilon) \in C^0(\mathbb{R}_+, \mathcal{D}^2) \cap C^1(\mathbb{R}_+, X^2)$. ■

Now that the existence and uniqueness of solutions of (33) is proved, let us show the boundedness of trajectories.

Lemma 4.19. *For all $\mu > 0$, all $R_2 \in (0, \frac{j}{\mu})$ and all $R_1 \in (0, R_2)$, there exist $R_0 \in (0, R_1)$ and $\delta_0 > 0$ such that for all $x_0, \hat{x}_0$ in $B_{\mathbb{R}^2}(0, R_0)$, all $\alpha > 0$ and all $\delta \in (0, \delta_0)$, the unique solution $(x, \hat{z}) \in C^1(\mathbb{R}_+, \mathbb{R}^2) \times (C^0(\mathbb{R}_+, \mathcal{D}) \cap C^1(\mathbb{R}_+, X))$ of (33) such that $x(0) = x_0$ and $\hat{z}(0) = \tau(\hat{x}_0)$ satisfies $|x(t)| < R_1$, $|\langle \hat{z}(t), e_1 \rangle_X| < J_1(\mu R_2)$ and $|\pi(\hat{z}(t))| < R_2$ for all $t \in \mathbb{R}_+$.*

Proof. Recall that $\kappa = |K|$. Denote by ℓ_π the global Lipschitz constant of π. Let $R_0 \in (0, R_1)$ and $\delta_0 > 0$ satisfying the following inequalities:

$$R_0 + M\left(2\kappa\ell_\pi\sqrt{2(1 - J_0(\mu R_0))} + 16\nu^2\delta_0\right) < R_1, \tag{41}$$

$$2\sqrt{2(1 - J_0(\mu R_0))} + J_1(\mu R_1) < J_1(\mu R_2). \tag{42}$$

This is always possible by choosing R_0 and δ_0 small enough since $J_0(0) = 1$.

Let $\delta \in (0, \delta_0)$, $x_0, \hat{x}_0 \in B_{\mathbb{R}^2}(0, R_0)$, $(x, \hat{z})$ as in Lemma 4.18, $z = \tau(x)$, $\varepsilon = \hat{z} - z$ and $u = \phi(\pi(\hat{z})) + \delta\mathcal{N}^2(\hat{z} - \mathbb{1})$. Set $e = b(u - Kx)$. Then $\dot{x} = (A + bK)x + e$. According to the variation of constants formula, and since $A + bK$ is Hurwitz, we get that

$$|x(t)| \leqslant |x_0| + M \sup_{s\in[0,t]} |e(s)| \qquad \forall t \in \mathbb{R}_+, \tag{43}$$

for some $M > 0$. Note that

$$\|\tau(x_0) - \mathbb{1}\|_X = \left(\|\tau(x_0)\|_X^2 + 1 - 2\langle \tau(x_0), 1\rangle_X\right)^{\frac{1}{2}} = \sqrt{2(1 - J_0(\mu|x_0|))} \leqslant \sqrt{2(1 - J_0(\mu R_0))}.$$

Then

$$\|\varepsilon_0\|_X \leqslant \|\hat{z}_0 - \mathbb{1}\|_X + \|z_0 - \mathbb{1}\|_X \leqslant 2\sqrt{2(1 - J_0(\mu R_0))}. \tag{44}$$

Let $t \in [0, T]$. Then

$$|e(t)| \leqslant \kappa|\pi(\hat{z}(t)) - x(t)| + \delta\mathcal{N}^2(\hat{z}(t) - \mathbb{1}). \tag{45}$$

On one hand,

$$\begin{aligned}
\mathcal{N}^2(\hat{z}(t) - \mathbb{1}) &\leqslant \nu^2 \|\hat{z}(t) - \mathbb{1}\|_X^2 \\
&\leqslant \nu^2 \left(\|\varepsilon(t)\|_X + \|z(t) - \mathbb{1}\|_X\right)^2 && \text{(by triangular inequality)} \\
&\leqslant \nu^2 \left(\|\varepsilon_0\|_X + 2\right)^2 && \text{(since } \|\varepsilon\|_X \text{ is non-increasing and } \|\tau(x(t))\|_X = 1) \\
&\leqslant 16\nu^2. && \text{(since } \|z_0\|_X = \|\hat{z}_0\|_X = 1)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
|\pi(\hat{z}(t)) - x(t)| &= |\pi(\hat{z}(t)) - \pi(z(t))| \\
&\leqslant \ell_\pi \|\varepsilon(t)\|_X \leqslant \ell_\pi \|\varepsilon_0\|_X \leqslant 2\ell_\pi\sqrt{2(1 - J_0(\mu R_0))}. && \text{(by (44))}
\end{aligned}$$

Hence,

$$|e(t)| \leqslant 2\kappa\ell_\pi\sqrt{2(1 - J_0(\mu R_0))} + 16\nu^2\delta. \tag{46}$$

Thus, combining (43) and (41), $|x(t)| < R_1$ for all $t \in \mathbb{R}_+$. Then, for all $t \in \mathbb{R}_+$,

$$\begin{aligned}
|\langle \hat{z}(t), e_1\rangle_X| &\leqslant |\langle \varepsilon(t), e_1\rangle_X| + |\langle z(t), e_1\rangle_X| \\
&\leqslant \|\varepsilon(t)\|_X + |\langle \tau(x(t)), e_1\rangle_X| \\
&\leqslant \|\varepsilon_0\|_X + J_1(\mu|x|) \\
&\leqslant 2\sqrt{2(1 - J_0(\mu R_0))} + J_1(\mu R_1) \\
&< J_1(\mu R_2).
\end{aligned}$$

Thus, (42) yields $|\langle \hat{z}(t), e_1\rangle_X| < J_1(\mu R_2)$ for all $t \in \mathbb{R}_+$. Finally, since $J_1(\mu R_2) < J_1(j)$, $|\pi(\hat{z}(t))| = |\mathfrak{f}(\langle \hat{z}(t), e_1\rangle_X)| = \frac{1}{\mu} J_1^{-1}(\langle \hat{z}(t), e_1\rangle_X) \leqslant R_2$. ■

In particular, we have the following corollary, which shows that the compact set of initial conditions that ensures the boundedness of trajectories can be chosen as big as desired, as soon as μ and δ are sufficiently small.

Corollary 4.20. *For all $R_0 > 0$, there exist $\mu_0 > 0$, $\delta_0 > 0$ and $R_2 > R_1 > R_0$ such that for all $x_0, \hat{x}_0$ in $B_{\mathbb{R}^2}(0, R_0)$, all $\mu \in (0, \mu_0)$, all $\alpha > 0$ and all $\delta \in (0, \delta_0)$, the unique solution $(x, \hat{z}) \in C^1(\mathbb{R}_+, \mathbb{R}^2) \times (C^0(\mathbb{R}_+, \mathcal{D}) \cap C^1(\mathbb{R}_+, X))$ of (33) such that $x(0) = x_0$ and $\hat{z}(0) = \tau(\hat{x}_0)$ satisfies $|x(t)| < R_1$, $|\langle \hat{z}(t), e_1\rangle_X| < J_1(\mu R_2)$ and $|\pi(\hat{z}(t))| < R_2$ for all $t \in \mathbb{R}_+$.*

Proof. Let $\beta_2 > \beta_1 > 1$ to be fixed later, and let $R_1 = \beta_1 R_0$ and $R_2 = \beta_2 R_0$. Then there exist $\mu_0, \delta_0 > 0$ small enough such that (41) holds for all $\mu \in (0, \mu_0)$. Recall the following asymptotic expansions of the Bessel functions of the first kind at 0:

$$J_0(r) = 1 - \frac{r^2}{4} + o(r^2), \qquad J_1(r) = \frac{r}{2} + o(r).$$

Then for all $\mu > 0$,

$$2\sqrt{2(1 - J_0(\mu R_0))} + J_1(\mu R_1) = \mu R_0 \left(\sqrt{2} + \frac{\beta_1}{2}\right) + o(\mu), \qquad J_1(\mu R_2) = \mu R_0 \frac{\beta_2}{2} + o(\mu).$$

Hence, if $\beta_2 > 2\sqrt{2} + \beta_1$, then there exists $\mu_0 > 0$ such that (42) holds for all $\mu \in (0, \mu_0)$. Set $\beta_1 = 2$ and $\beta_2 = 2\sqrt{2} + 3$. Then there exist $\mu_0 > 0$ and $\delta_0 > 0$ such that $\mu_0 R_2 < j$ and (41) and (42) are satisfied for all $\mu \in (0, \mu_0)$. Reasoning as in the proof of Lemma 4.19, the result follows. ■

4.2.4 Observability analysis

In order to state the main result of Section 4.2, we need to introduce two last assumptions on the linear output map $\mathcal{C}$ obtained from the function h in Assumption 4.12. For each assumption, we give examples of output maps h satisfying these assumptions. Following Remark 4.15, we investigate the case where at least one of the components of $\mathcal{C}$ is in the linear span of a finite number of elements of the Hilbert basis. This component is used to ensure the two observability properties. The first one states that $\mathcal{C}$ distinguishes the target point in a neighborhood of it.

Assumption 4.21 (Short time 0-detectability)**.** Let $u : \mathbb{R}_+ \to \mathbb{R}$ be uniformly continuous and x be a solution of (34) bounded by $\frac{j}{\mu}$. If $u(t_n) \underset{n\to+\infty}{\longrightarrow} 0$ and if there exists $\Delta > 0$ such that $\mathcal{C}\tau(x(t_n + t)) \underset{n\to+\infty}{\longrightarrow} \mathcal{C}\tau(0)$ for all $t \in [0, \Delta]$, then $x(t_n) \underset{n\to+\infty}{\longrightarrow} 0$.

Remark 4.22. Assumption 4.21 implies the necessary Condition 2.8 (local). Indeed, if x is a solution of (34) with $u = 0$ and $h(x(t)) = 0$ for all $t \geqslant 0$, then for any positive increasing sequence $(t_n)_{n\in\mathbb{N}} \to +\infty$, $u(t_n) = 0$ and $C\tau(x(t_n + t)) = \mathfrak{h}(0)$ for all $n \in \mathbb{N}$ and all $t \geqslant 0$. Hence, according to Assumption 4.21, $x(t_n) \to 0$. Thus Condition 2.8 (local) is satisfied. Moreover, if $\mathfrak{h}$ has a continuous inverse in a neighborhood of 0, then Assumption 4.21 implies the input/output-to-state stability condition (see *e.g.,* [22]), which states that any solution x of (34) such that $u(t) \to 0$ and $y(t) \to 0$ is such that $x(t) \to 0$ as $t \to +\infty$. This condition has proved to be of interest in the context of output feedback stabilization.

Example 4.23. Let $\zeta \in X \setminus \{0\}$ be such that $\mathcal{C} = (\langle \cdot, \zeta \rangle_X, \dots) \in \mathscr{L}(X, \mathbb{C}^{\mathfrak{m}})$. Let $(c_k)_{k\in\mathbb{Z}}$ be such that $\zeta = \sum_{k\in I} c_k e_k$, $I \subset \mathbb{Z}$. If I is finite (see Remark 4.15), then Assumption 4.21 is satisfied.

Indeed if $\mathcal{C}\tau(x(t_n + t)) \underset{n\to+\infty}{\longrightarrow} \mathcal{C}\tau(0)$, then $\sum_{k\in I} c_k J_k(\mu r(t_n + t))e^{-ik\theta(t_n+t)} \underset{n\to+\infty}{\longrightarrow} c_0$ (see (37)), where $x = (r\cos(\theta), r\sin(\theta))$. Since u is uniformly continuous, there exists $\Delta > 0$ (small enough) such that

$$\int_{t_n}^{t_n+\Delta} |u(t)| \, \mathrm{d}t \underset{n\to+\infty}{\longrightarrow} 0. \tag{47}$$

Then, according to Duhamel's formula, for all $t \in [0, \Delta]$,

$$x(t_n + t) - e^{tA}x(t_n) \underset{n\to+\infty}{\longrightarrow} 0,$$

i.e.,

$$r(t_n + t) - r(t_n) \underset{n\to+\infty}{\longrightarrow} 0 \text{ and } e^{i\theta(t_n+t)} - e^{i(\theta(t_n)+t)} \underset{n\to+\infty}{\longrightarrow} 0,$$

Hence, $\sum_{k\in I} c_k J_k(\mu r(t_n))e^{-ik\theta(t_n)}e^{-ikt} \underset{n\to+\infty}{\longrightarrow} c_0$ for all $t \in [0, \Delta]$. Since I is finite, this limit implies that $c_0 J_0(\mu r(t_n)) \to c_0$ and $c_k J_k(\mu r(t_n)) \to 0$ for $k \in I$ as n goes to $+\infty$. Denote by j_0 the first zero of J_0. Then $J_k(r) \neq 0$ for any $r \in (-j_0, j_0) \setminus \{0\}$ and any $k \in \mathbb{Z}$. Since for some $k \in I$, $c_k \neq 0$, we have $J_k(\mu r(t_n)) \to J_k(0)$, hence $x(t_n) \to 0$ since $|r(t_n)| < \frac{j}{\mu}$, $j < j_1 < j_0$.

Moreover, if there exist $k_1, k_2 \in \mathbb{Z}$ with $|k_1| \neq |k_2|$, $c_{k_1} \neq 0$ and $c_{k_2} \neq 0$, then $j_0 = +\infty$ is a suitable choice due to the Bourgets's hypothesis, proved by Siegel in [33].

The second hypothesis is that the unobservable input $u \equiv 0$ is isolated from other singular inputs of the infinite-dimensional system. Let us recall the usual definition of approximate observability of (26) (see, *e.g.*, [39]).

Definition 4.24 (Approximate observability)**.** System (26) is said to be *approximately observable* in some time $T > 0$ for some input $u \in C^1(\mathbb{R}_+, \mathbb{R})$ if and only if

$$(\forall t \in [0, T],\ \mathcal{C}\mathbb{T}_t(z_0, u) = 0) \Longrightarrow z_0 = 0. \tag{48}$$

Since (26) is a linear system, Definition 4.24 coincides with Definition 2.13 in the finite-dimensional context.

Assumption 4.25 (Isolated observability singularity)**.** Let $u \in [-u_{\max}, u_{\max}]$ where $u_{\max} = \kappa\frac{j}{\mu} + 16\nu^2\delta$. If $u \neq 0$, then the constant input u makes (26) approximately observable in some time $T > 0$.

Example 4.26. In [11, Example 1], the authors investigate the observability of (26) in the case where $\mu = 1$ and $\mathcal{C} = \langle \cdot, \mathbb{1} \rangle_X$ (*i.e.*, $\mathfrak{h} \circ h(x) = J_0(|x|)$, see Remark 4.14). Using a similar method, we prove that if $\mathcal{C} = (\langle \cdot, \zeta \rangle_X, \dots)$ for some $\zeta = \sum_{k\in I} c_k e_k$ in $X \setminus \{0\}$, where $I \subset \mathbb{Z}$ is finite, if $\mu u_{\max} < j_0$ for some $j_0 > 0$, then Assumption 4.25 holds.

Note that it is always possible to make $\mu u_{\max} < j_0$ by choosing κj and $\mu\delta$ small enough. Moreover, the considered set of such maps $\mathcal{C}$ is sufficient to approximate any output map h, as explained in Remark 4.15.

Let $z_0 \in X$, $u \in \mathbb{R} \setminus \{0\}$ and $z(t) = \mathbb{T}_t(z_0, u)$ be the unique corresponding solution of (26). We have

$$\begin{aligned}\langle z(t), \zeta\rangle_X &= \frac{1}{2\pi}\int_0^{2\pi} e^{-i\mu u \int_0^t \sin(s-\sigma)\,\mathrm{d}\sigma} z_0(s-t) \sum_{k\in I} \bar{c}_k e^{-iks}\,\mathrm{d}s \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{(by the method of characteristics)} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left(\sum_{k\in I} \bar{c}_k e^{-i\mu u\cos(s) - iks}\right) e^{i\mu u \cos(s-t)} z_0(s-t)\,\mathrm{d}s \\ &= (\psi * \psi_0)(t)\end{aligned}$$

where $*$ denotes the convolution product over X, $\psi : s \mapsto \sum_{k\in I} \bar{c}_k e^{-i\mu u \cos(s) - iks}$ and $\psi_0 : s \mapsto e^{i\mu u \cos(s)} z_0(s)$. Hence, according to Parseval's theorem,

$$\frac{1}{2\pi}\int_0^{2\pi} |\langle z(t), \zeta\rangle_X|^2\,\mathrm{d}t = \|\psi * \psi_0\|_X^2 = \left\|\hat{\psi}\cdot\hat{\psi}_0\right\|_{\hat{X}}^2 = \sum_{\ell\in\mathbb{Z}} |\langle \psi, e_\ell\rangle_X|^2 |\langle \psi_0, e_\ell\rangle_X|^2.$$

where $\hat{\psi}$ (resp. $\hat{\psi}_0$) denotes the Fourier series coefficients of ψ (resp ψ_0) in $X = L^2(\mathcal{S}^1, \mathbb{C}) \subset L^1(\mathcal{S}^1, \mathbb{C})$ and $\hat{X} = l^2(\mathbb{Z}, \mathbb{C})$. Hence, it is sufficient to show that there exists $j_0 > 0$ such that, if $\mu u < j_0$, then $\langle \psi, e_\ell\rangle_X \neq 0$ for all $\ell \in \mathbb{Z}$. Indeed, it yields that if $\mathcal{C}z(t) = 0$ for all $t \in [0, 2\pi]$, then $\psi_0 = 0$, *i.e.,* $z_0 = 0$, and thus u makes (26) approximately observable in time 2π.

Note that

$$\langle \psi, e_\ell\rangle_X = \frac{1}{2\pi}\int_0^{2\pi} \sum_{k\in I} \bar{c}_k e^{-i\mu u\cos(s) - i(k+\ell)s}\,\mathrm{d}s = \sum_{k\in I} \bar{c}_k i^k J_{k+\ell}(\mu u). \qquad \text{(by (37))}$$

Set $d_k = \bar{c}_k i^k$ and $F_\ell(r) = \sum_{k\in I} d_k J_{k+\ell}(r)$ for all $r \in \mathbb{R}$. Since F_ℓ is analytic for each $\ell \in \mathbb{Z}$, its zeros are isolated. Hence, for all $L > 0$, there exists $j_0 > 0$ such that, if $|\ell| < L$, then $F_\ell(r) \neq 0$ for all $r \in (-j_0, j_0) \setminus \{0\}$. Now, let $k_{\min} = \min\{k \in I \ : \ d_k \neq 0\}$ and let us prove that there exists $j_0 > 0$ such that $F_\ell(r) \neq 0$ for all $r \in (-j_0, j_0) \setminus \{0\}$ and all $\ell \geqslant -k_{\min}$. (One can reason similarly for $\ell \leqslant \max\{k \in I \ : \ d_k \neq 0\}$). We have $F_\ell(r) = d_{k_{\min}} J_{k_{\min}+\ell}(r) \left(1 + \sum_{k\in I} \frac{d_k}{d_{k_{\min}}} \frac{J_{k+\ell}(r)}{J_{k_{\min}+\ell}(r)}\right)$. According to [27], $|J_{k+\ell}(r)| \leqslant \frac{1}{(k+\ell)!}\left(\frac{|r|}{2}\right)^{k+\ell}$ for all $r \in \mathbb{R}$. Moreover, according to [23], if $|r| \leqslant 1$, then

$$|J_{k_{\min}+\ell}(r)| \geqslant |r|^{k_{\min}+\ell} J_{k_{\min}+\ell}(1) \geqslant \frac{|r|^{k_{\min}+\ell}}{(k_{\min}+\ell)!2^{k_{\min}+\ell}}\left(1 - \frac{1}{2(k_{\min}+\ell+1)}\right).$$

Hence

$$\begin{aligned}|F_\ell(r)| &\geqslant |d_{k_{\min}}|\,|J_{k_{\min}+\ell}(r)| \left(1 - \sum_{k\in I} \frac{|d_k|}{|d_{k_{\min}}|}\frac{|J_{k+\ell}(r)|}{|J_{k_{\min}+\ell}(r)|}\right) \\ &\geqslant |d_{k_{\min}}|\,|J_{k_{\min}+\ell}(r)| \left(1 - 2\sum_{k\in I} \frac{|d_k|}{|d_{k_{\min}}|}\left(\frac{|r|}{2}\right)^{k-k_{\min}}\right).\end{aligned}$$

Hence, there exits $j_0 > 0$ such that, if $0 < |r| < j_0$, $|F_\ell(r)| \geqslant \frac{|d_{k_{\min}}|}{2}|J_{k_{\min}+\ell}(r)|$ for all $\ell \in \mathbb{Z}$. Choosing $j_0 \leqslant \min\{r > 0 \ : \ J_0(r) = 0\}$, one has $J_{k_{\min}+\ell}(r) \neq 0$ for all $\ell \in \mathbb{Z}$, hence $F_\ell(r) \neq 0$.

In particular, if $\zeta = e_k$ for some $k \in I$, then $j_0 = \min\{r > 0 \ : \ J_0(r) = 0\}$ is a suitable choice. Indeed, $J_k(r) \neq 0$ for all $r \in (-j_0, j_0) \setminus \{0\}$ and all $k \in \mathbb{Z}$. Hence, if $\mu u_{\max} < j_0$, then u makes (26) approximately observable in time 2π.

Moreover, if $\mathcal{C} = (\langle \cdot, e_{k_1}\rangle_X, \langle \cdot, e_{k_2}\rangle_X, \dots)$ with $|k_1| \neq |k_2|$, then $j_0 = +\infty$ is a suitable choice due to the Bourgets's hypothesis, proved by Siegel in [33].

We are now in position to state the main result of Section 4.2.

Theorem 4.27. *Let $\mathcal{K} \subset \mathbb{R}^2$ be a compact set. Let $R_0 > 0$ be such that $\mathcal{K} \subset B_{\mathbb{R}^2}(0, R_0)$. Let $\mu_0 > 0$ and $\delta_0 > 0$ be as in Corollary 4.20. Suppose that there exists $\mu \in (0, \mu_0)$ and $\delta \in (0, \delta_0)$ such that Assumptions 4.12, 4.21 and 4.25 are satisfied.*

Then system (34) *is stabilizable over $\mathcal{K}$ by means of an infinite-dimensional embedding-based dynamic output feedback. Moreover, the closed-loop system is explicitly given by* (33) *for any $\alpha > 0$ and with τ as in* (35) *and π as in* (39).

According to Examples 4.14, 4.23 and 4.26, we obtain the following corollary.

Corollary 4.28. *If $\mathfrak{h}(h(r\cos(\theta), r\sin(\theta))) = \sum_{k \in I} c_k J_k(\mu r) e^{-ik\theta}$ for some map $\mathfrak{h} : \mathbb{R}^m \to \mathbb{C}$, $\mu > 0$, $(c_k)_{k \in I} \in \mathbb{C}^I$ and $I \subset \mathbb{Z}$ finite, then there exists $\delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$, system* (34) *is stabilizable over $\bar{B}(0, \frac{j}{\mu})$ for all $j \in (0, j_1)$ by means of an infinite-dimensional embedding-based dynamic output feedback.*

Example 4.29. As as application of Corollary 4.28, we provide the following examples of output maps for which our embedding based strategy allows to conclude to the stabilizability.

- If $h(x) = |x|$, then system (34) is semi-globally stabilizable by means of an infinite-dimensional embedding-based dynamic output feedback.

  Let $\mathcal{K} \subset \mathbb{R}^2$ be a compact set. Let $R_0 > 0$ be such that $\mathcal{K} \subset B_{\mathbb{R}^2}(0, R_0)$. Let $\mu_0 > 0$ and $\delta_0 > 0$ be as in Corollary 4.20. According to Examples 4.14 and 4.23, Assumptions 4.12 and 4.21 are satisfied for any $\mu \in (0, \mu_0)$ by considering $\mathfrak{h} : y \mapsto J_0(\mu y)$. Moreover, by choosing $\kappa j < j_0$ and $\delta < \frac{j_0 - \kappa j}{16\nu^2 \mu}$, Assumption 4.25 is also satisfied according to Example 4.26. Hence, Theorem 4.27 does apply on $\mathcal{K}$.

- Naturally, this last example was treated via a finite-dimensional strategy in Section 3. Furthermore, for radially symmetric output maps, one can devise a strategy where $|x|$ is extracted (at least locally around the target by inversion) and apply the same finite-dimensional method. However, this is impossible if the output is not radially symmetric. For instance, if $h(r\cos(\theta), r\sin(\theta)) = J_2(\mu r)\cos(2\theta)$ for some $\mu > 0$, then system (34) is stabilizable over $\bar{B}(0, \frac{j}{\mu})$ for all $j \in (0, j_1)$ by means of an infinite-dimensional embedding-based dynamic output feedback. To our knowledge there doesn't exist a strategy that achieves the same result with a finite-dimensional time-independent approach.

The two following sections are devoted to the proof of Theorem 4.27. Let $\mathcal{K}$ be a compact subset of $\mathbb{R}^2$. Since Lemma 4.18 implies the statement *(i)* of Definition 4.10, it remains to show *(ii)* and *(iii)*. Let $R_0 > 0$ be such that $\mathcal{K} \subset B_{\mathbb{R}^2}(0, R_0)$, $\mu \in (0, \mu_0)$ and $\delta \in (0, \delta_0)$ be as in Corollary 4.20, $\alpha > 0$, τ be as in (35) and π be as in (39). Let x_0 and $\hat{x}_0$ be in $\mathcal{K}$, $(x, \hat{z})$ be the corresponding solution of (26), $z = \tau(x)$, $\varepsilon = \hat{z} - z$ and $u = \phi(\pi(\hat{z})) + \delta\mathcal{N}^2(\hat{z} - \mathbb{1})$. Remark that

$$\|\varepsilon\|_X - \ell_\tau |x| \leqslant \|\hat{z} - \mathbb{1}\|_X + \|z - \mathbb{1}\|_X - \ell_\tau |x| = \|\hat{z} - \mathbb{1}\|_X + \|\tau(x) - \tau(0)\|_X - \ell_\tau |x| \leqslant \|\hat{z} - \mathbb{1}\|_X$$

and

$$\|\hat{z} - \mathbb{1}\|_X \leqslant \|\varepsilon\|_X + \|z - \mathbb{1}\|_X \leqslant \|\varepsilon\|_X + \ell_\tau |x|$$

where ℓ_τ is the Lipschitz constant of τ over $\mathcal{K}$. Hence proving statement *(ii)* of Definition 4.10 reduces to prove *(ii')* For all $R_x, R_\varepsilon > 0$, there exists $r_x, r_\varepsilon > 0$ such that for all $(x_0, \hat{z}_0) \in \mathcal{K} \times \tau(\widehat{\mathcal{K}})$, if $|x_0| < r_x$ and $\|\varepsilon_0\|_X < r_\varepsilon$, then $|x(t)| < R_x$ and $\|\varepsilon(t)\|_X < R_\varepsilon$ for all $t \geqslant 0$.

Since τ is continuous, if $x \to 0$, then $\tau(x) \to \mathbb{1}$, and, a fortiori, $\tau(x) \xrightarrow{w} \mathbb{1}$. Hence proving statement *(iii)* of Definition 4.10 reduces to prove

(iii') $x(t) \to 0$ and $\varepsilon(t) \xrightarrow{w} 0$ as t goes to infinity.

We prove *(ii')* in Section 4.2.5 and *(iii')* in Section 4.2.6.

4.2.5 Stability

Let $R_x, R_\varepsilon > 0$. We seek $r_x, r_\varepsilon > 0$ such that for all $(x_0, \hat{z}_0) \in \mathcal{K} \times \tau(\widehat{\mathcal{K}})$, if $|x_0| < r_x$ and $\|\varepsilon_0\|_X < r_\varepsilon$, then $|x(t)| < R_x$ and $\|\varepsilon(t)\|_X < R_\varepsilon$ for all $t \geqslant 0$. Since $\|\varepsilon\|_X$ is non-increasing, choose $r_\varepsilon \leqslant R_\varepsilon$. Recall that x satisfies the following dynamics:

$$\dot{x} = (A + bK)x + bK(\pi(\hat{z}) - x) + \delta\mathcal{N}^2(\hat{z} - \mathbb{1})b.$$

Moreover,

$$|\pi(\hat{z}) - x| \leqslant \ell_\pi \|\varepsilon\|_X \leqslant \ell_\pi r_\varepsilon$$

where ℓ_π is the global Lipschitz constant of π and

$$\mathcal{N}(\hat{z} - \mathbb{1}) \leqslant \mathcal{N}(\varepsilon) + \mathcal{N}(z - \mathbb{1}) \leqslant \nu \|\varepsilon\|_X + \nu \|z - \mathbb{1}\|_X \leqslant \nu r_\varepsilon + \nu \ell_\tau |x|$$

where ℓ_τ is the Lipschitz constant of τ over $\mathcal{K}$. Since $A + bK$ is Hurwitz, there exists $P \in \mathbb{R}^{2\times 2}$ positive definite such that $P(A + bK) + P'(A + bK)' < -2\mathrm{I}_{\mathbb{R}^2}$. Denote by $\sigma_{\min}$ (resp. $\sigma_{\max}$) the smallest (resp. largest) eigenvalue of P. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} x'Px &\leqslant -2|x|^2 + 2|x||Pb|\kappa|\pi(\hat{z}) - x| + 2|x||Pb|\delta\mathcal{N}^2(\hat{z} - \mathbb{1}) \\ &\leqslant -2|x|^2 + 2\kappa|Pb|\ell_\pi r_\varepsilon|x| + 4|Pb|\delta\nu^2(r_\varepsilon^2 + \ell_\tau^2|x|^2)|x|.\end{aligned}$$

Set $r_x = \min\left(\frac{R_x}{2}, \sqrt{\frac{\sigma_{\min}}{\sigma_{\max}}}, \frac{1}{4|Pb|\delta\nu^2\ell_\tau^2}\right)$ and $r_\varepsilon = \min\left(R_\varepsilon, \frac{r_x}{8\kappa|Pb|\ell_\pi}, \frac{\sqrt{r_x}}{4\nu\sqrt{\delta|Pb|}}\right)$. If $|x(t)| = r_x$ for some $t \in \mathbb{R}_+$, then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} x'(t)Px(t) &\leqslant (-2 + 4|Pb|\delta\nu^2\ell_\tau^2 r_x)r_x^2 + 2\kappa|Pb|\ell_\pi r_\varepsilon r_x + 4|Pb|\delta\nu^2 r_\varepsilon^2 r_x \\ &\leqslant -r_x^2 + \frac{1}{4}r_x^2 + \frac{1}{4}r_x^2 \\ &< 0.\end{aligned}$$

Hence, for all $t \in \mathbb{R}_+$, $|x(t)| \leqslant \sqrt{\frac{\sigma_{\max}}{\sigma_{\min}}} r_x < R_x$ and $|\varepsilon(t)| < r_\varepsilon \leqslant R_\varepsilon$.

4.2.6 Attractivity

Step 1: Show that $\boldsymbol{\varepsilon \xrightarrow{w} 0}$. Let Ω be the set of limit points of $(\varepsilon(t))_{t\in\mathbb{R}_+}$ for the weak topology of X, that is, the set of points $\varepsilon^\star \in X$ such that there exists an increasing sequence $(t_n)_{n\in\mathbb{N}}$ such that $\varepsilon(t_n) \xrightarrow{w} \varepsilon^\star$ as $n \to +\infty$. According to (31), ε is bounded. Hence, by Kakutani's theorem, Ω is not empty. It remains to show that $\Omega = \{0\}$. Let $\varepsilon^\star \in \Omega$ and an increasing sequence $(t_n)_{n\in\mathbb{N}}$ such that $\varepsilon(t_n) \xrightarrow{w} \varepsilon^\star$ as $n \to +\infty$. Combining (31) and (30), $\hat{z}$

is also bounded. Then, after passing to a subsequence, we may assume that $\hat{z}(t_n)$ converges weakly to some $\hat{z}^\star \in X$. According to Corollary 4.20, $|u|$ is bounded by $\kappa\frac{j}{\mu} + 16\nu^2\delta$.

If $u(t_n)$ does not converge to 0, then according to Assumption 4.25, the sequence $(u(t_n))_{n\in\mathbb{N}}$ has an accumulation point $u^\star$ that makes the system (26) approximately observable. Hence, by [8, Theorem 3.5], $\varepsilon^\star = 0$. Similarly, if $u(t_n + t')$ does not converge to 0 for some $t' \geqslant 0$ then $\varepsilon^\star = 0$, following the same reasoning as in the proof of [8, Theorem 3.5] (see also [11, Theorem 7, Step 4]).

Now, assume that $u(t_n + t') \to 0$ as $n \to +\infty$ for all $t' > 0$. Passing to the limit in the expression of $u(t_n)$, we get the existence of $\mathcal{N}^2_\infty \in \mathbb{R}_+$ such that $\mathcal{N}^2(\hat{z}(t_n) - \mathbb{1}) \to \mathcal{N}^2_\infty$ and

$$K\mathfrak{f}(\langle \hat{z}^\star, e_1\rangle_X) + \delta\mathcal{N}^2_\infty = 0. \tag{49}$$

Since u is uniformly continuous (because f is globally Lipschitz and $\dot{\hat{z}}$ and $\dot{\varepsilon}$ are bounded), there exists $\Delta > 0$ (small enough) such that

$$\int_{t_n}^{t_n+\Delta} |u(t)|\,\mathrm{d}t \underset{n\to+\infty}{\longrightarrow} 0. \tag{50}$$

Using the method of characteristics, one can show that for all $t, t' \in \mathbb{R}_+$ and almost all $s \in \mathcal{S}^1$,

$$z(t+t', s) = \mathcal{I}(t+t', t, s) z(t, s-t'). \tag{51}$$

where $\mathcal{I}(t+t', t, s) = e^{-i\mu\int_t^{t+t'} u(\sigma)\sin(s-\sigma)\,\mathrm{d}\sigma}$. Then, according to Duhamel's formula,

$$\hat{z}(t+t', s) = \mathcal{I}(t+t', t, s)\hat{z}(t, s-t') - \alpha\int_t^{t+t'} \mathcal{I}(t+t', \sigma, s)\big((\mathcal{C}^*\mathcal{C}\varepsilon(\sigma))\,(s-t')\big)\,\mathrm{d}\sigma. \tag{52}$$

Let $t' \in [0, \Delta]$. By (50), $\mathcal{I}(t_n + t', t_n, s) \to 1$ as $n \to +\infty$, uniformly in $s \in \mathcal{S}^1$. Then

$$\begin{aligned}\left\|\hat{z}(t_n+t', \cdot) - \hat{z}(t_n, \cdot - t')\right\|_X &\leqslant \sup_{s\in\mathcal{S}^1} |\mathcal{I}(t_n+t', t_n, s) - 1|\,\|\hat{z}(t_n)\|_X \\ &\quad + \alpha \sup_{\sigma\in[t_n, t_n+t'],\, s\in\mathcal{S}^1} |\mathcal{I}(t_n+t', \sigma, s)| \left\|\int_{t_n}^{t_n+t'} \mathcal{C}^*\mathcal{C}\varepsilon(\sigma)\,\mathrm{d}\sigma\right\|_X\end{aligned} \tag{53}$$

tends towards 0 as n goes to $+\infty$ since $\hat{z}$ and ε are bounded, $t \mapsto \|\mathcal{C}\varepsilon(t)\|_X$ is integrable over $\mathbb{R}_+$ (see (31)) and $t' \leqslant \Delta$. Hence

$$\langle \hat{z}(t_n+t', \cdot), e_1\rangle_X \to \langle \hat{z}^\star(\cdot - t'), e_1\rangle_X = e^{it'}\langle \hat{z}^\star, e_1\rangle_X \tag{54}$$

and

$$\mathcal{N}^2(\hat{z}(t_n+t', \cdot) - \mathbb{1}) \to \mathcal{N}^2(\hat{z}^\star(\cdot - t') - \mathbb{1}) = \mathcal{N}^2_\infty. \tag{55}$$

as n goes to $+\infty$. Combining (54) and (55) with $u(t_n + t') \to 0$ for $t' \in \{0, \frac{\Delta}{3}, \frac{2\Delta}{3}\}$, we get

$$\begin{cases} K\mathfrak{f}(\langle \hat{z}^\star, e_1\rangle_X) + \delta\mathcal{N}^2_\infty = 0 \\ K\mathfrak{f}(e^{i\frac{\Delta}{3}}\langle \hat{z}^\star, e_1\rangle_X) + \delta\mathcal{N}^2_\infty = 0 \\ K\mathfrak{f}(e^{i\frac{2\Delta}{3}}\langle \hat{z}^\star, e_1\rangle_X) + \delta\mathcal{N}^2_\infty = 0 \end{cases} \tag{56}$$

For all $t \in \mathbb{R}$ and all $\zeta \in B_{\mathbb{C}}(0, J_1(j))$, we have by (38), $K\mathfrak{f}(e^{it}\zeta) = K\mathfrak{R}(t)\mathfrak{f}(\zeta)$ where $\mathfrak{R}(t) = \begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t)\end{pmatrix}$. Let $\mathfrak{P} = \begin{pmatrix} K & 1 \\ K\mathfrak{R}(\frac{\Delta}{3}) & 1 \\ K\mathfrak{R}(\frac{2\Delta}{3}) & 1\end{pmatrix}$. Then (56) yields $\mathfrak{P}\begin{pmatrix} K\mathfrak{f}(\langle \hat{z}^\star, e_1\rangle_X) \\ \delta\mathcal{N}^2_\infty\end{pmatrix} = 0$.

Since $\mathfrak{P}$ is invertible for $\Delta \in (0, 3\pi)$, we get that $\mathcal{N}_\infty^2 = 0$ *i.e.* $\hat{z}(t_n) \xrightarrow{w} \mathbb{1}$. Combining it with (53), we have $\hat{z}(t_n + t') \xrightarrow{w} \mathbb{1}$ as n goes to $+\infty$. In particular, $\mathcal{C}\hat{z}(t_n + t') \to \mathcal{C}\tau(0)$. Since $\mathcal{C}\varepsilon \to 0$ by (31), we obtain $\mathcal{C}\tau(x(t_n + t')) \to \mathcal{C}\tau(0)$. Hence, by Assumption 4.21, $x(t_n) \to 0$ *i.e.* $z(t_n) \to \mathbb{1}$. Thus $\varepsilon(t_n) \xrightarrow{w} 0$ *i.e.* $\varepsilon^\star = 0$.

Step 2: Show that $x \to 0$. Recall that x satisfies the following dynamics:

$$\dot{x} = (A + bK)x + bK(\pi(\hat{z}) - x) + \delta\mathcal{N}^2(\hat{z} - \mathbb{1})b.$$

Since $A+bK$ is Hurwitz, there exists $P \in \mathbb{R}^{2\times 2}$ positive definite such that $P(A+bK) + P'(A+bK)' < -2\mathrm{I}_{\mathbb{R}^2}$. Set $V : \mathbb{R}^2 \ni x \mapsto x'Px$. Then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V(x) &\leqslant -2|x|^2 + 2|x||Pb|\kappa|\pi(\hat{z}) - x| + 2|x||Pb|\delta\mathcal{N}^2(\hat{z} - \mathbb{1}) \\ &\leqslant -2|x|^2 + 2|Pb|\frac{j}{\mu}\left(\kappa|\pi(\hat{z}) - x| + \delta\mathcal{N}^2(\hat{z} - \mathbb{1})\right).\end{aligned}$$

We have

$$\mathcal{N}(\hat{z} - \mathbb{1}) \leqslant \mathcal{N}(\varepsilon) + \mathcal{N}(z - \mathbb{1}) \leqslant \mathcal{N}(\varepsilon) + \nu\left\|z - \mathbb{1}\right\|_X \leqslant \mathcal{N}(\varepsilon) + \nu\ell_\tau|x|$$

where ℓ_τ is the Lipschitz constant of τ over $\mathcal{K}$. Hence, if $\delta \leqslant \frac{\mu}{4|Pb|j\nu^2\ell_\tau^2}$ (which we can assume without loss of generality by replacing δ_0 by $\min(\delta_0, \frac{\mu}{4|Pb|j\nu^2\ell_\tau^2})$, since diminishing δ), then

$$\frac{\mathrm{d}}{\mathrm{d}t}V(x) \leqslant -|x|^2 + 2|Pb|\frac{j}{\mu}\left(\kappa|\pi(\hat{z}) - x| + 2\delta\mathcal{N}^2(\varepsilon)\right).$$

Recall that $|x|$ and $|\pi(\hat{z})|$ are bounded by $\frac{j}{\mu}$. Moreover, $\mathcal{N}(\varepsilon(t)) \to 0$ as $t \to +\infty$ by Step 1, and $\pi(\hat{z}) - x \to 0$ as $t \to +\infty$ since π is a strong left-inverse of τ (see Corollary 4.6).

For all $r > 0$, set $D(r) = \{x \in \mathbb{R}^2 \ : \ V(x) \leqslant r\}$. In order to prove that $x \to 0$, we show that for all $r > 0$, there exists $T(r) \geqslant 0$ such that $x(t) \in D(r)$ for all $t \geqslant T(r)$. If $r > 0$ is such that $\bar{B}_{\mathbb{R}^2}(0, \frac{j}{\mu}) \subset D(r)$ then $T(r) = 0$ satisfies the statement. Let $0 < r < R$ be such that $\bar{B}_{\mathbb{R}^2}(0, \frac{j}{\mu}) \not\subset D(r)$ and $\bar{B}_{\mathbb{R}^2}(0, \frac{j}{\mu}) \subset D(R)$. Since $\mathcal{N}(\varepsilon(t)) \to 0$ and $\pi(\hat{z}(t)) - x(t) \to 0$, there exists $T_1(r) > 0$ such that for all $t \geqslant T_1(r)$, if $x(t) \notin D(r)$, then $\frac{\mathrm{d}}{\mathrm{d}t}V(x) < -\bar{m}$, for some $\bar{m} > 0$. First, this implies that if $x(t) \in D(r)$ for some $t \geqslant T_1(r)$, then $x(s) \in D(r)$ for all $s \geqslant t$. Second, for all $t \geqslant 0$,

$$\begin{aligned}V(x(T_1(r) + t)) &= V(x(T_1(r))) + \int_0^t \frac{\mathrm{d}}{\mathrm{d}t}V(x(T_1(r) + \tau))\,\mathrm{d}\tau \\ &\leqslant R - \bar{m}t \qquad \text{while } x(T_1(r) + t) \notin D(r).\end{aligned}$$

Set $T_2(r) = \frac{R-r}{m}$ and $T(r) = T_1(r) + T_2(r)$. Then for all $t \geqslant T(r)$, $x(t) \in D(r)$, which concludes the proof.

5 Conclusion

The objective of the paper is to illustrate a new approach to tackle the issue of stabilization at unobservable points. As we showed, we can use arguments from Lie group representation theory to obtain embeddings of finite-dimensional systems into unitary systems of infinite dimension. These contractive systems allow to design observers that appear to be resilient to singular inputs issues.

Beyond the method we explored in the present article, we wish to stress that topological obstructions to output feedback stabilization can be lifted when infinite-dimensional observers are considered. More precisely, the obstruction brought up in [13] regarding the stabilizability of $\dot{x} = u$, $y = x^2$ and extended in Corollary 3.2 vanishes if one extends the usual definition of dynamic output feedback stabilizability by allowing infinite-dimensional states fed by the output, as in Definition 4.10.

Therefore, new infinite-dimensional embedding techniques for output feedback stabilization, either based on the more general framework of [11], or on other infinite-dimensional observers, need to be investigated.

Acknowledgments: The authors would like to thank Vincent Andrieu for many fruitful discussions.

This research was partially funded by the French Grant ANR ODISSE (ANR-19-CE48-0004-01) and by the ANR SRGI (ANR-15-CE40-0018).

A The matrix $\mathcal{Q}$ is invertible

Let us compute the determinant of $\mathcal{Q}$.

$$\det \mathcal{Q} = \begin{vmatrix} K & \delta & 0 \\ KA & 0 & -\delta\alpha \\ \vdots & \vdots & \vdots \\ KA^{n+1} & 0 & \delta(-\alpha)^{n+1} \end{vmatrix} = (-1)^{n+1}\delta^2\alpha \begin{vmatrix} KA & 1 \\ \vdots & \vdots \\ KA^{n+1} & (-\alpha)^n \end{vmatrix} = -\delta^2\alpha \sum_{k=0}^{n} \alpha^k Q(k)$$

where

$$Q(k) = \begin{vmatrix} \tilde{K}A^0 \\ \vdots \\ \tilde{K}A^{k-1} \\ \tilde{K}A^{k+1} \\ \vdots \\ \tilde{K}A^n \end{vmatrix}, \qquad \tilde{K} = KA, \qquad k \in \{0, \dots, n\}.$$

Let $P(X) = \sum_{k=0}^{n} c_k X^k$ be the characteristic polynomial of A. Since A is skew-symmetric and invertible, it holds that n is even, P is minimal for A, positive on $\mathbb{R}$, $c_n = 1$. Then,

$$A^n = -\sum_{k=0}^{n-1} c_k A^k.$$

Let Δ be the determinant of the Kalman observability matrix of $(\tilde{K}, A)$. Since (K, A) is observable and A is invertible, $\Delta \neq 0$. Then for $k < n$,

$$Q(k) = \begin{vmatrix} \tilde{K}A^0 \\ \vdots \\ \tilde{K}A^{k-1} \\ \tilde{K}A^{k+1} \\ \vdots \\ \sum_{i=0}^{n-1} c_i \tilde{K}A^i. \end{vmatrix} = \begin{vmatrix} \tilde{K}A^0 \\ \vdots \\ \tilde{K}A^{k-1} \\ \tilde{K}A^{k+1} \\ \vdots \\ -c_k \tilde{K}A^k. \end{vmatrix} = -c_k(-1)^{n-k}\Delta.$$

The case $k = n$ simply yields $Q(n) = \Delta$. Then

$$\det \mathcal{Q} = \delta^2 \alpha \Delta \sum_{k=0}^{n} c_k (-1)^k \alpha^k = \delta^2 \alpha \Delta P(-\alpha).$$

Since P is positive on $\mathbb{R}$, $\det \mathcal{Q} > 0$ as soon as $\alpha > 0$.

References

[1] A. Ajami, M. Brouche, J. P. Gauthier, and L. Sacchelli. Output stabilization of military UAV in the unobservable case. In *2020 IEEE Aerospace Conference*, pages 1–6, 2020.

[2] A. Ajami, J. P. Gauthier, and L. Sacchelli. Dynamic output stabilization of control systems: an unobservable kinematic drone model. working paper or preprint, 2020.

[3] V. Andrieu and L. Praly. On the existence of a kazantzis–kravaris/luenberger observer. *SIAM J. Control and Optimization*, 45:432–456, 02 2006.

[4] V. Andrieu and L. Praly. A unifying point of view on output feedback designs for global asymptotic stabilization. *Automatica. A Journal of IFAC, the International Federation of Automatic Control*, 45(8):1789–1798, 2009.

[5] A. O. Barut and R. Rą̧czka. *Theory of group representations and applications*. World Scientific Publishing Co., Singapore, second edition, 1986.

[6] P. Bernard, L. Praly, V. Andrieu, and H. Hammouri. On the triangular canonical form for uniformly observable controlled systems. *Automatica. A Journal of IFAC, the International Federation of Automatic Control*, 85:293–300, 2017.

[7] H. Brezis. *Functional analysis, Sobolev spaces and partial differential equations*. Universitext. Springer, New York, 2011.

[8] L. Brivadis, V. Andrieu, U. Serres, and J.-P. Gauthier. Luenberger observers for infinite-dimensional systems, back and forth nudging and application to a crystallization process, 2020.

[9] L. Brivadis, J.-P. Gauthier, L. Sacchelli, and U. Serres. Avoiding observability singularities in output feedback bilinear systems, 2019.

[10] L. Brivadis, L. Sacchelli, V. Andrieu, J.-P. Gauthier, and U. Serres. From local to global asymptotic stabilizability for weakly contractive control systems. *Automatica*, page 109308, 2020.

[11] F. Celle, J.-P. Gauthier, D. Kazakos, and G. Sallet. Synthesis of nonlinear observers: a harmonic-analysis approach. *Mathematical Systems Theory. An International Journal on Mathematical Computing Theory*, 22(4):291–322, 1989.

[12] P. Combes, A. K. Jebai, F. Malrait, P. Martin, and P. Rouchon. Adding virtual measurements by signal injection. In *2016 American Control Conference (ACC)*, pages 999–1005, 2016.

[13] J.-M. Coron. On the stabilization of controllable and observable systems by an output feedback law. *Mathematics of Control, Signals, and Systems*, 7(3):187–216, 1994.

[14] J.-M. Coron. Relations entre commandabilité et stabilisations non linéaires. In *Nonlinear partial differential equations and their applications. Collège de France Seminar, Vol. XI (Paris, 1989–1991)*, volume 299 of *Pitman Research Notes In Mathematics Series*, pages 68–86. Longman Scientific & Technical, Harlow, 1994.

[15] K.-J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.

[16] E. Flayac. *Coupled methods of nonlinear estimation and control applicable to terrain-aided navigation*. PhD thesis, Université Paris-Saclay (ComUE), 2019. Thèse de doctorat dirigée par Jean, Frédéric Mathématiques appliquées.

[17] J.-P. Gauthier and I. Kupka. *Deterministic observation theory and applications*. Cambridge University Press, Cambridge, 2001.

[18] T.-B. Hoang, W. Pasillas-Lépine, and W. Respondek. A switching observer for systems with linearizable error dynamics via singular time-scaling. In *MTNS 2014*, Groningen, Netherlands, July 2014.

[19] K. Ito and F. Kappel. *Evolution equations and approximations*, volume 61 of *Series on Advances in Mathematics for Applied Sciences*. World Scientific Publishing Co., Inc., River Edge, NJ, 2002.

[20] P. Jouan and J. . Gauthier. Finite singularities of nonlinear systems. output stabilization, observability and observers. In *Proceedings of 1995 34th IEEE Conference on Decision and Control*, volume 4, pages 3295–3299 vol.4, 1995.

[21] M. A. Krasnosel'skiĭ and P. P. Zabreĭko. *Geometrical methods of nonlinear analysis*, volume 263 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1984. Translated from the Russian by Christian C. Fenske.

[22] M. Krichman, E. D. Sontag, and Y. Wang. Input-output-to-state stability. *SIAM Journal on Control and Optimization*, 39(6):1874–1928, 2001.

[23] A. Laforgia. Inequalities for Bessel functions. *Journal of Computational and Applied Mathematics*, 15(1):75–81, 1986.

[24] M. Lagache, U. Serres, and J. Gauthier. Exact output stabilization at unobservable points: Analysis via an example. In *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*, pages 6744–6749, Dec 2017.

[25] R. Narasimhan. *Introduction to the theory of analytic spaces*. Lecture Notes in Mathematics, No. 25. Springer-Verlag, Berlin-New York, 1966.

[26] D. Nešić and E. Sontag. Input-to-state stabilization of linear systems with positive outputs. *Systems & Control Letters*, 35(4):245 – 255, 1998.

[27] E. Neuman. Inequalities involving Bessel functions of the first kind. *JIPAM. Journal of Inequalities in Pure and Applied Mathematics*, 5(4):Article 94, 4, 2004.

[28] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[29] A. Rapaport and D. Dochain. A robust asymptotic observer for systems that converge to unobservable states—a batch reactor case study. *IEEE Transactions on Automatic Control*, 65(6):2693–2699, 2020.

[30] L. Sacchelli, L. Brivadis, V. Andrieu, U. Serres, and J.-P. Gauthier. Dynamic output feedback stabilization of non-uniformly observable dissipative systems, 2020.

[31] I. Segal. Non-linear semi-groups. *Annals of Mathematics. Second Series*, 78:339–364, 1963.

[32] H. Shim and A. Teel. Asymptotic controllability and observability imply semiglobal practical asymptotic stabilizability by sampled-data output feedback. *Automatica*, 39(3):441 – 454, 2003.

[33] C. L. Siegel. Über einige Anwendungen diophantischer Approximationen [reprint of Abhandlungen der Preußischen Akademie der Wissenschaften. Physikalisch-mathematische Klasse 1929, Nr. 1]. In *On some applications of Diophantine approximations*, volume 2 of *Quad./Monogr.*, pages 81–138. Ed. Norm., Pisa, 2014.

[34] E. D. Sontag. Conditions for abstract nonlinear regulation. *Information and Control*, 51(2):105 – 127, 1981.

[35] D. Surroop, P. Combes, P. Martin, and P. Rouchon. Adding virtual measurements by pwm-induced signal injection. In *2020 American Control Conference (ACC)*, pages 2692–2698, 2020.

[36] A. Teel and L. Praly. Global stabilizability and observability imply semi-global stabilizability by output feedback. *Systems & Control Letters*, 22(5):313–325, 1994.

[37] A. Teel and L. Praly. Tools for semiglobal stabilization by partial state and output feedback. *SIAM Journal on Control and Optimization*, 33(5):1443–1488, 1995.

[38] A. R. Teel and L. Praly. A smooth Lyapunov function from a class-$\mathcal{KL}$ estimate involving two positive semidefinite functions. *ESAIM. Control, Optimisation and Calculus of Variations. European Series in Applied and Industrial Mathematics*, 5:313–367, 2000.

[39] M. Tucsnak and G. Weiss. *Observation and control for operator semigroups*. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Verlag, Basel, 2009.

[40] N. J. Vilenkin. *Special functions and the theory of group representations*. Translated from the Russian by V. N. Singh. Translations of Mathematical Monographs, Vol. 22. American Mathematical Society, Providence, R. I., 1968.