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SHORT GEODESICS LOSING OPTIMALITY IN CONTACT SUB-RIEMANNIAN MANIFOLDS AND STABILITY OF THE 5-DIMENSIONAL CAUSTIC*

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Abstract. We study the sub-Riemannian exponential for contact distributions on manifolds of dimension greater than or equal to 5. We compute an approximation of the sub-Riemannian Hamiltonian flow and show that the conjugate time can have multiplicity 2 in this case. We obtain an approximation of the first conjugate locus for small radii and introduce a geometric invariant to show that the metric for contact distributions typically exhibits an original behavior, different from the classical 3-dimensional case. We apply these methods to the case of 5-dimensional contact manifolds. We provide a stability analysis of the sub-Riemannian caustic from the Lagrangian point of view and classify the singular points of the exponential map.

Key words. sub-Riemannian geometry, conjugate locus, contact distributions, Lagrangian singularities

AMS subject classifications. 93D05, 53C17, 58K05, 58K40, 34C20

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1. Introduction.

Let M be a smooth (C^∞) manifold of dimension $2n+1$ with $n \geq 1$ integer. A *contact distribution* is a $2n$-dimensional vector subbundle $\Delta \subset TM$ that locally coincides with the kernel of a smooth 1-form ω on M such that $\omega \wedge (\mathrm{d}\omega)^n \neq 0$. The sub-Riemannian structure on M is given by a smooth scalar product g on Δ, and we call (M, Δ, g) a *contact sub-Riemannian manifold* (see, for instance, [1, 2]).

The small scale geometry of general 3-dimensional contact sub-Riemannian manifolds is well understood but not much can be said for dimension 5 and beyond, apart from the particular case of Carnot groups. We are interested in giving a qualitative description of the local geometry of contact sub-Riemannian manifolds by describing the family of short locally-length-minimizing curves (or geodesics) originating from a given point. In the case of contact sub-Riemannian manifolds, all length-minimizing curves are projections of integral curves of an intrinsic Hamiltonian vector field on T^*M and, as such, geodesics are characterized by their initial point and initial covector.

By analogy with the Riemannian case, for all $q \in M$, we denote by $\mathcal{E}_q$ the *sub-Riemannian exponential*, that maps a covector $p \in T_q^*M$ to the evaluation at time 1 of the geodesic curve starting at q with initial covector p. An essential observation on length minimizing curves in sub-Riemannian geometry is that there exist locally-length-minimizing curves that lose local optimality arbitrarily close to their starting point [18, 21, 24]. Hence the geometry of sub-Riemannian balls of small radii is inherently linked with the geometry of the conjugate locus, that is, at q, the set of points $\mathcal{E}_q(p)$ such that p is a critical point of $p \mapsto \mathcal{E}_q(p)$, [7, 8, 9].

The sub-Riemannian exponential has a natural structure of a Lagrangian map, since it is the projection of a Hamiltonian flow over T^*M, and its conjugate locus is a Lagrangian caustic. In small dimensions, this observation allows for the study of

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the stability of the caustic and the classification of singular points of the exponential from the point of view of Lagrangian singularities (see, for instance, [6]).

In the 3-dimensional case, this analysis has been initially conducted with different approaches in [4] and [19]. These works describe asymptotics of the sub-Riemannian exponential, the conjugate, and cut loci near the starting point (see also [5] and recently [15] for later developments on the subject). The aim of the present work is to extend this study to higher-dimensional contact sub-Riemannian manifolds, following the methodology developed in [19] and [17] (the latter focusing on a similar study of quasi-contact sub-Riemannian manifolds). More precisely, we use a perturbative approach to compute approximations of the Hamiltonian flow. This is made possible by using a general well-suited normal form for contact sub-Riemannian structures. The normal form we use has been obtained in [3]. (We recall its properties in [25, Supplementary Materials].

Finally, it can be noted that classical behaviors displayed by 3-dimensional contact sub-Riemannian structures may not be typical in larger dimensions. The 3-dimensional case is very rigid in the class of sub-Riemannian manifolds and appears to be so even in regard to contact sub-Riemannian manifolds of arbitrary dimension. Therefore, part of our focus is dedicated to highlighting the differences between this classical case and those of larger dimension.

1.1. Approximation of short geodesics. Let (M, Δ, g) be a contact sub-Riemannian manifold of dimension $2n+1$, $n \geq 1$ integer. The central idea we follow in this paper is that the sub-Riemannian structure at a point $q \in M$ can be expressed as a small perturbations of the nilpotent structure at $q_0 \in M$ for points q sufficiently close to q_0. As such, geodesics starting at q_0 are expected to be small perturbations of geodesics in the nilpotent approximation of M at q_0, which proves to be fundamental. Indeed, for a given $q_0 \in M$, the nilpotent approximation at q_0, or metric tangent to the sub-Riemannian manifold at q_0 (see [13]), admits both a structure of the Carnot group and contact manifold. Geodesics of such contact Carnot groups can be computed explicitly, and the features of these sub-Riemannian manifolds have been extensively studied (see, for instance, [11, 12, 20, 23]).

Let $H : T^*M \to \mathbb{R}$ be the sub-Riemannian Hamiltonian. Geodesics are the projection on M of integral flow curves in T^*M of the Hamiltonian vector field $\vec{H}$. This implies that they are indexed by their starting point q and their initial covector $p \in T_q^*M$. Since $\vec{H}$ is a quadratic Hamiltonian vector field, its integral curves satisfy the symmetry

$$\mathrm{e}^{t\vec{H}}(p,q) = \mathrm{e}^{\vec{H}}(tp,q) \qquad \forall q \in M, p \in T_q^*M, t \in \mathbb{R}.$$

Hence it is useful for us to consider the time-dependent exponential at q that maps the pair $(t,p) \in \mathbb{R} \times T_q^*M$ to the geodesic of initial covector p and evaluated at time t. For a given $q \in M$, the conjugate time $t_c(p)$ is the smallest positive time such that $\mathcal{E}_q(t_c(p), \cdot)$ is critical at p. Notably, this notion is key in the study of the critical set of the exponential, as computing the conjugate locus follows once the conjugate time is known.

For a contact sub-Riemannian manifold, $H(\cdot, q)$ is a corank 1 positive quadratic form on T_q^*M for all $q \in M$. This implies that the level set

$$\mathcal{C}_q(r) = \{H(p,q) = r \mid p \in T_q^*M\}$$

has the topology of a cylinder $\mathbb{S}^{2n-1} \times \mathbb{R}$ for all $r > 0$ (see, for instance, [1, 2]). We can endow T_q^*M with coordinates $p = (h, h_0)$ respecting this topology, choosing h_0 to denote the coordinate along the 1-dimensional subspace $\ker H(\cdot, q)$.

A crucial observation is that in the contact Carnot group case, geodesics that lose optimality near their starting point correspond to initial covectors in $\mathcal{C}_q(r)$ such that $|h_0|/r$ is very large (see, for instance, [10, 20]). The expansions obtained in this paper rely on applying this fact in the framework of a sub-Riemannian structure expressed as a perturbation of its nilpotent approximation. This is best exemplified by examining the 3-dimensional case, which has already been thoroughly studied (see, for instance, [1, Chapter 19]).

Consider indeed the case $n = 1$. For an initial covector $(\cos\theta, \sin\theta, h_0) \in \mathcal{C}_q(1/2)$, the conjugate time in the nilpotent structure is simply $t_c = 2\pi/|h_0|$ if $h_0 \neq 0$. Moreover, it is proven in [4, 19] that the conjugate time at q satisfies, as $h_0 \to \pm\infty$,

$$t_c(\cos\theta, \sin\theta, h_0) = \frac{2\pi}{|h_0|} - \frac{\pi\kappa}{|h_0|^3} + O\left(\frac{1}{|h_0|^4}\right), \tag{1.1}$$

and the first conjugate point satisfies (in well-chosen adapted coordinates at q)

$$\begin{aligned}\mathcal{E}_q(t_c(\cos\theta, \sin\theta, h_0), (\cos\theta, \sin\theta, h_0)) \\ = \pm\frac{\pi}{|h_0|^2}(0,0,1) \pm \frac{2\pi\chi}{|h_0|^3}(-\sin^3\theta, \cos^3\theta, 0) + O\left(\frac{1}{|h_0|^4}\right).\end{aligned}$$

The analysis we use in sections 2 to 4 aims at generalizing such expansions. (We focus only on the case $h_0 \to +\infty$ but the case $h_0 \to -\infty$ is similar.)

Our results provide important distinctions between the classical 3-dimensional (3D) contact case and higher-dimensional ones. Notably, a very useful fact in the analysis of the geometry of the 3D case is that a 3D sub-Riemannian contact structure is very well approximated by its nilpotent approximation (as exemplified in [7], for instance).

This can be illustrated by using the 3D version of the Agrachev–Gauthier normal form, as introduced in [19]. Denoting by $\widehat{\mathcal{E}}_q$ the exponential of the nilpotent approximation of the sub-Riemannian structure at q_0 in normal form, we have the expansion as $h_0 \to +\infty$,

$$\mathcal{E}_q(\tau/h_0, (h_1, h_2, h_0)) = \widehat{\mathcal{E}}_q(\tau/h_0, (h_1, h_2, h_0)) + O\left(\frac{1}{h_0^3}\right). \tag{1.2}$$

As a result, one immediately obtains a rudimentary version of expansion (1.1),

$$t_c(\cos\theta, \sin\theta, h_0) = \frac{2\pi}{|h_0|} + O\left(\frac{1}{|h_0|^3}\right). \tag{1.3}$$

However, expansion (1.3) is not true in general when we consider contact manifolds of dimension larger than 3 (that is, the conjugate time is not a third order perturbation of the nilpotent conjugate time $2\pi/|h_0|$). As an application of Theorem 3.7, which gives a general second order approximation of the conjugate time in dimension greater than or equal to 5, we are able to prove that the expansion (1.2) does not hold generically (see section 5).

In the rest of this paper, statements refer to generic (d-dimensional) sub-Riemannian contact manifolds in the following sense: such statements hold for contact sub-Riemannian metrics in a countable intersection of open and dense sets of the space of smooth (d-dimensional) sub-Riemannian contact metrics endowed with the C^3-

Whitney topology. As an application of transversality theory, we then prove statements holding on the complementary of stratified subsets of codimension d' of the manifolds, locally unions of finitely many submanifolds of codimension d', at least.

THEOREM 1.1. *Let (M, Δ, g) be a generic contact sub-Riemannian manifold of dimension $2n + 1 > 3$. There exists a codimension 1 stratified subset $\mathfrak{S}$ of M such that for all $q \in M \setminus \mathfrak{S}$, for all linearly adapted coordinates at q, and for all $T > 0$,*

(1.4)
$$\limsup_{h_0 \to +\infty} \left(h_0^2 \sup_{\tau \in (0,T)} \left| \mathcal{E}_q \left(\frac{\tau}{h_0}, (h_1, \dots, h_{2n}, h_0) \right) - \widehat{\mathcal{E}}_q \left(\frac{\tau}{h_0}, (h_1, \dots, h_{2n}, h_0) \right) \right| \right) > 0.$$

This observation needs to be put in perspective with some already observed differences between 3D contact sub-Riemannian manifolds and those of greater dimension. For a given 1-form ω such that $\ker \omega = \Delta$ and $\omega \wedge (\mathrm{d}\omega)^n \neq 0$, the Reeb vector field is the unique vector field X_0 such that $\omega(X_0) = 1$ and $\iota_{X_0} \mathrm{d}\omega = 0$. The contact form ω is not unique (for any smooth nonvanishing function f, $f\omega$ is also a contact form), and neither is X_0. In 3 dimensions however, the conjugate locus lies tangent to a single line that carries a Reeb vector field that is deemed canonical. In larger dimensions, this uniqueness property is not true in general. For this reason, we introduce in section 5 a geometric invariant that plays a similar role in measuring how the conjugate locus lies with respect to the nilpotent conjugate locus and use it to prove Theorem 1.1.

The main difference seems to be a lack of symmetry in greater dimensions. Indeed the existence of a unique Reeb vector field (up to rescaling) points toward the idea of a natural $\mathrm{SO}(2n)$ symmetry of the nilpotent structure. However the actual symmetry of a contact sub-Riemannian manifold (or rather its nilpotent approximation) is $\mathrm{SO}(2)^n$ (on the subject, see, for instance, [3]). Of course, when $n = 1$, $\mathrm{SO}(2)^n = \mathrm{SO}(2n)$. More discussions on this issue can also be found in [16].

1.2. Stability in the 5-dimensional case. We wish to apply these asymptotics to the study of the stability of the caustic in the 5-dimensional case. This study has been carried for 3D contact sub-Riemannian manifolds in [19] and for 4-dimensional quasi-contact sub-Riemannian manifolds in [17].

As stated before, the sub-Riemannian exponential has a natural structure of a Lagrangian map. Hence, in small dimensions, we can rest the analysis of the sub-Riemannian caustic, the set of singular values of the sub-Riemannian exponential, on the classical study of singularities of Lagrangian maps. (See, for instance, [6, Chapters 18, 21] and also [14, 22].) Indeed, for dimensions $d \leq 5$, there exists only a finite number of equivalence classes for stable singularities of Lagrangian maps (for instance, one can find a summary in [9, Theorem 2]) and, critically, for us, if two Lagrangian maps are Lagrange equivalent then their caustics are diffeomorphic.

THEOREM 1.2 (Lagrangian stability in dimension 5). *A generic Lagrangian map $f : \mathbb{R}^5 \to \mathbb{R}^5$ has only stable singularities of type $\mathcal{A}_2, \dots, \mathcal{A}_6$, $\mathcal{D}_4^{\pm}, \mathcal{D}_5^{\pm}, \mathcal{D}_6^{\pm}$, and $\mathcal{E}_6^{\pm}$.*

Sub-Riemannian exponential maps form a subclass of Lagrangian maps and we can define sub-Riemannian stability as Lagrangian stability restricted to the class of sub-Riemannian exponential maps. Notoriously, the point q_0 is an unstable critical value of the sub-Riemannian exponential $\mathcal{E}_{q_0}$, as the starting point of the geodesics defining $\mathcal{E}_{q_0}$.

We focus our study of the stability of the sub-Riemannian caustic on the first conjugate locus. This work can be summarized in the following theorem (see also Figures 1, 2).

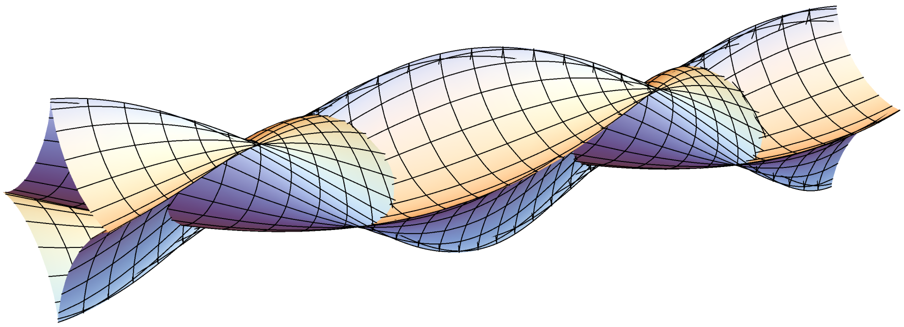


FIG. 1. *Section of the caustic of a* 5*-dimensional sub-Riemannian manifold, at a point of the manifold chosen so that it exhibits* $\mathcal{A}_4$ *singularities. This representation is obtained after sectioning by the hyperplanes* $\{z = z_0\}$, $\{x_3 = R_2 \cos\omega\}$, $\{x_4 = R_2 \sin\omega\}$ *(all in Agrachev–Gauthier normal form coordinates), and plotting for all* $\omega \in [0, 2\pi)$ *with fixed* $z_0, R_2 > 0$.

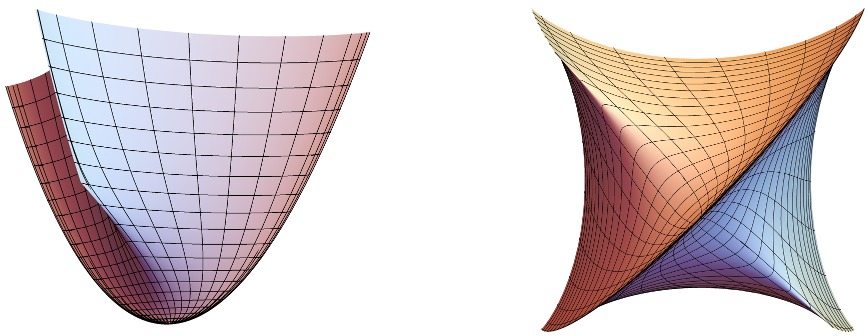


FIG. 2. *Section of the caustic of a* 5*-dimensional sub-Riemannian manifold, at a point of the manifold chosen so that it exhibits* $\mathcal{D}_4^+$ *singularities. This representation is obtained after sectioning by the hyperplanes* $\{z = z_0\}$, $\{x_3 = R_2 \cos\omega\}$, $\{x_4 = R_2 \sin\omega\}$, *and plotting for all* $z_0 \in [0, \bar{z}_0]$, *with fixed* $\bar{z}_0, R_2, \omega > 0$.

THEOREM 1.3 (sub-Riemannian stability in dimension 5). *Let* (M, Δ, g) *be a generic* 5*-dimensional contact sub-Riemannian manifold. There exists a stratified set* $\mathfrak{S} \subset M$ *of codimension* 1 *for which all* $q_0 \in M \setminus \mathfrak{S}$ *admit an open neighborhood* V_{q_0} *such that for all* U *open neighborhood of* q_0 *small enough, the intersection of the interior of the first conjugate locus at* q_0 *with* $V_{q_0} \setminus U$ *is (sub-Riemannian) stable and has only Lagrangian singularities of type* $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$, $\mathcal{D}_4^+$, *and* $\mathcal{A}_5$.

This result stands on two foundations. On one hand, a careful study of the problem of conjugate points in contact sub-Riemannian manifolds, and on the other hand, a stability analysis from the point of view of Lagrangian singularities in small dimensions.

1.3. Content. In section 2, we compute an approximation of the exponential map for small time and large h_0 (Proposition 2.2). Using the Agrachev–Gauthier normal form, the exponential appears to be a small perturbation of the standard nilpotent exponential.

Sections 3–5 are dedicated to the problem of approximating the conjugate time from which an approximation of the conjugate locus can be obtained. The result of this analysis is summarized in Theorem 3.7. Noticeably, a careful analysis of the conjugate time for the nilpotent approximation shows that, under some conditions, the second conjugate time accumulates on the first (section 3.2) and different cases should be treated separately.

Section 4 is specifically dedicated to the computation of higher order approximations of the conjugate time. We first treat the direct case (section 4.1), and treat the problem of a double conjugate time via blowup (section 4.2). With the aim of proving stability of the caustic, we conclude the section by computing a third order approximation of the conjugate time for a small subset of initial covectors (section 4.3).

With the asymptotics of section 4 at hand, we are able in section 5 to prove the two main statements of this paper on approximations of the sub-Riemannian exponential, Theorems 1.1 and 3.7.

Finally, in section 6 we carry a stability analysis of the conjugate locus in the 5-dimensional case. We first observe that we can tackle this analysis relying on a Lagrangian equivalence classification (section 6.1) and show that only stable Lagrangian singularities appear on three domains relevant to this study (section 6.3).

2. Normal extremals.

Notations. In the following, for any two integers $m, n \in \mathbb{N}$, $m \leq n$, we denote by $[\![m, n]\!]$ the set of integers $k \in \mathbb{N}$ such that $m \leq k \leq n$.

Let $(x_1, \ldots, x_{2n}, z) : M \to \mathbb{R}^{2n+1}$ be a set of privileged coordinates at $q \in M$. For any vector field Y, for all $i \in [\![1, 2n+1]\!]$, we denote by $(Y)_i$ the ith coordinate of Y written in the basis $(\partial_{x_1}, \ldots, \partial_{x_{2n}}, \partial_z)$.

2.1. The local sub-Riemannian structure as a perturbation of the nilpotent approximation.

Let (M, Δ, g) be a $(2n+1)$-dimensional contact sub-Riemannian manifold.

Consider a 1-form ω such that $\ker \omega = \Delta$ and $\omega \wedge (\mathrm{d}\omega)^n \neq 0$. For all $q \in M$, there exists a linear map $A(q) : \Delta_q \to \Delta_q$, skew-symmetric with respect to g_q, such that for all $X, Y \in \Delta$, $\mathrm{d}\omega(X, Y)(q) = g_q(A(q)X(q), Y(q))$. Neither ω nor A are unique, but the eigenvalues of $A(q)$, $\{\pm ib_1, \ldots, \pm ib_n\}$, are invariants of the sub-Riemannian structure at q up to a multiplicative constant. In the following, we will assume that the invariants $\{b_1, \ldots, b_n\} \in \mathbb{R}^+$ are rescaled so that $b_1 \cdots b_n = \frac{1}{n!}$.

These invariants are parameters of the nilpotent approximation at q. For instance, their contribution to the metric can be made explicit via the introduction of a normal form of the nilpotent approximation. There exists a set of coordinates $(x_1, \ldots, x_{2n}, z) : \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$ such that a frame $(\widehat{X}_1, \ldots, \widehat{X}_{2n})$ of the nilpotent approximation at q can be written in the form

$$\widehat{X}_{2i-1} = \partial_{x_{2i-1}} + \frac{b_i}{2} x_{2i} \partial_z, \quad \widehat{X}_{2i} = \partial_{x_{2i}} - \frac{b_i}{2} x_{2i-1} \partial_z \qquad \forall i \in [\![1, n]\!].$$

Notice in particular that the nilpotent approximations of a contact sub-Riemannian structure at two points $q_1, q_2 \in M$ may not be isometric if the dimension $2n+1$ is larger than 3.

An important tool we use is the Agrachev–Gauthier normal form, introduced in [3], which endows the structure with normal coordinates and a frame displaying useful symmetries.

THEOREM 2.1 (see [3, section 6]). *Let (M, Δ, g) be a contact sub-Riemannian manifold of dimension $2n+1$ and $q \in M$. There exist privileged coordinates at q, $(x_1, \ldots, x_{2n}, z) : M \to \mathbb{R}^{2n+1}$, and a frame of (Δ, g), $(X_1, \ldots, X_{2n})$, that satisfy the following properties on a small neighborhood of $q = (0, \ldots, 0)$.*

(1) *The horizontal components of the vector fields $X_1, \dots, X_{2n}$ satisfy the two symmetries*

$$(X_i)_j = (X_j)_i \qquad \textit{and} \qquad \sum_{j=1}^{2n} (X_j)_i \, x_j = x_i \qquad \forall i, j \in [\![1, 2n]\!].$$

(2) *The vertical components of $X_1, \dots, X_{2n}$ satisfy the symmetry*

$$\sum_{j=1}^{2n} (X_j)_{2n+1} \, x_j = 0.$$

(3) *Denoting $X_0 = \frac{\partial}{\partial z}$ and ω the contact form such that $(\mathrm{d}\omega)^n_{|\Delta}$ coincides with the volume form induced by g on Δ, we have*

$$\omega(X_0) = 1 \qquad \textit{and} \qquad \iota_{X_0} \, \mathrm{d}\omega = 0.$$

Finer relations can be obtained using these symmetries, which is the object of [3]. However, the computations present in this paper essentially rely on the following consequence of Theorem 2.1 (the method is further discussed in [25, Supplementary Materials]. Let $q \in M$, let $(x_1, \dots, x_{2n}, z)$ and $(X_1, \dots, X_{2n}, X_0)$ be in the Agrachev–Gauthier normal form centered at q, that is, as described in Theorem 2.1. For all $i \in [\![1, 2n]\!]$, there exists a smooth vector field R_i such that on a small neighborhood of q, for all $i, j \in [\![1, 2n]\!]$, for all $k \in \mathbb{N}$,

$$\partial_z^k R_i(0) = \partial_z^k \partial_{x_j} R_i(0) = 0 \qquad \text{and} \qquad X_i(x, z) = \widehat{X}_i(x, z) + R_i(x, z). \tag{2.1}$$

Or, equivalently, uniformly on a small neighborhood of q,

$$X_i(x, z) = \widehat{X}_i(x, z) + O\left(|x|^2\right).$$

2.2. Geodesic equation in perturbed form. In this section we establish the dynamical system satisfied by geodesics in terms of small perturbations of the nilpotent structure.

Let V be an open subset of M and $(X_1, \dots, X_{2n})$ be a frame of (Δ, g) on V, that is, a family of vector fields such that $g_q(X_i(q), X_j(q)) = \delta_i^j$ for all $i, j \in [\![1, 2n]\!]$ and all $q \in V$ (such a family always exists for V sufficiently small). The sub-Riemannian Hamiltonian can be written

$$H(p, q) = \frac{1}{2} \sum_{i=1}^{2n} \langle p, X_i(q) \rangle^2.$$

In the case of contact distributions, locally-length-minimizing curves are projections of normal extremals, the integral curves of the Hamiltonian vector field $\vec{H}$ on T^*M (see, for instance, [1, 2]). In other words, a normal extremal $t \mapsto (p(t), q(t))$ satisfies in coordinates the Hamiltonian ordinary differential equation

$$\begin{cases} \dfrac{\mathrm{d}q}{\mathrm{d}t} = \displaystyle\sum_{i=1}^{2n} \langle p, X_i(q) \rangle X_i(q), \\ \dfrac{\mathrm{d}p}{\mathrm{d}t} = -\displaystyle\sum_{i=1}^{2n} \langle p, X_i(q) \rangle \, {}^t p \, D_q X_i(q). \end{cases} \tag{2.2}$$

For V sufficiently small, we can arbitrarily choose a nonvanishing vector field X_0 transverse to Δ in order to complete $(X_1(q), \ldots, X_{2n}(q))$ into a basis of T_qM at any point q of V. We use the family $(X_1, \ldots, X_{2n}, X_0)$ to endow T^*M with dual coordinates $(h_1, \ldots, h_{2n}, h_0)$ such that

$$h_i(p, q) = \langle p, X_i(q)\rangle \qquad \forall i \in [\![0, 2n]\!] \ \forall q \in V \ \forall p \in T_q^*M.$$

We also introduce the structural constants $(c_{ij}^k)_{i,j,k\in[\![0,2n]\!]}$ on V, defined by the relations

$$[X_i, X_j](q) = \sum_{k=0}^{2n} c_{ij}^k(q) X_k(q) \qquad \forall i, j \in [\![0, 2n]\!] \ \forall q \in V.$$

In terms of the coordinates $(h_i)_{i\in[\![0,2n]\!]}$, along a normal extremal, (2.2) yields (see [1, Chapter 4])

$$\frac{\mathrm{d}h_i}{\mathrm{d}t} = \{H, h_i\} = \sum_{j=0}^{2n}\sum_{k=0}^{2n} c_{ji}^k h_j h_k \quad \forall i \in [\![0, 2n]\!].$$

We set $J : V \to \mathcal{M}_{2n}(\mathbb{R})$ to be the matrix such that $J_{ij} = c_{ji}^0$ for all $i, j \in [\![1, 2n]\!]$, and $Q : V \longrightarrow \left(\mathbb{R}^{2n} \to \mathbb{R}^{2n}\right)$ to be the map such that for all $i \in [\![1, 2n]\!]$,

$$Q_i(h_1, \ldots, h_{2n}) = \sum_{j=1}^{2n}\sum_{k=1}^{2n} c_{ji}^k h_j h_k.$$

By denoting $h = (h_1, \ldots, h_{2n})$ we then have $\frac{\mathrm{d}h}{\mathrm{d}t} = h_0 J h + Q(h)$.

As stated in section 1, we want an approximation of the geodesics for small time when $h_0(0) \to +\infty$, thus we introduce $w = \frac{h_0(0)}{h_0}$ and $\eta = h_0(0)^{-1}$. Then $\frac{\mathrm{d}w}{\mathrm{d}t} = -\eta w^2 \frac{\mathrm{d}h_0}{\mathrm{d}t}$.

We separate the terms containing h_0 in the derivative of w to obtain an equation similar to the one of h. We set $L : V \to \mathcal{M}_{1\times 2n}(\mathbb{R})$ to be the line matrix such that $L_i = c_{i0}^0$, for all $i \in [\![1, 2n]\!]$, and $Q_0 : V \to \left(\mathbb{R}^{2n} \to \mathbb{R}\right)$ to be the map such that

$$Q_0(h_1, \ldots, h_{2n}) = \sum_{j=1}^{2n}\sum_{k=1}^{2n} c_{j0}^k h_j h_k,$$

so that $\frac{\mathrm{d}w}{\mathrm{d}t} = -wLh - \eta w^2 Q_0(h)$.

Finally, rescaling time with $\tau = t/\eta$, we obtain

$$\begin{cases} \dfrac{\mathrm{d}q}{\mathrm{d}\tau} = \eta \displaystyle\sum_{i=1}^{2n} h_i X_i(q), \\ \dfrac{\mathrm{d}h}{\mathrm{d}\tau} = \dfrac{1}{w} J h + \eta\, Q(h), \\ \dfrac{\mathrm{d}w}{\mathrm{d}\tau} = -\eta w L h - \eta^2 w^2 Q_0(h). \end{cases} \tag{2.3}$$

Hence to the solution of (2.2) with initial condition $(q_0, (h(0), \eta^{-1}))$ corresponds the solution of the parameter depending differential equation (2.3) of initial condition $(q_0, h(0), w(0))$ and parameter η. Since $w(0) = 1$, the flow of this ODE is well defined (at least for τ small enough), and smooth with respect to $\eta \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$.

This motivates in the following a power series study of its solutions as $\eta \to 0$.

2.3. Approximation of the Hamiltonian flow. We now use the elements we introduced in the previous two sections to compute an approximations of the geodesics starting from a point $q_0 \in M$. In the rest of the paper, except when explicitly stated otherwise, we assume the structure on a neighborhood V of q_0 has been put in the Agrachev–Gauthier normal form discussed in 2.1, where we denote the coordiantes by $(x_1, \dots, x_{2n}, z) : V \to \mathbb{R}^{2n+1}$ and the frame by $(X_1, \dots, X_{2n})$, locally completed as a basis of TM with $X_0 = \frac{\partial}{\partial z}$.

Let us introduce a few notations. Let $\bar{J} = J(q_0)$. As a consequence of the choice of frame, (in particular, see (2.1)), $\bar{J}$ is already in reduced form $\mathrm{diag}(\bar{J}_1, \dots, \bar{J}_n)$, that is, block diagonal with 2×2 blocks

$$\bar{J}_i = \begin{pmatrix} 0 & b_i \\ -b_i & 0 \end{pmatrix} \qquad \forall i \in [\![1, n]\!],$$

where $(b_i)_{i \in [\![1,n]\!]}$, are the nilpotent invariants of the contact structure at q_0. Then let $\hat{h} : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, $\hat{x} : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$, and $\widehat{z} : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be defined by

$$\hat{h}(t,h) = \mathrm{e}^{t\bar{J}} h, \qquad \hat{x}(t,h) = \bar{J}^{-1}(\mathrm{e}^{t\bar{J}} - I_{2n})h,$$
$$\widehat{z}(t,h) = \sum_{i=1}^{n} \left(h_{2i-1}^2 + h_{2i}^2\right) \frac{b_i t - \sin(b_i t)}{2b_i}$$

for all $t \in \mathbb{R}$ and all $h \in \mathbb{R}^{2n}$.

We also set $J^{(1)} : \mathbb{R}^{2n} \to \mathcal{M}_{2n}(\mathbb{R})$ such that

$$J^{(1)}_{i,j}(y) = \sum_{k=1}^{2n} \left(\frac{\partial^2 (X_i)_{2n+1}}{\partial x_j \partial x_k} - \frac{\partial^2 (X_j)_{2n+1}}{\partial x_i \partial x_k} \right) y_k \qquad \forall i, j \in [\![1, 2n]\!],$$

where for any vector field Y, we denote by $(Y)_i$, $1 \le i \le 2n+1$, the ith coordinate of Y, written in the basis $(\partial_{x_1}, \dots, \partial_{x_{2n}}, \partial_z)$.

Finally, let us denote $B_R = \{h \in \mathbb{R}^{2n} \mid \sum_{i=1}^{2n} h_i^2 \le R\}$.

PROPOSITION 2.2. *For all $T, R > 0$, normal extremals with initial covector $(h(0), \eta^{-1})$ have the following order 2 expansion at time $\eta\tau$, as $\eta \to 0^+$, uniformly with respect to $\tau \in [0, T]$ and $h(0) \in B_R$. In normal form coordinates, we denote*

$$\mathrm{e}^{\eta\tau\vec{H}} \left((0,0), \left(h(0), \eta^{-1}\right)\right) = \left((x(\tau), z(\tau)), \left(h(\tau), \eta w(\tau)^{-1}\right)\right).$$

Then

$$x(\tau) = \eta\hat{x}(\tau, h(0)) + \eta^2 \int_0^\tau \int_0^\sigma \mathrm{e}^{(\sigma-\rho)\bar{J}} J^{(1)}\left(\hat{x}(\rho, h(0))\right) \hat{h}(\rho, h(0))\, \mathrm{d}\rho\, \mathrm{d}\sigma + O(\eta^3),$$
$$z(\tau) = \eta^2 \widehat{z}(\tau, h(0)) + O(\eta^3),$$

and

$$h(\tau) = \hat{h}(\tau, h(0)) + \eta \int_0^\tau \mathrm{e}^{(\tau-\sigma)\bar{J}} J^{(1)}\left(\hat{x}(\sigma, h(0))\right) \hat{h}(\sigma, h(0))\, \mathrm{d}\sigma + O(\eta^2),$$
$$w(\tau) = 1 + O(\eta^2).$$

Proof. This is a consequence of the integration of the time-rescaled system (2.3). Since the system smoothly depends on η near 0, we prove this result by successive integration of the terms of the power series in η of $x = \sum \eta^k x^{(k)}$, $z = \sum \eta^k z^{(k)}$, $h = \sum \eta^k h^{(k)}$, and $w = \sum \eta^k w^{(k)}$.

Let $T, R > 0$. All asymptotic expressions are to be understood uniform with respect to $\tau \in [0, T]$ and $h(0) \in B_R$. Solutions of (2.3) are integral curves of a Hamiltonian vector field $\vec{H}$, hence, H is preserved along the trajectory, that is, for all $\tau \in [0, T]$,

$$\sum_{i=1}^{2n} h_i(\tau)^2 = \sum_{i=1}^{2n} h_i(0)^2.$$

Furthermore, we have by (2.3) $\frac{\mathrm{d}x}{\mathrm{d}\tau} = O(\eta)$, $\frac{\mathrm{d}z}{\mathrm{d}\tau} = O(\eta)$, and since $x(0) = 0$ and $z(0) = 0$, we have $x(\tau) = O(\eta)$ and $z(\tau) = O(\eta)$.

As a consequence of the choice of frame (see, in particular, (2.1)), $c_{ij}^k(q_0) \neq 0$ if and only if $k = 0$ and there exists $l \in [\![1, n]\!]$ such that $\{i, j\} = \{2l-1, 2l\}$.

Hence for all $j \in [\![1, 2n]\!]$, $c_{j0}^0(q(\tau)) = O(\eta)$ and $Lh = O(\eta)$. Similarly, $Q_i(h) = O(\eta)$ for all $i \in [\![0, 2n]\!]$ and, since $w(0) = 1$, we have that $\frac{\mathrm{d}w}{\mathrm{d}\tau} = O(\eta^2)$ and $w(\tau) = 1 + O(\eta^2)$.

Since $J(q_0) = \bar{J}$, we have $J(q) = \bar{J} + O(\eta)$ and thus $\frac{\mathrm{d}h}{\mathrm{d}\tau} = \bar{J}h + O(\eta)$. Hence h is a small perturbation of the solution of $\frac{\mathrm{d}h}{\mathrm{d}\tau} = \bar{J}h$ with initial condition $h(0)$, that is, $h(\tau) = \hat{h}(\tau, h(0)) + O(\eta)$.

Since $X_i(q_0) = \frac{\partial}{\partial x_i}$ for all $i \in [\![1, 2n]\!]$ (as a consequence of (2.1)),

$$\frac{\mathrm{d}x^{(1)}}{\mathrm{d}\tau} = h^{(0)}(\tau) = \hat{h}(\tau, h(0)), \quad \frac{\mathrm{d}z^{(1)}}{\mathrm{d}\tau} = 0,$$

and since $x(0) = 0$, $z(0) = 0$, $x(\tau) = \eta\hat{x}(\tau, h(0)) + O(\eta^2)$, and $z(\tau) = O(\eta^2)$.

The definition of $J^{(1)}$ implies $J^{(1)}(x^{(1)}) = \frac{\partial J(q)}{\partial \eta}\big|_{\eta=0}$. Then, since $Q(h) = O(\eta)$, $h^{(1)}$ is solution of $\frac{\mathrm{d}h^{(1)}}{\mathrm{d}\tau} = \bar{J}h^{(1)} + J^{(1)}\left(x^{(1)}\right)$ with initial condition $h^{(1)}(0) = 0$. Hence

$$h^{(1)}(\tau) = \int_0^\tau \mathrm{e}^{(\tau-\sigma)\bar{J}} J^{(1)}\left(\hat{x}(\sigma, h(0))\right) \hat{h}(\sigma, h(0))\, \mathrm{d}\sigma.$$

Since $\frac{\partial (X_i)_j}{\partial x_k} = 0$ for all $i, j, k \in [\![1, 2n]\!]$ (as stated in (2.1)),

$$X_{2i-1}(q(\tau)) = \partial_{x_{2i-1}} + \eta\, \hat{x}_{2i}(\tau, h(0)) \frac{b_i}{2} \partial_z + O(\eta^2),$$

$$X_{2i}(q(\tau)) = \partial_{x_{2i}} - \eta\, \hat{x}_{2i-1}(\tau, h(0)) \frac{b_i}{2} \partial_z + O(\eta^2).$$

Thus $\frac{\mathrm{d}x^{(2)}}{\mathrm{d}\tau} = h^{(1)}$, $\frac{\mathrm{d}z^{(2)}}{\mathrm{d}\tau} = \sum_{i=1}^n \frac{b_i}{2}(\hat{h}_{2i-1}\hat{x}_{2i} - \hat{h}_{2i}\hat{x}_{2i-1})$. Hence we have the statement by integration. □

3. Conjugate time.

3.1. Singularities of the sub-Riemannian exponential.

DEFINITION 3.1. *Let $q_0 \in M$. We call the* sub-Riemannian exponential at q_0 *the map*

$$\begin{array}{rccc} \mathcal{E}_{q_0}: & \mathbb{R}^+ \times T_{q_0}^* M & \longrightarrow & M, \\ & (t, p_0) & \longmapsto & \mathcal{E}_{q_0}(t, p_0) = \pi \circ \mathrm{e}^{t\vec{H}}(p_0, q_0), \end{array}$$

where $\pi : T^*M \to M$ is the canonical fiber projection.

Recall that the flow of the Hamiltonian vector field $\vec{H}$ satisfies the equality

$$\mathrm{e}^{t\vec{H}}(p_0, q_0) = \mathrm{e}^{\vec{H}}(tp_0, q_0) \qquad \forall q_0 \in M, p_0 \in T^*_{q_0}M, t \in \mathbb{R}.$$

We use this property to our advantage to compute the sub-Riemannian caustic. Indeed, the caustic at q_0 is defined as the set of critical values of $\mathcal{E}_{q_0}(1,\cdot)$. But for any time $t > 0$, the caustic is also the set of critical values of $\mathcal{E}_{q_0}(t,\cdot)$. Hence instead of classifying the covectors p_0 such that $\mathcal{E}_{q_0}(1,\cdot)$ is critical at p_0, we compute for a given p_0 the conjugate time $t_c(p_0)$ such that $\mathcal{E}_{q_0}(t_c(p_0),\cdot)$ is critical at p_0.

Definition 3.2. *Let $q_0 \in M$ and $p_0 \in T^*_{q_0}M$. A* conjugate time for p_0 *is a positive time $t > 0$ such that the map $\mathcal{E}_{q_0}(t,\cdot)$ is critical at p_0. The* conjugate locus of q_0 *is the subset of M,*

$$\{\mathcal{E}_{q_0}(t,p_0) \mid t \textit{ is a conjugate time for } p_0 \in T_{q_0}M\}.$$

The first conjugate time for p_0*, denoted $t_c(p_0)$, is the minimum of conjugate times for p_0. The* first conjugate locus of q_0 *is the subset of M*

$$\{\mathcal{E}_{q_0}(t,p_0) \mid t \textit{ is the first conjugate time for } p_0 \in T_{q_0}M\}.$$

In the following, we restrict our study of the sub-Riemannian caustic to the first conjugate locus.

From now on, let us index the nilpotent invariants in descending order $b_1 \geq b_2 \geq \cdots \geq b_n > 0$. Let $\mathfrak{S}_1 \subset M$ be the set of points of M such that two invariants coincide, $b_i = b_j$, with $i \neq j$. Assuming genericity of the sub-Riemannian manifold, $\mathfrak{S}_1$ is a stratified subset of M of codimension 3 (see [17] for instance).

Remark 3.3. This is a consequence of Thom's transversality theorem applied to the jets of the sub-Riemannian structure, seen as a smooth map.

Furthermore, for a given $q_0 \in M$, if the sub-Riemannian structure at q_0 is in Agrachev–Gauthier normal form then the jets of order k at q_0 of the sub-Riemannian structure are given by the jets at 0 of the vector fields $X_1, \ldots, X_{2n}$ (see [3]).

As stated previously, to study the sub-Riemannian caustic near its starting point, we consider asymptotic expansions for initial covectors $p = (h, h_0)$ in $\mathcal{C}_{q_0}(1/2)$ such that $|h_0| \to \infty$.

Let us recall that the family of geodesics with initial covectors in $\mathcal{C}_{q_0}(1/2)$ are parametrized by arc length, hence, $t_c(p)$ is an upper bound on the distance between q_0 and the critical value $\mathcal{E}_{q_0}(t_c(p), p)$. We show in the following that we have the relation

$$\lim_{h_0 \to +\infty} t_c(h, h_0) = 0.$$

However this approach is justified because the converse also holds: a short conjugate time implies h_0 is large. Formally, we have the following fact.

Proposition 3.4. *Let (M, Δ, g) be a contact sub-Riemannian manifold and $q_0 \in M$. For all $\bar{h}_0 > 0$, there exists $\varepsilon > 0$ such that all $p \in \mathcal{C}_{q_0}(1/2)$ with $t_c(p) < \varepsilon$ have $|h_0(p)| > \bar{h}_0$.*

A proof of this classical observation is given in [25, Supplementary Materials, Proposition SM1.4], as an application of the Agrachev–Gauthier normal form.

In coordinates, conjugate points satisfy the following equality

$$\det\left(\frac{\partial \mathcal{E}_{q_0}}{\partial h_1}, \ldots, \frac{\partial \mathcal{E}_{q_0}}{\partial h_{2n}}, \frac{\partial \mathcal{E}_{q_0}}{\partial h_0}\right)\Bigg|_{(t,p_0)} = 0. \tag{3.1}$$

To use this equation in relation to the results of Proposition 2.2, we introduce

$$F(\tau, h, \eta) = \mathcal{E}_{q_0}(\eta\tau; (h, \eta^{-1})) \quad \forall \tau > 0, h \in \mathbb{R}^{2n}, \eta > 0.$$

Then

$$\frac{\partial \mathcal{E}_{q_0}}{\partial h_0}(\eta\tau; (h, \eta^{-1})) = -\eta \left(\eta \frac{\partial F}{\partial \eta}(\tau, h, \eta) - \tau \frac{\partial F}{\partial \tau}(\tau, h, \eta) \right)$$

and (3.1) equates to

$$\det \left(\frac{\partial F}{\partial h_1}, \dots, \frac{\partial F}{\partial h_{2n}}, \eta \frac{\partial F}{\partial \eta} - \tau \frac{\partial F}{\partial \tau} \right) \Big|_{(\tau, h, \eta)} = 0. \tag{3.2}$$

We have shown in Proposition 2.2, as $\eta \to 0$, that the map F is a perturbation of the map $(\tau, h, \eta) \mapsto (\hat{x}, \widehat{z})$, the nilpotent exponential map. Hence the conjugate time is expected to be a perturbation of the conjugate time for $(\hat{x}, \widehat{z})$. To get an approximation of the conjugate time for a covector (h, η^{-1}) as $\eta \to 0$, we use expansions from Proposition 2.2 to derive equations on a power series expansion of the conjugate time.

3.2. Nilpotent order and doubling of the conjugate time. Let us define

$$\Phi(\tau, h, \eta) = \det \left(\frac{\partial F}{\partial h_1}, \dots, \frac{\partial F}{\partial h_{2n}}, \eta \frac{\partial F}{\partial \eta} - \tau \frac{\partial F}{\partial \tau} \right) \Big|_{(\tau, h, \eta)} \tag{3.3}$$

and its power series expansion $\Phi(\tau, h, \eta) = \sum_{k \geq 0} \eta^k \Phi^{(k)}(\tau, h)$.

As a first application of Proposition 2.2, notice that $F_i = O(\eta)$ for all $i \in [\![1, 2n]\!]$, while $F_{2n+1} = O(\eta^2)$. Hence, one gets $\Phi^{(k)} = 0$ for all $k \in [\![0, 2n+1]\!]$, and $\Phi^{(2n+2)}$ is the first nontrivial term in the power series.

To study $\Phi^{(2n+2)}$, let us introduce the set $Z = \{2k\pi/b_i \mid i \in [\![1, n]\!], k \in \mathbb{N}\}$ and the map $\psi : (\mathbb{R}^+ \setminus Z) \times \mathbb{R}^n \to \mathbb{R}$ defined by

$$\psi(\tau, r) = \sum_{i=1}^{n} \frac{r_i^2}{2} \left(3\tau - b_i \tau^2 \frac{\cos(b_i \tau/2)}{\sin(b_i \tau/2)} - \frac{\sin(b_i \tau)}{b_i} \right) \quad \forall (\tau, r) \in (\mathbb{R}^+ \setminus Z) \times \mathbb{R}^n.$$

We first need the following result on the zeros of ψ (see, for instance, [25, Supplementary Materials].

LEMMA 3.5. *Assume $b_1 > b_2 \geq \dots \geq b_n$. For all $r \in (\mathbb{R}^+)^n$, let $\tau_1(r)$ be the first positive time in $\mathbb{R}^+ \setminus Z$ such that $\psi(\tau_1, r) = 0$. Then $\tau_1(r_1, \dots, r_n) > 2\pi/b_1$ and there exists $f(r_2, \dots, r_n) > 0$ such that, as $r_1 \to 0^+$,*

$$\tau_1(r_1, \dots, r_n) = 2\pi/b_1 + f(r_2, \dots, r_n) r_1^2 + o(r_1^2). \tag{3.4}$$

The zeros of $\Phi^{(2n+2)}$ can be deduced from the zeros of ψ, as shown in the following proposition.

PROPOSITION 3.6. *Assume $b_1 > b_2 > \dots > b_n$. Let $h \in \mathbb{R}^{2n} \setminus \{0\}$ and $r \in \mathbb{R}^n$ be such that $r_i = \sqrt{h_{2i-1}^2 + h_{2i}^2}$ for all $i \in [\![1, n]\!]$. Then $\Phi^{(2n+2)}(\tau, h) = 0$ if and only if $\tau \in Z$ or $\psi(\tau, r) = 0$. In particular*

$$\Phi^{(2n+2)}(\tau, h) \neq 0 \quad \forall \tau \in (0, 2\pi/b_1)\ \forall h \in \mathbb{R}^{2n} \setminus \{0\}.$$

Proof. By factorizing powers of η in Φ, we obtain that $\Phi^{(2n+2)}$ is given by the determinant of the matrix

$$M = \left(\begin{array}{c|c} D_h\hat{x}(\tau) & \hat{x}(\tau) - \tau\hat{h}(\tau) \\ \hline D_h\widehat{z}(\tau) & \widehat{z}(\tau) - \tau\frac{\mathrm{d}}{\mathrm{d}\tau}\widehat{z}(\tau) \end{array} \right).$$

The Jacobian matrix $D_h\hat{x} = \bar{J}^{-1}(\mathrm{e}^{\tau\bar{J}} - I_{2n})$ is invertible for $\tau \in \mathbb{R}^+ \setminus Z$ and of rank $2n-2$ for $\tau \in Z$. Hence, the matrix M is not invertible for $\tau \in \mathbb{R}^+ \setminus Z$ if and only if we have the linear dependance of the family

$$\left\{ \frac{\partial}{\partial h_1}\left(\frac{\hat{x}(\tau)}{\widehat{z}(\tau)}\right), \dots, \frac{\partial}{\partial h_{2n}}\left(\frac{\hat{x}(\tau)}{\widehat{z}(\tau)}\right), \left(\frac{\hat{x}(\tau) - \tau\hat{h}(\tau)}{\widehat{z}(\tau) - \tau\frac{\mathrm{d}}{\mathrm{d}\tau}\widehat{z}(\tau)}\right) \right\}.$$

This implies the existence of $\mu \in \mathbb{R}^{2n}$ such that both $D_h\hat{x}(\tau)\mu = \hat{x}(\tau) - \tau\hat{h}(\tau)$ and $D_h\widehat{z}(\tau)\mu = \widehat{z}(\tau) - \tau\frac{\mathrm{d}}{\mathrm{d}\tau}\widehat{z}(\tau)$. That is

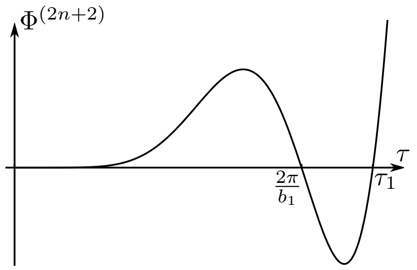
$$D_h\widehat{z}(\tau)\left(D_h\hat{x}(\tau)\right)^{-1}\left(\hat{x}(\tau) - \tau\hat{h}(\tau)\right) = \widehat{z}(\tau) - \tau\frac{\mathrm{d}}{\mathrm{d}\tau}\widehat{z}(\tau).$$

We explicitly have $\widehat{z}(\tau) - \tau\frac{\mathrm{d}}{\mathrm{d}\tau}\widehat{z}(\tau) = \sum_{i=1}^n \frac{r_i^2}{2}\left(\tau\cos b_i\tau - \frac{\sin b_i\tau}{b_i}\right)$ and

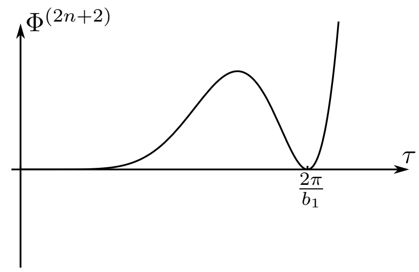
$$D_h\widehat{z}(\tau)\left(D_h\hat{x}(\tau)\right)^{-1}\left(\hat{x}(\tau) - \tau\hat{h}(\tau)\right) = \sum_{i=1}^n r_i^2\left(\sin b_i\tau - b_i\tau\right)\frac{b_i\tau\cos(b_i\tau/2) - 2\sin(b_i\tau/2)}{2b_i\sin(b_i\tau/2)}.$$

Hence $D_h\widehat{z}(D_h\hat{x})^{-1}(\hat{x} - \tau\hat{h}) - (\widehat{z} - \tau\frac{\mathrm{d}\widehat{z}}{\mathrm{d}\tau}) = \psi(\tau, r)$, and times $\tau \in \mathbb{R}^+$ such that $\Phi_k^{(2n+2)}(\tau, h) = 0$ are either multiples of $2\pi b_i$, $i \in [\![1, n]\!]$, or zeros of ψ. Under the assumption that $h \in \mathbb{R}^{2n} \setminus \{0\}$ and $\tau \in (0, 2b_i\pi)$, we have $\psi(\tau, r) > 0$, hence, the result. ∎

We can draw some conclusions regarding our analysis of the conjugate locus via a perturbative approach. From Proposition 3.6, we have that $2\pi/b_1$ is the first zero of $\Phi^{(2n+2)}(\cdot, h)$ for all $h \in \mathbb{R}^{2n} \setminus \{0\}$. From Lemma 3.5 we also know that $2\pi/b_1$ is a simple zero if $\sqrt{h_1^2 + h_2^2} = r_1 > 0$ and a double zero otherwise (see Figure 3). Zeros of order 2 or more can be unstable under perturbation and this case requires a separate analysis, either by high order approximation or by blowup. We choose the latter for computational reasons.



(a) $\Phi^{(2n+2)}$ as $r_1 = \frac{r_2}{4}$.



(b) $\Phi^{(2n+2)}$ as $r_1 = 0$.

FIG. 3. *Representation of $\Phi^{(2n+2)}$ as a function of τ in the case $n = 2$, as $r_1 \neq 0$ and $r_1 = 0$ (with $b_1 = 2$, $b_2 = 1/4$, and $r_2 = 1$).*

From (3.4) in Lemma 3.5, the blowup $r_1 \leftarrow \eta^\alpha r_1$ corresponds to

$$\tau_1(\eta^\alpha r_1, r_2, \dots, r_n) = 2\pi/b_1 + \eta^{2\alpha} f(r_2, \dots, r_n) r_1^2 + o(\eta^{2\alpha}).$$

Since we have an approximation of the exponential that is a perturbation of order η of the nilpotent exponential, we expect the conjugate time to be a perturbation of order η of the nilpotent conjugate time. Hence it is natural to chose $\alpha = 1/2$ in hopes of capturing a perturbation of comparable order in η.

We separate the cases in the following way:

- We can compute the conjugate time assuming $\sqrt{h_1^2 + h_2^2} = r_1 > \varepsilon$ for some arbitrary ε (in section 4.1);
- we use the blowup $r_1 \leftarrow \sqrt{\eta} r_1$ to get the conjugate time near $r_1 = 0$ (in section 4.2).

3.3. Statement of the conjugate time asymptotics. The focus of this paper is now devoted to the proof of the following asymptotic expansion theorem for the conjugate time on $M \setminus \mathfrak{S}_1$, that is, at points such that $b_1 > b_2 > \cdots > b_n$. Let S_1 be the subspace of $T_{q_0}^* M$ defined by

$$S_1 = \left\{(h_1, \dots, h_{2n}, h_0) \in T_{q_0}^* M \setminus \mathcal{C}_{q_0}(0) \mid h_1 = h_2 = 0, H \neq 0\right\}$$

and, for all $\varepsilon > 0$, let us denote by S_1^ε the subset of $T_{q_0}^* M$ containing S_1:

$$S_1^\varepsilon = \left\{(h_1, \dots, h_{2n}, h_0) \in T_{q_0}^* M \setminus \mathcal{C}_{q_0}(0) \mid h_1^2 + h_2^2 < \varepsilon H(h_1, \dots, h_{2n}, h_0)\right\}.$$

Abusing notations, for $V \subset \mathbb{R}^+$, we denote $\mathcal{C}_{q_0}(V) = \cup_{r \in V} \mathcal{C}_{q_0}(r)$.

THEOREM 3.7. *Let $q_0 \in M \setminus \mathfrak{S}_1$. There exist real valued invariants $(\kappa_k^{ij})_{\substack{i,k \in [\![1,2]\!],\\ j \in [\![3,2n]\!]}}$, α, β, such that we have the following asymptotic behavior for initial covectors $p_0 \in T_{q_0}^* M$ with $h_0 \to +\infty$.*

Away from S_1. For all $R > 0, \varepsilon \in (0,1)$, uniformly with respect to $p_0 = (h_1, \dots, h_{2n}, h_0)$ in $\mathcal{C}_{q_0}((0,R)) \setminus S_1^\varepsilon$, we have as $h_0 \to +\infty$

$$t_c(h_1, \dots, h_{2n}, h_0) = \frac{2\pi}{b_1 h_0} + \frac{1}{h_0^2} t_c^{(2)}(h_1, \dots, h_{2n}) + O\left(\frac{1}{h_0^3}\right),$$

where $t_c^{(2)}$ satisfies

$$(h_1^2 + h_2^2) t_c^{(2)}(h) = -2(\alpha h_1 + \beta h_2)\left(h_1^2 + h_2^2\right) + (\gamma_{12} + \gamma_{21}) h_1 h_2 - \gamma_{22} h_1^2 - \gamma_{11} h_2^2, \quad (3.5)$$

denoting

$$\gamma_{ij} = \sum_{k=3}^{2n} \kappa_i^{jk} h_k \qquad \forall i, j \in [\![1, 2n]\!].$$

Near S_1. The asymptotic expansion

$$t_c\left(\frac{h_1}{\sqrt{h_0}}, \frac{h_2}{\sqrt{h_0}}, h_3, \dots, h_{2n}, h_0\right) = \frac{2\pi}{b_1 h_0} + O\left(\frac{1}{h_0^2}\right)$$

holds if and only if the quadratic polynomial equation in X

$$\begin{aligned} X^2 K - X &\left[\frac{2\pi}{b_1}(h_1^2 + h_2^2) - K(\gamma_{11} + \gamma_{22})\right] \\ &+ \frac{2\pi}{b_1}\left[(\gamma_{12} + \gamma_{21}) h_1 h_2 - \gamma_{22} h_1^2 - \gamma_{11} h_2^2\right] + K(\gamma_{11}\gamma_{22} - \gamma_{12}\gamma_{21}) = 0 \end{aligned}$$

admits a real solution, where $K = \sum_{i=2}^n (h_{2i-1}^2 + h_{2i}^2)(1 - \frac{b_i}{b_1}\pi \cot \frac{b_i \pi}{b_1}) > 0$.

If that is the case, denote by $\tilde{t}_c^{(2)}(h_1,\ldots,h_{2n})$ *the smallest of its two (possibly double) solutions. Then, for all* $R>0, \varepsilon\in(0,1)$*, uniformly with respect to* $p_0 = (\frac{h_1}{\sqrt{h_0}}, \frac{h_2}{\sqrt{h_0}}, h_3,\ldots,h_{2n},h_0) \in \mathcal{C}_{q_0}((0,R))\cap S_1^\varepsilon$*, we have*

$$t_c\left(\frac{h_1}{\sqrt{h_0}}, \frac{h_2}{\sqrt{h_0}}, h_3,\ldots,h_{2n},h_0\right) = \frac{2\pi}{b_1 h_0} + \frac{1}{h_0^2}\tilde{t}_c^{(2)}(h_1,\ldots,h_{2n}) + O\left(\frac{1}{h_0^3}\right).$$

4. Perturbations of the conjugate time. Thanks to the previous section, we have a sufficiently precise picture of the behavior of the conjugate time for the nilpotent approximation. We now introduce small perturbations of the exponential map in accordance with Proposition 2.2. As stated previously, we treat separately the case of initial covectors away from S_1 and near S_1 since S_1 corresponds to the set of covectors such that $r_1 = \sqrt{h_1^2+h_2^2} = 0$. Recall also that we assumed $q_0 \in M\setminus\mathfrak{S}_1$.

However, rather than computing t_c, we compute $\tau_c = t_c/\eta$, the rescaled conjugate time, since we use asymptotics in rescaled time from Proposition 2.2.

4.1. Asymptotic expansions for covectors in $T^*_{q_0}M\setminus S_1$. In this section we assume that $(h_1,h_2)\neq(0,0)$. Recall that $F(\tau,h,\eta) = \mathcal{E}(\eta\tau;(h,\eta^{-1}))$, for all $\tau>0$, $h\in\mathbb{R}^{2n}$, $\eta>0$. The function F admits a power series expansion

$$F(\tau,h,\eta) = \sum_{k\geq 0}\eta^k F^{(k)}(\tau,h),$$

and for $\delta\tau\in\mathbb{R}$, $h\in\mathbb{R}^{2n}$, evaluating F at the perturbed conjugate time $\frac{2\pi}{b_1}+\eta\delta\tau$ yields

$$F\left(\frac{2\pi}{b_1}+\eta\delta\tau,h,\eta\right) = \eta\, F^{(1)}\Big|_{\tau=\frac{2\pi}{b_1}} + \eta^2\left[F^{(2)} + \delta\tau\frac{\partial F^{(1)}}{\partial\tau}\right]\Bigg|_{\tau=\frac{2\pi}{b_1}} + O(\eta^3). \tag{4.1}$$

In the previous section, we highlighted the role of the function Φ defined by (3.3). Observe that τ_c must annihilate every term in the Taylor expansion of $\Phi(\tau_c(\cdot,\eta),\cdot,\eta)$. This first nontrivial term is obtained by straightforward algebraic computations (provided, for instance, in [25, Supplementary Materials, in particular, Lemma SM3.3]).

Proposition 4.1. *Let* $\tau_c(h,\eta) = \sum_{k=0}^{+\infty}\eta^k\tau_c^{(k)}(h)$ *be the formal power series expansion of* τ_c *for all* $(h,\eta^{-1})\in T^*_{q_0}M$*. Then* $\tau_c^{(0)} = 2\pi/b_1$ *and* $\tau_c^{(1)}$ *must satisfy*

$$(h_1^2+h_2^2)\tau_c^{(1)}(h) = -h_1^2\frac{\partial\left(F^{(2)}\right)_2}{\partial h_2} - h_2^2\frac{\partial\left(F^{(2)}\right)_1}{\partial h_1} + h_1h_2\left(\frac{\partial\left(F^{(2)}\right)_1}{\partial h_2} + \frac{\partial\left(F^{(2)}\right)_2}{\partial h_1}\right). \tag{4.2}$$

Proof. As discussed in the previous section, $\tau_c^{(0)} = 2\pi/b_1$ is a consequence of Proposition 3.6. The first nontrivial term of the expansion of the determinant $\Phi(2\pi/b_1+\eta\delta\tau,h,\eta)$, that is, the term of order $2n+3$, is obtained by algebraic computations. As a consequence of Proposition 2.2, notice that $\left(F^{(2)}\right)_{2n+1} = \widehat{z}$, $\frac{\partial F^{(1)}}{\partial\tau} = \hat{h}$, and that $\partial_{h_1}\widehat{z} = 2\pi h_1/b_1$, $\partial_{h_2}\widehat{z} = 2\pi h_2/b_1$. Hence we get the stated result by solving for $\delta\tau$:

$$\Phi^{(2n+3)}(2\pi/b_1+\eta\delta\tau,h,\eta) \propto \begin{vmatrix} \frac{\partial}{\partial h_1}\left(F^{(2)}\right)_1 + \delta\tau & \frac{\partial}{\partial h_2}\left(F^{(2)}\right)_1 & h_1 \\ \frac{\partial}{\partial h_1}\left(F^{(2)}\right)_2 & \frac{\partial}{\partial h_2}\left(F^{(2)}\right)_2 + \delta\tau & h_2 \\ h_1 & h_2 & 0\end{vmatrix}_{\tau=2\pi/b_1} = 0.$$

(Where we denote, for $f,g:\mathbb{R}^n\to\mathbb{R}$, $f\propto g$ if there exists $h:\mathbb{R}^n\to\mathbb{R}\setminus\{0\}$ such that $f=gh$.) ∎

Remark 4.2. Relation (4.2) is degenerate at $h_1 = h_2 = 0$. This is another illustration of the behavior we highlighted in the previous section, that is, $\tau_c^{(1)}$ can be a zero of order 2 at $r_1 = 0$.

As a consequence of Proposition 2.2, it appears that for all $k \in [\![1, 2n]\!]$ and all $\tau > 0$, each function $h \mapsto x_k^{(2)}(\tau)$ can be seen as a quadratic form on $(h_1, \ldots, h_{2n})$. Hence we introduce the invariants $(\kappa_k^{ij})_{i,j,k\in[\![1,2n]\!]}$ such that

$$F_k^{(2)}\left(\frac{2\pi}{b_1}, h\right) = \sum_{1\le i\le j\le 2n} \kappa_k^{ij} h_i h_j \qquad \forall k \in [\![1, 2n]\!].$$

These invariants satisfy some useful properties (of which a proof can be found in [25, Supplementary Materials, Lemmas SM2.1 through SM2.4]. We give the following summary.

PROPOSITION 4.3. *The invariants* $\left(\kappa_k^{ij}\right)_{i,j,k\in[\![1,2n]\!]}$ *depend linearly on the family*

$$\left(\frac{\partial^2 (X_i)_{2n+1}}{\partial x_j \partial x_k}(q_0)\right)_{i,j,k\in[\![1,2n]\!]}.$$

There exist $\alpha, \beta \in \mathbb{R}$ *such that we have the symmetries*

$$\kappa_1^{1,1} = 3\alpha, \quad \kappa_1^{2,2} = \alpha, \quad \kappa_2^{1,2} = 2\alpha, \qquad \kappa_2^{1,1} = \beta, \quad \kappa_2^{2,2} = 3\beta, \quad \kappa_1^{1,2} = 2\beta,$$

and for all $i \in [\![2, n]\!]$, $\left(\kappa_m^{kl}\right)_{\substack{k,m\in[\![1,2]\!] \\ l\in[\![2i-1,2i]\!]}}$ *only depend on the family*

$$\left\{\left(\frac{\partial^2 (X_k)_{2n+1}}{\partial x_l \partial x_m}(q_0)\right) \mid (k, l, m) \in [\![2i-1, 2i]\!] \times [\![1, 2]\!]^2 \cup [\![1, 2]\!]^2 \times [\![2i-1, 2i]\!]\right\}.$$

Furthermore, the corresponding linear map $\zeta_i : \mathbb{R}^{15} \to \mathbb{R}^8$ *such that*

$$\zeta_i\left(\left(\frac{\partial^2 (X_k)_{2n+1}}{\partial x_l \partial x_m}(q_0)\right)_{k,l,m\in\{1,2\}\cup\{2i-1,2i\}}\right) = \left(\kappa_m^{kl}\right)_{\substack{k,m\in\{1,2\} \\ l\in\{2i-1,2i\}}}$$

is of rank at least 7 *(and of rank* 8 *on the complementarity of a codimension* 1 *subset* $\mathfrak{S}_3$ *of* M*).*

Remark 4.4. A consequence of the rank of ζ_i being 7, for all $2 \le i \le n$, is that a single condition of codimension $k \ge 2$ on $\left(\kappa_m^{kl}\right)_{\substack{k,m\in[\![1,2]\!] \\ l\in[\![2i-1,2i]\!]}}$ is then a condition of codimension at least $k - 1$ on the jets of order 2 of the sub-Riemannian structure at q_0.

Using this notation, we can give a first approximation of the conjugate locus.

PROPOSITION 4.5. *Let* $q_0 \in M \setminus \mathfrak{S}_1$. *As* $\eta \to 0^+$, *uniformly with respect to* $p_0 = (h_1, \ldots, h_{2n}, \eta^{-1}) \in \mathcal{C}_{q_0}((0, R)) \setminus S_1^\varepsilon$ *for all* $R > 0, \varepsilon \in (0, 1)$, *we have (in normal form coordinates)*

$$(F(\tau_c(h, \eta), h, \eta))_1 = \eta^2 \frac{(\gamma_{11} - \gamma_{22})h_1^3 + \gamma_{12}h_2^3 + (\gamma_{21} + 2\gamma_{12})h_1^2 h_2 + \delta_1}{h_1^2 + h_2^2} + O(\eta^3),$$

$$(F(\tau_c(h, \eta), h, \eta))_2 = \eta^2 \frac{\gamma_{12}h_1^3 - (\gamma_{11} - \gamma_{22})h_2^3 + (\gamma_{12} + 2\gamma_{21})h_1 h_2^2 + \delta_2}{h_1^2 + h_2^2} + O(\eta^3)$$

with

$$\gamma_{ij} = \sum_{k=3}^{2n} \kappa_i^{jk} h_k \qquad \forall i,j \in [\![1,2n]\!],$$

$$\delta_1 = \alpha(h_1^2+h_2^2)^2 + \sum_{3\le i<j\le 2n} \kappa_1^{ij} h_i h_j, \qquad \delta_2 = \beta(h_1^2+h_2^2)^2 + \sum_{3\le i<j\le 2n} \kappa_2^{ij} h_i h_j.$$

If there exists a covector such that $\gamma_{11}-\gamma_{22}=\gamma_{12}=\gamma_{21}=0$ then this first order approximation of the conjugate locus is not sufficient to prove stability and higher orders of approximation are necessary. This occurs for instance when $h_3=\dots=h_{2n}=0$, and

$$(F(\tau_c(h,\eta),h,\eta))_1 = \eta^2\alpha(h_1^2+h_2^2)+O(\eta^3),$$
$$(F(\tau_c(h,\eta),h,\eta))_2 = \eta^2\beta(h_1^2+h_2^2)+O(\eta^3).$$

PROPOSITION 4.6. *Let M be a generic contact sub-Riemannian manifold of dimension $2n+1\ge 5$. Let $\mathfrak{S}_2 \subset M$ be the set of points at which the linear system in $(h_3,\dots,h_{2n})$,*

$$\begin{cases} \sum_{i=3}^{2n}(\kappa_1^{1,i}-\kappa_2^{2,i})h_i = 0,\\ \sum_{i=3}^{2n}\kappa_2^{1,i}h_i = 0,\\ \sum_{i=3}^{2n}\kappa_1^{2,i}h_i = 0,\end{cases}$$

admits nontrivial solutions. If $\dim M \ge 7$ *then $M=\mathfrak{S}_2$. However if* $\dim M = 5$*, the set $\mathfrak{S}_2$ is a codimension* 1 *stratified subset of M.*

Proof. If we assume $(r_2,\dots,r_n)\neq 0$ then $\gamma_{11}-\gamma_{22}=\gamma_{12}=\gamma_{21}=0$ reduces to the existence of a nonzero vector of $\mathbb{R}^{2n-2}$ in the intersection

$$\mathrm{Span}\{(\kappa_1^{1,3}-\kappa_2^{2,3},\dots,\kappa_1^{1,2n}-\kappa_2^{2,2n})\}^{\perp} \cap \mathrm{Span}\{(\kappa_1^{2,3},\dots,\kappa_1^{2,2n})\}^{\perp} \cap \mathrm{Span}\{(\kappa_2^{1,3},\dots,\kappa_2^{1,2n})\}^{\perp}.$$

This space is never reduced to a single point for $n>2$, hence, $M=\mathfrak{S}_2$. However for $n=2$, this requires the three vectors

$$(\kappa_1^{1,3}-\kappa_2^{2,3},\kappa_1^{1,4}-\kappa_2^{2,4}), \quad (\kappa_1^{2,3},\kappa_1^{2,4}), \quad (\kappa_2^{1,3},\kappa_2^{1,4}), \tag{4.3}$$

to be colinear, which is a constraint of codimension 2 on the family $\left(\kappa_m^{kl}\right)_{\substack{k,m\in\{1,2\}\\ l\in\{3,4\}}}$. By Remark 4.4, this is a codimension 1 (at least) constraint on the jets of order 2 of the sub-Riemannian structure at q_0, hence, the result. ∎

4.2. Asymptotics for covectors near S_1. We repeat the previous construction for a special class of initial covector in the vicinity of

$$S_1 = \{(h_1,\dots,h_{2n},h_0)\in T_{q_0}^*M \mid h_1=h_2=0\},$$

in accordance with the discussion of section 3.2.

Let $\bar h\in\mathbb{R}^{2n}$ be such that $(\bar h_3,\dots,\bar h_{2n})\neq(0,\dots,0)$. We blow up the singularity at $h_1=h_2=0$ by computing an approximation of the conjugate locus for

$$h(0) = (\sqrt{\eta}\bar h_1, \sqrt{\eta}\bar h_2, \bar h_3,\dots,\bar h_{2n}). \tag{4.4}$$

Let Λ be the square $2n\times 2n$ matrix such that

$$\Lambda_{i,j} = \begin{cases} 1 & \text{if } i=j=1 \text{ or } i=j=2,\\ 0 & \text{otherwise,}\end{cases} \tag{4.5}$$

so that $h(0)=\sqrt{\eta}\Lambda\bar h + (I_{2n}-\Lambda)\bar h$.

Recall the power series notation $f(\eta\tau, h(0)) = \sum \eta^k f^{(k)}(\tau, h(0))$. As a consequence of Proposition 2.2, we can give a new expansion of the Hamiltonian flow for the special class of initial covectors of type (4.4) in terms of coefficients of the power series of x, z, h, w. (Recall that for all $R > 0$, B_R denotes the set $\{h \in \mathbb{R}^{2n} \mid \sum_{i=1}^{2n} h_i^2 \leq R\}$.)

PROPOSITION 4.7. *For all $T, R > 0$, normal extremals with initial covector*

$$(\sqrt{\eta}\Lambda\bar{h} + (I_{2n} - \Lambda)\bar{h}, \eta^{-1})$$

have the following order 3 *expansion at time $\eta\tau$, as $\eta \to 0^+$, uniformly with respect to $\tau \in [0, T]$ and $\bar{h} \in B_R$:*

$$\begin{aligned} x(\eta\tau) = {} & \eta\hat{x}(\tau, (I_{2n} - \Lambda)\bar{h}) + \eta^{3/2}\left[\hat{x}\left(\tau, \Lambda\bar{h}\right)\right] + \eta^2\left[x^{(2)}\left(\tau, (I_{2n} - \Lambda)\bar{h}\right)\right] \\ & + \eta^{5/2}\left[x^{(2)}\left(\tau, \bar{h}\right) - x^{(2)}\left(\tau, (I_{2n} - \Lambda)\bar{h}\right) - x^{(2)}\left(\tau, \Lambda\bar{h}\right)\right] + O(\eta^3), \end{aligned}$$

$$z(\eta\tau) = \eta^2\widehat{z}(\tau, (I_{2n} - \Lambda)\bar{h}) + \eta^3\left[z^{(3)}(\tau, (I_{2n} - \Lambda)\bar{h}) + \widehat{z}(\tau, \Lambda\bar{h})\right] + O(\eta^4).$$

Likewise, the associated covector has the expansion

$$\begin{aligned} h(\eta\tau) = {} & \hat{h}(\tau, (I_{2n} - \Lambda)\bar{h}) + \sqrt{\eta}\left[\hat{h}(\tau, \Lambda\bar{h})\right] + \eta\left[h^{(1)}(\tau, (I_{2n} - \Lambda)\bar{h})\right] \\ & + \eta^{3/2}\left[h^{(1)}(\tau, \bar{h}) - h^{(1)}(\tau, \Lambda\bar{h}) - h^{(1)}(\tau, (I_{2n} - \Lambda)\bar{h})\right] + O(\eta^2), \end{aligned}$$

$$w(\eta\tau) = 1 + O(\eta^2).$$

Proof. Let $h, h' \in \mathbb{R}^{2n}$ and let $\psi : \mathbb{R}^{2n} \to \mathbb{R}$ be a quadratic form; we have by the polarization identity $\psi\left(h + \sqrt{\eta}h'\right) = \psi(h) + \sqrt{\eta}\left[\psi(h + h') - \psi(h) - \psi(h')\right] + \eta\psi(h')$. Applying this identity with $h = \Lambda\bar{h}$ and $h' = (I_{2n} - \Lambda)\bar{h}$, we get the statement since we proved in Proposition 2.2 that $x^{(1)}(\eta\tau, \cdot)$, $h^{(0)}(\eta\tau, \cdot)$ are linear and $x^{(2)}(\eta\tau, \cdot)$, $h^{(1)}(\eta\tau, \cdot)$, $z^{(2)}(\eta\tau, \cdot)$ are quadratic, coordinatewise. The case of w comes from the fact that $w^{(1)} = 0$. ◻

We set $G(\tau, \bar{h}, \eta) = F\left(\tau, \sqrt{\eta}\Lambda\bar{h} + (I_{2n} - \Lambda)\bar{h}, \eta\right)$ for all $\tau > 0$, $\bar{h} \in \mathbb{R}^{2n}$, and $\eta > 0$. The function G admits a power series expansion in $\sqrt{\eta}$:

$$G(\tau, \bar{h}, \eta) = \sum_{k \geq 0} \eta^{k/2} G^{(k/2)}(\tau, \bar{h}).$$

We prove the following proposition on the conjugate time for such initial covectors.

PROPOSITION 4.8. *Let us define the quadratic polynomial in $\delta\tau$:*

$$\begin{aligned} P(\delta\tau) = {} & -\delta\tau^2 K + \delta\tau\left(\frac{2\pi}{b_1}\left(\bar{h}_1^2 + \bar{h}_2^2\right) - K\left(\frac{\partial G_1^{(5/2)}}{\partial\bar{h}_1} + \frac{\partial G_2^{(5/2)}}{\partial\bar{h}_2}\right)\right) \\ & + \frac{2\pi}{b_1}\left(\bar{h}_2^2\frac{\partial G_1^{(5/2)}}{\partial\bar{h}_1} + \bar{h}_1^2\frac{\partial G_2^{(5/2)}}{\partial\bar{h}_2} - \bar{h}_1\bar{h}_2\left(\frac{\partial G_2^{(5/2)}}{\partial\bar{h}_1} + \frac{\partial G_1^{(5/2)}}{\partial\bar{h}_2}\right)\right) \\ & + K\left(\frac{\partial G_2^{(5/2)}}{\partial\bar{h}_1}\frac{\partial G_1^{(5/2)}}{\partial\bar{h}_2} - \frac{\partial G_1^{(5/2)}}{\partial\bar{h}_1}\frac{\partial G_2^{(5/2)}}{\partial\bar{h}_2}\right), \end{aligned}$$

and let $\Delta(\bar{h})$ be its discriminant. We have the following cases:

- *If* $\Delta(\bar{h}) \geq 0$, *let* $\delta\tau^*$ *be the smallest of the (possibly equal) two roots of* P. *Then*
$$\tau_c(\sqrt{\eta}\Lambda\bar{h} + (I_{2n} - \Lambda)\bar{h}) = 2\pi/b_1 + \eta\delta t^* + o(\eta).$$
- *If* $\Delta(\bar{h}) < 0$,
$$\limsup_{\eta\to 0} \left|\tau_c(\sqrt{\eta}\Lambda\bar{h} + (I_{2n} - \Lambda)\bar{h}) - 2\pi/b_1\right| > 0,$$
that is, the first conjugate time is not a perturbation of $2\pi/b_1$.

Proof. We first have to check that the conjugate time is not a perturbation of order $\sqrt{\eta}$ of the nilpotent conjugate time $2\pi/b_1$. We apply the same method as before to evaluate $\Phi\left(2\pi/b_1 + \sqrt{\eta}\delta\tau, \sqrt{\eta}\Lambda\bar{h} + (I_{2n} - \Lambda)\bar{h}, \eta\right)$, $\delta\tau \in \mathbb{R}$, $\bar{h} \in \mathbb{R}^{2n}$. Notice that

$$\frac{\partial F}{\partial h_i} = \frac{1}{\sqrt{\eta}}\frac{\partial G}{\partial \bar{h}_i} \quad \forall i \in [\![1,2]\!] \quad \text{and} \quad \frac{\partial F}{\partial h_i} = \frac{\partial G}{\partial \bar{h}_i} \quad \forall i \in [\![3,2n]\!].$$

With $\delta\tau \in \mathbb{R}$, $\bar{h} \in \mathbb{R}^{2n}$, we have

$$G\left(\frac{2\pi}{b_1} + \sqrt{\eta}\delta\tau, \bar{h}, \eta\right) = \eta\, G^{(1)}\Big|_{\tau=\frac{2\pi}{b_1}} + \eta^{3/2}\left(G^{(3/2)} + \delta\tau\frac{\partial G^{(1)}}{\partial\tau}\right)\Bigg|_{\tau=\frac{2\pi}{b_1}} + O(\eta^{5/2}). \tag{4.6}$$

Hence $\Phi\left(2\pi/b_1 + \sqrt{\eta}\delta\tau, \sqrt{\eta}\Lambda\bar{h} + (I_{2n} - \Lambda)\bar{h}, \eta\right) = O(\eta^{2n+3})$ (see, for instance, [25, Supplementary Materials]). By capturing the first nontrivial term in the expansion of Φ, one has

$$\Phi^{(2n+3)}\left(2\pi/b_1 + \sqrt{\eta}\delta\tau, \sqrt{\eta}\Lambda\bar{h} + (I_{2n} - \Lambda)\bar{h}, \eta\right) \propto \delta\tau^2$$

[25, see also Lemma SM3.4 in Supplementary Materials]). Hence perturbations of the nilpotent conjugate time $2\pi/b_1$ must be of order 1 in η at least for Φ to vanish.

Computing the perturbation of the conjugate time is then a matter of computing Φ at time $2\pi/b_1 + \eta\delta\tau$. Regarding G, we have

$$\begin{aligned} G\left(\frac{2\pi}{b_1} + \eta\delta\tau, \bar{h}, \eta\right) =& \eta\, G^{(1)}\Big|_{\tau=\frac{2\pi}{b_1}} + \eta^{3/2}\, G^{(3/2)}\Big|_{\tau=\frac{2\pi}{b_1}} + \eta^2\left[G^{(2)} + \delta\tau\frac{\partial G^{(1)}}{\partial\tau}\right]\Bigg|_{\tau=\frac{2\pi}{b_1}} \\ &+ \eta^{5/2}\left[G^{(5/2)} + \delta\tau\frac{\partial G^{(3/2)}}{\partial\tau}\right]\Bigg|_{\tau=\frac{2\pi}{b_1}} + O(\eta^3). \end{aligned} \tag{4.7}$$

Thus $\Phi\left(2\pi/b_1 + \eta\delta\tau, \sqrt{\eta}\Lambda\bar{h} + (I_{2n} - \Lambda)\bar{h}, \eta\right) = O(\eta^{2n+5})$. Again, computing the first nontrivial term in the expansion yields (for instance, [25, see Lemma SM3.5 in Supplementary Materials])

$$\Phi^{(2n+5)}\left(2\pi/b_1 + \eta\delta\tau, h, \eta\right) \propto P(\delta\tau).$$

This implies the statement: either P admits real roots, of which the smallest is $\tau_c^{(1)}$, or the system does not admit a perturbation of $2\pi/b_1$ as a first conjugate time. ∎

Remark 4.9. Contrary to (4.2), the equation $P(\delta\tau) = 0$ is not degenerate at $\bar{h}_1 = \bar{h}_2 = 0$.

4.3. Next order perturbations. As observed in section 4.1, there exists a subset of initial covectors in $T^*_{q_0}\backslash S_1$ for which our approximation of the conjugate locus is degenerate (this makes the second order approximation unstable as a Lagrangian map). In particular, for all $q_0 \in M$, this set contains

$$S_2 = \{(h_1, h_2, 0, \dots, 0, \eta^{-1}) \in T^*_{q_0}M\}.$$

As proved in Proposition 4.6, this set is reduced to S_2 at points q_0 in the complement of a startified codimension 1 subset $\mathfrak{S}_2$ of M if $n = 2$.

Hence in preparation of the stability analysis of section 6, we compute here a third order approximation of the conjugate time in the case of covectors near S_2. When $n = 2$, we get a complete description of the sub-Riemannian caustic at points of $M \setminus \mathfrak{S}_2$ as a result.

We use a blowup technique similar to the one of section 4.2. Let $\bar{h} \in \mathbb{R}^{2n}$ be such that $(\bar{h}_1, \bar{h}_2) \neq (0,0)$. We blow up the singularity at $(\bar{h}_1, \bar{h}_2, 0, \dots, 0)$ by computing an approximation of the conjugate locus with

$$h(0) = (\bar{h}_1, \bar{h}_2, \eta\bar{h}_3, \dots, \eta\bar{h}_{2n}).$$

With Λ the square $2n \times 2n$ matrix defined in (4.5), $h(0) = \Lambda\bar{h} + \eta(I_{2n} - \Lambda)\bar{h}$.

We give an equivalent of Proposition 4.7 for this case.

Proposition 4.10. *For all $T, R > 0$, normal extremals with initial covector $(\Lambda\bar{h} + \eta(I_{2n} - \Lambda)\bar{h}, \eta^{-1})$ have the following order 3 expansion at time $\eta\tau$, as $\eta \to 0^+$, uniformly with respect to $\tau \in [0, T]$ and $h(0) \in B_R$:*

$$\begin{aligned} x(\eta\tau, \Lambda\bar{h} + \eta(I_{2n} - \Lambda)\bar{h}) &= \eta\hat{x}(\tau, \Lambda\bar{h}) + \eta^2 \left[x^{(2)}\left(\tau, \Lambda\bar{h}\right) + \hat{x}\left(\tau, (I_{2n} - \Lambda)\bar{h}\right)\right] \\ &+ \eta^3 \left[x^{(3)}\left(\tau, \Lambda\bar{h}\right) + x^{(2)}\left(\tau, \bar{h}\right) - x^{(2)}\left(\tau, \Lambda\bar{h}\right) - x^{(2)}\left(\tau, (I_{2n} - \Lambda)\bar{h}\right)\right] + O(\eta^4), \end{aligned}$$

$$z(\eta\tau) = \eta^2\hat{z}(\tau, \Lambda\bar{h}) + \eta^3 z^{(3)}(\tau, \Lambda\bar{h}) + O(\eta^4).$$

Likewise, the associated covector has the following expansion:

$$\begin{aligned} h(\eta\tau, \Lambda\bar{h} + \eta(I_{2n} - \Lambda)\bar{h}) &= \hat{h}(\tau, \Lambda\bar{h}) + \eta \left[h^{(1)}(\tau, \Lambda\bar{h}) + \hat{h}(\tau, (I_{2n} - \Lambda)\bar{h})\right] \\ &+ \eta^2 \left[h^{(2)}(\tau, \Lambda\bar{h}) + h^{(1)}(\tau, \bar{h}) - h^{(1)}(\tau, \Lambda\bar{h}) - h^{(1)}(\tau, (I_{2n} - \Lambda)\bar{h})\right] + O(\eta^3), \end{aligned}$$

$$w(\eta\tau) = 1 + \eta^2 w^{(2)}(\tau, \Lambda\bar{h}) + O(\eta^4).$$

Proof. The proof relies on the same arguments as that of Proposition 4.7. ∎

We aim to obtain a second order approximation of τ_c in the case of an initial covector of the form $(\Lambda\bar{h} + \eta(I_{2n} - \Lambda)\bar{h}, \eta^{-1})$, for $\bar{h} \in \mathbb{R}^{2n}$. The previous section, together with Proposition 4.10, applies to give us

$$\tau_c^{(1)}(\Lambda\bar{h} + \eta(I_{2n} - \Lambda)\bar{h}) = \tau_c^{(1)}(\Lambda\bar{h}) \qquad \forall \bar{h} \in \mathbb{R}^{2n}.$$

Similarly to section 4.2, for all $\tau > 0$, $h \in \mathbb{R}^{2n}$, and $\eta > 0$, we denote $F(\tau, h, \eta) = \mathcal{E}(\eta\tau; (h, \eta^{-1}))$, and we set

$$G(\tau, \bar{h}, \eta) = F\left(\tau, \Lambda\bar{h} + \eta(I_{2n} - \Lambda)\bar{h}, \eta\right) \quad \forall \tau > 0, \bar{h} \in \mathbb{R}^{2n}, \eta > 0.$$

The function G admits a formal power series expansion in η: $G(\tau, \bar{h}, \eta) = \sum_{k \geq 0} \eta^k G^{(k)}(\tau, \bar{h})$. Techniques similar to those introduced in sections 4.1 and 4.2 yield the following statement on second order approximations of the conjugate time τ_c.

PROPOSITION 4.11. *The second order perturbation of* τ_c *with initial covector* $h(0) = \Lambda\bar{h} + \eta(I_{2n} - \Lambda)\bar{h}$ *satisfies the equation*

$$(\bar{h}_1^2+\bar{h}_2^2)\tau_c^{(2)}(h(0)) = -\bar{h}_1^2\frac{\partial\left(G^{(3)}\right)_2}{\partial\bar{h}_2} - \bar{h}_2^2\frac{\partial\left(G^{(3)}\right)_1}{\partial\bar{h}_1} + \bar{h}_1\bar{h}_2\left(\frac{\partial\left(G^{(3)}\right)_1}{\partial\bar{h}_2} + \frac{\partial\left(G^{(3)}\right)_2}{\partial\bar{h}_1}\right)$$
$$+ (\bar{h}_1^2+\bar{h}_2^2)(\alpha\bar{h}_2 - \beta\bar{h}_1)\left(\frac{b_1}{2\pi}(\beta\bar{h}_1 - \alpha\bar{h}_2) + 4b_1(\alpha\bar{h}_1 + \beta\bar{h}_2)\right) + \sum_{i=3}^{2n} d_i,$$

where α *and* β *are the second order invariants introduced in Proposition* 4.3 *and*

$$d_k = \frac{2\pi^2}{b_1^2} e_k\left(-h_2\partial_{h_k}\left(G^{(3)}\right)_1 + h_1\partial_{h_k}\left(G^{(3)}\right)_2\right) \qquad \forall k \in [\![3, 2n]\!]$$

with $e \in \mathbb{R}^{2n-2}$ *the vector such that* $Ae = \left(h_2\partial_{h_1}G^{(2)} - h_1\partial_{h_2}G^{(2)}\right)_{3,\dots,2n}$, *where* $A \in \mathcal{M}_{2n-2}(\mathbb{R})$ *is the matrix introduced in* [25, Lemma SM3.3, Supplementary Materials] *and where we denote* $(v)_{3,\dots,2n} = (v_3, \dots, v_{2n}) \in \mathbb{R}^{2n-2}$ *for all* $v \in \mathbb{R}^{2n+1}$.

Proof. With $\delta\tau_1, \delta\tau_2 \in \mathbb{R}$, $\bar{h} \in \mathbb{R}^{2n}$, we have

$$G\left(\frac{2\pi}{b_1} + \eta\delta\tau_1 + \eta^2\delta\tau_2, \bar{h}, \eta\right) = \eta\, G^{(1)}\Big|_{\tau=\frac{2\pi}{b_1}} + \eta^2\left(G^{(2)} + \delta\tau_1\frac{\partial G^{(1)}}{\partial\tau}\right)\Bigg|_{\tau=\frac{2\pi}{b_1}}$$
$$+ \eta^3\left[G^{(3)} + \delta\tau_2\frac{\partial G^{(1)}}{\partial\tau} + \frac{\delta\tau_1^2}{2}\frac{\partial^2 G^{(1)}}{\partial\tau^2} + \delta\tau_1\frac{\partial G^{(2)}}{\partial\tau}\right]\Bigg|_{\tau=\frac{2\pi}{b_1}} + O(\eta^3).$$

To evaluate $\Phi\left(2\pi/b_1 + \eta\delta\tau_1 + \eta^2\delta\tau_2, \Lambda\bar{h} + \eta(I_{2n} - \Lambda)\bar{h}, \eta\right)$, $\delta\tau_1, \delta\tau_2 \in \mathbb{R}$, $\bar{h} \in \mathbb{R}^{2n}$, notice that

$$\frac{\partial F}{\partial h_i} = \frac{\partial G}{\partial \bar{h}_i} \quad \forall i \in [\![1, 2]\!] \quad \text{ and } \quad \frac{\partial F}{\partial h_i} = \frac{1}{\eta}\frac{\partial G}{\partial \bar{h}_i} \quad \forall i \in [\![3, 2n]\!].$$

Hence with $\delta\tau_1 = \tau_c^{(1)}(\Lambda\bar{h})$, one has $\Phi\left(2\pi/b_1 + \eta\delta\tau_1 + \eta^2\delta\tau_2, \Lambda\bar{h} + \eta(I_{2n} - \Lambda)\bar{h}, \eta\right) = O(\eta^{4n+2})$. The result is again obtained by computing the first nontrivial term in the expansion of the determinant Φ (see [25, Lemma SM3.6, Supplementary Materials]). We obtain the stated result by refining this evaluation thanks to [25, Lemma SM3.7, Supplementary Materials]. ∎

Up to the computation of $G^{(3)}$, which is carried out in [25, Supplementary Materials], we have enough information to compute the conjugate time, similarly to Proposition 4.1.

Remark 4.12. By definition of the invariants $\chi_{11}, \chi_{12}, \chi_{22}$ introduced in [25, Supplementary Materials], the third dimensional case would correspond to the case $\kappa_k^{ij} = 0$ if $3 \le i, j, k \le 2n$, $\alpha = \beta = 0$. Under these conditions, one has $\tau_c^{(1)}(\bar{h}) = 0$, $\tau_c^{(2)}(\bar{h}) = -3(\chi_{11} + \chi_{22})(\bar{h}_1^2 + \bar{h}_2^2)$, and

$$\left[\mathcal{E}(\eta\tau_c; (h, \eta^{-1}))\right]_1 = \eta^3\left(2\bar{h}_1^3(\chi_{22} - \chi_{11}) + 3\bar{h}_1^2\bar{h}_2\chi_{12} + \bar{h}_2^3\chi_{12}\right) + O(\eta^4),$$
$$\left[\mathcal{E}(\eta\tau_c; (h, \eta^{-1}))\right]_2 = \eta^3\left(2\bar{h}_2^3(\chi_{11} - \chi_{22}) + 3\bar{h}_1\bar{h}_2^2\chi_{12} + \bar{h}_1^3\chi_{12}\right) + O(\eta^4).$$

This expression corresponds to the classical astroidal caustic expansion observed in the 3D contact case.

5. Proof of the asymptotic expansion theorems. This short section is devoted to the proof of Theorems 1.1 and 3.7. It appears now that proving Theorem 3.7 is a matter of summarizing what we know about the conjugate time from the previous results of section 4.

Proof of Theorem 3.7. In the previous section we computed the rescaled conjugate time τ_c. We have for all covector $p_0 = (\bar{h}_1, \dots, \bar{h}_{2n}, \eta^{-1}) \in T^*_{q_0}M$,

$$t_c(\bar{h}, \eta^{-1}) = \eta\tau_c(\bar{h}, \eta^{-1}).$$

From Proposition 3.6, we deduce that under the assumption $(\bar{h}_1, \bar{h}_2) \neq (0,0)$, we have as $\eta \to 0^+$ that $\tau_c(\bar{h}, \eta^{-1}) = 2\pi/b_1 + O(\eta)$. From Proposition 4.1, we deduce the existence of $t_c^{(2)} = \eta\tau_c^{(1)}$ that satisfies the given equation, using the invariants introduced in Proposition 4.3.

On the other hand, by performing the blow up at $(0, 0, \bar{h}_3 \dots, \bar{h}_{2n})$, we compute an approximation of

$$t_c(\sqrt{\eta}\bar{h}_1, \sqrt{\eta}\bar{h}_2, \bar{h}_3, \dots, \bar{h}_{2n}, \eta^{-1}) = \eta\tau_c(\sqrt{\eta}\bar{h}_1, \sqrt{\eta}\bar{h}_2, \bar{h}_3, \dots, \bar{h}_{2n}, \eta^{-1}).$$

Again, from Proposition 3.6, we deduce that under the assumption $(\bar{h}_1, \bar{h}_2) \neq (0,0)$, a possible approximation is $\tau_c(\sqrt{\eta}\Lambda\bar{h} + (I_{2n} - \Lambda)\bar{h}, \eta^{-1}) = 2\pi/b_1 + O(\eta)$. However from Lemma 3.5, we now know that in the nilpotent case, $2\pi/b_1$ is a zero of order two at $(\bar{h}_1, \bar{h}_2) = (0,0)$. Thus computing a perturbation of the conjugate time, one gets the statement for $\tilde{t}_c^{(2)}$ from Proposition 4.8 and the expression in terms of invariants from Proposition 4.7. ☐

Having proved Theorem 3.7, we can introduce a geometric invariant that will help us prove Theorem 1.1. For all $q \in M \setminus \mathfrak{S}_1$, let

$$\mathcal{A}_q = \overline{\{t_c(p)p \mid H(p,q) = 1/2\}}.$$

By the usual property of the Hamiltonian flow, the first conjugate locus at q is given by $\mathcal{E}_q(1, \mathcal{A}_q)$. Furthermore, the set $\mathcal{A}_q$ is an immersed hypersurface of T_q^*M and $\mathcal{A}_q \cap \mathcal{C}_q(0)$ is reduced to the two points $p^+ = (0, \dots, 0, 2\pi/b_1)$, $p^- = (0, \dots, 0, -2\pi/b_1)$. Then let $\mathcal{A}_q^+$ be the tangent cone to $\mathcal{A}_q$ at p^+.

Observe that $\mathcal{A}_q^+$ is a geometric invariant independent of the choice of coordinates on M. It can be computed once the asymptotics of the conjugate time are known.

Proof of Theorem 1.1. We prove the theorem by contradiction. Assume there exists a set of coordinates for which (1.4) does not hold, i.e.,

$$\lim_{h_0 \to +\infty} \left(h_0^2 \sup_{\tau \in (0,T)} \left| \mathcal{E}_q\left(\frac{\tau}{h_0}, (h_1, \dots, h_{2n}, h_0)\right) - \widehat{\mathcal{E}}_q\left(\frac{\tau}{h_0}, (h_1, \dots, h_{2n}, h_0)\right) \right| \right) = 0.$$

Then we have that uniformly with respect to $\tau \in (0,T)$,

$$\mathcal{E}_q\left(\eta\tau, (h_1, \dots, \bar{h}_{2n}, \eta^{-1})\right) = \widehat{\mathcal{E}}_q\left(\eta\tau, (h_1, \dots, \bar{h}_{2n}, \eta^{-1})\right) + o(\eta^2).$$

That is, the exponential is a second order perturbation of the nilpotent exponential. If that is the case, as a consequence of section 4, and in particular Proposition 4.1, we have that for $p_0 = (h_1, \dots, h_{2n}, \eta^{-1}) \in T_q^*M$,

$$t_c(p_0) = \frac{2\pi}{b_1}\eta + o(\eta^2).$$

Then

$$t_c(p_0)p_0 = \left(0, \ldots, 0, \frac{2\pi}{b_1}\right) + \eta\left(\frac{2\pi}{b_1}h_1, \ldots, \frac{2\pi}{b_1}h_{2n}, 0\right) + o(\eta)$$

and the cone $\mathcal{A}_q^+$ is the affine plane $\{h_0 = 2\pi/b_1\}$.

However, as a consequence of Theorem 3.7, the cone $\mathcal{A}_q^+$ can be computed using the Agrachev–Gauthier frame, where we have for $p_0 = (h_1, \ldots, h_{2n}, \eta^{-1}) \in T_q^*M \setminus S$,

$$t_c(p_0)p_0 = \left(0, \ldots, 0, \frac{2\pi}{b_1}\right) + \eta\left(\frac{2\pi}{b_1}h_1, \ldots, \frac{2\pi}{b_1}h_{2n}, t_c^{(2)}(h_1, \ldots, h_{2n})\right) + o(\eta).$$

For $\mathcal{A}_q^+$ to be planar, the following symmetry for $t_c^{(2)}$ is needed (with $r_1^2 = h_1^2 + h_2^2$):

$$\lim_{r_1 \to 0^+} t_c^{(2)}(h_1, h_2, h_3, \ldots, h_{2n}) = -\lim_{r_1 \to 0^+} t_c^{(2)}(-h_1, -h_2, h_3, \ldots, h_{2n})$$

for all $(h_3, \ldots, h_{2n}) \in \mathbb{R}^{2n-2}$. Given the expression (3.5), we have rather

$$\lim_{r_1 \to 0^+} t_c^{(2)}(h_1, h_2, h_3, \ldots, h_{2n}) = \lim_{r_1 \to 0^+} t_c^{(2)}(-h_1, -h_2, h_3, \ldots, h_{2n}),$$

which is not everywhere zero unless $\gamma_{11} = \gamma_{22} = \gamma_{12} + \gamma_{21} = 0$ for all $(h_3, \ldots, h_{2n}) \in \mathbb{R}^{2n-2}$. That is $\kappa_1^{1i} = \kappa_2^{2i} = \kappa_1^{2i} + \kappa_2^{1i} = 0$ for all $i \in [\![3, 2n]\!]$, which is not generic with respect to the sub-Riemannian structure at $q \in M \setminus (\mathfrak{S}_1 \cup \mathfrak{S}_3)$ (see Proposition 4.3 and [25, Supplementary Materials]).

In consequence, we have proven that generically with respect to the sub-Riemannian structure at $q \in M \setminus \mathfrak{S}$, there does not exist a set of privileged coordinates at q and $T > 0$ such that the limit (1.4) holds. □

Remark 5.1. Regarding the nongenericity of $\kappa_1^{1i} = \kappa_2^{2i} = \kappa_1^{2i} + \kappa_2^{1i} = 0$, notice that it constitutes $6(n-1)$ independent conditions on the family $(\kappa_k^{ij})_{\substack{i,k \in [\![1,2]\!], \\ j \in [\![3,2n]\!]}}$ and thus a codimension $5(n-1)$ condition (at least) on the 2-jets of the sub-Riemannian structure at q.

Notice that $5n - 5 > 2n + 1$ if $n > 2$ and $5n - 5 = 2n + 1$ when $n = 2$. Hence in the $n = 2$ case, assuming $q \in M \setminus \mathfrak{S}_3$ (see Proposition 4.3), we ensure the codimension of the condition on the 2-jets of the sub-Riemannian structure to be 6.

6. Stability of the sub-Riemannian caustic.

6.1. Sub-Riemannian to Lagrangian stability. The aim of the classification is to prove Theorem 1.3 using tools from low-dimensional Lagrangian singularity theory.

The sub-Riemannian exponential at time 1, $\mathcal{E}_{q_0}^1 : T_{q_0}^*M \to M$, has a structure of a Lagrangian map, hence, sub-Riemannian stability can be defined as the restriction of Lagrangian stability to the class of sub-Riemannian exponential maps (see, for instance, [22] for an introduction to Lagrangian stability). Observe the following immediate fact.

PROPOSITION 6.1. *Let (M, Δ, g) be a sub-Riemannian manifold and let $q_0 \in M$. If the exponential map at time 1, $\mathcal{E}_{q_0}^1 : T_{q_0}^*M \to M$, is Lagrange stable at $p \in T_{q_0}M$, then $\mathcal{E}_{q_0}^1$ is sub-Riemannian stable at p.*

The chosen method to prove the stability of the sub-Riemannian exponential in dimension 5 is to show that the singular points of the exponential map are all

Lagrange stable according to the classification of generic Lagrange stable singularities of Theorem 1.2.

The approximations of the sub-Riemannian exponential we carried in sections 2 to 5 are suited for the time-dependent exponential at q_0 with initial covectors in $\mathcal{C}_{q_0}(1/2)$, hence, we will prove the stability statements in this framework. As a consequence of Proposition 3.4, classifying Lagrangian stable singularities of the sub-Riemannian exponential near the starting point q_0 requires considering inital covectors in $\mathcal{C}_{q_0}(1/2)$ such that h_0 is very large. As stated in the previous sections, some restrictions on the starting point are necessary to prove stability. Hence we consider points on the complementary of a codimension 1 stratified subset $\mathfrak{S}$ of M, containing $\mathfrak{S}_1$, $\mathfrak{S}_2$, and $\mathfrak{S}_3$, introduced in section 3.1, Propositions 4.6 and 4.3, respectively. The aim of section 6.3 is to prove the following theorem.

THEOREM 6.2. *Let (M, Δ, g) be a generic 5-dimensional contact sub-Riemannian manifold and let $q_0 \in M \setminus \mathfrak{S}$. There exist $\bar{\eta} > 0$ such that for all $(h_1, h_2, h_3, h_4, h_0) \in \mathcal{C}_{q_0}(1/2) \cap \{|h_0| > \bar{\eta}^{-1}\}$, the first conjugate point of $\mathcal{E}_{q_0}$ with initial covector $(h_1, h_2, h_3, h_4, h_0)$ is a Lagrange stable singular point of type $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$, $\mathcal{D}_4^+$, or $\mathcal{A}_5$.*

We can check that the time-dependent framework is indeed sufficient by showing that Theorem 1.3 is a corollary of Theorem 6.2.

Proof of Theorem 1.3. As a consequence of Proposition 6.1, we prove the Lagrange stability of the singular points of $\mathcal{E}^1_{q_0}$. For all $t > 0$, $p_0 \in T^*_{q_0}M$, $\mathcal{E}^1_{q_0}(tp_0) = \mathcal{E}_{q_0}(t, p_0)$. Hence for a given covector $p_0 \in \{H \neq 0\}$, $t_c(p_0)p_0$ is a critical point of $\mathcal{E}^1_{q_0}$.

Recall that for all $q \in M$, we have set $\mathcal{A}_{q_0} = \overline{\{t_c(p_0)p_0 \mid H(p_0, q_0) = 1/2\}}$, and the caustic is the set $\mathcal{E}^1_{q_0}(\mathcal{A}_{q_0})$.

Since $\mathcal{E}^1_{q_0}(\mathcal{C}_{q_0}(0)) = q_0$, to prove the statement it is sufficient to show the existence of a V_{q_0} neighborhood of q_0 such that $\mathcal{E}^1_{q_0}$ is Lagrange stable at every point of $\mathcal{A}_{q_0} \cap \left(\mathcal{E}^1_{q_0}\right)^{-1}(V_{q_0}) \cap \{H > 0\}$ (and satisfies the stated classification). As a result of Theorem 6.2, what remains to be checked is that there exists $R > 0$ such that for all covectors $p \in \mathcal{A}_{q_0} \cap \mathcal{C}_{q_0}((0, R))$,

$$\frac{p}{\sqrt{2H(p, q_0)}} \in \mathcal{C}_{q_0}(1/2) \cap \{|h_0| > \bar{\eta}^{-1}\}$$

with $\bar{\eta} > 0$ as in the statement of Theorem 6.2, but this is Proposition 3.4. □

6.2. Classification methodology. We first recall normal forms for the stable singularities that appear in Theorem 6.2.

DEFINITION 6.3. *Let $f : \mathbb{R}^5 \to \mathbb{R}^5$ be a smooth map singular at $q \in \mathbb{R}^5$. Assume there exist variables* x *centered at q and and variables centered at $f(q)$ such that*

- $f(\mathrm{x}_1, \dots, \mathrm{x}_5) = (\mathrm{x}_1^2, \mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4, \mathrm{x}_5)$, *then the singularity is of type $\mathcal{A}_2$;*
- $f(\mathrm{x}_1, \dots, \mathrm{x}_5) = (\mathrm{x}_1^3 + \mathrm{x}_1\mathrm{x}_2, \mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4, \mathrm{x}_5)$, *then the singularity is of type $\mathcal{A}_3$;*
- $f(\mathrm{x}_1, \dots, \mathrm{x}_5) = (\mathrm{x}_1^4 + \mathrm{x}_1^2\mathrm{x}_2 + \mathrm{x}_1\mathrm{x}_3, \mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4, \mathrm{x}_5)$, *then the singularity is of type $\mathcal{A}_4$;*
- $f(\mathrm{x}_1, \dots, \mathrm{x}_5) = (\mathrm{x}_1^5 + \mathrm{x}_1^3\mathrm{x}_2 + \mathrm{x}_1^2\mathrm{x}_3 + \mathrm{x}_1\mathrm{x}_4, \mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4, \mathrm{x}_5)$, *then the singularity is of type $\mathcal{A}_5$;*
- $f(\mathrm{x}_1, \dots, \mathrm{x}_5) = (\mathrm{x}_1^2 + \mathrm{x}_2^2 + \mathrm{x}_1\mathrm{x}_3, \mathrm{x}_1\mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4, \mathrm{x}_5)$, *then the singularity is of type $\mathcal{D}_4^+$.*

We use these normal forms to characterize the singularities in terms of jets. Let M be a 5-dimensional manifold, let $q_0 \in M$, and let $g : T^*_{q_0}M \to M$ be a Lagrangian map. Let p_0 be a critical point of g. We transpose the normal form definition of stable singularities to condition on the jets of g. Given a set of coordinates x on $T^*_{q_0}M$, let us introduce the functions (depending on whether the kernel of the Jacobian matrix of g is of dimension 1 or 2)

$$\phi_{i_1\dots i_k}(p_0) = \det\left(\partial_{\mathrm{x}_{i_1}}\dots\partial_{\mathrm{x}_{i_k}} g, V_2, V_3, V_4, V_5\right) \quad \text{if } \dim\ker \mathrm{Jac}_{p_0} g = 1,$$

$$\phi'_{i_1\dots i_k}(p_0) = \det\left(\partial_{\mathrm{x}_{i_1}}\dots\partial_{\mathrm{x}_{i_k}} g, \partial_{\mathrm{x}_1}\partial_{\mathrm{x}_2} g, V'_3, V'_4, V'_5\right) \quad \text{if } \partial_{\mathrm{x}_1} g = \partial_{\mathrm{x}_2} g = 0.$$

(Here we denote by V_2, V_3, V_4, V_5, linearly independent vectors, depending smoothly on p_0, generating $\mathrm{imJac}_{p_0} g$ if $\dim\ker\mathrm{Jac}_{p_0} g = 1$ and likewise V'_3, V'_4, V'_5, linearly independent vectors, depending smoothly on p_0, generating $\mathrm{imJac}_{p_0} g$ if $\dim\ker\mathrm{Jac}_{p_0} g = 2$.)

In terms of $\phi_{i_1,\dots,i_k}$, we have the following characterization of Lagrangian equivalence classes.

PROPOSITION 6.4. *Let M be a 5-dimensional manifold, let $g : T^*_{q_0}M \to M$ be a Lagrangian map, and let $p_0 \in T^*_{q_0}M$. Assume* $\ker\mathrm{Jac}_{p_0} g$ *is 1 dimensional, if there exist coordinates* $(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4, \mathrm{x}_5)$ *such that $\partial_{\mathrm{x}_1} g(p_0) = 0$ and the following hold at p_0:*

- $\phi_{11} \neq 0$, *then p_0 is a singular point of type $\mathcal{A}_2$;*
- $\phi_{11} = 0$, $\phi_{111} \cdot \phi_{12} \neq 0$, *then p_0 is a singular point of type $\mathcal{A}_3$;*
- $\phi_{11} = \phi_{111} = \phi_{12} = 0$, $\phi_{1111} \cdot \phi_{112} \cdot \phi_{13} \neq 0$, *then p_0 is a singular point of type $\mathcal{A}_4$;*
- $\phi_{11} = \phi_{111} = \phi_{12} = \phi_{1111} = \phi_{112} = \phi_{13} = 0$, $\phi_{11111} \cdot \phi_{1112} \cdot \phi_{113} \cdot \phi_{14} \neq 0$, *then p_0 is a singular point of type $\mathcal{A}_5$.*

Assume $\ker\mathrm{Jac}_{p_0} g$ *is 2-dimensional, if there exist coordinates* $(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4, \mathrm{x}_5)$ *such that $\partial_{\mathrm{x}_1} g = \partial_{\mathrm{x}_2} g = 0$ and $\phi'_{11} \cdot \phi'_{22}(p_0) > 0$, $\phi'_{13}(p_0) \neq 0$ then p_0 is a singular point of type $\mathcal{D}_4^+$.*

Proof. This is a matter of proving that g has the same k-jets as the normal form for $\mathcal{A}_k$ singularities, $k \in [\![2, 5]\!]$, and 2-jet for $\mathcal{D}_4^+$. For each of the stated cases, the existence of changes of coordinates at p_0 and $g(p_0)$ such that it is the case is then justified by the stated conditions. ∎

Remark 6.5. The condition $\phi'_{11} \cdot \phi'_{22}(p_0) > 0$ corresponds to the distinction between $\mathcal{D}_4^+$ and $\mathcal{D}_4^-$ singularities, the latter corresponding to the opposite sign.

Recall that we are considering points $q_0 \in M \setminus (\mathfrak{S}_1 \cup \mathfrak{S}_2)$, where $\mathfrak{S}_1$ (introduced at the beginning of section 3) and $\mathfrak{S}_2$ (introduced in Proposition 4.6) are both stratified subsets of M of codimension 1 at most.

Let (M, Δ, g) be a contact sub-Riemannian manifold of dimension 5 and let $q_0 \in M$. To study the sub-Riemannian caustic at q_0, we study for a given p_0 the stability at $p_0 \in \mathcal{C}_{q_0}(1/2)$ of $\mathcal{E}_{q_0}(t_c(p_0), \cdot)$. To apply Proposition 6.4, we first compute an approximation of the linear spaces $\ker\mathrm{Jac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0))$ and $\mathrm{imJac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0))$. Then we compute approximations of the functions $\phi_{i_1\dots i_k}$ by approximating the map

$$v \mapsto \det\left(v, \mathrm{imJac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0))\right)$$

for a well-chosen representation of $\mathrm{imJac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0))$.

Remark 6.6. Precisely checking the conditions of Proposition 6.4 requires explicit computations executed in the computer algebra system Mathematica.

6.3. Classification of singular points of the caustic. We compute approximations of the sub-Riemannian exponential evaluated at the conjugate time according to the expansions obtained in section 4. In this section, three domains of initial covectors $p = (h_1, h_2, h_3, h_4, h_0)$ naturally appear, depending on the respective values of $r_1 = \sqrt{h_1^2 + h_2^2}$ and $r_2 = \sqrt{h_3^2 + h_4^2}$. If r_1 and r_2 have the same amplitude, direct approximations from section 4.1 are sufficient. On the other hand, if either r_1 or r_2 is greatly smaller than the other, then it is preferable to use the expansions obtained by blowups in sections 4.2 and 4.3, respectively. In order to perform the blowups necessary for the expansions, we define for $\varepsilon > 0$

$$S_1^\varepsilon = \left\{p \in \mathcal{C}_{q_0}(1/2) \mid h_1^2 + h_2^2 < \varepsilon\right\} \text{ and } S_2^{\varepsilon^2} = \left\{p \in \mathcal{C}_{q_0}(1/2) \mid h_3^2 + h_4^2 < \varepsilon^2\right\}.$$

For $\varepsilon > 0$ small enough, we classify on the following three domains: first on $\mathcal{C}_{q_0}(1/2) \setminus (S_1^\varepsilon \cup S_2^{\varepsilon^2}), S_1^\varepsilon$, then $S_2^{\varepsilon^2}$.

Notice that only singularities of corank 1 are expected, apart from singularities of type $\mathcal{D}_4^+$ which can only appear on the second domain S_1^ε. Hence gauging the degree of the singularities is sufficient to classify them, provided that singularities of degree k effectively correspond to singularities of type $\mathcal{A}_k$.

6.3.1. First domain: $\boldsymbol{\mathcal{C}_{q_0}(1/2) \setminus (S_1^\varepsilon \cup S_2^{\varepsilon^2})}$. We consider initial covectors of the form $(h_1, h_2, h_3, h_4, \eta^{-1})$ and build on expansions computed in section 4.1. Algebraic computations, similar to those of the previous sections and left as supplementary materials, lead to the following proposition on the ϕ functions. (See [25, Supplementary Materials].)

(With $n = 2$, recall that B_R denotes the set $\{h \in \mathbb{R}^4 \mid \sum_{i=1}^4 h_i^2 \le R\}$ for all $R > 0$.)

PROPOSITION 6.7. *Let us denote $p_0 = (h_1, h_2, h_3, h_4, \eta^{-1})$. There exists a family of vectors (V_2, V_3, V_4, V_5), smoothly depending on p_0, generating* $\mathrm{imJac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0))$ *for which we have the following. For all $R > 0$, uniformly with respect to $h \in B_R$, as $\eta \to 0$*

$$\phi_{11}(p_0) = O(\eta^8), \qquad \phi_{111}(p_0) = O(\eta^8), \qquad \phi_{1111}(p_0) = O(\eta^8).$$

Furthermore, there exists a function $\Psi : \mathbb{R}^4 \times \mathbb{R}^5 \to \mathbb{R}$ such that for all $V \in \mathbb{R}^5$, $\Psi(h, V) \neq 0$ implies $V \notin \mathrm{imJac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0))$ and with

$$\Psi_k(h) = \Psi\left(h, \partial_{x_1}^k \mathcal{E}_{q_0}(t_c(p_0))\right)^{(2)} \qquad \forall k \in [\![2, 4]\!],$$

we have

$$\Psi_2(h) = \phi_{11}^{(8)}(h), \qquad \Psi_3(h) = \phi_{111}^{(8)}(h), \qquad \Psi_4(h) = \phi_{1111}^{(8)}(h).$$

As a consequence of this proposition we obtain that for η small enough

$$\Psi_2(h) \neq 0 \Rightarrow \phi_{11}(p_0) \neq 0, \qquad \Psi_3(h) \neq 0 \Rightarrow \phi_{111}(p_0) \neq 0,$$

$$\Psi_4(h) \neq 0 \Rightarrow \phi_{1111}(p_0) \neq 0.$$

We can further numerically check as an application of Proposition 6.4 that

- if $\Psi_2 \neq 0$ then the singularity is of type $\mathcal{A}_2$;
- if $\Psi_3 \neq 0$ and the singularity is not of type $\mathcal{A}_2$ then the singularity is of type $\mathcal{A}_3$;
- if $\Psi_4 \neq 0$ and the singularity is not of type $\mathcal{A}_2, \mathcal{A}_3$ then the singularity is of type $\mathcal{A}_4$.

Then we have the following conclusion.

Proposition 6.8. *Let (M, Δ, g) be a generic sub-Riemannian structure and let $q_0 \in M \setminus \mathfrak{S}$. There exists $\bar{\eta} > 0$ such that for all covectors p_0 in*

$$(\mathcal{C}_{q_0}(1/2) \cap \{h_0 > \bar{\eta}^{-1}\}) \setminus (S_1 \cup S_2),$$

the singularity at p_0 of $\mathcal{E}_{q_0}(t_c(p_0))$ is a Lagrange stable singular point of type $\mathcal{A}_2$, $\mathcal{A}_3$, or $\mathcal{A}_4$.

Proof. As a consequence of our discussion, what remains to be proved is that generically with respect to the sub-Riemannian structure, there are no points $(h_1, h_2, h_3, h_4) \in (\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}^2 \setminus \{0\})$ such that

$$\Psi_2(h_1, h_2, h_3, h_4) = \Psi_3(h_1, h_2, h_3, h_4) = \Psi_4(h_1, h_2, h_3, h_4) = 0.$$

However, one can check that this equation admits solutions in $(\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}^2 \setminus \{0\})$ only if $q_0 \in \mathfrak{S}_2$. By assumption $\mathfrak{S}_2 \subset \mathfrak{S}$, hence, the statement. ∎

6.3.2. Second domain: S_1^ϵ. We now consider initial covectors of the form $(\sqrt{\eta}h_1, \sqrt{\eta}h_2, h_3, h_4, \eta^{-1})$ and build on expansions computed in section 4.2. Again, algebraic computations left as supplementary materials lead to the following proposition on the ϕ functions. (See [25, Supplementary Materials].)

Proposition 6.9. *Let us denote $p_0 = (\sqrt{\eta}h_1, \sqrt{\eta}h_2, h_3, h_4, \eta^{-1})$. Let S^+ be the subset of $T^*_{q_0}M$, where $\dim\ker \mathrm{Jac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0)) = 2$. If $\dim\ker \mathrm{Jac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0)) = 1$, then there exists a family of vectors (V_2, V_3, V_4, V_5), smoothly depending on p_0, generating $\mathrm{imJac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0))$ for which we have the following. For all $R > 0$, uniformly with respect to $h \in B_R$, as $\eta \to 0$*

$$\phi_{11}(p_0) = O(\eta^{10}), \quad \phi_{111}(p_0) = O(\eta^{10}), \quad \phi_{1111}(p_0) = O(\eta^{10}), \quad \phi_{11111}(p_0) = O(\eta^{10}).$$

Furthermore, there exists a function $\Phi : \mathbb{R}^4 \times \mathbb{R}^5 \to \mathbb{R}$ such that for all $V \in \mathbb{R}^5$, $\Phi(h, V) \neq 0$ implies $V \notin \mathrm{imJac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0))$ and with

$$\Phi_k(h) = \Phi\left(h, \partial^k_{\mathrm{x}_1}\mathcal{E}_{q_0}(t_c(p_0))\right)^{(5/2)} \qquad \forall k \in [\![2, 4]\!],$$

we have

$$\phi_{11}^{(10)}(h) = \Phi_2(h), \quad \phi_{111}^{(10)}(h) = \Phi_3(h), \quad \phi_{1111}^{(10)}(h) = \Phi_4(h), \quad \phi_{11111}^{(10)}(h) = \Phi_5(h).$$

As a consequence of [25, Remark SM4.7, Supplementary Materials]. we can check that the singularity is of type $\mathcal{D}_4^+$ if $p_0 \in S^+$ and that singular points of the exponential are such that $(h_1, h_2) = (0, 0)$ are of type $\mathcal{A}_3$.

As an application of Proposition 6.9, we obtain that for η small enough, if $p_0 \notin S^+$,

$$\Phi_2(h) \neq 0 \Rightarrow \phi_{11}(p_0) \neq 0, \qquad \Phi_3(h) \neq 0 \Rightarrow \phi_{111}(p_0) \neq 0,$$

$$\Phi_4(h) \neq 0 \Rightarrow \phi_{1111}(p_0) \neq 0, \qquad \Phi_5(h) \neq 0 \Rightarrow \phi_{11111}(p_0) \neq 0.$$

We can further numerically check as an application of Proposition 6.4 that

- if $\Phi_2 \neq 0$ then the singularity is of type $\mathcal{A}_2$;
- if $\Phi_3 \neq 0$ and the singularity is not of type $\mathcal{A}_2$ then the singularity is of type $\mathcal{A}_3$;
- if $\Phi_4 \neq 0$ and the singularity is not of type $\mathcal{A}_2, \mathcal{A}_3$ then the singularity is of type $\mathcal{A}_4$;
- if $\Phi_5 \neq 0$and the singularity is not of type $\mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ then the singularity is of type $\mathcal{A}_5$.

Then we have the following conclusion.

PROPOSITION 6.10. *Let* (M,Δ,g) *be a generic sub-Riemannian structure and let* $q_0 \in M \setminus \mathfrak{S}$. *There exists* $\bar\eta > 0$ *such that for all covectors* p_0 *in*

$$\mathcal{C}_{q_0}(1/2) \cap \{h_0 > \bar\eta^{-1}\} \cap \{h_1^2 + h_2^2 < \bar\eta\},$$

the singularity at p_0 *of* $\mathcal{E}_{q_0}(t_c(p_0))$ *is a Lagrange stable singular point of type* $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$, $\mathcal{A}_5$, *or* $\mathcal{D}_4^+$.

Proof. As a consequence of our discussion and Proposition 6.9, what remains to be proved is that there are no elements $(h_1,h_2,h_3,h_4) \in (\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}^2 \setminus \{0\})$ such that $\Phi_2(h) = \Phi_3(h) = \Phi_4(h) = \Phi_5(h) = 0$.

Similarly to the proof of Proposition 6.8, this is excluded on the complementarity of $\mathfrak{S}$. ∎

Remark 6.11. An intuition can be given to the reason $\mathcal{A}_5$ singularities can appear on the second (and third) domain but not the first one. In the first domain, our approximation of the exponential presents symmetries that do not appear in the other domains. For instance these symmetries appear in the computations of the approximations of the ϕ functions of Proposition 6.4.

Indeed, we have on the first domain a two-parameter symmetry: for all $\lambda,\mu > 0$, $h \in \mathbb{R}^4$,

$$\Psi_i(\lambda h_1,\lambda h_2,\mu h_3,\mu h_4) = \lambda^2\mu\Psi_i(h_1,h_2,h_3,h_4), \qquad i \in [\![2,4]\!].$$

On the second domain, on the other hand, we only have a one-parameter symmetry:

$$\Phi_i(\lambda^3 h_1,\lambda^3 h_2,\lambda^2 h_3,\lambda^2 h_4) = \lambda^{14}\Phi_i(h_1,h_2,h_3,h_4), \qquad i \in [\![2,5]\!].$$

In other words, the exponential map reduces to a 3D Lagrangian map on the first domain and only singularities of type $\mathcal{A}_2$ to $\mathcal{A}_4$ should appear. Conversely, the symmetry on the second domain implies that the exponential reduces to a 4-dimensional Lagrangian map and $\mathcal{A}_5$ singularities can be expected.

A similar argument can be made in the 3D contact case for the presence of $\mathcal{A}_2$ and $\mathcal{A}_3$ singularities (see [1] for instance).

6.3.3. Third domain: $S_2^{\varepsilon^2}$. Finally, we consider initial covectors of the form $(h_1,h_2,\eta h_3,\eta h_4,\eta^{-1})$ and apply expansions computed in section 4.3. Again, algebraic computations left as supplementary materials lead to the following proposition on the ϕ functions. (See [25, Supplementary Materials].)

PROPOSITION 6.12. *Let us denote* $p_0 = (h_1,h_2,\eta h_3,\eta h_4,\eta^{-1})$. *There exist a family of vectors* (V_2,V_3,V_4,V_5), *smoothly depending on* p_0, *generating* $\mathrm{imJac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0))$ *for which we have the following. For all* $R > 0$, *uniformly with respect to* $h \in B_R$, *as* $\eta \to 0$,

$$\phi_{11}(p_0) = O(\eta^{11}), \quad \phi_{111}(p_0) = O(\eta^{11}), \quad \phi_{1111}(p_0) = O(\eta^{11}), \quad \phi_{11111}(p_0) = O(\eta^{11}).$$

Furthermore, there exists a function $\Gamma : \mathbb{R}^4 \times \mathbb{R}^5 \to \mathbb{R}$ *such that for all* $V \in \mathbb{R}^5$, $\Gamma(h,V) \neq 0$ *implies* $V \notin \mathrm{imJac}_{p_0}\mathcal{E}_{q_0}(t_c(p_0))$ *and with*

$$\Gamma_k(h) = \Gamma\left(h, \partial_{x_1}^k \mathcal{E}_{q_0}(t_c(p_0))\right)^{(3)} \qquad \forall k \in [\![2,5]\!],$$

we have

$$\phi_{11}^{(11)}(h) = \Gamma_2(h), \quad \phi_{111}^{(11)}(h) = \Gamma_3(h), \quad \phi_{1111}^{(11)}(h) = \Gamma_4(h), \quad \phi_{11111}^{(11)}(h) = \Gamma_5(h).$$

As a consequence of this proposition we obtain that for η small enough

$$\Gamma_2(h) \neq 0 \Rightarrow \phi_{11}(p_0) \neq 0, \qquad \Gamma_3(h) \neq 0 \Rightarrow \phi_{111}(p_0) \neq 0,$$

$$\Gamma_4(h) \neq 0 \Rightarrow \phi_{1111}(p_0) \neq 0, \qquad \Gamma_5(h) \neq 0 \Rightarrow \phi_{11111}(p_0) \neq 0.$$

We can further numerically check as an application of Proposition 6.4 that

- if $\Gamma_2 \neq 0$ then the singularity is of type $\mathcal{A}_2$;
- if $\Gamma_3 \neq 0$ and the singularity is not of type $\mathcal{A}_2$ then the singularity is of type $\mathcal{A}_3$;
- if $\Gamma_4 \neq 0$ and the singularity is not of type $\mathcal{A}_2, \mathcal{A}_3$ then the singularity is of type $\mathcal{A}_4$;
- if $\Gamma_5 \neq 0$ and the singularity is not of type $\mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ then the singularity is of type $\mathcal{A}_5$.

Then we have the following conclusion.

PROPOSITION 6.13. *Let (M, Δ, g) be a generic sub-Riemannian structure and let $q_0 \in M \setminus \mathfrak{S}$. There exists $\bar{\eta} > 0$ such that for all covectors p_0 in*

$$\mathcal{C}_{q_0}(1/2) \cap \{h_0 > \bar{\eta}^{-1}\} \cap \{h_3^2 + h_4^2 < \bar{\eta}^2\},$$

the singularity at p_0 of $\mathcal{E}_{q_0}(t_c(p_0))$ is a Lagrange stable singular point of type $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$, or $\mathcal{A}_5$.

Proof. The argument is the same as in the other two cases, that is, as a consequence of our discussion, there are no points $h \in (\mathbb{R}^2 \setminus \{0\}) \times (\mathbb{R}^2)$ such that $\Gamma_2(h) = \Gamma_3(h) = \Gamma_4(h) = \Gamma_5(h) = 0$. Again, this is excluded on the complementarity of $\mathfrak{S}$. ∎

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REFERENCES

[1] A. Agrachev, D. Barilari, and U. Boscain, *A Comprehensive Introduction to Sub-Riemannian Geometry*, Cambridge University Press, Cambridge, 2019.

[2] A. Agrachev, D. Barilari, and L. Rizzi, *Sub-Riemannian curvature in contact geometry*, J. Geom. Anal., 27 (2016), pp. 366–408, https://doi.org/10.1007/s12220-016-9684-0.

[3] A. Agrachev and J. Gauthier, *Sub-Riemannian metrics and isoperimetric problems in the contact case*, J. Math. Sci., 103 (2001), pp. 639–663.

[4] A. A. Agrachev, *Exponential mappings for contact sub-Riemannian structures*, J. Dyn. Control Syst., 2 (1996), pp. 321–358, https://doi.org/10.1007/BF02269423.

[5] A. A. Agrachev, G. Charlot, J. P. A. Gauthier, and V. M. Zakalyukin, *On sub-Riemannian caustics and wave fronts for contact distributions in the three-space*, J. Dyn. Control Syst., 6 (2000), pp. 365–395, https://doi.org/10.1023/A:1009566723037.

[6] V. I. Arnold, S. M. Guseĭ n Zade, and A. N. Varchenko, *Singularities of Differentiable Maps.* Vol. I, Monogr. Math. 82, Birkhäuser, Boston, 1985, https://doi.org/10.1007/978-1-4612-5154-5.

[7] D. Barilari, *Trace heat kernel asymptotics in 3D contact sub-Riemannian geometry*, J. Math. Sci., 195 (2013), pp. 391–411, https://doi.org/10.1007/s10958-013-1585-1.

[8] D. Barilari, I. Beschastnyi, and A. Lerario, *Volume of small balls and sub-Riemannian curvature in 3D contact manifolds*, J. Symplectic Geom., 18 (2010), 2, to appear.

[9] D. Barilari, U. Boscain, G. Charlot, and R. W. Neel, *On the heat diffusion for generic Riemannian and sub-Riemannian structures*, Int. Math. Res. Not. IMRN, 2017 (2016), pp. 4639–4672.

[10] D. Barilari, U. Boscain, and R. W. Neel, *Small-time heat kernel asymptotics at the sub-Riemannian cut locus*, J. Differential Geom., 92 (2012), pp. 373–416, https://doi.org/10.4310/jdg/1354110195.

[11] D. Barilari, U. Boscain, and M. Sigalotti, eds., *Geometry, Analysis and Dynamics on sub-Riemannian Manifolds*, Vol. 1, EMS Ser. Lect. Math., European Mathematical Society, Zürich, 2016, https://doi.org/10.4171/162.

[12] D. Barilari, U. Boscain, and M. Sigalotti, eds., *Geometry, Analysis and Dynamics on sub-Riemannian Manifolds*, Vol. 2, EMS Ser. Lect. Math., European Mathematical Society, Zürich, 2016, https://doi.org/10.4171/163.

[13] A. Bellaïche, *The tangent space in sub-Riemannian geometry*, in Sub-Riemannian Geometry, Progr. Math. 144, Birkhäuser, Basel, 1996, pp. 1–78, https://doi.org/10.1007/978-3-0348-9210-0_1.

[14] D. Bennequin, *Caustique mystique*, in Séminaire Bourbaki 27, 1984–1985, pp. 19–56, http://eudml.org/doc/110042.

[15] B. Bonnet, J.-P. Gauthier, and F. Rossi, *Generic singularities of the 3D-contact sub-Riemannian conjugate locus*, C. R. Math. Acad. Sci. Paris., hal-01936019, to appear.

[16] U. Boscain, R. Neel, and L. Rizzi, *Intrinsic random walks and sub-Laplacians in sub-Riemannian geometry*, Adv. Math. 314 (2017), pp. 124–184, https://doi.org/10.1016/j.aim.2017.04.024.

[17] G. Charlot, *Quasi-contact S-R metrics: Normal form in* $\mathbf{R}^{2n}$*, wave front and caustic in* $\mathbf{R}^4$, Acta Appl. Math., 74 (2002), pp. 217–263, https://doi.org/10.1023/A:1021199303685.

[18] M. M. Diniz and J. M. M. Veloso, *Regions where the exponential map at regular points of sub-Riemannian manifolds is a local diffeomorphism*, J. Dyn. Control Syst., 15 (2009), pp. 133–156, https://doi.org/10.1007/s10883-008-9054-8.

[19] E.-H. C. El-Alaoui, J. Gauthier, and I. Kupka, *Small sub-Riemannian balls on* $\mathbf{R}^3$, J. Dyn. Control Systems, 2 (1996), pp. 359–421, https://doi.org/10.1007/BF02269424.

[20] B. Gaveau, *Principe de moindre action, propagation de la chaleur et estimées sous elliptiques sur certains groupes nilpotents*, Acta Math., 139 (1977), pp. 95–153.

[21] W. K. Hughen, *The Sub-Riemannian Geometry of Three-Manifolds*, Ph.D. thesis, Duke University, Durham, NC, 1995.

[22] S. Izumiya, M. d. C. R. Fuster, M. A. S. Ruas, F. Tari, *Differential Geometry from a Singularity Theory Viewpoint*, World Scientific, Hackensack, NJ, 2016.

[23] A. Lerario and L. Rizzi, *How many geodesics join two points on a contact sub-Riemannian manifold?*, J. Symplectic Geom., 15 (2017), pp. 247–305.

[24] R. Montgomery, *A tour of SubRiemannian geometries, their geodesics and applications*, Math. Surveys Monogr. 91, American Mathematical Society, Providence, RI, 2006.

[25] L. Sacchelli, *Short Geodesics Losing Optimality in Contact Sub-Riemannian Manifolds and Stability of the 5-Dimensional Caustic*, preprint, hal-01961959, version 3, 2019, https://hal.archives-ouvertes.fr/hal-01961959.