SDE4DVar

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1 Formulation

SDE and observation model

$$x^{(1)} \sim N(\mu, E_1) \tag{1}$$

$$x^{(t)} = M(x^{(t-1)}, p) + \xi^{(t)}, \xi^{(t)} \sim N(0, E), t = 2, \cdots, T \tag{2}$$

$$y^{(t)} = H(x^{(t)}, r) + \xi'^{(t)}, \xi'^{(t)} \sim N(0, \Sigma), t = 1, \cdots, T \tag{3}$$

Maximul path

$$\mathrm{Max}_{x^{(1:T)}} p(x^{(1:T)}, y^{(1:T)}|\mu) = p(x^{(1)}|\mu)p(y^{(1)}|x^{(1)}) \prod_{t=2}^{T} p(x^{(t)}|x^{(t-1)})p(y^{(t)}|x^{(t)}) \tag{4}$$

$$\begin{aligned}
&\mathrm{Min}_{x^{(1:T)}} l(x^{(1:T)}, y^{(1:T)}|\mu) && (5)\\
&= -\ln p(x^{(1:T)}, y^{(1:T)}|\mu) && (6)\\
&= \frac{1}{2}\ln|E_1| + \frac{T-1}{2}\ln|E| + \frac{T}{2}\ln|\Sigma| + \frac{1}{2}(x^{(1)} - \mu)^{\mathrm{T}} E_1^{-1}(x^{(1)} - \mu) \\
&+ \frac{1}{2}\sum_{t=2}^{T}(x^{(t)} - M(x^{(t-1)}, p))^{\mathrm{T}} E^{-1}(x^{(t)} - M(x^{(t-1)}, p)) \\
&+ \frac{1}{2}\sum_{t=1}^{T}(H(x^{(t)}, r) - y^{(t)})^{\mathrm{T}} \Sigma^{-1}(H(x^{(t)}, r) - y^{(t)}) && (7)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial l}{\partial x^{(1)}} &= E_1^{-1}(x^{(1)} - \mu) + \left(\frac{\partial H}{\partial x^{(1)}}\right)^{\mathrm{T}} \Sigma^{-1}(H(x^{(1)}, r) - y^{(1)}) = 0 \\
&+ \left(\frac{\partial M}{\partial x^{(1)}}\right)^{\mathrm{T}} E^{-1}(M(x^{(1)}, p) - x^{(2)}) = 0
\end{aligned} \tag{8}$$

$$\begin{aligned}
\frac{\partial l}{\partial x^{(t)}} &= E^{-1}(x^{(t)} - M(x^{(t-1)}, p)) + \left(\frac{\partial H}{\partial x^{(t)}}\right)^{\mathrm{T}} \Sigma^{-1}(H(x^{(t)}, r) - y^{(t)}) \\
&+ \left(\frac{\partial M}{\partial x^{(t)}}\right)^{\mathrm{T}} E^{-1}(M(x^{(t)}, p) - x^{(t+1)}) = 0 \ (t = 2, \cdots, T-1)
\end{aligned} \tag{9}$$

Cost function

$$J(x^{(1)},p,r)=\mathcal{J}(x^{(1:T)},p,r)=l(x^{(1:T)},y^{(1:T)}|\mu)+\text{Regularization} \tag{10}$$

$$\begin{aligned}
&\text{Min}_{x^{(1:T)},p,r}L(x^{(1:T)},p,r;\lambda^{(2:T)})=\mathcal{J}(x^{(1:T)},p,r)\\
&+\lambda^{(2)\mathrm{T}}\Big(E_1^{-1}(x^{(1)}-\mu)+\left(\frac{\partial H}{\partial x^{(1)}}\right)^{\mathrm{T}}\Sigma^{-1}(H(x^{(1)},r)-y^{(1)})\\
&\quad+\left(\frac{\partial M}{\partial x^{(1)}}\right)^{\mathrm{T}}E^{-1}(M(x^{(1)},p)-x^{(2)})\Big)\\
&+\sum_{t=2}^{T-1}\lambda^{(t+1)\mathrm{T}}\Big(E^{-1}(x^{(t)}-M(x^{(t-1)},p))+\left(\frac{\partial H}{\partial x^{(1)}}\right)^{\mathrm{T}}\Sigma^{-1}(H(x^{(1)},r)-y^{(1)})\\
&\quad+\left(\frac{\partial M}{\partial x^{(t)}}\right)^{\mathrm{T}}E^{-1}(M(x^{(t)},p)-x^{(t+1)})\Big)
\end{aligned} \tag{11}$$

Orbit calculation

$$x^{(2)}-M(x^{(1)},p)=\left(\frac{\partial M}{\partial x^{(1)}}\right)^{\mathrm{T}}E^{-1}\backslash\left\{E_1^{-1}(x^{(1)}-\mu)+\left(\frac{\partial H}{\partial x^{(1)}}\right)^{\mathrm{T}}\Sigma^{-1}(H(x^{(1)},r)-y^{(1)})\right\} \tag{12}$$

$$x^{(t+1)}-M(x^{(t)},p)=\left(\frac{\partial M}{\partial x^{(t)}}\right)^{\mathrm{T}}E^{-1}\backslash\left\{E^{-1}(x^{(t)}-M(x^{(t-1)},p))+\left(\frac{\partial H}{\partial x^{(t)}}\right)^{\mathrm{T}}\Sigma^{-1}(H(x^{(t)},r)-y^{(t)})\right\}$$
$$(t=2,\cdots,T-1) \tag{13}$$

Gradient calculation

$$\lambda^{(T+1)} = 0 \tag{14}$$

$$\lambda^{(T)} = E^{-1}\frac{\partial M}{\partial x^{(T-1)}} \backslash \frac{\partial \mathcal{J}(x^{(1:T)}, p, r)}{\partial x^{(T)}} \tag{15}$$

$$\lambda^{(t)} = E^{-1}\frac{\partial M}{\partial x^{(t-1)}} \backslash \Bigg[\left\{ E^{-1} + \left(\frac{\partial H}{\partial x^{(t)}}\right)^{\mathrm{T}} \Sigma^{-1}\frac{\partial H}{\partial x^{(t)}} + \left(\frac{\partial M}{\partial x^{(t)}}\right)^{\mathrm{T}} E^{-1}\frac{\partial M}{\partial x^{(t)}} \right\} \lambda^{(t+1)} - \left(\frac{\partial M}{\partial x^{(t)}}\right)^{\mathrm{T}} E^{-1}\lambda^{(t+2)}$$
$$+ \left(\frac{\partial^2 H}{\partial x^{(t)2}}\lambda^{(t+1)}\right)^{\mathrm{T}} \Sigma^{-1}(H(x^{(t)}, r) - y^{(t)}) - \left(\frac{\partial^2 M}{\partial x^{(t)2}}\lambda^{(t+1)}\right)^{\mathrm{T}} E^{-1}(x^{(t+1)} - M(x^{(t)}))\Bigg] \; (t = T-1, \cdots, 2) \tag{16}$$

$$\frac{\partial J(x^{(1)}, p, r)}{\partial x^{(1)}} = \left\{ E_1^{-1} + \left(\frac{\partial H}{\partial x^{(1)}}\right)^{\mathrm{T}} \Sigma^{-1}\frac{\partial H}{\partial x^{(1)}} + \left(\frac{\partial M}{\partial x^{(1)}}\right)^{\mathrm{T}} E^{-1}\frac{\partial M}{\partial x^{(1)}} \right\} \lambda^{(2)} - \left(\frac{\partial M}{\partial x^{(1)}}\right)^{\mathrm{T}} E^{-1}\lambda^{(3)}$$
$$+ \left(\frac{\partial^2 H}{\partial x^{(1)2}}\lambda^{(2)}\right)^{\mathrm{T}} \Sigma^{-1}(H(x^{(1)}, r) - y^{(1)}) - \left(\frac{\partial^2 M}{\partial x^{(1)2}}\lambda^{(2)}\right)^{\mathrm{T}} E^{-1}(x^{(2)} - M(x^{(1)})) \tag{17}$$

$$\frac{\partial J(x^{(1)}, p, r)}{\partial p} = \sum_{t=1}^{T-1} \Bigg\{ \left(\frac{\partial M^{(t)}}{\partial p}\right)^{\mathrm{T}} E^{-1}\left(\frac{\partial M}{\partial x^{(t)}}\lambda^{(t+1)} - \lambda^{(t+2)} + M(x^{(t)}, p) - x^{(t+1)}\right)$$
$$- \left(\frac{\partial^2 M}{\partial p \partial x^{(t)}}\lambda^{(t+1)}\right)^{\mathrm{T}} E^{-1}(x^{(t+1)} - M(x^{(t)}, p))\Bigg\} + \text{Regularizationgradient} \tag{18}$$

$$\frac{\partial J(x^{(1)}, p, r)}{\partial r} = \sum_{t=1}^{T} \Bigg\{ \left(\frac{\partial^2 H}{\partial r \partial x^{(t)}}\lambda^{(t+1)}\right)^{\mathrm{T}} \Sigma^{-1}(H(x^{(t)}, r) - y^{(t)})$$
$$+ \left(\frac{\partial H^{(t)}}{\partial r}\right)^{\mathrm{T}} \Sigma^{-1}\left(\frac{\partial H}{\partial x^{(t)}}\lambda^{(t+1)} + H(x^{(t)}, r) - y^{(t)}\right)\Bigg\} + \text{Regularizationgradient} \tag{19}$$

where

$$\frac{\partial \mathcal{J}(x^{(1:T)}, p, r)}{\partial x^{(T)}} = E^{-1}(x^{(T)} - M(x^{(T-1)}, p)) + \left(\frac{\partial H}{\partial x^{(T)}}\right)^{\mathrm{T}} \Sigma^{-1}(H(x^{(T)}, r) - y^{(T)}) \tag{20}$$

Neighboring orbit calculation

$$\begin{aligned}
&\delta x^{(2)} - \frac{\partial M}{\partial x^{(1)}}\delta x^{(1)} - \frac{\partial M^{(1)}}{\partial p}\delta p \\
= &\left(\frac{\partial M}{\partial x^{(1)}}\right)^{\mathrm{T}} E^{-1}\backslash\Big\{-\left(\frac{\partial^2 M}{\partial x^{(1)2}}\delta x^{(1)} + \frac{\partial^2 M}{\partial x^{(1)}\partial p}\delta p\right)^{\mathrm{T}} E^{-1}(x^{(2)} - M(x^{(1)}, p)) + E_1^{-1}\delta x^{(1)} \\
&+ \left(\frac{\partial^2 H}{\partial x^{(1)2}}\delta x^{(1)} + \frac{\partial^2 H}{\partial x^{(1)}\partial r}\delta r\right)^{\mathrm{T}} \Sigma^{-1}(H(x^{(1)}, r) - y^{(1)}) + \left(\frac{\partial H}{\partial x^{(1)}}\right)^{\mathrm{T}} \Sigma^{-1}\left(\frac{\partial H}{\partial x^{(1)}}\delta x^{(1)} + \frac{\partial H^{(1)}}{\partial r}\delta r\right)\Big\}
\end{aligned} \tag{21}$$

$$\begin{aligned}
&\delta x^{(t+1)} - \frac{\partial M}{\partial x^{(t)}}\delta x^{(t)} - \frac{\partial M^{(t)}}{\partial p}\delta p \\
= &\left(\frac{\partial M}{\partial x^{(t)}}\right)^{\mathrm{T}} E^{-1}\backslash\Big\{-\left(\frac{\partial^2 M}{\partial x^{(t)2}}\delta x^{(t)} + \frac{\partial^2 M}{\partial x^{(t)}\partial p}\delta p\right)^{\mathrm{T}} E^{-1}(x^{(t+1)} - M(x^{(t)}, p)) \\
&+E^{-1}\left(\delta x^{(t)} - \frac{\partial M}{\partial x^{(t-1)}}\delta x^{(t-1)} - \frac{\partial M^{(t-1)}}{\partial p}\delta p\right) \\
&+ \left(\frac{\partial^2 H}{\partial x^{(t)2}}\delta x^{(t)} + \frac{\partial^2 H}{\partial x^{(t)}\partial r}\delta r\right)^{\mathrm{T}} \Sigma^{-1}(H(x^{(t)}, r) - y^{(t)}) \\
&+ \left(\frac{\partial H}{\partial x^{(t)}}\right)^{\mathrm{T}} \Sigma^{-1}\left(\frac{\partial H}{\partial x^{(t)}}\delta x^{(t)} + \frac{\partial H^{(t)}}{\partial r}\delta r\right)\Big\} \\
&(t = 2, \cdots, T-1)
\end{aligned} \tag{22}$$

Hessian-vector product calculation

$$\delta\lambda^{(T+1)} = 0 \tag{23}$$

$$\begin{aligned}
\delta\lambda^{(T)} = E^{-1}\frac{\partial M}{\partial x^{(T-1)}}\backslash\Big\{ & -E^{-1}\left(\frac{\partial^2 M}{\partial x^{(T-1)2}}\delta x^{(T-1)} + \frac{\partial^2 M}{\partial x^{(T-1)}\partial p}\delta p\right)\lambda^{(T)} \\
& +E^{-1}\left(\delta x^{(T)} - \frac{\partial M}{\partial x^{(T-1)}}\delta x^{(T-1)} - \frac{\partial M^{(T-1)}}{\partial p}\delta p\right) \\
& +\left(\frac{\partial^2 H}{\partial x^{(T)2}}\delta x^{(T)} + \frac{\partial^2 H}{\partial x^{(T)}\partial r}\delta r\right)^{\mathrm{T}}\Sigma^{-1}(H(x^{(T)}, r) - y^{(T)}) \\
& +\left(\frac{\partial H}{\partial x^{(T)}}\right)^{\mathrm{T}}\Sigma^{-1}\left(\frac{\partial H}{\partial x^{(T)}}\delta x^{(T)} + \frac{\partial H^{(T)}}{\partial r}\delta r\right)\Big\}
\end{aligned} \tag{24}$$

$$\begin{aligned}
\delta\lambda^{(t)} = E^{-1}\frac{\partial M}{\partial x^{(t-1)}}\backslash\Big[& -E^{-1}\left(\frac{\partial^2 M}{\partial x^{(t-1)2}}\delta x^{(t-1)} + \frac{\partial^2 M}{\partial x^{(t-1)}\partial p}\delta p\right)\lambda^{(t)} \\
& \Big\{\left(\frac{\partial^2 H}{\partial x^{(t)2}}\delta x^{(t)} + \frac{\partial^2 H^{(t)}}{\partial x^{(t)}\partial r}\delta r\right)^{\mathrm{T}}\Sigma^{-1}\frac{\partial H}{\partial x^{(t)}} \\
& +\left(\frac{\partial H}{\partial x^{(t)}}\right)^{\mathrm{T}}\Sigma^{-1}\left(\frac{\partial^2 H}{\partial x^{(t)2}}\delta x^{(t)} + \frac{\partial^2 H^{(t)}}{\partial x^{(t)}\partial r}\delta r\right) \\
& +\left(\frac{\partial^2 M}{\partial x^{(t)2}}\delta x^{(t)} + \frac{\partial^2 M}{\partial x^{(t)}\partial p}\delta p\right)^{\mathrm{T}}E^{-1}\frac{\partial M}{\partial x^{(t)}} \\
& +\left(\frac{\partial M}{\partial x^{(t)}}\right)^{\mathrm{T}}E^{-1}\left(\frac{\partial^2 M}{\partial x^{(t)2}}\delta x^{(t)} + \frac{\partial^2 M^{(t)}}{\partial x^{(t)}\partial p}\delta p\right)\Big\}\lambda^{(t+1)} \\
& +\Big\{E^{-1} + \left(\frac{\partial H}{\partial x^{(t)}}\right)^{\mathrm{T}}\Sigma^{-1}\frac{\partial H}{\partial x^{(t)}} + \left(\frac{\partial M}{\partial x^{(t)}}\right)^{\mathrm{T}}E^{-1}\frac{\partial M}{\partial x^{(t)}}\Big\}\delta\lambda^{(t+1)} \\
& -\left(\frac{\partial^2 M}{\partial x^{(t)2}}\delta x^{(t)} + \frac{\partial^2 M^{(t)}}{\partial x^{(t)}\partial p}\delta p\right)^{\mathrm{T}}E^{-1}\lambda^{(t+2)} - \left(\frac{\partial M}{\partial x^{(t)}}\right)^{\mathrm{T}}E^{-1}\delta\lambda^{(t+2)} \\
& +\Big\{\left(\frac{\partial^3 H}{\partial x^{(t)3}}\delta x^{(t)} + \frac{\partial^3 H}{\partial x^{(t)2}\partial r}\delta r\right)\lambda^{(t+1)} + \frac{\partial^2 H}{\partial x^{(t)2}}\delta\lambda^{(t+1)}\Big\}^{\mathrm{T}}\Sigma^{-1}(H(x^{(t)}, r) - y^{(t)}) \\
& +\left(\frac{\partial^2 H}{\partial x^{(t)}}\lambda^{(t+1)}\right)^{\mathrm{T}}\Sigma^{-1}\left(\frac{\partial H}{\partial x^{(t)}}\delta x^{(t)} + \frac{\partial H^{(t)}}{\partial r}\delta r\right) \\
& -\Big\{\left(\frac{\partial^3 M}{\partial x^{(t)3}}\delta x^{(t)} + \frac{\partial^3 M}{\partial x^{(t)2}\partial p}\delta p\right)\lambda^{(t+1)} + \frac{\partial^2 M}{\partial x^{(t)2}}\delta\lambda^{(t+1)}\Big\}^{\mathrm{T}}E^{-1}(x^{(t+1)} - M(x^{(t)}, p)) \\
& -\left(\frac{\partial^2 M}{\partial x^{(t)}}\lambda^{(t+1)}\right)^{\mathrm{T}}E^{-1}\left(\delta x^{(t+1)} - \frac{\partial M}{\partial x^{(t)}}\delta x^{(t)} - \frac{\partial M^{(t)}}{\partial p}\delta p\right)\Big] \\
& (t = T-1, \cdots, 2)
\end{aligned} \tag{25}$$

$$
\begin{aligned}
&\delta\left(\frac{\partial J(x^{(1)},p,r)}{\partial x^{(1)}}\right)\\
&=\Bigg\{\left(\frac{\partial^2 H}{\partial x^{(1)2}}\delta x^{(1)}+\frac{\partial^2 H^{(1)}}{\partial x^{(1)}\partial r}\delta r\right)^{\mathrm{T}}\Sigma^{-1}\frac{\partial H}{\partial x^{(1)}}\\
&+\left(\frac{\partial H}{\partial x^{(1)}}\right)^{\mathrm{T}}\Sigma^{-1}\left(\frac{\partial^2 H}{\partial x^{(1)2}}\delta x^{(1)}+\frac{\partial^2 H^{(1)}}{\partial x^{(1)}\partial r}\delta r\right)\\
&+\left(\frac{\partial^2 M}{\partial x^{(1)2}}\delta x^{(1)}+\frac{\partial^2 M}{\partial x^{(1)}\partial p}\delta p\right)^{\mathrm{T}}E^{-1}\frac{\partial M}{\partial x^{(1)}}\\
&+\left(\frac{\partial M}{\partial x^{(1)}}\right)^{\mathrm{T}}E^{-1}\left(\frac{\partial^2 M}{\partial x^{(1)2}}\delta x^{(1)}+\frac{\partial^2 M^{(1)}}{\partial x^{(1)}\partial p}\delta p\right)\Bigg\}\lambda^{(2)}\\
&+\Bigg\{E^{-1}+\left(\frac{\partial H}{\partial x^{(1)}}\right)^{\mathrm{T}}\Sigma^{-1}\frac{\partial H}{\partial x^{(1)}}+\left(\frac{\partial M}{\partial x^{(1)}}\right)^{\mathrm{T}}E^{-1}\frac{\partial M}{\partial x^{(1)}}\Bigg\}\delta\lambda^{(2)}\\
&-\left(\frac{\partial^2 M}{\partial x^{(1)2}}\delta x^{(1)}+\frac{\partial^2 M^{(1)}}{\partial x^{(1)}\partial p}\delta p\right)^{\mathrm{T}}E^{-1}\lambda^{(3)}-\left(\frac{\partial M}{\partial x^{(1)}}\right)^{\mathrm{T}}E^{-1}\delta\lambda^{(3)}\\
&+\Bigg\{\left(\frac{\partial^3 H}{\partial x^{(1)3}}\delta x^{(1)}+\frac{\partial^3 H}{\partial x^{(1)2}\partial r}\delta r\right)\lambda^{(2)}+\frac{\partial^2 H}{\partial x^{(1)2}}\delta\lambda^{(2)}\Bigg\}^{\mathrm{T}}\Sigma^{-1}(H(x^{(1)},r)-y^{(1)})\\
&+\left(\frac{\partial^2 H}{\partial x^{(1)}}\lambda^{(2)}\right)^{\mathrm{T}}\Sigma^{-1}\left(\frac{\partial H}{\partial x^{(1)}}\delta x^{(1)}+\frac{\partial H^{(1)}}{\partial r}\delta r\right)\\
&-\Bigg\{\left(\frac{\partial^3 M}{\partial x^{(1)3}}\delta x^{(1)}+\frac{\partial^3 M}{\partial x^{(1)2}\partial p}\delta p\right)\lambda^{(2)}+\frac{\partial^2 M}{\partial x^{(1)2}}\delta\lambda^{(2)}\Bigg\}^{\mathrm{T}}E^{-1}(x^{(2)}-M(x^{(1)},p))\\
&-\left(\frac{\partial^2 M}{\partial x^{(1)}}\lambda^{(2)}\right)^{\mathrm{T}}E^{-1}\left(\delta x^{(2)}-\frac{\partial M}{\partial x^{(1)}}\delta x^{(1)}-\frac{\partial M^{(1)}}{\partial p}\delta p\right)
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
&\delta\left(\frac{\partial J(x^{(1)},p,r)}{\partial p}\right)\\
&=\sum_{t=1}^{T-1}\Bigg[\left(\frac{\partial^2 M}{\partial p\partial x^{(t)}}\delta x^{(t)}+\frac{\partial^2 M^{(t)}}{\partial p^2}\delta p\right)^{\mathrm{T}}E^{-1}\left(\frac{\partial M}{\partial x^{(t)}}\lambda^{(t+1)}-\lambda^{(t+2)}+M(x^{(t)},p)-x^{(t+1)}\right)\\
&+\left(\frac{\partial M^{(t)}}{\partial p}\right)^{\mathrm{T}}E^{-1}\Bigg\{\left(\frac{\partial^2 M}{\partial x^{(t)2}}\delta x^{(t)}+\frac{\partial^2 M^{(t)}}{\partial x^{(t)}\partial p}\delta p\right)\lambda^{(t+1)}\\
&+\frac{\partial M}{\partial x^{(t)}}\left(\delta\lambda^{(t+1)}+\delta x^{(t)}\right)+\frac{\partial M^{(t)}}{\partial p}\delta p-\delta\lambda^{(t+2)}-\delta x^{(t+1)}\Bigg\}\\
&-\Bigg\{\left(\frac{\partial^3 M}{\partial p\partial x^{(t)2}}\delta x^{(t)}+\frac{\partial^3 M}{\partial p\partial x^{(t)}\partial p}\delta p\right)\lambda^{(t+1)}+\frac{\partial^2 M}{\partial p\partial x^{(t)}}\delta\lambda^{(t+1)}\Bigg\}^{\mathrm{T}}E^{-1}(x^{(t+1)}-M(x^{(t)},p))\\
&-\left(\frac{\partial^2 M}{\partial p\partial x^{(t)}}\lambda^{(t+1)}\right)^{\mathrm{T}}E^{-1}\left(\delta x^{(t+1)}-\frac{\partial M}{\partial x^{(t)}}\delta x^{(t)}-\frac{\partial M^{(t)}}{\partial p}\delta p\right)\Bigg]+\delta\mathrm{Regularizationgradient}
\end{aligned}
\tag{27}
$$

$$
\begin{aligned}
&\delta\left(\frac{\partial J(x^{(1)},p,r)}{\partial r}\right)\\
&=\sum_{t=1}^{T-1}\Bigg[\left(\frac{\partial^2 H}{\partial r\partial x^{(t)}}\delta x^{(t)}+\frac{\partial^2 H^{(t)}}{\partial r^2}\delta r\right)^{\mathrm{T}}\Sigma^{-1}\left(\frac{\partial H}{\partial x^{(t)}}\lambda^{(t+1)}+H(x^{(t)},r)-y^{(t)}\right)\\
&\quad+\left(\frac{\partial H^{(t)}}{\partial r}\right)^{\mathrm{T}}\Sigma^{-1}\Bigg\{\left(\frac{\partial^2 H}{\partial x^{(t)2}}\delta x^{(t)}+\frac{\partial^2 H^{(t)}}{\partial x^{(t)}\partial r}\delta r\right)\lambda^{(t+1)}\\
&\quad+\frac{\partial H}{\partial x^{(t)}}\left(\delta\lambda^{(t+1)}+\delta x^{(t)}\right)+\frac{\partial H^{(t)}}{\partial r}\delta r\Bigg\}\\
&\quad+\Bigg\{\left(\frac{\partial^3 H}{\partial r\partial x^{(t)2}}\delta x^{(t)}+\frac{\partial^3 H}{\partial r\partial x^{(t)}\partial r}\delta r\right)\lambda^{(t+1)}+\frac{\partial^2 H}{\partial r\partial x^{(t)}}\delta\lambda^{(t+1)}\Bigg\}^{\mathrm{T}}\Sigma^{-1}(H(x^{(t)},r)-y^{(t)})\\
&\quad+\left(\frac{\partial^2 H}{\partial r\partial x^{(t)}}\lambda^{(t+1)}\right)^{\mathrm{T}}\Sigma^{-1}\left(\frac{\partial H}{\partial x^{(t)}}\delta x^{(t)}+\frac{\partial H^{(t)}}{\partial r}\delta r\right)\Bigg]+\delta\mathrm{Regularizationgradient}
\end{aligned}
\tag{28}
$$