CHAPTER FOUR

Theory of Probability

4·1. Introduction. If an experiment is repeated under essentially homogeneous and similar conditions we generally come across two types of situations:

(i) The result or what is usually known as the *'outcome'* is unique or certain.

(ii) The result is not unique but may be one of the several possible outcomes.

The phenomena covered by *(i)* are known as *'deterministic'* or *'predictable'* phenomena. By a deterministic phenomenon we mean one in which the result can be predicted with certainty. For example :

(a) For a perfect gas,

$$V \propto \frac{1}{P} \quad i.e., \ PV = constant,$$

provided the temperature remains the same.

(b) The velocity 'v' of a particle after time 't' is given by

$$v = u + at$$

where u is the initial velocity and a is the acceleration. This equation uniquely determines v if the right-hand quantities are known.

(c) Ohm's Law, *viz.*, $C = \dfrac{E}{R}$

where C is the flow of current, E the potential difference between the two ends of the conductor and R the resistance, uniquely determines the value C as soon as E and R are given.

A deterministic model is defined as a model which stipulates that the conditions under which an experiment is performed determine the outcome of the experiment. For a number of situations the deterministic model suffices. However, there are phenomena [as covered by *(ii)* above] which do not lend themselves to deterministic approach and are known as *'unpredictable'* or *'probabilistic'* phenomena. For example :

(i) In tossing of a coin one is not sure if a head or tail will be obtained.

(ii) If a light tube has lasted for t hours, nothing can be said about its further life. It may fail to function any moment.

In such cases we talk of chance or probability which is taken to be a quantitative measure of certainty.

4·2. Short History. Galileo (1564-1642), an Italian mathematician, was the first to attempt at a quantitative measure of probability while dealing with some problems related to the theory of dice in gambling. But the first foundation of the mathematical theory if probability was laid in the mid-seventeenth century by two French mathematicians, B. Pascal (1623-1662) and P. Fermat (1601-1665), while solving a number of problems posed by French gambler and noble man Chevalier-De-Mere to Pascal. The famous *'problem of points'* posed by De-Mere to Pascal is : "Two persons play a game of chance. The person who first gains a certain number of points wins the stake. They stop playing before the game is completed. How is the stake to be decided on the basis of the number of points each has won?" The two mathematicians after a lengthy correspondence between themselves ultimately solved this problem and this correspondence laid the first foundation of the science of probability. Next stalwart in this field was J. Bernoulli (1654-1705) whose 'Treatise on Probability' was published posthumously by his nephew N. Bernoulli in 1713. De-Moivre (1667-1754) also did considerable work in this field and published his famous 'Doctrine of Chances' in 1718. Other main contributors are : T. Bayes (Inverse probability), P.S. Laplace (1749-1827) who after extensive research over a number of years finally published 'Theoric analytique des probabilities' in 1812. In addition to these, other outstanding contributors are Levy, Mises and R.A. Fisher.

Russian mathematicians also have made very valuable contributions to the modern theory of probability. Chief contributors, to mention only a few of them are: Chebyshev (1821-94) who founded the Russian School of Statisticians; A. Markoff (1856-1922); Liapounoff (Central Limit Theorem); A. Khintchine (Law of Large Numbers) and A. Kolmogorov, who axiomised the calculus of probability.

4·3. Definitions of Various Terms. In this section we will define and explain the various terms which are used in the definition of probability.

Trial and Event. Consider an experiment which, though repeated under essentially identical conditions, does not give unique results but may result in any one of the several possible outcomes. The experiment is known as a *trial* and the outcomes are known as *events* or *cases*. For example :

(i) Throwing of a die is a trial and getting 1(or 2 or 3, ...or 6) is an event.

(ii) Tossing of a coin is a trial and getting head (H) or tail (T) is an event.

(iii) Drawing two cards from a pack of well-shuffled cards is a trial and getting a king and a queen are events.

Exhaustive Events. The total number of possible outcomes in any trial is known as exhaustive events or exhaustive cases. For example :

(i) In tossing of a coin there are two exhaustive cases, *viz.*, head and tail (the possibility of the coin standing on an edge being ignored).

(ii) In throwing of a die, there are six exhaustive cases since any one of the 6 faces 1, 2, ...,6 may come uppermost.

(iii) In drawing two cards from a pack of cards the exhaustive number of cases is ${}^{52}C_2$, since 2 cards can be drawn out of 52 cards in ${}^{52}C_2$ ways.

(iv) In throwing of two dice, the exhaustive number of cases is $6^2 = 36$, since any of the 6 numbers 1 to 6 on the first die can be associated with any of the six numbers on the other die.

In general in throwing of n dice the exhaustive number of cases is 6^n.

Favourable Events or Cases. The number of cases favourable to an event in a trial is the number of outcomes which entail the happening of the event. For example,

(i) In drawing a card from a pack of cards the number of cases favourable to drawing of an ace is 4, for drawing a spade is 13 and for drawing a red card is 26.

(ii) In throwing of two dice, the number of cases favourable to getting the sum 5 is : (1,4) (4,1) (2,3) (3,2), *i.e.*, 4.

Mutually exclusive events. Events are said to be *mutually exclusive* or *incompatible* if the happening of any one of them precludes the happening of all the others, *i.e.*, if no two or more of them can happen simultaneously in the same trial. For example :

(i) In throwing a die all the 6 faces numbered 1 to 6 are mutually exclusive since if any one of these faces comes, the possibility of others, in the same trial, is ruled out.

(ii) Similarly in tossing a coin the events head and tail are mutually exclusive.

Equally likely events. Outcomes of a trial are set to be equally likely if taking into consideration all the relevant evidences, there is no reason to expect one in preference to the others. For example

(i) In tossing an unbiased or uniform coin, head or tail are equally likely events.

(ii) In throwing an unbiased die, all the six faces are equally likely to come.

Independent events. Several events are said to be independent if the happening (or non-happening) of an event is not affected by the supplementary knowledge concerning the occurrence of any number of the remaining events. For example

(i) In tossing an unbiased coin the event of getting a head in the first toss is independent of getting a head in the second, third and subsequent throws.

(ii) If we draw a card from a pack of well-shuffled cards and replace it before drawing the second card, the result of the second draw is independent of the first draw. But, however, if the first card drawn is not replaced then the second draw is dependent on the first draw.

4·3·1. Mathematical or Classical or 'a priori' Probabality

Definition. If a trial results in n exhaustive, mutually exclusive and equally likely cases and m of them are favourable to the happening of an event E, then the probability 'p' of happening of E is given by

$$p = P(E) = \frac{\text{Favourable number of cases}}{\text{Exhaustive number of cases}} = \frac{m}{n} \qquad \text{....(4·1)}$$

Sometimes we express (4·1) by saying that 'the odds in favour of E are m : $(n - m)$ or the odds against E are $(n - m) : n$.'

Since the number of cases favourable to the 'non-happening' of the event E are $(n - m)$, the probability 'q' that E will not happen is given by

$$q = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - p \quad \Rightarrow \quad p + q = 1 \qquad \text{....(4·1}a)$$

Obviously p as well as q are non-negative and cannot exceed unity, *i.e.*, $0 \leq p \leq 1$, $0 \leq q \leq 1$.

Remarks. 1. Probability 'p' of the happening of an event is also known as the probability of success and the probability 'q' of the non- happening of the event as the probability of failure.

2. If $P(E) = 1$, E is called a *certain event* and if $P(E) = 0$, E is called an *impossible* event.

3. **Limitations of Classical Definition.** This definition of Classical Probability breaks down in the following cases :

(i) If the various outcomes of the trial are not equally likely or equally probable. For example, the probability that a candidate will pass in a certain test is not 50% since the two possible outcomes, *viz.*, sucess and failure (excluding the possibility of a compartment) are not equally likely.

(ii) If the exhaustive number of cases in a trial is infinite.

4·3·2. Statistical or Empirical Probability

***Definition* (Von Mises).** *If a trial is repeated a number of times under essentially homogeneous and identical conditions, then the limiting value of the ratio of the number of times the event happens to the number of trials, as the number of trials become indefinitely large, is called the probability of happening of the event. (It is assumed that the limit is finite and unique).*

Symbolically, if in n trials an event E happens m times, then the probability 'p' of the happening of E is given by

$$p = P(E) = \lim_{n \to \infty} \frac{m}{n} \qquad \text{....(4·2)}$$

Example 4·1. *What is the chance that a leap year selected at random will contain 53 Sundays?*

Solution. In a leap year (which consists of 366 days) there are 52 complete weeks and 2 days over. The following are the possible combinations for these two 'over' days:

(i) Sunday and Monday, *(ii)* Monday and Tuesday, *(iii)* Tuesday and Wednesday, *(iv)* Wednesday and Thursday, *(v)* Thursday and Friday, *(vi)* Friday and Saturday, and *(vii)* Saturday and Sunday.

In order that a leap year selected at random should contain 53 Sundays, one of the two 'over' days must be Sunday. Since out of the above 7 possibilities, 2 *viz.*, *(i)* and *(vii)*, are favourable to this event,

$$\therefore \qquad \text{Required probability} = \frac{2}{7}$$

Example 4·2. *A bag contains 3 red, 6 white and 7 blue balls. What is the probability that two balls drawn are white and blue?*

Solution. Total number of balls = 3 + 6 + 7 = 16.

Now, out of 16 balls, 2 can be drawn in ${}^{16}C_2$ ways.

$$\therefore \quad \text{Exhaustive number of cases} = {}^{16}C_2 = \frac{16 \times 15}{2} = 120.$$

Out of 6 white balls 1 ball can be drawn in ${}^{6}C_1$ ways and out of 7 blue balls 1 ball can be drawn in ${}^{7}C_1$ ways. Since each of the former cases can be associated with each of the latter cases, total number of favourable cases is : ${}^{6}C_1 \times {}^{7}C_1$ $= 6 \times 7 = 42$.

$$\therefore \quad \text{Required probability} = \frac{42}{120} = \frac{7}{20}.$$

Example 4·3. (a) *Two cards are drawn at random from a well-shuffled pack of 52 cards. Show that the chance of drawing two aces is 1/221.*

(b) From a pack of 52 cards, three are drawn at random. Find the chance that they are a king, a queen and a knave.

(c) Four cards are drawn from a pack of cards. Find the probability that

(i) all are diamond, (ii) there is one card of each suit, and (iii) there are two spades and two hearts.

Solution. *(a)* From a pack of 52 cards 2 cards can be drawn in ${}^{52}C_2$ ways, all being equally likely.

$$\therefore \quad \text{Exhaustive number of cases} = {}^{52}C_2$$

In a pack there are 4 aces and therefore 2 aces can be drawn in ${}^{4}C_2$ ways.

$$\therefore \quad \text{Required probability} = \frac{{}^{4}C_2}{{}^{52}C_2} = \frac{4 \times 3}{2} \times \frac{2}{52 \times 51} = \frac{1}{221}$$

(b) Exhaustive number of cases $= {}^{52}C_3$

A pack of cards contains 4 kings, 4 queens and 4 knaves. A king, a queen and a knave can each be drawn in ${}^{4}C_1$ ways and since each way of drawing a king can be associated with each of the ways of drawing a queen and a knave, the total number of favourable cases $= {}^{4}C_1 \times {}^{4}C_1 \times {}^{4}C_1$

$$\therefore \quad \text{Required probability} = \frac{{}^{4}C_1 \times {}^{4}C_1 \times {}^{4}C_1}{{}^{52}C_3} = \frac{4 \times 4 \times 4 \times 6}{52 \times 51 \times 50} = \frac{16}{5525}$$

(c) Exhaustive number of cases $= {}^{52}C_4$.

(i) $\quad \text{Required probability} = \dfrac{{}^{13}C_4}{{}^{52}C_4}$

(ii) $\quad \text{Required probability} = \dfrac{{}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1}{{}^{52}C_4}$

(iii) $\quad \text{Required probability} = \dfrac{{}^{13}C_2 \times {}^{13}C_2}{{}^{52}C_4}$

Example 4·4. *What is the probability of getting 9 cards of the same suit in one hand at a game of bridge?*

Solution. One hand in a game of bridge consists of 13 cards.

$\therefore$ Exhaustive number of cases $= {}^{52}C_{13}$

Number of ways in which, in one hand, a particular player gets 9 cards of one suit are ${}^{13}C_9$ and the number of ways in which the remaining 4 cards are of some other suit are ${}^{39}C_4$. Since there are 4 suits in a pack of cards, total number of favourable cases $= 4 \times {}^{13}C_9 \times {}^{39}C_4$.

$$\therefore \quad \text{Required probability} = \frac{4 \times {}^{13}C_9 \times {}^{39}C_4}{{}^{52}C_{13}}$$

Example 4·5. *(a) Among the digits 1, 2, 3, 4, 5, at first one is chosen and then a second selection is made among the remaining four digits. Assuming that all twenty possible outcomes have equal probabilities, find the probability that an odd digit will be selected (i) the first time, (ii) the second time, and (iii) both times.*

(b) From 25 tickets, marked with the first 25 numerals, one is drawn at random. Find the chance that

(i) it is a multiple of 5 or 7,

(ii) it is a multiple of 3 or 7.

Solution. *(a)* Total number of cases $= 5 \times 4 = 20$

(i) Now there are 12 cases in which the first digit drawn is odd, *viz.*, (1, 2), (1, 3), (1, 4), (1, 5), (3, 1), (3, 2), (3, 4), (3, 5), (5, 1), (5, 2), (5, 3) and (5, 4).

$\therefore$ The probability that the first digit drawn is odd

$$= \frac{12}{20} = \frac{3}{5}$$

(ii) Also there are 12 cases in which the second digit drawn is odd, *viz.*, (2, 1), (3, 1), (4, 1), (5, 1), (1, 3), (2, 3), (4, 3), (5, 3), (1, 5), (2, 5), (3, 5) and (4, 5).

$\therefore$ The probability that the second digit drawn is odd

$$= \frac{12}{20} = \frac{3}{5}$$

(iii) There are six cases in which both the digits drawn are odd, *viz.*, (1, 3), (1, 5), (3, 1), (3, 5), (5, 1) and (5, 3).

$\therefore$ The probability that both the digits drawn are odd

$$= \frac{6}{20} = \frac{3}{10}$$

(b) (i) Numbers (out of the first 25 numerals) which are multiples of 5 are 5, 10, 15, 20 and 25, *i.e.*, 5 in all and the numbers which are multiples of 7 are 7, 14 and 21, *i.e.*, 3 in all. Hence required number of favourable cases are 5+3=8.

$$\therefore \quad \text{Required probability} = \frac{8}{25}$$

(ii) Numbers (among the first 25 numerals) which are multiples of 3 are 3, 6, 9, 12, 15, 18, 21, 24, *i.e.*, 8 in all, and the numbers which are multiples of 7 are 7,

14, 21, *i.e.*, 3 in all. Since the number 21 is common in both the cases, the required number of distinct favourable cases is $8 + 3 - 1 = 10$.

$$\therefore \quad \text{Required probability} = \frac{10}{25} = \frac{2}{5}$$

Example 4·6. *A committee of 4 people is to be appointed from 3 officers of the production department, 4 officers of the purchase department, two officers of the sales department and 1 chartered accountant. Find the probability of forming the committee in the following manner:*

(i) There must be one from each category.

(ii) It should have at least one from the purchase department.

(iii) The chartered accountant must be in the committee.

Solution. There are 3+4+2+1=10 persons in all and a committee of 4 people can be formed out of them in ${}^{10}C_4$ ways. Hence exhaustive number of cases is

$${}^{10}C_4 = \frac{10 \times 9 \times 8 \times 7}{4\ !} = 210$$

(i) Favourable number of cases for the committee to consist of 4 members, one from each category is :

$${}^4C_1 \times {}^3C_1 \times {}^2C_1 \times 1 = 4 \times 3 \times 2 = 24$$

$$\therefore \quad \text{Required probability} = \frac{24}{210} = \frac{8}{70}$$

(ii) P [Committee has at least one purchase officer]

$= 1 - P$ [Committee has no purchase officer]

In order that the committee has no purchase officer, all the 4 members are to be selected from amongst officers of production department, sales department and chartered accountant, *i.e.*, out of 3+2+1=6 members and this can be done in ${}^6C_4 = \dfrac{6 \times 5}{1 \times 2} = 15$ ways. Hence

$$P\,[\text{ Committee has no purchase officer }] = \frac{15}{210} = \frac{1}{14}$$

$$\therefore \quad P\,[\text{ Committee has at least one purchase officer }] = 1 - \frac{1}{14} = \frac{13}{14}$$

(iii) Favourable number of cases that the committee consists of a chartered accountant as a member and three others are :

$$1 \times {}^9C_3 = \frac{9 \times 8 \times 7}{1 \times 2 \times 3} = 84 \text{ ways,}$$

since a chartered accountant can be selected out of one chartered accountant in only 1 way and the remaining 3 members can be selected out of the remaining $10 - 1 = 9$ persons in ${}^9C_3$ ways. Hence the required probability $= \frac{84}{210} = \frac{2}{5}$.

Example 4·7. *(a) If the letters of the word 'REGULATIONS' be arranged at random, what is the chance that there will be exactly 4 letters between R and E?*

(b) What is the probability that four S's come consecutively in the word 'MISSISSIPPI'?

Solution. *(a)* The word '*REGULATIONS*' consists of 11 letters. The two letters R and E can occupy ${}^{11}P_2$, *i.e.*, $11 \times 10 = 110$ positions.

The number of ways in which there will be exactly 4 letters between R and E are enumerated below:

(i) R is in the 1st place and E is in the 6th place.

(ii) R is in the 2nd place and E is in the 7th place.

... ...

... ...

... ...

(vi) R is in the 6th place and E is in the 11th place.

Since R and E can interchange their positions, the required number of favourable cases is $2 \times 6 = 12$

$\therefore$ The required probability $= \frac{12}{110} = \frac{6}{55}$.

(b) Total number of permutations of the 11 letters of the word '*MISSISSIPPI*', in which 4 are of one kind *(viz., S)*, 4 of other kind *(viz., I)*, 2 of third kind *(viz., P)* and 1 of fourth kind *(viz., M)* are

$$\frac{11\,!}{4\,!\;4\,!\;2\,!\;1\,!}$$

Following are the 8 possible combinations of 4 S's coming consecutively:

(i) S S S S
(ii) — S S S S
(iii) — — S S S S
⋮ ⋮ ⋮
(viii) — — — — — — — S S S S

Since in each of the above cases, the total number of arrangements of the remaining 7 letters, *viz.*, *MIIIPPI* of which 4 are of one kind, 2 of other kind and one of third kind are $\frac{7\,!}{4\,!\;2\,!\;1\,!}$, the required number of favourable cases

$$= \frac{8 \times 7\,!}{4\,!\;2\,!\;1\,!}$$

$$\therefore \quad \text{Required probability} = \frac{8 \times 7\,!}{4\,!\;2\,!\;1\,!} \div \frac{11\,!}{4\,!\;4\,!\;2\,!\;1\,!}$$

$$= \frac{8 \times 7\,! \times 4\,!}{11\,!} = \frac{4}{165}$$

Example 4·8. *Each coefficient in the equation $ax^2 + bx + c = 0$ is determined by throwing an ordinary die. Find the probability that the equation will have real roots.* **[Madras Univ. B. Sc. (Stat. Main), 1992]**

Solution. The roots of the equation $ax^2+bx+c = 0$ ...(*)
will be real if its discriminant is non-negative, *i.e.*, if

$$b^2-4ac \geq 0 \quad \Rightarrow \quad b^2 \geq 4ac$$

Since each co-efficient in equation (*) is determined by throwing an ordinary die, each of the co-efficients a, b and c can take the values from 1 to 6.

$\therefore$ Total number of possible outcomes (all being equally likely)

$$= 6\times6\times6 = 216$$

The number of favourable cases can be enumerated as follows:

ac		*a*	*c*	*4ac*	*b* (so that $b^2 \geq 4ac$)	*No. of cases*
1		1	1	4	2, 3, 4, 5	$1\times5=5$
2	(*i*)	1	2	8	3, 4, 5, 6	$2\times4=8$
	(*ii*)	2	1			
3	(*i*)	1	3	12	4, 5, 6	$2\times3=6$
	(*ii*)	3	1			
4	(*i*)	1	4	16	4, 5, 6	$3\times3=9$
	(*ii*)	4	1			
	(*iii*)	2	2			
5	(*i*)	1	5	20	5, 6	$2\times2=4$
	(*ii*)	5	1			
6	(*i*)	1	6	24	5, 6	$4\times2=8$
	(*ii*)	6	1			
	(*iii*)	3	2			
	(*iv*)	2	3			
7	(*ac* = 7 *is not possible*)					
8	(*i*)	2	4	32	6	$2\times1=2$
	(*ii*)	4	2			
9		3	3	36	6	1
						Total = 43

Since $b^2 \geq 4ac$ and since the maximum value of b^2 is 36, $ac = 10, 11, 12, \ldots$ etc. is not possible.

Hence total number of favourable cases = 43.

$\therefore$ Required probability $= \dfrac{43}{216}$

Example 4·9. *The sum of two non-negative quantities is equal to 2n. Find the chance that their product is not less than $\frac{3}{4}$ times their greatest product.*

Solution. Let $x > 0$ and $y > 0$ be the given quantities so that $x + y = 2n$.

We know that the product of two positive quantities whose sum is constant is greatest when the quantities are equal. Thus the product of x and y is maximum when $x = y = n$.

$\therefore$ Maximum product $= n \, . \, n = n^2$

Now $P\left[xy \not< \frac{3}{4} n^2\right] = P\left[xy \geq \frac{3}{4} n^2\right] = P\left[x\,(2n - x) \geq \frac{3}{4} n^2\right]$

$$= P\,[(4x^2 - 8nx + 3n^2) \leq 0]$$
$$= P\,[(2x - 3n)(2x - n) \leq 0]$$
$$= P\left[x \text{ lies between } \frac{n}{2} \text{ and } \frac{3n}{2}\right]$$

$\therefore$ Favourable range $= \frac{3n}{2} - \frac{n}{2} = n$

Total range $= 2n$

$\therefore$ Required probability $= \frac{n}{2n} = \frac{1}{2}$

Example 4·10. *Out of $(2n+1)$ tickets consecutively numbered three are drawn at random. Find the chance that the numbers on them are in A.P.*

[Calicut Univ. B.Sc., 1991; Delhi Univ. B.Sc.(Stat. Hons.), 1992]

Solution. Since out of $(2n+1)$ tickets, 3 tickets can be drawn in ${}^{2n+1}C_3$ ways,

$$\text{Exhaustive number of cases} = {}^{2n+1}C_3 = \frac{(2n+1)\,2n\,(2n-1)}{3\,!}$$
$$= \frac{n\,(4n^2 - 1)}{3}$$

To find the favourable number of cases we are to enumerate all the cases in which the numbers on the drawn tickets are in A.P with common difference, (say $d = 1, 2, 3, \ldots, n-1, n$).

If $d = 1$, the possible cases are as follows:

$$\left.\begin{array}{ccc} 1, & 2, & 3 \\ 2, & 3, & 4 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 2n-1, & n, & 2n+1 \end{array}\right\}, \text{ i.e., } (2n-1) \text{ cases in all}$$

If $d = 2$, the possible cases are as follows :

$$\left.\begin{array}{ccc} 1, & 3, & 5 \\ 2, & 4, & 6 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 2n-3, & 2n-1, & 2n+1 \end{array}\right\}, \text{ i.e., } (2n-3) \text{ cases in all}$$

and so on.

If $d = n-1$, the possible cases are as follows:

$$\left.\begin{array}{lll} 1, n, & 2n-1 \\ 2, n+1, & 2n \\ 3, n+2, & 2n+1 \end{array}\right\}, \text{ i.e., 3 cases in all}$$

If $d = n$, there is only one case, *viz.*, $(1, n+1, 2n+1)$.

Hence total number of favourable cases

$$= (2n-1) + (2n-3)) + ... + 5 + 3 + 1$$
$$= 1 + 3 + 5 + ... + (2n-1),$$

which is a series in A.P. with $d = 2$ and n terms.

$\therefore$ Number of favourable cases $= \frac{n}{2}[1 + (2n-1)] = n^2$

$\therefore$ Required probability $= \dfrac{n^2}{n\ (4n^2-1)/3} = \dfrac{3n}{(4n^2-1)}$

EXCERCISE 4 (a)

1. (*a*) Give the classical and statistical definitions of probability. What are the objections raised in these definitions?

[Delhi Univ. B.Sc. (Stat. Hons.), 1988, 1985]

(*b*) When are a number of cases said to be equally likely? Give an example each of the following :

(*i*) the equally likely cases,
(*ii*) four cases which are not equally likely, and
(*iii*) five cases in which one case is more likely than the other four.

(*c*) What is meant by mutually exclusive events? Give an example of

(*i*) three mutually exclusive events,
(*ii*) three events which are not mutually exclusive.

[Meerut Univ. B.Sc. (Stat.), 1987]

(*d*) Can

(*i*) events be mutually exclusive and exhaustive?
(*ii*) events be exhaustive and indepenent?
(*iii*) events be mutually exclusive and independent?
(*iv*) events be mutually exhaustive, exclusive and independent?

2. (*a*) Prove that the probability of obtaining a total of 9 in a single throw with two dice is one by nine.

(*b*) Prove that in a single throw with a pair of dice the probability of getting the sum of 7 is equal to 1/6 and the probability of getting the sum of 10 is equal to 1/12.

(*c*) Show that in a single throw with two dice, the chance of throwing more than seven is equal to that of throwing less than seven.

Ans. 5/12 **[Delhi Univ. B.Sc., 1987, 1985]**

(*d*) In a single throw with two dice, what is the number whose probability is minimum?

(e) Two persons A and B throw three dice (six faced). If A throws 14, find B's chance of throwing a higher number. **[Meerut Univ. B.Sc.(Stat.), 1987]**

3. *(a)* A bag contains 7 white, 6 red and 5 black balls. Two balls are drawn at random. Find the probability that they will both be white.

Ans. 21/153

(b) A bag contains 10 white, 6 red, 4 black and 7 blue balls. 5 balls are drawn at random. What is the probability that 2 of them are red and one black?

Ans. ${}^6C_2 \times {}^4C_1 / {}^{27}C_5$

4. *(a)* From a set of raffle tickets numbered 1 to 100, three are drawn at random. What is the probability that all the tickets are odd-numbered?

Ans. ${}^{50}C_3 / {}^{100}C_3$

(b) A number is chosen from each of the two sets :

$$(1, 2, 3, 4, 5, 6, 7, 8, 9); \quad (4, 5, 6, 7, 8, 9)$$

If p_1 is the probability that the sum of the two numbers be 10 and p_2 the probability that their sum be 8, find $p_1 + p_2$.

(c) Two different digits are chosen at random from the set 1,2,3,...,8. Show that the probability that the sum of the digits will be equal to 5 is the same as the probability that their sum will exceed 13, each being 1/14. Also show that the chance of both digits exceeding 5 is 3/28. **[Nagpur Univ. B.Sc., 1992]**

5. What is the chance that *(i)* a leap year selected at random will contain 53 Sundays? *(ii)* a non-leap year selected at random would contain 53 Sundays.

Ans. (i) 2/7, (ii) 1/7

6. *(a)* What is the probability of having a knave and a queen when two cards are drawn from a pack of 52 cards?

Ans. 8/663

(b) Seven cards are drawn at random from a pack of 52 cards. What is the probability that 4 will be red and 3 black?

Ans. ${}^{26}C_4 \times {}^{26}C_3 / {}^{52}C_7$

(c) A card is drawn from an ordinary pack and a gambler bets that it is a spade or an ace. What are the odds against his winning the bet?

Ans. 9:4

(d) Two cards are drawn from a pack of 52 cards. What is the chance that

(i) they belong to the same suit?

(ii) they belong to different suits and different denominations.

[Bombay Univ. B.Sc., 1986]

7. *(a)* If the letters of the word RANDOM be arranged at random, what is the chance that there are exactly two letters between A and O.

(b) Find the probability that in a random arrangement of the leters of the word 'UNIVERSITY', the two I's do not come together.

(c) In random arrangements of the letters of the word 'ENGINEERING', what is the probability that vowels always occur together?

[Kurushetra Univ. B.E., 1991]

(d) Letters are drawn one at a time from a box containing the letters A, H, M, O, S, T. What is the probability that the letters in the order drawn spell the word 'Thomas'?

8. A letter is taken out at random out of 'ASSISTANT' and a letter out of 'STATISTIC'. What is the chance that they are the same letters?

9. *(a)* Twelve persons amongst whom are x and y sit down at random at a round table. What is the probability that there are two persons between x and y?

(b) A and B stand in a line at random with 10 other people. What is the probability that there will be 3 persons between A and B?

10. *(a)* If from a lot of 30 tickets marked 1, 2, 3, ..., 30 four tickets are drawn, what is the chance that those marked 1 and 2 are among them?

Ans. 2/145

(b) A bag contains 50 tickets numbered 1, 2, 3, ..., 50 of which five are drawn at random and arranged in ascending order of the magnitude ($x_1 < x_2 < x_3 < x_4 < x_5$). What is the probability that $x_3 = 30$?

Hint. *(a)* Exhaustive number of cases = ${}^{30}C_4$

If, of the four tickets drawn, two tickets bear the numbers 1 and 2, the remaining 2 must have come out of 28 tickets numbered from 3 to 30 and this can be done in ${}^{28}C_2$ ways.

$\therefore$ Favourable number of cases = ${}^{28}C_2$

(b) Exhaustive number of cases = ${}^{50}C_5$

If $x_3 = 30$, then the two tickets with numbers x_1 and x_2 must have come out of 29 tickets numbered from 1 to 29 and this can be done in ${}^{29}C_2$ ways, and the other two tickets with numbers x_4 and x_5 must have been drawn out of 20 tickets numbered from 31 to 50 and this can be done in ${}^{20}C_2$ ways.

$\therefore$ No. of favourable cases = ${}^{29}C_2 \times {}^{20}C_2$.

11. Four persons are chosen ar random from a group containing 3 men, 2 women and 4 children. Show that the chance that exactly two of them will be children is 10/21. **[Delhi Univ. B.A.1988]**

Ans. $\dfrac{{}^4C_2 \times {}^5C_2}{{}^9C_4} = \dfrac{10}{21}$

12. From a group of 3 Indians, 4 Pakistanis and 5 Americans a sub-committee of four people is selected by lots. Find the probability that the sub-committee will consist of

(i) 2 Indians and 2 Pakistanis

(ii) 1 Indian, 1 Pakistani and 2 Americans

(iii) 4 Americans [Madras Univ. B.Sc.(Main Stat.), 1987]

Ans. *(i)* $\dfrac{{}^3C_2 \times {}^4C_2}{{}^{12}C_4}$, *(ii)* $\dfrac{{}^3C_1 \times {}^4C_1 \times {}^5C_2}{{}^{12}C_4}$, *(iii)* $\dfrac{{}^5C_4}{{}^{12}C_4}$

13. In a box there are 4 granite stones, 5 sand stones and 6 bricks of identical size and shape. Out of them 3 are chosen at random. Find the chance that :

(i) They all belong to different varieties.

(ii) They all belong to the same variety.

(iii) They are all granite stones. (Madras Univ. B.Sc., Oct. 1992)

14. If n people are seated at a round table, what is the chance that two named individuals will be next to each other?

Ans. 2/(n-1)

15. Four tickets marked 00, 01, 10 and 11 respectively are placed in a bag. A ticket is drawn at random five times, being replaced each time. Find the probability that the sum of the numbers on tickets thus drawn is 23.

[Delhi Univ. B.Sc.(Subs.), 1988]

16. From a group of 25 persons, what is the probability that all 25 will have different birthdays? Assume a 365 day year and that all days are equally likely.

[Delhi Univ. B.Sc.(Maths Hons.), 1987]

Hint. $(365 \times 364 \times \ldots \times 341) \div (365)^{25}$

4·4. Mathematical Tools : Preliminary Notions of Sets. The set theory was developed by the German mathematician, G. Cantor (1845–1918).

4·4·1. Sets and Elements of Sets. A set is a well defined collection or aggregate of all possible objects having given properties and specified according to a well defined rule. The objects comprising a set are called elements, members or points of the set. Sets are often denoted by capital letters, *viz.*, A, B, C, etc. If x is an element of the set A, we write symbolically $x \in A$ (x belongs to A). If x is not a member of the set A, we write $x \notin A$ (x does not belong to A). Sets are often described by describing the properties possessed by their members. Thus the set A of all non-negative rational numbers with square less than 2 will be written as $A = \{x : x \text{ rational}, x \geq 0, x^2 < 2\}$.

If every element of the set A belongs to the set B, *i.e.*, if $x \in A \Rightarrow x \in B$, then we say that A is a subset of B and write symbolically $A \subseteq B$ (A is contained in B) or $B \supseteq A$ (B contains A). Two sets A and B are said to be *equal* or *identical* if $A \subseteq B$ and $B \subseteq A$ and we write $A = B$ or $B = A$.

A *null* or an *empty* set is one which does not contain any element at all and is denoted by ϕ.

Remarks. 1. Every set is a subset of itself.

2. An empty set is subset of every set.

3. A set containing only one element is conceptually distinct from the element itself, but will be represented by the same symbol for the sake of convenience.

4. As will be the case in all our applications of set theory, especially to probability theory, we shall have a fixed set S (say) given in advance, and we shall be concerned only with subsets of this given set. The underlying set S may vary from one application to another, and it will be referred to as *universal set* of each particular discourse.

4·4·2. Operation on Sets

The union of two given sets A and B, denoted by $A \cup B$, is defined as a set consisting of all those points which belong to either A or B or both. Thus symbolically,

$$A \cup B = \{ x : x \in A \text{ or } x \in B \}.$$

Similarly

$$\bigcup_{i=1}^{n} A_i = \{ x : x \in A_i \text{ for at least one } i = 1, 2, \ldots, n \}$$

The *intersection* of two sets A and B, denoted by $A \cap B$, is defined as a set consisting of all those elements which belong to both A and B. Thus

$$A \cap B = \{ x : x \in A \text{ and } x \in B \}.$$

Similarly

$$\bigcap_{i=1}^{n} A_i = \{ x : x \in A_i \text{ for all } i = 1, 2, \ldots, n \}$$

For example, if $A = \{1, 2, 5, 8, 10\}$ and $B = \{2, 4, 8, 12\}$, then

$A \cup B = \{1, 2, 4, 5, 8, 10, 12\}$ and $A \cap B = \{2, 8\}$.

If A and B have no common point, *i.e.*, $A \cap B = \phi$, then the sets A and B are said to be *disjoint, mutually exclusive* or *non-overlapping*.

The *relative difference* of a set A from another set B, denoted by $A-B$ is defined as a set consisting of those elements of A which do not belong to B. Symbolically,

$$A - B = \{ x : x \in A \text{ and } x \notin B \}.$$

The *complement* or *negative* of any set A, denoted by $\bar{A}$ is a set containing all elements of the universal set S, (say), that are not elements of A, *i.e.*, $\bar{A} = S - A$.

4·4·3. Algebra of Sets

Now we state certain important properties concerning operations on sets. If A, B and C are the subsets of a universal set S, then the following laws hold:

Commutative Law : $A \cup B = B \cup A$, $A \cap B = B \cap A$

Associative Law : $(A \cup B) \cup C = A \cup (B \cup C)$

$(A \cap B) \cap C = A \cap (B \cap C)$

Distributive Law : $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Complementary Law : $A \cup \bar{A} = S$, $A \cap \bar{A} = \phi$

$A \cup S = S$, $(\because A \subseteq S)$, $A \cap S = A$

$A \cup \phi = A$, $A \cap \phi = \phi$

Difference Law : $A - B = A \cap \bar{B}$

$A - B = A - (A \cap B) = (A \cup B) - B$

$A - (B - C) = (A - B) \cup (A - C)$.

$$(A \cup B) - C = (A - C) \cup (B - C)$$
$$A - (B \cup C) = (A - B) \cap (A - C)$$
$$(A \cap B) \cup (A - B) = A\,,\ (A \cap B) \cap (A - B) = \phi$$

De-Morgan's Law—Dualization Law.

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B} \text{ and } \overline{(A \cap B)} = \overline{A} \cup \overline{B}$$

More generally

$$\overline{(\bigcup_{i=1}^{n} A_i)} = \bigcap_{i=1}^{n} \overline{A_i} \quad \text{and} \quad \overline{(\bigcap_{i=1}^{n} A_i)} = \bigcup_{i=1}^{n} \overline{A_i}$$

Involution Law : $\overline{(\overline{A})} = A$

Idempotency Law : $A \cup A = A\,,\quad A \cap A = A$

4·4·4. Limit of Sequence of Sets

Let $\{A_n\}$ be a sequence of sets in S. The *limit supremum* or *limit superior* of the sequence, usually written as lim sup A_n, is the set of all those elements which belong to A_n for infinitely many n. Thus

$$\limsup_{n\to\infty} A_n = \{\, x : x \in A_n \text{ for infinitely many } n \,\} \qquad \text{...(4·3)}$$

The set of all those elements which belong to A_n for all but a finite number of n is called *limit infinimum or limit inferior* of the sequence and is denoted by lim inf A_n. Thus

$$\liminf_{n\to\infty} A_n = \{\, x : x \in A_n \text{ for all but a finite number of } n \,\} \qquad \text{...(4·3 a)}$$

The sequence $\{A_n\}$ is said to have a limit if and only if lim sup A_n = lim inf A_n and this common value gives the limit of the sequence.

Theorem 4·1. *lim sup* $A_n = \bigcap_{m=1}^{\infty} (\bigcup_{n=m}^{\infty} A_n)$

and *lim inf* $A_n = \bigcup_{m=1}^{\infty} (\bigcap_{n=m}^{\infty} A_n)$

Def. $\{A_n\}$ is a monotone (infinite) sequence of sets if either

(i) $A_n \subset A_{n+1}\ \forall\ n$ or *(ii)* $A_n \supset A_{n+1}\ \forall\ n$.

In the former case the sequence $\{A_n\}$ is said to be *non-decreasing sequence* and is usually expressed as $A_n \uparrow$ and in the latter case it is said to be *non-increasing sequence* and is expressed as $A_n \downarrow$.

For a monotone sequence (non-increasing or non-decreasing), the limit always exists and we have,

$$\lim_{n\to\infty} A_n = \begin{cases} \bigcup_{n=1}^{\infty} A_n \text{ in case } (i),\ i.e.,\ A_n\uparrow \\ \bigcap_{n=1}^{\infty} A_n \text{ in case } (ii),\ i.e.,\ A_n\downarrow \end{cases}$$

4·4·5. Classes of Sets. A group of sets will be termed as a *class* (of sets). Below we shall define some useful types of classes.

A ring **R** is a *non-empty* class of sets which is closed under the formation of 'finite unions' and 'difference',

i.e., if $A \in \mathbf{R}$, $B \in \mathbf{R}$, then $A \cup B \in \mathbf{R}$ and $A - B \in \mathbf{R}$.

Obviously ϕ is a member of every ring.

A *field* **F** (or an *algebra*) is a non-empty class of sets which is closed under the formation of finite unions and under complementation. Thus

(i) $A \in \mathbf{F}, B \in \mathbf{F} \Rightarrow A \cup B \in \mathbf{F}$ and

(ii) $A \in \mathbf{F} \Rightarrow \bar{A} \in \mathbf{F}$.

A *σ-ring* **C** is a non-empty class of sets which is closed under the formation of 'countable unions' and 'difference'. Thus

(i) $A_i \in \mathbf{C}, i = 1, 2, \ldots \quad \Rightarrow \quad \bigcup_{i=1}^{\infty} A_i \in \mathbf{C}$

(ii) $A \in \mathbf{C}, B \in \mathbf{C} \quad \Rightarrow \quad A - B \in \mathbf{C}$.

More precisely σ-ring is a ring which is closed under the formation of countable unions.

A σ field (or σ-algebra) **B** is a non-empty class of sets that is closed under the formation of 'countable unions' and complementations,

i.e.,

(i) $A_i \in \mathbf{B}, i = 1, 2, \ldots \quad \Rightarrow \quad \bigcup_{i=1}^{\infty} A_i \in \mathbf{B}$.

(ii) $A \in \mathbf{B} \quad \Rightarrow \quad \bar{A} \in \mathbf{B}$.

σ-field may also be defined as a field which is closed under the formation of countable unions.

4·5. Axiomatic Approach to Probability. The axiomatic approach to probability, which closely relates the theory of probability with the modern metric theory of functions and also set theory, was proposed by A.N. Kolmogorov, a Russian mathematician, in 1933. The axiomatic definition of probability includes 'both' the classical and the statistical definitions as particular cases and overcomes the deficiencies of each of them. On this basis, it is possible to construct a logically perfect structure of the modern theory of probability and at the same time to satisfy the enchanced requirements of modern natural science. The axiomatic development of mathematical theory of probability relies entirely upon the logic of deduction.

The diverse theorems of probability, as were available prior to 1933, were finally brought together into a unified axiomised system in 1933. It is important to remark that probability theory, in common with all axiomatic mathematical systems, is concerned solely with relations among undefined things.

The axioms thus provide a set of rules which define relationships between abstract entities. These rules can be used to deduce theorems, and the theorems can be brought together to deduce more complex theorems. These theorems have no empirical meaning although they can be given an interpretation in terms of empirical phenomenon. The important point, however, is that the mathematical development of probability theory is, in no way, conditional upon the interpretation given to the theory.

More precisely, under axiomatic approach, the probability can be deduced from mathematical concepts. To start with some concepts are laid down. Then some statements are made in respect of the properties possessed by these concepts. These properties, often termed as "*axioms*" of the theory, are used to frame some theorems. These theorems are framed without any reference to the real world and are deductions from the axioms of the theory.

4·5·1. Random Experiment, Sample Space. The theory of probability provides *mathematical models* for "real-world phenomenon" involving games of chance such as the tossing of coins and dice. We feel intuitively that statements such as

(i) "The probability of getting a "head" in one toss of an unbiased coin is 1/2"

(ii) "The probability of getting a "four" in a single toss of an unbiased die is 1/6",

should hold. We also feel that the probability of obtaining *either* a "5" or a "6" in a single throw of a die, should be the sum of the probabilities of a "5" and a "6", *viz.*, 1/6+1/6=1/3. That is, probabilities should have some kind of *additive* property. Finally, we feel that in a large number of repetitions of, for example, a coin tossing experiment, the proportion of heads should be approximately 1/2. That is, the probability should have a *frequency interpretation.*

To deal with these properties sensibly, we need a *mathematical description* or *model* for the probabilistic situation we have. Any such probabilistic situation is referred to as a *random experiment*, denoted by **E**. **E** may be a coin or die throwing experiment, drawing of cards from a well-shuffled pack of cards, an agricultural experiment to determine the effects of fertilizers on yield of a commodity, and so on.

Each performance in a random experiment is called a *trial*. That is, all the trials conducted under the same set of conditions form a random experiment. The result of a trial in a random experiment is called an outcome, an elementary event or a sample point. The totality of all possible outcomes (*i.e.*, sample points) of a random experiment constitutes the *sample space*.

Suppose $e_1, e_2, ..., e_n$ are the possible outcomes of a random experiment **E** such that no two or more of them can occur simultaneously and exactly one of the outcomes $e_1, e_2, ..., e_n$ must occur. More specifically, with an experiment **E**, we associated a set $S = (e_1, e_2, ..., e_n)$ of possible outcomes with the following properties:

(i) each element of S denotes a possible outcome of the experiement,

(ii) any trial results in an outcome that corresponds to one and only one element of the set S.

The set S associated with an experiment E, real or conceptual, satisfying the above two properties is called the *sample space* of the experiment.

Remarks. 1. The sample space serves as universal set for all questions concerned with the experiment.

2. A sample space S is said to be finite (infinite) sample sapce if the number of elements in S is finite (infinite). For example, the sample space associated with the experiment of throwing the coin until a head appears, is infinite, with possible *sample points*

$$\{\omega_1, \omega_2, \omega_3, \omega_4, ...\}$$

where $\omega_1 = H$, $\omega_2 = TH$, $\omega_3 = TTH$, $\omega_4 = TTTH$, and so on, H denoting a head and T a tail.

3. A sample space is called discrete if it contains only finitely or infinitely many points which can be arranged into a simple sequence ω_1, ω_2, ..., while a sample space containing non-denumerable number of points is called a continuous sample space. In this book, we shall restrict ourselves to discrete sample spaces only.

4·5·2. Event. Every non-empty subset A of S, which is a disjoint union of single element subsets of the sample space S of a random experiment E is called an event. The notion of an event may also be defined as follows:

"Of all the possible outcomes in the sample space of an experiment,some outcomes satisfy a specified description,which we call an event."

Remarks. 1. As the empty set ϕ is a subset of S, ϕ is also an event, known as *impossible* event.

2. An event A, in particular, can be a single element subset of S, in which case it is known as *elementary* event.

4·5·3. Some Illustrations — Examples. We discuss below some examples to illustrate the concepts of sample space and event.

1. Consider tossing of a coin singly. The possible outcomes for this experiment are (writing H for a "head" and T for a "tail") : H and T. Thus the sample space S consists of two points $\{\omega_1, \omega_2\}$, corresponding to each possible outcome or elementary event listed.

i.e., $S = \{\omega_1, \omega_2\} = \{H, T\}$ and $n(S) = 2$,

where $n(S)$ is the total number of sample points in S.

If we consider two tosses of a coin, the possible outcomes are HH, HT, TH, TT. Thus, in this case the sample space S consists of four points $\{\omega_1, \omega_2, \omega_3, \omega_4\}$, corresponding to each possible outcome listed and $n(S) = 4$. Combinations of these outcomes form what we call events. For example, the event of getting at least one head is the set of the outcomes $\{HH, HT, TH\} = \{\omega_1, \omega_2, \omega_3\}$. Thus, mathematically, the events are subsets of S.

2. Let us consider a single toss of a die. Since there are six possible outcomes, our sample space S is now a space of six points $\{\omega_1, \omega_2, \ldots, \omega_6\}$ where ω_i corresponds to the appearance of number i. Thus $S = \{\omega_1, \omega_2, \ldots, \omega_6\} = \{1, 2, \ldots, 6\}$ and $n(S) = 6$. The event that the outcome is even is represented by the set of points $\{\omega_2, \omega_4, \omega_6\}$.

3. A coin and a die are tossed together. For this experiment, our sample space consists of twelve points $\{\omega_1, \omega_2, \ldots, \omega_{12}\}$ where ω_i $(i = 1, 2, \ldots, 6)$ represents a head on coin together with appearance of ith number on the die and ω_i $(i = 7, 8, \ldots, 12)$ represents a tail on coin together with the appearance of ith number on die. Thus

$$S = \{\omega_1, \omega_2, \ldots, \omega_{12}\} = \{(H, T) \times (1, 2, \ldots, 6)\} \text{ and } n(S) = 12$$

Remark. If the coin and die are unbiased, we can see intuitively that in each of the above examples, the outcomes (sample points) are equally likely to occur.

4. Consider an experiment in which two balls are drawn one by one from an urn containing 2 white and 4 blue balls such that when the second ball is drawn, the first is *not* replaced.

Let us number the six balls as 1, 2, 3, 4, 5 and 6, numbers 1 and 2 representing a white ball and numbers 3, 4, 5, and 6 representing a blue ball. Suppose in a draw we pick up balls numbered 2 and 6. Then (2,6) is called an outcome of the experiment. It should be noted that the outcome (2,6) is different from the outcome (6,2) because in the former case ball No. 2 is drawn first and ball No.6 is drawn next while in the latter case, 6th ball is drawn first and the second ball is drawn next.

The sample space consists of thirty points as listed below:

$\omega_1 = (1,2)$	$\omega_2 = (1,3)$	$\omega_3 = (1,4)$	$\omega_4 = (1,5)$	$\omega_5 = (1,6)$
$\omega_6 = (2,1)$	$\omega_7 = (2,3)$	$\omega_8 = (2,4)$	$\omega_9 = (2,5)$	$\omega_{10} = (2,6)$
$\omega_{11} = (3,1)$	$\omega_{12} = (3,2)$	$\omega_{13} = (3,4)$	$\omega_{14} = (3,5)$	$\omega_{15} = (3,6)$
$\omega_{16} = (4,1)$	$\omega_{17} = (4,2)$	$\omega_{18} = (4,3)$	$\omega_{19} = (4,5)$	$\omega_{20} = (4,6)$,
$\omega_{21} = (5,1)$	$\omega_{22} = (5,2)$	$\omega_{23} = (5,3)$	$\omega_{24} = (5,4)$	$\omega_{25} = (5,6)$
$\omega_{26} = (6,1)$	$\omega_{27} = (6,2)$	$\omega_{28} = (6,3)$	$\omega_{29} = (6,4)$	$\omega_{30} = (6,5)$

Thus

$$S = \{\omega_1, \omega_2, \omega_3, \ldots, \omega_{30}\} \text{ and } n(S) = 30$$

$$\Rightarrow \quad S = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\} - \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$$

The event

(i) the first ball drawn is white

(ii) the second ball drawn is white

(iii) both the balls drawn are white

(iv) both the balls drawn are black

are represented respectively by the following sets of points:

$\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8, \omega_9, \omega_{10}\}$,

$\{\omega_1, \omega_6, \omega_{11}, \omega_{12}, \omega_{16}, \omega_{17}, \omega_{21}, \omega_{22}, \omega_{26}, \omega_{27}\}$,

$\{\omega_1, \omega_6\}$, and

$\{\omega_{13}, \omega_{14}, \omega_{15}, \omega_{18}, \omega_{19}, \omega_{20}, \omega_{23}, \omega_{24}, \omega_{25}, \omega_{28}, \omega_{29}, \omega_{30}\}$.

5. Consider an experiment in which two dice are tossed. The sample space S for this experiment is given by

$$S = \{1, 2, 3, 4, 5, 6\} \times \{1, 2, 3, 4, 5, 6\}$$

and $\quad n(S) = 6 \times 6 = 36.$

Let E_1 be the event that 'the sum of the spots on the dice is greater than 12', E_2 be the event that 'the sum of spots on the dice is divisible by 3', and E_3 be the event that 'the sum is greater than or equal to two and is less than or equal to 12'. Then these events are represented by the following subsets of S :

$$E_1 = \{\phi\},\ E_3 = S \text{ and}$$

$$E_2 = \{(1, 2), (1, 5), (2, 1), (2, 4), (3, 3), (3, 6), (4, 2), (4, 5), (5, 1), (5, 4), (6, 3), (6, 6)\}$$

Thus $\quad n(E_1)=0,\ n(E_2)=12,$ and $n(E_3)=36$

Here E is an 'impossible event' and E_3 a 'certain event'.

6. Let E denote the experiment of tossing a coin three times in succession or tossing three coins at a time. Then the sample space S is given by

$$\begin{aligned} S &= \{H, T\} \times \{H, T\} \times \{H, T\} \\ &= \{H, T\} \times \{HH, HT, TH, TT\} \\ &= \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\} \\ &= \{\omega_1, \omega_2, \omega_3, ..., \omega_8\}, \text{ say.} \end{aligned}$$

If E_1 is the event that 'the number of heads exceeds the number of tails', E_2, the event of 'getting two heads' and E_3, the event of getting 'head in the first trial' then these are represented by the following sets of points :

$$E_1 = \{\omega_1, \omega_2, \omega_3, \omega_5\},$$

$$E_2 = \{\omega_2, \omega_3, \omega_5\}$$

and $\quad E_3 = \{\omega_1, \omega_2, \omega_3, \omega_4\}.$

7. In the foregoing examples the sample sapce is finite. To construct an experiment in which the sample sapce is countably infinite, we toss a coin repeatedly until head or tail appears twice in succession. The sample space of all the possible outcomes may be represented as :

$$S = \{HH, TT, THH, HTT, HTHH, THTT, THTHH, HTHTT, ...\}.$$

4·5·4. Algebra of Events. For events A, B, C

(i) $A \cup B = \{\omega \in S : \omega \in A \text{ or } \omega \in B\}$

(ii) $A \cap B = \{\omega \in S : \omega \in A \text{ and } \omega \in B\}$

(iii) $\bar{A}$ (*A complement*) $= \{\omega \in S : \omega \notin A\}$

(iv) $A - B = \{\omega \in S : \omega \in A \text{ but } \omega \notin B\}$

(v) Similar generalisations for $\bigcup_{i=1}^{n} A_i,\ \bigcap_{i=1}^{n} A_i,\ \bigcup_i A_i$ etc.

(vi) $A \subset B \Rightarrow$ for every $\omega \in A,\ \omega \in B$.

(vii) $B \supset A \Rightarrow A \subset B$.

(viii) $A = B$ if and only if A and B have the same elements, *i.e.*, if $A \subset B$ and $B \subset A$.

(ix) A and B disjoint (mutually exclusive) $\Rightarrow A \cap B = \phi$ (*null set*).

(x) $A \cup B$ can be denoted by $A + B$ if A and B are disjoint.

(xi) $A \Delta B$ denotes those ω belonging to exactly one of A and B, *i.e.*,

$$A \Delta B = A \overline{B} \cup \overline{A} B$$

Remark. Since the events are subsets of S, all the laws of set theory *viz.*, commutative laws, associative laws, distributive laws, De-Morgan's law, etc., hold for algebra of events.

Table – *Glossary of Probability Terms*

Statement	*Meaning in terms of set theory*
1. At least one of the events A or B occurs.	$\omega \in A \cup B$
2. Both the events A and B occur.	$\omega \in A \cap B$
3. Neither A nor B occurs	$\omega \in \overline{A} \cap \overline{B}$
4. Event A occurs and B does not occur	$\omega \in A \cap \overline{B}$
5. Exactly one of the events A or B occurs.	$\omega \in A \Delta B$
6. Not more than one of the events A or B occurs.	$\omega \in (A \cap \overline{B}) \cup (\overline{A} \cap B) \cup (\overline{A} \cap \overline{B})$
7. If event A occurs, so does B	$A \subset B$
8. Events A and B are mutually exclusive.	$A \cap B = \phi$
9. Complementary event of A.	$\overline{A}$
10. Sample space	universal set S

Example 4·11. *A, B and C are three orbitrary events. Find expressions for the events noted below, in the context of A, B and C.*

(i) *only A occurs,*
(ii) *Both A and B, but not C, occur,*
(iii) *All three events occur,*
(iv) *At least one occurs,*
(v) *At least two occur,*
(vi) *One and no more occurs,*
(vii) *Two and no more occur,*
(viii) *None occurs.*

Solution.

(i) $A \cap \overline{B} \cap \overline{C}$, (ii) $A \cap B \cap \overline{C}$, (iii) $A \cap B \cap C$,
(iv) $A \cup B \cup C$,

(v) $(A \cap B \cap \bar{C}) \cup (A \cap \bar{B} \cap C) \cup (\bar{A} \cap B \cap C) \cup (A \cap B \cap C)$

(vi) $(A \cap \bar{B} \cap \bar{C}) \cup (\bar{A} \cap B \cap \bar{C}) \cup (\bar{A} \cap \bar{B} \cap C)$

(vii) $(A \cap B \cap \bar{C}) \cup (\bar{A} \cap B \cap C) \cup (A \cap \bar{B} \cap C)$

(viii) $\bar{A} \cap \bar{B} \cap \bar{C}$ or $\overline{A \cup B \cup C}$

EXERCISE 4(b)

1. *(i)* If A, B and C are any three events, write down the theoretical expressions for the following events:

(a) Only A occurs, *(b)* A and B occur but C does not,

(c) A, B, and C all the three occur, *(d)* at least one occurs

(e) At least two occur, *(f)* one does not occur,

(g) Two do not occurs, and *(h)* None occurs.

(ii) A, B and C are three events. Express the following events in appropriate symbols:

(a) Simultaneous occurrence of A, B and C.

(b) C currence of at least one of them.

(c) A, B and C are mutually exclusive events.

(d) Every point of A is contained in B.

(e) The event B but not A occurs. **[Gauhati Univ. B.Sc., Oct.1990]**

2. A sample space S contains four points x_1, x_2, x_3 and x_4 and the values of a set function $P(A)$ are known for the following sets :

$$A_1 = (x_1, x_2) \text{ and } P(A_1) = \frac{4}{10}\,;\; A_2 = (x_3, x_4) \text{ and } P(A_2) = \frac{6}{10}\,;$$

$$A_3 = (x_1, x_2, x_3) \text{ and } P(A_3) = \frac{4}{10}\,;\; A_4 = (x_2, x_3, x_4) \text{ and } P(A_4) = \frac{7}{10}$$

Show that :

(i) the total number of sets (including the "null" set of number points) of points of x is 16.

(ii) Although the set containing no sample point has zero probability, the converse is not always true, *i.e.*, a set may have zero probability and yet it may be the set of a number of points.

3. Describe explicitly the sample spaces for each of the following experiments:

(i) The tossing of four coins.

(ii) The throwing of three dice.

(iii) The tossing of ten coins with the aim of observing the numbers of tails coming up.

(iv) Two cards are selected from a standard deck of cards.

(v) Four successive draws *(a)* with replacement, and *(b)* without replacement, from a bag containing fifty coloured balls out of which ten are white, twenty blue and twenty red.

(vi) A survey of families with two children is conduced and the sex of the children (the older child first) is recorded.

(vii) A survey of families with three children is made and the sex of the children (in order of age, oldest child first) are recorded.

(viii) Three distinguishable objects are distributed in three numbered cells.

(ix) A poker hand (five cards) is dealt from an ordinary deck of cards.

(x) Selecting r screws from the lot produced by a machine, a screw can be defective or non-defective.

4. In an experiment a coin is thrown five times. Write down the sample space. How many points are there in the sample space?

5. Describe sample space appropriate in each of the following cases :

(i) n-tosses of a coin with head or tails as outcome in each toss.

(ii) Successive tosses of a coin until a head turns up.

(iii) A survey of families with two children is conducted and the sex of the children (the older child first) is recorded.

(iv) Two successive draws, *(a)* with replacement *(b)* without replacement, from a bag containing 4 coloured toys out of which one is white, one black and 2 red toys. **[M.S.Baroda Univ. B.Sc., 1991]**

6. *(a)* An experiment consists of tossing an unbiased coin until the same result appears twice on succession for the first time. To every possible outcome requiring n tosses attribute probability $1/2^n$. Describe the sample space.

(b) A coin is tossed until there are either two consecutive heads or two consecutive tails or the number of tosses becomes five. Describe the sample space along with the probability associated with each sample point, if every sequence of n tosses has probabilty 2^{-n}. **[Civil Services (main), 1983]**

7. Urn 1 contains two white, one red and 3 black balls. Urn 2 contains one white, 3 red and 2 black balls. An experiment consists of first selecting an urn and then drawing a ball from this urn. Define a suitable sample space for this experiment.

8. Suppose an experiment has n outcomes $A_1, A_2, ..., A_n$ and that it is repeated r times. Let $x_1, x_2, ..., x_n$ record the number of occurrences of $A_1, A_2, ..., A_n$. Describe the sample space. Show that the number of sample points is

$$\binom{n+r-1}{r-1}$$

9. A manufacturer buys parts from four different vendors numbered 1, 2, 3 and 4. Referring to orders placed on two successive days, (1,4) denotes the event that on the first day, the order was given to vendor 1 and on the second day it was given to vendor 4. Letting A represent the event that vendor 1 gets at least one of these two orders, B the event that the same vendor gets both orders and C the event that vendors 1 and 3 do not get either order. List the elements of :

(a) entire sample space, *(b)* A, *(c)* B, *(d)* C, *(e)* $\bar{A}$, *(f)* $\bar{B}$, *(g)* $B \cup C$, *(h)* $A \cap B$, *(i)* $A \cap C$, *(j)* $\overline{A \cup B}$, and *(k)* $A - B$

[Hint. *(a)* The elements of entire sample space are

(1,1); (1, 2); (1, 3); (1, 4); (2, 1); (2, 2); (2, 3); (2, 4);
(3, 1); (3, 2); (3, 3); (3, 4); (4, 1); (4, 2); (4, 3); (4, 4).

(b) The elements of A are
(1, 1); (1, 2); (1, 3); (1, 4); (2, 1); (3, 1); (4, 1).

(c) The elements of B are (1, 1); (2, 2); (3, 3) and (4, 4).

(d) The elements of C are (2, 2); (2, 4); (4, 2); (4, 4).

(e) The element of $\overline{A}$ are :
(2, 2); (2, 3); (2, 4); (3, 2); (3, 3); (3, 4); (4, 2); (4, 3); (4, 4).

(f) The elements of $\overline{B}$ are : (1, 2); (1, 3); (1, 4); (2, 1); (2, 3); (2, 4); (3, 1); (3, 2); (3, 4); (4, 1); (4, 2); (4, 3).

(g) The elements of $B \cup C$ are (1, 1); (2, 2); (3, 3); (4, 4); (2, 4); (4, 2).

(h) The elements of $A \cap B$ are (1, 1).

(i) $A \cap C = \phi$

(j) Since $\overline{A \cup B} = \overline{A} \cap \overline{B}$. The elements of $\overline{A \cup B}$ are (2, 3); (2, 4); (3, 2); (3, 4); (4, 2); (4, 3).

(k) The elements of $A - B$ are (1, 2); (1, 3); (1, 4); (2, 1); (3, 1); (4, 1).

4·6. Probability — Mathematical Notion. We are now set to give the mathematical notion of the occurrence of a random phenomenon and the mathematical notion of probability. Suppose in a large number of trials the sample space S contains N sample points. The event A is defined by a description which is satisfied by N_A of the occurrences. The frequency interpretation of the probability $P(A)$ of the event A, tells us that $P(A)=N_A/N$.

A purely mathematical definition of probability cannot give us the actual value of $P(A)$ and this must be considered as a function defined on all events. With this in view, a mathematical definition of probability is enunciated as follows:

"Given a sample description space, probability is a function which assigns a non-negative real number to every event A, denoted by P(A) and is called the probability of the event A."

4·6·1. Probability Function. $P(A)$ is the probability function defined on a σ-field **B** of events if the following properties or axioms hold :

1. For each $A \in$ **B**, $P(A)$ is defined, is real and $P(A) \geq 0$
2. $P(S) = 1$
3. If $\{A_n\}$ is any finite or infinite sequence of disjoint events in **B**, then

$$P\left(\bigcup_{i=1}^{n} A_n\right) = \sum_{i=1}^{n} P(A_i) \qquad ...(4\cdot4)$$

The above three axioms are termed as the axiom of positiveness, certainty and union (additivity), respectively.

Remarks. 1. The set function **P** defined on σ-field **B**, taking its values in the real line and satisfying the above three axioms is called the probability measure.

2. The same definition of probability applies to *uncountable sample space* except that special restrictions must be placed on S and its subsets. It is important to realise that for a complete description of a probability measure, three things must

be specified, *viz.*, the sample space S, the σ-field (σ-algebra) **B** formed from certain subset of S and set function **P**. The triplet (**S**, **B**, **P**,) is often called the *probability space*. In most elementary applications, S is finite and the σ-algebra **B** is taken to be the collection of all subsets of S.

3. It is interesting to see that there are some formal statements of the properties of events derived from the frequency approach. Since $P(A)=N_A/N$, it is easy to see that $P(A) \geq 0$, as in Axiom 1. Next since $N_S = N$, $P(S)=1$, as in Axiom 2. In case of two mutually exclusive (or disjoint) events A and B defined by sample points N_A and N_B, the sample points belonging to $A \cup B$ are $N_A + N_B$. Therefore,

$$P(A \cup B) = \frac{N_A + N_B}{N} = \frac{N_A}{N} + \frac{N_B}{N} = P(A) + P(B), \text{ as in axiom 3.}$$

Extended Axiom of Addition. If an event A can materialise in the occurrence of any one of the pairwise disjoint events $A_1, A_2, \ldots$ so that

$$A = \bigcup_{i=1}^{\infty} A_i \, ; \; A_i \cap A_j = \phi \;\; (i \neq j)$$

then

$$P(A) = P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i) \qquad \ldots(1)$$

Axiom of Continuity. If $B_1, B_2, \ldots, B_n, \ldots$ be a countable sequences of events such that

$$(i) \; B_i \supset B_{i+1}, \; (i = 1, 2, 3, \ldots)$$

and

$$(ii) \; \bigcap_{n=1}^{\infty} B_n = \phi$$

i.e., if each succeeding event implies the preceeding event and if their simultaneous occurrence is an impossible event then

$$\lim_{n \to \infty} P(B_n) = 0 \qquad \ldots(2)$$

We shall now prove that these two axioms, *viz.*, the extended axiom of addition and axiom of continuity are equivalent, *i.e.*, each implies the other, *i.e.*, (1) ⇔ (2).

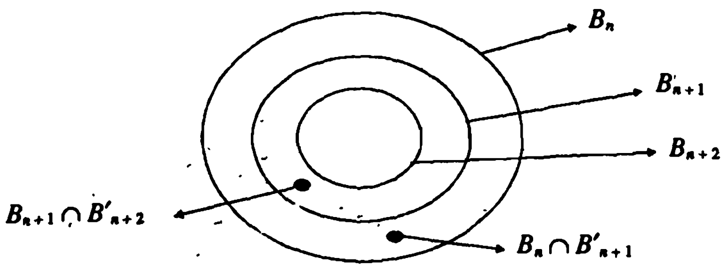
Theorem 4·1. *Axiom of continuity follows from the extended axiom of addition and vice versa.*

Proof. *(a)* (1) ⇒(2). Let $\{B_n\}$ be a countable sequence of events such that

$$B_1 \supset B_2 \supset B_3 \supset \ldots, \supset B_n \supset B_{n+1} \supset \ldots$$

and let for any $n \geq 1$,

$$\bigcap_{k \geq n} B_k = \phi \qquad (*)$$



Then it is obvious from the diagram that

$$B_n = B_n B'_{n+1} \cup B_{n+1} B'_{n+2} \cup \ldots \cup \left(\bigcap_{k \geq n} B_k \right)$$

$$\Rightarrow \qquad B_n = \left(\bigcup_{k=n}^{\infty} B_k B'_{k+1} \right) \cup \left(\bigcap_{k \geq n} B_k \right),$$

where the events $B_k B'_{k+1}$; $(k=n, n+1, \ldots)$ are pairwise disjoint and each is disjoint with $\bigcap_{k \geq n} B_k$.

Thus B_n has been expressed as the countable union of pairwise disjoint events and hence by the extended axiom of addition, we get

$$P(B_n) = \sum_{k=n}^{\infty} P(B_k B'_{k+1}) + P\left(\bigcap_{k \geq n} B_k \right)$$

$$= \sum_{k=n}^{\infty} P(B_k B'_{k+1}), \qquad (**)$$

since, from (*)

$$P\left(\bigcap_{k \geq n} B_k \right) = P(\phi) = 0$$

Further, from (**), since

$$\sum_{k=1}^{\infty} P(B_k B'_{k+1}) = P(B_1) \leq 1,$$

the right hand sum in (**), being the remainder after n terms of a convergent series tends to zero as $n \to \infty$.

Hence

$$\lim_{n \to \infty} P(B_n) = \lim_{n \to \infty} \sum_{k=n}^{\infty} P(B_k B'_{k+1}) = 0$$

Thus the extended axiom of addition implies the axiom of continuity.

(b) **Conversely (2) $\Rightarrow$ (1),** *i.e., the extended axiom of addition follows from the axiom of continuity.*

Let $\{A_n\}$ be a countable sequence of pairwise disjoint events and let

$$A = \bigcup_{i=1}^{\infty} A_i$$

$$= \left(\bigcup_{i=1}^{n} A_i\right) \cup \left(\bigcup_{i=n+1}^{\infty} A_i\right) \quad ...(3)$$

Let us define a countable sequence $\{B_n\}$ of events by

$$B_n = \bigcup_{i=n}^{\infty} A_i \quad ...(4)$$

Obviously B_n is a decreasing sequence of events, *i.e.*,

$$B_1 \supset B_2 \supset ... \supset B_n \supset B_{n+1} \supset .... \quad ...(5)$$

Also we have

$$A = \left(\bigcup_{i=1}^{n} A_i\right) \cup B_{n+1} \quad ...(6)$$

Since A_i 's are pairwise disjoint, we get

$$A_i \cap B_{n+1} = \phi, \ (i = 1, 2, ..., n) \quad ...(6a)$$

From (4) we see that if the event B_n has occurred it implies the occurrence of any one of the events $A_{n+1}, A_{n+2},...$ Without loss of generality let us assume that this event is A_i $(i = n+1, n+2,...)$. Further since A_i 's are pairwise disjoint, the occurrence of A_i implies that events $A_{i+1}, A_{i+2}, ...$ do not occur leading to the conclusion that $B_{i+1}, B_{i+2}, ...$ will not occur.

$$\Rightarrow \quad \bigcap_{i=n}^{\infty} B_i = \phi \quad ...(7)$$

From (5) and (7), we observe that both the conditions of axiom of continuity are satisfied and hence we get

$$\lim_{n \to \infty} P(B_n) = 0 \quad ...(8)$$

From (6), we get

$$P(A) = P\left[\left(\bigcup_{i=1}^{n} A_i\right) \cup B_{n+1}\right]$$

$$= \sum_{i=1}^{n} P(A_i) + P(B_{n+1})$$

(By axiom of Additivity)

$$\Rightarrow \quad P\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n \to \infty} \sum_{i=1}^{n} P(A_i) + \lim_{n \to \infty} (B_{n+1})$$

$$= \sum_{i=1}^{\infty} P(A_i), \qquad \text{[From (8)]}$$

which is the extended axiom of addition.

THEOREMS ON PROBABILITIES OF EVENTS

Theorem 4·2. *Probability of the impossible event is zero, i.e., $P(\phi) = 0$.*

Proof. Impossible event contains no sample point and hence the certain event S and the impossible event ϕ are mutually exclusive.

Hence $\qquad S \cup \phi = S$

$\therefore \qquad P(S \cup \phi) = P(S)$

$\Rightarrow \qquad P(S) + P(\phi) = P(S) \qquad$ [By Axiom 3]

$\Rightarrow \qquad P(\phi) = 0$

Remark. It may be noted $P(A)=0$, does not imply that A is necessarily an empty set. In practice, probability '0' is assigned to the events which are so rare that they happen only once in a lifetime. For example, if a person who does not know typing is asked to type the manuscript of a book, the probability of the event that he will type it correctly without any mistake is 0.

As another illustration, let us consider the random tossing of a coin. The event that the coin will stand erect on its edge, is assigned the probability 0.

The study of continuous random variable provides another illustration to the fact that $P(A)=0$, does not imply $A=\phi$, because in case of continous random variable X, the proability at a point is always zero, *i.e.*, $P(X=c)=0$ [See Chapter 5].

Theorem 4·3. Probability of the complementary event $\bar{A}$ of A is given by

$$P(\bar{A}) = 1 - P(A)$$

Proof. A and $\bar{A}$ are disjoint events.

Moreover, $\qquad A \cup \bar{A} = S$

From axioms 2 and 3 of probability, we have

$$P(A \cup \bar{A}) = P(A) + P(\bar{A}) = P(S) = 1$$

$\Rightarrow \qquad P(\bar{A}) = 1 - P(A)$

Cor. 1. We have $\qquad P(A) = 1 - P(\bar{A})$

$\Rightarrow \qquad P(A) \le 1 \qquad (\because P(\bar{A}) \ge 0)$

Cor. 2. $P(\phi) = 0$, since $\phi = \bar{S}$

and $\qquad P(\phi) = P(\bar{S}) = 1 - P(S) = 1 - 1 = 0.$

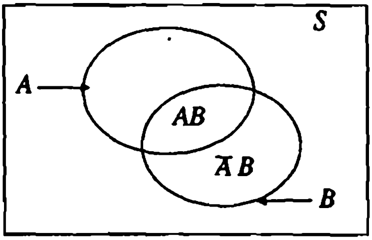
Theorem 4·4. *For any two events A and B,*

$$P(\bar{A} \cap B) = P(B) - P(A \cap B) \qquad \textbf{[Mysore Univ. B.Sc., 1992]}$$

Proof.

$\bar{A} \cap B$ and $A \cap B$ are disjoint events and

$(A \cap B) \cup (\bar{A} \cap B) = B$



Hence by axiom 3, we get

$$P(B) = P(A \cap B) + P(\bar{A} \cap B)$$
$$\Rightarrow P(\bar{A} \cap B) = P(B) - P(A \cap B)$$

Remark. Similarly, we shall get $P(A \cap \bar{B}) = P(A) - P(A \cap B)$

Theorem 4·5. *Probability of the union of any two events A and B is given by*

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof. $A \cup B$ can be written as the union of the two mutually disjoint events, A and $B \cap \bar{A}$.

$$\therefore \quad P(A \cup B) = P[A \cup (B \cap \bar{A})] = P(A) + P(B \cap \bar{A})$$
$$= P(A) + P(B) - P(A \cap B) \qquad (c.f. \text{ Theorem } 4{\cdot}4)$$

Theorem 4·6. *If $B \subset A$, then*

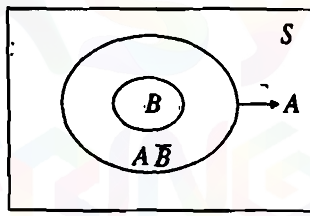
(i) $P(A \cap \bar{B}) = P(A) - P(B)$,

(ii) $P(B) \le P(A)$

Proof. *(i)* When $B \subset A$, B and $A \cap \bar{B}$ are mutually exclusive events and their union is A

Therefore

$$P(A) = P[B \cup (A \cap \bar{B})]$$
$$= P(B) + P(A \cap \bar{B}) \qquad \text{[By axiom 3]}$$
$$\Rightarrow \quad P(A \cap \bar{B}) = P(A) - P(B)$$



(ii) Using axiom 1,

$$P(A \cap \bar{B}) \ge 0 \quad \Rightarrow \quad P(A) - P(B) \ge 0$$

Hence $\quad P(B) \le P(A)$

Cor. Since $(A \cap B) \subset A$ and $(A \cap B) \subset B$,

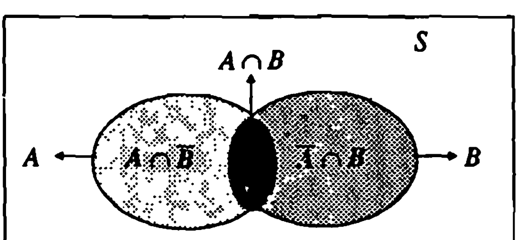
$$P(A \cap B) \le P(A) \quad \text{and} \quad P(A \cap B) \le P(B)$$

4·6·2. Law of Addition of Probabilities

Statement. *If A and B are any two events [subsets of sample space S] and are not disjoint, then*

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \qquad ...(4{\cdot}5)$$

Proof.



We have

$$A \cup B = A \cup (\bar{A} \cap B)$$

Since A and $(\bar{A} \cap B)$ are disjoint,

$$P(A \cup B) = P(A) + P(\bar{A} \cap B)$$

$$= P(A) + [P(\bar{A} \cap B) + P(A \cap B)] - P(A \cap B)$$

$$= P(A) + P[(\bar{A} \cap B) \cup (A \cap B)] - P(A \cap B)$$

[$\because$ $(\bar{A} \cap B)$ and $(A \cap B)$ *are disjoint*]

$$\Rightarrow \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Remark. An alternative proof is provided by Theorems 4·4 *and* 4·5.

4·6·3. Extention of General Law of Addition of Probabilities. *For n events* $A_1, A_2, \ldots, A_n$, *we have*

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum\sum_{1 \le i < j \le n} P(A_i \cap A_j) + \sum\sum\sum_{1 \le i < j < k \le n} P(A_i \cap A_j \cap A_k) - \ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap \ldots \cap A_n) \qquad \ldots(4\cdot6)$$

Proof. For two events A_1 and A_2, we have

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \qquad \ldots(*)$$

Hence (4·6) is true for $n = 2$.

Let us now suppose that (4·6) is true for $n = r$, (say). Then

$$P\left(\bigcup_{i=1}^{r} A_i\right) = \sum_{i=1}^{r} P(A_i) - \sum\sum_{1 \le i < j \le r} P(A_i \cap A_j) + \ldots + (-1)^{r-1} P(A_1 \cap A_2 \cap \ldots \cap A_r) \qquad \ldots(**)$$

Now

$$P\left(\bigcup_{i=1}^{r+1} A_i\right) = P\left[\left(\bigcup_{i=1}^{r} A_i\right) \cup A_{r+1}\right]$$

$$= P\left(\bigcup_{i=1}^{r} A_i\right) + P(A_{r+1}) - P\left[\left(\bigcup_{i=1}^{r} A_i\right) \cap A_{r+1}\right] \qquad \ldots[\text{Using } (*)]$$

$$= P\left(\bigcup_{i=1}^{r} A_i\right) + P(A_{r+1}) - P\left[\bigcup_{i=1}^{r} (A_i \cap A_{r+1})\right] \qquad \text{(Distributive Law)}$$

$$= \sum_{i=1}^{r} P(A_i) - \sum\sum_{1 \le i < j \le r} P(A_i \cap A_j) + \ldots$$

$$\ldots + (-1)^{r-1} P(A_1 \cap A_2 \cap \ldots \cap A_r) + P(A_{r+1})$$

$$- P\left[\bigcup_{i=1}^{r} (A_i \cap A_{r+1})\right] \qquad \ldots[\text{From } (**)]$$

$$= \sum_{i=1}^{r+1} P(A_i) - \sum\sum_{1 \le i < j \le r} P(A_i \cap A_j) + \ldots$$

$$+ (-1)^{r-1} P(A_1 \cap A_2 \cap \ldots \cap A_r)$$

$$- [\sum_{i=1}^{r} P(A_i \cap A_{r+1}) - \sum\sum_{1 \le i < j \le r} P(A_i \cap A_j \cap A_{r+1})$$
$$+ ... + (-1)^{r-1} P(A_1 \cap A_2 \cap ... \cap A_r \cap A_{r+1})] \quad \text{...[From (**)]}$$

$$\Rightarrow \quad P(\bigcup_{i=1}^{r+1} A_i) = \sum_{i=1}^{r+1} P(A_i) - [\sum\sum_{1 \le i < j \le r} P(A_i \cap A_j) + \sum_{i=1}^{r} P(A_i \cap A_{r+1})]$$
$$+ ... + (-1)^r P[(A_1 \cap A_2 \cap ... \cap A_{r+1})]$$
$$= \sum_{i=1}^{r+1} P(A_i) - \sum\sum_{1 \le i < j \le (r+1)} P(A_i \cap A_j)$$
$$+ ... + (-1)^r P(A_1 \cap A_2 \cap ... \cap A_{r+1})$$

Hence if (4·6) is true for $n=r$, it is also true for $n = (r+1)$. But we have proved in (*) that (4·6) is true for $n=2$. Hence by the principle of mathematical induction, it follows that (4·6) is true for all positive integral values of n.

Remarks. 1. If we write

$$P(A_i) = p_i, P(A_i \cap A_j) = p_{ij}, P(A_i \cap A_j \cap A_k) = p_{ijk}$$

and so on and

$$S_1 = \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} P(A_i)$$
$$S_2 = \sum\sum_{1 \le i < j \le n} p_{ij} = \sum\sum_{1 \le i < j \le n} P(A_i \cap A_j)$$
$$S_3 = \sum\sum\sum_{1 \le i < j < k \le n} p_{ijk} \quad \text{and so on,}$$

then

$$P(\bigcup_{i=1}^{n} A_i) = S_1 - S_2 + S_3 - ... + (-1)^{n-1} S_n \quad \text{...(4·6a)}$$

2. If all the events A_i, ($i = 1, 2, ..., n$) are mutually disjoint then (4·6) gives

$$P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$$

3. From practical point of view the theorem can be restated in a slightly different form. Let us suppose that an event A can materialise in several mutually exclusive forms, *viz.*, $A_1, A_2, ..., A_n$ which may be regarded as that many mutually exclusive events. If A happens then any one of the events A_i, ($i = 1, 2, ..., n$) must happen and conversely if any one of the events A_i, ($i = 1, 2, ..., n$) happens, then A happens. Hence the probability of happening of A is the same as the probability of happening of any one of its (unspecified) mutually exclusive forms. From this point of view, the total probability theorem can be restated as follows:

The probability of happening of an event A is the sum of the probabilities of happening of its mutually exclusive forms $A_1, A_2, ..., A_n$. Symbolically,

$$P(A) = P(A_1) + P(A_2) + ... + P(A_n) \quad \text{(4·6b)}$$

The probabilities $P(A_1), P(A_2), ..., P(A_n)$ of the mutually exclusive forms of A are known as the *partial probabilities*. Since $P(A)$ is their sum, it may be called the *total probability* of A. Hence the name of the theorem.

Theorem 4·7. *(Boole's inequality). For n events $A_1, A_2, ..., A_n$, we have*

(a) $$P\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - (n-1) \qquad ...(4·7)$$

(b) $$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i) \qquad ...(4·7a)$$

[Delhi Univ. B.Sc. (Stat Hons.), 1992, 1989]

Proof. (a) $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \leq 1$

$\Rightarrow \quad P(A_1 \cap A_2) \geq P(A_1) + P(A_2) - 1 \qquad (*)$

Hence (4·7) is true for $n = 2$.

Let us now suppose that (4·7) is true for $n=r$ (say), such that

$$P\left(\bigcap_{i=1}^{r} A_i\right) \geq \sum_{i=1}^{r} P(A_i) - (r-1) \qquad (**)$$

Then

$$P\left(\bigcap_{i=1}^{r+1} A_i\right) = P\left(\bigcap_{i=1}^{r} A_i \cap A_{r+1}\right)$$

$$\geq P\left(\bigcap_{i=1}^{r} A_i\right) + P(A_{r+1}) - 1 \qquad \text{[From (*)]}$$

$$\geq \sum_{i=1}^{r} P(A_i) - (r-1) + P(A_{r+1}) - 1 \qquad \text{[From (**)]}$$

$$\Rightarrow \quad P\left(\bigcap_{i=1}^{r+1} A_i\right) \geq \sum_{i=1}^{r+1} P(A_i) - r$$

$\Rightarrow$ (4·7) is true for $n = r + 1$ also.

The result now follows by the principle of mathematical induction.

(b) Applying the inequality (4·7) to the events $\bar{A}_1, \bar{A}_2, ..., \bar{A}_n$, we get

$$P(\bar{A}_1 \cap \bar{A}_2 \cap ... \cap \bar{A}_n) \geq [P(\bar{A}_1) + P(\bar{A}_2) + ... + P(\bar{A}_n)] - (n-1)$$
$$= [1 - P(A_1)] + [1 - P(A_2)] + ... + [1 - P(A_n)] - (n-1)$$
$$= 1 - P(A_1) - P(A_2) - ... - P(A_n)$$

$$\Rightarrow \quad P(A_1) + P(A_2) + ... + P(A_n) \geq 1 - P(\bar{A}_1 \cap \bar{A}_2 \cap ... \cap \bar{A}_n).$$
$$= 1 - P(\overline{A_1 \cup A_2 \cup ... \cup A_n}).$$
$$= P(A_1 \cup A_2 \cup ... \cup A_n)$$

$$\Rightarrow \quad P(A_1 \cup A_2 \cup ... \cup A_n) \leq P(A_1) + P(A_2) + ... + P(A_n)$$

as desired.

Aliter for (b) i.e., (4·7a). We have

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

$$\leq P(A_1) + P(A_2) \qquad [\because P(A_1 \cap A_2) \geq 0] \qquad \ldots(***)$$

Hence (4·7*a*) is true for $n = 2$.

Let us now suppose that (4·7*a*) is true for $n=r$, (say), so that

$$P(\bigcup_{i=1}^{r} A_i) \leq \sum_{i=1}^{r} P(A_i) \qquad \ldots(****)$$

Now

$$P(\bigcup_{i=1}^{r+1} A_i) = P(\bigcup_{i=1}^{r} A_i \cup A_{r+1})$$

$$\leq P(\bigcup_{i=1}^{r} A_i) + P(A_{r+1}) \qquad \text{[Using (***)]}$$

$$\leq \sum_{i=1}^{r} P(A_i) + P(A_{r+1}) \qquad \text{[Using (****)]}$$

$$\Rightarrow \quad P(\bigcup_{i=1}^{r+1} A_i) \leq \sum_{i=1}^{r+1} P(A_i)$$

Hence if (4·7*a*) is true for $n=r$, then it is also true for $n=r+1$. But we have proved in (***) that (4·7*a*) is true for $n=2$. Hence by mathematical induction we conclude that (4·7*a*) is true for all positive integral values of n.

Theorem 4·8. *For n events* $A_1, A_2, \ldots, A_n$,

$$P[\bigcup_{i=1}^{n} A_i] \geq \sum_{i=1}^{n} P(A_i) - \sum_{1 \leq i < j \leq n} P(A_i \cap A_j)$$

[Delhi Univ. B.Sc. (Stat Hons.), 1986]

Proof. We shall prove this theorem by the method of induction.

We know that

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - [P(A_1 \cap A_2) + P(A_2 \cap A_3) + P(A_3 \cap A_1)] + P(A_1 \cap A_2 \cap A_3)$$

$$\Rightarrow P(\bigcup_{i=1}^{3} A_i) \geq \sum_{i=1}^{3} P(A_i) - \sum\sum_{1 \leq i < j \leq 3} P(A_i \cap A_j)$$

Thus the result is true for $n=3$. Let us now suppose that the result is true for $n=r$ (say), so that

$$P(\bigcup_{i=1}^{r} A_i) \geq \sum_{i=1} P(A_i) - \sum\sum_{1 \leq i < j \leq r} P(A_i \cap A_j) \qquad \ldots(*)$$

Now

$$P(\bigcup_{i=1}^{r+1} A_i) = P(\bigcup_{i=1}^{r} A_i \cup A_{r+1})$$

$$= P(\bigcup_{i=1}^{r} A_i) + P(A_{r+1}) - P[(\bigcup_{i=1}^{r} A_i) \cap A_{r+1}]$$

$$= P\left(\bigcup_{i=1}^{r} A_i\right) + P(A_{r+1}) - P\left[\bigcup_{i=1}^{r} (A_i \cap A_{r+1})\right]$$

$$\geq \left[\sum_{i=1}^{r} P(A_i) - \sum\sum_{1 \leq i < j < r} P(A_i \cap A_j)\right]$$

$$+ P(A_{r+1}) - P\left[\bigcup_{i=1}^{r} (A_i \cap A_{r+1})\right] \qquad ...(**)$$

[From (*)]

From Boole's inequality (*c.f.* Theorem 4·7 page 4·33), we get

$$P\left[\bigcup_{i=1}^{r} (A_i \cap A_{r+1})\right] \leq \sum_{i=1}^{r} P(A_i \cap A_{r+1})$$

$$\Rightarrow \quad -P\left[\bigcup_{i=1}^{r} (A_i \cap A_{r+1})\right] \geq -\sum_{i=1}^{r} P(A_i \cap A_{r+1})$$

$\therefore$ From (**), we get

$$P\left(\bigcup_{i=1}^{r+1} A_i\right) \geq \sum_{i=1}^{r+1} P(A_i) - \sum\sum_{1 \leq i < j \leq r} P(A_i \cap A_j) - \sum_{i=1}^{r} P(A_i \cap A_{r+1})$$

$$\Rightarrow \quad P\left(\bigcup_{i=1}^{r+1} A_i\right) \geq \sum_{i=1}^{r+1} P(A_i) - \sum\sum_{1 \leq i < j \leq r+1} P(A_i \cap A_j)$$

Hence, if the theorem is true for $n = r$, it is also true for $n = r + 1$. But we have seen that the result is true for $n = 3$. Hence by mathematical induction, the result is true for all positive integral values of n.

4·7. Multiplication Law of Probability and Conditional Probability

Theorem 4·8. *For two events A and B*

$$\left.\begin{aligned} P(A \cap B) &= P(A) \, . \, P(B \mid A), \; P(A) > 0 \\ &= P(B) \, . \, P(A \mid B), \; P(B) > 0 \end{aligned}\right\} \qquad ...(4·8)$$

where P(B | A) represents the conditional probability of occurrence of B when the event A has already happened and P(A | B) is the conditional probability of happening of A, given that B has already happened.

Proof.

$$P(A) = \frac{n(A)}{n(S)} \; ; \; P(B) = \frac{n(B)}{n(S)} \text{ and } P(A \cap B) = \frac{n(A \cap B)}{n(S)} \qquad (*)$$

For the conditional event $A \mid B$, the favourable outcomes must be one of the sample points of B, *i.e.*, for the event $A \mid B$, the sample space is B and out of the $n(B)$ sample points, $n(A \cap B)$ pertain to the occurrence of the event A. Hence

$$P(A \mid B) = \frac{n(A \cap B)}{n(B)}$$

Rewriting (*), we get

$$P(A \cap B) = \frac{n(B)}{n(S)} \cdot \frac{n(A \cap B)}{n(B)} = P(B) \, . \, P(A \mid B)$$

Similarly we can prove :

$$P(A \cap B) = \frac{n(A)}{n(S)} \cdot \frac{n(A \cap B)}{n(A)} = P(A) \cdot P(B \mid A)$$

Remarks. 1. $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)}$ and $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

Thus the conditional probabilities $P(B \mid A)$ and $P(A \mid B)$ are defined if and only if $P(A) \neq 0$ and $P(B) \neq 0$, respectively.

2. *(i)* For $P(B) > 0$, $P(A \mid B) \leq P(A)$

(ii) The conditional probability $P(A \mid B)$ is not defined if $P(B) = 0$.

(iii) $P(B \mid B) = 1$.

3. Multiplication Law of Probability for Independent Events. If A and B are independent then

$$P(A \mid B) = P(A) \quad \text{and} \quad P(B \mid A) = P(B)$$

Hence (4·8) gives :

$$P(A \cap B) = P(A)\, P(B) \qquad \ldots(4{\cdot}8a)$$

provided A and B are independent.

4·7·1. Extension of Multiplication Law of Probability. *For n events $A_1, A_2, \ldots, A_n$, we have*

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1)\, P(A_2 \mid A_1)\, P(A_3 \mid A_1 \cap A_2) \ldots \times P(A_n \mid A_1 \cap A_2 \cap \ldots \cap A_{n-1}) \qquad \ldots(4{\cdot}8b)$$

where $P(A_i \mid A_j \cap A_k \cap \ldots \cap A_l)$ represents the conditional probability of the event A_i given that the events $A_j, A_k, \ldots, A_l$ have already happened.

Proof. We have for three events A_1, A_2, and A_3

$$\begin{aligned} P(A_1 \cap A_2 \cap A_3) &= P[A_1 \cap (A_2 \cap A_3)] \\ &= P(A_1)\, P(A_2 \cap A_3 \mid A_1) \\ &= P(A_1)\, P(A_2 \mid A_1)\, P(A_3 \mid A_1 \cap A_2) \end{aligned}$$

Thus we find that (4·8*b*) is true for $n=2$ and $n=3$. Let us suppose that (4·8*b*) is true for $n=k$, so that

$$P(A_1 \cap A_2 \cap \ldots \cap A_k) = P(A_1)\; P(A_2 \mid A_1)\; P(A_3 \mid A_1 \cap A_2) \ldots P(A_k \mid A_1 \cap A_2 \cap \ldots \cap A_{k-1})$$

Now

$$\begin{aligned} P[(A_1 \cap A_2 \cap \ldots \cap A_k) \cap A_{k+1}] &= P(A_1 \cap A_2 \cap \ldots \cap A_k) \\ &\quad \times P(A_{k+1} \mid A_1 \cap A_2 \cap \ldots \cap A_k) \\ &= P(A_1)\; P(A_2 \mid A_1) \ldots P(A_k \mid A_1 \cap A_2 \cap \ldots \cap A_{k-1}) \\ &\quad \times P(A_{k+1} \mid A_1 \cap A_2 \cap \ldots \cap A_k) \end{aligned}$$

Thus (4·8*b*) is true for $n=k+1$ also. Since (4·8*b*) is true for $n=2$ and $n=3$, by the principle of mathematical induction, it follows that (4·8*b*) is true for all positive integral values of n.

Remark. If $A_1, A_2, \ldots, A_n$ are independent events then

$$P(A_2 \mid A_1) = P(A_2),\; P(A_3 \mid A_1 \cap A_2) = P(A_3) \ldots P(A_n \mid A_1 \cap A_2 \cap \ldots \cap A_{n-1}) = P(A_n)$$

Hence (4·8*b*) gives :

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_1) P(A_2) \ldots \cap P(A_n), \qquad \ldots(4\cdot8c)$$

provided $A_1, A_2, \ldots, A_n$ are independent.

Remark. Mutually Exclusive (Disjoint) Events and Independent Events. Let A and B be mutually exclusive (disjoint) events with positive probabilities ($P(A) > 0, P(B) > 0$), *i.e.*, both A and B are possible events such that

$$A \cap B = \phi \quad \Rightarrow \quad P(A \cap B) = P(\phi) = 0 \qquad \ldots(i)$$

Further, by compound probability theorem we have

$$P(A \cap B) = P(A) \cdot P(B \mid A) = P(B) \cdot P(A \mid B) \qquad \ldots(ii)$$

Since $P(A) \neq 0$; $P(B) \neq 0$, from *(i)* and *(ii)* we get

$$P(A \mid B) = 0 \neq P(A), \quad P(B \mid A) = 0 \neq P(B) \qquad \ldots(iii)$$

$\Rightarrow$ A and B are dependent events.

Hence two possible mutually disjoint events are always dependent (not independent) events.

However, if A and B are independent events with $P(A) > 0$ and $P(B) > 0$, then

$$P(A \cap B) = P(A)\, P(B) \neq 0$$

$\Rightarrow$ A and B cannot be mutually exclusive.

Hence two independent events (both of which are possible events), cannot be mutually disjoint.

4·7·2. Given n independent events A_i, (i = 1,2,...,n) with respective probabilities of occurrence p_i, to find the probability of occurrence of at least one of them.

We have

$$P(A_i) = p_i \quad \Rightarrow \quad P(\bar{A}_i) = 1 - p_i\,;\; i = 1, 2, \ldots, n$$

$$[\because \overline{(A_1 \cup A_2 \cup \ldots \cup A_n)} = (\bar{A}_1 \cap \bar{A}_2 \cap \ldots \cap \bar{A}_n) \quad (\textit{De-Morgan's Law})]$$

Hence the probability of happening of at least one of the events is given by

$$\begin{aligned} P(A_1 \cup A_2 \cup \ldots \cup A_n) &= 1 - P\,\overline{(A_1 \cup A_2 \cup \ldots \cup A_n)} && \ldots(*) \\ &= 1 - P(\bar{A}_1 \cap \bar{A}_2 \cap \ldots \cap \bar{A}_n) \\ &= 1 - P(\bar{A}_1)\, P(\bar{A}_2) \ldots P(\bar{A}_n) && \ldots(**) \end{aligned}$$

[*c.f.* Theorem 4·14 page 4·41]

$$\begin{aligned} &= 1 - [(1-p_1)(1-p_2)\ldots(1-p_n)] \\ &= \Bigg[\sum_{i=1}^{n} p_i - \sum\sum_{\substack{i,j=1 \\ i<j}}^{n} (p_i p_j) + \sum\sum\sum_{\substack{i,j,k=1 \\ i<j<k}}^{n} (p_i p_j p_k) \\ &\qquad \ldots + (-1)^{n-1} (p_1 p_2 \ldots p_n)\Bigg] \end{aligned}$$

Remark. The results in (*) and (**) are very important and are used quite often in numerical problems. Result (*) stated in words gives:

P [happening of at least one of the events $A_1, A_2, \ldots, A_n$]

$= 1 - P$ (none of the events $A_1, A_2, \ldots, A_n$ happens) or equivalently,

$$P \text{ \{none of the given events happens\}} = 1 - P \text{ \{at least one of them happens\}}.$$

Theorem 4·9. *For any three events A, B and C*

$$P(A \cup B \mid C) = P(A \mid C) + P(B \mid C) - P(A \cap B \mid C)$$

Proof. We have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow P[(A \cap C) \cup (B \cap C)] = P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$$

Dividing both sides by $P(C)$, we get

$$\frac{P[(A \cap C) \cup (B \cap C)]}{P(C)} = \frac{P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)}{P(C)}, P(C) > 0$$

$$= \frac{P(A \cap C)}{P(C)} + \frac{P(B \cap C)}{P(C)} - \frac{P(A \cap B \cap C)}{P(C)}$$

$$\Rightarrow \quad \frac{P[(A \cup B) \cap C]}{P(C)} = P(A \mid C) + P(B \mid C) - P(A \cap B \mid C)$$

$$\Rightarrow \quad P[(A \cup B) \mid C] = P(A \mid C) + P(B \mid C) - P(A \cap B \mid C)$$

Theorem 4·10. *For any three events A, B and C*

$$P(A \cap \overline{B} \mid C) + P(A \cap B \mid C) = P(A \mid C)$$

Proof. $P(A \cap \overline{B} \mid C) + P(A \cap B \mid C)$

$$= \frac{P(A \cap \overline{B} \cap C)}{P(C)} + \frac{P(A \cap B \cap C)}{P(C)}$$

$$= \frac{P(A \cap \overline{B} \cap C) + P(A \cap B \cap C)}{P(C)}$$

$$= \frac{P(A \cap C)}{P(C)} = P(A \mid C)$$

Theorem 4·11. *For a fixed B with $P(B) > 0$, $P(A \mid B)$ is a probability function.* **[Delhi Univ. B.Sc. (Stat. Hons.), 1991; (Maths Hons.), 1992]**

Proof.

(i) $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)} \geq 0$

(ii) $P(S \mid B) = \dfrac{P(S \cap B)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$

(iii) If $\{A_n\}$ is any finite or infinite sequences of disjoint events, then

$$P[\bigcup_n A_n \mid B] = \frac{P[(\bigcup_n A_n) \cap B]}{P(B)} = \frac{P[(\bigcup_n A_n . B)]}{P(B)}$$

$$= \frac{\sum_n P(A_n B)}{P(B)} = \sum_n \left[\frac{P(A_n B)}{P(B)}\right] = \sum_n P(A_n \mid B)$$

Hence the theorem.

Remark. For given B satisfying $P(B) > 0$, the conditional probability $P[\cdot|B]$ also enjoys the same properties as the unconditional probability.

For example, in the usual notations, we have:

(*i*) $P[\phi \mid B] = 0$

(*ii*) $P[\overline{A} \mid B] = 1 - P[A \mid B]$

(*iii*) $P[\bigcup_{i=1}^{n} A_i \mid B] = \sum_{i=1}^{n} P[A_i \mid B]$,

where $A_1, A_2, ..., A_n$ are mutually disjoint events.

(*iv*) $P(A_1 \cup A_2 \mid B) = P(A_1 \mid B) + P(A_2 \mid B) - P(A_1 A_2 \mid B)$

(*v*) If $E \subset F$, then $P(E \mid B) \le P(F \mid B)$

and so on.

The proofs of results (*iv*) and (*v*) are given in theorems 4·9 and 4·13 respectively. Others are left as exercises to the reader.

Theorem 4·12. *For any three events, A, B and C defined on the sample space S such that $B \subset C$ and $P(A) > 0$,*

$$P(B \mid A) \le P(C \mid A)$$

Proof.

$$P(C \mid A) = \frac{P(C \cap A)}{P(A)} \qquad \text{(By definition)}$$

$$= \frac{P[B \cap C \cap A) \cup (\overline{B} \cap C \cap A)}{P(A)}$$

$$= \frac{P[B \cap C \cap A)}{P(A)} + \frac{P(\overline{B} \cap C \cap A)}{P(A)} \qquad \text{(Using axiom 3)}$$

$$= P[(B \cap C|A) + (\overline{B} \cap C \cap A)]$$

Now $\quad B \subset C \quad \Rightarrow \quad B \cap C = B$

$\therefore \quad P(C \mid A) = P(B \mid A) + P(\overline{B} \cap C|A)$

$\Rightarrow \quad P(C \mid A) \ge P(B \mid A)$

4·7·3. Independent Events. *An event B is said to be independent (or statistically independent) of event A, if the conditional probability of B given A i.e., $P(B \mid A)$ is equal to the unconditional probability of B, i.e., if*

$$P(B \mid A) = P(B)$$

Since

$$P(A \cap B) = P(B \mid A)\, P(A) = P(A \mid B)\, P(B)$$

and since $P(B \mid A) = P(B)$ when B is independent of A, we must have $P(A \mid B) = P(A)$ or it follows that A is also independent of B. Hence the events A and B are independent if and only if

$$P(A \cap B) = P(A)\, P(B) \qquad ...(4·9)$$

4·7·4. Pairwise Independent Events

Definition. *A set of events $A_1, A_2, ..., A_n$ are said to be pair-wise independent if*

$$P(A_i \cap A_j) = P(A_i)\, P(A_j) \quad \forall \; i \ne j \qquad ...(4·10)$$

4·7·5. Conditions for Mutual Independence of n Events. Let S denote the sample space for a number of events. The events in S are said to be mutually independent if the probability of the simultaneous occurrence of (any) finite number of them is equal to the product of their separate probabilities.

If $A_1, A_2, ..., A_n$ are n events, then for their mutual independence, we should have

(i) $P(A_i \cap A_j) = P(A_i) P(A_j), \ (i \neq j \ ; i, j = 1, 2, ..., n)$

(ii) $P(A_i \cap A_j \cap A_k) = P(A_i) P(A_j) P(A_k), (i \neq j \neq k \ ; i \ ; j \ ; k = 1, 2, ..., n)$

$$\vdots \qquad\qquad \vdots$$

$$P(A_1 \cap A_2 \cap ... \cap A_n) = P(A_1) P(A_2) ... P(A_n)$$

It is interesting to note that the above equations give respectively ${}^nC_2, {}^nC_3, ..., {}^nC_n$ conditions to be satisfied by $A_1, A_2, ..., A_n$.

Hence the total number of conditions for the mutual independence of $A_1, A_2, ..., A_n$ is ${}^nC_2 + {}^nC_3 + ... + {}^nC_n$.

Since ${}^nC_0 + {}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n$, we get the required number of conditions as $(2^n - 1 - n)$.

In particular for three events A_1, A_2 and A_3, $(n = 3)$, we have the following $2^3 - 1 - 3 = 4$, conditions for their mutual independence.

$$P(A_1 \cap A_2) = P(A_1) \ P(A_2)$$
$$P(A_2 \cap A_3) = P(A_2) \ P(A_3)$$
$$P(A_1 \cap A_3) = P(A_1) \ P(A_3)$$
$$P(A_1 \cap A_2 \cap A_3) = P(A_1) \ P(A_2) \ P(A_3) \qquad ...(4{\cdot}11)$$

Remarks. 1. It may be observed that pairwise or mutual independence of events $A_1, A_2, ..., A_n$, is defined only when $P(A_i) \neq 0$, for $i = 1, 2, ..., n$.

2. If the events A and B are such that $P(A_i) \neq 0$, $P(B) \neq 0$ and A is independent of B, then B is independent of A.

Proof. We are given that

$$P(A \mid B) = P(A)$$

$$\Rightarrow \quad \frac{P(A \cap B)}{P(B)} = P(A)$$

$$\Rightarrow \quad P(A \cap B) = P(A) \ P(B)$$

$$\Rightarrow \quad \frac{P(B \cap A)}{P(A)} = P(B) \qquad [\because P(A) \neq 0 \text{ and } A \cap B = B \cap A]$$

$$\Rightarrow \quad P(B \mid A) = P(B),$$

which by definition of independent events, means that B is independent of A.

3. It may be noted that pairwise independence of events does not imply their mutual independence. For illustrations, see Examples 4·50 and 4·51.

Theorem 4·13. *If A and B are independent events then A and $\overline{B}$ are also independent events.*

Proof. By theorem 4·4, we have

$$P(A \cap \bar{B}) = P(A) - P(A \cap B)$$
$$= P(A) - P(A)P(B) \quad [\because A \text{ and } B \text{ are independent}]$$
$$= P(A)[1 - P(B)]$$
$$= P(A)\, P(\bar{B})$$

$\Rightarrow$ A and $\bar{B}$ are independent events.

Aliter. $P(A \cap B) = P(A)\,P(B) = P(A)\,P(B \mid A) = P(B)\,P(A \mid B)$

i.e., $P(B \mid A) = P(B) \Rightarrow B$ is independent of A.

also $P(A \mid B) = P(A) \Rightarrow A$ is independent of B.

Also $P(B \mid A) + P(\bar{B} \mid A) = 1 \Rightarrow P(B) + P(\bar{B} \mid A) = 1$

or $P(\bar{B} \mid A) = 1 - P(B) = P(\bar{B})$

$\therefore$ $\bar{B}$ is independent of A and by symmetry we say that A is independent of $\bar{B}$. Thus A and $\bar{B}$ are independent events.

Remark. Similarly, we can prove that if A and B are independent events then $\bar{A}$ and B are also independent events.

Theorem 4·14. *If A and B are independent events then $\bar{A}$ and $\bar{B}$ are also independent events.*

Proof. We are given $P(A \cap B) = P(A)\,P(B)$

Now $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B)$

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$
$$= 1 - [P(A) + P(B) - P(A)P(B)]$$
$$= 1 - P(A) - P(B) + P(A)P(B)$$
$$= [1 - P(B)] - P(A)[1 - P(B)]$$
$$= [1 - P(A)][1 - P(B)] = P(\bar{A})P(\bar{B})$$

$\therefore$ $\bar{A}$ and $\bar{B}$ are independent events.

Aliter. We know

$$P(\bar{A} \mid \bar{B}) + P(A \mid \bar{B}) = 1$$

$\Rightarrow \quad P(\bar{A} \mid \bar{B}) + P(A) = 1$ (*c.f.* Theorem 4·13)

$\Rightarrow \quad P(\bar{A} \mid \bar{B}) = 1 - P(A) = P(\bar{A})$

$\therefore$ $\bar{A}$ and $\bar{B}$ are independent events.

Theorem 4·15. *If A, B, C are mutually independent events then $A \cup B$ and C are also independent.*

Proof. We are required to prove:

$P[(A \cup B) \cap C] = P(A \cup B)P(C)$

L.H.S. $= P[(A \cap C) \cup (B \cap C)]$ [Distributive Law]

$$= P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)$$
$$= P(A)P(C) + P(B)P(C) - P(A)P(B)P(C)$$

[$\because A, B$ and C are mutually independent]

$$= P(C)[P(A) + P(B) - P(A \cap B)]$$

$$= P(C)\,P(A \cup B) = \text{R.H.S.}$$

Hence $(A \cup B)$ and C are independent.

Theorem 4·16. *If A, B and C are random events in a sample space and if A, B and C are pairwise independent and A is independent of (B ∪ C), then A, B and C are mutually independent.*

Proof. We are given

$$\left.\begin{aligned} P(A \cap B) &= P(A)P(B) \\ P(B \cap C) &= P(B)P(C) \\ P(A \cap C) &= P(A)P(C) \\ P[A \cap (B \cup C)] &= P(A)P(B \cup C) \end{aligned}\right\} \quad \ldots (*)$$

Now $\quad P[A \cap (B \cup C)] = P[(A \cap B) \cup (A \cap C)]$

$$= P(A \cap B) + P(A \cap C) - P[(A \cap B) \cap (A \cap C)]$$

$$= P(A).P(B) + P(A).P(C) - P(A \cap B \cap C) \quad \ldots(**)$$

and $P(A)\,P(B \cup C) = P(A)\,[P(B) + P(C) - P(B \cap C)]$

$$= P(A).P(B) + P(A)\,P(C) - P(A)\,P(B \cap C) \quad \ldots(***)$$

From (**) and (***), on using (*), we get

$$P(A \cap B \cap C) = P(A)\,P(B \cap C) = P(A)\,P(B)\,P(C)$$

Hence A, B, C are mutually independent.

Theorem 4·17. *For any two events A and B,*

$$P(A \cap B) \le P(A) \le P(A \cup B) \le P(A) + P(B)$$

[Patna Univ. B.A.(Stat. Hons.), 1992; Delhi Univ. B.Sc.(Stat. Hons.), 1989]

Proof. We have

$$A = (A \cap \bar{B}) \cup (A \cap B)$$

Using axiom 3, we have

$$P(A) = P[(A \cap \bar{B}) \cup (A \cap B)] = P(A \cap \bar{B}) + P(A \cap B)$$

Now $\quad P[(A \cap \bar{B}) \ge 0 \quad$ (From axiom 1)

$\therefore \quad P(A) \ge P(A \cap B) \quad \ldots(*)$

Similarly $\quad P(B) \ge P(A \cap B)$

$\Rightarrow \quad P(B) - P(A \cap B) \ge 0$

Now $\quad P(A \cup B) = P(A) + [P(B) - P(A \cap B)] \quad \ldots(**)$

$\therefore \quad P(A \cup B) \ge P(A) \quad \Rightarrow \quad P(A) \le P(A \cup B) \quad \ldots(***)$

Also $\quad P(A \cup B) \le P(A) + P(B) \quad$ [From (**)]

Hence from (*), (**) and (***), we get

$$P(A \cap B) \le P(A) \le P(A \cup B) \le P(A) + P(B)$$

Aliter. Since $A \cap B \subset A$, by Theorem 4·6 *(ii)* page 4·30, we get

$$P(A \cap B) \le P(A).$$

Also $A \subset (A \cup B) \quad \Rightarrow \quad P(A) \le P(A \cup B)$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\le P(A) + P(B) \qquad [\because P(A \cap B) \ge 0]$$

Combining the above results, we get

$$P(A \cap B) \le P(A) \le P(A \cup B) \le P(A) + P(B)$$

Example 4·12. *Two dice, one green and the other red, are thrown. Let A be the event that the sum of the points on the faces shown is odd, and B be the event of at least one ace (number '1').*

(a) Describe the (i) complete sample space, (ii) events A, B, $\overline{B}$, $A \cap B$, $A \cup B$, and $A \cap \overline{B}$ and find their probabilities assuming that all the 36 sample points have equal probabilities.

(b) Find the probabilities of the events :

(i) $(\overline{A} \cup \overline{B})$ *(ii)* $(\overline{A} \cap \overline{B})$ *(iii)* $(A \cap \overline{B})$ *(iv)* $(\overline{A} \cap B)$ *(v)* $(\overline{A \cap B})$ *(vi)* $(\overline{A} \cup B)$ *(vii)* $(\overline{A \cup B})$ *(viii)* $\overline{A} \cap (A \cup B)$ *(ix)* $A \cup (\overline{A} \cap B)$ *(x)* $(A \mid B)$ and $(B \mid A)$, and *(xi)* $(\overline{A} \mid \overline{B})$ *and* $(\overline{B} \mid \overline{A})$.

Solution. *(a)* The sample space consists of the 36 elementary events .

$$\begin{array}{llllll}
(1,1) & ;(1,2) & ;(1,3) & ;(1,4) & ;(1,5) & ;(1,6)\\
(2,1) & ;(2,2) & ;(2,3) & ;(2,4) & ;(2,5) & ;(2,6)\\
(3,1) & ;(3,2) & ;(3,3) & ;(3,4) & ;(3,5) & ;(3,6)\\
(4,1) & ;(4,2) & ;(4,3) & ;(4,4) & ;(4,5) & ;(4,6)\\
(5,1) & ;(5,2) & ;(5,3) & ;(5,4) & ;(5,5) & ;(5,6)\\
(6,1) & ;(6,2) & ;(6,3) & ;(6,4) & ;(6,5) & ;(6,6)
\end{array}$$

where, for example, the ordered pair (4, 5) refers to the elementary event that the green die shows 4 and and the red die shows 5.

A = The event that the sum of the numbers shown by the two dice is odd.

$= \{(1,2);(2,1);(1,4);(2,3);(3,2);(4,1);(1,6);(2,5)$
$(3,4);(4,3);(5,2);(6,1);(3,6);(4,5);(5,4);(6,3)$
$(5,6);(6,5)\}$ and therefore

$$P(A) = \frac{n(A)}{n(S)} = \frac{18}{36}$$

B = The event that at least one face is 1,

$= \{(1,1);(1,2);(1,3);(1,4);(1,5);(1,6)$
$(2,1);(3,1);(4,1);(5,1);(6,1)\}$ and therefore

$$P(B) = \frac{n(B)}{n(S)} = \frac{11}{36}$$

$\overline{B}$ = The event that each of the face obtained is not an ace.

$= \{(2,2);(2,3);(2,4);(2,5);(2,6);(3,2);(3,3);$
$(3,4);(3,5);(3,6);(4,2);(4,3);(4,4);(4,5);$
$(4,6);(5,2);(5,3);(5,4);(5,5);(5,6);(6,2);$
$(6,3);(6,4);(6,5);(6,6)\}$ and therefore

$$P(\overline{B}) = \frac{n(\overline{B})}{n(S)} = \frac{25}{36}$$

$A \cap B$ = The event that sum is odd and at least one face is an ace.

$= \{(1,2);(2,1);(1,4);(4,1);(1,6);(6,1)\}$

$$\therefore \quad P(A \cap B) = \frac{n(A \cap B)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

$A \cup B = \{(1,2);(2,1);(1,4);(2,3);(3,2);(4,1);(1,6);(2,5)$
$(3,4);(4,3);(5,2);(6,1);(3,6);(4,5);(5,4);(6,3)$
$(5,6);(6,5);(1,1);(1,3);(1,5);(3,1);(5,1)\}$

$$\therefore \quad P(A \cup B) = \frac{n(A \cup B)}{n(S)} = \frac{23}{36}$$

$A \cap \bar{B} = \{(2,3);(3,2);(2,5);(3,4);(3,6);(4,3);(4,5);(5,2)$
$(5,4);(5,6);(6,3);(6,5)\}$

$$P(A \cap \bar{B}) = \frac{n(A \cap \bar{B})}{n(S)} = \frac{12}{36} = \frac{1}{3}$$

(b) (i) $P(\bar{A} \cup \bar{B}) = P(\overline{A \cap B}) = 1 - P(A \cap B) = 1 - \frac{1}{6} = \frac{5}{6}$

(ii) $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - \frac{23}{36} = \frac{13}{36}$

(iii) $P(A \cap \bar{B}) = P(A) - P(A \cap B) = \frac{18}{36} - \frac{6}{36} = \frac{12}{36} = \frac{1}{3}$

(iv) $P(\bar{A} \cap B) = P(B) - P(A \cap B) = \frac{11}{36} - \frac{6}{36} = \frac{5}{36}$

(v) $P(\overline{A \cap B}) = 1 - P(A \cap B) = 1 - \frac{1}{6} = \frac{5}{6}$

(vi) $P(\bar{A} \cup B) = P(\bar{A}) + P(B) - P(\bar{A} \cap B)$

$$= \left(1 - \frac{18}{36}\right) + \frac{11}{36} - \frac{5}{36} = \frac{2}{3}$$

(vii) $P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - \frac{23}{36} = \frac{13}{36}$

(viii) $P[\bar{A} \cap (A \cup B)] = P[(A \cap \bar{A}) \cup (\bar{A} \cap B)]$

$$= P(\bar{A} \cap B) = \frac{5}{36} \qquad [\because A \cap \bar{A} = \phi]$$

(ix) $P[A \cup (\bar{A} \cap B)] = P(A) + P(\bar{A} \cap B) - P(A \cap \bar{A} \cap B)$

$$= P(A) + P(\bar{A} \cap B) = \frac{18}{36} + \frac{5}{36} = \frac{23}{36}$$

(x) $P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{6/36}{11/36} = \frac{6}{11}$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{6/36}{18/36} = \frac{6}{18} = \frac{1}{3}$$

(xi) $P(\bar{A} \mid \bar{B}) = \frac{P(\bar{A} \cap \bar{B})}{P(\bar{B})} = \frac{13/36}{25/36} = \frac{13}{25}$

$$P(\bar{B} \mid \bar{A}) = \frac{P(\bar{A} \cap \bar{B})}{P(\bar{A})} = \frac{13/36}{18/36} = \frac{13}{18}$$

Example 4·13. *If two dice are thrown, what is the probability that the sum is (a) greater than 8, and (b) neither 7 nor 11?*

Solution. *(a)* If S denotes the sum on the two dice, then we want $P(S > 8)$.

The required event can happen in the following mutually exclusive ways:

(i) $S = 9$ *(ii)* $S = 10$ *(iii)* $S = 11$ *(iv)* $S = 12$.

Hence by addition theorem of probability

$$P(S > 8) = P(S = 9) + P(S = 10) + P(S = 11) + P(S = 12)$$

In a throw of two dice, the sample space contains $6^2 = 36$ points.

The number of favourable cases can be enumerated as follows:

$S = 9$: (3, 6), (6, 3), (4, 5), (5, 4), *i.e.*, 4 sample points.

$$\therefore \quad P(S = 9) = \frac{4}{36}$$

$S = 10$: (4,6), (6,4), (5,5), *i.e.*, 3 sample points.

$$\therefore \quad P(S = 10) = \frac{3}{36}$$

$S = 11$: (5, 6), (6, 5), *i.e.*, 2 sample points.

$$\therefore \quad P(S = 11) = \frac{2}{36}$$

$S = 12$: (6, 6), *i.e.*, 1 sample point.

$$\therefore \quad P(S = 12) = \frac{1}{36}$$

$$\therefore \quad P(S > 8) = \frac{4}{36} + \frac{3}{36} + \frac{2}{36} + \frac{1}{36} = \frac{10}{36} = \frac{5}{18}$$

(b) Let A denote the event of getting the sum of 7 and B denote the event of getting the sum of 11 with a pair of dice.

$S = 7$: (1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3), *i.e.*, 6 distinct sample points.

$$\therefore \quad P(A) = P(S = 7) = \frac{6}{36} = \frac{1}{6}$$

$S = 11$: (5, 6), (6, 5), $P(B) = P(S = 11) = \dfrac{2}{36} = \dfrac{1}{18}$

$$\therefore \text{ Required probability} = P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$$
$$= 1 - [P(A) + P(B)]$$

$(\because A \text{ and } B \text{ are disjoint events})$

$$= 1 - \frac{1}{6} - \frac{1}{18} = \frac{7}{9}$$

Example 4·14. *An urn contains 4 tickets numbered 1, 2, 3, 4 and another contains 6 tickets numbered 2, 4, 6, 7, 8, 9. If one of the two urns is chosen at random and a ticket is drawn at random from the chosen urn, find the probabilities that the ticket drawn bears the number (i) 2 or 4, (ii) 3, (iii) 1 or 9*

[Calicut Univ. B.Sc.,1992]

Solution. *(i)* Required event can happen in the following mutually exclusive ways:

(I) First urn is chosen and then a ticket is drawn.

(II) Second urn is chosen and then a ticket is drawn.

Since the probability of choosing any urn is ½, the required probability 'p' is given by

$$p = P(I) + P(II)$$
$$= \frac{1}{2} \times \frac{2}{4} + \frac{1}{2} \times \frac{2}{6} = \frac{5}{12}$$

(ii) Required probability $= \frac{1}{2} \times \frac{1}{4} + \frac{1}{2} \times 0 = \frac{1}{8}$

($\because$ in the 2nd urn there is no ticket with number 3)

(iii) Required probability $= \frac{1}{2} \times \frac{1}{4} + \frac{1}{2} \times \frac{1}{6} = \frac{5}{24}$

Example 4·15. *A card is drawn from a well-shuffled pack of playing cards. What is the probability that it is either a spade or an ace ?*

Solution. The equiprobable sample space S of drawing a card from a well-shuffled pack of playing cards consists of 52 sample points.

If A and B denote the events of drawing a 'spade card' and 'an ace' respectively then A consists of 13 sample points and B consists of 4 sample points so that,

$$P(A) = \frac{13}{52} \quad \text{and} \quad P(B) = \frac{4}{52}$$

The compound event $A \cap B$ consists of only one sample point, *viz.*, ace of spade so that,

$$P(A \cap B) = \frac{1}{52}$$

The probability that the card drawn is either a spade or an ace is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{4}{13}$$

Example 4·16. *A box contains 6 red, 4 white and 5 black balls. A person draws 4 balls from the box at random. Find the probability that among the balls drawn there is at least one ball of each colour.* **(Nagpur Univ. B.Sc., 1992)**

Solution. The required event E that 'in a draw of 4 balls from the box at random there is at least one ball of each colour', can materialise in the following mutually disjoint ways :

(i) 1 Red, 1 White, 2 Black balls

(ii) 2 Red, 1 White, 1 Black balls

(iii) 1 Red, 2 White, 1 Black balls.

Hence by the addition theorem of probability, the required probability is given by

$$P(E) = P(i) + P(ii) + P(iii)$$
$$= \frac{{}^6C_1 \times {}^4C_1 \times {}^5C_2}{{}^{15}C_4} + \frac{{}^6C_2 \times {}^4C_1 \times {}^5C_1}{{}^{15}C_4} + \frac{{}^6C_1 \times {}^4C_2 \times {}^5C_1}{{}^{15}C_4}$$
$$= \frac{1}{{}^{15}C_4} \left[6 \times 4 \times 10 + 15 \times 4 \times 5 + 6 \times 6 \times 5 \right]$$
$$= \frac{4!}{15 \times 14 \times 13 \times 12} \left[240 + 300 + 180 \right]$$
$$= \frac{24 \times 720}{15 \times 14 \times 13 \times 12} = 0{\cdot}5275$$

Example 4·17. *Why does it pay to bet consistently on seeing 6 at least once in 4 throws of a die, but not on seeing a double six at least once in 24 throws with two dice? (de Mere's Problem).*

Solution. The probability of getting a '6' in a throw of die $=1/6$.

$\therefore$ The probability of not getting a '6' in a throw of die

$$=1-1/6=5/6.$$

By compound probability theorem, the probability that in 4 throws of a die no '6' is obtained $=(5/6)^4$

Hence the probability of obtaining '6' at least once in 4 throws of a die $=1-(5/6)^4=0{\cdot}516$

Now, if a trial consists of throwing two dice at a time, then the probability of getting a 'double' of '6' in a trial $=1/36$.

Thus the probability of not getting a 'double of 6' in a trial $=35/36$.

The probability that in 24 throws, with two dice each, no 'double of 6' is obtained $=(35/36)^{24}$

Hence the probability of getting a 'double of 6' at least once in 24 throws $=1-(35/36)^{24}=0{\cdot}491$.

Since the probability in the first case is greater than the probability in the second case, the result follows.

Example 4·18. *A problem in Statistics is given to the three students A, B and C whose chances of solving it are 1/2, 3/4, and 1/4 respectively.*

What is the probability that the problem will be solved if all of them try independently? **[Madurai Kamraj Univ. B.Sc.,1986; Delhi Univ. B.A.,1991]**

Solution. Let A, B, C denote the events that the problem is solved by the students A, B, C respectively. Then

$$P(A)=\frac{1}{2},\ P(B)=\frac{3}{4}\ \text{ and }\ P(C)=\frac{1}{4}$$

The problem will be solved if at least one of them solves the problem. Thus we have to calculate the probability of occurrence of at least one of the three events A, B, C, *i.e.*, $P(A\cup B\cup C)$.

$$\begin{aligned}P(A\cup B\cup C)&=P(A)+P(B)+P(C)-P(A\cap B)-P(A\cap C)\\&\qquad -P(B\cap C)+P(A\cap B\cap C)\\&=P(A)+P(B)+P(C)-P(A)P(B)-P(A)P(C)\\&\qquad -P(B)P(C)+P(A)P(B)P(C)\end{aligned}$$

($\because A, B, C$ are independent events.)

$$\begin{aligned}&=\frac{1}{2}+\frac{3}{4}+\frac{1}{4}-\frac{1}{2}\cdot\frac{3}{4}-\frac{3}{4}\cdot\frac{1}{4}\\&\qquad-\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{4}\\&=\frac{29}{32}\end{aligned}$$

Aliter.
$$\begin{aligned} P(A \cup B \cup C) &= 1 - P(\overline{A \cup B \cup C}) \\ &= 1 - P(\overline{A} \cap \overline{B} \cap \overline{C}) \\ &= 1 - P(\overline{A})P(\overline{B})P(\overline{C}) \\ &= 1 - \left(1 - \frac{1}{2}\right)\left(1 - \frac{3}{4}\right)\left(1 - \frac{1}{4}\right) \\ &= \frac{29}{32} \end{aligned}$$

Example 4·19. *If* $A \cap B = \phi$, *then show that* ...(*)
$$P(A) \leq P(\overline{B})$$
[Delhi Univ. B.Sc. (Maths Hons.) 1987]

Solution. We have
$$\begin{aligned} A &= (A \cap B) \cup (A \cap \overline{B}) \\ &= \phi \cup (A \cap \overline{B}) \qquad \text{[Using *]} \\ &= A \cap \overline{B} \end{aligned}$$
$$\Rightarrow \quad A \subseteq \overline{B}$$
$$\Rightarrow \quad P(A) \leq P(\overline{B})$$
as desired.

Aliter. Since $A \cap B = \phi$, we have $A \subset \overline{B}$, which implies that $P(A) \leq P(\overline{B})$.

Example 4·20. *Let A and B be two events such that*
$$P(A) = \frac{3}{4} \text{ and } P(B) = \frac{5}{8}$$
show that

(a) $P(A \cup B) \geq \frac{3}{4}$

(b) $\frac{3}{8} \leq P(A \cap B) \leq \frac{5}{8}$

[Delhi Univ. B.Sc. Stat (Hons.) 1986,1988]

Solution. *(i)* We have
$$A \subset (A \cup B)$$
$$\Rightarrow \quad P(A) \leq P(A \cup B)$$
$$\Rightarrow \quad \frac{3}{4} \leq P(A \cup B)$$
$$\Rightarrow \quad P(A \cup B) \geq \frac{3}{4}$$

(ii)
$$A \cap B \subseteq B$$
$$\Rightarrow \quad P(A \cap B) \leq P(B) = \frac{5}{8} \qquad ...(i)$$
Also
$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq 1$$
$$\Rightarrow \quad \frac{3}{4} + \frac{5}{8} - 1 \leq P(A \cap B)$$

$$\Rightarrow \qquad \frac{6+5-8}{8} \leq P(A \cap B)$$

$$\Rightarrow \qquad \frac{3}{8} \leq P(A \cap B) \qquad ...(ii)$$

From *(i)* and *(ii)* we get

$$\frac{3}{8} \leq P(A \cap B) \leq \frac{5}{8}$$

Example 4·21. *(Chebychev's Problem). What is the chance that two numbers, chosen at random, will be prime to each other ?*

Solution. If any number 'a' is divided by a prime number 'r', then the possible remainders are 0, 1, 2, ...r–1. Hence the chance that 'a' is divisible by r is $1/r$ (because the only case favourable to this is remainder being 0). Similarly, the probability that any number 'b' chosen at random is divisible by r is $1/r$. Since the numbers a and b are chosen at random, the probability that none of them is divisible by 'r' is given (by compound probability theorem) by :

$$\left(1-\frac{1}{r}\right) \times \left(1-\frac{1}{r}\right) = \left(1-\frac{1}{r}\right)^2; \quad r = 2, 3, 5, 7, \ldots$$

Hence the required probability that the two numbers chosen at random are prime to each other is given by

$$P = \prod_r \left(1-\frac{1}{r}\right)^2, \text{ where } r \text{ is a prime number.}$$

$$= \frac{6}{\pi^2} \qquad \text{(From trigonometry)}$$

Example 4·22. *A bag contains 10 gold and 8 silver coins. Two successive drawings of 4 coins are made such that : (i) coins are replaced before the second trial, (ii) the coins are not replaced before the second trial. Find the probability that the first drawing will give 4 gold and the second 4 silver coins.*

[Allahabad Univ. B.Sc., 1987]

Solution. Let A denote the event of drawing 4 gold coins in the first draw and B denote the event of drawing 4 silver coins in the second draw. Then we have to find the probability of $P(A \cap B)$.

(i) Draws with replacement. If the coins drawn in the first draw are replaced back in the bag before the second draw then the events A and B are independent and the required probability is given (using the multiplication rule of probability) by the expression

$$P(A \cap B) = P(A) \,.\, P(B) \qquad ...(*)$$

Ist draw. Four coins can be drawn out of 10+8=18 coins in ${}^{18}C_4$ ways, which gives the exhaustive number of cases. In order that all these coins are of gold, they must be drawn out of the 10 gold coins and this can be done in ${}^{10}C_4$ ways. Hence

$$P(A) = {}^{10}C_4 \,/\, {}^{18}C_4$$

2nd draw. When the coins drawn in the first draw are replaced before the 2nd draw, the bag contains 18 coins. The probability of drawing 4 silver coins in the 2nd draw is given by $P(B) = {}^{8}C_4 \;/\; {}^{18}C_4$.

Substituting in (*), we have

$$P(A \cap B) = \frac{{}^{10}C_4}{{}^{18}C_4} \times \frac{{}^{8}C_4}{{}^{18}C_4}$$

(ii) Draws without replacement. If the coins drawn are not eplaced back before the second draw, then the events A and B are not independent and the required probability is given by

$$P(A \cap B) = P(A) \,.\, P(B \mid A) \qquad \ldots(**)$$

As discussed in part *(i)*, $P(A) = {}^{10}C_4 \;/\; {}^{18}C_4$.

Now, if the 4 gold coins which were drawn in the first draw are not replaced back, there are 18 – 4=14 coins left in the bag and $P(B \mid A)$ is the probability of drawing 4 silver coins from the bag containing 14 coins out of which 6 are gold coins and 8 are silver coins.

Hence $\qquad P(B \mid A) = {}^{8}C_4 \;/\; {}^{14}C_4$

Substituting in (**) we get

$$P(A \cap B) = \frac{{}^{10}C_4}{{}^{18}C_4} \times \frac{{}^{8}C_4}{{}^{14}C_4}$$

Example 4·23. *A consignment of 15 record players contains 4 defectives. The record players are selected at random, one by one, and examined. Those examined are not put back. What is the probability that the 9th one examined is the last defective?*

Solution. Let A be the event of getting exactly 3 defectives in examination of 8 record players and let B the event that the 9th piece examined is a defective one.

Since it is a problem of sampling without replacement and since there are 4 defectives out of 15 record players, we have

$$P(A) = \frac{\binom{4}{3} \times \binom{11}{5}}{\binom{15}{8}}$$

$P(B \mid A)$ = Probability that the 9th examined record player is defective given that there were 3 defectives in the first 8 pieces examined.

$= 1/7,$

since there is only one defective piece left among the remaining 15 – 8 = 7 record players.

Hence the required probability is

$$P(A \cap B) = P(A) \,.\, P(B \mid A)$$

$$= \frac{\binom{4}{3} \times \binom{11}{5}}{\binom{15}{8}} \times \frac{1}{7} = \frac{8}{195}$$

Example 4·24. *p is the probability that a man aged x years will die in a year. Find the probability that out of n men $A_1, A_2, ..., A_n$ each aged x, A_1 will die in a year and will be the first to die.* **[Delhi Univ. B.Sc., 1985]**

Solution. Let E_i, $(i = 1, 2, ..., n)$ denote the event that A_i dies in a year. Then

$$P(E_i) = p, (i = 1, 2, ..., n) \quad \text{and} \quad P(\overline{E}_i) = 1 - p.$$

The probability that none of n men $A_1, A_2, ..., A_n$ dies in a year

$$= P(\overline{E}_1 \cap \overline{E}_2 \cap ... \cap \overline{E}_n) = P(\overline{E}_1) P(\overline{E}_2) ... P(\overline{E}_n)$$

(By compound probability theorem)

$$= (1 - p)^n$$

$\therefore$ The probability that at least one of $A_1, A_2, ..., A_n$, dies in a year

$$= 1 - P(\overline{E}_1 \cap \overline{E}_2 \cap ... \cap \overline{E}_n) = 1 - (1 - p)^n$$

The probability that among n men, A_1 is the first to die is $1/n$ and since this event is independent of the event that at least one man dies in a year, required probability is

$$\frac{1}{n}\left[1 - (1 - p)^n\right]$$

Example 4·25. *The odds against Manager X settling the wage dispute with the workers are 8:6 and odds in favour of manager Y settling the same dispute are 14:16.*

(i) What is the chance that neither settles the dispute, if they both try, independently of each other?

(ii) What is the probability that the dispute will be settled?

Solution. Let A be the event that the manager X will settle the dispute and B be the event that the Manager Y will settle the dispute. Then clearly

$$P(\overline{A}) = \frac{8}{8+6} = \frac{4}{7} \quad \Rightarrow \quad P(A) = 1 - P(\overline{A}) = \frac{6}{14} = \frac{3}{7}$$

$$P(B) = \frac{14}{14+16} = \frac{7}{15} \quad \Rightarrow \quad P(\overline{B}) = 1 - P(B) = \frac{16}{14+16} = \frac{8}{15}$$

The required probability that neither settles the dispute is given by :

$$P(\overline{A} \cap \overline{B}) = P(\overline{A}) \times P(\overline{B}) = \frac{4}{7} \times \frac{8}{15} = \frac{32}{105}$$

[Since A and B are independent $\Rightarrow$ $\overline{A}$ and $\overline{B}$ are also independent]

(ii) The dispute will be settled if at least one of the managers X and Y settles the dispute. Hence the required probability is given by:

$$P(A \cup B) = \text{Prob. [At least one of X and Y settles the dispute]}$$
$$= 1 - \text{Prob. [None settles the dispute]}$$
$$= 1 - P(\overline{A} \cap \overline{B}) = 1 - \frac{32}{105} = \frac{73}{105}$$

Example 4·26. *The odds that person X speaks the truth are 3:2 and the odds that person Y speaks the truth are 5:3. In what percentage of cases are they likely to contradict each other on an identical point.*

Solution. Let us define the events:

$A : X$ speaks the truth, $\quad B : Y$ speaks the truth

Then $\overline{A}$ and $\overline{B}$ represent the complementary events that X and Y tell a lie respectively. We are given:

$$P(A) = \frac{3}{3+2} = \frac{3}{5} \quad \Rightarrow \quad P(\overline{A}) = 1 - \frac{3}{5} = \frac{2}{5}$$

and $$P(B) = \frac{5}{5+3} = \frac{5}{8} \quad \Rightarrow \quad P(\overline{B}) = 1 - \frac{5}{8} = \frac{3}{8}$$

The event E that X and Y contradict each other on an identical point can happen in the following mutually exclusive ways:

(i) X speaks the truth and Y tells a lie, *i.e.*, the event $A \cap \overline{B}$ happens,

(ii) X tells a lie and Y speaks the truth, *i.e.*, the event $\overline{A} \cap B$ happens.

Hence by addition theorem of probability the required probability is given by:

$$P(E) = P(i) + P(ii) = P(A \cap \overline{B}) + P(\overline{A} \cap B)$$

$$= P(A) \,.\, P(\overline{B}) + P(\overline{A}) \,.\, P(B),$$

[Since A and B are independent]

$$= \frac{3}{5} \times \frac{3}{8} + \frac{2}{5} \times \frac{5}{8} = \frac{19}{40} = 0{\cdot}475$$

Hence A and B are likely to contradict each other on an identical point in 47·5% of the cases.

Example 4·27. *A special dice is prepared such that the probabilities of throwing 1, 2, 3, 4, 5 and 6 points are :*

$$\frac{1-k}{6}, \frac{1+2k}{6}, \frac{1-k}{6}; \frac{1+k}{6}, \frac{1-2k}{6}, \text{ and } \frac{1+k}{6}$$

respectively. If two such dice are thrown, find the probability of getting a sum equal to 9. **[Delhi Univ. B.Sc. (Stat. Hons.), 1988]**

Solution. Let (x, y) denote the numbers obtained in a thrown of two dice, x denoting the number on the first dice and y denoting the number on the second dice. The sum $S = x+y = 9$, can be obtained in the following mutually disjoint ways:

(i) (3, 6), *(ii)* (6, 3), *(iii)* (4, 5), *(iv)* (5, 4)

Hence by addition theorem of probability:

$$P(S=9) = P(3,6) + P(6,3) + P(4,5) + P(5,4)$$
$$= P(x=3)\,P(y=6) + P(x=6)\,P(y=3) + P(x=4)\,P(y=5) + P(x=5)\,P(y=4),$$

since the number on one dice is independent of the number on the other dice.

$$\therefore \quad P(S=9) = \frac{(1-k)}{6}\cdot\frac{(1+k)}{6} + \frac{(1+k)}{6}\cdot\frac{(1-k)}{6} + \frac{(1+k)}{6}\cdot\frac{(1-2k)}{6} + \frac{(1-2k)}{6}\cdot\frac{(1+k)}{6}$$

$$= 2\left(\frac{1+k}{36}\right)[(1-k) + (1-2k)]$$

$$= \frac{1}{18}(1+k)(2-3k)$$

Example 4·28. *(a) A and B alternately cut a pack of cards and the pack is shuffled after each cut. If A starts and the game is continued until one cuts a diamond, what are the respective chances of A and B first cutting a diamond?*

(b) One shot is fired from each of the three guns. E_1, E_2, E_3 denote the events that the target is hit by the first, second and third gun respectively. If $P(E_1) = 0{\cdot}5$, $P(E_2) = 0{\cdot}6$ and $P(E_3) = 0{\cdot}8$ and E_1, E_2, E_3 are independent events, find the probability that (a) exactly one hit is registered, (b) at least two hits are registered.

Solution. *(a)* Let E_1 and E_2 denote the events of A and B cutting a diamond respectively. Then

$$P(E_1) = P(E_2) = \frac{13}{52} = \frac{1}{4} \quad\Rightarrow\quad P(\overline{E_1}) = P(\overline{E_2}) = \frac{3}{4}$$

If A starts the game, he can first cut the diamond in the following mutually exclusive ways:

(i) E_1 happens, *(ii)* $\overline{E_1} \cap \overline{E_2} \cap E_1$ happens, *(iii)* $\overline{E_1} \cap \overline{E_2} \cap \overline{E_1} \cap \overline{E_2} \cap E_1$ happens, and so on. Hence by addition theorem of probability, the probability 'p' that A first wins is given by

$$p = P(i) + P(ii) + P(iii) + \ldots\ldots$$

$$= P(E_1) + P(\overline{E_1} \cap \overline{E_2} \cap E_1) + P(\overline{E_1} \cap \overline{E_2} \cap \overline{E_1} \cap \overline{E_2} \cap E_1) + \ldots$$

$$= P(E_1) + P(\overline{E_1})\,P(\overline{E_2})\,P(E_1) + P(\overline{E_1})\cdot P(\overline{E_2})\,P(\overline{E_1})\,P(\overline{E_2})\,P(E_1) + \ldots$$

(By Compound Probability Theorem)

$$= \frac{1}{4} + \frac{3}{4}\cdot\frac{3}{4}\cdot\frac{1}{4} + \frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{1}{4} + \ldots\ldots$$

$$= \frac{\frac{1}{4}}{1-\frac{9}{16}} = \frac{4}{7}$$

The probability that B first cuts a diamond

$$= 1 - p = 1 - \frac{4}{7} = \frac{3}{7}$$

(b) We are given

$$P(\overline{E_1}) = 0{\cdot}5,\ P(\overline{E_2}) = 0{\cdot}4 \text{ and } P(\overline{E_3}) = 0{\cdot}2$$

(a) Exactly one hit can be registered in the following mutually exclusive ways: (i) $E_1 \cap \overline{E_2} \cap \overline{E_3}$ happens, (ii) $\overline{E_1} \cap E_2 \cap \overline{E_3}$ happens, (iii) $\overline{E_1} \cap \overline{E_2} \cap E_3$ happens.

Hence by addition probability theorem, the required probability 'p' is given by :

$$p = P(E_1 \cap \overline{E_2} \cap \overline{E_3}) + P(\overline{E_1} \cap E_2 \cap \overline{E_3}) + P(\overline{E_1} \cap \overline{E_2} \cap E_3)$$

$$= P(E_1)\, P(\overline{E_2})\, P(\overline{E_3}) + P(\overline{E_1})\, P(E_2)\, P(\overline{E_3}) + P(\overline{E_1})\, P(\overline{E_2})\, P(E_3)$$

(Since E_1, E_2 and E_3 are independent)

$$= 0{\cdot}5 \times 0{\cdot}4 \times 0{\cdot}2 + 0{\cdot}5 \times 0{\cdot}6 \times 0{\cdot}2 + 0{\cdot}5 \times 0{\cdot}4 \times 0{\cdot}8 = 0{\cdot}26.$$

(b) At least two hits can be registered in the following mutually exclusive ways:

(i) $E_1 \cap E_2 \cap \overline{E_3}$ happens (ii) $E_1 \cap \overline{E_2} \cap E_3$ happens, (iii) $\overline{E_1} \cap E_2 \cap E_3$ happens. (iv) $E_1 \cap E_2 \cap E_3$ happens.

Required probability

$$= P(E_1 \cap E_2 \cap \overline{E_3}) + P(E_1 \cap \overline{E_2} \cap E_3) + P(\overline{E_1} \cap E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$$

$$= 0{\cdot}5 \times 0{\cdot}6 \times 0{\cdot}2 + 0{\cdot}5 \times 0{\cdot}4 \times 0{\cdot}8 + 0{\cdot}5 \times 0{\cdot}6 \times 0{\cdot}8 + 0{\cdot}5 \times 0{\cdot}6 \times 0{\cdot}8$$

$$= 0{\cdot}06 + 0{\cdot}16 + 0{\cdot}24 + 0{\cdot}24 = 0{\cdot}70$$

Example 4·29. *Three groups of children contain respectively 3 girls and 1 boy, 2 girls and 2 boys, and 1 girl and 3 boys. One child is selected at random from each group. Show that the chance that the three selected consist of 1 girl and 2 boys is 13/32.* **[Madurai Univ. B.Sc.,1988; Nagpur Univ. B.Sc.,1991]**

Solution. The required event of getting 1 girl and 2 boys among the three selected children can materialise in the following three mutually disjoint cases:

Group No. →	I	II	III
(i)	Girl	Boy	Boy
(ii)	Boy	Girl	Boy
(iii)	Boy	Boy	Girl

Hence by addition theorem of probability,

$$\text{Required probability} = P\,(i) + P\,(ii) + P\,(iii) \qquad \ldots(*)$$

Since the probability of selecting a girl from the first group is 3/4, of selecting a boy from the second is 2/4, and of selecting a boy from the third group is 3/4, and since these three events of selecting children from three groups are independent of each other, by compound probability theorem, we have

$$P\,(i) = \frac{3}{4} \times \frac{2}{4} \times \frac{3}{4} = \frac{9}{32}$$

Similarly, we have

$$P\,(ii) = \frac{1}{4} \times \frac{2}{4} \times \frac{3}{4} = \frac{3}{32}$$

and $$P\,(iii) = \frac{1}{4} \times \frac{2}{4} \times \frac{1}{4} = \frac{1}{32}$$

Substituting in (*), we get

Required probability = $\frac{9}{32} + \frac{3}{32} + \frac{1}{32} = \frac{13}{32}$

EXERCISE 4 (b)

1. *(a)* Which function defines a probability space on $S = (e_1, e_2, e_3)$

(i) $P(e_1) = \frac{1}{4}$, $P(e_2) = \frac{1}{3}$, $P(e_3) = \frac{1}{2}$

(ii) $P(e_1) = \frac{2}{3}$, $P(e_2) = -\frac{1}{3}$, $P(e_3) = \frac{2}{3}$

(iii) $P(e_1) = \frac{1}{4}$, $P(e_2) = \frac{1}{3}$, $P(e_3) = \frac{1}{2}$, and

(iv) $P(e_1) = 0$, $P(e_2) = \frac{1}{3}$, $P(e_3) = \frac{2}{3}$

Ans. *(i)* No, *(ii)* No, *(iii)* No, and *(iv)* Yes

(b) Let $S = (e_1, e_2, e_3, e_4)$, and let P be a probability function on S.

(i) Find $P(e_1)$, if $P(e_2) = \frac{1}{3}, P(e_3) = \frac{1}{6}, P(e_4) = \frac{1}{9}$,

(ii) Find $P(e_1)$ and $P(e_2)$ if $P(e_3) = P(e_4) = \frac{1}{4}$ and $P(e_1) = 2P(e_2)$, and

(iii) Find $P(e_1)$ if $P[(e_2, e_3)] = \frac{2}{3}, P[(e_2, e_4)] = \frac{1}{2}$ and $P(e_2) = \frac{1}{3}$.

Ans. *(i)* $P(e_1) = \frac{7}{18}$, *(ii)* $P(e_1) = \frac{1}{3}$, $P(e_2) = \frac{1}{6}$, and *(iii)* $P(e_1) = \frac{1}{6}$

2. *(a)* With usual notations, prove that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Deduce a similar result for $P(A \cup B \cup C)$, where C is one more event.

(b) For any event : E_i, $P(E_i) = p_i$, $(i = 1, 2, 3)$; $P(E_i \cap E_j) = p_{ij}$, $(i, j = 1, 2, 3)$ and $P(E_1 \cap E_2 \cap E_3) = p_{123}$, find the probability that of the three events, *(i)* at least one, and *(ii)* exactly one happens.

(c) Discuss briefly the axiomatic approach to probability, illustrating by examples how it meets the deficiencies of the classical approach.

(d) If A and B are any two events, state the results giving

(i) $P(A \cup B)$ and *(ii)* $P(A \cap B)$.

A and B are mutually exclusive events and $P(A) = \frac{1}{2}, P(B) = \frac{1}{3}$. Find $P(A \cup B)$ and $P(A \cap B)$.

3. Let $S = \left\{1, \frac{1}{2}, \left(\frac{1}{2}\right)^2, \ldots, \left(\frac{1}{2}\right)^n\right\}$, be a classical event space and A, B be events given by

$$A = \left\{1, \frac{1}{2}\right\}, \quad B = \left\{\left(\frac{1}{2}\right)^k \mid k \text{ is an even positive integer}\right\}$$

Find $P(\bar{A} \cap \bar{B})$ **[Calcutta Univ. B.Sc. (Stat Hons.), 1986]**

4. What is a 'probability space'? State *(i)* the 'law of total probability' and *(ii)* Boole's inequality for events not necessarily mutually exclusive.

5. *(a)* Explain the following with examples:

(i) random experiment, *(ii)* an event, *(iii)* an event space. State the axioms of probability and explain their frequency interpretations.

A man forgets the last digit of a telephone number, and dials the last digit at random. What is the probability of calling no more than three wrong numbers?

(b) Define conditional probability and give its frequency interpretation. Show that conditional probabilities satisfy the axioms of probability.

6. Prove the following laws, in each case assuming the conditional probabilities being defined.

(a) $P(E \mid E) = 1$, *(b)* $P(\phi \mid F) = 0$

(c) If $E_1 \subseteq E_2$, then $E(E_1 \mid F) < P(E_2 \mid F)$

(d) $P(\bar{E} \mid F) = 1 - P(E \mid F)$

(e) $P(E_1 \cup E_2 \mid F) = P(E_1 \mid F) + P(E_2 \mid F) - P(P(E_1 \cap E_2 \mid F)$

(f) If $P(F) = 1$ then $P(E \mid F) = P(E)$

(g) $P(E - F) = P(E) - P(E \cap F)$

(h) If $P(F) > 0$, and E and F are mutully exclusive then $P(E \mid F) = 0$

(i) If $P(E \mid F) = P(E)$, then $P(E \mid \bar{F}) = P(E)$ and $P(\bar{E} \mid F) = P(\bar{E})$

7. *(a)* If $P(\bar{A}) = a$, $P(\bar{B}) = b$, then prove that $P(A \cap B) \geq 1 - a - b$.

(b) If $P(A) = \alpha$, $P(B) = \beta$, then prove that $P(A \mid B) \geq (\alpha + \beta - 1)/\beta$.

Hint. In each case use $P(A \cup B) \leq 1$

8. Prove or disprove:

(a) *(i)* If $P(A \mid B) \geq P(A)$, then $P(B \mid A) \geq P(B)$

(ii) If $P(A) = P(\bar{B})$, then $A = \bar{B}$.

[Delhi Univ. B.Sc. (Maths Hons.), 1988]

(b) If $P(A) = 0$, then $A = \phi$

[Delhi Univ. B.Sc. (Maths Hons.), 1990]

Ans. Wrong.

(c) For possible events A, B, C,

(i) If $P(A) > P(B)$, then $P(A \mid C) > P(B \mid C)$

(ii) If $P(A \mid C) \geq P(B \mid C)$ and $P(A \mid \bar{C}) \geq P(B \mid \bar{C})$, then $P(A) \geq P(B)$. **[Delhi Univ. B.Sc.(Maths Hons),1989]**

(d) If $P(A) = 0$, then $P(A \cap B) = 0$.

[Delhi Univ. B.Sc. (Maths Hons.), 1986]

(e) *(i)* If $P(A) = P(B) = p$, then $P(A \cap B) \leq p^2$

(ii) If $P(B \mid \bar{A}) = P(B \mid A)$, then A and B are independent.

[Delhi Univ. B.Sc. (Maths Hons.), 1990]

(f) If $P(A) > 0, P(B) > 0$ and $P(A \mid B) = P(B \mid A)$, then $P(A) = P(B)$.

9. (a) Let A and B be two events, neither of which has probability zero. Then if A and B are disjoint, A and B are independent.

[Delhi Univ. B.Sc.(Stat. Hons.), 1986]

(b) Under what conditions does the following equality hold?

$$P(A) = P(A \mid B) + P(A \mid \overline{B})$$

[Punjab Univ. B.Sc. (Maths Hons.), 1992]

Ans. $B = S$ or $\overline{B} = S$

10. (a) If A and B are two events and the probability $P(B) \neq 1$, prove that

$$P(A \mid \overline{B}) = \frac{[P(A) - P(A \cap B)]}{[1 - P(B)]}$$

where $\overline{B}$ denotes the event complementary to B and hence deduce that

$$P(A \cap B) \geq P(A) + P(B) - 1$$

[Delhi Univ. B.Sc. (Stat. Hons.), 1989]

Also show that $P(A) >$ or $< P(A \mid B)$ according as

$$P(A \mid \overline{B}) > \text{ or } < P(A).$$

[Sri Venkat. Univ. B.Sc. 1992 ; Karnatak Univ. B.Sc. 1991]

Hint. (i)

$$P(A \mid \overline{B}) = \frac{P(A \cap \overline{B})}{P(\overline{B})} = \frac{[P(A) - P(A \cap B)]}{[1 - P(B)]}$$

(ii) Since $P(A \mid \overline{B}) \leq 1,\ P(A) - P(A \cap B) \leq 1 - P(B)$

$\Rightarrow \quad P(A) + P(B) - 1 \leq P(A \cap B)$

(iii) $$\frac{P(A \mid \overline{B})}{P(A)} = \frac{P(\overline{B} \mid A)}{P(\overline{B})} = \frac{1 - P(B \mid A)}{1 - P(B)}$$

Now $\quad P(A \mid \overline{B}) > P(A)$ if $\{1 - P(B \mid A)\} > \{1 - P(B)\}$

i.e., if $\quad P(B \mid A) < P(B)$

i.e., if $\quad \dfrac{P(B \mid A)}{P(B)} < 1$

i.e., if $\quad \dfrac{P(A \mid B)}{P(A)} < 1$ i.e., if $P(A) > P(A \mid B)$

(b) If A and B are two mutually exclusive events show that

$$P(A \mid \overline{B}) = P(A)/[1 - P(B)]$$

[Delhi Univ. B.Sc. (Stat. Hons.), 1987]

(c) If A and B are two mutually exclusive events and $P(A \cup B) \neq 0$, then

$$P(A \mid A \cup B) = \frac{P(A)}{P(A) + P(B)}$$ **[Guahati Univ. B.Sc. 1991]**

(d) If A and B are two independent events show that

$$P(A \cup B) = 1 - P(\overline{A})\, P(\overline{B})$$

(e) If $\overline{A}$ denotes the non-occurrence of A, then prove that

$$P(\overline{A}_1 \cup \overline{A}_2 \cup \overline{A}_3) = 1 - P(A_1)\ P(A_2 \mid A_1)\ P(A_3 \mid A_1 \cap A_2)$$

[Agra Univ. B.Sc., 1987]

11. If A, B and C are three arbitrary events and

$$S_1 = P(A) + P(B) + P(C)$$
$$S_2 = P(A \cap B) + P(B \cap C) + P(C \cap A)$$
$$S_3 = P(A \cap B \cap C).$$

Prove that the probability that exactly one of the three events occurs is given by $S_1 - 2S_2 + 3S_3$.

12. (a) For the events $A_1, A_2, \ldots, A_n$ assuming

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} P(A_i), \text{ prove that}$$

(i) $P\left(\bigcap_{i=1}^{n} A_i\right) \geq 1 - \sum_{i=1}^{n} P(\overline{A}_i)$ and that

(ii) $P\left(\bigcap_{i=1}^{n} A_i\right) \geq \sum_{i=1}^{n} P(A_i) - (n-1)$

[Sardar Patel Univ. B.Sc. Nov.1992]

(b) Let A, B and C denote events. If $P(A \mid C) \geq P(B \mid C)$ and $P(A \mid \overline{C}) \geq P(B \mid \overline{C})$, then show that $P(A) \geq P(B)$.

[Calcutta Univ. B.Sc. (Maths Hons.), 1992]

13. (a) If A and B are independent events defined on a given probability space $(\Omega, \mathcal{A}, P(.))$, then prove that A and $\overline{B}$ are independent, $\overline{A}$ and $\overline{B}$ are independent. **[Delhi Univ. B.Sc. (Maths Hons.), 1988]**

(b) A, B and C are three events such that A and B are independent, $P(C) = 0$. Show that A, B and C are independent.

(c) An event A is known to be independent of the events B, $B \cup C$ and $B \cap C$. Show that it is also independent of C. **[Nagpur Univ. B.Sc.1992]**

(d) Show that if an event C is independent of two mutually exclusive events A and B, then C is also independent of $A \cup B$.

(e) The outcome of an experiment is equally likely to be one of the four points in three-dimensional space with rectangular coordinates $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and $(1, 1, 1)$. Let E, F and G be the events : x-coordinate$=1$, y-coordinate$=1$ and z-coordinate$=1$, respectively. Check if the events E, F and G are independent. **(Calcutta Univ. B.Sc., 1988)**

14. Explain what is meant by "Probability Space". You fire at a target with each of the three guns; A, B and C denote respectively the event — hit the target with the first, second and third gun. Assuming that the events are independent and have probabilities $P(A) = a$, $P(B) = b$ and $P(C) = c$, express in terms of A, B and C the following events:

(i) You will not hit the target at all.

(*ii*) You will hit the target at least twice. Find also the probabilities of these events. **[Sardar Patel Univ. B.Sc., 1990)**

15. (*a*) Suppose A and B are any two events and that $P(A) = p_1$, $P(B) = p_2$ and $P(A \cap B) = p_3$. Show that the formula of each of the following probabilities in terms of p_1, p_2 and p_3 can be expressed as follows :

(*i*) $P(\overline{A} \cup \overline{B}) = 1 - p_2$ (*ii*) $P(\overline{A} \cap \overline{B}) = 1 - p_1 - p_2 + p_3$

(*iii*) $P(A \cap \overline{B}) = p_1 - p_3$ (*iv*) $P(\overline{A} \cap B) = p_2 - p_3$

(*v*) $P(\overline{A \cap B}) = 1 - p_3$ (*vi*) $P(\overline{A} \cup B) = 1 - p_1 + p_3$

(*vii*) $P(\overline{A \cup B}) = 1 - p_1 - p_2 + p_3$ (*viii*) $P[\overline{A} \cap (A \cup B)] = p_2 - p_3$

(*ix*) $P[A \cup (\overline{A} \cap B)] = p_1 + p_2 - p_3$

(*x*) $P(A \mid B) = \dfrac{p_3}{p_2}$ and $P(B \mid A) = \dfrac{p_3}{p_1}$

(*xi*) $P(\overline{A} \mid \overline{B}) = \dfrac{1 - p_1 - p_2 + p_3}{1 - p_2}$ and $P(\overline{B} \mid \overline{A}) = \dfrac{1 - p_1 - p_2 + p_3}{1 - p_1}$

[Allahabad Univ. B.Sc. (Stat.), 1991]

(*b*) If $P(A) = 1/3$, $P(B) = 3/4$ and $P(A \cup B) = 11/12$, find $P(A \mid B)$ and $P(B \mid A)$.

(*c*) Let $P(A) = p$, $P(A \mid B) = q$, $P(B \mid A) = r$. Find the relation between the numbers p, q and r such that $\overline{A}$ and $\overline{B}$ are mutually exclusive.

[Delhi Univ. B.Sc. (Maths Hons.), 1985]

Hint. $P(AB) = P(A)\, P(B \mid A) = P(B) \,.\, P(A \mid B)$

$\Rightarrow \quad P(AB) = pr = P(B).\, q \quad \Rightarrow \quad P(B) = pr/q$

If $\overline{A}$ and $\overline{B}$ are mutually disjoint, then $P(\overline{A} \cap \overline{B}) = 0$.

$\Rightarrow \quad 1 - P(A \cup B) = 0 \Rightarrow 1 - [p + (pr/q) - pr] = 0$

16. (*a*) In terms of probabilities, $p_1 = P(A)$, $p_2 = P(B)$ and $p_3 = P(A \cap B)$;

Express (*i*) $P(A \cup B)$, (*ii*) $P(A \mid B)$, (*iii*) $P(\overline{A} \cap B)$ under the condition that (*i*) A and B are mutually exclusive, (*ii*) A and B are mutually independent.

(*b*) Let A and B be the possible outcomes of an experiment and suppose

$P(A) = 0{\cdot}4$, $P(A \cup B) = 0{\cdot}7$ and $P(B) = p$

(*i*) For what choice of p are A and B mutually exclusive ?

(*ii*) For what choice of p are A and B independent ?

[Aligarh Univ. B.Sc., 1988 ; Guwahati Univ. B.Sc., 1991]

Ans. (*i*) 0·3, (*ii*) 0·5

(*c*) Let A_1, A_2, A_3, A_4 be four independent events for which $P(A_1) = p$, $p(A_2) = q$, $P(A_3) = r$ and $P(A_4) = s$. Find the probability that

(*i*) at least one of the events occurs, (*ii*) exactly two of the events occur, and (*iii*) at most three of the events occur. **[Civil Services (Main), 1985]**

17. (*a*) Two six-faced unbiased dice are thrown. Find the probability that the sum of the numbers shown is 7 or their product is 12.

Ans. 2/9

(b) Defects are classifed as A, B or C, and the following probabilities have been determined from available production data :

$P(A) = 0{\cdot}20, P(B) = 0{\cdot}16, P(C) = 0{\cdot}14, P(A \cap B) = 0{\cdot}08, P(A \cap C) = 0{\cdot}05,$ $P(B \cap C) = 0{\cdot}04,$ and $P(A \cap B \cap C) = 0{\cdot}02.$

What is the probability that a randomly selected item of product will exhibit at least one type of defect ? What is the probability that it exhibits both A and B defects but is free from type C defect ? **[Bombay Univ. B.Sc., 1991]**

(c) A language class has only three students A, B, C and they independently attend the class. The probabilities of attendance of A, B and C on any given day are $1/2, 2/3$ and $3/4$ respectively. Find the probability that the total number of attendances in two consecutive days is exactly three.

[Lucknow Univ. B.Sc. 1990; Calcutta Univ. B.Sc.(Maths Hons.), 1986]

18. *(a)* Cards are drawn one by one from a full deck. What is the probability that exactly 10 cards will precede the first ace. **[Delhi Univ. B.Sc.,1988]**

Ans. $\left(\frac{48}{52} \times \frac{47}{51} \times \frac{46}{50} \times \ldots \times \frac{39}{43}\right) \times \frac{4}{42} = \frac{164}{4165}$

(b) Each of two persons tosses three fair coins. What is the probability that they obtain the same number of heads.

Ans. $\left(\frac{1}{8}\right)^2 + \left(\frac{3}{8}\right)^2 + \left(\frac{3}{8}\right)^2 + \left(\frac{1}{8}\right)^2 = \frac{5}{16}.$

19. *(a)* Given that A, B and C are mutually exclusive events, explain why each of the following is not a permissible assignment of probabilities.

(i) $P(A) = 0{\cdot}24, \quad P(B) = 0{\cdot}4 \quad$ and $\quad P(A \cup C) = 0{\cdot}2,$
(ii) $P(A) = 0{\cdot}7, \quad P(B) = 0{\cdot}1 \quad$ and $\quad P(B \cap C) = 0{\cdot}3$
(iii) $P(A) = 0{\cdot}6, \quad P(A \cap \overline{B}) = 0{\cdot}5$

(b) Prove that for n arbitrary independent events $A_1, A_2, \ldots, A_n$

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n) + P(\overline{A_1})\, P(\overline{A_2}) \ldots P(\overline{A_n}) = 1.$$

(c) $A_1, A_2, \ldots, A_n$ are n independent events with

$$P(A_i) = 1 - \frac{1}{\alpha^i}, \quad i = 1, 2, \ldots, n.$$

Find the value of $P(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n)$. **(Nagpur Univ. B.Sc., 1987)**

Ans. $1 - \frac{1}{\alpha^{n(n+1)/2}}$

(d) Suppose the events $A_1, A_2, \ldots, A_n$ are independent and that

$P(A_i) = \frac{1}{i+1}$ for $1 \le i \le n$. Find the probability that none of the n events occurs, justifying each step in your calculations.

Ans. $1/(n+1)$

20. *(a)* A denotes getting a heart card, B denotes getting a face card (King, Queem or Jack), $\overline{A}$ and $\overline{B}$ denote the complementary events. A card is drawn at random from a full deck. Compute the following probabilities.

(*i*) $P(A)$, (*ii*) $P(A \cap \bar{B})$, (*iii*) $P(A \cup B)$, (*iv*) $P(A \cap B)$,

(*v*) $P(\bar{A} \cup B)$.

Assume natural assignment of probabilities.

Ans. (*i*) 1/4, (*ii*) 5/26, (*iii*) 11/26, (*iv*) 3/5, (*v*) 21/26.

(*b*) A town has two doctors X and Y operating independently. If the probability that doctor X is available is 0·9 and that for Y is 0·8, what is the probability that at least one doctor is available when needed? **[Gorakhpur Univ. B.Sc., 1988]**

Ans. 0·98

21. (*a*) The odds that a book will be favourably reviewed by 3 independent critics are 5 to 2, 4 to 3 and 3 to 4 respectively. What is the probability that, of the three reviews, a majority will be favourable? **[Gauhati Univ. BSc., 1987]**

Ans. 209/343.

(*b*) A, B and C are independent witnesses of an event which is known to have occurred. A speaks the truth three times out of four, B four times out of five and C five times out of six. What is the probability that the occurrence will be reported truthfully by majority of three witnesses?

Ans. 31/60.

(*c*) A man seeks advice regarding one of two possible courses of action from three advisers who arrived at their recommendations independently. He follows the recommendation of the majority. The probability that the individual advisers are wrong are 0·1, 0·05 and 0·05 respectively. What is the probability that the man takes incorrect advise ? **[Gujarat Univ. B.Sc., 1987]**

22. (*a*) The odds against a certain event are 5 to 2 and odds in favour of another (independent) event are 6 to 5. Find the chance that at least one of the events will happen. **(Madras Univ. B.Sc.,1987)**

Ans. 52/77.

(*b*) A person takes four tests in succession. The probability of his passing the first test is p, that of his passing each succeeding test is p or $p/2$ according as he passes or fails the preceding one. He qualifies provided he passes at least three tests. What is his chance of qualifying. **[Gauhati Univ. B.Sc. (Hons.) 1988]**

23. (*a*) The probability that a 50-years old man will be alive at 60 is 0·83 and the probability that a 45-years old woman will be alive at 55 is 0·87. What is the probability that a man who is 50 and his wife who is 45 will both be alive 10 years hence?

Ans. 0·7221.

(*b*) It is 8:5 against a husband who is 55 years old living till he is 75 and 4:3 against his wife who is now 48, living till she is 68. Find the probability that (*i*) the couple will be alive 20 years hence, and (*ii*) at least one of them will be alive 20 years hence.

Ans (*i*) 15/91, (*ii*) 59/91.

(c) A husband and wife appear in an interview for two vacancies in the same post. The probability of husband's selection is 1/7 and that of wife's selection is 1/5. What is the probability that only one of them will be selected ?

Ans. 2/7 **[Delhi Univ. B.Sc., 1986]**

24. *(a)* The chances of winning of two race-horses are 1/3 and 1/6 respectively. What is the probability that at least one will win when the horses are running *(a)* in different races, and *(b)* in the same race?

Ans. *(a)* 8/18 *(b)* 1/2

(b) A problem in statistics is given to three students whose chances of solving it are 1/2, 1/3 and 1/4. What is the probability that the problem will be solved?

Ans. 3/4 **[Meerut Univ. B.Sc., 1990]**

25. *(a)* Ten pairs of shoes are in a closet. Four shoes are selected at random. Find the probability that there will be at least one pair among the four shoes selected?

Ans. $1-\dfrac{{}^{10}C_4 \times 2^4}{{}^{20}C_4}$

(b) From 100 tickets numbered 1, 2, ..., 100 four are drawn at random. What is the probability that 3 of them will bear number from 1 to 20 and the fourth will bear any number from 21 to 100 ?

Ans. $\dfrac{{}^{20}C_3 \times {}^{80}C_1}{{}^{100}C_4}$

26. A six faced die is so biased that it is twice as likely to show an even number as an odd number when thrown. It is thrown twice. What is the probability that the sum of the two numbers thrown is odd?

Ans. 4/9

27. From a group of 8 children, 5 boys and 3 girls, three children are selected at random. Calculate the probabilities that selected group contains *(i)* no girl, *(ii)* only one girl, *(iii)* one particular girl, *(iv)* at least one girl, and *(v)* more girls than boys.

Ans. *(i)* 5/28, *(ii)* 15/28, *(iii)* 5/28, *(iv)* 23/28, *(v)* 2/7.

28. If three persons, selected at random, are stopped on a street, what are the probabilities that :

(a) all were born on a Friday;

(b) two were born on a Friday and the other on a Tuesday;

(c) none was born on a Monday.

Ans. *(a)* 1/343, *(b)* 3/343, *(c)* 216/343.

29. *(a)* A and B toss a coin alternately on the understanding that the first who obtains the head wins. If A starts, show that their respective chances of winning are 2/3 and 1/3.

(b) A, B and C, in order, toss a coin. The first one who throws a head wins. If A starts, find their respectivre chances of winning. (Assume that the game may

continue indefinitely.)

Ans. 4/7, 2/7, 1/7.

(c) A man alternately tosses a coin and throws a die, beginning with coin. What is the probability that he will get a head before he gets a '5 or 6' on die?

Ans. 3/4.

30. *(a)* Two ordinary six-sided dice are tossed.

(i) What is the probability that both the dice show the number 5.

(ii) What is the probability that both the dice show the same number.

(iii) Given that the sum of two numbers shown is 8, find the conditional probability that the number noted on the first dice is larger than the number noted on the second dice.

(b) Six dice are thrown simultaneously. What is the probability that all will show different faces?

31. *(a)* A bag contains 10 balls, two of which are red, three blue and five black. Three balls are drawn at random from the bag, that is every ball has an equal chance of being included in the three. What is the probability that

(i) the three balls are of different colours,

(ii) two balls are of the same colour, and

(iii) the balls are all of the same colour?

Ans. *(i)* 30/120, *(ii)* 79/120, *(iii)* 11/120.

(b) A is one of six horses entered for a race and is to be ridden by one of the two jockeys B and C. It is 2 to 1 that B rides A, in which case all the horses are equally likely to win, with rider C, A's chance is trebled.

(i) Find the probability that A wins.

(ii) What are odds against A's winning?

[Shivaji Univ. B.Sc. (Stat. Hons.), 1992]

Hint. Probability of A's winning

$$= P(B \text{ rides } A \text{ and } A \text{ wins}) + P(C \text{ rides } A \text{ and } A \text{ wins})$$

$$= \frac{2}{3} \times \frac{1}{6} + \frac{1}{3} \times \frac{3}{6} = \frac{5}{18}$$

$\therefore$ Probability of A's losing $= 1 - 5/18 = 13/18$.

Hence odds against A's winning are : $13/18 : 5/18$, *i.e.*, $13 : 5$.

32. *(a)* Two-third of the students in a class are boys and the rest girls. It is known that the probability of a girl getting a first class is 0·25 and that of boy getting a first class is 0·28. Find the probability that a student chosen at random will get first class marks in the subject.

Ans. 0·27

(b) You need four eggs to make omelettes for breakfast. You find a dozen eggs in the refrigerator but do not realise that two of these are rotten. What is the probability that of the four eggs you choose at random

(i) none is rotten,

(ii) exactly one is rotten?

Ans. *(i)* 625/1296 ; *(ii)* 500/1296.

(c) The probability of occurrence of an event A is 0·7, the probability of non-occurrence of another event B is 0·5 and that of at least one of A or B not occurring is 0·6. Find the probability that at least one of A or B occurs.

[Mysore Univ. B.Sc., 1991]

33. *(a)* The odds against A solving a certain problem are 4 to 3 and odds in favour of B solving the same problem are 7 to 5. What is the probability that the problem is solved if they both try independently? **[Gujarat Univ.B.Sc., 1987]**

Ans. 16/21

(b) A certain drug manufactured by a company is tested chemically for its toxic nature. Let the event 'the drug is toxic' be denoted by E and the event 'the chemical test reveals that the drug is toxic' be denoted by F. Let $P(E) = \theta$, $P(F \mid E) = P(\bar{F} \mid \bar{E}) = 1 - \theta$. Then show that probability that the drug is not toxic given that the chemical test reveals that it is toxic is free from θ.

Ans. 1/2 **[M.S. Baroda Univ. B.Sc., 1992]**

34. A bag contains 6 white and 9 black balls. Four balls are drawn at a time. Find the probability for the first draw to give 4 white and the second draw to give 4 black balls in each of the following cases :

(i) The balls are replaced before the second draw.

(ii) The balls are not replaced before the second draw.

[Jammu Univ. B.Sc., 1992]

Ans. *(i)* $\dfrac{{}^6C_4}{{}^{15}C_4} \times \dfrac{{}^9C_4}{{}^{15}C_4}$ *(ii)* $\dfrac{{}^6C_4}{{}^{15}C_4} \times \dfrac{{}^9C_4}{{}^{11}C_4}$

35. The chances that doctor A will diagnose a disease X correctly is 60%. The chances that a patient will die by his treatment after correct diagnosis is 40% and the chance of death by wrong diagnosis is 70%. A patient of doctor A, who had disease X, died. What is the chance that his disease was diagnosed correctly?

Hint. Let us define the following events:

E_1 : Disease X is diagnosed correctly by doctor A.

E_2 : A patient (of doctor A) who has disease X dies.

Then we are given :

$$P(E_1) = 0{\cdot}6 \quad \Rightarrow \quad P(\bar{E}_1) = 1 - 0{\cdot}6 = 0{\cdot}4$$

$$\text{and} \quad P(E_2 \mid E_1) = 0{\cdot}4 \quad \text{and} \quad P(E_2 \mid \bar{E}_1) = 0{\cdot}7$$

$$\text{We want } P(E_1 \mid E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} = \frac{P(E_1 \cap E_2)}{P(E_1 \cap E_2) + P(\bar{E}_1 \cap E_2)} = \frac{6}{13}$$

36. The probability that at least 2 of 3 people A, B and C will survive for 10 years is 247/315. The probability that A alone will survive for 10 years is 4/105 and the probability that C alone will die within 10 years is 2/21. Assuming that the events of the survival of A, B and C can be regarded as independent, calculate the probability of surviving 10 years for each person.

Ans. 3/5, 5/7, 7/9.

37. A and B throw alternately a pair of unbiased dice, A beginning. A wins if he throws 7 before B throws 6, and B wins if he throws 6 before A throws 7. If A and B respectively denote the events that A wins and B wins the series, and a and b respectively denote the events that it is A's and B's turn to throw the dice, show that

(*i*) $P(A \mid a) = \dfrac{1}{6} + \dfrac{5}{6} P(A \mid b)$, (*ii*) $P(A \mid b) = \dfrac{31}{36} P(A \mid a)$,

(*iii*) $P(B \mid a) = \dfrac{5}{6} P(B \mid b)$, and (*iv*) $P(B \mid b) = \dfrac{5}{36} + \dfrac{13}{36} P(B \mid a)$,

Hence or otherwise, find $P(A \mid a)$ and $P(B \mid a)$. Also comment on the result that $P(A \mid a) + P(B \mid a) = 1$.

38. A bag contains an assortment of blue and red balls. If two balls are drawn at randon, the probability of drawing two red balls is five times the probability of drawing two blue balls. Furthermore, the probability of drawing one ball of each colour is six times the probabililty of drawing two blue balls. How many red and blue balls are there in the bag?

Hint. Let number of red and blue balls in the bag be r and b respectively. Then

$$p_1 = \text{Prob. of drawing two red balls} = \frac{r(r-1)}{(r+b)(r+b-1)}$$

$$p_2 = \text{Prob. of drawing two blue balls} = \frac{b(b-1)}{(r+b)(r+b-1)}$$

$$p_3 = \text{Prob. of drawing one red and one blue ball} = \left[\frac{2br}{(r+b)(r+b-1)}\right]$$

Now $p_1 = 5p_2$ and $p_3 = 6p_2$

$\therefore$ $r(r-1) = 5b(b-1)$ and $2br = 6b(b-1)$

Hence $b = 3$ and $r = 6$.

39. Three newspapers A, B and C are published in a certain city. It is estimated from a survey that 20% read A, 16% read B, 14% read C, 8% read A and B, 5% read A and C, 4% read B and C and 2% read all the three newspapers. What is the probability that a normally chosen person

(*i*) does not read any paper, (*ii*) does not read C

(*iii*) reads A but not B, (*iv*) reads only one of these papers, and

(*v*) reads only two of these papers.

Ans. (*i*) 0·65, (*ii*) 0·86, (*iii*) 0·12, (*iv*) 0·22, (*v*) 0·11.

40. (*a*) A die is thrown twice, the event space S consisting of the 36 possible pairs of outcomes (a,b) each assigned probability 1/36. Let A, B and C denote the following events :

$A = \{(a,b) \mid a \text{ is odd}\}$, $B = \{(a,b) \mid b \text{ is odd}\}$, $C = \{(a,b) \mid a+b \text{ is odd}\}$

Check whether A, B and C are independent or independent in pairs only.

[Calcutta Univ. B.Sc. Hons., 1985]

(b) Eight tickets numbered 111, 121, 122, 122, 211, 212, 212, 221 are placed in a hat and stirred. One of them is then drawn at random. Show that the event A : "the first digit on the ticket drawn will be 1", B : "the second digit on the ticket drawn will be 1," and C : "the third digit on the ticket drawn will be 1", are not pairwise independent although

$$P(A \cap B \cap C) = P(A)\, P(B)\, P(C)$$

41. *(a)* Four identical marbles marked 1, 2, 3 and 123 respectively are put in an urn and one is drawn at random. Let A_i, $(i = 1, 2, 3)$, denote the event that the number i appears on the drawn marble. Prove that the events A_1, A_2 and A_3 are pairwise independent but not mutually independent.

[Gauhati Univ. B.Sc. (Hons.), 1988]

Hint. $P(A_1) = \frac{1}{2} = P(A_2) = P(A_3)$; $P(A_1 A_2) = P(A_1 A_3) = P(A_2 A_3) = \frac{1}{4}$

$$P(A_1 A_2 A_3) = \frac{1}{4}.$$

(b) Two fair dice are thrown independently. Define the following events :

A : Even number on the first dice

B : Even number on the second dice.

C : Same number on both dice.

Discuss the independence of the events A, B and C.

(c) A die is of the shape of a regular tetrahedron whose faces bear the numbers 111, 112, 121, 122. A_1, A_2, A_3 are respectively the events that the first two, the last two and the extreme two digits are the same, when the die is tossed at random. Find whether or not the events A_1, A_2, A_3 are *(i)* pairwise independent, *(ii)* mutually (*i.e.* completely) independent. Determine $P(A_1 \mid A_2 A_3))$ and explain its value by by argument. **[Civil Services (Main), 1983]**

42. *(a)* For two events A and B we have the following probabilities:

$$P(A) = P(A \mid B) = \frac{1}{4} \text{ and } P(B \mid A) = \frac{1}{2}.$$

Check whether the following statements are true or false :

(i) A and B are mutually exclusive, *(ii)* A and B are independent, *(iii)* A is a subevent of B, and *(iv)* $P(\bar{A} \mid B) = \frac{3}{4}$

Ans. *(i)* False, *(ii)* True, *(iii)* False, and *(iv)* True.

(b) Consider two events A and B such that $P(A) = 1/4$, $P(B \mid A) = 1/2$, $P(A \mid B) = 1/4$. For each of the following statements, ascertain whether it is true or false :

(i) A is a sub-event of B, *(ii)* $P(\bar{A} \mid \bar{B}) = 3/4$,

(iii) $P(A \mid B) + P(A \mid \bar{B}) = 1$

43. *(a)* Let A and B be two events such that $P(A) = \frac{3}{4}$ and $P(B) = \frac{5}{8}$.

Show that

(i) $P(A \cup B) \geq \frac{3}{4}$, (ii) $\frac{3}{8} \leq P(A \cap B) \leq \frac{5}{8}$, and (iii) $\frac{1}{8} \leq P(A \cap \overline{B}) \leq \frac{3}{8}$.

[Coimbatore Univ. B.E., Nov. 1990; Delhi Univ. B.Sc.(Stat. Hons.),1986]

(b) Given two events A and B. If the odds against A are 2 to 1 and those in favour of $A \cup B$ are 3 to 1, show that

$$\frac{5}{12} \leq P(B) \leq \frac{3}{4}$$

Give an example in which $P(B) = 3/4$ and one in which $P(B) = 5/12$.

44. Let A and B be events, neither of which has probability zero. Prove or disprove the following events :

(i) If A and B are disjoint, A and B are independent,

(ii) If A and B are independent, A and B are disjoint.

45. *(a)* It is given that $P(A_1 \cup A_2) = \frac{5}{6}$, $P(A_1 \cap A_2) = \frac{1}{3}$ and $P(\overline{A}_2) = \frac{1}{2}$, where $P(\overline{A}_2)$ stands for the probability that A_2 does not happen. Determine $P(A_1)$ and $P(A_2)$.

Hence show that A_1 and A_2 are independent.

Ans. $P(A_1) = \frac{2}{3}$, $P(A_2) = \frac{1}{2}$

(b) A and B are events such that

$$P(A \cup B) = \frac{3}{4},\ P(A \cap B) = \frac{1}{4},\ \text{and}\ P(\overline{A}) = \frac{2}{3}.$$

Find *(i)* $P(A)$, *(ii)* $P(B)$ and *(iii)* $P(A \cap \overline{B})$.

(Madras Univ. B.E., 1989)

Ans. *(i)* 1/3, *(ii)* 2/3 *(iii)* 1/12.

46. A thief has a bunch of n keys, exactly one of which fits a lock. If the thief tries to open the lock by trying the keys at random, what is the probability that he requires exactly k attempts, if he rejects the keys already tried? Find the same probability if he does not reject the keys already tried.

(Aligarh Univ. B.Sc., 1991)

Ans. *(i)* $\frac{1}{n}$, *(ii)* $\left(\frac{n-1}{n}\right)^{k-1} \cdot \frac{1}{n}$

(b) There are M urns numbered 1 to M and M balls numbered 1 to M. The balls are inserted randomly in the urns with one ball in each urn. If a ball is put into the urn bearing the same number as the ball, a match is said to have occurred. Find the probability that no match has occurred. **[Civil Services (Main), 1984]**

Hint. See Example 4·54 page 4·97.

47. If n letters are placed at random in n correctly addressed envelopes, find the probability that

(i) none of the letters is placed in the correct envelope,

(ii) At least one letter goes to the correct envelope,
(iii) All letters go to the correct envelopes.

[Delhi Univ. B.Sc. (Stat Hons.), 1987, 1984]

48. An urn contains n white and m black balls, a second urn contains N white and M black balls. A ball is randomly transferred from the first to the second urn and then from the second to the first urn. If a ball is now selected randomly from the first urn, prove that the probability that it is white is

$$\frac{n}{n+m}+\frac{mN-nM}{(n+m)^2\,(N+M+1)}$$

[Delhi Univ. B.Sc. (Stat.Hons.) 1986]

Hint. Let us define the following events :

B_i : Drawing of a black ball from the ith urn, $i = 1, 2$.

W_i : Drawing of a white ball from the ith urn, $i = 1, 2$.

The four distinct possibilities for the first two exchanges are $B_1 W_2, B_1 B_2, W_1 B_2, W_1 W_2$. Hence if E denotes the event of drawing a white ball from the first urn after the exchanges, then

$$P(E) = P(B_1 W_2 E) + P(B_1 B_2 E) + P(W_1 B_2 E) + P(W_1 W_2 E) \qquad \ldots(*)$$

We have :

$$P(B_1 W_2 E) = P(B_1)\,.\,P(W_2 \mid B_1)\,P(E \mid B_1 W_2) = \frac{m}{m+n} \times \frac{N}{M+N+1} \times \frac{n+1}{m+n}$$

$$P(B_1 B_2 E) = P(B_1)\,.\,P(B_2 \mid B_1)\,.\,P(E \mid B_1 B_2) = \frac{m}{m+n} \times \frac{M+1}{M+N+1} \times \frac{n}{m+n}$$

$$P(W_1 B_2 E) = P(W_1)\,.\,P(B_2 \mid W_1)\,.\,P(E \mid W_1 B_2) = \frac{n}{m+n} \times \frac{M}{M+N+1} \times \frac{n-1}{m+n}$$

$$P(W_1 W_2 E) = P(W_1)\,.\,P(W_2 \mid W_1)\,.\,P(E \mid W_1 W_2) = \frac{n}{m+n} \times \frac{N+1}{M+N+1} \times \frac{n}{m+n}$$

Substituting in (*) and simplifying we get the result.

49. A particular machine is prone to three similar types of faults A_1, A_2 and A_3. Past records on breakdowns of the machine show the following : the probability of a breakdown (*i.e.*, at least one fault) is 0·1; for each i, the probability that fault A_i occurs and the others do not is 0·02 ; for each pair i, j the probability that A_i and A_j occur but the third fault does not is 0·012. Determine the probabilities of

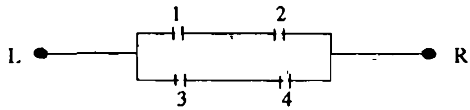
(a) the fault of type A_1 occurring irrespective of whether the other faults occur or not,

(b) a fault of type A_1 given that A_2 has occurred,

(c) faults of type A_1 and A_2 given that A_3 has occurred.

[London U. B.Sc. 1976]

50. The probability of the closing of each relay of the circuit shown below is given by p. If all the relays function independently, what is the probability that a circuit exists between the terminals L and R?



Ans. $p^2(2-p^2)$.

4·9. Bayes Theorem. *If $E_1, E_2, ..., E_n$ are mutually disjoint events with $P(E_i) \neq 0$, $(i = 1, 2, \ldots, n)$ then for any orbitrary event A which is a subset of $\bigcup_{i=1}^{n} E_i$ such that $P(A) > 0$, we have*

$$P(E_i \mid A) = \frac{P(E_i)\, P(A \mid E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A \mid E_i)}, \quad i = 1, 2, ..., n. \qquad ...(4\cdot12)$$

Proof. Since $A \subset \bigcup_{i=1}^{n} E_i$, we have

$$A = A \cap \left(\bigcup_{i=1}^{n} E_i\right) = \bigcup_{i=1}^{n} (A \cap E_i) \qquad \text{[By distributive law]}$$

Since $(A \cap E_i) \subset E_i$, $(i = 1, 2, ..., n)$ are mutually disjoint events, we have by addition theorem of probability (or Axiom 3 of probability)

$$P(A) = P\left[\bigcup_{i=1}^{n} (A \cap E_i)\right] = \sum_{i=1}^{n} P(A \cap E_i) = \sum_{i=1}^{n} P(E_i)\, P(A \mid E_i), \qquad ...(*)$$

by compound theorem of probability.

Also we have

$$P(A \cap E_i) = P(A)\, P(E_i \mid A)$$

$$P(E_i \mid A) = \frac{P(A \cap E_i)}{P(A)} = \frac{P(E_i)\, P(A \mid E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A \mid E_i)} \qquad \text{[From (*)]}$$

Remarks. 1. The probabilities $P(E_1), P(E_2), ..., P(E_n)$ are termed as the '*a priori probabilities*' because they exist before we gain any information from the experiment itself.

2. The probabilities $P(A \mid E_i)$, $i = 1, 2, ..., n$ are called '*likelihoods*' because they indicate how likely the event A under consideration is to occur, given each and every *a priori* probability.

3. The probabilities $P(E_i \mid A)$, $i = 1, 2, ..., n$ are called '*posterior probabilities*' because they are determined after the results of the experiment are known.

4. From (*) we get the following important result:

"If the events $E_1, E_2, ..., E_n$ constitute a partition of the sample space S and $P(E_i) \neq 0$, $i = 1, 2, \ldots, n$, then for any event A in S we have

$$P(A) = \sum_{i=1}^{n} P(A \cap E_i) = \sum_{i=1}^{n} P(E_i)\, P(A \mid E_i) \qquad \ldots(4\cdot 12\ a)$$

Cor. *(Bayes theorem for future events)*

The probability of the materialisation of another event C, given

$$P(C \mid A \cap E_1),\ P(C \mid A \cap E_2),\ \ldots,\ P(C \mid A \cap E_n) \quad \textit{is}$$

$$P(C \mid A) = \frac{\sum_{i=1}^{n} P(E_i)\, P(A \mid E_i)\, P(C \mid E_i \cap A)}{\sum_{i=1}^{n} P(E_i)\, P(A \mid E_i)} \qquad \ldots(4\cdot 12\ b)$$

Proof. Since the occurrence of event A implies the occurrence of one and only one of the events $E_1, E_2, \ldots, E_n$, the event C (granted that A has occurred) can occu in the following mutually exclusive ways:

$$C \cap E_1,\ C \cap E_2,\ \ldots,\ C \cap E_n$$

i.e., $\qquad C = (C \cap E_1) \cup (C \cap E_2) \cup \ldots \cup (C \cap E_n)$

$\Rightarrow \qquad C \mid A = [(C \cap E_1) \mid A] \cup [(C \cap E_2) \mid A] \cup \ldots \cup [(C \cap E_n) \mid A]$

$\therefore \quad P(C \mid A) = P[(C \cap E_1) \mid A] + P[(C \cap E_2) \mid A] + \ldots + P[(C \cap E_n) \mid A]$

$$= \sum_{i=1}^{n} P[(C \cap E_i) \mid A]$$

$$= \sum_{i=1}^{n} P(E_i \mid A)\ P[C \mid (E_i \cap A)]$$

Substituting the value of $P(E_i \mid A)$ from (*), we get

$$P(C \mid A) = \frac{\sum_{i=1}^{n} P(E_i)\, P(A \mid E_i)\ P(C \mid E_i \cap A)}{\sum_{i=1}^{n} P(E_i)\, P(A \mid E_i)}$$

Remark. It may happen that the materialisation of the event E_i makes C independent of A, then we have

$$P(C \mid E_i \cap A) = P(C \mid E_i),$$

and the above formula reduces to

$$P(C \mid A) = \frac{\sum_{i=1}^{n} P(E_i)\, P(A \mid E_i)\ P(C \mid E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A \mid E_i)} \qquad \ldots(4\cdot 12\ c)$$

The event C can be considered in regard to A, as Future Event.

Example 4·30. *In 1989 there were three candidates for the position of principal – Mr. Chatterji, Mr. Ayangar and Dr. Singh – whose chances of getting the appointment are in the proportion 4:2:3 respectively. The probability that Mr. Chatterji if selected would introduce co-education in the college is 0·3. The probabilities of Mr. Ayangar and Dr. Singh doing the same are respectively 0·5 and 0·8. What is the probability that there was co-education in the college in 1990?*

[Delhi Univ. B.Sc.(Stat. Hons.), 1992; Gorakhpur Univ. B.Sc., 1992]

Solution. Let the events and probabilities be defined as follows:

A : Introduction of co-education

E_1 : Mr. Chatterji is selected as principal

E_2 : Mr. Ayangar is selected as principal

E_3 : Dr. Singh is selected as principal.

Then

$$P(E_1)=\frac{4}{9},\quad P(E_2)=\frac{2}{9}\quad \text{and}\quad P(E_3)=\frac{3}{9}$$

$$P(A \mid E_1)=\frac{3}{10},\quad P(A \mid E_2)=\frac{5}{10}\quad \text{and}\quad P(A \mid E_3)=\frac{8}{10}$$

$$\begin{aligned}\therefore\quad P(A) &= P[(A\cap E_1)\cup(A\cap E_2)\cup(A\cap E_3)]\\ &= P(A\cap E_1)+P(A\cap E_2)+P(A\cap E_3)\\ &= P(E_1)P(A \mid E_1)+P(E_2)P(A \mid E_2)+P(E_3)P(A \mid E_3)\\ &= \frac{4}{9}\cdot\frac{3}{10}+\frac{2}{9}\cdot\frac{5}{10}+\frac{3}{9}\cdot\frac{8}{10}=\frac{23}{45}\end{aligned}$$

Example. 4·31. *The contents of urns I, II and III are as follows:*

1 white, 2 black and 3 red balls,

2 white, 1 black and 1 red balls, and

4 white, 5 black and 3 red balls.

One urn is chosen at random and two balls drawn. They happen to be white and red. What is the probability that they come from urns I, II or III ?

[Delhi Univ. B.Sc. (Stat. Hons.), 1988]

Solution. Let E_1, E_2, and E_3 denote the events that the urn I, II and III is chosen, respectively, and let A be the event that the two balls taken from the selected urn are white and red. Then

$$P(E_1)=P(E_2)=P(E_3)=\frac{1}{3}$$

$$P(A \mid E_1)=\frac{1\times 3}{{}^6C_2}=\frac{1}{5},\quad P(A \mid E_2)=\frac{2\times 1}{{}^4C_2}=\frac{1}{3},$$

and $$P(A \mid E_3)=\frac{4\times 3}{{}^{12}C_2}=\frac{2}{11}$$

Hence

$$P(E_2 \mid A) = \frac{P(E_2)\ P(A \mid E_2)}{\sum_{i=1}^{3} P(E_i)\, P(A \mid E_i)}$$

$$= \frac{\frac{1}{3} \times \frac{1}{3}}{\frac{1}{3} \times \frac{1}{5} + \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{11}} = \frac{55}{118}$$

Similarly

$$P(E_3 \mid A) = \frac{\frac{1}{3} \times \frac{2}{11}}{\frac{1}{3} \times \frac{1}{5} + \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{11}} = \frac{30}{118}$$

$$\therefore \quad P(E_1 \mid A) = 1 - \frac{55}{118} - \frac{30}{118} = \frac{33}{118}$$

Example. 4·32. *In answering a question on a multiple choice test a student either knows the answer or he guesses. Let p be the probability that he knows the answer and 1–p the probability that he guesses. Assume that a student who guesses at the answer will be correct with probability 1/5, where 5 is the number of multiple-choice alternatives. What is the conditional probability that a student knew the answer to a question given that he answered it correctly?*

[Delhi Univ. B.Sc. (Maths Hons.), 1985]

Solution. Let us define the following events:

E_1 : The student knew the right answer.

E_2 : The student guesses the right answer.

A : The student gets the right answer.

Then we are given

$P(E_1) = p, \quad P(E_2) = 1 - p, \quad P(A \mid E_2) = 1/5$

$P(A \mid E_1) = P$ [student gets the right answer given that he knew the right answer] $= 1$

We want $P(E_1 \mid A)$.

Using Bayes' rule, we get :

$$P(E_1 \mid A) = \frac{P(E_1)\, P(A \mid E_1)}{P(E_1)\, P(A \mid E_1) + P(E_2)\, P(A \mid E_2)} = \frac{p \times 1}{p \times 1 + (1-p) \times \frac{1}{5}} = \frac{5p}{4p+1}$$

Example 4·33. *In a bolt factory machines A, B and C manufacture respectively 25%, 35% and 40% of the total. Of their output 5, 4, 2 per cent are defective bolts. A bolt is drawn at random from the product and is found to be defective. What are the probabilities that it was manufactured by machines A, B and C?*

Solution. Let E_1, E_2 and E_3 denote the events that a bolt selected at random is manufactured by the machines A, B and C respectively and let E denote the event of its being defective. Then we have

$$P(E_1) = 0{\cdot}25,\ P(E_2) = 0{\cdot}35,\ P(E_3) = 0{\cdot}40$$

The probability of drawing a defective bolt manufactured by machine A is $P(E \mid E_1) = 0{\cdot}05$.

Similarly, we have

$$P(E \mid E_2) = 0{\cdot}04, \text{ and } P(E \mid E_3) = 0{\cdot}02$$

Hence the probability that a defective bolt selected at random is manufactured by machine A is given by

$$P(E_1 \mid E) = \frac{P(E_1)\,P(E \mid E_1)}{\sum\limits_{i=1}^{3} P(E_i)\,P(E \mid E_i)}$$

$$= \frac{0{\cdot}25 \times 0{\cdot}05}{0{\cdot}25 \times 0{\cdot}05 + 0{\cdot}35 \times 0{\cdot}04 + 0{\cdot}40 \times 0{\cdot}02} = \frac{125}{345} = \frac{25}{69}$$

Similarly

$$P(E_2 \mid E) = \frac{0{\cdot}35 \times 0{\cdot}04}{0{\cdot}25 \times 0{\cdot}05 + 0{\cdot}35 \times 0{\cdot}04 + 0{\cdot}40 \times 0{\cdot}02} = \frac{140}{345} = \frac{28}{69}$$

and

$$P(E_3 \mid E) = 1 - [P(E_1 \mid E) + P(E_2 \mid E)] = 1 - \frac{25}{69} - \frac{28}{69} = \frac{16}{69}$$

This example illustrates one of the chief applications of Bayes Theorem.

EXERCISE 4 (d)

1. *(a)* State and prove Baye's Theorem.

(b) The set of events A_k, $(k = 1, 2, ..., n)$ are *(i)* exhaustive and *(ii)* pairwise mutually exclusive. If for all k the probabilities $P(A_k)$ and $P(E \mid A_k)$ are known, calculate $P(A_k \mid E)$, where E is an arbitrary event. Indicate where conditions *(i)* and *(ii)* are used.

(c) The events $E_1, E_2, ..., E_n$ are mutually exclusive and $E = E_1 \cup E_2 \cup \ldots \cup E_n$. Show that if $P(A \mid E_i) = P(B \mid E_i)$; $i = 1, 2, ..., n$, then $P(A \mid E) = P(B \mid E)$. Is this conclusion true if the events E_i are not mutually exclusive?

[Calcutta Univ. B.Sc. (Maths Hons.), 1990]

(d) What are the criticisms against the use of Bayes theorem in probability theory. **[Sri. Venketeswara Univ. B.Sc., 1991]**

(e) Using the fundamental addition and multiplication rules of probability, show that

$$P(B \mid A) = \frac{P(B)\,P(A \mid B)}{P(B)\,P(A \mid B) + P(\bar{B})\,P(A \mid \bar{B})}$$

where $\bar{B}$ is the event complementary to the event B.

[Delhi Univ. M.A. (Econ.), 1987]

2. *(a)* Two groups are competing for the positions on the Board of Directors of a corporation. The probabilities that the first and second groups will win are 0·6 and 0·4 respectively. Furthermore, if the first group wins the probability of introducing a new product is 0·8 and the corresponding probability if the second group wins is 0·3. What is the probability that the new product will be introduced?

Ans. $0{\cdot}6 \times 0{\cdot}8 + 0{\cdot}4 \times 0{\cdot}3 = 0{\cdot}6$

(b) The chances of X, Y, Z becoming managers of a certain company are 4:2:3. The probabilities that bonus scheme will be introduced if X, Y, Z become managers, are 0·3, 0·5 and 0·8 respectively. If the bonus scheme has been introduced, what is the probability that X is appointed as the manager.

Ans. 0·51

(c) A restaurant serves two special dishes, A and B to its customers consisting of 60% men and 40% women. 80% of men order dish A and the rest B. 70% of women order dish B and the rest A. In what ratio of A to B should the restaurant prepare the two dishes? **(Bangalore Univ. B.Sc., 1991)**

Ans. $P(A) = P[(A \cap M) \cup (A \cap W)] = 0{\cdot}6 \times 0{\cdot}8 + 0{\cdot}4 \times 0{\cdot}3 = 0{\cdot}6$

Similarly $P(B) = 0{\cdot}4$. Required ratio $= 0{\cdot}6 : 0{\cdot}4 = 3 : 2$.

3. *(a)* There are three urns having the following compositions of black and white balls.

Urn 1 : 7 white, 3 black balls
Urn 2 : 4 white, 6 black balls
Urn 3 : 2 white, 8 black balls.

One of these urns is chosen at random with probabilities 0·20, 0·60 and 0·20 respectively. From the chosen urn two balls are drawn at random without replacement. Calculate the probability that both these balls are white.

Ans. 8/45. **(Madurai Univ. B.Sc., 1991)**

(b) Bowl I contain 3 red chips and 7 blue chips, bowl II contain 6 red chips and 4 blue chips. A bowl is selected at random and then 1 chip is drawn from this bowl. *(i)* Compute the probability that this chip is red, *(ii)* Relative to the hypothesis that the chip is red, find the conditional probability that it is drawn from bowl II. **[Delhi Univ. B.Sc. (Maths Hons.)1987]**

(c) In a factory machines A and B are producing springs of the same type. Of this production, machines A and B produce 5% and 10% defective springs, respectively. Machines A and B produce 40% and 60% of the total output of the factory. One spring is selected at random and it is found to be defective. What is the possibility that this defective spring was produced by machine A ? **[Delhi Univ. M.A. (Econ.),1986]**

(d) Urn A contains 2 white, 1 black and 3 red balls, urn B contains 3 white, 2 black and 4 red balls and urn C contains 4 white, 3 black and 2 red balls. One urn is chosen at random and 2 balls are drawn. They happen to be red and black. What

is the probability that both balls came from urn 'B' ?

[Madras U. B.Sc. April; 1989]

(e) Urn X_1, X_2, X_3, each contains 5 red and 3 white balls. Urns Y_1, Y_2, each contain 2 red and 4 white balls. An urn is selected at random and a ball is drawn. It is found to be red. Find the probability that the ball comes out of the urns of the first type. **[Bombay U. B.Sc., April 1992]**

(f) Two shipments of parts are received. The first shipment contains 1000 parts with 10% defectives and the second shipment contains 2000 parts with 5% defectives. One shipment is selected at random. Two parts are tested and found good. Find the probability (*a posterior*) that the tested parts were selected from the first shipment. **[Burdwan Univ. B.Sc. (Hons.), 1988]**

(g) There are three machines producing 10,000 ; 20,000 and 30,000 bullets per hour respectively. These machines are known to produce 5%, 4% and 2% defective bullets respectively. One bullet is taken at random from an hour's production of the three machines. What is the probability that it is defective? If the drawn bullet is defective, what is the probability that this was produced by the second machine? **[Delhi Univ. B.Sc. (Stat. Hons.), 1991]**

4. *(a)* Three urns are given each containing red and white chips as indicated.

Urn 1 : 6 red and 4 white.
Urn 2 : 2 red and 6 white.
Urn 3 : 1 red and 8 white.

(i) An urn is chosen at random and a ball is drawn from this urn. The ball is red. Find the probability that the urn chosen was urn I.

(ii) An urn is chosen at random and two balls are drawn without replacement from this urn. If both balls are red, find the probability that urn I was chosen. Under these conditions, what is the probability that urn III was chosen.

Ans. 108/173, 112/12, 0 **[Gauhati Univ. B.Sc., 1990]**

(b) There are ten urns of which each of three contains 1 white and 9 black balls, each of other three contains 9 white and 1 black ball, and of the remaining four each contains 5 white and 5 black balls. One of the urns is selected at random and a ball taken blindly from it turns out to be white. What is the probabililty that an urn containing 1 white and 9 black balls was selected? **(Agra Univ. B.Sc., 1991)**

Hint : $P(E_1) = \frac{3}{10}$, $P(E_2) = \frac{3}{10}$ and $P(E_3) = \frac{4}{10}$.

Let A be the event of drawing a white ball.

$$P(A) = \frac{3}{10} \times \frac{1}{10} + \frac{3}{10} \times \frac{9}{10} + \frac{4}{10} \times \frac{5}{10} = \frac{1}{2}$$

$$P(A \mid E_1) = \frac{1}{10} \text{ and } P(E_1 \mid A) = \frac{3}{50}$$

(c) It is known that an urn containing altogether 10 balls was filled in the following manner: A coin was tossed 10 times, and according as it showed heads or tails, one white or one black ball was put into the urn. Balls are drawn from this

urn one at a time, 10 times in succession (with replacement) and every one turns out to be white. Find the chance that the urn contains nothing but white balls.

Ans. 0·0702.

5. (*a*) From a vessel containing 3 white and 5 black balls, 4 balls are transferred into an empty vessel. From this vessel a ball is drawn and is found to be white. What is the probability that out of four balls transferred, 3 are white and 1 black. **[Delhi Uni. B.Sc. (Stat. Hons.), 1985]**

Hint. Let the five mutually exclusive events for the four balls transferred be E_0, E_1, E_2, E_3, and E_4, where E_i denotes the event that i white balls are transferred and let A be the event of drawing a white ball from the new vessel.

Then $$P(E_0) = \frac{{}^5C_4}{{}^8C_4},\ P(E_1) = \frac{{}^3C_1 \times {}^5C_3}{{}^8C_4},\ P(E_2) = \frac{{}^3C_2 \times {}^5C_2}{{}^8C_4}$$

$$P(E_3) = \frac{{}^3C_3 \times {}^5C_1}{{}^8C_4} \text{ and } P(E_4) = 0$$

Also $P(A \mid E_0) = 0,\ P(A \mid E_1) = \frac{1}{4},\ P(A \mid E_2) = \frac{2}{4},\ (A \mid E_3) = \frac{3}{4},$

and $P(A \mid E_4) = 1$. Hence $P(E_3 \mid A) = \frac{1}{7}$.

(*b*) The contents of the urns 1 and 2 are as follows :

Urn 1 : 4 white and 5 black balls.
Urn 2 : 3 white and 6 black balls.

One urn is chosen at random and a ball is drawn and its colour noted and replaced back to the urn. Again a ball is drawn from the same urn, colour noted and replaced. The process is repeated 4 times and as a result one ball of white colour and three balls of black colour are obtained. What is the probability that the urn chosen was the urn 1 ? **(Poona Univ. B.E., 1989)**

Hint. $P(E_1) = P(E_2) = 1/2,$

$P(A \mid E_1) = 4/9, \quad 1 - P(A \mid E_1) = 5/9$
$P(A \mid E_2) = 1/3, \quad 1 - P(A \mid E_2) = 2/3$

The probability that the urn chosen was the urn 1

$$= \frac{\frac{1}{2} \cdot \frac{4}{9} \left(\frac{5}{9}\right)^3}{\frac{1}{2} \cdot \frac{4}{9} \cdot \left(\frac{5}{9}\right)^3 + \frac{1}{2} \cdot \frac{1}{3} \cdot \left(\frac{2}{3}\right)^3}$$

(*c*) There are five urns numbered 1 to 5. Each urn contains 10 balls. The ith urn has i defective balls and $10 - i$ non-defective balls; $i = 1, 2, \ldots 5$. An urn is chosen at random and then a ball is selected at random from that urn. (*i*) What is the probability that a defective ball is selected ?

(*ii*) If the selected ball is defective, find the probability that it came from urn i, $(i = 1, 2, \ldots, 5)$. **[Delhi Univ. B.Sc. (Maths Hons.), 1987]**

Hint.: Define the following events :

E_i : ith urn is selected at random.

A : Defective ball is selected.

$P(E_i) = 1/5;\ i = 1, 2, ..., 5.$

$P(A \mid E_i) = P$ [Defective ball from ith urn] $= i/10,\ (i = 1, 2, ..., 5)$

$$P(E_i) \cdot P(A \mid E_i) = \frac{1}{5} \times \frac{i}{10} = \frac{i}{50}\ ,\quad (i = 1, 2, ..., 5).$$

(i) $$P(A) = \sum_{i=1}^{5} P(E_i)\, P(A \mid E_i) = \sum_{i=1}^{5} \left(\frac{i}{50}\right) = \frac{1+2+3+4+5}{50} = \frac{3}{10}$$

(ii) $$P(E_i \mid A) = \frac{P(E_i)\, P(A \mid E_i)}{\sum_i P(E_i)\, P(A \mid E_i)} = \frac{i/50}{3/10} = \frac{i}{15}\ ;\ i = 1, 2, ..., 5.$$

For example, the probability that the defective ball came from 5th urn $= (5/15) = 1/3$.

6. *(a)* A bag contains six balls of different colours and a ball is drawn from it at random. A speaks truth thrice out of 4 times and B speaks truth 7 times out of 10 times. If both A and B say that a red ball was drawn, find the probability of their joint statement being true.

[Delhi Univ. B.Sc. (Stat. Hons.),1987; Kerala Univ. B.Sc.1988]

(b) A and B are two very weak students of Statistics and their chances of solving a problem correctly are 1/8 and 1/12 respectively. If the probability of their making a common mistake is 1/1001 and they obtain the same answer, find the chance that their answer is correct. **[Poona Univ. B.Sc., 1989]**

Ans. Reqd. Probability $= \dfrac{\frac{1}{8} \times \frac{1}{12}}{\frac{1}{8} \times \frac{1}{12} + (1 - \frac{1}{8}) \cdot (1 - \frac{1}{12}) \cdot \frac{1}{1001}} = \dfrac{13}{14}$

7. *(a)* Three boxes, practically indistinguishable in appearance, have two drawers each. Box I contains a gold coin in one and a silver coin in the other drawer, box II contains a gold coin in each drawer and box III contains a silver coin in each drawer. One box is chosen at random and one of its drawers is opened at random and a gold coin found. What is the probability that the other drawer contains a coin of silver? **(Gujarat Univ. B.Sc., 1992)**

Ans. 1/3, 1/3.

(b) Two cannons No. 1 and 2 fire at the same target. Cannon No. 1 gives on an average 9 shots in the time in which Cannon No. 2 fires 10 projectiles. But on an average 8 out of 10 projectiles from Cannon No. 1 and 7 out of 10 from Cannon No. 2 strike the target. In the course of shooting, the target is struck by one projectile. What is the probability of a projectile which has struck the target belonging to Cannon No. 2 ? **(Lucknow Univ. B.Sc., 1991)**

Ans. 0·493

(c) Suppose 5 men out of 100 and 25 women out of 10,000 are colour blind. A colour blind person is chosen at random. What is the probability of his being male? (Assume males and females to be in equal number.)

Hint. E_1 = Person is a male, E_2 = Person is a female.

A = Person is colour blind.

Then $P(E_1) = P(E_2) = ½$, $P(A \mid E_1) = 0·05$, $P(A \mid E_2) = 0·0025$.

Hence find $P(E_1 \mid A)$.

8. *(a)* Three machines X, Y, Z with capacities proportional to 2:3:4 are producting bullets. The probabilities that the machines produce defective are 0·1, 0·2 and 0·1 respectively. A bullet is taken from a day's production and found to be defective What is the probability that it came from machine X?

[Madras Univ. B.Sc., 1988]

(b) In a factory 2 machines M_1 and M_2 are used for manufacturing screws which may be uniquely classified as good or bad. M_1 produces per day n_1 boxes of screws, of which on the average, p_1% are bad while the corresponding numbers for M_2 are n_2 and p_2. From the total production of both M_1 and M_2 for a certain day, a box is chosen at random, a screw taken out of it and it is found to be bad. Find the chance that the selected box is manufactured *(i)* by M_1, *(ii)* M_2.

Ans. *(i)* $n_1 p_1/(n_1 p_1 + n_2 p_2)$, *(ii)* $n_2 p_2/(n_1 p_1 + n_2 p_2)$.

9. *(a)* A man is equally likley to choose any one of three routes A, B, C from his house to the railway station, and his choice of route is not influenced by the weather. If the weather is dry, the probabilities of missing the train by routes A, B, C are respectively 1/20, 1/10, 1/5. He sets out on a dry day and misses the train. What is the probability that the route chosen was C?

On a wet day, the respective probabilities of missing the train by routes A, B, C are 1/20, 1/5, 1/2 respectively. On the average, one day in four is wet. If he misses the train, what is the probability that the day was wet?

[Allahabad Univ. B.Sc., 1991]

(b) A doctor is to visit the patient and from past experience it is known that the probabilities that he will come by train, bus or scooter are respectively 3/10, 1/5, and 1/10, the probabililty that he will use some other means of transport being, therefore, 2/5. If he comes by train, the probability that he will be late is 1/4, if by bus 1/3 and if by scooter 1/12, if he uses some other means of transport it can be assumed that he will not be late. When he arrives he is late. What is the probability that *(i)* he comes by train *(ii)* he is not late?

[Burdwan Univ. B.Sc. (Hons.), 1990]

Ans. *(i)* 1/2, *(ii)* 9/34

10. State and prove Bayes rule and expalin why, in spite of its easy deductibility from the postulates of probability, it has been the subject of such extensive controversy.

In the chest X-ray tests, it is found that the probability of detection when a person has actually T.B. is 0·95 and probability of diagnosing incorrectly as having T.B. is 0·002. In a certain city 0·1% of the adult population is suspected to be suffering from T.B. If an adult is selected at random and is diagnosed as having

T.B. on the basis of the X-ray test, what is the probability of his actually having a T.B.? **(Nagpur Univ. B.E., 1991)**

Ans. 0·97

11. A certain transistor is manufactured at three factories at Barnsley, Bradford and Bristol. It is known that the Barnsley factory produces twice as many transistors as the Bradford one, which produces the same number as the Bristol one (during the same period). Experience also shows that 0·2% of the transistors produced at Barnsley and Bradford are faulty and so are 0·4% of those produced at Bristol.

A service engineer, while maintaining an electronic equipment, finds a defective transistor. What is the probability that the Bradford factory is to blame?

(Bangalore Univ. B.E., Oct. 1992)

12. The sample space consists of integers from 1 to $2n$ which are assigned probabilities proportional to their logarithms. Find the probabilities and show that the conditional probability of the integer 2, given that an even integer occurs, is

$$\frac{\log 2}{[n \log 2 + \log(n!)]}$$ **(Lucknow Univ. M.A., 1992)**

[**Hint.** Let E_i : the event that the integer $2i$ is drawn, $(i = 1, 2, 3, \ldots, n)$.

A : the event of drawing an even integer.

$$\Rightarrow \quad A = E_1 \cup E_2 \cup \ldots \cup E_n \quad \Rightarrow \quad P(A) = \sum_{i=1}^{n} P(E_i)$$

But $\quad P(E_i) = k \log(2i) \quad$ (Given)

$$\therefore \quad P(A) = k \sum_{i=1}^{n} \log(2i) = k \log \prod_{i=1}^{n} (2i) = k[n \log 2 + \log(n!)]$$

$$\therefore \quad P(E_i \mid A) = \frac{\log(2i)}{[n \log 2 + \log(n!)]}$$

13. In answering a question on a multiple choice test, an examinee either knows the answer (with probability p), or he guesses (with probability $1-p$). Assume that the probability of answering a question correctly is unity for an examinee who knows the answer and $1/m$ for the examinee who guesses, where m is the number of multiple choice alternatives. Supposing an examinee answers a question correctly, what is the probability that he really knows the answer?

[Delhi Univ. M.C.A., 1990; M.Sc. (Stat.), 1989]

Hint. Let $E_1 =$ The examinee knows the answer,

$E_2 =$ The examinee guesses the answer,

and $A =$ The examinee answers correctly.

Then $P(E_1) = p$, $P(E_2) = 1 - p$, $P(A \mid E_1) = 1$ and $P(A \mid E_2) = 1/m$

Now use Bayes theorem to prove

$$P(E_1 \mid A) = \frac{mp}{1 + (m-1)p}$$

14. Die A has four red and two white faces whereas die B has two red and four white faces. A biased coin is flipped once. If it falls heads, the game continues by

throwing die A, if it falls tails die B is to be used.

(i) Show that the probability of getting a red face at any throw is 1/2.

(ii) If the first two throws resulted in red faces, what is the probability of red face at the 3rd throw?

(iii) If red face turns up at the first n throws, what is the probability that die A is being used?

Ans. *(ii)* 3/5 *(iii)* $\dfrac{2^n}{2^n+1}$

15. A manufacturing firm produces steel pipes in three plants with daily production volumes of 500, 1,000 and 2,000 units respectively. According to past experience it is known that the fraction of defective outputs produced by the three plants are respectively 0.005, 0.008 and 0.010. If a pipe is selected at random from a day's total production and found to be defective, from which plant does that pipe come?

Ans. Third plant.

16. A piece of mechanism consists of 11 components, 5 of type A, 3 of type B, 2 of type C and 1 of type D. The probability that any particular component will function for a period of 24 hours from the commencement of operations without breaking down is independent of whether or not any other component breaks down during that period and can be obtained from the following table:

Component type:ABCD

Probability:0·60·70·30·2

(i) Calculate the probability that 2 components chosen at random from the 11 components will both function for a period of 24 hours from the commencement of operations without breaking down.

(ii) If at the end of 24 hours of operations neither of the 2 components chosen in, *(i)* has broken down, what is the probability that they are both type C components.

Hint.

(i) Required probability $= \dfrac{1}{{}^{11}C_2} [\,{}^5C_2 \times (0{\cdot}6)^2 + {}^3C_2\,(0{\cdot}7)^2 + {}^2C_2\,(0{\cdot}3)^2$

$+\, {}^5C_1 \times {}^3C_1 \times 0{\cdot}6 \times 0{\cdot}7 + {}^5C_1 \times {}^2C_1 \times (0{\cdot}6) \times (0{\cdot}3)$

$+\, {}^5C_1 \times {}^1C_1 \times (0{\cdot}6) \times (0{\cdot}2) + {}^3C_1 \times {}^2C_1 \times 0{\cdot}7 \times 0{\cdot}3$

$+\, {}^3C_1 \times {}^1C_1 \times 0{\cdot}7 \times 0{\cdot}2 + {}^2C_1 \times {}^1C_1 \times 0{\cdot}3 \times 0{\cdot}2]$

$= p$ (Say).

(ii) Required probability (By Bayes theorem)

$$= \frac{{}^2C_2 \times (0{\cdot}3)^2}{p} = \frac{0{\cdot}09}{p}$$

4·10. Geometric probability. In remark 3, § 4·3·1 it was pointed out that the classical definition of probability fails if the total number of outcomes of an experiment is infinite. Thus, for example, if we are interested in finding the probability that a point selected at random in a given region will lie in a specified part of it, the classical definition of probability is modified and extended to what is called *geometrical probability or probability in continuum*. In this case, the general expression for probability 'p' is given by

$$p = \frac{\text{Measure of specified part of the region}}{\text{Measure of the whole region}}$$

where 'measure' refers to the length, area or volume of the region if we are dealing with one, two or three dimensional space respectively.

Example 4·34. *Two points are taken at random on the given straight line of length a. Prove that the probability of their distance exceeding a given length* $c\ (< a)$ is equal to $\left(1 - \frac{c}{a}\right)^2$.

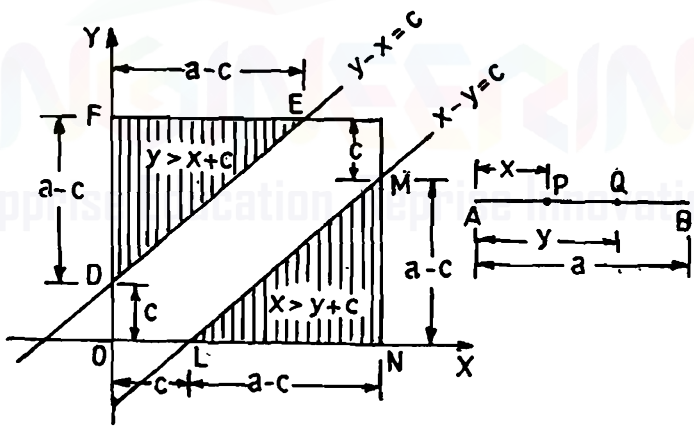
[Burdwan Univ. B.Sc. (Hons.), 1992; Delhi Univ. M.A. (Econ.), 1987]

Solution. Let P and Q be any two points taken at random on the given straight line AB of length 'a'. Let $AP = x$ and $AQ = y$,

$$(0 \le x \le a, 0 \le y \le a).$$

Then we want $P\{|x - y| > c\}$.

The probability can be easily calculated geometrically. Plotting the lines $x - y = c$ and $y - x = c$ along the co-ordinate axes, we get the following diagram:



Since $0 \le x \le a, 0 \le y \le a$, total area $= a \,.\, a = a^2$.

Area favourable to the event $|x - y| > c$ is given by

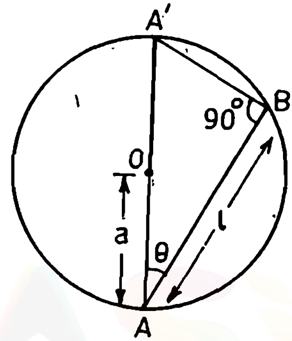
$$\Delta LMN + \Delta DEF = \frac{1}{2} LN \,.\, MN + \frac{1}{2} EF \,.\, DF$$

$$= \frac{1}{2}(a-c)^2 + \frac{1}{2}(a-c)^2 = (a-c)^2$$

$$P(|x - y| > c) = \frac{(a-c)^2}{a^2} = \left(1 - \frac{c}{a}\right)^2$$

Example 4·35. (Bertrand's Problem). *If a chord is taken at random in a circle, what is the chance that its length l is not less than 'a', the radius of the circle?*

Solution. Let the chord AB make an angle θ with the diameter AOA' of the circle with centre O and radius $OA=a$. Obviously θ lies betwseen $-\pi/2$ and $\pi/2$. Since all the positions of the chord AB and consequently all the values of θ are equally likely, θ may be regarded as a random variable which is uniformly distributed *c.f.* § 8·1 over $(-\pi/2, \pi/2)$ with probability density function



$$f(\theta) = \frac{1}{\pi}\ ;\ -\pi/2 < \theta \le \pi/2$$

$\angle ABA'$, being the angle in a semi-circle, is a right angle. From $\Delta ABA'$ we have

$$\frac{AB}{AA'} = \cos\theta$$

$$\Rightarrow \qquad l = 2a\cos\theta$$

The required probability 'p' is given by

$$p = P(l \ge a) = P(2a\cos\theta \ge a)$$

$$= P(\cos\theta \ge 1/2) = P(|\theta| \le \pi/3)$$

$$= \int_{-\pi/3}^{\pi/3} f(\theta)\, d\theta = \frac{1}{\pi}\int_{-\pi/3}^{\pi/3} d\theta = \frac{2}{3}$$

Example 4·36. *A rod of length 'a' is broken into three parts at random. What is the probability that a triangle can be formed from these parts?*

Solution. Let the lengths of the three parts of the rod be x, y and $a-(x+y)$. Obviously, we have

$$x > 0\ ;\ y > 0 \text{ and } x + y < a \quad \Rightarrow \quad y < a - x \qquad \ldots(*)$$

In order that these three parts form the sides of a triangle, we should have

$$\left.\begin{aligned} & x + y > a - (x+y) && \Rightarrow && y > \frac{a}{2} - x \\ \text{and}\quad & x + a - (x+y) > y && \Rightarrow && y < \frac{a}{2} \\ & y + a - (x+y) > x && \Rightarrow && y < \frac{a}{2} \end{aligned}\right\} \qquad \ldots(**)$$

since in a triangle, the sum of any two sides is greater than the third. Equivalently, (**) can be written as

$$\frac{a}{2} - x < y < \frac{a}{2} \quad \wedge \quad 0 < x < \frac{a}{2} \qquad \ldots(***)$$

Hence, on using (*) and (***), the required probability is given by

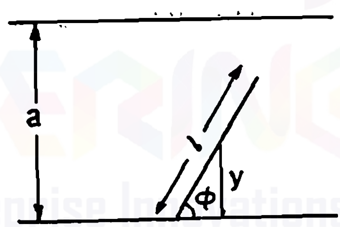
$$\frac{\int_0^{a/2}\int_{(a/2)-x}^{a/2} dy\,dx}{\int_0^{a}\int_0^{a-x} dy\,dx} = \frac{\int_0^{a/2}\left[\frac{a}{2}-\left(\frac{a}{2}-x\right)\right]dx}{\int_0^{a}(a-x)\,dx}$$

$$= \frac{\left|\frac{x^2}{2}\right|_0^{a/2}}{\left|\frac{-(a-x)^2}{2}\right|_0^{a}} = \frac{a^2/8}{a^2/2} = \frac{1}{4}$$

Example 4·37. (Buffon's Needle Problem). *A vertical board is ruled with horizontal parallel lines at constant distance 'a' apart. A needle of length $l\ (< a)$ is thrown at random on the table. Find the probability that it will intersect one of the lines.*

Solution. Let y denote the distance from the centre of the needle to the nearest parallel and ϕ be angle formed by the needle with this parallel. The quantities y and ϕ fully determine the position of the needle. Obviously y ranges from 0 to $a/2$ (since $l < a$) and ϕ from 0 to π.

Since the needle is dropped randomly, all possible values of y and ϕ may be regarded as equally likely and consequently the joint probability density function $f(y, \phi)$ of y and ϕ is given by the uniform distribution (c.f. § 8.1) by



$$f(y, \phi) = k; \quad 0 \le \phi \le \pi, \quad 0 \le y \le a/2, \quad ...(*)$$

where k is a constant.

The needle will intersect one of the lines if the distance of its centre from the line is less than $\frac{1}{2} l \sin\phi$, *i.e.*, the required event can be represented by the inequality $0 < y < \frac{1}{2} l \sin\phi$. Hence the required probability p is given by

$$p = \frac{\int_0^{\pi}\int_0^{(l\sin\phi)/2} f(y, \phi)\,dy\,d\phi}{\int_0^{\pi}\int_0^{a/2} f(y, \phi)\,dy\,d\phi}$$

$$= \frac{\frac{l}{2}\int_0^{\pi}\sin\phi\,d\phi}{(a/2)\,.\,\pi}$$

$$= \frac{l}{a\pi}\left|-\cos\phi\right|_0^{\pi} = \frac{2l}{a\pi}$$

EXERCISE 4 (e)

1. Two points are selected at random in a line AC of length 'a' so as to lie on the opposite sides of its mid-point O. Find the probability that the distance between them is less than $a/3$.

2. *(a)* Two points are selected at random on a line of length a. What is the probability that none of three sections in which the line is thus divided is less than $a/4$?

Ans. 1/16.

(b) A rectilinear segment AB is divided by a point C into two parts $AC=a$, $CB=b$. Points X and Y are taken at random on AC and CB respectively. What is the probability that AX, XY and BY can form a triangle?

(c) ABG is a straight line such that AB is 6 inches and BG is 5 inches. A point Y is chosen at random on the BG part of the line. If C lies between B and G in such a way that $AC=t$ inches, find

(i) the probability that Y will lie in BC.

(ii) the probability that Y will lie in CG.

What can you say about the sum of these probabilities?

(d) The sides of a rectangle are taken at random each less than a and all lengths are equally likely. Find the chance that the diagonal is less than a.

3. *(a)* Three points are taken at random on the circumference of a circle. Find the chance that they lie on the same semi- circle.

(b) A chord is drawn at random in a given circle. What is the probability that it is greater than the side of an equilateral triangle inscribed in that circle?

(c) Show that the probability of choosing two points randomly from a line segment of length 2 inches and their being at a distance of at least 1 inch from each other is 1/4. **[Delhi Univ. M.A. (Econ.), 1985]**

4. A point is selected at random inside a circle. Find the probability that the point is closer to the centre of the circle than to its circumference.

5. One takes at random two points P and Q on a segment AB of length a

(i) What is the probability for the distance PQ being less than $b\ (<a)$?

(ii) Find the chance that the distance between them is greater than a given length b.

6. Two persons A and B, make an appointment to meet on a certain day at a certain place, but without fixing the time further than that it is to be between 2 p.m. and 3 p.m and that each is to wait not longer than ten minutes for the other. Assuming that each is independently equally likely to arrive at any time during the hour, find the probability that they meet.

Third person C, is to be at the same place from 2·10 p.m. until 2·40 p.m. on the same day. Find the probabilities of C being present when A and B are there together *(i)* When A and B remain after they meet, *(ii)* When A and B leave as soon as they meet.

Hint. Denote the times of arrival of A by x and of B by y. For the meeting to take place it is necessary and sufficient that

$$| x - y | < 10$$

We depict x and y as Cartesian coordinates in the plane; for the scale unit we take one minute. All possible outcomes can be described as points of a aquare with side 60. We shall finally get [*c.f.* Example 4·34, with $a = 60$, $c = 10$]

$$P [| x - y | < 10] = 1 - (5/6)^2 = 11/36$$

7. The outcome of an experiment are represented by points in the square bounded by $x = 0$, $x = 2$ and $y = 2$ in the xy–plane. If the probability is distributed uniformly, determine the probability that $x^2 + y^2 > 1$

Hint.

Required probability $P(E) = \int_E \frac{1}{4}\, dx\, dy = 1 - \int_{E'} \frac{1}{4}\, dx\, dy$

where E is the region for which $x^2 + y^2 > 1$ and E' is the region for which $x^2 + y^2 \leq 1$.

$$\therefore \quad 4P(E) = 4 - \int_0^1 \int_0^1 dx\, dy = 3 \qquad \Rightarrow \qquad P(E) = \frac{3}{4}$$

8. A floor is paved with tiles, each tile being a parallelogram such that the distance between pairs of opposite sides are a and b respectively, the length of the diagonal being l. A stick of length c falls on the floor parallel to the diagonal. Show that the probability that it will lie entirely on one tile is

$$\left(1 - \frac{c}{l}\right)^2$$

If a circle of diameter d is thrown on the floor, show that the probability that it will lie on one tile is

$$\left(1 - \frac{d}{a}\right)\left(1 - \frac{d}{b}\right)$$

9. Circular discs of radius r are thrown at random on to a plane circular table of radius R which is surrounded by a border of uniform width r lying in the same plane as the table. If the discs are thrown independently and at random, and N stay on the table, show that the probability that a fixed point on the table but not on the border, will be covered is

$$1 - \left(1 - \frac{r^2}{(R + r)^2}\right)^N$$

SOME MISCELLANEOUS EXAMPLES

Example 4·38. *A die is loaded in such a manner that for n=1, 2, 3, 4, 5, 6 the probability of the face marked n, landing on top when the die is rolled is proportional to n. Find the probability that an odd number will appear on tossing the die.*

[Madras Univ. B.Sc. (Stat. Main), 1987]

Solution. Here we are given

$P(n) \propto n$ or $P(n) = kn$, where k is the constant of proportionality.

Also $P(1) + P(2) + ... P(6) = 1 \Rightarrow k(1+2+3+4+5+6) = 1$ or $k = 1/21$

Required Probability $= P(1) + P(3) + P(5) = \dfrac{1+3+5}{21} = \dfrac{3}{7}$

Example 4·39. *In terms of probability :*

$p_1 = P(A)$, $p_2 = P(B)$, $p_3 = P(A \cap B)$, $(p_1, p_2, p_3 > 0)$

Express the following in terms of p_1, p_2, p_3

(a) $P(\overline{A \cup B})$, (b) $P(\bar{A} \cup \bar{B})$, (c) $P(\bar{A} \cap B)$, (d) $P(\bar{A} \cup B)$, (e) $P(\bar{A} \cap \bar{B})$

(f) $P(A \cap \bar{B})$, (g) $P(A \mid B)$, (h) $P(B \mid \bar{A})$, (i) $P[\bar{A} \cap (A \cup B)]$

Solution.

(a) $P(\overline{A \cup B}) = 1 - P(A \cup B) = 1 - [P(A) + P(B) - P(AB)]$
$= 1 - p_1 - p_2 + p_3$.

(b) $P(\bar{A} \cup \bar{B}) = P(\overline{A \cap B}) = 1 - P(A \cap B) = 1 - p_3$

(c) $P(\bar{A} \cap B) = P(B - AB) = P(B) - P(A \cap B) = p_2 - p_3$

(d) $P(\bar{A} \cup B) = P(\bar{A}) + P(B) - P(\bar{A} \cap B) = 1 - p_1 + p_2 - (p_2 - p_3)$
$= 1 - p_1 + p_3$

(e) $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B}) = 1 - p_1 - p_2 + p_3$. [Part (*a*)]

(f) $P(A \cap \bar{B}) = P(A - A \cap B) = P(A) - P(A \cap B) = p_1 - p_3$

(g) $P(A \mid B) = P(A \cap B)/P(B) = p_3/p_2$

(h) $P(B \mid \bar{A}) = P(\bar{A} \cap B)/P(\bar{A}) = (p_2 - p_3)/(1 - p_1)$

(i) $P[\bar{A} \cap (A \cup B)] = P[(\bar{A} \cap A) \cup (\bar{A} \cap B)]$

$= P(\bar{A} \cap B) = p_2 - p_3$ $[\because A \cap \bar{A} = \phi]$

Example 4·40. *Let $P(A) = p$, $P(A \mid B) = q$, $P(B \mid A) = r$. Find relations between the numbers p, q, r for the following cases :*

(a) Events A and B are mutually exclusive.

(b) A and B are mutually exclusive and collectively exhaustive.

(c) A is a subevent of B; B is a subevent of A.

(d) $\bar{A}$ and $\bar{B}$ are mutually exclusive.

[Delhi Univ. B.Sc. (Maths Hons.) 1985]

Solution. From given data : $P(A) = p$, $P(A \cap B) = P(A)\,P(B \mid A) = rp$

$$\therefore \quad P(B) = \frac{P(A \cap B)}{P(A \mid B)} = \frac{rp}{q}$$

(a) $P(A \cap B) = 0 \quad \Rightarrow \quad rp = 0$.

(b) $P(A \cap B) = 0$ and $P(A) + P(B) = 1$
$\Rightarrow \quad p(q + r) = q;\ rp = 0 \quad \Rightarrow \quad pq = q \quad \Rightarrow \quad p = 1 \vee q = 0$.

(c) $A \subseteq B \Rightarrow A \cap B = A$ or $P(A \cap B) = P(A) \Rightarrow rp = p \Rightarrow r = 1 \vee p = 0$.
$B \subseteq A \Rightarrow A \cap B = B$ or $P(A \cap B) = P(B)$
$\Rightarrow \quad rp = (rp/q)$ or $rp(q - 1) = 0 \quad \Rightarrow \quad q = 1$

(d) $P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B) \quad \Rightarrow \quad 0 = 1 - [P(A) + P(B) - P(A \cap B)]$

So $\quad P(A) + P(B) = 1 + P(A \cap B) \quad \Rightarrow \quad p[1 + (r/q)] = 1 + rp$

$\therefore \qquad p(q + r) = q(1 + pr).$

Example 4·41. *(a) Twelve balls are distributed at random among three boxes. What is the probability that the first box will contain 3 balls?*

(b) If n biscuits be distributed among N persons, find the chance that a particular person receives r (< n) biscuits. **[Marathwada Univ. B.Sc. 1992]**

Solution. *(a)* Since each ball can go to any one of the three boxes, there are 3 ways in which a ball can go to any one of the three boxes. Hence there are 3^{12} ways in which 12 balls can be placed in the three boxes.

Number of ways in which 3 balls out of 12 can go to the first box is ${}^{12}C_3$. Now the remaining 9 balls are to be placed in 2 boxes and this can be done in 2^9 ways. Hence the total number of favourable cases $= {}^{12}C_3 \times 2^9$.

$$\therefore \quad \text{Required probability} = \frac{{}^{12}C_3 \times 2^9}{3^{12}}$$

(b) Take any one biscuit. This can be given to any one of the N beggars so that there are N ways of distributing any one biscuit. Hence the total number of ways in which n biscuit can be distributed at random among N beggars

$$= N \,.\, N \ldots N \,(\,n \text{ times}\,) = N^n.$$

'r' biscuits can be given to any particular beggar in ${}^nC_r$ ways. Now we are left with $(n - r)$ biscuits which are to be distributed among the remaining $(N - 1)$ beggars and this can be done in $(N - 1)^{n-r}$ ways.

$$\therefore \quad \text{Number of favourable cases} = {}^nC_r \,.\, (N - 1)^{n-r}$$

$$\text{Hence, required probability} = \frac{{}^nC_r\,(N - 1)^{n-r}}{N^n}$$

Example 4·42. *A car is parked among N cars in a row, not at either end. On his return the owner finds that exactly r of the N places are still occupied. What is the probability that both neighbouring places are empty?*

Solution. Since the owner finds on return that exactly r of the N places (including owner's car) are occupied, the exhaustive number of cases for such an arrangement is ${}^{N-1}C_{r-1}$ [since the remaining $r - 1$ cars are to be parked in the remaining $N - 1$ places and this can be done in ${}^{N-1}C_{r-1}$ ways].

Let A denote the event that both the neighbouring places to owner's car are empty. This requires the remaining $(r - 1)$ cars to be parked in the remaining $N - 3$ places and hence the number of cases favourable to A is ${}^{N-3}C_{r-1}$. Hence

$$P(A) = \frac{{}^{N-3}C_{r-1}}{{}^{N-1}C_{r-1}} = \frac{(N - r)(N - r - 1)}{(N - 1)(N - 2)}$$

Example 4·43. *What is the probability that at least two out of n people have the same birthday? Assume 365 days in a year and that all days are equally likely.*

Solution. Since the birthday of any person can fall on any one of the 365 days, the exhaustive number of cases for the birthdays of n persons is 365^n.

If the birthdays of all the n persons fall on different days, then the number of favourable cases is

$$365\,(365-1)\,(365-2)\,....\,[365-(n-1)],$$

because in this case the birthday of the first person can fall on any one of 365 days, the birthday of the second person can fall on any one of the remaining 364 days and so on.

Hence the probability (p) that birthdays of all the n persons are different is given by :

$$p = \frac{365\,(365-1)\;(365-2)\ldots[\,365-(n-1)\,]}{365^n}$$

$$= \left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)\left(1-\frac{3}{365}\right)\cdots\left(1-\frac{n-1}{365}\right)$$

Hence the required probability that at least two persons have the same birthday is

$$1-p = 1-\left(1-\frac{1}{365}\right)\left(1-\frac{2}{365}\right)\left(1-\frac{3}{365}\right)\cdots\left(1-\frac{n-1}{365}\right)$$

Example 4·44. *A five-figure number is formed by the digits 0, 1, 2, 3, 4 (without repetition). Find the probability that the number formed is divisible by 4.*

[Delhi Univ. B.Sc. (Stat. Hons.), 1990]

Solution. The total number of ways in which the five digits 0, 1, 2, 3, 4 can be arranged among themselves is 5!. Out of these, the number of arrangements which begin with 0 (and, therefore, will give only 4-digited numbers) is 4!. Hence the total number of five digited numbers that can be formed from the digits 0, 1, 2, 3, 4 is

$$5\,!-4\,!=120-24=96$$

The number formed will be divisible by 4 if the number formed by the two digits on extreme right (*i.e.*, the digits in the unit and tens places) is divisible by 4. Such numbers are :

$$04\;,\;12\;,\;20\;,\;24\;,\;32\;,\text{ and }40$$

If the numbers end in 04, the remaining three digits, *viz.*, 1, 2 and 3 can be arranged among themselves in 3 ! ways. Similarly, the number of arrangements of the numbers ending with 20 and 40 is 3 ! in each case.

If the numbers end with 12, the remaining three digits 0 , 3 , 4 can be arranged in 3 ! ways. Out of these we shall reject those numbers which start with 0 (*i.e.*, have 0 as the first digit). There are (3 – 1) ! = 2 ! such cases. Hence, the number of five digited numbers ending with 12 is

$$3\,!-2\,!=6-2=4$$

Similarly the number of 5 digited numbers ending with 24 and 32 each is 4. Hence the total number of favourable cases is

$$3 \times 3\,! + 3 \times 4 = 18 + 12 = 30$$

Hence required probability $= \dfrac{30}{96} = \dfrac{5}{16}$

Example 4·45. *(Huyghen's problem). A and B throw alternately with a pair of ordinary dice. A wins if he throws 6 before B throws 7, and B wins if he throws 7 before A throws 6. If A begins, show that his chance of winning is 30 / 61*

[Delhi Univ. B.Sc. (Stat. Hons.), 1991; Delhi Univ. B.Sc.,1987]

Solution. Let E_1 denote the event of A's throwing '6' and E_2 the event of B's throwing '7 with a pair of dice. Then $\bar{E}_1$ and $\bar{E}_2$ are the complementary events.

'6' can be obtained with two dice in the following ways:

(1, 5), (5, 1), (2, 4), (4, 2), (3, 3), *i.e.*, in 5 distinct ways.

$$\therefore \qquad P(E_1) = \frac{5}{36} \quad \text{and} \quad P(\bar{E}_1) = 1 - \frac{5}{36} = \frac{31}{36}$$

'7' can be obtained with two dice as follows:

(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3), *i.e.*, in 6 distinct ways.

$$\therefore \qquad P(E_2) = \frac{6}{36} = \frac{1}{6} \quad \text{and} \quad P(\bar{E}_2) = 1 - \frac{1}{6} = \frac{5}{6}$$

If A starts the game, he will win in the following mutually exclusive ways:

(i) E_1 happens *(ii)* $\bar{E}_1 \cap \bar{E}_2 \cap E_1$ happens

(iii) $\bar{E}_1 \cap \bar{E}_2 \cap \bar{E}_1 \cap \bar{E}_2 \cap E_1$ happens, and so on.

Hence by addition theorem of probability, the required probability of A's winning, (say), $P(A)$ is given by

$$P(A) = P(i) + P(ii) + P(iii) + \ldots$$
$$= P(E_1) + P(\bar{E}_1 \cap \bar{E}_2 \cap E_1) + P(\bar{E}_1 \cap \bar{E}_2 \cap \bar{E}_1 \cap \bar{E}_2 \cap E_1) + \ldots$$
$$= P(E_1) + P(\bar{E}_1)\,P(\bar{E}_2)\,P(E_1) + P(\bar{E}_1)\,P(\bar{E}_2)\,P(\bar{E}_1)\,P(\bar{E}_2)\,P(E_1) + \ldots$$

(By compound probability theorem)

$$= \frac{5}{36} + \frac{31}{36} \times \frac{5}{6} \times \frac{5}{36} + \frac{31}{36} \times \frac{5}{6} \times \frac{31}{36} \times \frac{5}{6} \times \frac{5}{36} + \ldots$$
$$= \frac{5/36}{1 - \dfrac{31}{36} \times \dfrac{5}{6}} = \frac{30}{61}$$

Example 4·46. *A player tosses a coin and is to score one point for every head and two points for every tail turned up. He is to play on until his score reaches or passes n. If p_n is the chance of attaining exactly n score, show that*

$$p_n = \frac{1}{2}\,[\,p_{n-1} + p_{n-2}\,],$$

and hence find the value of p_n. **[Delhi Univ. B.Sc. (Stat. Hons.),1992]**

Solution. The score n can be reached in the following two mutually exclusive ways:

(i) By throwing a tail when score is $(n-2)$, and

(ii) By throwing a head when score is $(n-1)$.

Hence by addition theorem of probability, we get

$$p_n = P(i) + P(ii) = \frac{1}{2} \cdot p_{n-2} + \frac{1}{2} \cdot p_{n-1} = \frac{1}{2}(p_{n-1} + p_{n-2}) \qquad ...(*)$$

To find p_n explicitly, (*) may be re-written as

$$\begin{aligned} p_n + \frac{1}{2} p_{n-1} &= p_{n-1} + \frac{1}{2} p_{n-2} \\ &= p_{n-2} + \frac{1}{2} p_{n-3} \\ &\cdots \quad \cdots \\ &\cdots \quad \cdots \\ &= p_2 + \frac{1}{2} p_1 \end{aligned} \qquad ...(**)$$

Since the score 2 can be obtained as

(i) Head in first throw and head in 2nd throw,

(ii) Tail in the first throw, we have

$$p_2 = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} = \frac{1}{4} + \frac{1}{2} = \frac{3}{4} \text{ and obviously } p_1 = \frac{1}{2}$$

Hence, from (**), we get

$$p_n + \frac{1}{2} p_{n-1} = \frac{3}{4} + \frac{1}{2} \cdot \frac{1}{2} = 1 = \frac{2}{3} + \frac{1}{3} = \frac{2}{3} + \frac{1}{2} \cdot \frac{2}{3}$$

$$\left.\begin{aligned} p_n - \tfrac{2}{3} &= (-\tfrac{1}{2})(p_{n-1} - \tfrac{2}{3}) \\ p_{n-1} - \tfrac{2}{3} &= (-\tfrac{1}{2})(p_{n-2} - \tfrac{2}{3}) \\ \vdots \quad & \quad \vdots \\ p_2 - \tfrac{2}{3} &= (-\tfrac{1}{2})(p_1 - \tfrac{2}{3}) \end{aligned}\right\}$$

Multiplying all the above equations, we get

$$\begin{aligned} p_n - \tfrac{2}{3} &= (-\tfrac{1}{2})^{n-1} (p_1 - \tfrac{2}{3}) \\ &= (-\tfrac{1}{2})^{n-1} (\tfrac{1}{2} - \tfrac{2}{3}) = (-1)^n \cdot \frac{1}{2^n} \cdot \frac{1}{3} \end{aligned}$$

$$\Rightarrow \qquad \begin{aligned} p_n &= \frac{2}{3} + (-1)^n \frac{1}{2^n} \cdot \frac{1}{3} \\ &= \frac{1}{3}\left[2 + (-1)^n \frac{1}{2^n}\right] \end{aligned}$$

Example 4·47. *A coin is tossed $(m+n)$ times, (mn). Show that the probability of at least m consecutive heads is* $\dfrac{n+2}{2^{m+1}}$.

[Kurukshetra Univ. M.Sc. 1990; Calcutta Univ. B.Sc.(Hons.),1986]

Solution. Since $m > n$, only one sequence of m consecutive heads is possible. This sequence may start either with the first toss or second toss or third toss, and so on, the last one will be starting with $(n+1)th$ toss.

Let E_i denote the event that the sequence of m consecutive heads starts with ith toss. Then the required probability is

$$P(E_1) + P(E_2) + \ldots + P(E_{n+1}) \qquad \ldots(*)$$

Now $P(E_1) = P$ [Consecutive heads in first m tosses and head or tail in the rest]

$$= \left(\frac{1}{2}\right)^m$$

$P(E_2) = P$ [Tail in the first toss, followed by m consecutive heads and head or tail in the next]

$$= \frac{1}{2}\left(\frac{1}{2}\right)^m = \frac{1}{2^{m+1}}$$

In general,

$P(E_r) = P$ [tail in the $(r-1)$th trial followed by m consecutive heads and head or tail in the next]

$$= \frac{1}{2}\left(\frac{1}{2}\right)^m = \frac{1}{2^{m+1}}, \quad \forall \quad r = 2, 3, \ldots, n+1.$$

Substituting in $(*)$,

$$\text{Required probability} = \frac{1}{2^m} + \frac{n}{2^{m+1}} = \frac{2+n}{2^{m+1}}$$

Example 4·48. *Cards are dealt one by one from a well-shuffled pack until an ace appears. Show that the probability that exactly n cards are dealt before the first ace appears is*

$$\frac{4(51-n)(50-n)(49-n)}{52.51.50.49}$$ **[Delhi Univ. B.Sc. 1992]**

Solution. Let E_i denote the event that an ace appears when the ith card is dealt. Then the required probability 'p' is given by

$$\begin{aligned} p &= P\text{[Exactly } n \text{ cards are dealt before the first ace appears]}\\ &= P\ \text{[The first ace appears at the } (n+1)\text{th dealing]}\\ &= P(\overline{E}_1 \cap \overline{E}_2 \cap \overline{E}_3 \cap \ldots \cap \overline{E}_{n-1} \cap \overline{E}_n \cap E_{n+1})\\ &= P(\overline{E}_1)\ P(\overline{E}_2 \mid \overline{E}_1)\ P(\overline{E}_3 \mid \overline{E}_1 \cap \overline{E}_2) \ldots\\ &\quad \times P(\overline{E}_n \mid \overline{E}_1 \cap \overline{E}_2 \cap \ldots \cap \overline{E}_{n-1}) \times P(E_{n+1} \mid \overline{E}_1 \cap \overline{E}_2 \cap \ldots \cap \overline{E}_n) \end{aligned} \qquad \ldots(*)$$

Now

$$P(E_1) = \frac{4}{52} \quad \Rightarrow \quad P(\overline{E}_1) = \frac{48}{52}$$

$$P(E_2 \mid \overline{E}_1) = \frac{4}{51} \quad \Rightarrow \quad P(\overline{E}_2 \mid \overline{E}_1) = \frac{47}{51}$$

$$P(E_3 \mid \bar{E}_1 \cap \bar{E}_2) = \frac{4}{50} \quad \Rightarrow \quad P(\bar{E}_3 \mid \bar{E}_1 \cap \bar{E}_2) = \frac{46}{50}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$P(E_{n-1} \mid \bar{E}_1 \cap \bar{E}_2 \cap \ldots \cap \bar{E}_{n-2}) = \frac{4}{52-(n-2)}$$

$$\therefore \quad P(\bar{E}_{n-1} \mid \bar{E}_1 \cap \bar{E}_2 \cap \ldots \cap \bar{E}_{n-2}) = \frac{50-n}{52-(n-2)}$$

$$P(E_n \mid \bar{E}_1 \cap \bar{E}_2 \cap \ldots \cap \bar{E}_{n-1}) = \frac{4}{52-(n-1)}$$

$$\therefore \quad P(\bar{E}_n \mid \bar{E}_1 \cap \bar{E}_2 \cap \ldots \cap \bar{E}_{n-1}) = \frac{49-n}{52-(n-1)}$$

and $$P(E_{n+1} \mid \bar{E}_1 \cap \bar{E}_2 \cap \ldots \cap \bar{E}_n) = \frac{4}{52-n}$$

Hence, from (*) we get

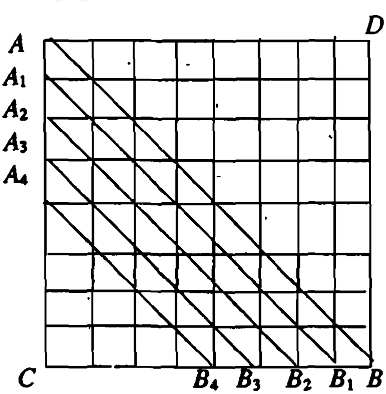
$$p = \left[\frac{48}{52} \times \frac{47}{51} \times \frac{46}{50} \times \frac{45}{49} \times \frac{44}{48} \times \frac{43}{47} \times \ldots \times \frac{52-n}{52-(n-4)} \right.$$
$$\left. \times \frac{51-n}{52-(n-3)} \times \frac{50-n}{52-(n-2)} \times \frac{49-n}{52-(n-1)} \times \frac{4}{52-n}\right]$$

$$= \frac{(51-n)(50-n)(49-n)\,4}{52 \times 51 \times 50 \times 49}$$

Example 4·49. *If four squares are chosen at random on a chess-board, find the chance that they should be in a diagonal line.*

[Delhi Univ. B.Sc. (Stat. Hons.), 1988]

Solution. In a chess-board there are $8 \times 8 = 64$ squares as shown in the following diagram.



Let us consider the number of ways in which the 4 squares selected at random are in a diagonal line parallel to AB. Consider the ΔABC. Number of ways in which 4 selected squares are along the lines A_4B_4, A_3B_3, A_2B_2, A_1B_1 and AB are ${}^4C_4$, ${}^5C_4$, ${}^6C_4$, ${}^7C_4$ and ${}^8C_4$ respectively.

Similarly, in ΔABD there are an equal number of ways of selecting 4 squares in a diagonal line parallel to AB.

Hence, total number of ways in which the 4 selected squares are in a diagonal line parallel to AB are $2({}^4C_4 + {}^5C_4 + {}^6C_4 + {}^7C_4) + {}^8C_4$.

Since there is an equal number of ways in which 4 selected squares are in a diagonal line parallel to CD, the required number of favourable cases is given by

$$2\,[\,2(\,^4C_4 + {}^5C_4 + {}^6C_4 + {}^7C_4) + {}^8C_4\,]$$

Since 4 squares can be selected out of 64 in $^{64}C_4$ ways, the required probability is

$$= \frac{2\,[\,2(\,^4C_4 + {}^5C_4 + {}^6C_4 + {}^7C_4) + {}^8C_4]}{^{64}C_4}$$

$$= \frac{[4\,(\,1 + 5 + 15 + 35\,) + 140\,] \times 4\,!}{64 \times 63 \times 62 \times 61} = \frac{91}{158844}$$

Example 4·50. *An urn contains four tickets marked with numbers 112, 121, 211, 222 and one ticket is drawn at random. Let A_i, $(i=1, 2, 3)$ be the event that ith digit of the number of the ticket drawn is 1. Discuss the independence of the events A_1, A_2 and A_3.* **[Delhi Univ. B.Sc.(Stat. Hons.),1987; Poona Univ. B.Sc.,1986]**

Solution. We have

$$P(A_1) = \frac{2}{4} = \frac{1}{2} = P(A_2) = P(A_3)$$

$A_1 \cap A_2$ is the event that the first two digits in the number which the selected ticket bears are each equal to unity and the only favourable case is ticket with number 112.

$$\therefore \qquad P(A_1 \cap A_2) = \frac{1}{4} = \frac{1}{2}\cdot\frac{1}{2}$$
$$= P(A_1)\,P(A_2)$$

Similarly,

$$P(A_2 \cap A_3) = \frac{1}{4} = P(A_2)\,P(A_3)$$

and
$$P(A_3 \cap A_1) = \frac{1}{4} = P(A_3)\,P(A_1)$$

Thus we conclude that the events A_1, A_2 and A_3 are pairwise independent.

Now $P(A_1 \cap A_3 \cap A_3) = P$ {all the three digits in the number are 1's}

$$= P(\phi)$$
$$= 0 \neq P(A_1)\,P(A_2)\,P(A_3)$$

Hence A_1, A_2 and A_3 though pairwise independent are not mutually independent.

Example 4·51. *Two fair dice are thrown independently. Three events A, B and C are defined as follows:*

A : Odd face with first dice
B : Odd face with second dice
C : Sum of points on two dice is odd.

Are the events A, B and C mutually independent?

[Delhi Univ. B.Sc. (Stat. Hons.) 1983; M.S. Baroda Univ. B.Sc.1987]

Solution. Since each of the two dice can show any one of the six faces 1, 2, 3, 4, 5, 6, we get :

$$P(A) = \frac{3\times 6}{36} = \frac{1}{2} \qquad [\because A = \{1, 3, 5\} \times \{1, 2, 3, 4, 5, 6\}]$$

$$P(B) = \frac{3\times 6}{36} = \frac{1}{2} \qquad [\because B = \{1, 2, 3, 4, 5, 6\} \times \{1, 3, 5\}]$$

The sum of points on two dice will be odd if one shows odd number and the other shows even number. Hence favourable cases for C are :

(1, 2), (1, 4), (1, 6) ; (4, 1), (4, 3), (4, 5)
(2, 1), (2, 3), (2, 5) ; (5, 2), (5, 4), (5, 6)
(3, 2), (3, 4), (3, 6) ; (6, 1), (6, 3), (6, 5)

i.e., 18 cases in all.

Hence $P(C) = \frac{18}{36} = \frac{1}{2}$.

Cases favourable to the events $A \cap B, A \cap C, B \cap C$ and $A \cap B \cap C$ are given below :

Event	*Favourable cases*
$A\cap B$	(1,1), (1, 3), (1, 5), (3, 1), (3, 3), (3, 5), (5, 1) (5, 3) (5, 5), *i.e.*, 9 in all.
$A\cap C$	(1, 2), (1, 4), (1, 6), (3, 2), (3, 4), (3, 6), (5, 2), (5, 4) (5, 6), *i.e.*, 9 in all.
$B\cap C$	(2, 1), (4, 1), (6, 1) (2, 3), (4, 3), (6, 3), (2, 5), (4, 5), (6, 5), *i.e.*, 9 in all
$A\cap B\cap C$	Nil, because $A\cap B$ implies that sum of points on two dice is even and hence $(A\cap B)\cap C = \phi$

$$\therefore \qquad P(A\cap B) = \frac{9}{36} = \frac{1}{4} = P(A)\cdot P(B)$$

$$P(A\cap C) = \frac{9}{36} = \frac{1}{4} = P(A)\,P(C)$$

$$P(B\cap C) = \frac{9}{36} = \frac{1}{4} = P(B)\,P(C)$$

and $\qquad P(A\cap B\cap C) = P(\phi) = 0 \neq P(A)\,P(B)\,P(C)$

Hence the events A, B and C are pairwise independent but not mutually independent.

Example 4·52. *Let $A_1, A_2, \ldots, A_n$ be independent events and $P\{A_k\} = p_k$. Further, let p be the probability that none of the events occurs; then show that*

$$p \leq e^{-\Sigma p_k}$$

[Agra Univ. M.Sc., 1987]

Solution. We have

$$p = P(\bar{A}_1 \cap \bar{A}_2 \cap \ldots \cap \bar{A}_n)$$

$$= \prod_{i=1}^{n} P(\bar{A}_i) = \prod_{i=1}^{n} [1 - P(A_i)] = \prod_{i=1}^{n} (1 - p_i)$$

[since A_i 's are independent]

$$\leq \prod_{i=1}^{n} e^{-p_i} \qquad [\because 1 - x \leq e^{-x} \text{ for } 0 \leq x \leq 1 \text{ and } 0 \leq p_i \leq 1]$$

$$\Rightarrow \qquad p \leq \exp\left[-\sum_{i=1}^{n} p_i\right],$$

as desired.

Remark. We have

$$1 - x \leq e^{-x} \text{ for } 0 \leq x \leq 1 \qquad \ldots(*)$$

Proof. The inequality (*) is obvious for $x = 0$ and $x = 1$. Consider $0 < x < 1$. Then

$$\log(1-x)^{-1} = -\log(1-x)$$

$$= \left[x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots\right], \qquad \ldots(**)$$

the expansion being valid since $0 < x < 1$. Further since $x > 0$, we get from (**)

$$\log(1-x)^{-1} > x$$

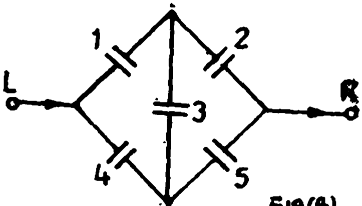
$$\Rightarrow \qquad -\log(1-x) > x$$

$$\Rightarrow \qquad \log(1-x) < -x$$

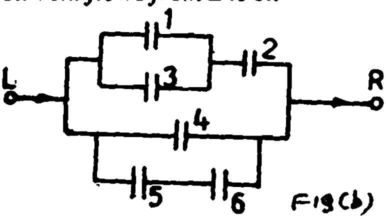
$$\Rightarrow \qquad 1 - x < e^{-x},$$

as desired.

Example 4·53. *In the following Fig.(a) and (b) assume that the probability of a relay being closed is P and that a relay is open or closed independently of any other. In each case find the probability that current flows from L to R.*



Fig(a)



Fig(b)

Solution. Let A_i denote the event that the relay i, ($i = 1, 2, \ldots, 6$) is closed. Let E be the event that current flows from L to R.

In Fig. (a) the current will flow from L to R if at least one of the circuits from L to R is closed. Thus for the current to flow from L to R we have the following favourable cases:

(i) $A_1 \cap A_2 = B_1$, (ii) $A_4 \cap A_5 = B_2$,

(iii) $A_1 \cap A_3 \cap A_5 = B_3$, (iv) $A_4 \cap A_3 \cap A_2 = B_4$,

The probability p_1 that current flows from L to R is given by

$$p_1 = P(B_1 \cup B_2 \cup B_3 \cup B_4) = \sum_i P(B_i) - \sum_{i<j} P(B_i \cap B_j) + \sum_{i<j<k} P(B_i \cap B_j \cap B_k) - P(B_1 \cap B_2 \cap B_3 \cap B_4) \qquad ...(*)$$

Since the relays operate independently of each other, we have

$$P(B_1) = P(A_1 \cap A_2) = P(A_1) \cdot P(A_2) = p \cdot p = p^2$$
$$P(B_2) = P(A_4 \cap A_5) = P(A_4) \cdot P(A_5) = p \cdot p = p^2$$
$$P(B_3) = P(A_1)\, P(A_3)\, P(A_5) = p^3$$
$$P(B_4) = P(A_4)\, P(A_3)\, P(A_2) = p^3$$

Similarly

$$P(B_1 \cap B_2) = P(A_1 \cap A_2 \cap A_4 \cap A_5) = P(A_1)\, P(A_2)\, P(A_4)\, P(A_5) = p^4$$
$$P(B_1 \cap B_2 \cap B_3) = P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5) = p^5$$

and so on. Finally, substituting in (*), we get

$$p_1 = (p^2 + p^2 + p^3 + p^3) - (p^4 + p^4 + p^4 + p^4 + p^4 + p^5) + (p^5 + p^5 + p^5 + p^5) - p^5$$
$$= 2p^2 + 2p^3 - 5p^4 + 2p^5$$

In Fig. (b). Arguing as in the above case, the required probability p_2 that the current flows from L to R is given by

$$p_2 = P(E_1 \cup E_2 \cup E_3 \cup E_4)$$

where

$$E_1 = A_1 \cap A_2,\ E_2 = A_3 \cap A_2,\ E_3 = A_4,\ E_4 = A_5 \cap A_6$$

$$\therefore \quad p_2 = \sum_i P(E_i) - \sum_{i<j} P(E_i \cap E_j) + \sum_{i<j<k} P(E_i \cap E_j \cap E_k) - P(E_1 \cap E_2 \cap E_3 \cap E_4)$$
$$= (p^2 + p^2 + p + p^2) - (p^3 + p^3 + p^4 + p^3 + p^4 + p^3) + (p^4 + p^5 + p^5 + p^5) - p^6$$
$$= p + 3p^2 - 4p^3 - p^4 + 3p^5 - p^6$$

Matching Problem. Let us have n letters corresponding to which there exist n envelopes bearing different addresses. Considering various letters being put in various envelopes, *a match is said to occur* if a letter goes into the right envelope. (Alternatively, if in a party there are n persons with n different hats, *a match is said to occur* if in the process of selecting hats *at random*, the ith person rightly gets the ith hat.)

A match at the kth position for k=1, 2, ..., n. Let us first consider the event A_k when a match occurs at the kth place. For better understanding let us put the envelopes bearing numbners 1, 2, ..., n in ascending order. When A_k occurs, k th letter goes to the kth envelope but $(n-1)$ letters can go to the remaining $(n-1)$ envelopes in $(n-1)\,!$ ways.

Hence $P(A_k) = \dfrac{(n-1)\,!}{n\,!} = \dfrac{1}{n}$,

where $P(A_k)$ denotes the probability of the kth match. It is interesting to see that *$P(A_k)$ does not depend on k.*

Example 4·54. (a) *'n' different objects 1, 2, ..., n are distributed at random in n places marked 1, 2, ..., n. Find the probability that none of the objects occupies the place corresponding to its number.* **[Calcutta Univ. B.A.(Stat.Hons.)1986; Delhi Univ. B.Sc.(Maths Hons.), 1990; B.Sc.(Stat.Hons.) 1988]**

(b) If n letters are randomly placed in correctly addressed envelopes, prove that the probability that exactly r leters are placed in correct envelopes is given by

$$\frac{1}{r\,!}\sum_{k=0}^{n-r}(-1)^k\frac{1}{k\,!}\,;\quad r = 1, 2, ..., n$$

[Bangalore Univ. B.Sc., 1987]

Solution (Probability of no match). Let E_i, $(i = 1, 2, ..., n)$ denote the event that the ith object occupies the place corresponding to its number so that $\overline{E}_i$, is the complementary event. Then the probability 'p' that none of the objects occupies the place corresponding to its number is given by

$$\begin{aligned} p &= P(\overline{E}_1 \cap \overline{E}_2 \cap \overline{E}_3 \cap \ldots \overline{E}_n) \\ &= 1 - P\ \{\text{at least one of the objects occupies the place corresponding to its number}\} \\ &= 1 - P(E_1 \cup E_2 \cup E_3 \cup \ldots \cup E_n) \\ &= 1 - \Big[\sum_{i=1}^{n} P(E_i) - \sum\sum_{\substack{i,j=1\\ i<j}}^{n} P(E_i \cap E_j) + \sum\sum\sum_{\substack{i,j,k=1\\ i<j<k}}^{n} P(E_i \cap E_j \cap E_k) - \ldots \\ &\qquad + (-1)^{n-1} P(E_1 \cap E_2 \cap \ldots \cap E_n)\Big] \qquad \ldots(*) \end{aligned}$$

Now $\quad P(E_i) = \dfrac{1}{n}, \ \forall\ i$

$$\begin{aligned} P(E_i \cap E_j) &= P(E_i)\,P(E_j \mid E_i) \\ &= \frac{1}{n}\cdot\frac{1}{n-1}, \ \forall\ i, j\ (i<j) \\ P(E_i \cap E_j \cap E_k) &= P(E_i)\,P(E_j \mid E_i)\,P(E_k \mid E_i \cap E_j) \\ &= \frac{1}{n}\cdot\frac{1}{n-1}\cdot\frac{1}{n-2}, \ \forall\ i, j, k\ (i<j<k) \end{aligned}$$

and so on. Finally,

$$P(E_1 \cap E_2 \cap E_3 \cap \ldots \cap E_n) = \frac{1}{n}\cdot\frac{1}{n-1}\cdot\frac{1}{n-2}\cdots\frac{1}{2}\cdot 1$$

Substituting in (*), we get

$$p = 1 - \left[{}^nC_1 \frac{1}{n} - {}^nC_2 \frac{1}{n(n-1)} + {}^nC_3 \frac{1}{n(n-1)(n-2)} - \dots + (-1)^{n-1} \frac{1}{n(n-1)\dots 3 \cdot 2 \cdot 1} \right]$$

$$= 1 - \left[1 - \frac{1}{2!} + \frac{1}{3!} - \dots + (-1)^{n-1} \frac{1}{n!} \right]$$

$$= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots + (-1)^n \frac{1}{n!}$$

$$= \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

Remark. For large n,

$$p = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots$$

$$= e^{-1} = 0{\cdot}36787$$

Hence the *probability of at least one match is*

$$1 - p = 1 - \frac{1}{2!} + \frac{1}{3!} - \dots + \frac{(-1)^n}{n!}$$

$$= 1 - \frac{1}{e}, \text{ (for large } n)$$

(b) **[Probability of exactly r matches $\{r \leq (n-2)\}$]** Let A_i, $(i = 1, 2, \dots, n)$ denote the event that ith letter goes to the correct envelope. Then the probability that none of the n letters goes to the correct envelope is

$$P(\bar{A}_1 \cap \bar{A}_2 \cap \dots \cap \bar{A}_n) = \sum_{k=0}^{n} (-1)^k / k! \qquad \dots(**) \text{ [(c.f. part (a)]}$$

The probability that each of the 'r' letters is in the right envelope is $\dfrac{1}{n(n-1)(n-2)\dots(n-r+1)}$, and the probability that none of the remaining $(n-r)$ letters goes in the correct envelope is obtained by replacing n by $(n-r)$ in (**) and is thus given by $\displaystyle\sum_{k=0}^{n-r} \frac{(-1)^k}{k!}$. Hence by compound probability theorem, the probability that out of n letters exactly r letters go to correct envelopes, (in a specified order), is

$$\frac{1}{n(n-1)(n-2)\dots(n-r+1)} \sum_{k=0}^{n-r} \frac{(-1)^k}{k!}\ ; \quad r \leq n-2.$$

Since r letters can go to n envelopes in ${}^nC_r$ mutually exclusive ways, the required probability of exactly r letters going to correct envelopes, (in any order, whatsoever), is given by

$$^nC_r \times \frac{1}{n(n-1)(n-2)\dots(n-r+1)} \sum_{k=0}^{n-r} \frac{(-1)^k}{k!} = \frac{1}{r!} \sum_{k=0}^{n-r} (-1)^k \frac{1}{k!}$$

Example 4·55. *Each of the n urns contains 'a' white balls and 'b' black balls. One ball is transferred from the first urn to the second, then one ball from the latter into the third, and so on. If p_k is the probability of drawing a white ball from the kth urn, show that*

$$p_{k+1} = \frac{a+1}{a+b+1} p_k + \frac{a}{a+b+1} (1-p_k)$$

Hence for the last urn, prove that

$$p_n = \frac{a}{a+b}$$ **[Punjab Univ. B.Sc.(Maths Hons.),1988]**

Solution. The event of drawing a white ball from the *k*th urn can materialise in the following two ways:

(i) The ball transferred from the $(k-1)$th urn is white and then a white ball is drawn from the kth urn.

(ii) The ball transferred from the $(k-1)$th urn is black and then a white ball is drawn from the *k*th urn.

The probability of case *(i)* is $p_{k-1} \times \dfrac{a+1}{a+b+1}$,

since the probability of drawing a white ball from the $(k-1)$th urn is p_{k-1} and then the probability of drawing white ball from the *k*th urn is

$$\frac{a+1}{a+b+1}.$$

Since the probability of drawing a black ball from the $(k-1)$th urn is $[1-p_{k-1}]$ and then the probability of drawing a white ball from the *k*th urn is

$$\frac{a}{a+b+1},$$

the probability of case *(ii)* is given by

$$\frac{a}{a+b+1} [1-p_{k-1}]$$

Since the cases *(i)* and *(ii)* are mutually exclusive, we have by addition theorem of probability

$$p_k = \frac{a+1}{a+b+1} p_{k-1} + \frac{a}{a+b+1} [1-p_{k-1}] \quad \dots(*)$$

$$\therefore \quad p_k = \frac{1}{a+b+1} p_{k-1} + \frac{a}{a+b+1} \quad \dots(1)$$

Replacing k by $k+1$ in (*) we get the required result.

Changing k to $k-1, k-2, \dots$ and so on, we get

$$p_{k-1} = \frac{1}{a+b+1} p_{k-2} + \frac{a}{a+b+1} \quad \dots(2)$$

$$p_{k-2} = \frac{1}{a+b+1}\, p_{k-3} + \frac{a}{a+b+1} \qquad \text{...(3)}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$p_2 = \frac{1}{a+b+1}\, p_1 + \frac{a}{a+b+1} \qquad \text{...}(k-1)$$

But p_1 = Probability of drawing a white ball from the first urn $= \dfrac{a}{a+b}$.

Multiplying (1) by 1, (2) by $\dfrac{1}{a+b+1}$, (3) by $\left(\dfrac{1}{a+b+1}\right)^2$, ..., and $(k-1)$th equation by $\left(\dfrac{1}{a+b+1}\right)^{k-2}$ and adding, we get

$$p_k = \left(\frac{1}{a+b+1}\right)^{k-1} p_1 + \frac{a}{a+b+1}\left[1 + \frac{1}{a+b+1} + \frac{1}{(a+b+1)^2} + \dots + \left(\frac{1}{a+b+1}\right)^{k-2}\right]$$

$$= \left(\frac{1}{a+b+1}\right)^{k-1} \times \frac{a}{(a+b)} + \frac{a}{a+b+1}\left[\frac{1 - \left(\dfrac{1}{a+b+1}\right)^{k-1}}{\left(1 - \dfrac{1}{a+b+1}\right)}\right]$$

$$= \frac{a}{a+b}\left(\frac{1}{a+b+1}\right)^{k-1} + \frac{a}{a+b}\left[1 - \left(\frac{1}{a+b+1}\right)^{k-1}\right]$$

$$= \frac{a}{a+b}\left[\left(\frac{1}{a+b+1}\right)^{k-1} + \left\{1 - \left(\frac{1}{a+b+1}\right)^{k-1}\right\}\right]$$

$$= \frac{a}{a+b}, \quad (k = 1, 2, \dots, n)$$

Since the probability of drawing a white ball from the kth urn is independent of k, we have

$$p_n = \frac{a}{a+b}.$$

Example 4·56. *(i) Let the probability p_n that a family has exactly n children be αp^n when $n \geq 1$ and $p_o = 1 - \alpha p\,(1 + p + p^2 + \dots)$. Suppose that all sex distributions of n children have the same probability. Show that for $k \geq 1$, the probability that a family contains exactly k boys is $2\alpha \,.\, p^k/(2-p)^{k+1}$.*

(ii) Given that a family includes at least one boy, show that the probability that there are two or more boys is $p/(2-p)$.

Solution. We are given

$$p_n = P\,[\text{that a family has exactly } n \text{ children}]$$
$$= \alpha\, p^n,\ n \geq 1$$

and $\quad p_0 = 1 - \alpha p\,(1 + p + p^2 + \ldots)$

Let E_j be the event that the number of children in a family is j and let A be the event that a family contains exactly k boys. Then

$$P(E_j) = p_j;\ j = 0, 1, 2, \ldots$$

Now, since each child can have any of the two sex distributions (either boy or girl), the total number of possible distributions for a family to have 'j' children is 2^j.

$$\therefore \qquad P(A \mid E_j) = \frac{^jC_k}{2^j},\ j \geq k$$

and
$$P(A) = \sum_{j=k}^{\infty} P(E_j)\, P(A \mid E_j) = \sum_{j=k}^{\infty} p_j P(A \mid E_j)$$
$$= \sum_{j=k}^{\infty} \alpha\, p^j \left[\frac{^jC_k}{2^j}\right],\quad j \geq k \geq 1$$
$$= \alpha \sum_{j=k}^{\infty} \left(\frac{p}{2}\right)^j {}^jC_k$$
$$= \alpha \sum_{r=0}^{\infty} {}^{k+r}C_k \left(\frac{p}{2}\right)^{k+r} \qquad [\text{Put } j - k = r\,]$$
$$= \alpha \left(\frac{p}{2}\right)^k \sum_{r=0}^{\infty} {}^{k+r}C_r \left(\frac{p}{2}\right)^r \qquad [\because\ {}^nC_r = {}^nC_{n-r}]$$

We know that

$${}^{-n}C_r = (-1)^r \cdot {}^{n+r-1}C_r \quad \Rightarrow \quad (-1)^r \cdot {}^{-n}C_r = {}^{n+r-1}C_r$$
$$\therefore \qquad (-1)^r \cdot {}^{-(k+1)}C_r = {}^{k+r}C_r$$

Hence

$$P(A) = \alpha \left(\frac{p}{2}\right)^k \sum_{r=0}^{\infty} (-1)^r \cdot {}^{-(k+1)}C_r \left(\frac{p}{2}\right)^r$$
$$= \alpha \left(\frac{p}{2}\right)^k \sum_{r=0}^{\infty} {}^{-(k+1)}C_r \left(-\frac{p}{2}\right)^r$$
$$= \alpha \left(\frac{p}{2}\right)^k \left(1 - \frac{p}{2}\right)^{-(k+1)}$$
$$= \alpha \left(\frac{p}{2}\right)^k \frac{2^{k+1}}{(2-p)^{k+1}} = \frac{2\,\alpha\, p^k}{(2-p)^{k+1}}$$

(b) Let B denote the event that a family includes at least one boy and C denote the event that a family has two or more boys. Then

$$P(B) = \sum_{k=1}^{\infty} P\,[\text{family has exactly } k \text{ boys}]$$

$$= \sum_{k=1}^{\infty} \frac{2\,\alpha\, p^k}{(2-p)^{k+1}} = \frac{2\,\alpha}{2-p} \sum_{k=1}^{\infty} \left(\frac{p}{2-p}\right)^k$$

$$= \frac{2\,\alpha}{2-p} \times \frac{p/(2-p)}{1-[p/(2-p)]} = \frac{\alpha\, p}{(1-p)(2-p)}$$

$$P(C) = \sum_{k=2}^{\infty} P\,[\text{family has exactly } k \text{ boys}]$$

$$= \sum_{k=2}^{\infty} \frac{2\,\alpha\, p^k}{(2-p)^{k+1}} = \frac{2\,\alpha}{2-p} \sum_{k=2}^{\infty} \left(\frac{p}{2-p}\right)^k$$

$$= \frac{2\,\alpha}{2-p} \cdot \frac{[p/(2-p)]^2}{1-[p/(2-p)]} = \frac{\alpha\, p^2}{(2-p)^2\,(1-p)}$$

Since $C \subset B$ and $B \cap C = C, P(B \cap C) = P(C) \Rightarrow P(B)\,P(C \mid B) = P(C)$

Therefore,

$$P(C \mid B) = \frac{P(C)}{P(B)} = \frac{\alpha\, p^2}{(2-p)^2\,(1-p)} \times \frac{(1-p)(2-p)}{\alpha\, p} = \frac{p}{2-p}$$

Example 4·57. *A slip of paper is given to person A who marks it either with a plus sign or a minus sign; the probability of his writing a plus sign is 1/3. A passes the slip to B, who may either leave it alone or change the sign before passing it to C. Next C passes the slip to D after perhaps changing the sign. Finally D passes it to a referee after perhaps changing the sign. The referee sees a plus sign on the slip. It is known that B, C and D each change the sign with probability 2/3. Find the probability that A originally wrote a plus.*

Solution. Let us define the following events :

E_1 : A wrote a plus sign; E_2 : A wrote a minus sign

E : The referee observes a plus sign on the slip.

We are given : $P(E_1) = 1/3$, $P(E_2) = 1 - 1/3 = 2/3$

We want $P(E_1 \mid E)$, which by Bayes rule is given by :

$$P(E_1 \mid E) = \frac{P(E_1)\,P(E \mid E_1)}{P(E_1)\,P(E \mid E_1) + P(E_2)\,P(E \mid E_2)} \qquad ...(i)$$

$P(E \mid E_1) = P$ [Referee observes the plus sign given that '*A*' wrote the plus sign an the slip]

$= P$ [(Plus sign was not changed at all) $\cup$ (Plus sign was changed exactly twice in passing from '*A*' to referee through B, C and D)]

$= P(E_3 \cup E_4)$, (say).

$= P(E_3) + P(E_4)$ ...(ii)

Let A_1, A_2 and A_3 respectively denote the events that B, C and D change the sign on the slip. Then we are given

$P(A_1) = P(A_2) = P(A_3) = 2/3 \quad ; \quad P(\bar{A}_1) = P(\bar{A}_2) = P(\bar{A}_3) = 1/3$

We have

$$P(E_3) = P(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3) = P(\bar{A}_1)\, P(\bar{A}_2)\, P(\bar{A}_3) = (1/3)^3 = 1/27$$

$$P(E_4) = P\,[(A_1 A_2 \bar{A}_3) \cup (A_1 \bar{A}_2 A_3) \cup (\bar{A}_1 A_2 A_3)]$$

$$= P(A_1 A_2 \bar{A}_3) + P(A_1 \bar{A}_2 A_3) + P(\bar{A}_1 A_2 A_3)$$

$$= P(A_1)\, P(A_2)\, P(\bar{A}_3) + P(A_1)\, P(\bar{A}_2)\, P(A_3) + P(\bar{A}_1)\, P(A_2)\, P(A_3)$$

$$= \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} + \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9},$$

Substituting in *(ii)* we get

$$P(E \mid E_1) = \frac{1}{27} + \frac{4}{9} = \frac{13}{27} \qquad ...(iii)$$

Similarly,

$P(E \mid E_2) = P$ [Referee observes the plus sign given that 'A' wrote minus sign on the slip]

$= P$ [(Minus sign was changed exactly once) $\cup$ (Minus sign was changed thrice)]

$= P(E_5 \cup E_6)$, (say),

$$= P(E_5) + P(E_6) \qquad ...(iv)$$

$$P(E_5) = P\,[(A_1 \bar{A}_2 \bar{A}_3) \cup (\bar{A}_1 A_2 \bar{A}_3) \cup (\bar{A}_1 \bar{A}_2 A_3)]$$

$$= P(A_1)\, P(\bar{A}_2)\, P(\bar{A}_3) + P(\bar{A}_1)\, P(A_2)\, P(\bar{A}_3) + P(\bar{A}_1)\, P(\bar{A}_2)\, P(A_3)$$

$$= \frac{2}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$$

$$P(E_6) = P(A_1 A_2 A_3) = P(A_1)\, P(A_2)\, P(A_3) = \frac{2}{3} \cdot \frac{2}{3} \cdot \frac{2}{3} = \frac{8}{27}$$

Substituting in *(iv)* we get :

$$P(E \mid E_2) = \frac{2}{9} + \frac{8}{27} = \frac{14}{27}. \qquad ...(v)$$

Substituting from *(iii)* and *(v)* in *(i)* we get :

$$P(E_1 \mid E) = \frac{\frac{1}{3} \times \frac{13}{27}}{\frac{1}{3} \times \frac{13}{27} + \frac{2}{3} \times \frac{14}{27}} = \frac{13}{13 + 28} = \frac{13}{41}$$

Example 4·58. *Three urns of the same appearance have the following proportion of balls.*

First urn	:	*2 black*	*1 white*
Second Urn	:	*1 black*	*2 white*
Third urn	:	*2 black*	*2 white*

One of the urns is selected and one ball is drawn. It turns out to be white. What is the probability of drawing a white ball again, the first one not having been returned?

Solution. Let us define the events:

E_i = The event of selection of ith urn, $(i = 1,2,3)$

and A = The event of drawing a white ball.

Then

$$P(E_1) = P(E_2) = P(E_3) = 1/3$$

and $P(A \mid E_1) = 1/3, P(A \mid E_2) = 2/3$ and $P(A \mid E_3) = 1/2$

Let C denote the future event of drawing another white ball from the urns. Then

$$P(C \mid E_1 \cap A) = 0, P(C \mid E_2 \cap A) = 1/2, \text{ and } P(C \mid E_3 \cap A) = 1/3$$

$$\therefore \quad P(C \mid A) = \frac{\sum\limits_{i=1}^{3} P(E_i) P(A \mid E_i) P(C \mid E_i \cap A)}{\sum\limits_{i=1}^{3} P(E_i) P(A \mid E_i)}$$

$$= \frac{\frac{1}{3}\cdot\frac{1}{3}\cdot 0 + \frac{1}{3}\cdot\frac{2}{3}\cdot\frac{1}{2} + \frac{1}{3}\cdot\frac{1}{2}\cdot\frac{1}{3}}{\frac{1}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{2}{3} + \frac{1}{3}\cdot\frac{1}{2}} = \frac{1}{3}$$

MISCELLANEOUS EXERCISE ON CHAPTER IV

1. Probabilities of occurrence of n independent events $E_1, E_2, ..., E_n$ are $p_1, p_2, ..., p_n$ respectively. Find the probability of occurrence of the compound event in which $E_1, E_2, ..., E_r$ occur and $E_{r+1}, E_{r+2}, ..., E_n$ do not occur.

Ans. $\prod\limits_{i=1}^{r} p_i \times \prod\limits_{i=r+1}^{n} (1 - p_i)$

2. Prove that for any integer $m \geq 1$,

(a) $P\left(\bigcap\limits_{i=1}^{m} A_i\right) \leq P(A_i) \leq P\left(\bigcup\limits_{i=1}^{m} A_i\right) \leq \sum\limits_{i=1}^{m} P(A_i)$

(b) $P\left(\bigcap\limits_{i=1}^{m} A_i\right) \geq 1 - \sum\limits_{i=1}^{m} P(\overline{A}_i)$

3. Establish the inequalities :

$$P(A \cap B \cap C) \leq P(A \cap B) \leq P(A \cup B) \leq P(A \cup B \cup C) \leq P(A) + P(B) + P(C)$$

4. Let $A_1, A_2, ..., A_n$ be mutually independent events with $P(A_k) = p_k$, $k = 1, 2, ..., n$.

Let p be the probability that none of the events $A_1, A_2, ..., A_n$ occurs. Show that

$$p = \prod_{k=1}^{n} (1 - p_k) \leq \exp\left\{-\sum_{k=1}^{n} p_k\right\}$$

Use the above relation to compute the probability that in six tosses of a fair die, no "aces are obtained". Compare this with the upper bound given above. Show that if each p_k is small compared with n, the upper bound is a good approximation.

5. A and B play a match, the winner being the one who first wins two games in succession, no games being drawn. Their respective chances of winning a particular game are $p : q$. Find

(i) A's initial chance of winning.

(ii) A's chance of winning after having won the first game.

6. A carpenter has a tool chest with two compartments, each one having a lock. He has two keys for each lock, and he keeps all four keys in the same ring. His habitual procedure in opening a compartment is to select a key at random and try it. If it fails, he selects one of the remaining three and tries it and so on. Show that the probability that he succeeds on the first, second and third try is 1/2, 1/3, 1/6 respectively. **(Lucknow Univ. B.Sc., 1990)**

7. Three players A, B and C agree to play a series of games observing the following rules : two players participate in each game, while third is idle, and the game is to be won by one of them. The loser in each game quits and his place in the next game is taken by the player who was idle. The player who succeeds in winning over both of his opponents without interruption wins the whole series of games.

Supposing the probability for each player to win a single game is 1/2, and that the first game is played by A and B, find the probability for A, B and C respectively to win the whole series if the number of games is unlimited.

Ans. 5/14, 5/14, 2/7

8. In a certain group of mathematicians, 60 per cent have insufficient background of modern Algebra, 50 per cent have inadequate knowledge of Mathematical Statistics and 80 per cent are in either one or both of the two categories. What is the percentage of people who know Mathematical Statistics among those who have a sufficient background of Modern Algebra? **(Ans. 0·50)**

9. (a) If A has $(n+1)$ and B has n fair coins, which they flip, show that the probability that A gets more heads than B is $\frac{1}{2}$.

(b) A student is given a column of 10 dates and column of 10 events and is asked to match the correct date to each event. He is not allowed to use any item more than once. Consider the case where the student knows how to match four of the items but he is very doubtful of the remaining six. He decides to match these at random. Find the probabilities that he will correctly match (i) all the items, (ii) at least seven of the items, and (iii) at least five.

Ans. (a) $\dfrac{1}{6\,!}$, (b) $\dfrac{10}{6\,!}$, (c) $1-\dfrac{1}{6\,!}$

10. An astrologer claims that he can predict before birth the sex of a baby just to be born. Suppose that the astrologer has no real power but he tosses a coin just

once before every birth and if the head turns up he predicts a boy for that birth and if the tail turns up he predicts a girl. Let p be the probability of the event that at a certain birth a male child is born, and p' the probability of a head turning up in a single toss with astrologer's coin. Find the probability of a correct prediction and that of at least one correct prediction in n predictions.

11. From a pack of 52 cards an even number of cards is drawn. Show that the probability of half of these cards being red is

$$[52\,!/(26\,!)^2 - 1]\,/\,(2^{51} - 1)$$

12. A sportsman's chance of shooting an animal at a distance $r\,(> a)$ is a^2/r^2. He fires when $r = 2a$, and if he misses he reloads and fires when $r = 3a, 4a...$ If he misses at distance na, the animal escapes. Find the odds against the sportsman.

Ans. $n + 1 : n - 1$

Hint. P [Sportsman shoots at a distance ia] $= \dfrac{a^2}{(ia)^2} = \dfrac{1}{i^2}$

$\Rightarrow$ P [Sportsman misses the shot at a distance ia] $= 1 - \dfrac{1}{i^2}$

$$\therefore \quad P\,[\text{Animal escapes}] = \prod_{i=2}^{n}\left(1 - \frac{1}{i^2}\right) = \prod_{i=2}^{n}\left[\left(\frac{i-1}{i}\right)\left(\frac{i+1}{i}\right)\right]$$

$$= \prod_{i=2}^{n}\left(\frac{i-1}{i}\right)\prod_{i=2}^{n}\left(\frac{i+1}{i}\right) = \frac{n+1}{2n}$$

$$\text{Required ratio} = \frac{n+1}{2n} : \left(1 - \frac{n+1}{2n}\right) = (n+1) : (n-1)$$

13. *(a)* Pataudi, the captain of the Indian team, is reported to have observed the rule of calling 'heads' every time the toss was made during the five matches of the Test series with the Australian team. What is the probability of his winning the toss in all the five matches?

Ans. $(1/2)^5$

How will the probability be affected if

(i) he had made a rule of tossing a coin privately to decide whether to call "heads" or "tails" on each occasion.

(ii) the factors determining his choice were not predetermined but he called out whatever occurred to him on the spur of the moment?

(b) A lot contains 50 defective and 50 non-defective bulbs. Two bulbs are drawn at random one at a time, with replacement. The events A, B, C are defined as

$A =$ {The first bulb is defective}
$B =$ {The second bulb is non-defective}
$C =$ {The two bulbs are both defective or both non-defective}

Determine whether

(i) A, B, C are pairwise independent,

(ii) A, B, C are independent.

14. A, B and C are three urns which contain 2 white, 1 black, 3 white, 2 black and 2 white and 2 black balls, respectively. One ball is drawn from urn A and put into the urn B; then a ball is drawn from urn B and put into the urn C. Then a ball is drawn from urn C. Find the probability that the ball drawn is white.

Ans. 4/15.

15. An urn contains a white and b black balls and a series of drawings of one ball at a time is made, the ball removed being retrurned to the urn immediately after the next drawing is made. If p_n denotes the probability that the nth ball drawn is black, show that

$$p_n = (b - p_{n-1}) / (a + b - 1).$$

Hence find p_n.

16. A person is to be tested to see whether he can differentiate between the taste of two brands of cigarettes. If he cannot differentiate, it is assumed that the probability is one-half that he will identify a cigarette correctly. Under which of the following two procedures is there less chance that he will make all correct identifications when he actually cannot differentiate between the two brands?

(i) The subject is given four pairs each containing both brands of cigarettes (this is known to the subject), he must identify for each pair which cigarette represents each brand.

(ii) The subject is given eight cigarettes and is told that the first four are of one brand and the last four of the other brand.

How do you explain the difference in results despite the fact that eight cigarettes are tested in each case?

Ans. *(i)* 1/16 *(ii)* 1/2

17. (Sampling with replacement). A sample of size r is taken from a population of n people. Find the probability U_r that N given people will be included in the sample.

Ans. $U_r = \sum_{m=0}^{N} (-1)^m \binom{N}{m} \left(1 - \frac{m}{n}\right)^r$

18. In a lottery m tickest are drawn at a time out of the total number of n tickets, and returned before the next drawing is made. Show that the chance that in k drawings, each of the numbers 1, 2, 3, ..., n will appear at least once is given by

$$P_k = 1 - \binom{n}{1}\left(1 - \frac{m}{n}\right)^k + \binom{n}{2}\left(1 - \frac{m}{n}\right)^k \left(1 - \frac{m}{n-1}\right)^k - \dots$$

[Nagpur Univ. M.Sc. 1987]

19. In a certain book of N pages, no page contains more than four errors, n_1 of them contain one error, n_2 contain two errors, n_3 contain three errors and n_4 contain four errors. Two copies of the book are opened at any two given pages. Show the probability that the number of errors in these two pages shall not exceed five is

$$1 - \frac{1}{N^2}\,(n_3^2 + n_4^2 + 2n_2 n_4 + 2n_3 n_4)$$

Hint. Let E_i I : the event that a page of first book contains i errors.
and E_i II : the event that a page of second book contains i errors.

$$P\,(\text{No. of errors in the two pages shall not exceed 5})$$
$$= 1 - P\,[E_2\,\text{I}\,E_4\,\text{II} + E_3\,\text{I}\,E_4\,\text{II} + E_4\,\text{I}\,E_4\,\text{II} + E_3\,\text{I}\,E_3\,\text{II} + E_4\,\text{I}\,E_3\,\text{II} + E_4\,\text{I}\,E_2\,\text{II}\,].$$

20. *(a)* Of three independent events, the chance that the first only should happens is a, the chance of the second only is b and the chance of the third only is c. Show that the independent chances of the three events are respectively

$$\frac{a}{a+x},\ \frac{b}{b+x},\ \frac{c}{c+x}$$

where x is the root of the equation

$$(a+x)\,(b+x)\,(c+x) = x^2$$

Hint. $P\,(E_1 \cap \bar{E}_2 \cap \bar{E}_3) = P\,(E_1)\,[1 - P\,(E_2)]\,[1 - P\,(E_3)] = a \quad \ldots(*)$

$P\,(\bar{E}_1 \cap E_2 \cap \bar{E}_3) = [1 - P\,(E_1)]\,P\,(E_2)\,[1 - P\,(E_3)] = b \quad \ldots(**)$

$P\,(\bar{E}_1 \cap \bar{E}_2 \cap E_3) = [1 - P\,(E_1)]\,[1 - P\,(E_2)]\,P\,(E_3) = c \quad \ldots(***)$

Multiplying (*), (**) and (***), we get

$$P\,(E_1)\,P\,(E_2)\,P\,(E_3)\,x^2 = abc,$$

where $x = [1 - P\,(E_1)]\,[1 - P\,(E_2)]\,[1 - P\,(E_3)]$

Multiplying (*) by $[1 - P\,(E_1)]$, we get

$$P\,(E_1) = \frac{a}{a+x}, \text{ and so on.}$$

(b) Of three independent events, the probability that the first only should happens is 1/4, the probability that the second only should happen is 1/8, and the probability that the third only should happen is 1/12. Obtain the unconditional probabilities of the three events.

Ans. 1/2, 1/3, 1/4.

(c) A total of n shells are fired at a target. The probability of the ith shell hitting the target is p_i ; $i = 1, 2, 3, \ldots, n$. Assuming that the n firings are n mutually independent events, find the probability that at least two shells out of n hit the target. **[Calcutta Univ. B.Sc.(Maths Hons.), 1988]**

(d) An urn contains M balls numbered 1 to M, where the first K balls are defective and the remaining $M - K$ are non-defective. A sample of n balls is drawn from the urn. Let A_k be the event that the sample of n balls contains exactly k defectives. Find $P(A_k)$ when the sample is drawn *(i)* with replacement and, *(ii)* without replacement. **[Delhi Univ. B.Sc. (Maths Hons.), 1989]**

21. For three independent events A, B and C, the probability for A to occur is a, the probability that A, B and C will not occur is b, and the probability that at least one of the three events will not occur is c. If p denotes the probability that C occurs but neither A nor B occurs, prove that p satisfies the quadratic equation

$$ap^2 + [ab - (1-a)(a+c-1)]\,p + b(1-a)(1-c) = 0$$

and hence deduce that $c > \dfrac{(1-a)^2 + ab}{(1-a)}$

Further show that the probability of occurrence of C is $p/(p+b)$, and that of B's happening is $(1-c)(p+b)/ap$.

Hint. Let $P(A) = x$, $P(B) = y$ and $P(C) = z$

Then $\quad x = a,\ (1-x)(1-y)(1-z) = b,\ 1 - xyz = c$

and $\quad p = z(1-x)(1-y)$

Elimination of x, y and z gives quadratic equation in p.

22. *(a)* The chance of success in each trial is p. If p_k is the probability that there are even number of successes in k trials, prove that

$$p_k = p + p_{k-1}(1-2p)$$

Deduce that $\quad p_k = \frac{1}{2}[1 + (1-2p)^k]$

(b) If a day is dry, the conditional probability that the following day will also be dry is p; if a day is wet, the conditional probability that the following day will be dry is p'. If u_n is the probability that the nth day will be dry, prove that

$$u_n - (p - p')\,u_{n-1} - p' = 0\ ;\ n \geq 2$$

If the first day is dry, $p = 3/4$ and $p' = 1/4$, find u_n.

23. There are n similar biased dice such that the probability of obtaining a 6 with each one of them is the same and equal to p. If all the dice are rolled once, show that p_n, the probability that an odd number of 6's is obtained satisfies the difference equation

$$p_n + (2p-1)\,p_{n-1} = p$$

and hence derive an explicit expression for p_n.

Ans. $p_n = \frac{1}{2}[1 + (1-2p)^n]$

24. Suppose that each day the weather can be uniquely classified as 'fine' or 'bad'. Suppose further that the probability of having fine weather on the last day of a certain year is P_0 and we have the probability p that the weather on an arbitrary day will be of the same kind as on the preceding day. Let the probability of having fine weather on the nth day of the following year be P_n. Show that

$$P_n = (2p-1)\,P_{n-1} + (1-p)$$

Deduce that

$$P_3 = (2p-1)^3\left(P_0 - \frac{1}{2}\right) + \frac{1}{2}$$

25. A closet contains n pairs of shoes. If $2r$ shoes are chosen at random (with $2r < n$), what is the probability that there will be *(i)* no complete pair,

(ii) exactly one complete pair, *(iii)* exactly two complete pairs among them?

Hint. *(i)* $P(\text{no complete pair}) = \binom{n}{2r} 2^{2r} \div \binom{2n}{2r}$

(ii) $P(\text{exactly one complete pair}) = n\binom{n-1}{2r-2} 2^{2r-2} \div \binom{2n}{2r}$

and *(iii)* $P(\text{exactly two complete pairs}) = \binom{n}{2}\binom{n-2}{2r-4} 2^{2r-4} \div \binom{2n}{2r}$

26. Show that the probability of getting no right pair out of n, when the left foot shoes are paired randomly with the rigth foot shoes, is the sum of the first $(n+1)$ terms in the expansion of e^{-1}.

27. *(a)* In a town consisting of $(n+1)$ inhabitants, a person narrates a rumour to a second person, who in turn narrates it to a third person, and so on. At each step the recipient of the rumour is chosen at random from the n available persons, excluding the narrator himself. Find the probability that the rumour will be told r times without:

(i) returning to the originator,

(ii) being narrated to any person more than once.

(b) Do the above problem when, at each step the rumour is told by one person to a gathering of N randomly chosen people.

Ans. *(a)* *(i)* $\dfrac{n(n-1)^{r-1}}{n^r} = \left(1-\dfrac{1}{n}\right)^{r-1}$; *(ii)* $\dfrac{n(n-1)(n-2)\ldots(n-r+1)}{n^r}$

(b) *(i)* $\left(1-\dfrac{N}{n}\right)^{r-1}$; *(ii)* $\dfrac{\binom{n}{rN}}{\left[\binom{n}{N}\right]^r}$

28. What is the probability that *(i)* the birthdays of twelve people will fall in twelve different calendar months (assume equal probabilities for the twelve months) and *(ii)* the birthdays of six people will fall in exactly two calendar months?

Hint. *(i)* The birthday of the first person, for instance, can fall in 12 different ways and so for the second, and so on.

$\therefore$ The total number of cases $= 12^{12}$.

Now there are 12 months in which the birthday of one person can fall and 11 months in which the birthday of the second person can fall and 10 months for another third person, and so on.

$\therefore$ The total number of favourable cases $= 12.11.10\ldots3.2.1$

Hence the required probability $= \dfrac{12!}{12^{12}}$

(ii) The total number of ways in which the birthdays of 6 persons can fall in any of the month $= 12^6$.

$$\therefore \qquad \text{The required probability} = \frac{\binom{12}{2}\left(2^6-2\right)}{12^6}$$

29. An elevator starts with 7 passengers and stops at 10 floors. What is the probability p that no two passengers leave at the same floor?

[Delhi Univ. M.C.A., 1988]

30. A bridge player knows that his two opponents have exactly five hearts between two of them. Each opponent has thirteen cards. What is the probability that there is three-two split on the hearts (that is one player has three hearts and the other two)? **[Delhi Univ. B.Sc.(Maths Hons.), 1988]**

31. An urn contains 2 white and 2 black balls. A ball is drawn at random. If it is white, it is not replaced into the urn. Otherwise it is replaced along with another ball of the same colour. The process is repeated. Find the probability that the third ball drawn is black. **[Burdwan Univ. B.Sc. (Hons.), 1990]**

Ans. $\dfrac{23}{30}$

32. There is a series of n urns. In the ith urn there are i white and $(n-i)$ black balls, $i = 1, 2, 3, \ldots, k$. One urn is chosen at random and 2 balls are drawn from it. Both turn out to be white. What is the probability that the jth urn was chosen, where j is a particular number between 3 and n.

Hint. Let E_j denote the event of selection of jth urn, $j = 3, 4, \ldots, n$ and A denote the event of drawing of 2 white balls, then

$$P(A \mid E_j) = \left(\frac{j}{n}\right)\left(\frac{j-1}{n-1}\right),\ P(E_j) = \frac{1}{n},\ P(A) = \sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)\left(\frac{i-1}{n-1}\right)$$

$$\therefore \qquad P(E_j \mid A) = \frac{\frac{1}{n}\left(\frac{j}{n}\right)\left(\frac{j-1}{n-1}\right)}{\sum_{i=1}^{n}\left(\frac{1}{n}\right)\left(\frac{i}{n}\right)\left(\frac{i-1}{n-1}\right)}$$

33. There are $(N+1)$ identical urns marked $0, 1, 2, \ldots, N$ each of which contains N white and red balls. The kth urn contains k red and $N-k$ white balls, $(k = 0, 1, 2, \ldots N)$. An urn is chosen at random and n random drawings of a ball are made from it, the ball drawn being replaced after each draw. If the balls drawn are all red, show that the probability that the next drawing will also yield a red ball is approximately $(n+1)(n+2)$ when N is large.

34. A printing machine can print n letters, say $\alpha_1, \alpha_2, \ldots, \alpha_n$. It is operated by electrical impulses, each letter being produced by a different impulse. Assume that p is the constant probability of printing the correct letter and the impulses are independent. One of the n impulses, chosen at random, was fed into the machine twice and both times the letter α_1 was printed. Compute the probability that the impulse chosen was meant to print α_1. **[Delhi Univ. M.Sc.(Stat.), 1981]**

Ans. $(n-1)p^2/(np^2 - 2p + 1)$

35. Two players A and B agree to contest a match consisting of a series of games, the match to be won by the player who first wins three games, with the provision that if the players win two games each, the match is to continue until it

is won by one player winning two games more than his opponent. The probabililty of A winning any given game is p, and the games cannot be drawn.

(i) Prove that $f(p)$, the initial probability of A winning the match is given by:

$$f(p) = p^3 (4 - 5p + 2p^2)/(1 - 2p + 2p^2)$$

(ii) Show that the equation $f(p) = p$ has five real roots, of which three are admissible values of p. Find these three roots and explain their significance.

[Civil Services (Main), 1986]

36. Two players A and B start playing a series of games with Rs. a and b respectively. The stake is Re. 1 on a game and no game can be drawn. If the probability of A winning any game is a constant p, find the initial probability of his exhausting the funds of B or his own. Also show that if the resources of B are unlimited then

(i) A is certain to be ruined if $p = ½$, and

(ii) A has an even chance of escaping ruin if $p = 2^{1/a}/(1 + 2^{1/a})$.

Hint. Let u_n be the probability of A's final win when he has Rs. n.

Then $\quad u_n = pu_{n+1} + (1-p)u_{n-1}$ where $u_0 = 0$ and $u_{a+b} = 1$

$$\therefore \quad u_{n+1} - u_n = \left(\frac{1-p}{p}\right)(u_n - u_{n-1})$$

Hence $\quad u_{n+1} - u_n = \left(\frac{1-p}{p}\right)^n u_1$, by repeated application,

so that $$u_n = u_1\left[1 - \left(\frac{1-p}{p}\right)^n\right] \Big/ \left[1 - \left(\frac{1-p}{p}\right)\right]$$

Hence using $u_{a+b} = 1$, $\quad u_n = \left[1 - \left(\frac{1-p}{p}\right)^n\right] \Big/ \left[1 - \left(\frac{1-p}{p}\right)^{a+b}\right]$

$\therefore$ Initial probability of A's win is $u_a = \dfrac{p^a - (1-p)^a}{p^{a+b} - (1-p)^{a+b}} \cdot p^b$

Probability of A's ruin $= 1 - u_a$.

For $p = ½$, $u_a = \dfrac{a}{a+b} \to 0$ as $b \to \infty$ and for $p \neq ½$, $u_a = ½$ if $p = 2^{1/a}/(1 + 2^{1/a})$.

37. In a game of skill a player has probability 1/3, 5/12 and 1/4 of scoring 0, 1 and 2 points respectively at each trial, the game terminating on the first realization of a zero score at a trial. Assuming that the trials are independent, prove that the probability of the player obtaining a total score of n points is

$$u_n = \frac{3}{13}\left(\frac{3}{4}\right)^n + \frac{4}{39}\left(-\frac{1}{3}\right)^n$$

Hint. Event can materialize in the two mutually exclusive ways:

(i) at the $(n-1)$th trial, a score of $(n-1)$ points is obtained and a score of 1 point is obtained at the nth trial.

(ii) at the $(n-2)$th trial, a score of $(n-2)$ points is obtained and a score of 2 points is obtained at the last two trials.

Hence $u_n = \frac{5}{12} u_{n-1} + \frac{1}{4} u_{n-2}$ where $u_0 = \frac{1}{3}$, $u_1 = \frac{1}{3} \cdot \frac{5}{12} = \frac{5}{36}$

Also $u_n = \left(\frac{3}{4} - \frac{1}{3}\right) u_{n-1} + \frac{1}{4} u_{n-2} \Rightarrow u_n + \frac{1}{3} u_{n-1} = \frac{3}{4}\left(u_{n-1} + \frac{1}{3} u_{n-2}\right)$

This equation can be solved as a homogeneous difference equation of second order with the initial conditions

$$u_0 = \frac{1}{3}, \quad u_1 = \frac{1}{3} \cdot \frac{5}{12} = \frac{5}{36}$$

38. The following weather forecasting is used by an amateur forecaster. Each day is classified as 'dry' or 'wet' and the probability that any given day is same as the preceding one is assumed to be a constant p, $(0 < p < 1)$. Based on past records, it is supposed that January 1 has a probability β of being dry. Letting

$\beta_n =$ Probability that nth day of the year is dry, obtain an expression for β_n in terms of β and p. Also evaluate $\lim\limits_{n \to \infty} \beta_n$.

Hint. $\quad \beta_n = p \cdot \beta_{n-1} + (1-p)(1-\beta_{n-1})$

$\Rightarrow \quad \beta_n = (2p-1)\beta_{n-1} + (1-p)\ ; \quad n = 2, 3, 4, \ldots$

Ans. $\quad \beta_n = (2p-1)^{n-1}(\beta - ½) + ½\ ; \quad \lim\limits_{n \to \infty} \beta_n = ½$

39. Two urns contain respectively 'a white and b black' and 'b white and a black' balls. A series of drawings is made according to the following rules:

(i) Each time only one ball is drawn and immediately returned to the same urn it came from.

(ii) If the ball drawn is white, the next drawing is made from the first urn.

(iii) If it is black, the next drawing is made from the second urn.

(iv) The first ball drawn comes from the first urn.

What is the probability that nth ball drawn will be white?

Hint. $p_r = P$ [Drawing a white ball at the rth draw].

$$p_r = \frac{a}{a+b} p_{r-1} + \frac{b}{a+b} \cdot \left(1 - p_{r-1}\right)$$

$$\Rightarrow \quad p_r = \frac{a-b}{a+b} \cdot p_{r-1} + \frac{b}{a+b}$$

Ans. $$p_n = \frac{1}{2} + \frac{1}{2}\left(\frac{a-b}{a+b}\right)^n$$

40. If a coin is tossed repeatedly, show that the probability of getting m heads before n tails is :

$$\frac{1}{2^{m+n-1}} \sum_{i=m}^{m+n-1} {}^{m+n-1}C_i.$$

[Burdwan Univ. (Maths Hons.), 1991]

OBJECTIVE TYPE QUESTIONS

I. Find out the correct answer from group Y for each item of group X.

Group X	Group Y
(a) At least one of the events A or B occurs.	(i) $(\bar{A} \cap B) \cup (A \cap \bar{B}) \cup (\bar{A} \cap \bar{B})$
(b) Neither A nor B occurs.	(ii) $(A \cup B) - (A \cap B)$
(c) Exactly one of the events A or B occurs.	(iii) $A \subset B$
(d) If event A occurs, so does B.	(iv) $B \not\subset A$
(e) Not more than one of the events A or B occur.	(v) $[A - (A \cap B)] \cup [B - (A \cap B)]$
	(vi) $A \cap \bar{B}$
	(vii) $1 - (A \cup \bar{B})$
	(viii) $A \cup B$
	(ix) $1 - (A \cup B)$

II. Match the correct expression of probabilities on the left :

(a) $P(\phi)$, where ϕ is null set	(i) $1 - P(A)$
(b) $P(A \mid B) P(B)$	(ii) $P(A \cap B)$
(c) $P(\bar{A})$	(iii) $P(A) - P(A \cap B)$
(d) $P(\bar{A} \cap \bar{B})$	(iv) 0
(e) $P(A \sim B)$	(v) $1 - P(A) - P(B) + P(A \cap B)$
	(vi) $P(A) + P(B) - P(A \cap B)$.

III. Given that A, B and C are mutually exclusive events, explain why the following are not permissible assignments of probabilities:

(i) $P(A) = 0{\cdot}24$, $P(B) = 0{\cdot}4$ and $P(A \cup C) = 0{\cdot}2$

(ii) $P(A) = 0{\cdot}4$, $P(B) = 0{\cdot}61$

(iii) $P(A) = 0{\cdot}6$, $P(A \cap \bar{B}) = 0{\cdot}5$

IV. In each of the following, indicate whether events A and B are : (i) independent, (ii) mutually exclusive, (iii) dependent but not mutually exclusive.

(a) $P(A \cap B) = 0$ (b) $P(A \cap B) = 0{\cdot}3$, $P(A) = 0{\cdot}45$

(c) $P(A \cup B) = 0{\cdot}85$, $P(A) = 0{\cdot}3$, $P(B) = 0{\cdot}6$

(d) $P(A \cup B) = 0{\cdot}70$, $P(A) = 0{\cdot}5$, $P(B) = 0{\cdot}4$

(e) $P(A \cup B) = 0{\cdot}90$, $P(A \mid B) = 0{\cdot}8$, $P(B) = 0{\cdot}5$.

V. Give the correct label as answer like a or b etc., for the following questions:

(i) The probability of drawing any one spade card from a pack of cards is

(a) $\frac{1}{52}$ (b) $\frac{1}{13}$ (c) $\frac{4}{13}$ (d) $\frac{1}{4}$

(ii) The probability of drawing one white ball from a bag containing 6 red, 8 black, 10 yellow and 1 green balls is

(a) $\frac{1}{25}$ (b) 0 (c) 1 (d) $\frac{24}{25}$ (e) $\frac{15}{20}$

(iii) A coin is tossed three times in succession, the number of sample points in sample space is

(a) 6 *(b)* 8 *(c)* 3

(iv) In the simultaneous tossing of two perfect coins, the probability of having at least one head is

(a) $\frac{1}{2}$ *(b)* $\frac{1}{4}$ *(c)* $\frac{3}{4}$ *(d)* 1

(v) In the simultaneous tossing of two perfect dice, the probability of obtaining 4 as the sum of the resultant faces is

(a) $\frac{4}{12}$ *(b)* $\frac{1}{12}$ *(c)* $\frac{3}{12}$ *(d)* $\frac{2}{12}$

(vi) A single letter is selected at random from the word 'probability'. The probability that it is a vowel is

(a) $\frac{3}{11}$ *(b)* $\frac{2}{11}$ *(c)* $\frac{4}{11}$ *(d)* 0

(vii) An urn contains 9 balls, two of which are red, three blue and four black. Three balls are drawn at random. The chance that they are of the same colour is

(a) $\frac{5}{84}$ *(b)* $\frac{3}{9}$ *(c)* $\frac{3}{7}$ *(d)* $\frac{7}{17}$

(viii) A number is chosen at random among the first 120 natural numbers. The probability of the number chosen being a multiple of 5 or 15 is

(a) $\frac{1}{5}$ *(b)* $\frac{1}{8}$ *(c)* $\frac{1}{16}$

(ix) If A and B are mutually exclusive events, then

(a) $P(A \cup B) = P(A) \cdot P(B)$

(b) $P(A \cup B) = P(A) + P(B)$, *(c)* $P(A \cup B) = 0$.

(x) If A and B are two independent events, the probability that both A and B occur is $\frac{1}{8}$ and the probability that neither of them occurs is $\frac{3}{8}$. The probability of the occurrence of A is :

(a) $\frac{1}{2}$, *(b)* $\frac{1}{3}$, *(c)* $\frac{1}{4}$, *(d)* $\frac{1}{5}$.

VI. Fill in the blanks :

(i) Two events are said to be equally likely if ......

(ii) A set of events is said to be independent if ......

(iii) If $P(A) \cdot P(B) \cdot P(C) = P(A \cap B \cap C)$, then the events A, B, C are ......

(iv) Two events A and B are mutually exclusive if $P(A \cap B) = \ldots$ and are independent if $P(A \cap B) = \ldots$

(v) The probability of getting a multiple of 2 in a throw of a dice is 1/2 and of getting a multiple of 3 is 1/3. Hence probability of getting a multiple of 2 or 3 is ......

(vi) Let A and B be independent events and suppose the event C has probability 0 or 1. Then A, B and C are ...... events.

(vii) If A, B, C are pairwise independent and A is independent of $B \cup C$, then A, B, C are ...... independent.

(viii) A man has tossed 2 fair dice. The conditional probability that he has tossed two sixes, given that he has tossed at least one six is ......

(ix) Let A and B be two events such that $P(A) = 0{\cdot}3$ and $P(A \cup B) = 0{\cdot}8$. If A and B are independent events then $P(B) = \ldots$

VII. Each of following statements is either true or false. If it is true prove it, otherwise, give a counter example to show that it is false.

(i) The probability of occurrence of at least one of two events is the sum of the probability of each of the two events.

(ii) Mutually exclusive events are independent.

(iii) For any two events A and B, $P(A \cap B)$ cannot be less than either $P(A)$ or $P(B)$.

(iv) The conditional probability of A given B is always greater than $P(A)$.

(v) If the occurrence of an event A implies the occurrence of another event B then $P(A)$ cannot exceed $P(B)$.

(vi) For any two events A and B, $P(A \cup B)$ cannot be greater then either $P(A)$ or $P(B)$.

(vii) Mutually exclusive events are not independent.

(viii) Pairwise independence does not necessarily imply mutual independence.

(ix) Let A and B be events neither of which has probability zero. Then if A and B are disjoint, A and B are independent.

(x) The probability of any event is always a proper fraction.

(xi) If $0 < P(B) < 1$ so that $P(A \mid B)$ and $P(A \mid \overline{B})$ are both defined, then $P(A) = P(B)\, P(A \mid B) + P(\overline{B})\, P(A \mid \overline{B})$.

(xii) For two events A and B if

$P(A) = P(A \mid B) = 1/4$ and $P(A \mid \overline{B}) = 1/2$, then

(a) A and B are mutually exclusive.

(b) A and B are independent.

(c) A is a sub-event of B.

(d) $P(\overline{A} \mid B) = 3/4$. **[Delhi Univ. B.Sc.(Stat. Hons.), 1992]**

(xiii) Two events can be independent and mutually exclusive simultaneously.

(xiv) Let A and B be events, neither of which has probability zero. Prove or disprove the following :

(a) If A and B are disjoint, A and B are independent.

(b) If A and B are independent, A and B are disjoint.

(xv) If $P(A) = 0$, then $A = \phi$.

CHAPTER FIVE

Random Variables — Distribution Functions

5·1. Random Variable. Intuitively by a *random variable (r.v)* we mean a real number X connected with the outcome of a random experiment E. For example, if E consists of two tosses of a coin, we may consider the random variable which is the number of heads (0, 1 or 2).

Outcome :	*HH*	*HT*	*TH*	*TT*
Value of X :	2	1	1	0

Thus to each outcome ω , there corresponds a real number $X(\omega)$. Since the points of the sample space S correspond to outcomes, this means that a real number, which we denote by $X(\omega)$, is defined for each $\omega \in S$. From this standpoint, we define random variable to be a real function on S as follows:

" Let S be the sample space associated with a given random experiment. A real-valued function defined on S and taking values in R $(-\infty, \infty)$ is called a one-dimensional random variable. If the function values are ordered pairs of real numbers (i.e., vectors in two-space) the function is said to be a two- dimensional random variable. More generally, an n-dimensional random variable is simply a function whose domain is S and whose range is a collection of n-tuples of real numbers (vectors in n- space)."

For a mathematical and rigorous definition of the random variable, let us consider the probability space, the triplet (**S**, **B**, **P**), where S is the sample space, *viz.*, space of outcomes, **B** is the σ-field of subsets in S, and **P** is a probability function on **B**.

Def. A random variable (r.v.) is a function $X(\omega)$ with domain S and range $(-\infty, \infty)$ such that for every real number a, the event $[\omega : X(\omega) \leq a] \in \mathbf{B}$.

Remarks: 1. The refinement above is the same as saying that the function $X(\omega)$ is *measurable* real function on $(S, \mathbf{B})$.

2. We shall need to make probability statements about a random variable X such as $P\{X \leq a\}$. For the simple example given above we should write $P\{X \leq 1\} = P\{HH, HT, TH\} = 3/4$. That is, $P(X \leq a)$ is simply the probability of the set of outcomes ω for which $X(\omega) \leq a$ or

$$P(X \leq a) = P\{\omega : X(\omega) \leq a\}$$

Since **P** is a measure on $(S, \mathbf{B})$ *i.e.*, **P** is defined on subsets of **B**, the above probability will be defined *only* if $\{\omega : X(\omega) \leq a\} \in \mathbf{B}$, which implies that $X(\omega)$ is a measurable function on $(S, \mathbf{B})$.

3. One-dimensional random variables will be denoted by capital letters, $X, Y, Z, \ldots$ etc. A typical outcome of the experiment (*i.e.*, a typical element of the sample space) will be denoted by ω or e. Thus $X(\omega)$ represents the real number which the random variable X associates with the outcome ω . The values which $X, Y, Z, \ldots$ etc., can assume are denoted by lower case letters *viz.*, $x, y, z, \ldots$ etc.

4. *Notations.* If x is a real number, the set of all ω in S such that $X(\omega) = x$ is denoted briefly by writing $X = x$. Thus

$$P(X = x) = P\{\omega : X(\omega) = x\}$$

Similarly $$P(X \leq a] = P\{\omega : X(\omega) \in [-\infty, a]\}$$

and $$P[a < X \leq b) = P(\omega : X(\omega) \in (a, b])$$

Analogous meanings are given to

$$P(X = a \text{ or } X = b) = P\{(X = a) \cup (X = b)\},$$

$$P(X = a \text{ and } X = b) = P\{(X = a) \cap (X = b)\}, \quad \text{etc.}$$

Illustrations : 1. If a coin is tossed, then

$$S = \{\omega_1, \omega_2\} \textit{ where } \omega_1 = H,\ \omega_2 = T$$

$$X(\omega) = \begin{cases} 1, & \text{if } \omega = H \\ 0, & \text{if } \omega = T \end{cases}$$

$X(\omega)$ is a Bernoulli random variable. Here $X(\omega)$ takes only two values. A random variable which takes only a finite number of values is called *single.*

2. An experiment consists of rolling a die and reading the number of points on the upturned face. The most natural random variable X to consider is

$$X(\omega) = \omega\ ;\ \omega = 1, 2, \ldots, 6$$

If we are interested in whether the number of points is even or odd, we consider a random variable Y defined as follows :

$$Y(\omega) = \begin{cases} 0, & \text{if } \omega \textit{ is even} \\ 1, & \text{if } \omega \textit{ is odd} \end{cases}$$

3. If a dart is thrown at a circular target, the sample space S is the set of all points w on the target. By imagining a coordinate system placed on the target with the origin at the centre, we can assign various random variables to this experiment. A natural one is the two dimensional random variable which assigns to the point ω, its rectangular coordinates (x,y). Another is that which assigns ω its polar coordinates (r, θ). A one dimensional random variable assigns to each ω only one of the coordinates x or y (for cartesian system), r or θ (for polar system). The event **E**, "that the dart will land in the first quadrant" can be described by a random variable which assigns to each point w its polar coordinate θ so that $X(\omega) = \theta$ and then $\mathbf{E} = \{\omega : 0 \leq X(\omega) \leq \pi/2\}$.

4. If a pair of fair dice is tossed then $S = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$ and $n(S) = 36$. Let X be a random variable with image set

$$X(S) = \{1,2,3,4,5,6\}$$

$$P(X = 1) = P\{1,1\} = 1/36$$

$$P(X = 2) = P\{(2,1),(2,2),(1,2)\} = 3/36$$

$$P(X = 3) = P\{(3,1),(3,2),(3,3),(2,3),(1,3)\} = 5/36$$

$P(X=4) = P[(4,1), (4,2), (4,3), (4,4), (3,4), (2,4), (1,4)] = 7/36$

Similarly $P(X=5) = 9/36$ and $P(X=6) = 11/36$

Some theorems on Random Variables. Here we shall state (without proof) some of the fundamental results and theorems on random variables.

Theorem 5·1. A function $X(\omega)$ from S to $R(-\infty, \infty)$ is a random variable if and only if

$$\{\omega : X(\omega) < a\} \in \mathbf{B}$$

Theorem 5·2. If X_1 and X_2 are random variables and C is a constant then CX_1, $X_1 + X_2$, X_1X_2 are also random variables.

Remark. It will follow that $C_1X_1 + C_2X_2$ is a random variable for constants C_1 and C_2. In particular $X_1 - X_2$ is a r.v.

Theorem 5·3. If $\{X_n(\omega), n \geq 1\}$ are random variables then $\sup_n X_n(\omega)$, $\inf_n X_n(\omega)$, $\lim_{n\to\infty} \sup X_n(\omega)$ and $\lim_{n\to\infty} \inf X_n(\omega)$ are all random variables, whenever they are finite for all ω.

Theorem 5·4. If X is a random variable then

(i) $\frac{1}{X}$ *where* $\left(\frac{1}{X}\right)(\omega) = \infty$ if $X(\omega) = 0$

(ii) $X_+(\omega) = \max[0, X(\omega)]$

(iii) $X_-(\omega) = -\min[0, X(\omega)]$

(iv) $|X|$

are random variables.

Theorem 5·5. If X_1 and X_2 are random variables then *(i)* $\max[X_1, X_2]$ and *(ii)* $\min[X_1, X_2]$ are also random variables.

Theorem 5·6. If X is a r.v. and $f(\cdot)$ is a continuous function, then $f(X)$ is a r.v.

Theorem 5·7. If X is a r.v. and $f(\cdot)$ is an increasing function, then $f(X)$ *is a r.v.*

Corollary. If f is a function of bounded variations on every finite interval [a,b], and X is a r.v. then $f(X)$ is a r.v.

(*proofs of the above theorems are beyond the scope of this book*)

EXERCISE 5 (a)

1. Let X be a one dimensional random variable. *(i)* If $a < b$, show that the two events $a < X \leq b$ and $X \leq a$ are disjoint, *(ii)* Determine the union of the two events in part *(i)*, *(iii)* show that $P(a < X \leq b) = P(X \leq b) - P(X \leq a)$.

2. Let a sample space S consist of three elements ω_1, ω_2, and ω_3. Let $P(\omega_1) = 1/4$, $P(\omega_2) = 1/2$ and $P(\omega_3) = 1/4$. If X is a random variable defined on S by $X(\omega_1) = 10$, $X(\omega_2) = -3$, $X(\omega_3) = 15$, find $P(-2 \leq X \leq 2)$.