

SDE4DVar

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1 Formulation

SDE and observation model

$$x^{(1)} \sim N(0, E_1) \quad (1)$$

$$x^{(t)} = M(x^{(t-1)}, p) + \xi^{(t)}, \xi^{(t)} \sim N(0, E), t = 2, \dots, T \quad (2)$$

$$y^{(t)} = H(x^{(t)}, r) + \xi'^{(t)}, \xi'^{(t)} \sim N(0, \Sigma), t = 1, \dots, T \quad (3)$$

Maximul path

$$\text{Max}_{x^{(1:T)}} p(x^{(1:T)}, y^{(1:T)} | \mu) = p(x^{(1)} | \mu) p(y^{(1)} | x^{(1)}) \prod_{t=2}^T p(x^{(t)} | x^{(t-1)}) p(y^{(t)} | x^{(t)}) \quad (4)$$

$$\text{Min}_{x^{(1:T)}} l(x^{(1:T)}, y^{(1:T)} | \mu) \quad (5)$$

$$= -\ln p(x^{(1:T)}, y^{(1:T)} | \mu) \quad (6)$$

$$\begin{aligned} &= \frac{1}{2} \ln |E_1| + \frac{T-1}{2} \ln |E| + \frac{T}{2} \ln |\Sigma| + \frac{1}{2} (x^{(1)} - \mu)^T E_1^{-1} (x^{(1)} - \mu) \\ &+ \frac{1}{2} \sum_{t=2}^T (x^{(t)} - M(x^{(t-1)}, p))^T E^{-1} (x^{(t)} - M(x^{(t-1)}, p)) \\ &+ \frac{1}{2} \sum_{t=1}^T (H(x^{(t)}, r) - y^{(t)})^T \Sigma^{-1} (H(x^{(t)}, r) - y^{(t)}) \end{aligned} \quad (7)$$

$$\begin{aligned} \frac{\partial l}{\partial x^{(1)}} &= E_1^{-1} (x^{(1)} - \mu) + \left(\frac{\partial H}{\partial x^{(1)}} \right)^T \Sigma^{-1} (H(x^{(1)}, r) - y^{(1)}) = 0 \\ &+ \left(\frac{\partial M}{\partial x^{(1)}} \right)^T E^{-1} (M(x^{(1)}, p) - x^{(2)}) = 0 \end{aligned} \quad (8)$$

$$\begin{aligned} \frac{\partial l}{\partial x^{(t)}} &= E^{-1} (x^{(t)} - M(x^{(t-1)}, p)) + \left(\frac{\partial H}{\partial x^{(t)}} \right)^T \Sigma^{-1} (H(x^{(t)}, r) - y^{(t)}) \\ &+ \left(\frac{\partial M}{\partial x^{(t)}} \right)^T E^{-1} (M(x^{(t)}, p) - x^{(t+1)}) = 0 \quad (t = 2, \dots, T-1) \end{aligned} \quad (9)$$

Cost function

$$J(x^{(1)}, p, r) = \mathcal{J}(x^{(1:T)}, p, r) = l(x^{(1:T)}, y^{(1:T)} | \mu) + \text{Regularization} \quad (10)$$

$$\begin{aligned} \text{Min}_{x^{(1:T)}, p, r} L(x^{(1:T)}, p, r; \lambda^{(2:T)}) &= \mathcal{J}(x^{(1:T)}, p, r) \\ &+ \lambda^{(2)\text{T}} \left(E_1^{-1}(x^{(1)} - \mu) + \left(\frac{\partial H}{\partial x^{(1)}} \right)^{\text{T}} \Sigma^{-1}(H(x^{(1)}, r) - y^{(1)}) \right. \\ &\quad \left. + \left(\frac{\partial M}{\partial x^{(1)}} \right)^{\text{T}} E^{-1}(M(x^{(1)}, p) - x^{(2)}) \right) \\ &+ \sum_{t=2}^{T-1} \lambda^{(t+1)\text{T}} \left(E^{-1}(x^{(t)} - M(x^{(t-1)}, p)) + \left(\frac{\partial H}{\partial x^{(t)}} \right)^{\text{T}} \Sigma^{-1}(H(x^{(t)}, r) - y^{(t)}) \right. \\ &\quad \left. + \left(\frac{\partial M}{\partial x^{(t)}} \right)^{\text{T}} E^{-1}(M(x^{(t)}, p) - x^{(t+1)}) \right) \end{aligned} \quad (11)$$

Orbit calculation

$$x^{(2)} - M(x^{(1)}, p) = \left(\frac{\partial M}{\partial x^{(1)}} \right)^{\text{T}} E^{-1} \setminus \left\{ E_1^{-1}(x^{(1)} - \mu) + \left(\frac{\partial H}{\partial x^{(1)}} \right)^{\text{T}} \Sigma^{-1}(H(x^{(1)}, r) - y^{(1)}) \right\} \quad (12)$$

$$\begin{aligned} x^{(t+1)} - M(x^{(t)}, p) &= \left(\frac{\partial M}{\partial x^{(t)}} \right)^{\text{T}} E^{-1} \setminus \left\{ E^{-1}(x^{(t)} - M(x^{(t-1)}, p)) + \left(\frac{\partial H}{\partial x^{(t)}} \right)^{\text{T}} \Sigma^{-1}(H(x^{(t)}, r) - y^{(t)}) \right\} \\ &\quad (t = 2, \dots, T-1) \end{aligned} \quad (13)$$

Gradient calculation

$$\lambda^{(T+1)} = 0 \quad (14)$$

$$\lambda^{(T)} = E^{-1} \frac{\partial M}{\partial x^{(T-1)}} \setminus \frac{\partial \mathcal{J}(x^{(1:T)}, p, r)}{\partial x^{(T)}} \quad (15)$$

$$\begin{aligned} \lambda^{(t)} = E^{-1} \frac{\partial M}{\partial x^{(t-1)}} \setminus \left[\left\{ E^{-1} + \left(\frac{\partial H}{\partial x^{(t)}} \right)^T \Sigma^{-1} \frac{\partial H}{\partial x^{(t)}} + \left(\frac{\partial M}{\partial x^{(t)}} \right)^T E^{-1} \frac{\partial M}{\partial x^{(t)}} \right\} \lambda^{(t+1)} - \left(\frac{\partial M}{\partial x^{(t)}} \right)^T E^{-1} \lambda^{(t+2)} \right. \\ \left. + \left(\frac{\partial^2 H}{\partial x^{(t)2}} \lambda^{(t+1)} \right)^T \Sigma^{-1} (H(x^{(t)}, r) - y^{(t)}) - \left(\frac{\partial^2 M}{\partial x^{(t)2}} \lambda^{(t+1)} \right)^T E^{-1} (x^{(t+1)} - M(x^{(t)})) \right] \quad (t = T-1, \dots, 2) \end{aligned} \quad (16)$$

$$\begin{aligned} \frac{\partial J(x^{(1)}, p, r)}{\partial x^{(1)}} = \left\{ E_1^{-1} + \left(\frac{\partial H}{\partial x^{(1)}} \right)^T \Sigma^{-1} \frac{\partial H}{\partial x^{(1)}} + \left(\frac{\partial M}{\partial x^{(1)}} \right)^T E^{-1} \frac{\partial M}{\partial x^{(1)}} \right\} \lambda^{(2)} - \left(\frac{\partial M}{\partial x^{(1)}} \right)^T E^{-1} \lambda^{(3)} \\ + \left(\frac{\partial^2 H}{\partial x^{(1)2}} \lambda^{(2)} \right)^T \Sigma^{-1} (H(x^{(1)}, r) - y^{(1)}) - \left(\frac{\partial^2 M}{\partial x^{(1)2}} \lambda^{(2)} \right)^T E^{-1} (x^{(2)} - M(x^{(1)})) \end{aligned} \quad (17)$$

$$\begin{aligned} \frac{\partial J(x^{(1)}, p, r)}{\partial p} = \sum_{t=1}^{T-1} \left\{ \left(\frac{\partial M^{(t)}}{\partial p} \right)^T E^{-1} \left(\frac{\partial M}{\partial x^{(t)}} \lambda^{(t+1)} - \lambda^{(t+2)} + M(x^{(t)}, p) - x^{(t+1)} \right) \right. \\ \left. - \left(\frac{\partial^2 M}{\partial p \partial x^{(t)}} \lambda^{(t+1)} \right)^T E^{-1} (x^{(t+1)} - M(x^{(t)}, p)) \right\} + \text{Regularization gradient} \end{aligned} \quad (18)$$

$$\begin{aligned} \frac{\partial J(x^{(1)}, p, r)}{\partial r} = \sum_{t=1}^T \left\{ \left(\frac{\partial^2 H}{\partial r \partial x^{(t)}} \lambda^{(t+1)} \right)^T \Sigma^{-1} (H(x^{(t)}, r) - y^{(t)}) \right. \\ \left. + \left(\frac{\partial H^{(t)}}{\partial r} \right)^T \Sigma^{-1} \left(\frac{\partial H}{\partial x^{(t)}} \lambda^{(t+1)} + H(x^{(t)}, r) - y^{(t)} \right) \right\} + \text{Regularization gradient} \end{aligned} \quad (19)$$

where

$$\frac{\partial \mathcal{J}(x^{(1:T)}, p, r)}{\partial x^{(T)}} = E^{-1} (x^{(T)} - M(x^{(T-1)}, p)) + \left(\frac{\partial H}{\partial x^{(T)}} \right)^T \Sigma^{-1} (H(x^{(T)}, r) - y^{(T)}) \quad (20)$$

Neighboring orbit calculation

$$\begin{aligned}
& \delta x^{(2)} - \frac{\partial M}{\partial x^{(1)}} \delta x^{(1)} - \frac{\partial M^{(1)}}{\partial p} \delta p \\
= & \left(\frac{\partial M}{\partial x^{(1)}} \right)^T E^{-1} \setminus \left\{ - \left(\frac{\partial^2 M}{\partial x^{(1)2}} \delta x^{(1)} + \frac{\partial^2 M}{\partial x^{(1)} \partial p} \delta p \right)^T E^{-1} (x^{(2)} - M(x^{(1)}, p)) + E_1^{-1} \delta x^{(1)} \right. \\
& \left. + \left(\frac{\partial^2 H}{\partial x^{(1)2}} \delta x^{(1)} + \frac{\partial^2 H}{\partial x^{(1)} \partial r} \delta r \right)^T \Sigma^{-1} (H(x^{(1)}, r) - y^{(1)}) + \left(\frac{\partial H}{\partial x^{(1)}} \right)^T \Sigma^{-1} \left(\frac{\partial H}{\partial x^{(1)}} \delta x^{(1)} + \frac{\partial H^{(1)}}{\partial r} \delta r \right) \right\} \\
& (21)
\end{aligned}$$

$$\begin{aligned}
& \delta x^{(t+1)} - \frac{\partial M}{\partial x^{(t)}} \delta x^{(t)} - \frac{\partial M^{(t)}}{\partial p} \delta p \\
= & \left(\frac{\partial M}{\partial x^{(t)}} \right)^T E^{-1} \setminus \left\{ - \left(\frac{\partial^2 M}{\partial x^{(t)2}} \delta x^{(t)} + \frac{\partial^2 M}{\partial x^{(t)} \partial p} \delta p \right)^T E^{-1} (x^{(t+1)} - M(x^{(t)}, p)) \right. \\
& \left. + E^{-1} \left(\delta x^{(t)} - \frac{\partial M}{\partial x^{(t-1)}} \delta x^{(t-1)} - \frac{\partial M^{(t-1)}}{\partial p} \delta p \right) \right. \\
& \left. + \left(\frac{\partial^2 H}{\partial x^{(t)2}} \delta x^{(t)} + \frac{\partial^2 H}{\partial x^{(t)} \partial r} \delta r \right)^T \Sigma^{-1} (H(x^{(t)}, r) - y^{(t)}) \right. \\
& \left. + \left(\frac{\partial H}{\partial x^{(t)}} \right)^T \Sigma^{-1} \left(\frac{\partial H}{\partial x^{(t)}} \delta x^{(t)} + \frac{\partial H^{(t)}}{\partial r} \delta r \right) \right\} \\
& (t = 2, \dots, T-1) \tag{22}
\end{aligned}$$

Hessian-vector product calculation

$$\delta\lambda^{(T+1)} = 0 \quad (23)$$

$$\begin{aligned} \delta\lambda^{(T)} = E^{-1} \frac{\partial M}{\partial x^{(T-1)}} \setminus \left\{ -E^{-1} \left(\frac{\partial^2 M}{\partial x^{(T-1)2}} \delta x^{(T-1)} + \frac{\partial^2 M}{\partial x^{(T-1)} \partial p} \delta p \right) \lambda^{(T)} \right. \\ \left. + E^{-1} \left(\delta x^{(T)} - \frac{\partial M}{\partial x^{(T-1)}} \delta x^{(T-1)} - \frac{\partial M^{(T-1)}}{\partial p} \delta p \right) \right. \\ \left. + \left(\frac{\partial^2 H}{\partial x^{(T)2}} \delta x^{(T)} + \frac{\partial^2 H}{\partial x^{(T)} \partial r} \delta r \right)^T \Sigma^{-1} (H(x^{(T)}, r) - y^{(T)}) \right. \\ \left. + \left(\frac{\partial H}{\partial x^{(T)}} \right)^T \Sigma^{-1} \left(\frac{\partial H}{\partial x^{(T)}} \delta x^{(T)} + \frac{\partial H^{(T)}}{\partial r} \delta r \right) \right\} \quad (24) \end{aligned}$$

$$\begin{aligned} \delta\lambda^{(t)} = E^{-1} \frac{\partial M}{\partial x^{(t-1)}} \setminus \left[-E^{-1} \left(\frac{\partial^2 M}{\partial x^{(t-1)2}} \delta x^{(t-1)} + \frac{\partial^2 M}{\partial x^{(t-1)} \partial p} \delta p \right) \lambda^{(t)} \right. \\ \left. \left\{ \left(\frac{\partial^2 H}{\partial x^{(t)2}} \delta x^{(t)} + \frac{\partial^2 H^{(t)}}{\partial x^{(t)} \partial r} \delta r \right)^T \Sigma^{-1} \frac{\partial H}{\partial x^{(t)}} \right. \right. \\ \left. + \left(\frac{\partial H}{\partial x^{(t)}} \right)^T \Sigma^{-1} \left(\frac{\partial^2 H}{\partial x^{(t)2}} \delta x^{(t)} + \frac{\partial^2 H^{(t)}}{\partial x^{(t)} \partial r} \delta r \right) \right. \\ \left. + \left(\frac{\partial^2 M}{\partial x^{(t)2}} \delta x^{(t)} + \frac{\partial^2 M}{\partial x^{(t)} \partial p} \delta p \right)^T E^{-1} \frac{\partial M}{\partial x^{(t)}} \right. \\ \left. + \left(\frac{\partial M}{\partial x^{(t)}} \right)^T E^{-1} \left(\frac{\partial^2 M}{\partial x^{(t)2}} \delta x^{(t)} + \frac{\partial^2 M^{(t)}}{\partial x^{(t)} \partial p} \delta p \right) \right\} \lambda^{(t+1)} \\ \left. + \left\{ E^{-1} + \left(\frac{\partial H}{\partial x^{(t)}} \right)^T \Sigma^{-1} \frac{\partial H}{\partial x^{(t)}} + \left(\frac{\partial M}{\partial x^{(t)}} \right)^T E^{-1} \frac{\partial M}{\partial x^{(t)}} \right\} \delta\lambda^{(t+1)} \right. \\ \left. - \left(\frac{\partial^2 M}{\partial x^{(t)2}} \delta x^{(t)} + \frac{\partial^2 M^{(t)}}{\partial x^{(t)} \partial p} \delta p \right)^T E^{-1} \lambda^{(t+2)} - \left(\frac{\partial M}{\partial x^{(t)}} \right)^T E^{-1} \delta\lambda^{(t+2)} \right. \\ \left. + \left\{ \left(\frac{\partial^3 H}{\partial x^{(t)3}} \delta x^{(t)} + \frac{\partial^3 H}{\partial x^{(t)2} \partial r} \delta r \right) \lambda^{(t+1)} + \frac{\partial^2 H}{\partial x^{(t)2}} \delta\lambda^{(t+1)} \right\}^T \Sigma^{-1} (H(x^{(t)}, r) - y^{(t)}) \right. \\ \left. + \left(\frac{\partial^2 H}{\partial x^{(t)}} \lambda^{(t+1)} \right)^T \Sigma^{-1} \left(\frac{\partial H}{\partial x^{(t)}} \delta x^{(t)} + \frac{\partial H^{(t)}}{\partial r} \delta r \right) \right. \\ \left. - \left\{ \left(\frac{\partial^3 M}{\partial x^{(t)3}} \delta x^{(t)} + \frac{\partial^3 M}{\partial x^{(t)2} \partial p} \delta p \right) \lambda^{(t+1)} + \frac{\partial^2 M}{\partial x^{(t)2}} \delta\lambda^{(t+1)} \right\}^T E^{-1} (x^{(t+1)} - M(x^{(t)}, p)) \right. \\ \left. - \left(\frac{\partial^2 M}{\partial x^{(t)}} \lambda^{(t+1)} \right)^T E^{-1} \left(\delta x^{(t+1)} - \frac{\partial M}{\partial x^{(t)}} \delta x^{(t)} - \frac{\partial M^{(t)}}{\partial p} \delta p \right) \right] \\ (t = T-1, \dots, 2) \quad (25) \end{aligned}$$

$$\begin{aligned}
& \delta \left(\frac{\partial J(x^{(1)}, p, r)}{\partial x^{(1)}} \right) \\
&= \left\{ \left(\frac{\partial^2 H}{\partial x^{(1)2}} \delta x^{(1)} + \frac{\partial^2 H^{(1)}}{\partial x^{(1)} \partial r} \delta r \right)^T \Sigma^{-1} \frac{\partial H}{\partial x^{(1)}} \right. \\
&+ \left(\frac{\partial H}{\partial x^{(1)}} \right)^T \Sigma^{-1} \left(\frac{\partial^2 H}{\partial x^{(1)2}} \delta x^{(1)} + \frac{\partial^2 H^{(1)}}{\partial x^{(1)} \partial r} \delta r \right) \\
&+ \left(\frac{\partial^2 M}{\partial x^{(1)2}} \delta x^{(1)} + \frac{\partial^2 M}{\partial x^{(1)} \partial p} \delta p \right)^T E^{-1} \frac{\partial M}{\partial x^{(1)}} \\
&+ \left. \left(\frac{\partial M}{\partial x^{(1)}} \right)^T E^{-1} \left(\frac{\partial^2 M}{\partial x^{(1)2}} \delta x^{(1)} + \frac{\partial^2 M^{(1)}}{\partial x^{(1)} \partial p} \delta p \right) \right\} \lambda^{(2)} \\
&+ \left\{ E^{-1} + \left(\frac{\partial H}{\partial x^{(1)}} \right)^T \Sigma^{-1} \frac{\partial H}{\partial x^{(1)}} + \left(\frac{\partial M}{\partial x^{(1)}} \right)^T E^{-1} \frac{\partial M}{\partial x^{(1)}} \right\} \delta \lambda^{(2)} \\
&- \left(\frac{\partial^2 M}{\partial x^{(1)2}} \delta x^{(1)} + \frac{\partial^2 M^{(1)}}{\partial x^{(1)} \partial p} \delta p \right)^T E^{-1} \lambda^{(3)} - \left(\frac{\partial M}{\partial x^{(1)}} \right)^T E^{-1} \delta \lambda^{(3)} \\
&+ \left\{ \left(\frac{\partial^3 H}{\partial x^{(1)3}} \delta x^{(1)} + \frac{\partial^3 H}{\partial x^{(1)2} \partial r} \delta r \right) \lambda^{(2)} + \frac{\partial^2 H}{\partial x^{(1)2}} \delta \lambda^{(2)} \right\}^T \Sigma^{-1} (H(x^{(1)}, r) - y^{(1)}) \\
&+ \left(\frac{\partial^2 H}{\partial x^{(1)}} \lambda^{(2)} \right)^T \Sigma^{-1} \left(\frac{\partial H}{\partial x^{(1)}} \delta x^{(1)} + \frac{\partial H^{(1)}}{\partial r} \delta r \right) \\
&- \left\{ \left(\frac{\partial^3 M}{\partial x^{(1)3}} \delta x^{(1)} + \frac{\partial^3 M}{\partial x^{(1)2} \partial p} \delta p \right) \lambda^{(2)} + \frac{\partial^2 M}{\partial x^{(1)2}} \delta \lambda^{(2)} \right\}^T E^{-1} (x^{(2)} - M(x^{(1)}, p)) \\
&- \left(\frac{\partial^2 M}{\partial x^{(1)}} \lambda^{(2)} \right)^T E^{-1} \left(\delta x^{(2)} - \frac{\partial M}{\partial x^{(1)}} \delta x^{(1)} - \frac{\partial M^{(1)}}{\partial p} \delta p \right) \quad (26)
\end{aligned}$$

$$\begin{aligned}
& \delta \left(\frac{\partial J(x^{(1)}, p, r)}{\partial p} \right) \\
&= \sum_{t=1}^{T-1} \left[\left(\frac{\partial^2 M}{\partial p \partial x^{(t)}} \delta x^{(t)} + \frac{\partial^2 M^{(t)}}{\partial p^2} \delta p \right)^T E^{-1} \left(\frac{\partial M}{\partial x^{(t)}} \lambda^{(t+1)} - \lambda^{(t+2)} + M(x^{(t)}, p) - x^{(t+1)} \right) \right. \\
&+ \left(\frac{\partial M^{(t)}}{\partial p} \right)^T E^{-1} \left\{ \left(\frac{\partial^2 M}{\partial x^{(t)2}} \delta x^{(t)} + \frac{\partial^2 M^{(t)}}{\partial x^{(t)} \partial p} \delta p \right) \lambda^{(t+1)} \right. \\
&+ \left. \frac{\partial M}{\partial x^{(t)}} (\delta \lambda^{(t+1)} + \delta x^{(t)}) + \frac{\partial M^{(t)}}{\partial p} \delta p - \delta \lambda^{(t+2)} - \delta x^{(t+1)} \right\} \\
&- \left\{ \left(\frac{\partial^3 M}{\partial p \partial x^{(t)2}} \delta x^{(t)} + \frac{\partial^3 M}{\partial p \partial x^{(t)} \partial p} \delta p \right) \lambda^{(t+1)} + \frac{\partial^2 M}{\partial p \partial x^{(t)}} \delta \lambda^{(t+1)} \right\}^T E^{-1} (x^{(t+1)} - M(x^{(t)}, p)) \\
&- \left. \left(\frac{\partial^2 M}{\partial p \partial x^{(t)}} \lambda^{(t+1)} \right)^T E^{-1} \left(\delta x^{(t+1)} - \frac{\partial M}{\partial x^{(t)}} \delta x^{(t)} - \frac{\partial M^{(t)}}{\partial p} \delta p \right) \right] + \delta \text{Regularizationgradient} \quad (27)
\end{aligned}$$

$$\begin{aligned}
& \delta \left(\frac{\partial J(x^{(1)}, p, r)}{\partial r} \right) \\
= & \sum_{t=1}^{T-1} \left[\left(\frac{\partial^2 H}{\partial r \partial x^{(t)}} \delta x^{(t)} + \frac{\partial^2 H^{(t)}}{\partial r^2} \delta r \right)^T \Sigma^{-1} \left(\frac{\partial H}{\partial x^{(t)}} \lambda^{(t+1)} + H(x^{(t)}, r) - y^{(t)} \right) \right. \\
& + \left(\frac{\partial H^{(t)}}{\partial r} \right)^T \Sigma^{-1} \left\{ \left(\frac{\partial^2 H}{\partial x^{(t)2}} \delta x^{(t)} + \frac{\partial^2 H^{(t)}}{\partial x^{(t)} \partial r} \delta r \right) \lambda^{(t+1)} \right. \\
& \quad \left. + \frac{\partial H}{\partial x^{(t)}} \left(\delta \lambda^{(t+1)} + \delta x^{(t)} \right) + \frac{\partial H^{(t)}}{\partial r} \delta r \right\} \\
& + \left\{ \left(\frac{\partial^3 H}{\partial r \partial x^{(t)2}} \delta x^{(t)} + \frac{\partial^3 H}{\partial r \partial x^{(t)} \partial r} \delta r \right) \lambda^{(t+1)} + \frac{\partial^2 H}{\partial r \partial x^{(t)}} \delta \lambda^{(t+1)} \right\}^T \Sigma^{-1} (H(x^{(t)}, r) - y^{(t)}) \\
& \left. + \left(\frac{\partial^2 H}{\partial r \partial x^{(t)}} \lambda^{(t+1)} \right)^T \Sigma^{-1} \left(\frac{\partial H}{\partial x^{(t)}} \delta x^{(t)} + \frac{\partial H^{(t)}}{\partial r} \delta r \right) \right] + \delta \text{Regularization gradient} \\
& \tag{28}
\end{aligned}$$