Game Theory

Introduction

Game theory is the science that involves mathematical strategies to determine the winner at the end of a game. When we talk about game theory, we are generally referring to a **Combinatorial Game** (i.e. a game that has an outcome and doesn't get stuck in an infinite loop of moves or turns).

For example, in chess there may come a stage where the two players are left with no choice other than repeating their moves infinitely.

In combinatorial game theory there are two types of games:

1. **Impartial Games:** In an impartial game, the two players can play the same kind of moves.
2. **Partial Games:** In a partial game there is a restriction as to what moves the two players can play. For example, in the game of chess one player can only move pieces of a single color. That is, if player 1 is playing for the black side then he/she can't move the white pieces.

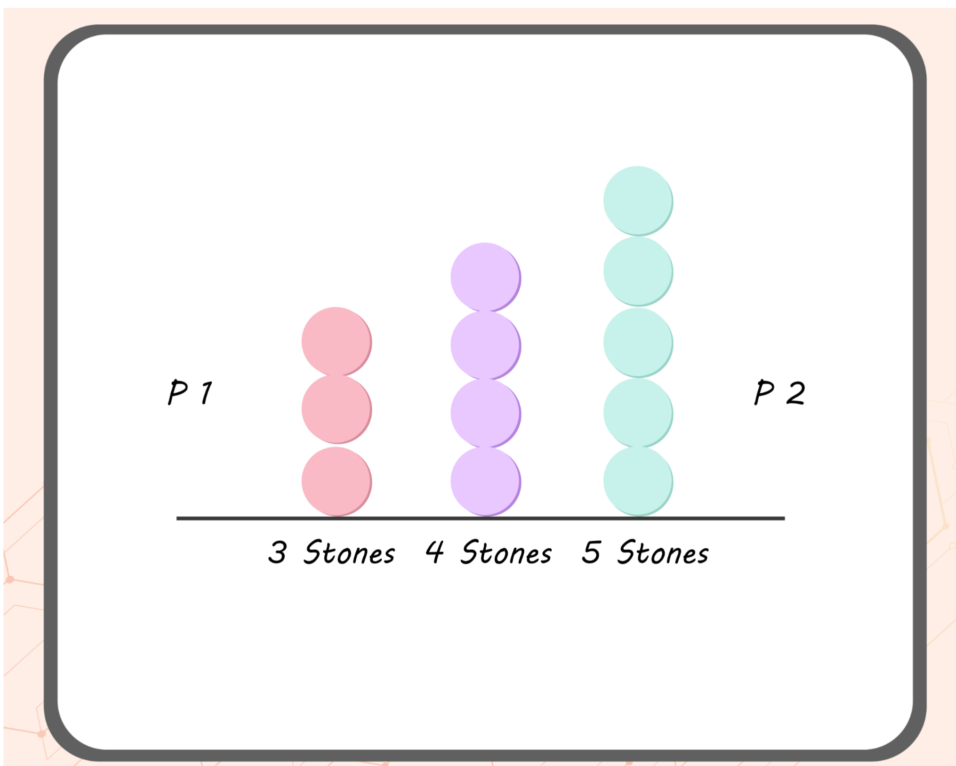
We have taken the example of chess above and seen how it is an impartial game and also that there may be a situation in the game where it is also not a combinatorial one.

Now, let's take a look at The game of Nims which is both a combinatorial game and also an impartial one.

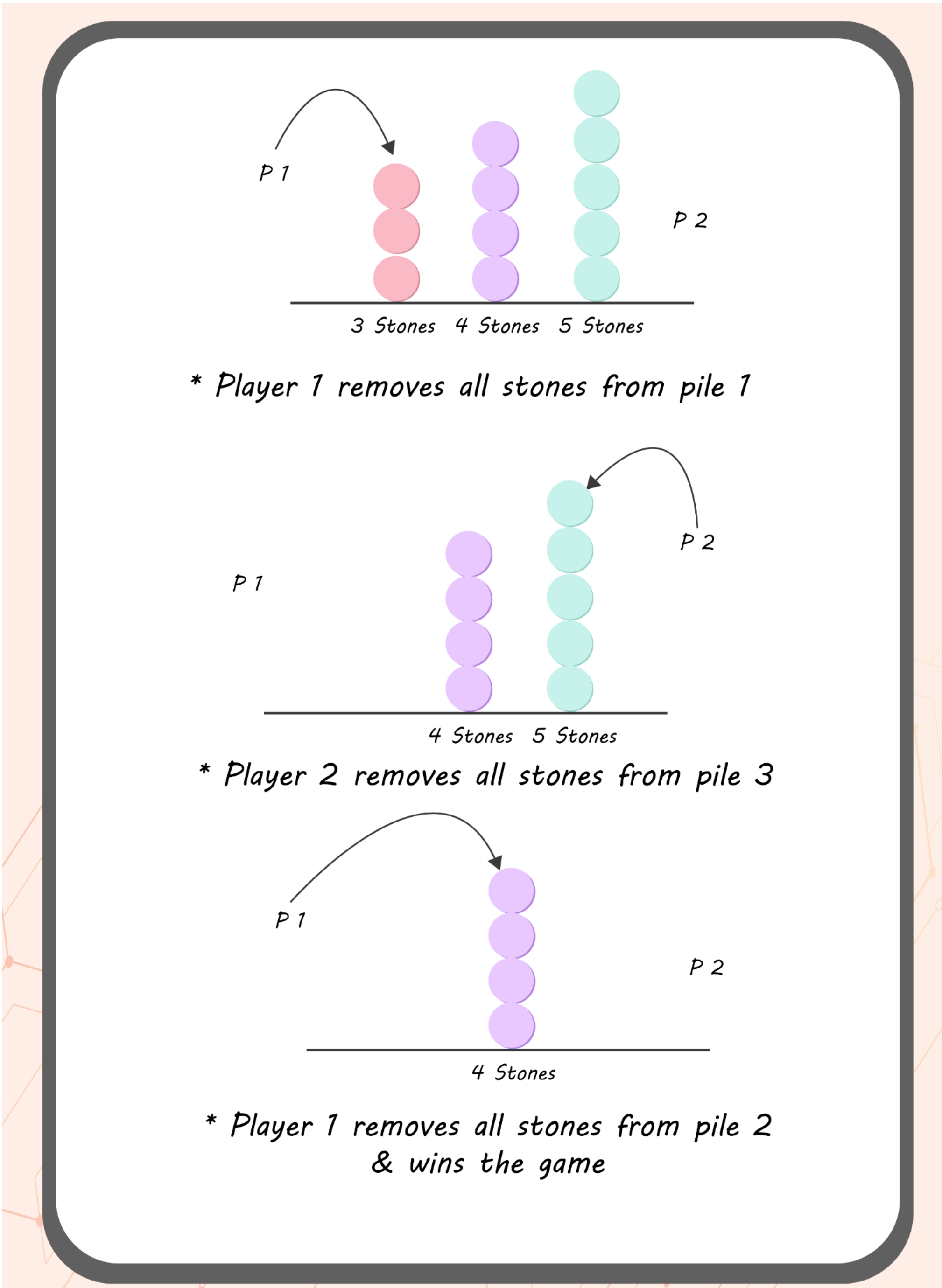
Game of Nims

In this game we are given numerous piles of stones and there are two players. The objective of the game is to remove a certain number of stones from one pile at a time in one move. The player may choose how many stones he/she wants to remove at their own convenience. The player who removes the last stone wins the game.

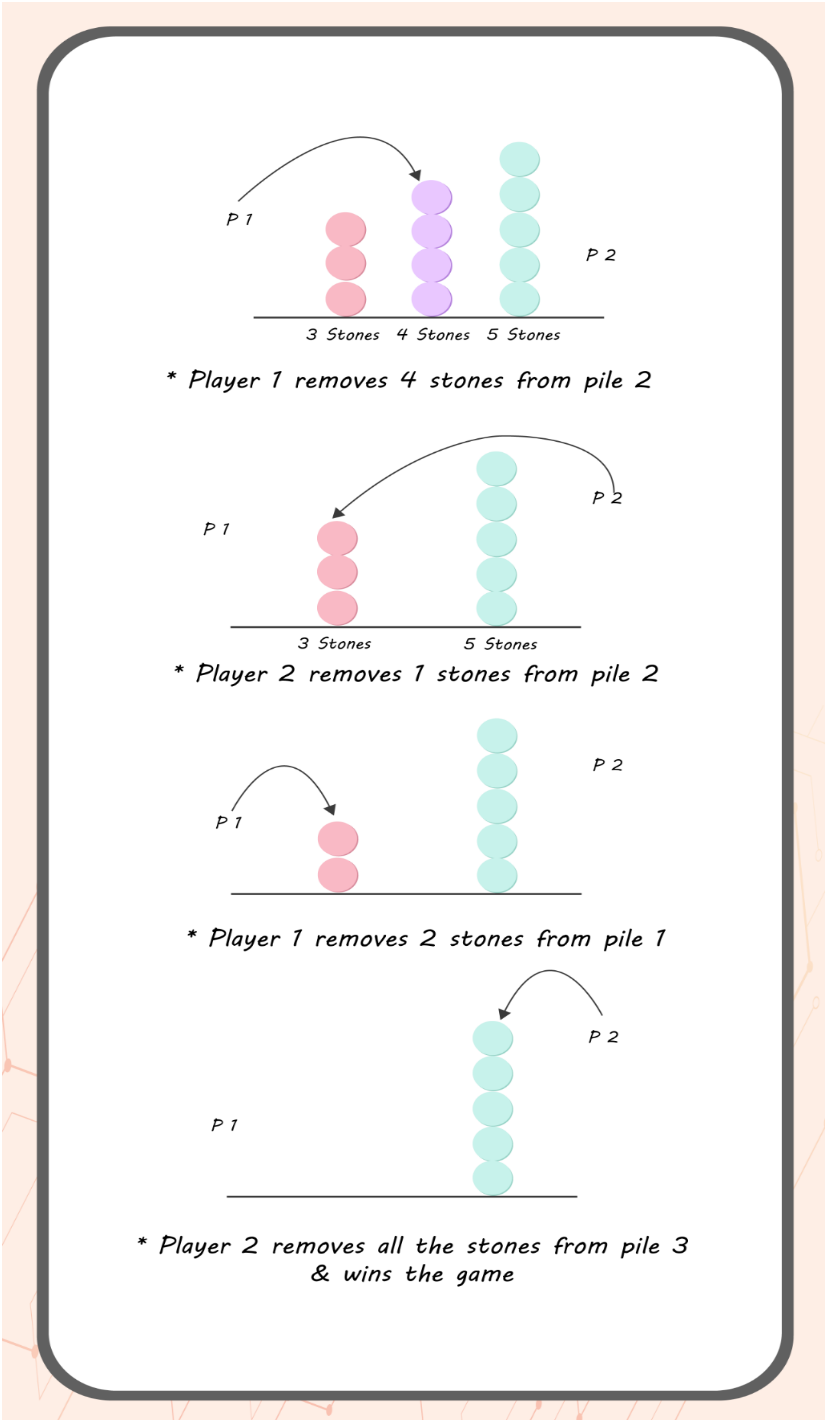
Let's take a look at the image given below for an example:



Now let's discuss one of the many possibilities in which **player 1 wins** the game.


P 1
P 2
3 Stones 4 Stones 5 Stones
* Player 1 removes all stones from pile 1
P 2
P 1
4 Stones 5 Stones
* Player 2 removes all stones from pile 3
P 1
P 2
4 Stones
* Player 1 removes all stones from pile 2
& wins the game

Now let's discuss one of the many possibilities in which **player 2 wins** the game.



Nim Formula:

Game theory says that we can predict the outcome of the game even before it is played.

We have to calculate a **Nim Sum** which is equal to the **cumulative XOR of the stones in each pile at the initial stage.** If the value of Nim Sum is **zero** and player 1 and player 2 are both playing optimally, then player 1 always loses. If the Nim Sum is **non zero** and player 1 and player 2 are both playing optimally then player 1 always wins.

In the above case Nim Sum = 3 ^ 4 ^ 5.

The proof of this formula is taught in the videos but the reader might not need to know the proof of concept to solve problems. So it is left for the reader to explore on their own.

mex

Before jumping on to what grundy numbers are, let's discuss a small mathematical concept called **mex.**

The mex or Minimum Excludent Set of a subset of a well ordered set is the smallest non-negative value from the whole set that does not belong to the subset.

For example, in a set of Non-Negative Integers, mex { 0, 1, 3 } = 2 because 2 is the smallest non-negative integer that is not present in the subset {0, 1, 3}.

Similarly, mex {1, 2, 3} = 0 because 0 is the smallest non-negative integer that is not present in the subset {1, 2, 3}.

Grundy Numbers

Let us consider a state **N** of a two player game and let **N-1** and **N-2** be the states reachable from N. We calculate the grundy value of N by

Grundy (N) = mex {Grundy(N-1), Grundy(N-2)}

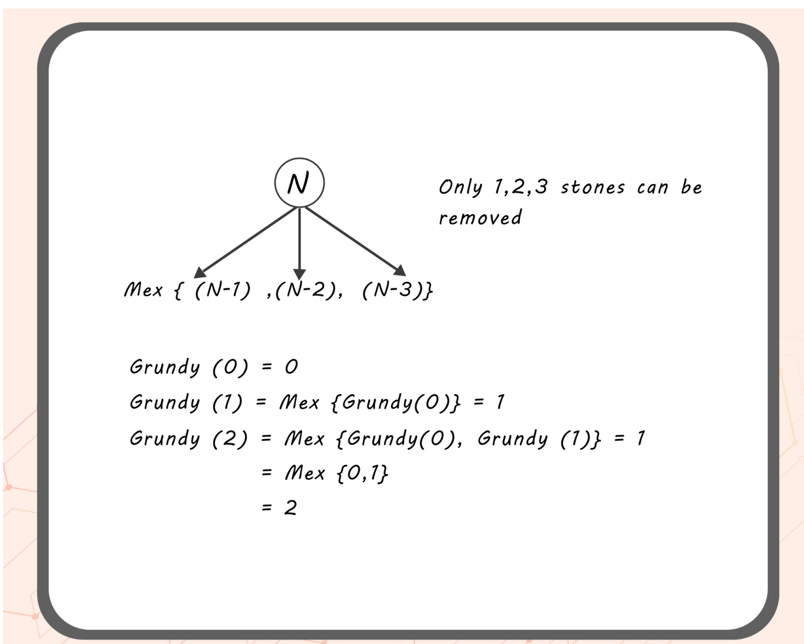
Also, Grundy (0) is always 0.

Let us take an example of the game of nims in which we have N number of stones initially and we can reach states from 0 to N-1.

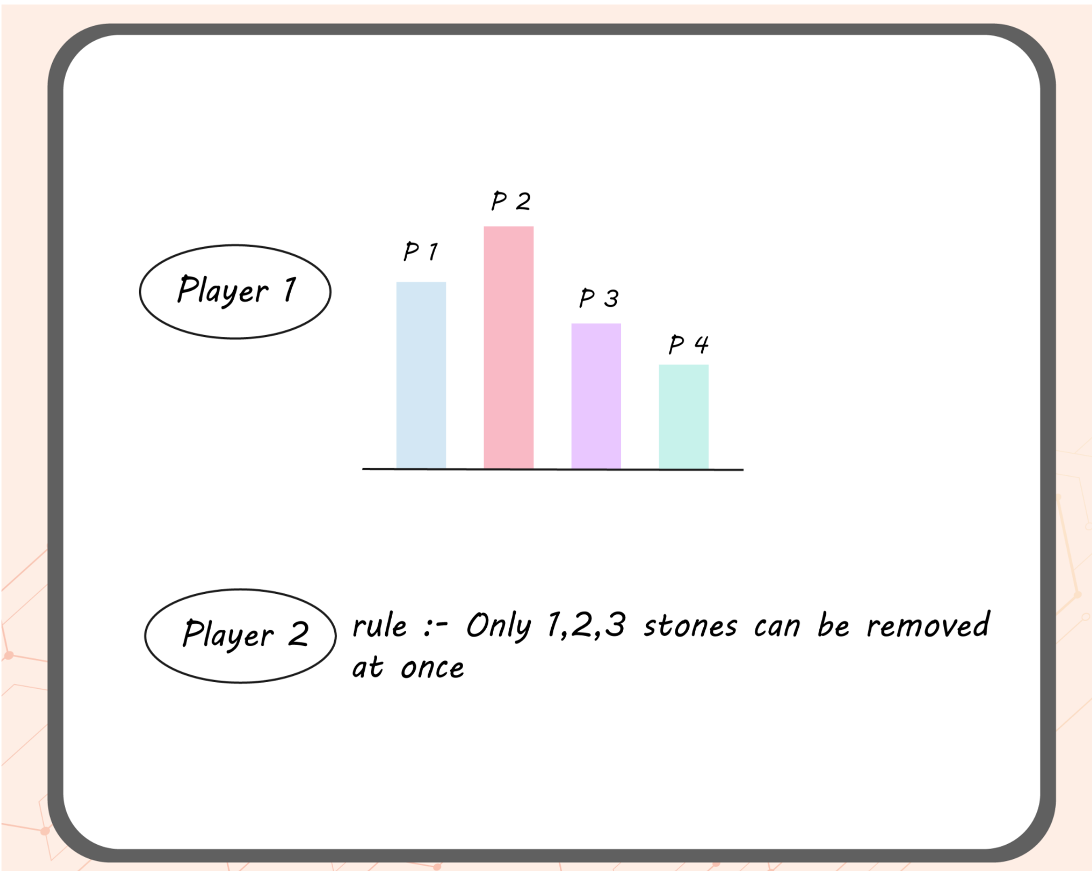
Then, **Grundy (N) = mex { Grundy (0), Grundy (1), Grundy (2) ...... Grundy (N-1) }**

Sprague Grundy Numbers

We have already studied the game of nims and how the value of Grundy(N) is calculated but what if the rules of the games are modified such that in one turn a player can only remove 1, 2 or 3 stones.



In such a game it would be very difficult to calculate the outcome of this game. So Sprague- Grundy Theorem states that instead of taking the cumulative XOR of the piles of stones initially present, we take the cumulative XOR of the grundy value of the piles.



If **Cumulative XOR (Grundy (p1), Grundy (p2), Grundy (p3), Grundy (p4))** is Non-Zero, then Player 1 always wins. Else, Player 2 always wins.

The proof of this concept is left for the reader to explore.

MiniMax Algorithm

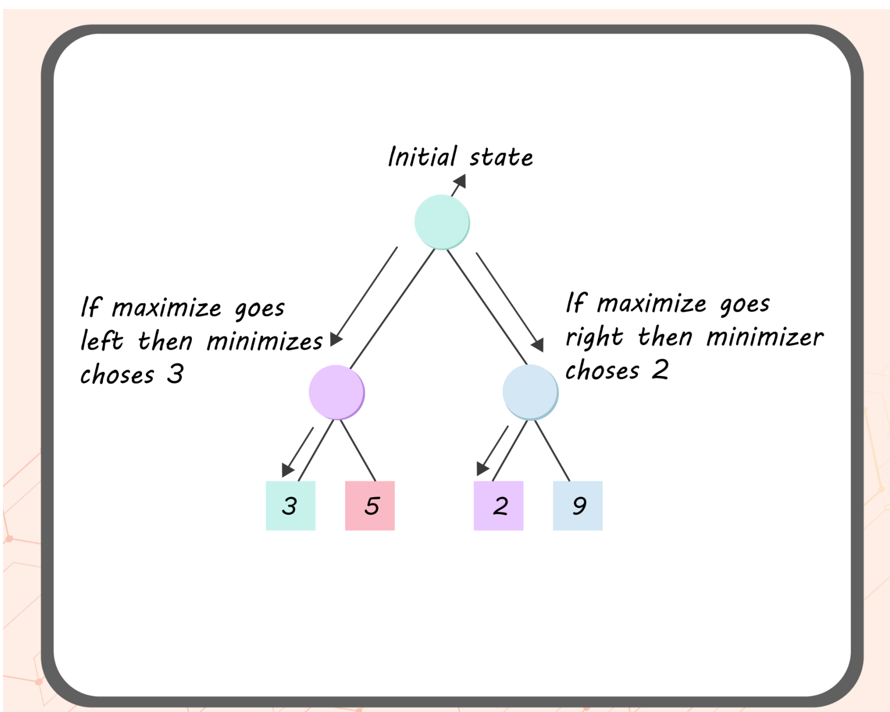
MiniMax is a kind of backtracking algorithm that is used in decision making to find the optimal move for a player, assuming that your opponent also plays optimally.

In MiniMax there are two players, Maximizer (who wants to maximize his score in the current state) and **Minimizer** (who wants to maximize his score in the current state).

Example:

Let's consider a game which has 4 final states and you are the maximizer and you have the first chance.

Since this is a backtracking algorithm, we have to explore all the paths and determine which would work the best.



Being the maximizer you would choose the larger value that is 3. Hence the optimal move for the maximizer is to go Left and the optimal value is 3.

Othello Evaluation Function

Rules of Othello Game:

- It is a two player game with a board size of 8X8 squares.
- At each round, the player has to play a disc on the board. One player plays white disc while the other plays black disc.
- The columns of the board are labeled A through H; the rows are labeled 1 through 8. A1 is the upper left square on the board; H8 is the lower right square.
- The initial configuration of the game is:
    - Two white discs in squares D4 and E5,
    - Two black discs in squares E4 and D5.
- If a player places a disc in a square (call it square X), and there is another square (call it square Y) on a row, column, or diagonal going through X, and all squares on the line between X and Y are filled with discs of the opponent's color, then the player is committing a legal move. After making a legal move, all discs between squares X and Y flip their color, i.e. the player making the legal move acquires the discs of the opponent. It is important to mention that a legal move can constitute disc flipping on more than one row, column, or diagonal.
- The game ends when all squares are filled, or when both players pass in two consecutive rounds.
- The winner is the player who has more discs on the board; the game is drawn when the disc count is even.
- It is a convention that all empty squares left on the board when the game ends are counted towards the winner.