CHAPTER FIVE

Random Variables — Distribution Functions

5·1. Random Variable. Intuitively by a *random variable (r.v)* we mean a real number X connected with the outcome of a random experiment E. For example, if E consists of two tosses of a coin, we may consider the random variable which is the number of heads (0, 1 or 2).

Outcome :	*HH*	*HT*	*TH*	*TT*
Value of X :	2	1	1	0

Thus to each outcome ω, there corresponds a real number $X(\omega)$. Since the points of the sample space S correspond to outcomes, this means that a real number, which we denote by $X(\omega)$, is defined for each $\omega \in S$. From this standpoint, we define random variable to be a real function on S as follows:

" Let S be the sample space associated with a given random experiment. A real-valued function defined on S and taking values in R $(-\infty, \infty)$ is called a one-dimensional random variable. If the function values are ordered pairs of real numbers (i.e., vectors in two-space) the function is said to be a two- dimensional random variable. More generally, an n-dimensional random variable is simply a function whose domain is S and whose range is a collection of n-tuples of real numbers (vectors in n- space)."

For a mathematical and rigorous definition of the random variable, let us consider the probability space, the triplet (**S, B, P**), where S is the sample space, *viz.*, space of outcomes, **B** is the σ-field of subsets in S, and **P** is a probability function on **B**.

Def. A random variable (r.v.) is a function $X(\omega)$ with domain S and range $(-\infty, \infty)$ such that for every real number a, the event $[\omega : X(\omega) \leq a] \in \mathbf{B}$.

Remarks: 1. The refinement above is the same as saying that the function $X(\omega)$ is *measurable* real function on $(S, \mathbf{B})$.

2. We shall need to make probability statements about a random variable X such as $P\{X \leq a\}$. For the simple example given above we should write $P\{X \leq 1\} = P\{HH, HT, TH\} = 3/4$. That is, $P(X \leq a)$ is simply the probability of the set of outcomes ω for which $X(\omega) \leq a$ or

$$P(X \leq a) = P\{\omega : X(\omega) \leq a\}$$

Since **P** is a measure on $(S, \mathbf{B})$ *i.e.*, **P** is defined on subsets of **B**, the above probability will be defined *only* if $\{\omega : X(\omega) \leq a\} \in$ **B**, which implies that $X(\omega)$ is a measurable function on $(S, \mathbf{B})$.

3. One-dimensional random variables will be denoted by capital letters, $X, Y, Z, \ldots$ etc. A typical outcome of the experiment (*i.e.*, a typical element of the sample space) will be denoted by ω or e. Thus $X(\omega)$ represents the real number which the random variable X associates with the outcome ω. The values which $X, Y, Z, \ldots$ etc., can assume are denoted by lower case letters *viz.*, $x, y, z, \ldots$ etc.

4. *Notations.* If x is a real number, the set of all ω in S such that $X(\omega) = x$ is denoted briefly by writing $X = x$. Thus

$$P(X = x) = P\{\omega : X(\omega) = x\}$$

Similarly $$P(X \le a) = P\{\omega : X(\omega) \in [-\infty, a]\}$$

and $$P[a < X \le b) = P(\omega : X(\omega) \in (a, b])$$

Analogous meanings are given to

$$P(X = a \text{ or } X = b) = P\{(X = a) \cup (X = b)\},$$

$$P(X = a \text{ and } X = b) = P\{(X = a) \cap (X = b)\}, \quad \text{etc.}$$

Illustrations : 1. If a coin is tossed, then

$$S = \{\omega_1, \omega_2\} \text{ where } \omega_1 = H,\ \omega_2 = T$$

$$X(\omega) = \begin{cases} 1, & \text{if } \omega = H \\ 0, & \text{if } \omega = T \end{cases}$$

$X(\omega)$ is a Bernoulli random variable. Here $X(\omega)$ takes only two values. A random variable which takes only a finite number of values is called *single.*

2. An experiment consists of rolling a die and reading the number of points on the upturned face. The most natural random variable X to consider is

$$X(\omega) = \omega \ ; \ \omega = 1, 2, \ldots, 6$$

If we are interested in whether the number of points is even or odd, we consider a random variable Y defined as follows :

$$Y(\omega) = \begin{cases} 0, & \text{if } \omega \text{ is even} \\ 1, & \text{if } \omega \text{ is odd} \end{cases}$$

3. If a dart is thrown at a circular target, the sample space S is the set of all points w on the target. By imagining a coordinate system placed on the target with the origin at the centre, we can assign various random variables to this experiment. A natural one is the two dimensional random variable which assigns to the point ω, its rectangular coordinates (x, y). Another is that which assigns ω its polar coordinates (r, θ). A one dimensional random variable assigns to each ω only one of the coordinates x or y (for cartesian system), r or θ (for polar system). The event E, "that the dart will land in the first quadrant" can be described by a random variable which assigns to each point w its polar coordinate θ so that $X(\omega) = \theta$ and then $E = \{\omega : 0 \le X(\omega) \le \pi/2\}$.

4. If a pair of fair dice is tossed then $S = \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$ and $n(S) = 36$. Let X be a random variable with image set

$$X(S) = \{1,2,3,4,5,6\}$$

$$P(X = 1) = P\{1,1\} = 1/36$$

$$P(X = 2) = P\{(2,1),(2,2),(1,2)\} = 3/36$$

$$P(X = 3) = P\{(3,1),(3,2),(3,3),(2,3),(1,3)\} = 5/36$$

$P(X=4) = P\{(4,1), (4,2), (4,3), (4,4), (3,4), (2,4), (1,4)\} = 7/36$

Similarly $P(X=5) = 9/36$ and $P(X=6) = 11/36$

Some theorems on Random Variables. Here we shall state (without proof) some of the fundamental results and theorems on random variables.

Theorem 5·1. A function $X(\omega)$ from S to $R(-\infty, \infty)$ is a random variable if and only if

$$\{\omega : X(\omega) < a\} \in \mathbf{B}$$

Theorem 5·2. If X_1 and X_2 are random variables and C is a constant then CX_1, $X_1 + X_2$, X_1X_2 are also random variables.

Remark. It will follow that $C_1X_1 + C_2X_2$ is a random variable for constants C_1 and C_2. In particular $X_1 - X_2$ is a r.v.

Theorem 5·3. If $\{X_n(\omega), n \geq 1\}$ are random variables then

$\sup_n X_n(\omega)$, $\inf_n X_n(\omega)$, $\lim \sup_{n\to\infty} X_n(\omega)$ and $\lim \inf_{n\to\infty} X_n(\omega)$ are all random variables, whenever they are finite for all ω.

Theorem 5·4. If X is a random variable then

(i) $\frac{1}{X}$ *where* $\left(\frac{1}{X}\right)(\omega) = \infty$ if $X(\omega) = 0$

(ii) $X_+(\omega) = \max\,[0, X(\omega)]$

(iii) $X_-(\omega) = -\min\,[0, X(\omega)]$

(iv) $|X|$

are random variables.

Theorem 5·5. If X_1 and X_2 are random variables then *(i)* max $[X_1, X_2]$ and *(ii)* min $[X_1, X_2]$ are also random variables.

Theorem 5·6. If X is a r.v. and $f(\cdot)$ is a continuous function, then $f(X)$ is a r.v.

Theorem 5·7. If X is a r.v. and $f(\cdot)$ is an increasing function, then $f(X)$ *is a r.v.*

Corollary. If f is a function of bounded variations on every finite interval [a,b], and X is a r.v. then $f(X)$ is a r.v.

(*proofs of the above theorems are beyond the scope of this book*)

EXERCISE 5 (a)

1. Let X be a one dimensional random variable. *(i)* If $a < b$, show that the two events $a < X \leq b$ and $X \leq a$ are disjoint, *(ii)* Determine the union of the two events in part *(i)*, *(iii)* show that $P(a < X \leq b) = P(X \leq b) - P(X \leq a)$.

2. Let a sample space S consist of three elements ω_1, ω_2, and ω_3. Let $P(\omega_1) = 1/4$, $P(\omega_2) = 1/2$ and $P(\omega_3) = 1/4$. If X is a random variable defined on S by $X(\omega_1) = 10$, $X(\omega_2) = -3$, $X(\omega_3) = 15$, find $P(-2 \leq X \leq 2)$.

3. Let $S = (e_1, e_2, ..., e_n)$ be the sample space of some experiment and let $E \subseteq S$ be some event associated with the experiment.

Define ψ_E, the *characteristic random variable* of E as follows :

$$\psi_E(e_i) = \begin{cases} 1 \text{ if } e_i \in E \\ 0 \text{ if } e_i \notin E. \end{cases}$$

In other words, ψ_E is equal to 1 if E occurs, and ψ_E is equal to 0 if E does not occur.

Verify the following properties of characteristic random variables :

(i) ψ_φ *is identically zero*, *i.e.*, $\psi_\varphi(e_i) = 0$; $i = 1, 2, \ldots, n$

(ii) ψ_S *is identically one*, *i.e.*, $\psi_S(e_i) = 1$; $i = 1, 2, \ldots, n$

(iii) $E = F \Rightarrow \psi_E(e_i) = \psi_F(e_i)$; $i = 1, 2, \ldots, n$ and *conversely*

(iv) If $E \subseteq F$ *then* $\psi_E(e_i) \le \psi_F(e_i)$; $i = 1, 2, \ldots, n$

(v) $\psi_E(e_i) + \psi_{\bar{E}}(e_i)$ *is identically* 1 : $i = 1, 2, ..., n$

(vi) $\psi_{E \cap F}(e_i) = \psi_E(e_i)\,\psi_F(e_i)$; $i = 1, 2, ..., n$

(vii) $\psi_{E \cup F}(e_i) = \psi_E(e_i) + \psi_F(e_i) - \psi_E(e_i)\,\psi_F(e_i)$, for $i = 1, 2, ..., n$.

5.2. Distribution Function. Let X be a r.v. on (**S,B,P**). Then the function :

$$F_X(x) = P(X \le x) = P\{\omega : X(\omega) \le x\}, \quad -\infty < x < \infty$$

is called the distribution function (d.f.) of X.

If clarity permits, we may write $F(x)$ instead of $F_X(x)$. ...(5·1)

5·2·1. Properties of Distribution Function. We now proceed to derive a number of properties common to all distribution functions.

Property 1. *If F is the d.f. of the r.v. X and if a < b, then*

$$P(a < X \le b) = F(b) - F(a)$$

Proof. The events '$a<X\le b$' and '$X\le a$' are disjoint and their union is the event '$X\le b$'. Hence by addition theorem of probability

$$P(a < X \le b) + P(X \le a) = P(X \le b)$$

$$\Rightarrow \quad P(a < X \le b) = P(X \le b) - P(X \le a) = F(b) - F(a) \qquad ...(5·2)$$

Cor. 1.

$$P(a \le X \le b) = P\{(X = a) \cup (a < X \le b)\}$$
$$= P(X = a) + P(a < X \le b)$$

(using additive property of P)

$$= P(X = a) + [F(b) - F(a)] \qquad ...(5·2\,a)$$

Similarly, we get

$$P(a < X < b) = P(a < X \le b) - P(X = b)$$
$$= F(b) - F(a) - P(X = b) \qquad ...(5·2\,b)$$

$$P(a \le X < b) = P(a < X < b) + P(X = a)$$

$$= F(b) - F(a) - P(X = b) + P(X = a) \quad ...(5\cdot 2\text{ c})$$

Remark. When $P(X = a) = 0$ and $P(X = b) = 0$, all four events $a \leq X \leq b$, $a < X < b$, $a \leq X < b$ and $a < X \leq b$ have the same probability $F(b) - F(a)$.

Property 2. *If F is the d.f. of one-dimensional r.v. X, then (i)* $0 \leq F(x) \leq 1$, *(ii)* $F(x) \leq F(y)$ *if* $x < y$.

In other words, all distribution functions are monotonically non-decreasing and lie between 0 and 1.

Proof. Using the axioms of certainty and non-negativity for the probability function **P**, part *(i)* follows triviality from the definition of $F(x)$.

For part *(ii)*, we have for $x < y$,

$$F(y) - F(x) = P(x < X \leq y) \geq 0 \qquad \text{(Property 1)}$$

$$\Rightarrow \quad F(y) \geq F(x)$$

$$\Rightarrow \quad F(x) \leq F(y) \text{ when } x < y \qquad ...(5\cdot 3)$$

Property 3. *If F is d.f. of one-dimensional r.v. X, then*

$$F(-\infty) = \lim_{x \to -\infty} F(x) = 0$$

and

$$F(\infty) = \lim_{x \to \infty} F(x) = 1$$

Proof. Let us express the whole sample space S as a countable union of disjoint events as follows :

$$S = \left[\bigcup_{n=1}^{\infty} (-n < X \leq -n+1) \right] \cup \left[\bigcup_{n=0}^{\infty} (n < X \leq n+1) \right]$$

$$\Rightarrow P(S) = \sum_{n=1}^{\infty} P(-n < X \leq -n+1) + \sum_{n=0}^{\infty} P(n < X \leq n+1)$$

$$(\because \ \mathbf{P} \text{ is additive})$$

$$\Rightarrow \quad 1 = \lim_{a \to \infty} \sum_{n=1}^{a} [F(-n+1) - F(-n)]$$

$$+ \lim_{b \to \infty} \sum_{n=0}^{b} [F(n+1) - F(n)]$$

$$= \lim_{a \to \infty} [F(0) - F(-a)] + \lim_{b \to \infty} [F(b+1) - F(0)]$$

$$= [F(0) - F(-\infty)] + [F(\infty) - F(0)]$$

$$\therefore \quad 1 = F(\infty) - F(-\infty) \qquad ...(*)$$

Since $-\infty < \infty$, $F(-\infty) \leq F(\infty)$. Also

$$F(-\infty) \geq 0 \text{ and } F(\infty) \leq 1 \qquad \text{(Property 2)}$$

$$\therefore \qquad 0 \le F(-\infty) \le F(\infty) \le 1 \qquad (**)$$

(*) and (**) give $F(-\infty) = 0$ and $F(\infty) = 1$.

Remarks. 1. Discontinuities of $F(x)$ are at most countable.

2. $$F(a) - F(a-0) = \lim_{h \to 0} P(a-h \le X \le a),\ h > 0$$

$$\therefore \qquad F(a) - F(a-0) = P(X = a)$$

and $$F(a+0) - F(a) = \lim_{h \to 0} P(a \le X \le a+h) = 0,\ h > 0$$

$$\Rightarrow \qquad F(a+0) = F(a)$$

5·3. Discrete Random Variable. If a random variable takes at most a countable number of values, it is called a discrete random variable. *In other words, a real valued function defined on a discrete sample space is called a discrete random variable.*

5·3·1. Probability Mass Function (*and probability distribution of a discrete random variable*).

Suppose X is a one-dimensional discrete random variable taking at most a countably infinite number of values $x_1, x_2, \ldots$ With each possible outcome x_i, we associate a number $p_i = P(X = x_i) = p(x_i)$, called the probability of x_i. The numbers $p(x_i)$; $i = 1, 2, \ldots$ must satisfy the following conditions :

$$(i) \quad p(x_i) \ge 0 \ \forall\ i, \quad (ii) \quad \sum_{i=1}^{\infty} p(x_i) = 1$$

This function p is called the probability mass function of the random variable X and the set $\{x_i, p(x_i)\}$ is called the probability distribution (p.d.) of the r.v. X.

Remarks: 1. The set of values which X takes is called the *spectrum* of the random variable.

2. For discrete random variable, a knowledge of the probability mass function enables us to compute probabilities of arbitrary events. In fact, if E is a set of real numbers, we have

$$P(X \in E) = \sum_{x \in E \cap S} p(x), \text{ where } S \text{ is the sample space.}$$

Illustration. Toss of coin, $S = \{H, T\}$. Let X be the random variable defined by

$$X(H) = 1, \text{ i.e., } X = 1, \text{ if 'Head' occurs.}$$
$$X(T) = 0, \text{ i.e., } X = 0, \text{ if 'Tail' occurs.}$$

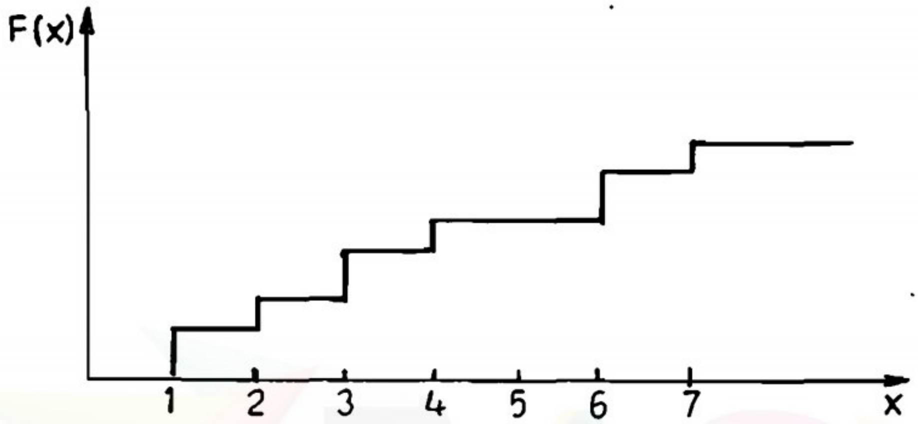
If the coin is 'fair' the probability function is given by

$$P(\{H\}) = P(\{T\}) = \tfrac{1}{2}$$

and we can speak of the probability distribution of the random variable X as

$$P(X = 1) = P(\{H\}) = \tfrac{1}{2},$$
$$P(X = 0) = P(\{T\}) = \tfrac{1}{2},$$

5·3·2. Discrete Distribution Function. In this case there are a countable number of points $x_1, x_2, x_3, \ldots$ and numbers $p_i \geq 0$, $\sum_{1}^{\infty} p_i = 1$ such that $F(X) = \sum_{(i : x_i \leq x)} p_i$. For example if x_i is just the integer i, $F(x)$ is a "*step function*" having jump p_i at i, and being constant between each pair of integers.



Theorem 5·5. $p(x_j) = P(X = x_j) = F(x_j) - F(x_{j-1})$, where F is the *d.f.* of X.

Proof. Let $x_1 < x_2 < \ldots$ We have

$$F(x_j) = P(X \leq x_j)$$

$$= \sum_{i=1}^{j} P(X = x_i) = \sum_{i=1}^{j} p(x_i)$$

and $$F(x_{j-1}) = P(X \leq x_{j-1}) = \sum_{i=1}^{j-1} p(x_i)$$

$$\therefore \quad F(x_j) - F(x_{j-1}) = p(x_j) \quad \ldots(5\cdot5)$$

Thus, given the distribution function of discrete random variable, we can compute its probability mass function.

Example 5·1. *An experiment consists of three independent tosses of a fair coin. Let*

X = The number of heads

Y = The number of head runs,

Z = The lenght of head runs,

a head run being defined as consecutive occurrence of at least two heads, its length then being the number of heads occurring together in three tosses of the coin.

Find the probability function of (i) X, (ii) Y, (iii) Z, (iv) $X+Y$ and (v) XY and construct probability tables and draw their probability charts.

Solution.

Table 1

S. No.	Elementary event	Random Variables				
		X	Y	Z	$X+Y$	XY
1	*HHH*	3	1	3	4	3
2	*HHT*	2	1	2	3	2
3	*HTH*	2	0	0	2	0
4	*HTT*	1	0	0	1	0
5	*THH*	2	1	2	3	2
6	*THT*	1	0	0	1	0
7	*TTH*	1	0	0	1	0
8	*TTT*	0	0	0	0	0

Here sample space is.

$$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$$

(i) Obviously X is a r.v. which can take the values 0, 1, 2, and 3

$$p(3) = P(HHH) = (1/2)^3 = 1/8$$

$$p(2) = P[HHT \cup HTH \cup THH]$$

$$= P(HHT) + P(HTH) + P(THH) = 1/8 + 1/8 + 1/8 = 3/8$$

Similarly $p(1) = 3/8$ and $p(0) = 1/8$.

These probabilities could also be obtained directly from the above table 1.

Table 2

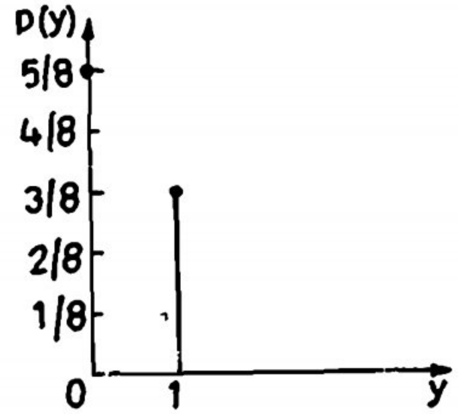
Probability table of X

Values of X (x)	0	1	2	3
$p(x)$	1/8	3/8	3/8	1/8

Table 3

***(ii)* Probability Table of Y**

Values of Y, (y)	0	1
$p(y)$	5/8	3/8



Probability chart of Y

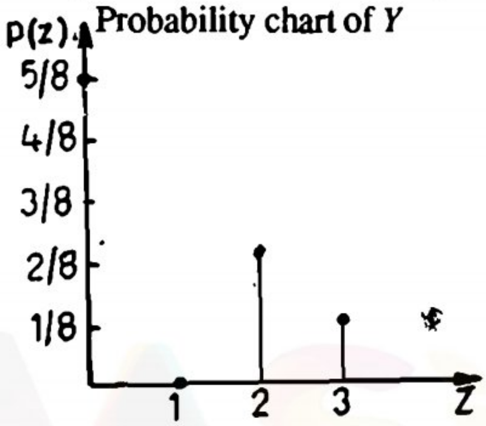
This is obvious from table 1.

(iii) From table 1 , we have

Table 4

Probability Table of

Values of Z, (z)	0	1	2	3
$p(z)$	5/8	0	2/8	1/8



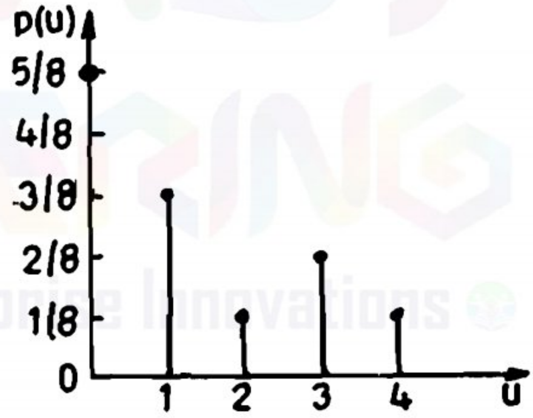
Probability chart of Z

(iv) Let $U = X + Y$. From table 1, we get

Table 5

Probability Table of U

Values of U, (u)	0	1	2	3	4
$p(u)$	1/8	3/8	1/8	2/8	1/8



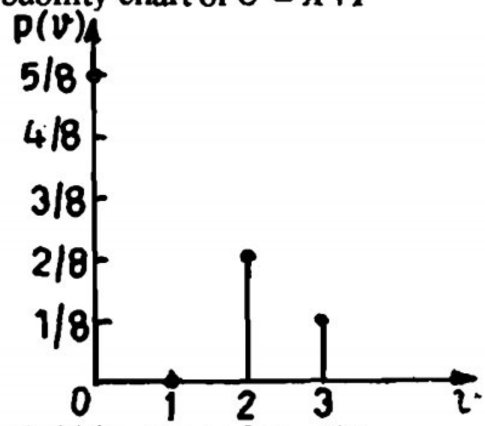
Probability chart of $U = X+Y$

(v) Let $V = XY$

Table 6

Probability Table of V

Values of V, (v)	0	1	2	3
$p(v)$	5/8	0	2/8	1/8



Probability chart of $V = XY$

Example 5·2. *A random variable X has the following probability distribution :*

x :	0	1	2	3	4	5	6	7
$p(x)$:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

(i) Find k, (ii) Evaluate $P(X < 6)$, $P(X \geq 6)$, *and* $P(0 < X < 5)$, *(iii) If* $P(X \leq c) > \frac{1}{2}$, *find the minimum value of c, and (iv) Determine the distribution function of X.* **[Madurai Univ. B.Sc., Oct. 1988]**

Solution. Since $\sum\limits_{x=0}^{7} p(x) = 1$, we have

$$\Rightarrow \quad k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$$

$$\Rightarrow \quad 10k^2 + 9k - 1 = 0$$

$$\Rightarrow \quad (10k - 1)(k + 1) = 0 \Rightarrow k = 1/10$$

[$\because k = -1$, is rejected, since probability canot be negative.]

(*ii*) $P(X < 6) = P(X = 0) + P(X = 1) + \ldots + P(X = 5)$

$$= \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} + \frac{1}{100} = \frac{81}{100}$$

$$P(X \geq 6) = 1 - P(X < 6) = \frac{19}{100}$$

$$P(0 < X < 5) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) = 8k = 4/5$$

(*iii*) $P(X \leq c) > \frac{1}{2}$. By trial, we get $c = 4$.

(*iv*)

X	$F_X(x) = P(X \leq x)$
0	0
1	$k = 1/10$
2	$3k = 3/10$
3	$5k = 5/10$
4	$8k = 4/5$
5	$8k + k^2 = 81/100$
6	$8k + 3k^2 = 83/100$
7	$9k + 10k^2 = 1$

EXERCISE 5 (b)

1. (*a*) A student is to match three historical events (Mahatma Gandhi's Birthday, India's freedom, and First World War) with three years (1947, 1914, 1896). If he guesses with no knowledge of the correct answers, what is the probability distribution of the number of answers he gets correctly ?

(*b*) From a lot of 10 items containing 3 defectives, a sample of 4 items is drawn at random. Let the random variable X denote the number of defective items in the sample. Answer the following when the sample is drawn without replacement.

(i) Find the probability distribution of X,

(ii) Find $P(X \leq 1)$, $P(X < 1)$ and $P(0 < X < 2)$

Ans. (a)

x	0	1	2	3
$p(x)$	$\frac{1}{3}$	$\frac{1}{2}$	0	$\frac{1}{6}$

(b) (i)

x	0	1	2	3
$p(x)$	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{30}$

(ii) 2/3, 5/6, 1/2

2. (a) A random variable X can take all non-negative integral values, and the probability that X takes the value r is proportional to α^r $(0 < \alpha < 1)$. Find $P(X = 0)$. **[Calcutta Univ. B.Sc. 1987]**

Ans. $P(X = r) = A\alpha^r$; $r = 0, 1, 2, \ldots$; $A = 1 - \alpha$; $P(X = 0) = A = 1 - \alpha$

(b) Suppose that the random variable X has possible values 1, 2, 3, ... and $P(X = j) = 1/2^j$, $j = 1, 2, \ldots$ (i) Compute $P(X \text{ is even})$, (ii) Compute $P(X \geq 5)$, and (iii) Compute $P(X \text{ is divisible by } 3)$.

Ans. (i) 1/3, (ii) 1/16, and (iii) 1/7

3. (a) Let X be a random variable such that

$$P(X = -2) = P(X = -1),\ P(X = 2) = P(X = 1) \text{ and}$$
$$P(X > 0) = P(X < 0) = P(X = 0).$$

Obtain the probability mass function of X and its distribution function.

Ans.

X	-2	-1	0	1	2
$p(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$
$F(x)$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1

(b) A random variable X assumes the values $-3, -2, -1, 0, 1, 2, 3$ such that

$$P(X = -3) = P(X = -2) = P(X = -1),$$
$$P(X = 1) = P(X = 2) = P(X = 3),$$
and $$P(X = 0) = P(X > 0) = P(X < 0),$$

Obtain the probability mass function of X and its distribution function, and find further the probability mass function of $Y = 2X^2 + 3X + 4$.

[Poona Univ. B.Sc., March 1991]

Ans.

X	-3	-2	-1	0	1	2	3
$p(x)$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$
Y	13	6	3	4	9	18	31
$p(y)$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$

4. *(a)* A random variable X has the following probability function :

Values of X, x :	-2	-1	0	1	2	3
$p(x)$:	$0{\cdot}1$	k	$0{\cdot}2$	$2k$	$0{\cdot}3$	k

(i) Find the value of k, and calculate mean and variance.

(ii) Construct the c.d.f. $F(X)$ and draw its graph.

Ans. *(i)* 0·1, 0·8 and 2·16, *(ii)* $F(X) = 0{\cdot}1, 0{\cdot}2, 0{\cdot}4, 0{\cdot}6, 0{\cdot}9, 1{\cdot}0$

(b) Given the probability function

x	0	1	2	3
$p(x)$	0·1	0·3	0·5	0·1

Let $Y = X^2 + 2X$, then find *(i)* the probability function of Y, *(ii)* mean and variance of Y.

Ans. *(i)*

y	0	3	8	15
$p(y)$	0·1	0·3	0·5	0·1

, *(ii)* 6·4 ,16·24

5. A random variable X has the following probability distribution :

Values of X, x	0	1	2	3	4	5	6	7	8
$p(x)$	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

(i) Determine the value of a.

(ii) Find $P(X < 3)$, $P(X \geq 3)$, $P(0 < X < 5)$.

(iii) What is the smallest value of x for which $P(X \leq x) > 0{\cdot}5$? and

(iv) Find out the distribution function of X ?

Ans. *(i)* $a = 1/81$, *(ii)* 9/81, 72/81, 24/81, *(iii)* 6

(iv) x	0	1	2	3	4	5	6	7	8
$F(x)$	a	$4a$	$9a$	$16a$	$25a$	$36a$	$49a$	$64a$	$81a$

6. *(a)* Let $p(x)$ be the probability function of a discrete random variable X which assumes the values x_1, x_2, x_3, x_4, such that $2p(x_1) = 3p(x_2) = p(x_3) = 5p(x_4)$. Find probability distribution and cumulative probability distribution of X. **(Sardar Patel Univ. B.Sc. 1987)**

Ans.

x	x_1	x_2	x_3	x_4
$p(x)$	15/16	10/16	30/16	6/16

(b) The following is the distribution function of a discrete random variable X :

x	-3	-1	0	1	2	3	5	8
$f(x)$	0·10	0·30	0·45	0·5	0·75	0·90	0·95	1·00

(i) Find the probability distribution of X.

(ii) Find $P(X \text{ is even})$ and $P(1 \leq X \leq 8)$.

(iii) Find $P(X = -3 \mid X < 0)$ and $P(X \geq 3 \mid X > 0)$.

[**Ans.** *(ii)* 0·30, 0·55, *(iii)* 1/3, 5/11].

7. If
$$p(x) = \frac{x}{15};\ x = 1, 2, 3, 4, 5$$
$$= 0, \text{ elsewhere}$$

Find *(i)* $P\{X = 1 \text{ or } 2\}$, and *(ii)* $P\left\{\frac{1}{2} < X < \frac{5}{2} \mid X > 1\right\}$

[Allahabad Univ. B.Sc., April 1992]

Hint. *(i)* $P\{X = 1 \text{ or } 2\} = P(X = 1) + P(X = 2) = \frac{1}{15} + \frac{2}{15} = \frac{1}{5}$

$$(ii)\ P\left\{\frac{1}{2} < X < \frac{5}{2} \,\Big|\, X > 1\right\} = \frac{P\left\{\left(\frac{1}{2} < X < \frac{5}{2}\right) \cap X > 1\right\}}{P(X > 1)}$$

$$= \frac{P\{(X = 1 \text{ or } 2) \cap X > 1\}}{P(X > 1)} = \frac{P(X = 2)}{1 - P(X = 1)} = \frac{2/15}{1 - (1/15)} = \frac{1}{7}$$

8. The probability mass function of a random variable X is zero except at the points $x = 0, 1, 2$. At these points it has the values $p(0) = 3c^3$, $p(1) = 4c - 10c^2$ and $p(2) = 5c - 1$ for some $c > 0$.

(i) Determine the value of c.

(ii) Compute the following probabilities, $P(X < 2)$ and $P(1 < X \leq 2)$.

(iii) Describe the distribution function and draw its graph.

(iv) Find the largest x such that $F(x) < 1/2$.

(v) Find the smallest x such that $F(x) \geq 1/3$. **[Poona Univ. B.Sc., 1987]**

Ans. *(i)* $\frac{1}{3}$, *(ii)* $\frac{1}{3}, \frac{2}{3}$, *(iv)* 1, *(v)* 1.

9. *(a)* Suppose that the random variable X assumes three values 0, 1 and 2 with probabilities $\frac{1}{3}$, $\frac{1}{6}$ and $\frac{1}{2}$ respectively. Obtain the distribution function of X.. **[Gujarat Univ. B.Sc., 1992]**

(b) Given that $f(x) = k(1/2)^x$ is a probability distribution for a random variable which can take on the values $x = 0, 1, 2, 3, 4, 5, 6$, find k and find an expression for the corresponding cumulative probabilities $F(x)$.

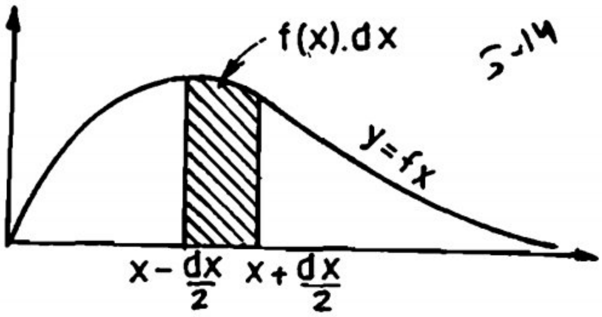
[Nagpur Univ. B.Sc., 1987]

5·4. Continuous Random Variable. A random variable X is said to be continuous if it can take all possible values between certain limits. *In other words, a random variable is said to be continuous when its different values cannot be put in 1-1 correspondence with a set of positive integers.*

A continuous random variable is a random variable that (at least conceptually) can be measured to any desired degree of accuracy. Examples of continuous random variables are age, height, weight etc.

5·4·1. Probability Density Function *(Concept and Definition)*. Consider the small interval $(x, x + dx)$ of length dx round the point x. Let $f(x)$ be any continuous function of x so that $f(x)\,dx$ represents the probability that X falls in the infinitesimal interval $(x, x + dx)$. Symbolically

$$P(x \le X \le x + dx) = f_X(x)\,dx \qquad \ldots (5{\cdot}5)$$



In the figure, $f(x)\,dx$ represents the area bounded by the curve $y = f(x)$, x–axis and the ordinates at the points x and $x + dx$. The function $f_X(x)$ so defined is known as *probability density function or simply density function of random variable* X *and is usually abbreviated as p.d.f.* The expression, $f(x)\,dx$, usually written as $dF(x)$, is known as the *probability differential* and the curve $y = f(x)$ is known as the *probability density curve* or simply *probability curve.*

Definition. p.d.f. $f_X(x)$ of the *r.v.* X is defined as :

$$f_X(x) = \lim_{\delta x \to 0} \frac{P(x \le X \le x + \delta x)}{\delta x} \qquad \ldots (5{\cdot}5\,a)$$

The probability for a variate value to lie in the interval dx is $f(x)\,dx$ and hence the probability for a variate value to fall in the finite interval $[\alpha, \beta]$ is :

$$P(\alpha \le X \le \beta) = \int_\alpha^\beta f(x)\,dx \qquad \ldots (5{\cdot}5\,b)$$

which represents the area between the curve $y = f(x)$, x–axis and the ordinates at $x = \alpha$ and $x = \beta$. Further since total probability is unity, we have $\int_a^b f(x)\,dx = 1$, where $[a, b]$ is the range of the random variable X. The range of the variable may be finite or infinite.

The probability density function (*p.d.f.*) of a random variable (*r.v.*) X usually denoted by $f_X(x)$ or simply by $f(x)$ has the following obvious properties

(i) $f(x) \ge 0,\ -\infty < x < \infty$ ... (5·5 *c*)

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$ ... (5·5 *d*)

(iii) The probability $P(E)$ given by

$$P(E) = \int_E f(x)\,dx \qquad \ldots (5{\cdot}5\,e)$$

is well defined for any event E.

Important Remark. In case of discrete random variable, the probability at a point, *i.e.*, $P(x = c)$ is not zero for some fixed c. However, in case of continuous random variables the probability at a point is always zero, *i.e.*, $P(x = c) = 0$ for all possible values of c. This follows directly from (5·5 *b*) by taking $\alpha = \beta = c$.

This also agrees with our discussion earlier that $P(E) = 0$ does not imply that the event E is null or impossible event. This property of continuous r.v., *viz.*,

$$P(X = c) = 0, \ \forall \ c \qquad \text{... (5·5 f)}$$

leads us to the following important result :

$$P(\alpha \le X \le \beta) = P(\alpha \le X < \beta) = P(\alpha < X \le \beta) = P(\alpha < X < \beta) \qquad \text{... (5·5 g)}$$

i.e., in case of continuous r.v., it does matter whether we include the end points of the interval from α to β.

However, this result is in general not true for discrete random variables.

5·4·2. Various Measures of Central Tendency, Dispersion, Skewness, and Kurtosis for Continuous Probability Distribution. The formulae for these measures in case of discrete frequency distribution can be easily extended to the case of continuous probability distribution by simply replacing $p_i = f_i/N$ by $f(x)\,dx$, x_i by x and the summation over 'i' by integration over the specified range of the variable X.

Let $f_X(x)$ *or* $f(x)$ be the *p.d.f.* of a random variable X where X is defined from a to b. Then

(*i*) *Arithmetic mean* $= \int_a^b x\, f(x)\, dx$ ...(5·6)

(*ii*) *Harmonic mean.* Harmonic mean H is given by

$$\frac{1}{H} = \int_a^b \left(\frac{1}{x}\right) f(x)\, dx \qquad \text{...(5·6 a)}$$

(*iii*) *Geometric mean.* Geometric mean G is given by

$$\log G = \int_a^b \log x\, f(x)\, dx \qquad \text{...(5·6 b)}$$

(*iv*) μ_r' (about origin) $= \int_a^b x^r\, f(x)\, dx$ ...(5·7)

μ_r' (about the point $x = A$) $= \int_a^b (x - A)^r\, f(x)\, dx$ ...(5·7 a)

and μ_r (about mean) $= \int_a^b (x - mean)^r\, f(x)\, dx$ ...(5·7 b)

In particular, from (5·7), we have

μ_1' (about origin) $=$ *Mean* $= \int_a^b x\, f(x)\, dx$

and $\mu_2' = \int_a^b x^2\, f(x)\, dx$

Hence $\mu_2 = \mu_2' - \mu_1'^2 = \int_a^b x^2 f(x)\, dx - \left(\int_a^b x f(x)\, dx\right)^2$ ... (5·7 c)

From (5·7), on putting $r=3$ and 4 respectively, we get the values of μ_3' and μ_4' and consequently the moments about mean can be obtained by using the relations :

and
$$\left.\begin{aligned}\mu_3 &= \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 \\ \mu_4 &= \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4\end{aligned}\right\} \quad \ldots (5\cdot 7\ d)$$

and hence β_1 and β_2 can be computed.

(v) Median. Median is the point which divides the entire distribution in two equal parts. In case of continuous distribution, median is the point which divides the total area into two equal parts. Thus if M is the median, then

$$\int_a^M f(x)\,dx = \int_M^b f(x)\,dx = \tfrac{1}{2} \quad \ldots (5\cdot 8)$$

Thus solving

$$\int_a^M f(x)\,dx = \tfrac{1}{2} \quad or \quad \int_M^b f(x)\,dx = \tfrac{1}{2} \quad \ldots (5\cdot 8\ a)$$

for M, we get the value of median.

(vi) Mean Deviation. Mean deviation about the mean μ_1' is given by

$$M.D. = \int_a^b |\, x - mean\,|\, f(x)\,dx \quad \ldots (5\cdot 9)$$

(vii) Quartiles and Deciles. Q_1 and Q_3 are given by the equations

$$\int_a^{Q_1} f(x)\,dx = \tfrac{1}{4} \quad \text{and} \quad \int_a^{Q_3} f(x)\,dx = \tfrac{3}{4} \quad \ldots (5\cdot 10)$$

D_i, i th decile is given by

$$\int_a^{D_i} f(x)\,dx = \frac{i}{10} \quad \ldots (5\cdot 10\ a)$$

(viii) Mode. Mode is the value of x for which $f(x)$ is maximum. Mode is thus the solution of

$$f'(x) = 0 \text{ and } f''(x) < 0 \quad \ldots (5\cdot 11)$$

provided it lies in $[a,b]$.

Example 5·3. *The diameter of an electric cable, say X, is assumed to be a continuous random variable with p.d.f.* $f(x) = 6x(1-x)$, $0 \le x \le 1$.

(i) Check that above is p.d.f.,

(ii) Determine a number b such that $P(X < b) = P(X > b)$

[Aligarh Univ. B.Sc.(Hons).1990]

Solution. Obviously, for $0 \le x \le 1$, $f(x) \ge 0$

Now
$$\int_0^1 f(x)\,dx = 6\int_0^1 x(1-x)\,dx$$

$$= 6\int_0^1 (x - x^2)\,dx = 6\left|\frac{x^2}{2} - \frac{x^3}{3}\right|_0^1 = 1$$

Hence $f(x)$ is the *p.d.f.* of *r.v.* X

(ii)
$$P(X < b) = P(X > b) \quad \ldots (*)$$

$$\Rightarrow \quad \int_0^b f(x)\,dx = \int_b^1 f(x)\,dx$$

$$\Rightarrow \quad 6\int_0^b x(1-x)\,dx = 6\int_b^1 x(1-x)\,dx$$

$$\Rightarrow \quad \left|\frac{x^2}{2}-\frac{x^3}{3}\right|_0^b = \left|\frac{x^2}{2}-\frac{x^3}{3}\right|_b^1$$

$$\Rightarrow \quad \left(\frac{b^2}{2}-\frac{b^3}{3}\right) = \left[\left(\frac{1}{2}-\frac{1}{3}\right)-\left(\frac{b^2}{2}-\frac{b^3}{3}\right)\right]$$

$$\Rightarrow \quad 3b^2-2b^3 = [\,1-3b^2+2b^3\,]$$

$$\Rightarrow \quad 4b^3-6b^2+1=0$$

$$(2b-1)(2b^2-2b-1)=0$$

$$\Rightarrow \quad 2b-1=0 \quad or \quad 2b^2-2b-1=0$$

Hence $b = 1/2$ is the only real value lying between 0 and 1 and satisfying (*).

Example 5·4. *A continuous random variable X has a p.d.f.*

$$f(x) = 3x^2,\ 0 \le x \le 1. \ \textit{Find a and b such that}$$

(i) $P\{X \le a\} = P\{X > a\}$, and

(ii) $P\{X > b\} = 0{\cdot}05$. **[Calicut Univ. B.Sc., Sept. 1988]**

Solution. *(i)* Since $P(X \le a) = P(X > a)$,
each must be equal to 1/2, because total probability is always one.

$$\therefore \quad P(X \le a) = \frac{1}{2} \quad \Rightarrow \quad \int_0^a f(x)\,dx = \frac{1}{2}$$

$$\Rightarrow \quad 3\int_0^a x^2\,dx = \frac{1}{2} \quad \Rightarrow \quad 3\left|\frac{x^3}{3}\right|_0^a = \frac{1}{2}$$

$$\Rightarrow \quad a^3 = \frac{1}{2} \quad \Rightarrow \quad a = \left(\frac{1}{2}\right)^{\frac{1}{3}}$$

(ii) $$P(X > b) = 0{\cdot}05 \quad \Rightarrow \quad \int_b^1 f(x)\,dx = 0{\cdot}05$$

$$\Rightarrow \quad 3\left|\frac{x^3}{3}\right|_b^1 = \frac{1}{20} \quad \Rightarrow \quad 1-b^3 = \frac{1}{20}$$

$$\Rightarrow \quad b^3 = \frac{19}{20} \quad \Rightarrow \quad b = \left(\frac{19}{20}\right)^{\frac{1}{3}}.$$

Example 5·5. *Let X be a continuous random variate with p.d.f.*

$$\begin{aligned} f(x) &= ax, \quad 0 \le x \le 1 \\ &= a, \quad 1 \le x \le 2 \\ &= -ax+3a, \quad 2 \le x \le 3 \\ &= 0, \ \textit{elsewhere} \end{aligned}$$

(i) Determine the constant a.

(ii) Compute $P(X \leq 1{\cdot}5)$. **[Sardar Patel Univ. B.Sc., Nov.1988]**

Solution. *(i)* Constant '*a*' is determined from the consideration that total probability is unity, *i.e.*,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{-\infty}^{0} f(x)\,dx + \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \int_3^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_0^1 ax\,dx + \int_1^2 a.dx + \int_2^3 (-ax + 3a)\,dx = 1$$

$$\Rightarrow a\left|\frac{x^2}{2}\right|_0^1 + a\left|\,x\,\right|_1^2 + a\left|-\frac{x^2}{2} + 3x\right|_2^3 = 1$$

$$\Rightarrow \frac{a}{2} + a + a\left[\left(-\frac{9}{2} + 9\right) - (-2 + 6)\right] = 1$$

$$\Rightarrow \frac{a}{2} + a + \frac{a}{2} = 1 \Rightarrow 2a = 1 \Rightarrow a = \frac{1}{2}$$

(ii) $P(X \leq 1{\cdot}5) = \int_{-\infty}^{1\cdot5} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_0^1 f(x)\,dx + \int_1^{1\cdot5} f(x)\,dx$

$$= a\int_0^1 x\,dx + \int_1^{1\cdot5} a.dx$$

$$= a\left|\frac{x^2}{2}\right|_0^1 + a\left|\,x\,\right|_1^{1\cdot5} = \frac{a}{2} + 0{\cdot}5\,a$$

$$= a = \frac{1}{2} \qquad \left[\because a = \tfrac{1}{2},\ \textit{Part (i)}\right]$$

Example 5·6. *A probability curve $y = f(x)$ has a range from 0 to ∞. If $f(x) = e^{-x}$, find the mean and variance and the third moment about mean.*

[Andhra Univ. B.Sc. 1988; Delhi Univ. B.Sc. Sept. 1987]

Solution.

μ_r' (rth moment about origin) $= \int_0^{\infty} x^r f(x)\,dx$

$$= \int_0^{\infty} x^r e^{-x}\,dx = \Gamma(r+1) = r\,!$$

(Using Gamma Integral)

Substituting $r = 1, 2$ and 3 successively, we get

$$\text{Mean} = \mu_1' = 1\,! = 1,\ \mu_2' = 2\,! = 2,\ \mu_3' = 3\,! = 6$$

Hence variance $= \mu_2 = \mu_2' - \mu_1'^2 = 2 - 1 = 1$

and $\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = 6 - 3 \times 2 + 2 = 2$

Example 5·7. *In a continuous distribution whose relative frequency density is given by*

$$f(x) = y_o \,.\, x(2-x)\,,\; 0 \le x \le 2,$$

find mean, variance, β_1, and β_2 and hence show that the distribution is symmetrical. Also (i) find mean deviation about mean and (ii) show that for this distribution $\mu_{2n+1} = 0$, (iii) find the mode, harmonic mean and median.

[Delhi Univ. B.Sc.(Stat. Hons.), 1992; B.Sc., Oct. 1992]

Solution. Since total probability is unity, we have

$$\int_0^2 f(x)\,dx = 1$$

$$\Rightarrow \qquad y_o \int_0^2 x(2-x)\,dx = 1 \quad \Rightarrow \quad y_o = 3/4$$

$$\therefore \qquad f(x) = \frac{3}{4} x(2-x)$$

$$\mu_r' = \int_0^2 x^r f(x)\,dx = \frac{3}{4}\int_0^2 x^{r+1}(2-x)\,dx = \frac{3 \cdot 2^{r+1}}{(r+2)(r+3)}$$

In particular

$$\text{Mean} = \mu_1' = \frac{3 \cdot 2^2}{3 \cdot 4} = 1, \quad \mu_2' = \frac{3 \cdot 2^3}{4 \cdot 5} = \frac{6}{5},$$

$$\mu_3' = \frac{3 \cdot 2^4}{5 \cdot 6} = \frac{8}{5}, \quad \text{and} \quad \mu_4' = \frac{3 \cdot 2^5}{6 \cdot 7} = \frac{16}{7}$$

$$\text{Hence varience} = \mu_2 = \mu_2' - \mu_1'^2 = \frac{6}{5} - 1 = \frac{1}{5}$$

$$\mu_3 = \mu_3' - 3\,\mu_2'\mu_1' + 2\mu_1'^3 = \frac{8}{5} - 3 \,.\, \frac{6}{5} \,.\, 1 + 2 = 0$$

$$\mu_4 = \mu_4' - 4\,\mu_3'\mu_1' + 6\,\mu_2'\mu_1'^2 - 3\,\mu_1'^4 = \frac{16}{7} - 4 \,.\, \frac{8}{5} \,.\, 1 + 6 \,.\, \frac{6}{5} \,.\, 1 - 3 \,.\, 1 = \frac{3}{35}$$

$$\therefore \qquad \beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0 \quad \text{and} \quad \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3/35}{(1/5)^2} = \frac{15}{7}$$

Since $\beta_1 = 0$, the distributior is *symmetrical*.

Mean deviation about mean

$$= \int_0^2 |\,x-1\,|\, f(x)\,dx$$

$$= \int_0^1 |\,x-1\,|\, f(x)\,dx + \int_1^2 |\,x-1\,|\, f(x)\,dx$$

$$= \frac{3}{4}\left[\int_0^1 (1-x)\,x\,(2-x)\,dx + \int_1^2 (x-1)\,x\,(2-x)\,dx\right]$$

$$= \frac{3}{4}\left[\int_0^1 (2x - 3x^2 + x^3)\,dx + \int_1^2 (3x^2 - x^3 - 2x)\,dx\right]$$

$$= \frac{3}{4}\left[\left| x^2 - \frac{3\,.\,x^3}{3} + \frac{x^4}{4}\right|_0^1 + \left| 3\,.\,\frac{x^3}{3} - \frac{x^4}{4} - \frac{2x^2}{2}\right|_1^2\right] = \frac{3}{8}$$

$$\mu_{2n+1} = \int_0^2 (x - mean)^{2n+1}\, f(x)\,dx$$

$$= \frac{3}{4}\int_0^2 (x-1)^{2n+1}\, x\,(2-x)\,dx$$

$$= \frac{3}{4}\int_{-1}^1 t^{2n+1}\,(t+1)\,(1-t)\,dt \qquad (x - 1 = t)$$

$$= \frac{3}{4}\int_{-1}^1 t^{2n+1}\,(1-t^2)\,dt$$

Since t^{2n+1} 1 is an odd function of t and $(1-t^2)$ is an even function of t, the integrand $t^{2n+1}(1-t^2)$ is an odd function of t.

Hence $\mu_{2n+1} = 0.$

Now $f'(x) = \frac{3}{4}(2 - 2x) = 0 \Rightarrow x = 1$

and $f''(x) = \frac{3}{4}(-2) = -\frac{3}{2} < 0$

Hence mode = 1

Harmonic mean H is given by

$$\frac{1}{H} = \int_0^2 \frac{1}{x}\, f(x)\,dx$$

$$= \frac{3}{4}\int_0^2 (2-x)\,dx = \frac{3}{2}$$

$$\Rightarrow \qquad H = \frac{2}{3}$$

If M is the median, then

$$\int_0^M f(x)\,dx = \frac{1}{2}$$

$$\Rightarrow \qquad \frac{3}{4}\int_0^M x\,(2-x)\,dx = \frac{1}{2}$$

$$\Rightarrow \qquad \left| x^2 - \frac{x^3}{3}\right|_0^M = \frac{2}{3}$$

$$\Rightarrow \qquad 3M^2 - M^3 = 2$$

$$\Rightarrow \qquad M^3 - 3M^2 + 2 = 0$$

$$\Rightarrow \qquad (M-1)(M^2 - 2M - 2) = 0$$

The only value of M lying in [0, 2] is $M = 1$. Hence median is 1.

Aliter. Since we have proved that distribution is symmetrical,

$$\text{Mode} = \text{Median} = \text{Mean} = 1$$

Example 5·8. *The elementary probability law of a continuous random variable X is*

$$f(x) = y_0 e^{-b(x-a)}, \quad a \le x < \infty, \; b > 0$$

where a, b and y_0 are constants.

Show that $y_0 = b = 1/\sigma$ and $a = m - \sigma$, where m and σ are respectively the mean and standard deviation of the distribution. Show also that $\beta_1 = 4$ and $\beta_2 = 9$. **[Gauhati Univ. B.Sc., 1992]**

Solution. Since total probability is unity,

$$\int_a^\infty f(x)\, dx = 1 \quad \Rightarrow \quad y_0 \int_a^\infty e^{-b(x-a)}\, dx = 1$$

$$\Rightarrow \quad y_0 \left| \frac{e^{-b(x-a)}}{-b} \right|_a^\infty = 1 \quad \Rightarrow \quad y_0 \frac{1}{b} = 1, \; (b > 0)$$

$$\Rightarrow \quad y_0 = b$$

μ_r' (rth moment about the point '$x = a$')

$$= \int_a^\infty (x-a)^r f(x)\, dx = b \int_a^\infty (x-a)^r e^{-b(x-a)}\, dx$$

$$= b \int_0^\infty t^r e^{-bt}\, dt \qquad \text{[On putting } x - a = t \text{]}$$

$$= b\, \frac{\Gamma(r+1)}{b^{r+1}} = \frac{r\,!}{b^r} \qquad \text{[Using Gamma Integral]}$$

In particular

$$\mu_1' = 1/b, \; \mu_2' = 2/b^2, \; \mu_3' = 6/b^3, \; \mu_4' = 24/b^4$$

$$\therefore \quad m = \text{Mean} = a + \mu_1' = a + (1/b)$$

and $\quad \sigma^2 = \mu_2 = \mu_2' - \mu_1'^2 = 1/b^2$

$$\Rightarrow \quad \sigma = \frac{1}{b} \text{ and } m = a + \frac{1}{b} = a + \sigma$$

Hence $\quad y_0 = b = \dfrac{1}{\sigma}$ and $a = m - \sigma$

Also $\quad \mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = \dfrac{1}{b^3}(6 - 3\cdot2 + 2) = \dfrac{2}{b^3} = 2\sigma^3$

and $\quad \mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

$$= \frac{1}{b^4}(24 - 4.6.1 + 6.2.1 - 3) = \frac{9}{b^4} = 9\sigma^4$$

Hence $\beta_1 = \mu_3^2/\mu_2^3 = 4\sigma^6/\sigma^6 = 4$ and $\beta_2 = \mu_4/\mu_2^2 = 9\sigma^4/\sigma^4 = 9$

Example 5·9. *For the following probability distribution*

$$dF = y_0 \,.\, e^{-|x|}\, dx\,, \quad -\infty < x < \infty$$

show that $y_0 = \frac{1}{2}$, $\mu_1' = 0$, $\sigma = \sqrt{2}$ *and mean deviation about mean = 1.*

Solution. We have $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\Rightarrow \quad y_0 \int_{-\infty}^{\infty} e^{-|x|}\,dx = 1 \quad \Rightarrow \quad 2y_0 \int_0^{\infty} e^{-|x|}\,dx = 1,$$

(since $e^{-|x|}$ is an even function of x)

$$\Rightarrow \quad 2y_0 \int_0^{\infty} e^{-x}\,dx = 1, \qquad (\text{since in } 0 \le x < \infty,\ |x| = x)$$

$$\Rightarrow \quad 2y_0 \left| \frac{e^{-x}}{-1} \right|_0^{\infty} = 1 \quad \Rightarrow \quad 2y_0 = 1, \ \text{i.e., } y_0 = \frac{1}{2}$$

$$\mu_1' \ (\text{about origin}) = \int_{-\infty}^{\infty} x\, f(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} x\, e^{-|x|}\,dx$$

$$= 0,$$

(since the integrand $x \,.\, e^{-|x|}$ is an odd function of x)

$$\mu_2' = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} x^2 e^{-|x|}\,dx$$

$$= \frac{1}{2}\, 2 \int_0^{\infty} x^2 e^{-|x|}\,dx$$

[since the integrand $x^2 e^{-|x|}$ is an even function of x]

$$\therefore \quad \mu_2' = \int_0^{\infty} x^2 e^{-x}\,dx = \Gamma(3) \qquad (\text{on using Gamma Integral})$$

$$\Rightarrow \quad \mu_2' = 2\,! = 2$$

Now $\sigma^2 = \mu_2 = \mu_2' - \mu_1'^2 = 2$

$$\text{M.D. about mean} = \int_{-\infty}^{\infty} |\,x - mean\,|\, f(x)\,dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} |\,x\,|\, e^{-|x|}\,dx \qquad (\because \text{Mean} = \mu_1' = 0)$$

$$= \frac{1}{2}\,.\,2\int_0^{\infty} |\,x\,|\, e^{-|x|}\,dx$$

$$= \int_0^{\infty} x\, e^{-x}\,dx = \Gamma(2) = 1$$

Example 5·10. *A random variable X has the probability law :*

$$dF(x) = \frac{x}{b^2} \cdot e^{-x^2/2b^2}\, dx\,, \quad 0 \le x < \infty$$

Find the distance between the quartiles and show that the ratio of this distance to the standard devation of X is independent of the patameter 'b'.

Solution. If Q_1 and Q_3 are the first and third quartiles respectively, we have

$$\int_0^{Q_1} f(x)\, dx = \frac{1}{4} \quad \Rightarrow \quad \frac{1}{b^2} \int_0^{Q_1} x\, e^{-x^2/2b^2}\, dx = \frac{1}{4}$$

Put $\quad y = \dfrac{x^2}{2b^2} \quad$ *then* $\quad dy = \dfrac{x}{b^2}\, dx$

$$\therefore \quad \int_0^{Q_1^2/2b^2} e^{-y}\, dy = \frac{1}{4} \quad \Rightarrow \quad \left| \frac{e^{-y}}{-1} \right|_0^{Q_1^2/2b^2} = \frac{1}{4}$$

$$\Rightarrow \quad 1 - e^{-Q_1^2/2b^2} = \frac{1}{4} \quad \Rightarrow \quad e^{-Q_1^2/2b^2} = \frac{3}{4}$$

$$\Rightarrow \quad Q_1 = \sqrt{2}\, b \sqrt{\log (4/3)}$$

Again we have $\int_0^{Q_3} f(x)\, dx = \dfrac{3}{4}$ which, on proceeding similarly, will give

$$1 - e^{-Q_3^2/2b^2} = 3/4 \quad \Rightarrow \quad e^{-Q_3^2/2b^2} = 1/4$$

$$\Rightarrow \quad Q_3 = \sqrt{2}\, b \sqrt{\log (4)}$$

The distance between the quartiles is given by

$$Q_3 - Q_1 = \sqrt{2}\, b\, [\sqrt{\log 4} - \sqrt{\log (4/3)}\,]$$

$$\mu_1' = \int_0^\infty x\, f(x)\, dx = \int_0^\infty x\, \frac{x}{b^2}\, e^{-x^2/2b^2}\, dx$$

$$= \int_0^\infty \sqrt{2}\, b y^{1/2}\, e^{-y}\, dy \qquad \left(y = \frac{x^2}{2b^2} \right)$$

$$= \sqrt{2}\, b \int_0^\infty e^{-y}\, y^{(3/2)-1}\, dy$$

$$= \sqrt{2}\, b\, \Gamma\left(\frac{3}{2}\right) = \sqrt{2} . b . \frac{1}{2}\, \Gamma\left(\frac{1}{2}\right) = \sqrt{2}\, b \frac{\sqrt{\pi}}{2} = b \sqrt{(\pi/2)}$$

$$\mu_2' = \int_0^\infty x^2 f(x)\, dx = \int_0^\infty x^2\, \frac{x}{b^2}\, e^{-x^2/2b^2}\, dx$$

$$= 2b^2 \int_0^\infty y\, e^{-y}\, dy \qquad \left(y = \frac{x^2}{2b^2} \right)$$

$$= 2b^2\, \Gamma(2) = 2b^2 . 1\,! = 2b^2$$

$$\therefore \qquad \sigma^2 = \mu_2 = \mu_2' - \mu_1'^2 = 2b^2 - b^2 \cdot \frac{\pi}{2} = b^2\left(2 - \frac{\pi}{2}\right)$$

$$\Rightarrow \qquad \sigma = b\sqrt{2 - (\pi/2)}$$

Hence
$$\frac{Q_3 - Q_1}{\sigma} = \frac{\sqrt{2}\,[\sqrt{\log 4} - \sqrt{\log (4/3)}\,]}{\sqrt{2 - (\pi/2)}},$$

which is independent of the parameter 'b'.

Example 5·11. *Prove that the geometric mean G of the distribution*

$$dF = 6(2-x)(x-1)\,dx, \quad 1 \le x \le 2$$

is given by $6 \log (16G) = 19$. **[Kanpur Univ. B.Sc., Oct. 1992]**

Solution. By definition, we have

$$\log G = \int_1^2 \log x\, f(x)\, dx = 6\int_1^2 \log x\ (2-x)(x-1)\, dx$$

$$= -6\int_1^2 (x^2 - 3x + 2) \log x\, dx$$

Integrating by parts, we get

$$\log G = -6\left[\left|\left(\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right)\log x\right|_1^2 - \int_1^2 \left(\frac{x^3}{3} - \frac{3x^2}{2} + 2x\right)\frac{1}{x}\, dx\right]$$

$$= -4 \log 2 + 6 \times \frac{19}{36} \qquad \text{(on simplification)}$$

$$\therefore \qquad \log G + 4 \log 2 = \frac{19}{6} \quad \Rightarrow \quad \log G + \log 2^4 = \frac{19}{6}$$

$$\Rightarrow \qquad \log G + \log 16 = \frac{19}{6} \quad \Rightarrow \quad \log (16\,G) = \frac{19}{6}$$

$$\Rightarrow \qquad 6 \log (16\,G) = 19$$

Example 5·12. *The time one has to wait for a bus at a downtown bus stop is observed to be random phenomenon (X) with the following probability density function :*

$$
\begin{aligned}
f_X(x) &= 0, && \text{for } x < 0 \\
&= \tfrac{1}{9}(x+1), && \text{for } 0 \le x < 1 \\
&= \tfrac{4}{9}\left(x - \tfrac{1}{2}\right), && \text{for } 1 \le x < \tfrac{3}{2} \\
&= \tfrac{4}{9}\left(\tfrac{5}{2} - x\right), && \text{for } \tfrac{3}{2} \le x < 2 \\
&= \tfrac{1}{9}(4 - x), && \text{for } 2 \le x < 3 \\
&= \tfrac{1}{9}, && \text{for } 3 \le x < 6
\end{aligned}
$$

$$= 0, \qquad \text{for} \quad 6 \le x,$$

Let the events A and B be defined as follows :

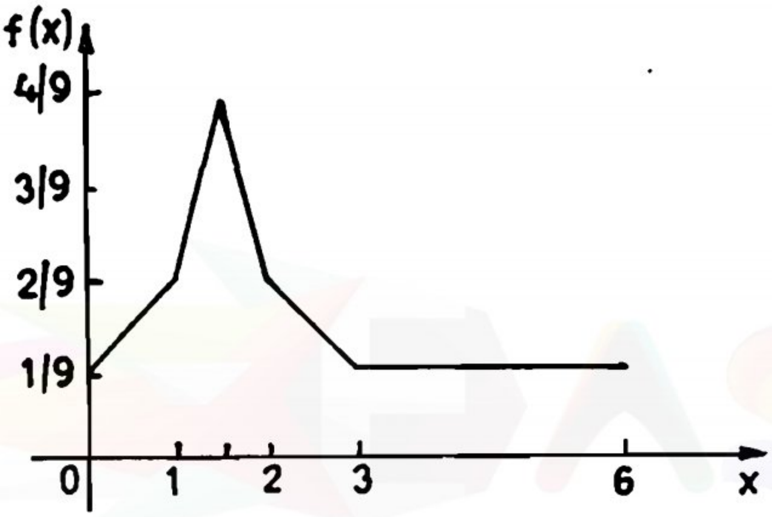
A : One waits between 0 to 2 minutes inclusive:

B : One waits between 0 to 3 minutes inclusive.

(i) Draw the graph of probability density function.

(ii) Show that (a) $P(B \mid A) = \frac{2}{3}$, *(b)* $P(\bar{A} \cap \bar{B}) = \frac{1}{3}$

Solution. *(i)* The graph of p.d.f. is given below.



(ii) (a) $P(A) = \int_0^2 f(x)\,dx = \int_0^1 \frac{1}{9}(x+1)\,dx + \int_1^{3/2} \frac{4}{9}\left(x - \frac{1}{2}\right)dx$

$$+ \int_{3/2}^{2} \frac{4}{9}\left(\frac{5}{2} - x\right)dx$$

$$= \frac{1}{2} \quad \text{(on simplification)}$$

$$P(A \cap B) = P(1 \le X \le 2) = \int_1^2 f(x)\,dx$$

$$= \int_1^{3/2} \frac{4}{9}\left(x - \frac{1}{2}\right)dx + \int_{3/2}^{2} \frac{4}{9}\left(\frac{5}{2} - x\right)dx$$

$$= \frac{4}{9}\left[\frac{x^2}{2} - \frac{x}{2}\right]_1^{3/2} + \frac{4}{9}\left[\frac{5}{2}x - \frac{x^2}{2}\right]_{3/2}^{2} = \frac{1}{3}$$

(on simplification)

$$\therefore \; P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{1/3}{1/2} = \frac{2}{3}$$

(b) $\bar{A} \cap \bar{B}$ means that waiting time is more than 3 minutes.

$$\therefore \; P(\bar{A} \cap \bar{B}) = P(X > 3) = \int_3^{\infty} f(x)\,dx = \int_3^6 f(x)\,dx + \int_6^{\infty} f(x)\,dx$$

$$= \int_3^6 \frac{1}{9}\,dx = \frac{1}{9}\,\Big| x \Big|_3^6 = \frac{1}{3}$$

Example 5·13. *The amount of bread (in hundreds of pounds) X that a certain bakery is able to sell in a day is found to be a numerical valued random phenomenon, with a probability function specified by the probability density function $f(x)$, given by*

$$f(x) = A \cdot x, \quad \text{for } 0 \le x < 5$$
$$= A(10 - x), \quad \text{for } 5 \le x < 10$$
$$= 0, \quad \text{otherwise}$$

(a) Find the value of A such that $f(x)$ is a probability density function.

(b) What is the probability that the number of pounds of bread that will be sold tomorrow is

(i) more than 500 pounds,

(ii) less than 500 pounds,

(iii) between 250 and 750 pounds? **[Agra Univ. B.Sc., 1989]**

(c) Denoting by A, B, C the events that the pounds of bread sold are as in b (i), b (ii) and b (iii) respectively, find $P(A \mid B), P(A \mid C)$. Are (i) A and B independent events? (ii) Are A and C independent events ?

Solution. *(a)* In order that $f(x)$ should be a probability density function

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

i.e.,
$$\int_0^5 A x\, dx + \int_5^{10} A(10 - x)\, dx = 1$$

$$\Rightarrow \quad A = \frac{1}{25} \qquad \text{(On simplification)}$$

(b) (i) The probability that the number of pounds of bread that will be sold tomorrow is more than 500 pounds, *i.e.,*

$$P(5 \le X \le 10) = \int_5^{10} \frac{1}{25}(10 - x)\, dx = \frac{1}{25}\left| 10x - \frac{x^2}{2} \right|_5^{10}$$
$$= \frac{1}{25}\left(\frac{25}{2}\right) = \frac{1}{2} = 0.5$$

(ii) The probability that the number of pounds of bread that will be sold tomorrow is less than 500 pounds, *i.e.,*

$$P(0 \le X \le 5) = \int_0^5 \frac{1}{25} \cdot x\, dx = \frac{1}{25}\left| \frac{x^2}{2} \right|_0^5 = \frac{1}{2} = 0.5$$

(iii) The required probability is given by

$$P(2.5 \le X \le 7.5) = \int_{2.5}^5 \frac{1}{25}\, x\, dx + \int_5^{7.5} \frac{1}{25}(10 - x)\, dx = \frac{3}{4}$$

(c) The events A, B and C are given by

$$A : 5 < X \le 10; \quad B : 0 \le X < 5; \quad C : 2.5 < X < 7.5$$

Then from parts *b (i), (ii)* and *(iii)*, we have

$$P(A) = 0{\cdot}5\ ,\ \ P(B) = 0{\cdot}5\ ,\ \ P(C) = \frac{3}{4}$$

The events $A \cap B$ and $A \cap C$ are given by

$$A \cap B = \phi \ \text{ and } \ A \cap C \ :\ 5 < X < 7{\cdot}5$$

$$\therefore \qquad P(A \cap B) = P(\phi) = 0$$

and $$P(A \cap C) = \int_5^{7\cdot5} f(x)\,dx = \frac{1}{25}\int_5^{7\cdot5} (10 - x)\,dx$$

$$= \frac{1}{25} \times \frac{75}{8} = \frac{3}{8}$$

$$P(A) \,.\, P(C) = \frac{1}{2} \times \frac{3}{4} = \frac{3}{8} = P(A \cap C)$$

$\Rightarrow$ A and C are independent.

Again $P(A) \,.\, P(B) = \dfrac{1}{4} \neq P(A \cap B)$

$\Rightarrow$ A and B are not independent.

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = 0$$

$$P(A \mid C) = \frac{P(A \cap C)}{P(C)} = \frac{3/8}{3/4} = \frac{1}{2}$$

Example 5·14. *The mileage C in thousands of miles which car owners get with a certain kind of tyre is a random variable having probability density function*

$$f(x) = \frac{1}{20}\, e^{-x/20}\ ,\ \text{for } x > 0$$
$$= 0\ ,\ \text{for } x \le 0$$

Find the probabilities that one of these tyres will last

(i) at most 10,000 miles,

(ii) anywhere from 16,000 to 24,000 miles.

(iii) at least 30,000 miles. **(Bombay Univ. B.Sc. 1989)**

Solution. Let *r.v.* X denote the mileage (in '000 miles) with a certain kind of tyre. Then required probability is given by:

(i) $$P(X \le 10) = \int_0^{10} f(x)\,dx = \frac{1}{20}\int_0^{10} e^{-x/20}\,dx$$

$$= \frac{1}{20}\left|\frac{e^{-x/20}}{-1/20}\right|_0^{10} = 1 - e^{-1/2}$$

$$= 1 - 0{\cdot}6065 = 0{\cdot}3935$$

(ii) $P(16 \leq X \leq 24) = \frac{1}{20} \int_{16}^{24} \exp\left(-\frac{x}{20}\right) dx = \left| -e^{-x/20} \right|_{16}^{24}$

$= e^{-16/20} - e^{-24/20} = e^{-4/5} - e^{-6/5}$

$= 0{\cdot}4493 - 0{\cdot}3012 = 0{\cdot}1481$

(iii) $P(X \geq 30) = \int_{30}^{\infty} f(x)\, dx = \frac{1}{20} \left| \frac{e^{-x/20}}{-1/20} \right|_{30}^{\infty}$

$= e^{-1.5} = 0{\cdot}2231$

EXERCISE 5 (c)

1. *(a)* A continuous random variable X follows the probability law

$$f(x) = A x^2,\ 0 \leq x \leq 1$$

Determine A and find the probability that *(i)* X lies between 0·2 and 0·5, *(ii)* X is less than 0·3, *(iii)* $1/4 < X < 1/2$ and *(iv)* $X > 3/4$ given $X > 1/2$.

Ans. $A = 0{\cdot}3$, *(i)* 0·117, *(ii)* 0·027, *(iii)* 15/256 and *(iv)* 27/56.

(b) If a random variable X has the density function

$$f(x) = \begin{Bmatrix} 1/4, & -2 < x < 2 \\ 0, & \textit{elsewhere}. \end{Bmatrix}$$

Obtain *(i)* $P(X < 1)$, *(ii)* $P(|X| > 1)$ *(iii)* $P(2X + 3 > 5)$

(Kerala Univ. B.Sc., Sept.1992)

Hint. *(ii)* $P(|X| > 1) = P(X > 1 \text{ or } X < -1) = \int_{-2}^{-1} f(x)\, dx + \int_{1}^{2} f(x)\, dx$

or $P(|X| > 1) = 1 - P(|X| \leq 1) = 1 - P(-1 \leq X \leq 1)$

Ans. *(i)* 3/4, *(ii)* 1/2 *(iii)* 1/4.

2. Are any of the following probability mass or density functions? Prove your answer in each case.

(a) $f(x) = x;\ x = \frac{1}{16}, \frac{3}{16}, \frac{1}{4}, \frac{1}{2}$

(b) $f(x) = \lambda e^{-\lambda x};\ x \geq 0;\ \lambda > 0$

(c) $f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 4 - 2x, & 1 < x < 2 \\ 0, & \textit{elsewhere}, \end{cases}$

(Calicut Univ. B. Sc., Oct. 1989)

Ans. *(a)* and *(b)* are p.m.f./p.d.f.'s, *(c)* is not.

3. If f_1 and f_2 are p.d.f.'s and $\theta_1 + \theta_2 = 1$, check if

$$g(x) = \theta_1 f_1(x) + \theta_2 f_2(x), \quad \text{is a p.d.f.}$$

Ans. $g(x)$ is a p.d.f. if $0 \leq (\theta_1, \theta_2) \leq 1$ and $\theta_1 + \theta_2 = 1$.

4. A continuous random variable X has the probability density function :

$$f(x) = A + Bx, \quad 0 \le x \le 1.$$

If the mean of the distribution is $\frac{1}{2}$, find A and B.

Hint : Solve $\int_0^1 f(x)\,dx = 1$ and $\int_0^1 x\,f(x)\,dx = \frac{1}{2}$. Find A and B.

5. For the following density function

$$f(x) = c\,x^2(1-x), \quad 0 < x < 1,$$

find *(i)* the constant c, and *(ii)* mean.

[Calicut Univ. B.Sc.(subs.), 1991]

Ans. *(i)* $c = 12$; *(ii)* mean $= 3/5$.

6. A continuous distribution of a variable X in the range $(-3, 3)$ is defined by

$$\begin{aligned} f(x) &= \tfrac{1}{16}(3+x)^2, \quad -3 \le x \le -1 \\ &= \tfrac{1}{16}(6-2x^2), \quad -1 \le x \le 1 \\ &= \tfrac{1}{16}(3-x)^2, \quad 1 \le x \le 3 \end{aligned}$$

(i) Verify that the area under the curve is unity.

(ii) Find the mean and variance of the above distribution.

(Madras Univ. B.Sc., Oct. 1992; Gujarat Univ. B.Sc., Oct. 1986)

Hint: $\int_{-3}^{3} f(x)\,dx = \int_{-3}^{-1} f(x)\,dx + \int_{-1}^{1} f(x)\,dx + \int_{1}^{3} f(x)\,dx$

Ans. Mean=0, Variance=1

7. If the random variable X has the p.d.f.,

$$\begin{aligned} f(x) &= \tfrac{1}{2}(x+1), \quad -1 < x < 1 \\ &= 0, \text{ elsewhere,} \end{aligned}$$

find the coefficient of skewness and kurtosis.

8. *(a)* A random variable X has the probability density function given by

$$f(x) = 6x(1-x), \quad 0 \le x \le 1$$

Find the mean μ, mode and S.D. σ, Compute $P(\mu - 2\sigma < X < \mu + 2\sigma)$.
Find also the mean deviation about the median.

(Lucknow Univ. B.Sc., 1988)

(b) For the continuous distribution

$$dF = y_o(x - x^2)\,dx \;; \quad 0 \le x \le 1, \quad y_o \text{ being a constant.}$$

Find *(i)* arithmetic mean, *(ii)* harmonic mean, *(iii)* Median, *(iv)* Mode and *(v)* rth moment about mean. Hence find β_1 and β_2 and show that the distribution is symmetrical. **(Delhi Univ. B.Sc., 1992; Karnatak Univ. B.Sc., 1991)**

Ans. Mean = Median = Mode = $\frac{1}{2}$

(c) Find the mean, mode and median for the distribution,

$$dF(x) = \sin x \, dx, \quad 0 \le x \le \pi/2$$

Ans. 1, $\pi/2$, $\pi/3$

9. If the function $f(x)$ is defined by

$$f(x) = c\, e^{-\alpha x}, \quad 0 \le x < \infty, \quad \alpha > 0$$

(i) Find the value of constant c.

(ii) Evaluate the first four moments about mean.

[Gauhati Univ. B.Sc. 1987]

Ans. *(i)* $c = \alpha$, *(ii)* 0, $1/\alpha^2$, $2/\alpha^3$, $9/\alpha^4$.

10. *(a)* Show that for the exponential distribution

$$dP = y_o \, . \, e^{-x/\sigma} \, dx, \quad 0 \le x < \infty, \quad \sigma > 0$$

the mean and S.D. are both equal to σ and that the interquartile range is $\sigma \log_e 3$. Also find μ_r' and show that $\beta_1 = 4$, $\beta_2 = 9$.

[Agra Univ. B.Sc., 1986 ; Madras Univ. B.Sc., 1987]

(b) Define the harmonic mean (H.M.) of variable X as the reciprocal of the expected value of $1/X$, show that the H.M. of variable which ranges from 0 to ∞ with probability density $\frac{1}{6} x^3 e^{-x}$ is 3.

11. *(a)* Find the mean, variance and the co-efficients β_1, β_2 of the distribution,

$$dF = k\, x^2 e^{-x} dx, \quad 0 < x < \infty.$$

Ans. $k = 1/2$; 3, 3, 4/3 and 5.

(b) Calculate β_1 for the distribution,

$$dF = k\, x\, e^{-x} dx, \quad 0 < x < \infty$$

Ans. 2 **[Delhi Univ. B.Sc. (Hons. Subs.), 1988]**

12. A continuous random variable X has a p.d.f. given by

$$f(x) = k\, x\, e^{\lambda x}, \quad x \ge 0, \quad \lambda > 0$$
$$= 0, \text{ otherwise}$$

Determine the constant k, obtain the mean and variance of X.

[Nagpur Univ. B.Sc. 1990]

13. For the probability density function,

$$f(x) = \frac{2(b+x)}{b(a+b)}, \quad -b \le x < 0$$
$$= \frac{2(a-x)}{a(a+b)}, \quad 0 \le x \le a$$

Find mean, median and variance. **[Calcutta Univ. B.Sc. 1984]**

Ans. Mean $= (a-b)/3$, Variance $= (a^2 + b^2 + ab)/18$,

Median $= a - \sqrt{a(a+b)/2}$

(ii) Show that, if terms of order $(a-b)^2/a^2$ are neglected, then

$$\text{mean} - \text{median} = (\text{mean} - \text{mode})/4$$

14. A variable X can assume values only between 0 and 5 and the equation of its frequency curve is

$$y = A \sin \tfrac{1}{5} \pi x, \quad 0 \le x \le 5$$

where A is a constant such that the area under the curve is unity. Determine the value of A and obtain the median and quartiles of the distribution.

Show also that the variance of the distribution is $50\left\{\frac{1}{8} - \frac{1}{\pi^2}\right\}$.

Ans. 1/10, 2·5, 4/3, 10/3

15. A continuous variable X is distributed over the interval [0, 1] with p.d.f. $ax^2 + bx$, where a, b are constants. If the arithmetic mean of X is 0·5, find the values of a and b.

Ans. $-6, 6$

16. A man leaves his house at the same time every morning and the time taken to journey to work has the following probability density function : less than 30 minutes, zero, between 30 minutes and 60 minutes, uniform with density k ; between 60 minutes and 70 minutes, uniform with density $2k$; and more than 70 minutes, zero. What is the probability that on one particular day he arrives at work later than on the previous day but not more than 5 minutes later.

17. The density function of sheer strength of spot welds is given by

$$f(x) = x/160{,}000 \quad \text{for } 0 \le x \le 400$$
$$= (800 - x)/160{,}000 \quad \text{for } 400 \le x \le 800$$

Find the number a such that

Prob. $(X < a) = 0{\cdot}50$ and the number b such that

Prob. $(X < b) = 0{\cdot}90$. Find the mean, median and variance of X.

[Delhi Univ. B.E., 1987]

18. A batch of small calibre ammunition is accepted as satisfactory if none of a sample of five shot falls more than 2 feet from the centre of the target at a given range. If X, the distance from the centre of the target to a given impact point, actually has the density

$$f(x) = k \,.\, 2x e^{-x^2}, \quad 0 < x < 3$$

where k is a number which makes it probability density function, what is the value of k and what is the probability that the batch will be accepted?

[Nagpur Univ. B.E., 1987]

Hint. $\int_0^3 f(x)\, dx = 1 \quad \Rightarrow \quad k = 1/(1 - e^{-9})$

Reqd. Prob. = P [Each of a sample of 5 shots falls within a distance of 2 ft. from the centre]

$$= [P(0 < X < 2)]^5 = \left[\int_0^2 f(x)\,dx\right]^5 = \left[\frac{1-e^{-4}}{1-e^{-9}}\right]^5$$

19. A random variable X has the p.d.f. :

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & otherwise \end{cases}$$

Find *(i)* $P\left(X < \frac{1}{2}\right)$, *(ii)* $P\left(\frac{1}{4} < X < \frac{1}{2}\right)$, *(iii)* $P\left(X > \frac{3}{4} \,\middle|\, X > \frac{1}{2}\right)$, and *(iv)* $P\left(X < \frac{3}{4} \,\middle|\, X > \frac{1}{2}\right)$.

(Gorakhpur Univ. B.Sc., 1988)

Ans. *(i)* 1/4 , *(ii)* 3/16 , *(iii)* $\dfrac{P(X > 3/4)}{P(X > 1/2)} = \dfrac{7/16}{3/4} = \dfrac{7}{12}$; *(iv)* $\dfrac{P(1/2 < X < 3/4)}{P(X > 1/2)}$

5·4·3. Continuous Distribution Function. If X is a continuous random variable with the p.d.f. $f(x)$, then the function

$$F_X(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt, \quad -\infty < x < \infty. \qquad \text{...(5·12)}$$

is called the *distribution function* (d.f.) or sometimes the *cumulative distribution function* (c.d.f.) of the random variable X.

Remarks 1. $0 \le F(x) \le 1, \quad -\infty < x < \infty.$

2. From analysis (Riemann integral), we know that

$$F'(x) = \frac{d}{dx}F(x) = f(x) \ge 0 \qquad [\because f(x) \text{ is p.d.f.}]$$

$\Rightarrow$ $F(x)$ is non-decreasing function of x.

3. $F(-\infty) = \lim\limits_{x \to -\infty} F(x) = \lim\limits_{x \to -\infty} \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{-\infty} f(x)\,dx = 0$

and $F(+\infty) = \lim\limits_{x \to \infty} F(x) = \lim\limits_{x \to \infty} \int_{-\infty}^{x} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx = 1$

4. $F(x)$ is a continuous function of x on the right.

5. The discontinuities of $F(x)$ are at the most countable.

6. It may be noted that

$$P(a \le X \le b) = \int_a^b f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx - \int_{-\infty}^{a} f(x)\,dx$$

$$= P(X \le b) - P(X \le a) = F(b) - F(a)$$

Similarly

$$P(a < X < b) = P(a < X \le b) = P(a \le X < b) = \int_a^b f(t)\,dt$$

7. Since $F'(x) = f(x)$, we have

$$\frac{d}{dx} F(x) = f(x) \quad \Rightarrow \quad dF(x) = f(x)\, dx$$

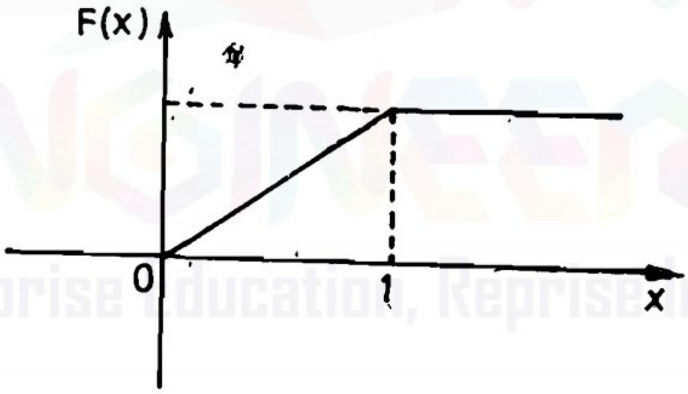
This is known as probability differential of X.

Remarks. 1. It may be pointed out that the properties (2), (3) and (4) above uniquely characterise the distribution functions. This means that any function $F(x)$ satisfying (2) to (4) is the distribution function of some random variable, and any function $F(x)$ violating any one or more of these three properties cannot be the distribution function of any random variable.

2. Often, one can obtain a p.d.f. from a distribution function $F(x)$ by differentiating $F(x)$, provided the derivative exists. For example, consider

$$F_X(x) = \begin{cases} 0, & \text{for } x < 0 \\ x, & \text{for } 0 \leq x \leq \\ 1, & \text{for } x > 1 \end{cases}$$

The graph of $F(x)$ is given by bold lines. Obviously we see that $F(x)$ is continuous from right as stipulated in (4) and we also see that $F(x)$ is not continuous at $x = 0$ and $x = 1$ and hence is not derivable at $x = 0$ and $x = 1$.



Differentiating $F(x)$ w.r.t. x, we get

$$\frac{d}{dx} F(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \textit{otherwise} \end{cases}$$

[Note the strict inequality in $0 < x < 1$, since $F(x)$ is not derivable at $x = 0$ and $x = 1$]

Let us define

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \textit{otherwise} \end{cases}$$

Then $f(x)$ is a p.d.f. for F.

Example 5·15. *Verify that the following is a distribution function:*

$$F(x)=\begin{cases} 0, & x<-a \\ \frac{1}{2}\left(\frac{x}{a}+1\right), & -a\leq x\leq a \\ 1, & x>a \end{cases}$$

(Madras Univ. B.Sc., 1992)

Solution. Obviously the properties *(i), (ii), (iii)* and *(iv)* are satisfied. Also we observe that $F(x)$ is continuous at $x=a$ and $x=-a$ as well.

Now

$$\frac{d}{dx}F(x)=\begin{cases} \frac{1}{2a}, & -a\leq x\leq a \\ 0, & otherwise \end{cases}$$

$$=f(x),\ say$$

In order that $F(x)$ is a distribution function, $f(x)$ must be a p.d.f. Thus we have to show that

$$\int_{-\infty}^{\infty} f(x)\ dx=1$$

Now $\int_{-\infty}^{\infty} f(x)\ dx=\int_{-a}^{a} f(x)\ dx=\frac{1}{2a}\int_{-a}^{a} 1\,.\,dx=1$

Hence $F(x)$ is a d.f.

Example 5·16. *Suppose the life in hours of a certain kind of radio tube has the probability density function:*

$$f(x)=\frac{100}{x^2},\quad when\ x\geq 100$$
$$=0,\quad when\ x<100$$

Find the distribution function of the distribution. What is the probabilty that none of three such tubes in a given radio set will have to be replaced during the first 150 hours of operation ? What is the probability that all three of the original tubes will have been replaced during the first 150 hours ? **(Delhi Univ. B.Sc, Oct. 1988)**

Solution. Probability that a tube will last for first 150 hours is given by

$$P(X\leq 150)=P(0<X<100)+P(100\leq X\leq 150)$$

$$=\int_{100}^{150} f(x)\,dx=\int_{100}^{150}\frac{100}{x^2}\,dx=\frac{1}{3}$$

Hence the probability that none of the three tubes will have to be replaced during the first 150 hours is $(1/3)^3=1/27$.

The probability that a tube will not last for the first 150 hours is $1-\frac{1}{3}=\frac{2}{3}$.

Hence the probability that all three of the original tubes will have to be replaced during the first 150 hours is $(2/3)^3 = 8/27$.

Example 5·17. *Suppose that the time in minutes that a person has to wait at a certain station for a train is found to be a random phenomenon, a probability function specified by the distribution function,*

$$F(x) = 0, \quad \text{for } x \le 0$$
$$= \frac{x}{2}, \quad \text{for } 0 \le x < 1$$
$$= \frac{1}{2}, \quad \text{for } 1 \le x < 2$$
$$= \frac{x}{4}, \quad \text{for } 2 \le x < 4$$
$$= 1, \quad \text{for } x \ge 4$$

(a) Is the Distribution Function continuous ? If so, give the formula for its probability density function ?

(b) What is the probability that a person will have to wait (i) more than 3 minutes, (ii) less than 3 minutes, and (iii) between 1 and 3 minutes ?

(c) What is the conditional probability that the person will have to wait for a train for (i) more than 3 minutes, given that it is more than 1 minute, (ii) less than 3 minutes given that it is more than 1 minute ? **(Calicut Univ. B.Sc., 1985)**

Solution. *(a)* Since the value of the distribution function is the same at the points $x = 0, x = 1, x = 2,$ and $x = 4$ given by the two forms of $F(x)$ for $x < 0$ and $0 \le x < 1$, $0 \le x < 1$ and $1 \le x < 2$, $1 \le x < 2$ and $2 \le x < 4$, $2 \le x < 4$ and $x \ge 4$, the distribution function is continuous.

$$\text{Probability density function} = f(x) = \frac{d}{dx} F(x)$$

$$\therefore \quad f(x) = 0, \quad \text{for } x < 0$$
$$= \frac{1}{2}, \quad \text{for } 0 \le x < 1.$$
$$= 0, \quad \text{for } 1 \le x < 2$$
$$= \frac{1}{4}, \quad \text{for } 2 \le x < 4$$
$$= 0, \quad \text{for } x \ge 4$$

(b) Let the random variable X represent the waiting time in minutes.

Then

(i) Required probability $= P(X > 3) = 1 - P(X \le 3) = 1 - F(3)$
$$= 1 - \frac{1}{4} . 3 = \frac{1}{4}$$

(ii) Required probability $= P(X < 3) = P(X \le 3) - P(X = 3)$
$$= F(3) = \frac{3}{4}$$

(Since, the probability that a continuous variable takes a fixed value is zero)

(iii) Required Probability $= P(1 < X < 3) = P(1 < X \le 3)$

$$= F(3) - F(1) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$$

(c) Let A denote the event that a person has to wait for more than 3 minutes and B the event that he has to wait for more than 1 minute. Then

$$P(A) = P(X > 3) = \frac{1}{4} \qquad [\textit{c.f.}\ (b), (i)]$$

$$P(B) = P(X > 1) = 1 - P(X \le 1) = 1 - F(1) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$P(A \cap B) = P(X > 3 \cap X > 1) = P(X > 3) = \frac{1}{4}$$

(i) Required probability is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)} = \frac{1/4}{1/2} = \frac{1}{2}$$

(ii) Required probability $= P(\bar{A} \mid B) = \dfrac{P(\bar{A} \cap B)}{P(B)}$

Now $P(\bar{A} \cap B) = P(X \le 3 \cap X > 1) = P(1 < X \le 3) = F(3) - F(1) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}$

$$\therefore \quad P(\bar{A} \mid B) = \frac{1/4}{1/2} = \frac{1}{2}$$

Example 5·18. *A petrol pump is supplied with petrol once a day. If its daily volume X of sales in thousands of litres is distributed by*

$$f(x) = 5(1-x)^4, \quad 0 \le x \le 1,$$

what must be the capacity of its tank in order that the probability that its supply will be exhausted in a given day shall be 0·01 ? **(Madras Univ. B.E., 1986)**

Solution. Let the capacity of the tank (in '000 of litres) be 'a' such that

$$P(X \ge a) = 0{\cdot}01 \quad \Rightarrow \quad \int_a^1 f(x)\,dx = 0{\cdot}01$$

$$\Rightarrow \quad \int_a^1 5(1-x)^4\,dx = 0{\cdot}01 \quad \textit{or} \quad \left[5 \cdot \frac{(1-x)^5}{(-5)}\right]_a^1 = 0{\cdot}01$$

$$\Rightarrow \quad (1-a)^5 = 1/100 \quad \textit{or} \quad 1 - a = (1/100)^{1/5}$$

$$\therefore \quad a = 1 - (1/100)^{1/5} = 1 - 0{\cdot}3981 = 0{\cdot}6019$$

Hence the capacity of the tank $= 0{\cdot}6019 \times 1000$ *litres* $= 601{\cdot}9$ *litres*.

Example 5·19. *Prove that mean deviation is least when measured from the median.* **[Delhi Univ. B.Sc. (Maths. Hons.), 1989]**

Solution. If $f(x)$ is the probability function of a random variable X, $a \le X \le b$, then mean deviation $M(A)$, say, about the point $x = A$ is given by

$$M(A) = \int_a^b |x - A| f(x)\,dx$$

$$= \int_a^A |x - A| f(x)\,dx + \int_A^b |x - A| f(x)\,dx$$

$$= \int_a^A (A - x) f(x)\,dx + \int_A^b (x - A) f(x)\,dx \qquad ...(1)$$

We want to find the value of 'A' so that $M(A)$ is minimum. From the principle of maximum and minimum in differential calculus, $M(A)$ will be minimum for variations in A if

$$\frac{\partial M(A)}{\partial A} = 0 \quad \text{and} \quad \frac{\partial^2 M(A)}{\partial A^2} > 0 \qquad ...(2)$$

Differentiating (1) w.r.t. 'A' under the integral sign, since the functions $(A - x) f(x)$ and $(x - A) f(x)$ vanish at the point $x = A$*, we get

$$\frac{\partial M(A)}{\partial A} = \int_a^A f(x)\,dx - \int_A^b f(x)\,dx \qquad ...(3)$$

Also $$\frac{\partial M(A)}{\partial A} = \int_a^A f(x)\,dx - \left[1 - \int_a^A f(x)\,dx\right],$$

$$\left[\because \int_a^b f(x)\,dx = 1\right]$$

$$= 2\int_a^A f(x)\,dx - 1 = 2F(A) - 1\,;$$

where $F(\cdot)$ is the distribution function of X. Differentiating again w.r.t. A, we get

$$\frac{\partial^2}{\partial A^2} M(A) = 2f(A) \qquad ...(4)$$

Now $\dfrac{\partial M(A)}{\partial A} = 0$, on using (3) gives

$$\int_a^A f(x)\,dx = \int_A^b f(x)\,dx$$

i.e., A is the median value.

Also from (4), we see that

$$\frac{\partial^2 M(A)}{\partial A^2} > 0,$$

assuming that $f(x)$ does not vanish at the median value. Thus mean deviation is least when taken from median.

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*If $f(x, \theta)$ is a continuous function of both variables x and θ, possessing continuous partial derivatives $\dfrac{\partial^2 f}{\partial x \partial \theta}$, $\dfrac{\partial^2 f}{\partial \theta \partial x}$ and a and b are differentiable functions of θ, then

$$\frac{\partial}{\partial \theta}\left[\int_a^b f(x, \theta)\,dx\right] = \int_a^b \frac{\partial f}{\partial \theta}\,dx + f(b, \theta)\frac{db}{d\theta} - f(a, \theta)\frac{da}{d\theta}$$

EXERCISE 5 (d)

1. *(a)* Explain the terms *(i)* probability differential, *(ii)* probability density function, and *(iii)* distribution function.

(b) Explain what is meant by a random variable. Distinguish between a discrete and a continuous random variable. Define distribution function of a random variable and show that it is monotonic non-decreasing everywhere and continuous on the right at every point.

[Madras Univ. B.Sc. (Stat Main), 1987]

(c) Show that the distribution function $F(x)$ of a random variable X is a non-decreasing function of x. Determine the jump of $F(x)$ at a point x_o of its discontinuity in terms of the probability that the random variable has the value x_o.

[Calcutta Univ. B.Sc. (Hons.), 1984]

2. The length (in hours) X of a certain type of light bulb may be supposed to be a continuous random variable with probability density function :

$$f(x) = \frac{a}{x^3},\ 1500 < x < 2500$$
$$= 0,\ \text{elsewhere.}$$

Determine the constant a, the distribution function of X, and compute the probability of the event $1{,}700 \le X \le 1{,}900$.

Ans. $a = 70{,}31{,}250$; $F(x) = \frac{a}{2}\left(\frac{1}{22{,}50{,}000} - \frac{1}{x^2}\right)$ and

$$P(1{,}700 < X < 1{,}900) = F(1{,}900) - F(1{,}700) = \frac{a}{2}\left(\frac{1}{28{,}90{,}000} - \frac{1}{36{,}10{,}000}\right)$$

3. Define the "distribution function" (or cumulative distribution function) of a random variable and state its essential properties.

Show that, whatever the distribution function $F(x)$ of a random variable X, $P[a \le F(x) \le b] = b - a,\ 0 \le a,\ b \le 1$.

4. *(a)* The distribution function of a random variable X is given by

$$F(x) = \begin{cases} 1 - (1 + x)e^{-x}, & \text{for } x \ge 0 \\ 0, & \text{for } x < 0 \end{cases}$$

Find the corresponding density function of random variable X.

(b) Consider the distribution for X defined by

$$F(x) = \begin{cases} 0, & \text{for } x < 0 \\ 1 - \frac{1}{4}e^{-x} & \text{for } x \ge 0 \end{cases}$$

Determine $P(x = 0)$ and $P(x > 0)$.

[Allahabad Univ. B.Sc., 1992]

5. *(a)* Let X be a continuous random variable with probability density function given by

$$f(x) = \begin{cases} ax, & 0 \le x \le 1 \\ a, & 1 \le x \le 2 \\ -ax + 3a, & 2 \le x \le 3 \\ 0, & elsewhere \end{cases}$$

(i) Determine the constant a.

(ii) Determine $F(x)$, and sketch its graph.

(iii) If three independent observations are made, what is the probability that exactly one of these three numbers is larger than 1·5 ?

[Rajasthan Univ. M.Sc., 1987]

Ans. *(i)* 1/2, *(iii)* 3/8.

(b) For the density $f_X(x) = k\, e^{-ax}(1 - e^{-ax})\; I_{0,\infty}(x)$, find the normalising constant k, $f_X(x)$ and evaluate $P(X > 1)$.

[Delhi Univ. B.Sc. (Maths Hons.), 1989]

Ans. $k = 2a$; $F(x) = 1 - 2e^{-ax} + e^{-2ax}$; $P(X > 1) = 2e^{-a} - e^{-2a}$

6. A random variable X has the density function :

$$f(x) = K \cdot \frac{1}{1 + x^2}, \text{ if } -\infty < x < \infty$$
$$= 0, \text{ otherwise}$$

Determine K and the distribution function.

Evaluate the probability $P(X \ge 0)$. Find also the mean and variance of X.

[Karnatak Univ. B.Sc. 1985]

Ans. $K = 1$, $F(x) = \dfrac{1}{\pi}\left\{\tan^{-1} x + \dfrac{\pi}{2}\right\}$, $P(x \ge 0) = 1/2$, Mean $= 0$, Variance does not exist.

7. A continuous random variable X has the distribution function

$$F(x) = \begin{cases} 0, & \text{if } x \le 1 \\ k(x-1)^4, & \text{if } 1 < x \le 3 \\ 1, & \text{if } x > 3 \end{cases}$$

Find *(i)* k, *(ii)* the probability density function $f(x)$, and *(iii)* the mean and the median of X.

Ans. *(i)* $k = \dfrac{1}{16}$, *(ii)* $f(x) = \dfrac{1}{4}(x - 1)^3$, $1 \le x \le 3$

8. Given $f(x) = \begin{cases} kx(1-x), & \text{for } 0 < x < 1 \\ 0, & elsewhere \end{cases}$

Show that

(i) $k = 1/5$, *(ii)* $F(x) = 0$ for $x \le 0$ and $F(x) = 1 - e^{-x/5}$, for $x > 0$

Using $F(x)$, show that

(iii) $P(3 < X < 5) = 0{\cdot}1809$, *(iv)* $P(X < 4) = 0{\cdot}5507$, *(v)* $P(X > 6) = 0{\cdot}3012$

9. A bombing plane carrying three bombs flies directly above a railroad track. If a bomb falls within 40 feet of track, the track will be sufficiently damaged to disrupt the traffic. Within a certain bomb site the points of impact of a bomb have the probability density function :

$$\begin{aligned} f(x) &= (100 + x)/10{,}000, \quad \textit{when } -100 \le x \le 0 \\ &= (100 - x)/10{,}000, \quad \textit{when } 0 \le x \le 100 \\ &= 0, \quad \textit{elsewhere} \end{aligned}$$

where x represents the vertical deviation (in feet) from the aiming point, which is the track in this case. Find the distribution function. If all the bombs are used, what is the probability that track will be damaged ?

Hint. Probability that track will be damaged by the bomb is given by

$$\begin{aligned} P(|X| < 40) &= P(-40 < X < 40) \\ &= \int_{-40}^{0} f(x)\,dx + \int_{0}^{40} f(x)\,dx \\ &= \int_{-40}^{0} \frac{100 + x}{10{,}000}\,dx + \int_{0}^{40} \frac{100 - x}{10{,}000}\,dx = \frac{16}{25} \end{aligned}$$

$\therefore$ Probability that a bomb will not damage the track $= 1 - \frac{16}{25} = \frac{9}{25}$

Probability that none of the three bombs damages the track $= \left(\frac{9}{25}\right)^3 = 0{\cdot}046656$

Required probability that the track will be damaged $= 1 - 0{\cdot}046656 = 0{\cdot}953344$.

10. The length of time (in minutes) that a certain lady speaks on the telephone is found to be random phenomenon, with a probability function specified by the probability density function $f(x)$ as

$$\begin{aligned} f(x) &= A\,e^{-x/5}, \quad \text{for } x \ge 0 \\ &= 0, \quad \textit{otherwise} \end{aligned}$$

(a) Find the value of A that makes $f(x)$ a p.d.f.

Ans. $A = 1/5$

(b) What is the probability that the number of minutes that she will talk over the phone is

(i) More than 10 minutes, *(ii)* less than 5 minutes, and *(iii)* between 5 and 10 minutes ? **[Shivaji Univ. B.Sc., 1990]**

Ans. *(i)* $\frac{1}{e^2}$, *(ii)* $\frac{e-1}{e}$, *(iii)* $\frac{e-1}{e^2}$.

11. The probability that a person will die in the time interval (t_1, t_2) is given by

$$A \int_{t_1}^{t_2} f(t)\, dt\,;$$

where A is a constant and the function $f(t)$ determined from long records, is

$$f(t) = \begin{cases} t^2 (100-t)^2, & 0 \le t \le 100 \\ 0, & elsewhere \end{cases}$$

Find the probability that a person will die between the ages 60 and 70 assuming that his age is ≥ 50. **[Calcutta Univ. B.A. (Hons.), 1987]**

5·5. Joint Probability Law. Two random variables X and Y are said to be jointly distributed if they are defined on the same probability space. The sample points consist of 2–tuples. If the joint probability function is denoted by $P_{XY}(x, y)$ then the probability of a certain event E is given by

$$P_{XY}(x, y) = P[(X, Y) \in E] \qquad \ldots (5·13)$$

(X, Y) is said to belong to E, if in the 2 dimensional space the 2–tuples lie in the Borel set B, representing the event E.

5·5·1. Joint Probability Mass Function and Marginal and Conditional Probability Functions. Let X and Y be random variables on a sample space S with respective image sets $X(S) = \{x_1, x_2, \ldots, x_n\}$ and $Y(S) = \{y_1, y_2, \ldots, y_m\}$. We make the product set

$$X(S) \times Y(S) = \{x_1, x_2, \ldots, x_n\} \times \{y_1, y_2, \ldots, y_m\}$$

into a probability space by defining the probability of the ordered pair (x_i, y_j) to be $P(X = x_i, Y = y_j)$ which we write $p(x_i, y_j)$. The function p on $X(S) \times Y(S)$ defined by

$$p_{ij} = P(X = x_i \cap Y = y_j) = p(x_i, y_j) \qquad \ldots (5·14)$$

is called the *joint probability function* of X and Y and is usually represented in the form of the following table :

X \ Y	y_1	y_2	y_3	...	y_j	...	y_m	Total
x_1	p_{11}	p_{12}	p_{13}	...	p_{1j}	...	p_{1m}	$p_{1.}$
x_2	p_{21}	p_{22}	p_{23}	...	p_{2j}	...	p_{2m}	$p_{2.}$
x_3	p_{31}	p_{32}	p_{33}	...	p_{3j}	...	p_{3m}	$p_{3.}$
$\vdots$	$\vdots$							$\vdots$
x_i	p_{i1}	p_{i2}	p_{i3}	...	p_{ij}	...	p_{im}	$p_{i.}$
$\vdots$	$\vdots$							$\vdots$
x_n	p_{n1}	p_{n2}	p_{n3}	...	p_{nj}	...	p_{nm}	$p_{n.}$
Total	$p_{.1}$	$p_{.2}$	$p_{.3}$	...	$p_{.j}$	...	$p_{.m}$	1

$$\therefore \qquad \sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i, y_j) = 1$$

Suppose the joint distribution of two random variables X and Y is given, then the probability distribution of X is determined as follows :

$$p_X(x_i) = P(X = x_i) = P[X = x_i \cap Y = y_1] + P[X = x_i \cap Y = y_2] + \ldots$$
$$+ P[X = x_i \cap Y = y_j] + \ldots + P[X = x_i \cap \quad y_m]$$
$$= p_{i1} + p_{i2} + \ldots + p_{ij} + \ldots + p_{im}$$
$$= \sum_{j=1}^{m} p_{ij} = \sum_{j=1}^{m} p(x_i, y_j) = p_{i\cdot} \qquad \text{.. (5·14 a)}$$

and is known as *marginal probability function of X.*

Also $\sum_{i=1}^{n} p_{i\cdot} = p_{1\cdot} + p_{2\cdot} + \ldots + p_{n\cdot} = \sum_{i=1}^{n} \sum_{j=1}^{m} p_{ij} = 1$

Similarly, we can prove that

$$p_Y(y_j) = P(Y = y_j) = \sum_{i=1}^{n} p_{ij} = \sum_{i=1}^{n} p(x_i, y_j) = p_{\cdot j} \qquad \text{... (5·14 b)}$$

which is the *marginal probability function of Y.*

Also

$$P[X = x_i \mid Y = y_j] = \frac{P[X = x_i \cap Y = y_j]}{P[Y = y_j]} = \frac{p(x_i, y_j)}{p(y_j)} = \frac{p_{ij}}{p_{\cdot j}}$$

This is known as *conditional probability function of X given* $Y = y_j$

Similarly

$$P[Y = y_j \mid X = x_i] = \frac{p(x_i, y_j)}{p(x_i)} = \frac{p_{ij}}{p_{i\cdot}} \qquad \text{... (5·14 c)}$$

is the *conditional probability function of Y given* $X = x_i$

Also $\sum_{i=1}^{n} \frac{p_{ij}}{p_{\cdot j}} = \frac{p_{1j} + p_{2j} + \ldots + p_{ij} + \ldots + p_{nj}}{p_{j}} = \frac{p_{\cdot j}}{p_{\cdot j}} = 1$

Similarly

$$\sum_{j=1}^{n} \frac{p_{ij}}{p_{i\cdot}} = 1$$

Two random variables X and Y are said to be *independent* if

$$P(X = x_i, Y = y_j) = P(X = x_i) \cdot P(Y = y_j), \qquad \text{... (5·14 d)}$$

otherwise they are said to be dependent.

5·5·2. Joint Probability Distribution Function. Let (X, Y) be a two-dimensional random variable then their joint distribution function is denoted by $F_{XY}(x, y)$ and it represents the probability that simultaneously the observation

(X, Y) will have the property $(X \le x$ and $Y \le y)$, *i.e.*,

$$F_{XY}(x, y) = P(-\infty < X \le x, -\infty < Y \le y)$$

$$= \int_{-\infty}^{x} \left[\int_{-\infty}^{y} f_{XY}(x, y)\, dx\, dy \qquad \ldots (5\cdot15)$$

(For continuous variables)

where $\qquad f_{XY}(x, y) \ge 0$

And $\displaystyle\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x, y)\; dx\, dy = 1 \;\; or \;\; \sum_x \sum_y f(x, y) = 1$

Properties of Joint Distribution Function

1. (*i*) For the real numbers a_1, b_1, a_2 and b_2

$$P(a_1 < X \le b_1, a_2 < Y \le b_2) = F_{XY}(b_1, b_2) + F_{XY}(a_1, a_2) - F_{XY}(a_1, b_2) - F_{XY}(b_1, a_2)$$

[For proof, See Example 5·29]

(*ii*) Let $a_1 < a_2$, $b_1 < b_2$. We have

$$(X \le a_1, Y \le a_2) + (a_1 < X \le b_1, Y \le a_2) = (X \le b_1, Y \le a_2)$$

and the events on the L.H.S. are mutually exclusive. Taking probabilities on both-sides, we get :

$$F(a_1, a_2) + P(a_1 < X \le b_1, Y \le a_2) = F(b_1, a_2)$$

$$\Rightarrow \quad F(b_1, a_2) - F(a_1, a_2) = P(a_1 < X \le b_1, Y \le a_2)$$

$$\therefore \quad F(b_1, a_2) \ge F(a_1, a_2) \qquad [\text{Since } P(a_1 < X \le b_1, Y \le a_2) \ge 0]$$

Similarly it follows that

$$F(a_1, b_2) - F(a_1, a_2) = P(X \le a_1, a_2 < Y \le b_2) \ge 0$$

$$\Rightarrow \quad F(a_1, b_2) \ge F(a_1, a_2),$$

which shows that $F(x, y)$ is monotonic non-decreasing function.

2. $\quad F(-\infty, y) = 0 = F(x, -\infty),\; F(+\infty, +\infty) = 1$

3. $\;$ If the density function $f(x, y)$ is continuous at (x, y) then

$$\frac{\partial^2 F}{\partial x\, \partial y} = f(x, y)$$

5·5·3. Marginal Distribution Functions. From the knowledge of joint distribution function $F_{XY}(x, y)$, it is possible to obtain the individual distribution functions, $F_X(x)$ and $F_Y(y)$ which are termed as marginal distribution function of X and Y respectively with respect to the joint distribution function $F_{XY}(x, y)$.

$$F_X(x) = P(X \le x) = P(X \le x, Y < \infty) = \lim_{y \to \infty} F_{XY}(x, y).$$

$$= F_{XY}(x, \infty) \qquad \ldots (5\cdot16)$$

Similarly, $F_Y(y) = P(Y \le y) = P(X < \infty, Y \le y)$

$$= \lim_{x \to \infty} F_{XY}(x, y) = F_{XY}(\infty, y)$$

$F_X(x)$ is termed as the marginal distribution function of X corresponding to the joint distribution function $F_{XY}(x, y)$ and similarly $F_Y(y)$ is called marginal distribution function of the random variable Y corresponding to the joint distribution function $F_{XY}(x, y)$.

In the case of jointly discrete random variables, the marginal distribution functions are given as

$$F_X(x) = \sum_y P(X \le x, Y = y),$$

$$F_Y(y) = \sum_x P(X = x, Y \le y)$$

Similarly in the case of jointly continuous random variable, the marginal distribution functions are given as

$$F_X(x) = \int_{-\infty}^{x} \left\{ \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy \right\} dx$$

$$F_Y(y) = \int_{-\infty}^{y} \left\{ \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx \right\} dy$$

5·5·4 Joint Density Function, Marginal Density Functions. From the joint distribution function $F_{XY}(x, y)$ of two dimensional continuous random variable we get the joint probabilty density function by differentiation as follows :

$$f_{XY}(x, y) = \partial^2 F(x, y) / \partial x \partial y$$

$$= \lim_{\delta x \to 0, \delta y \to 0} \frac{P(x \le X \le x + \delta x, y \le Y \le y + \delta y)}{\delta x\ \delta y}$$

Or it may be expressed in the following way also :

"The probability that the point (x, y) will lie in the infinitesimal rectangular region, of area $dx\, dy$ is given by

$$P\left\{ x - \tfrac{1}{2} dx \le X \le x + \tfrac{1}{2} dx, y - \tfrac{1}{2} dy \le Y \le y + \tfrac{1}{2} dy \right\} = dF_{XY}(x, y)$$

and is denoted by $f_{XY}(x, y)\, dx\, dy$, where the function $f_{XY}(x, y)$ is called the joint probability density function of X and Y.

The marginal probability function of Y and X are given respectively

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx \qquad \text{(for continuous variables)}$$

$$= \sum_x p_{XY}(x, y) \qquad \text{(for discrete variables)}$$

...(5·17)

and $$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy \qquad \text{(for continuous variables)}$$

$$= \sum_{y} p_{XY}(x, y) \qquad \text{(for discrete variables)}$$

(5·17*a*)

The marginal density functions of X and Y can be obtained in the following manner also.

$$\left.\begin{aligned} f_X(x) &= \frac{dF_X(x)}{dx} = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy \\ \text{and} \quad f_Y(y) &= \frac{dF_Y(y)}{dy} = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx \end{aligned}\right\} \qquad \text{... (5·17 } b)$$

Important Remark. If we know the joint p.d.f. (p.m.f.) $f_{XY}(x, y)$ of two random variables X and Y, we can obtain the individual distributions of X and Y in the form of their marginal p.d.f.'s (p.m.f's) $f_X(x)$ and $f_Y(y)$ by using (5·17) and (5·17a). However, the converse is not true *i.e., from the marginal distributions of two jointly distributed random variables, we cannot determine the joint distributions of these two random variables.*

To verify this, it will suffice to show that two different joint p.m.f's (p.d.f.'s) have the same marginal distribution for X and the same marginal distribution for Y. We give below two joint discrete probability distributions which have the same marginal distributions.

JOINT DISTRIBUTIONS HAVING SAME MARGINALS

Probability Distribution I

Y \ X	0	1	$f_Y(y)$
0	0·28	0·37	0·65
1	0·22	0·13	0·35
$f_X(x)$	0·50	0·50	1·00

Probability Distribution II

Y \ X	0	1	$f_Y(y)$
0	0·35	0·30	0·65
1	0·15	0·20	0·35
$f_X(x)$	0·50	0·50	1·00

As an illustration for continuous random variables, let (X, Y) be continuous r.v. with joint p.d.f.

$$f_{XY}(x, y) = x + y \; ; \quad 0 \le (x, y) \le 1 \qquad \text{...(5·17 } c)$$

The marginal p.d.f.'s of X and Y are given by :

$$f_X(x) = \int_0^1 f(x, y)\, dy = \int_0^1 (x + y)\, dy = \left| xy + \frac{y^2}{2} \right|_0^1$$

$$\left.\begin{array}{lll}\Rightarrow & f_X(x) = x + \frac{1}{2} \quad ; & 0 \le x \le 1 \\ \text{Similarly} & f_Y(y) = \int_0^1 f(x, y)\, dx = y + \frac{1}{2} \quad ; & 0 \le y \le 1\end{array}\right\} \quad \ldots (5\cdot17\, d)$$

Consider another continuous joint p.d.f.

$$g(x, y) = \left(x + \tfrac{1}{2}\right)\left(y + \tfrac{1}{2}\right) \quad ; \quad 0 \le (x, y) \le 1 \quad \ldots (5\cdot17\, e)$$

Then marginal p.d.f.'s of X and Y are given by :

$$g_1(x) = \int_0^1 g(x, y)\, dy = \left(x + \tfrac{1}{2}\right) \int_0^1 \left(y + \tfrac{1}{2}\right) dy$$

$$= \left(x + \tfrac{1}{2}\right) \left| \frac{y^2}{2} + \frac{1}{2} y \right|_0^1$$

$$\left.\begin{array}{lll}\Rightarrow & g_1(x) = x + \frac{1}{2} \quad ; & 0 \le x \le 1 \\ \text{Similarly} & g_2(y) = y + \frac{1}{2} \quad ; & 0 \le y \le 1\end{array}\right\} \quad \ldots (5\cdot17\, f)$$

(5·17 d) and (5·17 f) imply that the two joint p.d.f.'s in (5·17 c) and (5·17 e) have the same marginal p.d.f.'s (5·17 d) or (5·17 f).

Another illustration of continuous r.v.'s is given in Remark to Bivariate Normal Distribution, § 10·10·2.

5·5·5. The Conditional Distribution Function and Conditinal Probability Density Function. For two diamensional random variable (X, Y), the joint distribution function $F_{XY}(x, y)$ for any real numbers x and y is given by

$$F_{XY}(x, y) = P(X \le x, Y \le y)$$

Now let A be the event $(Y \le y)$ such that the event A is said to occur when Y assumes values up to and inclusive of y.

Using conditional probabilities we may now write

$$F_{XY}(x, y) = \int_{-\infty}^{x} P[A \mid X = x]\; dF_X(x) \quad \ldots (5\cdot18)$$

The *conditional distribution function* $F_{Y|X}(y \mid x)$ denotes the distribution function of Y when X has already assumed the particular value x. Hence

$$F_{Y|X}(y \mid x) = P[Y \le y \mid X = x) = P[A \mid X = x]$$

Using this expression, the joint distribution function $F_{XY}(x, y)$ may be expressed in terms of the conditional distribution function as follows :

$$F_{XY}(x, y) = \int_{-\infty}^{x} F_{Y|X}(y \mid x)\; dF_X(x) \quad \ldots (5\cdot18\, a)$$

Similarly

$$F_{XY}(x, y) = \int_{-\infty}^{y} F_{X|Y}(x \mid y)\; dF_Y(y) \quad \ldots (5\cdot18\, b)$$

The *conditional probability density function* of Y given X for two random variables X and Y which are jointly continuously distributed is defined as follows, for two real numbers x and y :

$$f_{Y|X}(y|x) = \frac{\partial}{\partial y} F_{Y|X}(y|x) \qquad \dots (5\cdot19)$$

Remarks : 1. $f_X(x) > 0$, then

$$f_{Y|X}(y|x) = \frac{f_{XY}(x, y)}{f_X(x)}$$

Proof. We have

$$F_{XY}(x, y) = \int_{-\infty}^{x} F_{Y|X}(y|x)\, dF_X(x)$$

$$= \int_{-\infty}^{x} F_{Y|X}(y|x)\, f_X(x)\, dx$$

Differentiating w.r.t. x, we get

$$\frac{\partial}{\partial x} F_{XY}(x, y) = F_{Y|X}(y|x)\, f_X(x)$$

Differentiating w.r.t. y, we get

$$\frac{\partial}{\partial y}\left[\frac{\partial}{\partial x} F_{XY}(x, y)\right] = f_{Y|X}(y|x)\, f_X(x)$$

$$\Rightarrow \qquad f_{XY}(x, y) = f_{Y|X}(y|x)\, f_X(x)$$

$$\Rightarrow \qquad f_{Y|X}(y|x) = \frac{f_{XY}(x, y)}{f_X(x)}$$

2. If $f_Y(y) > 0$, then

$$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)}$$

3. In terms of the differentials, we have

$$P(x < X \le x + dx \mid y < Y \le y + dy)$$

$$= \frac{P(x < X \le x + dx,\ y < Y \le y + dy)}{P(y < Y \le y + dy)}$$

$$= \frac{f_{XY}(x, y)\ dx\ dy}{f_Y(y)\ dy} = f_{X|Y}(x|y)\ dx$$

Whence $f_{X|Y}(x|y)$ may be interpreted as the conditional density function of X on the assumption $Y = y$.

5·5·6. Stochastic Independence. Let us consider two random variables X and Y (of discrete or continuous type) with joint p.d.f. $f_{XY}(x, y)$ and marginal p.d.f.'s $f_X(x)$ and $g_Y(y)$ respectively. Then by the compound probability theorem

$$f_{XY}(x, y) = f_X(x)\ g_Y(y|x)$$

where $g_Y(y \mid x)$ is the conditional p.d.f. of Y for given value of $X = x$.

If we assume that $g(y \mid x)$ does not depend on x, then by the definition of marginal p.d.f.'s, we get for continuous r.v.'s

$$g(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$

$$= \int_{-\infty}^{\infty} f_X(x)\, g(y \mid x)\, dx$$

$$= g(y \mid x) \int_{-\infty}^{\infty} f_X(x)\, dx$$

[since $g(y \mid x)$ does not depend on x]

$$= g(y \mid x) \qquad [\because f(.) \text{ is p.d.f. of } X]$$

Hence

$$g(y) = g(y \mid x)$$

and $\quad f_{XY}(x, y) = f_X(x)\, g_Y(y) \qquad \ldots (*)$

provided $g(y \mid x)$ does not depend on x. This motivates the following definition of independent random variables.

Independent Random variables. *Two r.v.'s X and Y with joint p.d.f. $f_{XY}(x, y)$ and marginal p.d.f.'s $f_X(x)$ and $g_Y(y)$ respectively are said to be stochastically independent if and only if*

$$f_{X,Y}(x, y) = f_X(x)\, g_Y(y) \qquad \ldots (5{\cdot}20)$$

Remarks. 1. In terms of the distribution function, we have the following definition :

Two jointly distributed random variables X and Y are stochastically independent if and only if their joint distribution function $F_{X,Y}(\,.\,,\,.\,)$ is the product of their marginal distribution functions $F_X(\cdot)$ and $G_Y(\cdot)$, i.e., if for real (x, y)

$$F_{X,Y}(x, y) = F_X(x)\, G_Y(y) \qquad \ldots (5{\cdot}20\, a)$$

2. The variables which are not stochastically independent are said to be stochastically dependent.

Theorem 5·8. *Two random variables X and Y with joint p.d.f. $f(x, y)$ are stochastically independent if and only if $f_{X,Y}(x, y)$ can be expressed as the product of a non-negative function of x alone and a non-negative function of y alone, i.e., if*

$$f_{X,Y}(x, y) = h_X(x)\, k_Y(y) \qquad \ldots (5{\cdot}21)$$

where $h(\cdot) \geq 0$ and $k(\cdot) \geq 0$.

Proof. If X and Y are independent then by definition, we have

$$f_{X,Y}(x, y) = f_X(x) \cdot g_Y(y)$$

where $f(x)$ and $g(y)$ are marginal p.d.f. of X and Y respectively. Thus condition (5·21) is satisfied.

Conversely if (5·21) holds, then we have to prove that X and Y are independent. For continuous random variables X and Y, the marginal p.d.f.'s are given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\ dy = \int_{-\infty}^{\infty} h(x)\ k(y)\ dy$$

$$= h(x) \int_{-\infty}^{\infty} k(y)\ dy = c_1\ h(x), \text{ say} \qquad \ldots (*)$$

and $$g_Y(y) = \int_{-\infty}^{\infty} f(x, y)\ dx = \int_{-\infty}^{\infty} h(x)\ k(y)\ dx$$

$$= k(y) \int_{-\infty}^{\infty} h(x)\ dx = c_2\ k(y), \text{ say} \qquad \ldots (**)$$

where c_1 and c_2 are constants independent of x and y. Moreover

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\ dx\ dy = 1$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x)\ k(y)\ dx\ dy = 1$$

$$\Rightarrow \quad \left(\int_{-\infty}^{\infty} h(x)\ dx\right)\left(\int_{-\infty}^{\infty} k(y)\ dy\right) = 1$$

$$\Rightarrow \quad c_2\ c_1 = 1 \qquad \ldots (***)$$

Finally, we get

$$f_{X,Y}(x, y) = h_X(x)\ k_Y(y) = c_1 c_2\ h_X(x)\ k_Y(y) \qquad [\text{using } (***)]$$
$$= (c_1 h_X(x))\ (c_2 k_Y(y))$$
$$= f_X(x)\ g_Y(y) \qquad [\text{from } (*) \text{ and } (**)]$$

$\Rightarrow$ X and Y are stochastically independent.

Theorem 5·9. *If the random variables X and Y are stochastically independent, then for all possible selections of the corresponding pairs of real numbers (a_1, b_1), (a_2, b_2) where $a_i \le b_i$ for all $i = 1, 2$ and where the values $\pm\infty$ are allowed, the events $(a_1 < X \le b_1)$ and $(a_2 < Y \le b_2)$ are independent, i.e.,*

$$P[(a_1 < X \le b_1) \cap (a_2 < Y \le b_2)] = P(a_1 < X \le b_1)\, P(a_2 < Y \le b_2)$$

Proof. Since X and Y are stochastically independent, we have in the usual notations

$$f_{X,Y}(x, y) = f_X(x)\ g_Y(y) \qquad \ldots (*)$$

In case of continuous r.v.'s, we have

$$P[(a_1 < X \le b_1) \cap (a_2 < Y \le b_2)] = \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(x, y)\ dx\ dy$$

$$= \left(\int_{a_1}^{b_1} f_X(x)\ dx \right) \left(\int_{a_2}^{b_2} g_Y(y)\ dy \right) \qquad [\text{from } (*)]$$

$$= P(a_1 < X \le b_1)\ P(a_2 < Y \le b_2)$$

as desired.

Remark. In case of discrete r.v.'s theorems 5·8 and 5·9 can be proved on replacing integration by summation over the given range of the variables.

Example 5.20. *For the following bivariate probability distribution of X and Y, find*

(i) $P(X \le 1, Y = 2)$, *(ii)* $P(X \le 1)$, *(iii)* $P(Y = 3)$, *(iv)* $P(Y \le 3)$ *and (v)* $P(X < 3, Y \le 4)$

X \ Y	1	2	3	4	5	6
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$

Solution. The marginal distributions are given below :

X \ Y	1	2	3	4	5	6	$p_X(x)$
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$	$\frac{8}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{10}{16}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$	$\frac{8}{64}$
$p_Y(y)$	$\frac{3}{32}$	$\frac{3}{32}$	$\frac{11}{64}$	$\frac{13}{64}$	$\frac{6}{32}$	$\frac{16}{64}$	$\Sigma p(x) = 1$ $\Sigma p(y) = 1$

(i) $P(X \le 1, Y = 2) = P(X = 0, Y = 2) + P(X = 1, Y = 2)$

$$= 0 + \frac{1}{16} = \frac{1}{16}$$

(ii) $P(X \le 1) = P(X = 0) + P(X = 1)$

$$= \frac{8}{32} + \frac{10}{16} = \frac{7}{8} \qquad (\text{From above table})$$

(iii) $P(Y = 3) = \frac{11}{64}$ (From above table)

(iv) $P(Y \le 3) = P(Y = 1) + P(Y = 2) + P(Y = 3)$

$$= \frac{3}{32} + \frac{3}{32} + \frac{11}{64} = \frac{23}{64}$$

(v) $P(X<3, Y\le 4) = P(X=0, Y\le 4) + P(X=1, Y\le 4)$

$$+ P(X=2, Y\le 4)$$

$$= \left(\frac{1}{32}+\frac{2}{32}\right)+\left(\frac{1}{16}+\frac{1}{16}+\frac{1}{8}+\frac{1}{8}\right)$$

$$+\left(\frac{1}{32}+\frac{1}{32}+\frac{1}{64}+\frac{1}{64}\right)=\frac{9}{16}$$

Example 5·21. *The joint probability distribution of two random variables X and Y is given by :*

$$p(x,y)=\frac{2}{n(n+1)},\quad \begin{array}{l} x=1,2,\ldots,n \\ y=1,2,\ldots,x \end{array}$$

Examine whether X and Y are independent. **(Calicut Univ. B.Sc., 1991)**

Solution. The joint probability distribution table along with the marginal distributions of X and Y is given below.

$Y \backslash X$	1	2	3	....		n	$p_Y(y)$
1	$\frac{2}{n(n+1)}$	$\frac{2}{n(n+1)}$	$\frac{2}{n(n+1)}$	....		$\frac{2}{n(n+1)}$	$\frac{2n}{n(n+1)}$
2	–	$\frac{2}{n(n+1)}$	$\frac{2}{n(n+1)}$	....		$\frac{2}{n(n+1)}$	$\frac{2(n-1)}{n(n+1)}$
3	–	–	$\frac{2}{n(n+1)}$	....		$\frac{2}{n(n+1)}$	$\frac{2(n-2)}{n(n+1)}$
$\vdots$	$\vdots$						
$n-1$	–	–	–		$\frac{2}{n(n+1)}$	$\frac{2}{n(n+1)}$	$\frac{2\times 2}{n(n+1)}$
n	–	–	–		–	$\frac{2}{n(n+1)}$	$\frac{2}{n(n+1)}$
$p_X(x)$	$\frac{2}{n(n+1)}$	$\frac{2\times 2}{n(n+1)}$	$\frac{2\times 3}{n(n+1)}$	....		$\frac{2\times n}{n(n+1)}$	

Note that $y=1, 2, \ldots, x$.

When $x=1$, $y=1$; when $x=2$, $y=1,2$; when $x=3$, $y=1,2,3$ and so on.

From the above table, we see that

$$p_{XY}(x,y)\neq p_X(x)\,p_Y(y)\ ;\quad \forall\ x, y$$

$\Rightarrow$ X and Y are not independent.

Example 5·22. *Given the following bivariate probability distribution, obtain (i) marginal distributions of X and Y, (ii) the conditional distribution of X given Y = 2.*

Y \ X	−1	0	1
0	1/15	2/15	1/15
1	3/15	2/15	1/15
2	2/15	1/15	2/15

(Mysore Univ. B.Sc., Oct. 1987)

Solution.

Y \ X	−1	0	1	$\sum_x p(x,y)$
0	1/15	2/15	1/15	4/15
1	3/15	2/15	1/15	6/15
2	2/15	1/15	2/15	5/15
$\sum_y p(x,y)$	6/15	5/15	4/15	1

(i) *Marginal distribution of X.* From the above table, we get

$$P(X=-1)=\frac{6}{15}=\frac{2}{5}\ ;\ P(X=0)=\frac{5}{15}=\frac{1}{3}\ ;\ P(X=1)=\frac{4}{15}$$

Marginal distribution of Y :

$$P(Y=0)=\frac{4}{15}\ ;\ P(Y=1)=\frac{6}{15}=\frac{2}{5}\ ;\ P(Y=2)=\frac{5}{15}=\frac{1}{3}$$

(ii) *Conditional distribution of X given Y = 2.* We have

$$P(X=x\cap Y=2)=P(Y=2).P(X=x\mid Y=2)$$

$$\Rightarrow \qquad P(X=x\mid Y=2)=\frac{P(X=x\cap Y=2)}{P(Y=2)}$$

$$\therefore \qquad P(X=-1\mid Y=2)=\frac{P(X=-1\cap Y=2)}{P(Y=2)}=\frac{2/15}{1/3}=\frac{2}{5}$$

Example 5·23. *X and Y are two random variables having the joint density function, $f(x,y)=\frac{1}{27}(2x+y)$, where x and y can assume only the integer values 0, 1 and 2. Find the conditional distribution of Y for X = x.*

[South Gujarat Univ. B.Sc., 1988]

Solution. The joint probability function

$$f(x,y)=\frac{1}{27}(2x+y)\,;\;\; x=0,1,2\,;\;\; y=0,1,2$$

gives the following table of joint probability distribution of X and Y.

JOINT PROBABILITY DISTRIBUTION $f(x,y)$ OF X AND Y

$X \downarrow$ \ $Y \rightarrow$	0	1	2	$f_X(x)$
0	0	1/27	2/27	3/27
1	2/27	3/27	4/27	9/27
2	4/27	5/27	6/27	15/27

For example $f(0,0)=\dfrac{1}{27}(0+2\times 0)=0$

$$f(1,0)=\frac{1}{27}(0+2\times 1)=\frac{2}{27}\;;\;\; f(2,0)=\frac{1}{27}(0+2\times 2)=\frac{4}{27}$$

and so on.

The marginal probability distribution of X is given by

$$f_X(x)=\sum_y f(x,y),$$

and is tabulated in last column of above table.

The conditional distribution of Y for $X=x$ is given by

$$f_{Y|X}(Y=y\,|\,X=x)=\frac{f(x,y)}{f_X(x)}$$

and is obtained in the following table.

CONDITIONAL DISTRIBUTION OF Y FOR $X=x$

X \ Y	0	1	2
0	0	1/3	2/3
1	2/9	3/9	4/9
2	4/15	5/15	6/15

Example 5·24. *Two discrete random variables X and Y have the joint probability density function :*

$$p(x,y)=\frac{\lambda^x e^{-\lambda} p^y (1-p)^{x-y}}{y\,!\,(x-y)\,!}\,,\; y=0,1,2,\ldots,x\;;\;\; x=0,1,2,\ldots$$

where λ, p are constants with $\lambda > 0$ and $0 < p < 1$.

Find (i) The marginal probability density functions of X and Y.

(ii) The conditional distribution of Y for a given X and of X for a given Y.

(Poona Univ. B.Sc., 1986 ; Nagpur Univ. M.Sc., 1989)

Solution. *(i)*

$$p_X(x) = \sum_{y=0}^{x} p(x,y) = \sum_{y=0}^{x} \frac{\lambda^x e^{-\lambda} p^y (1-p)^{x-y}}{y!\,(x-y)!}$$

$$= \frac{\lambda^x e^{-\lambda}}{x!} \sum_{y=0}^{x} \frac{x!\,p^y (1-p)^{x-y}}{y!\,(x-y)!} = \frac{\lambda^x e^{-\lambda}}{x!} \sum_{y=0}^{x} {}^xC_y p^y (1-p)^{x-y}$$

$$= \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots$$

which is the probability function of a Poisson distribution with parameter λ.

$$p_Y(y) = \sum_{x=0}^{\infty} p(x,y) = \sum_{x=y}^{\infty} \frac{\lambda^x e^{-\lambda} p^y (1-p)^{x-y}}{y!\,(x-y)!}$$

$$= \frac{(\lambda p)^y e^{-\lambda}}{y!} \sum_{x=y}^{\infty} \frac{[\lambda(1-p)]^{x-y}}{(x-y)!} = \frac{(\lambda p)^y e^{-\lambda}}{y!} e^{\lambda(1-p)}$$

$$= \frac{e^{-\lambda p} (\lambda p)^y}{y!}, \quad y = 0, 1, 2, \ldots$$

which is the probability function of a Poisson distribution with parameter λp.

(ii) The conditional distribution of Y for given X is

$$p_{Y|X}(y \mid x) = \frac{p_{XY}(x,y)}{p_X(x)} = \frac{\lambda^x e^{-\lambda} p^y (1-p)^{x-y}\, x!}{y!\,(x-y)!\,\lambda^x e^{-\lambda}}$$

$$= \frac{x!}{y!\,(x-y)!} p^y (1-p)^{x-y} = {}^xC_y\, p^y (1-p)^{x-y}, \quad x > y$$

The conditional probability distribution of X for given Y is

$$p_{X|Y}(x \mid y) = \frac{p_{XY}(x,y)}{p_Y(y)}$$

$$= \frac{\lambda^x e^{-\lambda} p^y (1-p)^{x-y}}{y!\,(x-y)!} \cdot \frac{y!}{e^{-\lambda p} (\lambda p)^y} \qquad [\text{ c.f. Part (i)}]$$

$$= \frac{e^{-\lambda q} (\lambda q)^{x-y}}{(x-y)!}\ ; \quad q = 1-p,\ x > y$$

Example 5·25. *The joint p.d.f. of two random variables X and Y is given by :*

$$f(x,y) = \frac{9(1+x+y)}{2(1+x)^4 (1+y)^4}\ ; \quad \begin{pmatrix} 0 \le x < \infty \\ 0 < y < \infty \end{pmatrix}$$

Find the marginal distributions of X and Y, and the conditional distribution of Y for X = x.

Solution. Marginal p.d.f. of X is given by

$$f_X(x) = \int_0^\infty f(x, y)\, dy$$

$$= \frac{9}{2(1+x)^4} \int_0^\infty \frac{(1+y)+x}{(1+y)^4}\, dy$$

$$= \frac{9}{2(1+x)^4} \cdot \int_0^\infty \left[(1+y)^{-3} + x\,(1+y)^{-4} \right] dy$$

$$= \frac{9}{2(1+x)^4} \left[\left| \frac{-1}{2(1+y)^2} \right|_0^\infty + x \left| \frac{-1}{3(1+y)^3} \right|_0^\infty \right]$$

$$= \frac{9}{2(1+x)^4} \cdot \left[\frac{1}{2} + \frac{x}{3} \right]$$

$$= \frac{3}{4} \cdot \frac{3+2x}{(1+x)^4} \; ; \; 0 < x < \infty$$

Since $f(x, y)$ is symmetric in x and y, the marginal p.d.f. of Y is given by

$$f_Y(y) = \int_0^\infty f(x, y)\, dx$$

$$= \frac{3}{4} \cdot \frac{3+2y}{(1+y)^4} \; ; \; 0 < y < \infty$$

The conditional distribution of Y for $X = x$ is given by

$$f_{XY}(Y = y \mid X = x) = \frac{f_{XY}(x, y)}{f_X(x)}$$

$$= \frac{9(1+x+y)}{2(1+x)^4 (1+y)^4} \cdot \frac{4(1+x)^4}{3(3+2x)}$$

$$= \frac{6(1+x+y)}{(1+y)^4 (3+2x)} \; ; \; 0 < y < \infty$$

Example 5·26. *The joint probability density function of a two-dimensional random variable (X,Y) is given by*

$$f(x, y) = 2 \; ; \; 0 < x < 1,\; 0 < y < x$$
$$= 0, \text{ elsewhere}$$

(i) Find the marginal density functions of X and Y,

(ii) find the conditional density function of Y given X = x and conditional density function of X given Y = y, and

(iii) check for independence of X and Y.

[M.S.Baroda Univ. B.Sc., 1987; Karnataka Univ. B.Sc., Oct. 1988]

Solution. Evidently $f(x, y) \geq 0$ and

$$\int_0^1 \int_0^x 2\, dx\, dy = 2 \int_0^1 x\, dx = 1$$

(i) The marginal p.d.f.'s of X and Y are given by

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dy = \int_0^x 2\, dy = 2x,\quad 0 < x < 1$$

$$= 0,\ \text{elsewhere}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)\, dx = \int_y^1 2dx = 2(1 - y),\quad 0 < y < 1$$

$$= 0,\ \text{elsewhere}$$

(ii) The conditional density function of Y given X is

$$f_{Y|X}(y\,|\,x) = \frac{f_{XY}(x, y)}{f_X(x)} = \frac{2}{2x} = \frac{1}{x},\ 0 < x < 1$$

The conditional density function of X given Y is

$$f_{X|Y}(x\,|\,y) = \frac{f_{XY}(x, y)}{f_Y(y)} = \frac{2}{2(1-y)} = \frac{1}{(1-y)},\ 0 < y < 1$$

(iii) Since $f_X(x)\, f_Y(y) = 2(2x)(1 - y) \neq f_{XY}(x, y)$, X and Y are not independent.

Example 5·27. *A gun is aimed at a certain point (origin of the coordinate system). Because of the random factors, the actual hit point can be any point (X,Y) in a circle of radius R about the origin. Assume that the joint density of X and Y is constant in this circle given by :*

$$f_{XY}(x, y) = k,\ \textit{for}\ x^2 + y^2 \leq R^2$$

$$= 0,\ \textit{otherwise}$$

(i) Compute k, (ii) show that

$$f_X(x) = \frac{2}{\pi R}\left\{1 - \left(\frac{x}{R}\right)^2\right\}^{1/2},\quad \text{for } -R \leq x \leq R$$

$$= 0,\ \textit{otherwise}$$

[Calcutta Univ. B.Sc.(Stat. Hons.),1987]

Solution. *(i)* The constant k is computed from the consideration that the total probability is 1, *i.e.*,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1 \quad \Rightarrow \quad \iint_{x^2 + y^2 \leq R^2} k\, dx\, dy = 1$$

$$\Rightarrow \quad 4 \iint_I k\, dx\, dy = 1$$

where region I is the first quadrant of the circle $x^2 + y^2 = R^2$.

$$\Rightarrow \quad 4k \int_0^R \left(\int_0^{\sqrt{R^2 - x^2}} 1 \,.\, dy \right) dx = 1$$

$$\Rightarrow \quad 4k \int_0^R \sqrt{R^2 - x^2}\, dx = 1$$

$$\Rightarrow \quad 4k \left| x \sqrt{R^2 - x^2} + \frac{R^2}{2} \sin^{-1}\left(\frac{x}{R}\right) \right|_0^R = 1$$

$$\Rightarrow \quad 4k \cdot \left(\frac{R^2}{2} \cdot \frac{\pi}{2} \right) = 1 \quad \Rightarrow \quad k = \frac{1}{\pi R^2}$$

$$\therefore \quad f_{XY}(x, y) = 1/(\pi R^2) \ ; \quad x^2 + y^2 \le R^2$$
$$= 0, \quad \textit{otherwise}$$

(ii) $$f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy = \frac{1}{\pi R^2} \int_{-\sqrt{R^2 - x^2}}^{\sqrt{R^2 - x^2}} 1 \cdot dy$$

$$\left[\textit{because } x^2 + y^2 \le R^2 \ \Rightarrow \ -\left(R^2 - x^2\right)^{1/2} \le y \le \left(R^2 - x^2\right)^{1/2} \right]$$

$$= \frac{2}{\pi R^2} \int_0^{\sqrt{R^2 - x^2}} 1 \cdot dy = \frac{2}{\pi R^2} (R^2 - x^2)^{1/2}$$

$$= \frac{2}{\pi R} \left[1 - \left(\frac{x}{R} \right)^2 \right]^{1/2}$$

Example 5·28. *Given:*

$$f(x, y) = e^{-(x+y)}\ I_{(0,\infty)}(x) \,.\, I_{(0,\infty)}(y),$$

find (i) $P(X > 1)$, (ii) $P(X < Y \mid X < 2Y)$, (iii) $P(1 < X + Y < 2)$

[Delhi Univ. B.Sc. (Maths Hons.), 1987]

Solution. We are given :

$$f(x, y) = e^{-(x+y)} \ ; \quad 0 \le x < \infty,\ 0 \le y < \infty \qquad \dots (1)$$
$$= \left(e^{-x}\right)\left(e^{-y}\right)$$
$$= f_X(x) \,.\, f_Y(y) \ ; \quad 0 \le x < \infty,\ 0 \le y < \infty$$

$\Rightarrow$ X and Y are independent and

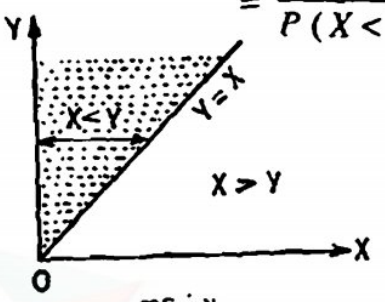
$$f_X(x) = e^{-x} \ ; \ x \ge 0 \quad \text{and} \quad f_Y(y) = e^{-y} \ ; \ y \ge 0 \qquad \dots (2)$$

(i) $$P(X > 1) = \int_1^\infty f_X(x)\,dx = \int_1^\infty e^{-x}\,dx$$

$$= \left| \frac{e^{-x}}{-1} \right|_1^\infty = \frac{1}{e}$$

(ii) $$P(X < Y \mid X < 2Y) = \frac{P(X < Y \cap X < 2Y)}{P(X < 2Y)}$$

$$= \frac{P(X < Y)}{P(X < 2Y)} \qquad \ldots (3)$$



$$P(X < Y) = \int_0^\infty \left[\int_0^y f(x, y)\,dx \right] dy$$

$$= \int_0^\infty \left[e^{-y} \left| \frac{e^{-x}}{-1} \right|_0^y \right] dy = -\int_0^\infty e^{-y}\left(e^{-y} - 1\right) dy$$

$$= -\left| \frac{e^{-2y}}{-2} + e^{-y} \right|_0^\infty = 1 - \frac{1}{2} = \frac{1}{2}$$

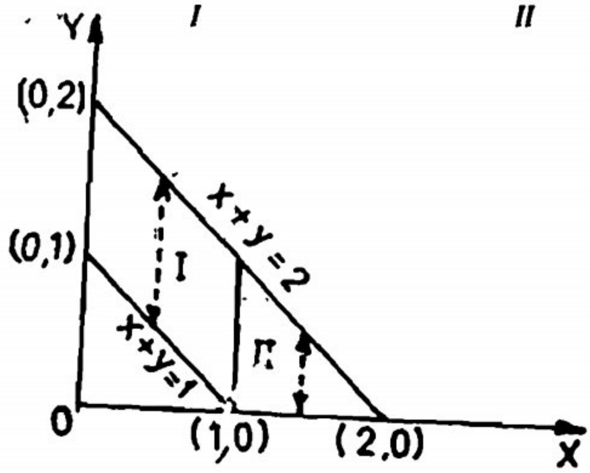
$$P(X < 2Y) = \int_0^\infty \left[\int_0^{2y} f(x, y)\,dx \right] dy = -\int_0^\infty e^{-y}(e^{-2y} - 1)\,dy$$

$$= -\left| \frac{e^{-3y}}{-3} + e^{-y} \right|_0^\infty = 1 - \frac{1}{3} = \frac{2}{3}$$

Substituting in (3),

$$P(X < Y \mid X < 2Y) = \frac{1/2}{2/3} = \frac{3}{4}$$

(iii) $$P(1 < X + Y < 2) = \iint_I f(x, y)\,dx\,dy = \iint_{II} f(x, y)\,dx\,dy$$



$$= \int_0^1 \left(\int_{1-x}^{2-x} f(x,y)\,dy \right) dx + \int_1^2 \left(\int_0^{2-x} f(x,y)\,dy \right) dx$$

$$= \int_0^1 \left(e^{-x} \int_{1-x}^{2-x} e^{-y}\,dy \right) dx + \int_1^2 \left(e^{-x} \int_0^{2-x} e^{-y}\,dy \right) dx$$

$$= \int_0^1 \frac{e^{-x}}{-1}\left(e^{x-2} - e^{x-1}\right) dx + \int_1^2 \frac{e^{-x}}{-1}\left(e^{x-2} - 1\right) dx$$

$$= -\left(e^{-2} - e^{-1}\right) \int_0^1 1 \cdot dx - \int_1^2 \left(e^{-2} - e^{-x}\right) dx$$

$$= -\left(e^{-2} - e^{-1}\right) \Big| x \Big|_0^1 - \Big| e^{-2} \cdot x + e^{-x} \Big|_1^2$$

$$= 2/e - 3/e^2$$

Example 5·29. *(i) Let $F(x,y)$ be the d.f. of X and Y. Show that* $P(a < X \le b,\ c < Y \le d) = F(b,d) - F(b,c) - F(a,d) + F(a,c)$ *where a, b, c, d are real constants $a < b$; $c < d$.*

Deduce that if : $F(x,y) = 1$, for $x + 2y \ge 1$

$F(x,y) = 0$, for $x + 2y < 1$,

then $F(x,y)$ cannot be joint distribution function of variables X and Y.

(ii) Show that, with usual notation : for all x, y,

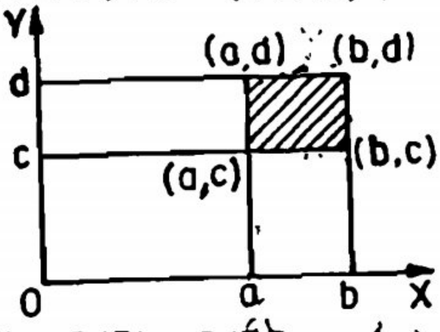
$$F_X(x) + F_Y(y) - 1 \le F_{XY}(x,y) \le \sqrt{F_X(x)\,F_Y(y)}$$

[Delhi Univ. B.Sc. (Maths Hons.), 1985]

Solution. *(i)* Let us define the events :

$$A : \{X \le a\} \ ; \ B : \{X \le b\} \ ; \ C = \{Y \le c\} \ ; \ D = \{Y \le d\} \ ;$$

for $a < b$; $c < d$.



$P(a < X \le b \cap c < Y \le d)$

$= P[(B - A) \cap (D - C)]$

$= P[B \cap (D - C) - A \cap (D - C)]$ ...(*)

(By distributive property of sets)

We know that if $E \subset F \Rightarrow E \cap F = E$, then

$P(F - E) = P(\bar{E} \cap F) = P(F) - P(E \cap F) = P(F) - P(E)$ ... (**)

Obviously $A \subset B \Rightarrow [A \cap (D - C)] \subset [B \cap (D - C)]$

Hence using (**), we get from (*)

$$P(a < X \le b \cap c < Y \le d) = P[B \cap (D - C)] - P[A \cap (D - C)]$$

$$= P[(B \cap D) - (B \cap C)] - P[(A \cap D) - (A \cap C)]$$

$$= P(B \cap D) - P(B \cap C) - P(A \cap D) + P(A \cap C) \quad ...(***)$$

[On using (**), since $C \subset D \Rightarrow (B \cap C) \subset (B \cap D)$ and $(A \cap C) \subset (A \cap D)$]

We have :

$P(B \cap D) = P[X \leq b \cap Y \leq d] = F(b, d)$.

Similarly

$P(B \cap C) = F(b, c)$; $P(A \cap D) = F(a, d)$ and $P(A \cap C) = F(a, c)$

Substituting in (***), we get :

$$P(a < X \leq b \cap c < Y \leq d) = F(b, d) - F(b, c) - F(a, d) + F(a, c) \quad ...(1)$$

We are given
$$\left.\begin{aligned} F(x, y) &= 1, \quad \text{for } x + 2y \geq 1 \\ &= 0, \quad \text{for } x + 2y < 1 \end{aligned}\right\} \quad ...(2)$$

In (1) let us take : $a = 0$, $b = 1/2$, ; $c = 1/4$, $d = 3/4$ s.t. $a < b$ and $c < d$. Then using (2) we get :

$F(b, d) = 1$; $F(b, c) = 1$; $F(a, d) = 1$; $F(a, c) = 0$.

Substituting in (1) we get :

$P(a < X \leq b \cap c < Y \leq d) = 1 - 1 - 1 + 0 = -1$;

which is not possible since $P(\,.\,) \geq 0$.

Hence $F(x, y)$ defined in (2) cannot be the distribution function of variates X and Y.

(ii) Let us define the events : $A = \{X \leq x\}$; $B = \{Y \leq y\}$

$$\left.\begin{aligned} &\textit{Then } P(A) = P(X \leq x) = F_X(x) ; \; P(B) = P(Y \leq y) = F_Y(y) \\ &\text{and } P(A \cap B) = P(X \leq x \cap Y \leq y) = F_{XY}(x, y) \end{aligned}\right\} \quad ...(3)$$

$(A \cap B) \subset A \Rightarrow P(A \cap B) \leq P(A) \Rightarrow F_{XY}(x, y) \leq F_X(x)$

$(A \cap B) \subset B \Rightarrow P(A \cap B) \leq P(B) \Rightarrow F_{XY}(x, y) \leq F_Y(y)$

Multiplying these inequalities we get :

$$F^2_{X,Y}(x, y) \leq F_X(x) F_Y(y) \Rightarrow F_{XY}(x, y) \leq \sqrt{F_X(x) F_Y(y)} \quad ...(4)$$

Also $P(A \cup B) \leq 1 \Rightarrow P(A) + P(B) - P(A \cap B) \leq 1$

$\Rightarrow P(A) + P(B) - 1 \leq P(A \cap B)$

$$\Rightarrow F_X(x) + F_Y(y) - 1 \leq F_{XY}(x, y) \quad ...(5)$$

From (4) and (5) we get :

$F_X(x) + F_Y(y) - 1 \leq F_{XY}(x, y) \leq \sqrt{F_X(x) F_Y(y)}$, as required.

Example 5·30. *If X and Y are two random variables having joint density function*

$$f(x, y) = \frac{1}{8}(6 - x - y) \; ; \; 0 < x < 2, \; 2 < y < 4$$

$$= 0, \quad \textit{otherwise}$$

Find (i) $P(X < 1 \cap Y < 3)$, *(ii)* $P(X + Y < 3)$ *and (iii)* $P(X < 1 \mid Y < 3)$

(Madras Univ. B.Sc., Nov. 1986)

Solution. We have

(i) $$P(X < 1 \cap Y < 3) = \int_{-\infty}^{1} \int_{-\infty}^{3} f(x, y)\, dx\, dy$$

$$= \int_0^1 \int_2^3 \frac{1}{8}(6 - x - y)\, dx\, dy = \frac{3}{8}$$

(ii) The probability that $X + Y$ will be less than 3 is

$$P(X + Y < 3) = \int_0^1 \int_2^{3-x} \frac{1}{8}(6 - x - y)\, dx\, dy = \frac{5}{24}$$

(iii) The probability that $X < 1$ when it is known that $Y < 3$ is

$$P(X < 1 \mid Y < 3) = \frac{P(X < 1 \cap Y < 3)}{P(Y < 3)} = \frac{3/8}{5/8} = \frac{3}{5}$$

$$\left[P(Y < 3) = \int_0^2 \int_2^3 \frac{1}{8}(6 - x - y)\, dx\, dy = \frac{5}{8} \right]$$

Example 5·31. *If the joint distribution function of X and Y is given by :*

$$F(x, y) = 1 - e^{-x} - e^{-y} + e^{-(x+y)}; \quad x > 0,\ y > 0$$
$$= 0; \qquad \textit{elsewhere}$$

(a) Find the marginal densities of X and Y.

(b) Are X and Y independent ?

(c) Find $P(X \le 1 \cap Y \le 1)$ and $P(X + Y \le 1)$. **(I.C.S., 1989)**

Solution. *(a)* & *(b)* The joint p.d.f. of the r.v.'s (X, Y) is given by :

$$f_{XY}(x, y) = \frac{\partial^2 F(x, y)}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left[e^{-y} - e^{-(x+y)} \right]$$
$$= e^{-(x+y)}; \quad x \ge 0,\ y \ge 0$$
$$= 0; \qquad \text{otherwise} \qquad \ldots (i)$$

We have

$$f_{XY}(x, y) = e^{-x} \cdot e^{-y} = f_X(x) f_Y(y) \qquad \ldots (ii)$$

where $$f_X(x) = e^{-x};\ x \ge 0; \qquad f_Y(y) = e^{-y};\ y \ge 0 \qquad \ldots (iii)$$

(ii) $\Rightarrow$ X and Y are independent,

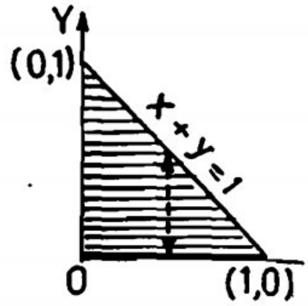
and *(iii)* gives the marginal p.d.f.'s of X and Y.

(c) $$P(X \le 1 \cap Y \le 1) = \int_0^1 \int_0^1 f(x, y)\, dx\, dy$$

$$= \left(\int_0^1 e^{-x}\, dx \right) \left(\int_0^1 e^{-y}\, dy \right)$$

$$= \left(1 - e^{-1} \right)^2$$

$$P(X+Y\le 1)=\iint\limits_{x+y\le 1} f(x,y)=\int_0^1\left(\int_0^{1-x} f(x,y)\,dy\right)dx$$



$$=\int_0^1\left[e^{-x}\int_0^{1-x} e^{-y}\,dy\right]dx$$

$$=\int_0^1 e^{-x}\left(1-e^{-(1-x)}\right)dx = 1-2e^{-1}$$

Example 5·32. *Joint distribution of X and Y is given by*

$$f(x,y)=4xy\,e^{-(x^2+y^2)};\quad x\ge 0,\ y\ge 0.$$

Test whether X and Y are independent.

For the above joint distribution, find the conditional density of X given Y = y. **(Calicut Univ. B.Sc., 1986)**

Solution. Joint p.d.f. of X and Y is

$$f(x,y)=4xy\,e^{-(x^2+y^2)};\quad x\ge 0,\ y\ge 0.$$

Marginal density of X is given by

$$f_1(x)=\int_0^\infty f(x,y)\,dy=\int_0^\infty 4xy\,e^{-(x^2+y^2)}\,dy$$

$$=4x\,e^{-x^2}\int_0^\infty y\,e^{-y^2}\,dy$$

$$=4x\,e^{-x^2}\cdot\int_0^\infty e^{-t}\cdot\frac{dt}{2}\qquad\left(\text{Put } y^2=t\right)$$

$$=2x\cdot e^{-x^2}\left|-e^{-t}\right|_0^\infty$$

$$\Rightarrow\quad f_1(x)=2x\,e^{-x^2};\quad x\ge 0$$

Similarly, the marginal p.d.f. of Y is given by

$$f_2(y)=\int_0^\infty f(x,y)\,dx=2y\,e^{-y^2};\quad y\ge 0$$

Since $f(x,y)=f_1(x)\cdot f_2(y)$, X and Y are independently distributed.

The conditional distribution of X for given Y is given by :

$$f(X=x \mid Y=y)=\frac{f(x,y)}{f_2(y)}$$
$$= 2x\, e^{-x^2}\,;\; x \ge 0\,.$$

EXERCISE 5(e)

1. (a) Two fair dice are tossed simultaneously. Let X denote the number on the first die and Y denote the number on the second die.

(i) Write down the sample space of this experiment.

(ii) Find the following probabilities :

(1) $P(X+Y=8)$, *(2)* $P(X+Y\ge 8)$, *(3)* $P(X=Y)$,
(4) $P(X+Y=6 \mid Y=4)$, *(5)* $P(X-Y=2)$.

(Sardar Patel Univ. B.Sc., 1991)

2. *(a)* Explain the concepts *(i)* conditional probability, *(ii)* random variable, *(iii)* independence of random variables, and *(iv)* marginal and conditional probability distributions.

(b) Explain the notion of the joint distribution of two random variables. If $F(x, y)$ be the joint distribution function of X and Y, what will be the distribution functions for *the marginal distribution of X and Y ?*

What is meant by the *conditional distribution* of Y under the condition that $X = x$? Consider separately the cases where *(i)* X and Y are both discrete and *(ii)* X and Y are both continuous.

3. The joint probability distribution of a pair of random variables is given by the following table :-

Y \ X	1	2	3
1	0.1	0.1	0.2
2	0.2	0.3	0.1

Find :

(i) The marginal distributions.

(ii) The conditional distribution of X given $Y = 1$.

(iii) $P\{(X+Y)<4\}$.

4. *(a)* What do you mean by marginal and conditional distributions ? The following table represents the joint probability distribution of the discrete random variable (X, Y)

Y \ X	1	2	3
1	1/12	1/6	0
2	0	1/9	1/5
3	1/18	1/4	2/15

(i) Evaluate marginal distribution of X.

(ii) Evaluate the conditional distribution of Y given $X = 2$.

(Aligarh Univ. B.Sc., 1992)

(b) Two discrete random variables X and Y have

$$P(X=0, Y=0)=\frac{2}{9} ;\ P(X=0, Y=1)=\frac{1}{9}$$

$$P(X=1, Y=0)=\frac{1}{9} ;\ P(X=1, Y=1)=\frac{5}{9}$$

Examine whether X and Y are independent.

(Kerala Univ. B.Sc., Oct. 1987)

5. *(a)* Let the joint p.m.f. of X_1 and X_2 be

$$p(x_1, x_2)=\frac{x_1+x_2}{21} ;\ x_1=1,2,3 ;\ x_2=1,2$$
$$= 0, \text{ otherwise}$$

Show that marginal p.m.f.'s of X_1 and X_2 are

$$p_1(x_1)=\frac{2x_1+3}{21} ;\ x_1=1,2,3 ; \quad p_2(x_2)=\frac{6+3x_2}{21} ;\ x_2=1,2$$

(b) Let

$$f(x_1, x_2)=C(x_1 x_2+e^{x_1}) ;\ 0<(x_1, x_2)<1$$
$$= 0, \text{ elsewhere}$$

(i) Determine C.

(ii) Examine whether X_1 and X_2 are stochastically independent.

Ans. *(i)* $C=\dfrac{4}{4e-3}$, *(ii)* $g(x_1)=C\left(\frac{1}{2}x_1+e^{x_1}\right)$, $g(x_2)=C\left(\frac{1}{2}x_2+e-1\right)$

Since $g(x_1) \cdot g(x_2) \neq f(x_1, x_2)$, X_1 and X_2 are not stochastically independent.

6. Find k so that $f(x, y)=kxy,\ 1 \leq x \leq y \leq 2$ will be a probability density function. **(Mysore Univ. B.Sc., 1986)**

Hint. $\iint f(x, y)\, dx\, dy=1 \Rightarrow k \int_1^2 x\left(\int_x^2 y\, dy\right) dx=1 \Rightarrow k=8/9$

7. *(a)* If $\quad f(x, y)=e^{-(x+y)} ;\ x \geq 0,\ y \geq 0$
$\qquad = 0$, *elsewhere*

is the joint probability density function of random variables X and Y, find *(i)* $P(X<1)$, *(ii)* $P(X>Y)$, and *(iii)* $P(X+Y<1)$.

Ans. *(i)* $1-\dfrac{1}{e}$, *(ii)* $\dfrac{1}{2}$ and *(iii)* $1-\dfrac{2}{e}$

(*b*) The joint frequency function of (X, Y) is given to be

$$f(x, y) = A\,e^{-x-y}\ ;\quad 0 \le x \le y,\ 0 \le y < +\infty$$
$$= 0 \qquad ;\quad \text{otherwise}$$

(*i*) Determine A.
(*ii*) Find the marginal density function of X.
(*iii*) Find the marginal density function of Y.
(*iv*) Examine if X and Y are independent.
(*v*) Find the conditional density function of Y given $X = 2$.

[Madras Univ. B.Sc. (Main Stat.), 1992]

(*c*) Suppose that the random variables X and Y have the joint p.d.f.

$$f(x, y) = \begin{cases} k\,x(x - y), & 0 < x < 2,\ -x < y < x \\ 0, & \textit{elsewhere}. \end{cases}$$

(*i*) Evaluate the constant k.
(*ii*) Find the marginal probability density functions of the random variables.

(South Gujarat Univ. B.Sc., 1988)

8. (*a*) Two-dimensional random variable (X, Y) have the joint density

$$f(x, y) = 8xy,\ 0 < x < y < 1$$
$$= 0,\ \text{otherwise}$$

(*i*) Find. $P(X < 1/2 \cap Y < 1/4)$.
(*ii*) Find the marginal and conditional distributions.
(*iii*) Are X and Y independent? Give reasons for your answer.

(South Gujarat Univ. B.Sc., 1992)

Ans. $f_1(x) = 4x(1 - x^2),\ 0 < x < 1$; $= 0$, *otherwise*; $f_2(y) = 4y^3,\ 0 < y < 1$ | $f_1(x \mid y) = 2x/y^2\ ;\ 0 < x < y,\ 0 < y < 1$; $f_2(y \mid x) = 2y/(1 - x^2)\ ;\ x < y < 1,\ 0 < x < 1$

9. (*a*) The random variables X and Y have the joint density function :

$$f(x, y) = 2,\ \text{if } x + y \le 1,\ x \ge 0 \text{ and } y \ge 0$$
$$= 0,\ \textit{otherwise}$$

Find the conditional distribution of Y, given $X = x$.

(Calcutta Univ. B.Sc. (Hons.), 1984)

(*b*) The random variables X and Y have the joint distribution given by the probability density function :

$$f(x, y) = \begin{cases} 6(1 - x - y), & \text{for } x > 0,\ y > 0,\ x + y < 1 \\ 0, & \textit{elsewhere} \end{cases}$$

Find the marginal distributions of X and Y. Hence examine if X and Y are independent. **[Calcutta Univ. B.Sc. (Hons.), 1986)**

10. If the joint distribution function of X and Y is given by

$$F(x, y) = (1 - e^{-x})(1 - e^{-y}) \quad \text{for } x > 0,\ y > 0$$
$$= 0,\ \text{elsewhere}$$

Find $P(1 < X < 3,\ 1 < Y < 2)$. **[Delhi Univ. M.A.(Econ.), 1988]**

Hint. Reqd. Prob. $= \left(\int_1^3 e^{-x}\,dx\right)\left(\int_1^2 e^{-y}\,dy\right) = \left(1 - e^{-3}\right)\left(1 - e^{-2}\right)$

11. Let X and Y be two random variables with the joint probability density function

$$f(x,y) = \begin{cases} 8xy, & 0 < x \leq y < 1 \\ 0, & otherwise \end{cases}$$

Obtain :

(i) the joint distribution function of X and Y.

(ii) the marginal probability density function of Y; and

(iii) $P(X \leq \frac{1}{4} \mid \frac{1}{2} < Y \leq 1)$.

12. Let X and Y be jointly distributed with p.d.f.

$$f(x,y) = \begin{cases} \frac{1}{4}(1+xy), & |x| < 1,\ |y| < 1 \\ 0, & otherwise \end{cases}$$

Show that X and Y are not independent but X^2 and Y^2 are independent.

Hint. $$f_1(x) = \int_{-1}^{1} f(x,y)\,dy = \frac{1}{2},\ -1 < x < 1;$$

$$f_2(y) = \int_{-1}^{1} f(x,y)\,dx = \frac{1}{2},\ -1 < y < 1$$

Since $f(x,y) \neq f_1(x) f_2(y)$, X and Y are not independent. However,

$$P(X^2 \leq x) = P(|X| \leq \sqrt{x}) = \int_{-\sqrt{x}}^{\sqrt{x}} f_1(x)\,dx = \sqrt{x}$$

$$\begin{aligned} P(X^2 \leq x \cap Y^2 \leq y) &= P(|X| \leq \sqrt{x} \cap |Y| \leq \sqrt{y}) \\ &= \int_{-\sqrt{x}}^{\sqrt{x}} \left[\int_{-\sqrt{y}}^{\sqrt{y}} f(u,v)\,dv \right] du \\ &= \sqrt{x} \cdot \sqrt{y} \\ &= P(X^2 \leq x) \cdot P(Y^2 \leq y) \end{aligned}$$

$\Rightarrow$ X^2 and Y^2 are independent.

13. *(a)* The joint probability density function of the two dimensional random variable (X, Y) is given by :

$$f(x,y) = \begin{cases} x^3 y^3/16 \ , & 0 \le x \le 2 \ , \ 0 \le y \le 2 \\ 0 \ , & elsewhere \end{cases}$$

Find the marginal densities of X and Y. Also find the cumulative distribution functions for X and Y. **(Annamalai Univ. B.E., 1986)**

Ans. $f_X(x) = \dfrac{x^3}{4} \ ; \ 0 \le x \le 2 \ ; \quad f_Y(y) = \dfrac{y^3}{4} \ ; \ 0 \le y \le 2$

$$F_X(x) = \begin{cases} 0 & ; \ x < o \\ x^4/16 & ; \ 0 \le x \le 2 \\ 1 & ; \ x > 2 \end{cases} \qquad F_Y(y) = \begin{cases} 0 & ; \ y < 0 \\ y^4/16 & ; \ 0 \le y \le 2 \\ 1 & ; \ y > 2 \end{cases}$$

(b) The joint probability density function of the two dimensional random variable (X, Y) is given by :

$$f(x,y) = \begin{cases} \frac{8}{9} xy \ , & 1 \le x \le y \le 2 \\ 0 \ , & elsewhere \end{cases}$$

(i) Find the marginal density functions of X and Y,

(ii) Find the conditional density function of Y given $X = x$, and conditional density funciton of X given $Y = y$.

[Madras Univ. B.Sc. (Stat. Main), 1987]

Ans. *(i)* $f_X(x) = \displaystyle\int_x^2 f(x,y)\,dy = \frac{4}{9} x\left(4 - x^2\right) \ ; \ 1 \le x \le 2$

$$= 0 \qquad ; \ \text{otherwise}$$

$$f_Y(y) = \int_1^y f(x,y)\,dx = \frac{4}{9} y\left(y^2 - 1\right) \ ; \ 1 \le y \le 2$$

$$f_{X|Y}(x \mid y) = \frac{2x}{y^2 - 1} \qquad ; \ 1 \le x \le y$$

$$f_{Y|X}(y \mid x) = \frac{f(x,y)}{f_X(y)} = \frac{2y}{4 - x^2} \ ; \ x \le y \le 2$$

14. The two random variables X and Y have, for $X = x$ and $Y = y$, the joint probability density function :

$$f(x,y) = \frac{1}{2x^2 y} \ , \ \text{for } 1 \le x < \infty \ \text{ and } \ \frac{1}{x} < y < x$$

Derive the marginal distributions of X and Y. Further obtain the conditional distribution of Y for $X = x$ and also that of X given $Y = y$.

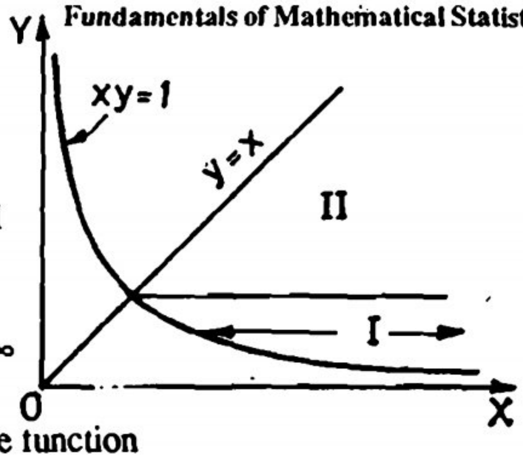
(Civil Services Main, 1986)

Hint. $f_X(x) = \displaystyle\int_y f(x,y)\,dy = \int_{1/x}^{x} f(x,y)\,dy$

$$f_Y(y) = \int_x f(x, y)\, dx$$

$$= \int_{1/y}^{\infty} f(x, y)\, dx\,; \quad 0 \le y \le 1$$

$$= \int_{y}^{\infty} f(x, y)\, dx\,; \quad 1 \le y < \infty$$



15. Show that the conditions for the function

$$f(x, y) = k \exp\left[A x^2 + 2\, H x y + B y^2\right], \; -\infty < (x, y) < \infty$$

to be a bivariate p.d.f. are

(i) $A \le 0$, *(ii)* $B \le 0$ *(iii)* $A B - H^2 \ge 0$.

Further show that under these conditions

$$k = \frac{1}{\pi}\left(A B - H^2\right)^{1/2}$$

Hint. $f(x, y)$ will represent the p.d.f. of a bivariate distriibution if and only if

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\; dx\; dy = 1$$

$$\Rightarrow \quad k \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[A x^2 + 2 H x y + B y^2\right] dx\; dy = 1 \qquad \dots (*)$$

We have

$$A x^2 + 2 H x y + B y^2 = A\left[x^2 + \frac{2H}{A} x y + \frac{B}{A} y^2\right]$$

$$= A\left[\left(x + \frac{H}{A} y\right)^2 + \frac{AB - H^2}{A^2} \cdot y^2\right] \qquad \dots (**)$$

Similarly, we can write

$$A x^2 + 2 H x y + B y^2 = B\left[\left(y + \frac{H}{B} x\right)^2 + \frac{AB - H^2}{B^2} x^2\right] \qquad \dots (***)$$

Substituting from (**) and (***) in (*) we observe that the double integral on the left hand side will converge if and only if

$$A \le 0, \; B \le 0 \text{ and } AB - H^2 \ge 0,$$

as desired.

Let us take $A = -a$; $B = -b$; $H = h$ so that $AB - H^2 = ab - h^2$, where $a > 0, b > 0$.

Substituting in (*), we get

$$k \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[-\frac{ab-h^2}{a} y^2 - \frac{1}{a}(-ax+hy)^2\right] dx\, dy = 1$$

$$\Rightarrow \quad k \int_{-\infty}^{\infty} \left[\exp\left(-\frac{ab-h^2}{a} y^2\right) \cdot \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{a}(ax-hy)^2\right\} dx\right] dy$$

$$= 1 \quad \ldots (****)$$

(By Fubini's theorem)

Now $\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{a}(ax-hy)^2\right\} dx = \int_{-\infty}^{\infty} \exp\left(-\frac{u^2}{a}\right)\frac{du}{a}$

$(ax - hy = u)$

$$= \frac{1}{a}\sqrt{\pi}\sqrt{a} = \sqrt{\frac{\pi}{a}}$$

$$\left(\because \int_{-\infty}^{\infty} e^{-c^2u^2}\, du = \frac{\sqrt{\pi}}{c}\right)$$

Hence from (****), we get

$$k\sqrt{\frac{\pi}{a}} \int_{-\infty}^{\infty} \exp\left\{-\frac{ab-h^2}{a} y^2\right\} dy = 1$$

$$\Rightarrow \quad k\sqrt{\frac{\pi}{a}} \cdot \sqrt{\frac{\pi a}{ab-h^2}} = 1$$

$$\Rightarrow \quad k = \frac{1}{\pi}\sqrt{ab-h^2} = \frac{1}{\pi}\sqrt{AB-H^2}.$$

OBJECTIVE TYPE QUESTIONS

I. Which of the following statements are TRUE or FALSE.

(i) Given a continuous random variable X with probability density function $f(x)$, then $f(x)$ cannot exceed unity.

(ii) A random variable X has the following probability density function :

$$f(x) = x, \; 0 < x < 1$$
$$= 0, \text{ elsewhere}$$

(iii) The function defined as

$$f(x) = |x|, \; -1 < x < 1$$
$$= 0, \text{ elsewhere}$$

is a possible probability density function.

(iv) The following represents joint probability distribution.

Y \ X	1	2	3
−1	1/9	1/18	1/18
0	1/18	2/9	3/9
1	1/8	1/18	1/18

II. Fill in the blanks :

(i) If $p_1(x)$ and $p_2(y)$ be the marginal probability functions of two independent discrete random variablies X and Y, then their joint probability function

$$p(x, y) = \ ...$$

(ii) The function $f(x)$ defined as

$$f(x) = |x|, \ -1 < x < 1$$
$$= 0, \text{ elsewhere}$$

is a possible ......

5·6. Transformation of One-dimensional Random Variable. Let X be a random variable defined on the event space S and let $g(\cdot)$ be a function such that $Y = g(X)$ is also a r.v. defined on S. In this section we shall deal with the following problem :

"Given the probability density of a r.v. X, to determine the density of a new r.v. $Y = g(X)$."

It can be proved in general that, if $g(\cdot)$ is any continuous function, then the distribution of $Y = g(X)$ is uniquely determined by that of X. The proof of this result is rather difficult and beyond the scope of this book. Here we shall consider the following, relatively simple theorem.

Theorem 5·9. *Let X be a continuous r.v. with p.d.f. $f_X(x)$. Let $y = g(x)$ be strictly monotonic (increasing or decreasing) function of x. Assume that $g(x)$ is differentiable (and hence continuous) for all x. Then the p.d.f. of the r.v. Y is given by*

$$h_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|,$$

where x is expressed in terms of y.

Proof. Case (i). $y = g(x)$ *is strictly increasing function of x (i.e., $dy/dx > 0$.* The *d.f.* of Y is given by

$$H_Y(y) = P(Y \leq y) = P[g(X) \leq y] = P(X \leq g^{-1}(y)],$$

the inverse exists and is unique, since $g(\cdot)$ is strictly increasing.

$$\therefore \quad H_Y(y) = F_X\left[g^{-1}(y)\right], \text{ where } F \text{ is the d.f. of } X$$

$$= F_X(x) \qquad \left[\because y = g(x) \Rightarrow g^{-1}(y) = x\right]$$

Differentiating w.r.t. y, we get

$$h_Y(y) = \frac{d}{dy}[F_X(x)] = \frac{d}{dx}\left(F_X(x)\right)\frac{dx}{dy}$$

$$= f_X(x)\frac{dx}{dy} \qquad ...(*)$$

Case (ii). $y = g(x)$ *is strictly monotonic decreasing.*

$$H_Y(y) = P(Y \le y) = P[g(X) \le y] = P[X \ge g^{-1}(y)]$$
$$= 1 - P\left[X \le g^{-1}(y)\right] = 1 - F_X\left[g^{-1}(y)\right] = 1 - F_X(x),$$

where $x = g^{-1}(y)$, the inverse exists and is unique. Differentiating w.r.t. y, we get

$$h_Y(y) = \frac{d}{dx}\left[1 - F_X(x)\right]\frac{dx}{dy} = -f_X(x)\cdot\frac{dx}{dy}$$

$$= f_X(x)\cdot\frac{-dx}{dy} \qquad \text{...(**)}$$

Note that the algebraic sign (–ive) obtained in (**) is correct, since y is a decreasing function of $x \Rightarrow x$ is a decreasing function of $y \Rightarrow dx/dy < 0$.

The results in (*) and (**) can be combined to give

$$h_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$

Example 5·33. *If the cumulative distribution function of X is $F(x)$, find the cumulative distribution function of*

(i) $Y = X + a$, *(ii)* $Y = X - b$, *(iii)* $Y = aX$,

(iv) $Y = X^3$, and *(v)* $Y = X^2$

What are the corresponding probability density functions ?

Solution. Let $G(\cdot)$ be the c.d.f. of Y. Then

(i) $G(x) = P(Y \le x) = P[X + a \le x] = P[X \le x - a] = F(x - a)$

(ii) $G(x) = P(Y \le x) = P[X - b \le x] = P[X \le x + b] = F(x + b)$

(iii) $G(x) = P[aX \le x] = P\left[X \le \dfrac{x}{a}\right],\ a > 0$

$$= F\left(\frac{x}{a}\right), \text{ if } a > 0$$

and $G(x) = P\left[X \ge \dfrac{x}{a}\right] = 1 - P\left[X < \dfrac{x}{a}\right]$

$$= 1 - F\left(\frac{x}{a}\right), \text{ if } a < 0$$

(iv) $G(x) = P[Y \le x] = P[X^3 \le x] = P\left[X \le x^{1/3}\right] = F\left(x^{1/3}\right)$

(v) $G(x) = P\left[X^2 \le x\right] = \left[-x^{1/2} \le X \le x^{1/2}\right]$

$$= P\left[X \le x^{1/2}\right] - P\left[X \le -x^{1/2}\right]$$

$$= 0, \qquad \text{if } x < 0$$
$$= F(\sqrt{x}) - F(-\sqrt{x} - 0), \qquad \text{if } x > 0$$

Variable	*d.f.*	*p.d.f.*
X	$F(x)$	$f(x)$
$X - a$	$F(x+a)$	$f(x+a)$
aX	$\left.\begin{array}{l} F(x/a) \quad a > 0 \\ 1 - F(x/a), a < 0 \end{array}\right\}$	$(1/a)\, f(x/a),\ a > 0$ $(-1/a)\, f(x/a),\ a < 0$
X^2	$\left.\begin{array}{l} F(\sqrt{x}) - F(-\sqrt{x} - 0) \\ \text{for } x > 0 \\ 0, \textit{ otherwise} \end{array}\right\}$	$\left.\begin{array}{l} \frac{1}{2(\sqrt{x})}[f\sqrt{x} + f(-\sqrt{x})] \\ \text{for } x > 0 \\ = 0 \text{ for } x \le 0 \end{array}\right\}$
X^3	$F(x^{1/3})$	$\frac{1}{3} f(x^{1/3}) \cdot \frac{1}{x^{2/3}}$

EXERCISE 5(f)

1. *(a)* A random variable X has $F(x)$ as its distribution function [$f(x)$ is the density function]. Find the distribution and the density functions of the random variable :

(i) $Y = a + bX$, a and b are real numbers, *(ii)* $Y = X^{-1}$, $[P(X = 0) = 0]$, *(iii)* $Y = \tan X$, and *(iv)* $Y = \cos X$.

(b) Let $f(x) = \begin{cases} 1/2, & -1 < x < 1 \\ 0, & \textit{elsewhere} \end{cases}$

be the p.d.f. of the r.v. X. Find the distribution function and the p.d.f. of $Y = X^2$.

[Delhi Univ. B.Sc. (Maths Hons.), 1988]

Hint. $F(x) = P(X \le x) = \int_{-1}^{x} f(x)\, dx = \frac{1}{2}(x + 1)$ ...(*)

Distribution function $G(\cdot)$ of $Y = X^2$ is given by :

$G_Y(x) = F(\sqrt{x}) - F(-\sqrt{x}) \quad ; \quad x > 0$ [c.f. Example 5·33 *(v)*]

$= \frac{1}{2}(\sqrt{x} + 1) - \frac{1}{2}(-\sqrt{x} + 1)$ [From (*)]

$= \sqrt{x} \quad ; \quad 0 < x < 1$

(As $-1 < x < 1$, $Y = X^2$ lies between 0 and 1)

p.d.f. of $Y = X^2$ is $g(x) = G'(x) = \dfrac{1}{2\sqrt{x}} \quad ; \quad 0 < x < 1$

2. Let X be a continuous random variable with p.d.f. $f(x)$. Let $Y = X^2$. Show that the random variable Y has p.d.f. given by

$$g(y) = \begin{cases} \dfrac{1}{2\sqrt{y}}\,[f(\sqrt{y}) + f(-\sqrt{y})], & y > 0 \\ 0, & y \leq 0 \end{cases}$$

3. Find the distribution and densitiy functions for *(i)* $Y = aX + b$, $a \neq 0$, b real, *(ii)* $Y = e^X$, assuming that $F(x)$ and $f(x)$, the distribution and the density of X are known.

Ans. *(i)* $\left.\begin{aligned} G(y) &= F[(y-b)/a], & \text{if } a > 0 \\ G(y) &= 1 - F[(y-b)/a], & \text{if } a < 0 \end{aligned}\right\}$ $g_1(y) = \dfrac{1}{|a|} f\left(\dfrac{y-b}{a}\right)$

(ii) $\left.\begin{aligned} G(y) &= F(\log y), & y > 0 \\ &= 0, & y \leq 0 \end{aligned}\right\}$ $\begin{aligned} g(y) &= \tfrac{1}{y} f(\log y), & y > 0 \\ &= 0, & y \leq 0 \end{aligned}$

4. *(a)* The random variable X has an exponential distribution

$$f(x) = e^{-x}, \quad 0 < x \leq \infty$$

Find the density function of the variable *(i)* $Y = 3X + 5$, *(ii)* $Y = X^3$.

(b) Suppose that X has p.d.f.,

$$\begin{aligned} f(x) &= 2x, \quad 0 < x < 1 \\ &= 0, \quad \textit{elsewhere} \end{aligned}$$

Find the p.d.f. of $Y = 3X + 1$.

Ans. $g(y) = \frac{2}{9}(y-1)$, $1 < y < 4$

5. Let X be a random variable with p.d.f.

$$\begin{aligned} f(x) &= \tfrac{2}{9}(x+1) \qquad -1 < x < 2 \\ &= 0, \quad \textit{elsewhere} \end{aligned}$$

Find the p.d.f. of $U = X^2$. **[Poona Univ. B.E., 1992]**

6. Let the p.d.f. of X be

$$\begin{aligned} f(x) &= \tfrac{1}{6}, \quad -3 \leq x \leq 3 \\ &= 0, \quad \textit{elsewhere} \end{aligned}$$

Find the p.d.f. of $Y = 2X^2 - 3$.

7. Let X be a random variable with the distribution function :

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

Determine the distribution function $F_Y(y)$ of the random variable $Y = \sqrt{X}$ and hence compute mean of Y. **[Calcutta Univ. B.A.(Hons.), 1986]**

5·7. Transformation of Two-dimensional Random Variable. In this section we shall consider the problem of change of variables in the two-dimensional case. Let the *r.v.'s* U and V by the transformation $u = u(x, y)$, $v = v(x, y)$, where u and v are continuously differentiable functions for which Jacobian of transformation

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}$$

is either > 0 or < 0 throughout the (x, y) plane so that the inverse transformation is uniquiely given by $x = x(u, v)$, $y = y(u, v)$.

Theorem 5·10. *The joint p.d.f.* $g_{UV}(u, v)$ *of the transformed variables* U *and* V *is given by*

$$g_{UV}(u, v) = f_{XY}(x, y) \, . \, |J|$$

where $|J|$ *is the modulus value of the Jacobian of transformation and* $f(x, y)$ *is expressed in terms of* u *and* v.

Proof. $P(x < X \leq x + dx,\ y < Y \leq y + dy)$

$$= P(u < U \leq u + du,\ v < V \leq v + dv)$$

$$\Rightarrow \quad f_{XY}(x, y)\, dx\, dy = g_{UV}(u, v)\, du\, dv$$

$$\Rightarrow \quad g_{UV}(u, v)\, du\, dv = f_{XY}(x, y) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv$$

$$\Rightarrow \quad g_{UV}(u, v) = f_{XY}(x, y) \left| \frac{\partial(x, y)}{\partial(u, v)} \right| = f_{XY}(x, y)\, |J|$$

Theorem 5·11. *If* X *and* Y *are independent continuous r.v.'s, then the p.d.f. of* $U = X + Y$ *is given by*

$$h(u) = \int_{-\infty}^{\infty} f_X(v) f_Y(u - v)\, dv$$

Proof. Let $f_{XY}(x, y)$ be the joint *p.d.f.* of independent continuous *r.v.'s* X and Y and let us make the transformation :

$$u = x + y,\ v = x \quad \Rightarrow \quad x = v,\ y = u - v$$

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix} = -1$$

Thus the joint p.d.f. of *r.v.'s* U and V is given by

$$g_{UV}(u, v) = f_{XY}(x, y)\, |J|$$

$$= f_X(x) \cdot f_Y(y)\, |J|$$

(Since X and Y are independent)

$$= f_X(v) \cdot f_Y(u - v)$$

The marginal density of U is given by

$$h(u) = \int_{-\infty}^{\infty} g_{UV}(u, v)\, dv$$

$$= \int_{-\infty}^{\infty} f_X(v)\, f_Y(u - v)\, dv$$

Remark. The function $h(\cdot)$ is given a special name and is said to be the *convolution* of $f_X(\cdot)$ and $f_Y(\cdot)$ and we write

$$h(\cdot) = f_X(\cdot) * f_Y(\cdot)$$

Example 5·34. *Let (X, Y) be a two-dimensional non-negative continuous r.v. having the joint density :*

$$f(x, y) = \begin{cases} 4xy\, e^{-(x^2+y^2)} & ; \; x \geq 0,\; y \geq 0 \\ 0 & ,\; \textit{elsewhere} \end{cases}$$

Prove that the densitiy function of $U = \sqrt{X^2 + Y^2}$ is

$$h(u) = \begin{cases} 2u^3\, e^{-u^2} & ,\; 0 \leq u < \infty \\ 0 & ,\; \textit{elsewhere} \end{cases}$$

[Meerut Univ. M.Sc., 1986]

Solution. Let us make the transformation :

$$u = \sqrt{x^2 + y^2} \quad \text{and} \quad v = x$$

$$\Rightarrow \quad v \geq 0,\; u \geq 0 \text{ and } u \geq v \quad \Rightarrow \quad u \geq 0 \text{ and } 0 \leq v \leq u$$

The Jacobian of transformation J is given by

$$\frac{1}{J} = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} = -\frac{y}{\sqrt{x^2 + y^2}}$$

The joint p.d.f. of U and V is given by

$$g(u, v) = f(x, y)\; |J|$$

$$= 4xy\, e^{-(x^2+y^2)} \left| -\frac{\sqrt{x^2+y^2}}{y} \right|$$

$$= 4x\sqrt{x^2 + y^2}\; e^{-(x^2+y^2)}$$

$$= \begin{cases} 4vu \,.\, e^{-u^2} \; ; \; u \geq 0,\; 0 \leq v \leq u \\ 0 \;, \quad \textit{otherwise} \end{cases}$$

Hence the density function of $U = \sqrt{X^2 + Y^2}$ is

$$h(u) = \int_0^u g(u, v)\, dv = 4u\, e^{-u^2} \int_0^u v\, dv$$

$$= \begin{cases} 2u^3 e^{-u^2}, & u \geq 0 \\ 0, & elsewhere \end{cases}$$

Example 5·35. *Let the probability density function of the random variable (X, Y) be*

$$f(x, y) = \begin{cases} \alpha^{-2} e^{-(x+y)/\alpha} & ; \quad x, y > 0,\ \alpha > 0 \\ 0 & ,\ elsewhere \end{cases}$$

Find the distribution of $\frac{1}{2}(X - Y)$. **[Nagpur Univ. B.E., 1988]**

Solution. Let us make the transformation :

$$u = \frac{1}{2}(x - y) \quad \text{and} \quad v = y$$

$$\Rightarrow \qquad x = 2u + v \quad \text{and} \quad y = v$$

The Jacobian of the transformation is :

$$J = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix} = 2$$

Thus, the joint p.d.f. of the random variables (U, V) is given by :

$$g(u, v) = \begin{cases} \dfrac{2}{\alpha^2} e^{-(2/\alpha)(u+v)}, & -\infty < u < \infty,\ v > -2u, \text{ if } u < 0 \\ & v > 0 \text{ if } u \geq 0 \text{ and } \alpha > 0 \\ 0, & elsewhere \end{cases}$$

The marginal p.d.f. of U is given by

$$g_U(u) = \begin{cases} \displaystyle\int_{-2u}^{\infty} \frac{2}{\alpha^2} \exp\{-(2/\alpha)(u+v)\}\, dv \\ \quad = \dfrac{1}{\alpha} e^{-2u/\alpha} & , u < 0 \\ \displaystyle\int_0^{\infty} \frac{2}{\alpha} \exp\{-(2/\alpha)(u+v)\}\, dv \\ \quad = \dfrac{1}{\alpha} e^{-2u/\alpha} & , u \geq 0 \end{cases}$$

Hence

$$g_U(u) = \frac{1}{\alpha} e^{-(2/\alpha)|u|} \quad ; \quad -\infty < u < \infty$$

Example 5·36. *Given the joint density function of X and Y as*

$$f(x,y) = \frac{1}{2} x\, e^{-y}\ ;\ 0 < x < 2\ ,\ y > 0$$

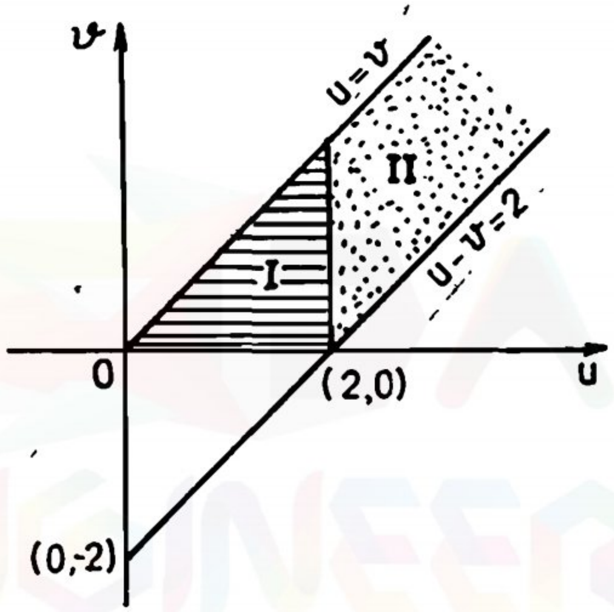
$$= 0\ ,\ \textit{elsewhere}$$

Find the distribution of $X + Y$.

Solution. Let us make the transformation :

$$u = x + y \text{ and } v = y \quad \Rightarrow \quad y = v\ ,\ x = u - v$$

The Jacobian of transformation $J = \dfrac{\partial\,(x,y)}{\partial\,(u,v)} = 1$ and the region $0 < x < 2$ and $y > 0$ transforms to $0 < u - v < 2$ and $v > 0$ as shown in the following figure.



The joint density function of U and V is given by

$$g(u,v) = \frac{1}{2}(u - v)\, e^{-v}\ ;\ 0 < v < u\ ,\ u > 0$$

To find the density of $U = X + Y$, we split the range of U into two parts (*i*) $0 < u \le 2$ (region I) (*ii*) $u > 2$ (region II) (which is suggested by the diagram).

For $0 < u \le 2$, (Region I) :

$$h(u) = \int_0^u g(u,v)\, dv = \frac{1}{2} \int_0^u (u - v)\, e^{-v}\, dv$$

$$= \frac{1}{2} \left| -e^{-v}(u - v) + e^{-v} \right|_{v=0}^{v=u} \quad \text{(Integration by parts)}$$

$$= \frac{1}{2}\left(e^{-u} + u - 1\right)$$

For $2 < u < \infty$, (Region II) :

$$h(u) = \frac{1}{2}\int_{u-2}^{u} (u-v)\, e^{-v}\, dv$$

$$= \frac{1}{2}\left| e^{-v}(1+v-u)\right|_{v=u-2}^{v=u}$$

$$= \frac{1}{2} e^{-u}\left(1+e^{2}\right) \qquad \text{(on simplification)}$$

Hence

$$g(u) = \begin{cases} \frac{1}{2}\left(e^{-u} + u - 1\right), & 0 < u \le 2 \\ \frac{1}{2} e^{-u}\left(1+e^{2}\right), & 2 < u < \infty \\ 0\ , & elsewhere \end{cases}$$

MISCELLANEOUS EXERCISE ON CHAPTER FIVE

1. 4 coins are tossed. Let X be the number of heads and Y be the number of heads minus the number of tails. Find the probability function of X, the probability function of Y and $P(-2 \le Y < 4)$.

Ans. Probability function of X is

Values of X, x	0	1	2	3	4
$p_1(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

Probability function of Y is

Values of Y, y	4	2	0	-2	-4
$p_2(y)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

$$P(-2 \le Y < 4) = \frac{4+6+4}{16} = \frac{7}{8}.$$

2. A random process gives measurements X between 0 and 1 with a probability density function

$$f(x) = 12x^3 - 21x^2 + 10x,\quad 0 \le x \le 1$$
$$= 0,\ elsewhere$$

(i) Find $P(X \le \frac{1}{2})$ and $P(X > \frac{1}{2})$

(ii) Find a number k such that $P(X \le k) = \frac{1}{2}$.

Ans. *(i)* $9/16$, $7/16$, *(ii)* $k = 0{\cdot}452$.

3. Show that for the distribution

$$dF = y_o \left[1 - \frac{|x-b|}{a} \right] dx, \quad b-a < x < b+a$$
$$= 0, \text{ otherwise,}$$
$$y_o = \frac{1}{a}, \text{ mean } = b \text{ and variance } = a^2/6$$

4. A ray of light is sent in a random direction towards the x-axis from a station Q (0, 1) on the y-axis and the ray meets the x-axis at a point P. Find the probability density function of the abscissa of P.

[Calcutta Univ. B.Sc.(Hons.), 1982]

5. Let X be a continuous variate with p.d.f.

$$f(x) = k(x - x^2) \; ; \; a < x < b, \; k > 0$$

What are the possible values of a and b and what is k ?

[Delhi Univ. B.Sc.(Maths Hons.), 1989]

6. Pareto distribution with parameters r and A is given by the probability density function

$$f(x) = rA^r \frac{1}{x^{r+1}}, \quad \text{for } x \geq A$$
$$= 0, \; x < A, \; r > 0$$

Show that it has a finite nth moment if and only if $n < r$. Find the mean and variance of the distribution.

7. For a continuous random variable X, defined in the range $(0 \leq x < \infty)$, the probability distribution is such that

$$P(X \leq x) = 1 - e^{-\beta x^2}, \text{ where } \beta > 0$$

Find the median of the distribution. Also if m, m_o and σ denote the mean, mode and standard deviation respectively of the distribution, prove that

$$2m_o^2 - m^2 = \sigma^2 \quad \text{and} \quad m_o = m\sqrt{2/\pi}$$

What is the sign of skewness of the distribution ?

8. *(a)* Two dice are rolled, $S = \{(a, b) \mid a, b = 1, 2 \ldots, 6\}$. Let X denote the sum of the two faces and Y the absolute value of their difference, *i.e.*, X is distributed over the integers 2, 3, ..., 12 and Y over 0, 1, 2, ..., 5. Assuming the dice are fair, find the probabilities that *(i)* $X = 5 \cap Y = 1$, *(ii)* $X = 7 \cap Y \geq 3$, *(iii)* $X = Y$, and *(iv)* $X + Y = 4 \cap X - Y = 2$.

Ans. *(i)* 1/18, *(ii)* 1/9, *(iii)* 0 and *(iv)* 1/18.

9. The joint probability density function of the two-dimensional variable (X, Y) is of the form

$$f(x, y) = k e^{-(x+y)}, \quad 0 \leq y < x < \infty$$
$$= 0, \text{ elsewhere}$$

(i) Determine the constant k. *(ii)* Find the conditional probability density function $f_1(x \mid y)$ and *(iii)* Compute $P(Y \geq 3)$.

[Sardar Patel Univ. B.Sc., 1986]

(iv) Find the marginal frequency function $f_1(x)$ of X.

(v) Find the marginal frequency function $f_2(y)$ of Y.

(vi) Examine if X, Y are independent.

(vii) Find the conditional frequency function of Y given $X = 2$.

Ans. (i) $k = 1$, (ii) $f_1(x \mid y) = e^{-x}$, (iii) e^{-3}.

10. Let

$$f(x, y) = \begin{cases} \binom{y}{x} p^x (1-p)^{y-x} \dfrac{e^{-\lambda} \lambda^y}{y!} \\ \qquad ; x = 0, 1, 2, \ldots; \ y = 0, 1, 2, \ldots; \ \textit{with } y \geq x \\ 0, \ \textit{elsewhere} \end{cases}$$

Find the marginal density function of X and the marginal density function of Y. Also determine whether the random variables X and Y are independent.

[I.S.I., 1987]

11. Consider the following function :

$$f(x \mid y) = \begin{cases} \dfrac{y^x e^{-y}}{x!}, & x = 0, 1, 2, \ldots \\ 0, & \textit{otherwise} \end{cases}$$

(i) Show that $f(x \mid y)$ is the conditional probability function of X given Y; $y \geq 0$.

(ii) If the marginal p.d.f. of Y is

$$f_Y(y) = \begin{cases} \lambda e^{-\lambda x}, & x > 0. \\ 0 & x \leq 0, \ \lambda > 0 \end{cases}$$

what is the joint p.d.f. of X and Y ?

(iii) Obtain the marginal probability function of X.

[Delhi Univ. M.A.(Econ.), 1989]

12. The probability density function of (x_1, x_2) is given as

$$f(x_1, x_2) = \begin{cases} \theta_1 \theta_2 e^{-\theta_1 x_1 - \theta_2 x_2} & \text{if } x_1, x_2 > 0 \\ 0 & \textit{otherwise}. \end{cases}$$

Find the density function of (y_1, y_2) where

$$y_1 = \frac{2x_1}{x_2} + 1, \quad y_2 = 3x_1 + x_2 \ \text{ almost everywhere.}$$

[Punjab Univ. M.A.(Econ.), 1992]

13. *(a)* Let X_1, X_2 be a random sample of size 2 from a distribution with probability density function,

$$f(x) = e^{-x}, \ 0 < x < \infty$$

$= 0$, *elsewhere*

Show

$$Y_1 = X_1 + X_2 \text{ and } Y_2 = \frac{X_1}{X_1 + X_2}$$

are independent. **[Sardar Patel Univ. B.Sc., Sept. 1986]**

(b) X_1, X_2, X_3 denote random sample of size 3 drawn from the distribution:

$$f(x) = e^{-x}, \ 0 < x < \infty$$
$$= 0, \ elsewhere$$

Show that

$$Y_1 = \frac{X_1}{X_1 + X_2}, \ Y_2 = \frac{X_1 + X_2}{X_1 + X_2 + X_3} \text{ and } Y_3 = X_1 + X_2 + X_3$$

are mutually independent.

14. If the probability density function of the random varaibles X and $Y \mid X$ is given by

$$f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & elsewhere \end{cases}$$

and $$f_{Y \mid X}(y \mid x) = \begin{cases} \dfrac{e^{-x} x^y}{y!}, & y \geq 0 \\ 0, & elsewhere \end{cases}$$

respectively, find the probability density function of the random variable Y.

[Jiwaji Univ. M.Sc., 1987]

15. *(a)* The random variable X and Y have a joint p.d.f. $f(x, y)$ given by

$$f(x, y) = g(x + y), \quad x > 0, \ y > 0$$
$$= 0, \quad \text{otherwise.}$$

Obtain the distribution function $H(z)$ of $Z = X + Y$ and hence show that its p.d.f. is

$$h(z) = z\, g(z), \quad z > 0$$
$$= 0 \quad z \leq 0.$$

(b) The joint density function of two random variables is given by

$f(x, y) = e^{-(x+y)}$; $x > 0$, $y > 0$. Show that the p.d.f. of

$$U = \frac{X + Y}{2} \text{ is } g(u) = 4\, u\, e^{-2u}$$

[Calicut Univ. B.Sc., 1986]

16. The time X taken by a garage to repair a car is a continuous random variable with probability density function

$$f_1(x) = \begin{cases} \frac{3}{4}x(2-x), & 0 \le x \le 2 \\ 0, & elsewhere \end{cases}$$

If, on leaving his car, a motorist goes to keep an engagement, lasting for a time Y, where Y is a continuous random variable, independent of X, with probability function

$$f_2(y) = \begin{cases} \frac{1}{2}y; & 0 \le y \le 2 \\ 0, & elsewhere; \end{cases}$$

determine the probability that the car will not be ready on his return.

[Calcutta Univ. B.A.(Hons.), 1988]

17. If X and Y are two independent random variables such that

$$f(x) = e^{-x}, \ x \ge 0 \quad \text{and} \quad g(y) = 3e^{-3y}, \ y \ge 0;$$

find the probability distribution of $Z = X/Y$.

[Madurai Univ. B.Sc., Oct. 1987]

18. The random variables X and Y are independent and their probability density functions are, respectively given by

$$f(x) = \frac{1}{\pi} \cdot \frac{1}{\sqrt{1-x^2}}, \ |x| < 1 \quad \text{and} \quad g(y) = y e^{-y^2/2}, \ y > 0.$$

Find the joint probability density of Z and W where $Z = XY$ and $W = X$. Deduce the probability density of Z. [Calcutta Univ. B.Sc.(Hons.), 1985]

CHAPTER SIX

Mathematical Expectation, Generating Functions and Law of Large Numbers

6·1. Mathematical Expectation. Let X be a random variable (r.v.) with p.d.f. (p.m.f.) $f(x)$. Then its mathematical expectation, denoted by $E(X)$ is given by :

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx, \quad (\text{ for continuous } r.v.) \qquad ...(6·1)$$

$$= \sum_x x\, f(x), \quad (\text{ for discrete } r.v.) \qquad ...(6·1a)$$

provided the righthand integral or series is absolutely convergent, *i.e.*, provided

$$\int_{-\infty}^{\infty} |x f(x)|\, dx = \int_{-\infty}^{\infty} |x| f(x)\, dx < \infty \qquad ...(6·2)$$

or
$$\sum_x |x f(x)| = \sum_x |x|\, f(x) < \infty \qquad ...(6·2a)$$

Remarks. 1. Since absolute convergence implies ordinary convergence, if (6·2) or (6·2a) holds then the integral or series in (6·1) and (6·1a) also exists, *i.e.*, has a finite value and in that case we define $E(X)$ by (6·1) or (6·1a). It should be clearly understood that although X has an expectation only if L.H.S. in (6·2) or (6·2a) exists, *i.e.*, converges to a finite limit, its value is given by (6·1) or (6·1a).

2. $E(X)$ exists iff $E|X|$ exists.

3. The expectation of a random variable is thought of as a long-term average. [See Remark to Example (6·2a), **page 6·19**.]

4. Expected value and variance of an Indicator Variable. Consider the indicator variable : $X = I_A$ so that

$$X = 1 \text{ if } A \textit{ happens}$$
$$= 0 \text{ if } \bar{A} \textit{ happens}$$

$\therefore \quad E(X) = 1 \; P(X=1) + 0 \,.\, P(X=0)$

$\Rightarrow \quad E(I_A) = 1 \; P[I_A = 1] + 0 \,.\, P[I_A = 0]$

$\Rightarrow \quad E(I_A) = P(A)$

This gives us a very useful tool to find $P(A)$, rather than to evaluate $E(X)$.

Thus $\qquad P(A) = E(I_A) \qquad ...(6·2b)$

For illustration of this result, see Example 6·14, page 6·27.

$$E(X^2) = 1^2 \,.\, P(X=1) + 0^2 \,.\, P(X=0) = P(I_A = 1) = P(A)$$

$$\therefore \quad Var\ X = E(X^2) - [E(X)]^2 = P(A) - [P(A)]^2$$

$$= P(A)\,[1 - P(A)]$$

$$= P(A)\ P(\overline{A}) \qquad ...(6{\cdot}2c)$$

Illustrations. If the r.v. X takes the values $0!, 1!, 2!, \ldots$ with probability law

$$P(X = x!) = \frac{e^{-1}}{x!}; \quad x = 0, 1, 2, \ldots$$

then $$\sum_{x=0}^{\infty} x!\ P(X = x!) = e^{-1} \sum_{x=0}^{\infty} 1$$

which is a divergent series. In this case $E(X)$ does not exist.

More rigorously, let us consider a random variable X which takes the values

$$x_i = (-1)^{i+1}\ (i+1); \quad i = 1, 2, 3, \ldots$$

with the probability law

$$p_i = P(X = x_i) = \frac{1}{i(i+1)}; \quad i = 1, 2, 3, \ldots$$

Here $$\sum_{i=1}^{\infty} x_i\ P(X = x_i) = \sum_{i=1}^{\infty} (-1)^{i+1} \left(\frac{1}{i}\right) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

Using Leibnitz test for alternating series the series on right hand side is conditionally convergent since the terms alternate in sign, are monotonically decreasing and converge to zero. By conditional convergence we mean that although $\sum_{i=1}^{\infty} p_i x_i$ converges, $\sum_{i=1}^{\infty} |p_i x_i|$ does not converge. So, rigorously speaking, in the above example $E(X)$ does not exist, although $\sum_{i=1}^{\infty} p_i x_i$ is finite, *viz.*, $\log_e 2$.

As another example, let us consider the r.v. X which takes the values

$$x_k = \frac{(-1)^k \cdot 2^k}{k}; \quad k = 1, 2, 3, \ldots$$

with probabilities $\quad p_k = 2^{-k}$.

Here also we get

$$\sum_{k=1}^{\infty} x_k p_k = \sum_{k=1}^{\infty} \frac{(-1)^k}{k}$$
$$= -\left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\right] = -\log_e 2$$

and $$\sum_{k=1}^{\infty} |x_k|\ p_k = \sum_{k=1}^{\infty} \frac{1}{k},$$

which is a divergent series. Hence in this case also expectation does not exist.

As an illustration of a continuous r.v. let us consider the r.v. X with p.d.f.

$$f(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2} \quad ; \quad -\infty < x < \infty$$

which is p.d.f. of Standard Cauchy distribution. [c.f.s. 8·9].

$$\int_{-\infty}^{\infty} |x| f(x)\, dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{|x|}{1+x^2}\, dx = \frac{2}{\pi} \int_{0}^{\infty} \frac{x}{1+x^2}\, dx$$

($\because$ *Integrand is an even function of x*)

$$= \frac{1}{\pi} \left| \log\left(1+x^2\right) \right|_0^{\infty} \rightarrow \infty$$

Since this integral does not converge to a finite limit, $E(X)$ does not exist.

6·2. Expectation of a Function of a Random Variable. Consider a r.v. X with p.d.f. (p.m.f.) $f(x)$ and distribution function $F(x)$. If $g(.)$ is a function such that $g(X)$ is a r.v. and $E[g(X)]$ exists (*i.e.*, is defined), then

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\, dF(x) = \int_{-\infty}^{\infty} g(x) f(x)\, dx \qquad ...(6·3)$$

(*For continuous r.v.*)

$$= \sum_x g(x) f(x) \qquad ...(6·3a)$$

(*For discrete r.v.*)

By definition, the expectation of $Y = g(X)$ is

$$E[g(X)] = E(Y) = \int y\, . dH_Y(y) = \int y\, h(y)\, dy \qquad ...(6·4)$$

or
$$E(Y) = \sum_y y\, h(y) \qquad ...(6·4a)$$

where $H_Y(y)$ is the distribution function of Y and $h(y)$ is p.d.f. of Y.

[The proof of equivalence of (6·3) and (6·4) is beyond the scope of the book.]

This result extends into higher dimensions. If X and Y have a joint p.d.f. $f(x, y)$ and $Z = h(x, y)$ is a random variable for some function h and if $E(Z)$ exists, then

$$E(Z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x, y) f(x, y)\, dx\, dy \qquad ...(6·5)$$

or
$$E(Z) = \sum_x \sum_y h(x, y)\, f(x, y) \qquad ...(6·5a)$$

Particular Cases. 1. If we take $g(X) = X^r$, r being a positive integer, in (6·3) we get :

$$E(X^r) = \int_{-\infty}^{\infty} x^r \cdot f(x)\, dx \qquad ...(6·5b)$$

which is defined as μ_r', the rth moment (about origin) of the probability distribution.

Thus μ_r' (*about origin*) $= E(X^r)$. In particular

μ_1' (*about origin*) $= E(X)$ and μ_2' (*about origin*) $= E(X^2)$.

Hence $\quad Mean = \bar{x} = \mu_1' \text{ (about origin) } = E(X)$ ...(6·6)

and $\quad \mu_2 = \mu_2' - \mu_1'^2 = E(X^2) - \{E(X)\}^2$ ...(6·6a)

2. If $g(X) = [X - E(X)]^r = (X - \bar{x})^r$, then from (6·3) we get :

$$E[X - E(X)]^r = \int_{-\infty}^{\infty} [x - E(X)]^r f(x)\, dx = \int_{-\infty}^{\infty} (x - \bar{x})^r f(x)\, dx \qquad ...(6·7)$$

which is μ_r, the rth moment about mean.

In particular, if $r = 2$, we get

$$\mu_2 = E[X - E(X)]^2 = \int_{-\infty}^{\infty} (x - \bar{x})^2 f(x)\, dx \qquad ...(6·8)$$

Formulae (6·6*a*) and (6·8) give the variance of the probability distribution of a r.v. X in terms of expectation.

3. Taking $g(x) = constant = c$, say in (6·3) we get

$$E(c) = \int_{-\infty}^{\infty} c \, . f(x)\, dx = c \int_{-\infty}^{\infty} f(x)\, dx = c \qquad ...(6·9)$$

$$E(c) = c \qquad ...(6·9a)$$

Remark. The corresponding results for a discrete r.v. X can be obtained on replacing integration by summation (Σ) over the given range of the variable X in the formulae (6·5) to (6·9).

In the following sections, we shall establish some more results on Expectation in the form of theorems, for continuous r.v.'s only. The corresponding results for discrete r.v.'s can be obtained similarly on replacing integration by summation (Σ) over the given range of the variable X and are left as an exercise to the reader.

6·3. Addition Theorem of Expectation

Theorem 6·1. *If X and Y are random variables then*

$$E(X + Y) = E(X) + E(Y), \qquad ...(6·10)$$

provided all the expectations exist.

Proof. Let X and Y be continuous r.v.'s with joint p.d.f. $f_{X,Y}(x, y)$ and marginal p.d.f 's $f_X(x)$ and $f_Y(y)$ respectively. Then by definition :

$$E(X) = \int_{-\infty}^{\infty} x \, f_X(x)\, dx \qquad ...(6·11)$$

$$E(Y) = \int_{-\infty}^{\infty} y \, f_Y(y)\, dy \qquad ...(6·12)$$

$$E(X + Y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x + y) f_{XY}(x, y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x\, f_{XY}(x, y)\, dx\, dy + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y\, f_{XY}(x, y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} x \left[\int_{-\infty}^{\infty} f_{XY}(x, y)\, dy \right] dx + \int_{-\infty}^{\infty} y \left[\int_{-\infty}^{\infty} f_{XY}(x, y)\, dx \right] dy$$

$$= \int_{-\infty}^{\infty} x\, f_X(x)\, dx + \int_{-\infty}^{\infty} y\, f_Y(y)\, dy$$

$$= E(X) + E(Y) \qquad \text{[On using (6·11) and (6·12)]}$$

The result in (6·10) can be extended to n variables as given below.

Theorem 6·1(a). *The mathematical expectation of the sum of n random variables is equal to the sum of their expectations, provided all the expectations exist.*

Symbolically, if $X_1, X_2, \ldots, X_n$ are random variables then

$$E(X_1 + X_2 + \ldots + X_n) = E(X_1) + E(X_2) + \ldots + E(X_n) \qquad \ldots(6·13)$$

or

$$E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i), \qquad \ldots(6·13a)$$

if all the expectations exist.

Proof. Using (6·10), for two r.v.'s X_1 and X_2 we get :

$$E(X_1 + X_2) = E(X_1) + E(X_2)$$

$\Rightarrow$ (6·13) is true for $n = 2$. ...(*)

Let us now suppose that (6·13) is true for $n = r$ (say), so that

$$E\left(\sum_{i=1}^{r} X_i\right) = \sum_{i=1}^{r} E(X_i) \qquad \ldots(6·14)$$

$$E\left(\sum_{i=1}^{r+1} X_i\right) = E\left[\sum_{i=1}^{r} X_i + X_{r+1}\right]$$

$$= E\left(\sum_{i=1}^{r} X_i\right) + E(X_{r+1}) \qquad \text{[Using (6·10)]}$$

$$= \sum_{i=1}^{r} E(X_i) + E(X_{r+1}) \qquad \text{[Using (6·14)]}$$

$$= \sum_{i=1}^{r+1} E(X_i)$$

Hence if (6·13) is true for $n = r$, it is also true for $n = r + 1$. But we have proved in (*) above that (6·13) is true for $n = 2$. Hence it is true for $n = 2 + 1 = 3$; $n = 3 + 1 = 4$; ... and so on. Hence by the principle of mathematical Introduction (6·13) is true for all positive integral values of n.

6·4. Multiplication Theorem of Expectation

Theorem 6·2. *If X and Y are independent random variables, then*

$$E(XY) = E(X) \cdot E(Y) \qquad \ldots(6\cdot15)$$

Proof. Proceeding as in Theorem 6·1, we have :

$$E(XY) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy \, f_{XY}(x, y) \, dx \, dy$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy \, f_X(x) f_Y(y) \, dx \, dy$$

[Since X and Y are independent]

$$= \int_{-\infty}^{\infty} x \, f_X(x) \, dx \int_{-\infty}^{\infty} y \, f_Y(y) \, dy$$

$$= E(X) E(Y), \qquad \textit{[Using (6·11) and (6·12)]}$$

provided X and Y are independent.

Generalisation to n-variables.

Theorem 6·2(a). *The mathematical expectation of the product of a number of independent random variables is equal to the product of their expectations. Symbolically, if $X_1, X_2, \ldots, X_n$ are n independent random variables, then*

$$\left.\begin{aligned} E(X_1 X_2 \ldots X_n) &= E(X_1) E(X_2) \ldots E(X_n) \\ \text{i.e.,} \quad E\left(\prod_{i=1}^{n} X_i\right) &= \prod_{i=1}^{n} E(X_i) \end{aligned}\right\} \qquad \ldots(6\cdot16)$$

provided all the expectations exist.

Proof. Using (6·15), for two *independent* random variables X_1 and X_2, we get:

$$E(X_1 X_2) = E(X_1) E(X_2)$$

$\Rightarrow$ (6·16) is true for $n = 2$. ...(*)

Let us now suppose that (6·16) is true for $n = r$, (say) so that :

$$E\left(\prod_{i=1}^{r} X_i\right) = \prod_{i=1}^{r} E(X_i) \qquad \ldots(6\cdot17)$$

$$E\left(\prod_{i=1}^{r+1} X_i\right) = E\left(\prod_{i=1}^{r} X_i \times X_{r+1}\right)$$

$$= E\left(\prod_{i=1}^{r} X_i\right) E\left(X_{r+1}\right) \qquad \text{[Using (6·15)]}$$

$$= \prod_{i=1}^{r} \left(E X_i\right) E\left(X_{r+1}\right) \qquad \text{[Using (6·17)]}$$

$$= \prod_{i=1}^{r+1} \left(E X_i\right)$$

Hence if (6·16) is true for $n = r$, it is also true for $n = r + 1$. Hence using (*), by the principle of mathematical induction we conclude that (6·16) is true for all positive integral values of n.

Theorem 6·3. *If X is a random variable and 'a' is constant, then*

(i) $E[a\Psi(X)] = a\, E[\Psi(X)]$ ...(6·18)

(ii) $E[\Psi(X) + a] = E[\Psi(X)] + a,$ ...(6·19)

where $\Psi(X)$, a function of X, is a r.v. and all the expectations exist.

Proof.

$$\text{(i)}\quad E[a\Psi(X)] = \int_{-\infty}^{\infty} a\Psi(x)\cdot f(x)\,dx = a\int_{-\infty}^{\infty}\Psi(x) f(x)\,dx = a\,E[\Psi(X)]$$

$$\text{(ii)}\quad E[\Psi(X) + a] = \int_{-\infty}^{\infty}[\Psi(x) + a]\, f(x)\,dx$$

$$= \int_{-\infty}^{\infty}\Psi(x)\, f(x)\,dx + a\int_{-\infty}^{\infty} f(x)\,dx$$

$$= E[\Psi(X)] + a \qquad \left(\because \int_{-\infty}^{\infty} f(x)\,dx = 1\right)$$

Cor. *(i)* If $\Psi(X) = X$, then

$E(aX) = a\,E(X)$ and $E(X + a) = E(X) + a$ ...(6·20)

(ii) If $\Psi(X) = 1$, then $E(a) = a$. ...(6·21)

Theroem 6·4. *If X is a random variable and a and b are constants, then*

$E(aX + b) = a\,E(X) + b$ ...(6·22)

provided all the expectations exist.

Proof. By definition, we have

$$E(aX + b) = \int_{-\infty}^{\infty}(ax + b)\, f(x)\,dx$$

$$= a\int_{-\infty}^{\infty} x\, f(x)\,dx + b\int_{-\infty}^{\infty} f(x)\,dx$$

$$= a\,E(X) + b$$

Cor. 1. If $b = 0$, then we get

$E(aX) = a\cdot E(X)$ ...(6·22a)

Cor. 2. Taking $a = 1$, $b = -\bar{X} = -E(X)$, we get

$$E(X - \bar{X}) = 0$$

Remark. If we write,

$g(X) = aX + b$ ...(6·23)

then $g[E(X)] = a\,E(X) + b$ ...(6·23a)

Hence from (6·22) and (6·23*a*) we get

$$E[g(X)] = g[E(X)] \qquad ...(6{\cdot}24)$$

Now (6·23) and (6·24) imply *that expectation of a linear function is the same linear function of the expectation.* The result, however, is not true if $g(\cdot)$ is not linear. For instance

$$E(1/X) \neq (1/EX) \quad ; \quad E(X^{\frac{1}{2}}) \neq [E(X)]^{\frac{1}{2}},$$
$$E[\log(X) \neq \log[E(X)] \quad ; \quad E(X^2) \neq [E(X)]^2,$$

since all the functions stated above are non-linear. As an illustration, let us consider a random varible X which assumes only two values +1 and –1, each with equal probability $\frac{1}{2}$. Then

$$E(X) = 1 \times \tfrac{1}{2} + (-1) \times \tfrac{1}{2} = 0.$$

and $$E(X^2) = 1^2 \times \tfrac{1}{2} + (-1)^2 \times \tfrac{1}{2} = 1.$$

Thus $$E(X^2) \neq [E(X)]^2$$

For a non-linear function $g(X)$, it is difficult to obtain expressions for $E[g(X)]$ in terms of $g[E(X)]$, say, for $E[\log(X)]$ or $E(X^2)$ in terms of $\log[E(X)]$ or $[E(X)]^2$. However, some results in the form of inequalities between $E[g(X)]$ and $g[E(X)]$ are available, as discussed in Theorem 6·12 (Jenson's Inequality) page 6·15.

6·5. Expectation of a Linear Combination of Random Variables

Let $X_1, X_2, ..., X_n$ be any n random variables and if $a_1, a_2, ..., a_n$ are any n constants, then

$$E\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i E(X_i) \qquad ...(6{\cdot}25)$$

provided all the expectations exist.

Proof. The result is obvious from (6·13) and (6·20).

Theorem 6·5 (a). *If $X \geq 0$ then $E(X) \geq 0$.*

Proof. If X is a continuous random variable s.t. $X \geq 0$ then

$$E(X) = \int_{-\infty}^{\infty} x \cdot p(x)\, dx = \int_{0}^{\infty} x \cdot p(x)\, dx > 0,$$

$$[\because \text{If } X \geq 0,\ p(x) = 0 \text{ for } x < 0]$$

provided the expectation exists.

Theorem 6·5 (b). *Let X and Y be two random variables such that $Y \leq X$ then*

$$E(Y) \leq E(X),$$

provided the expectations exist.

Proof. Since $Y \leq X$, we have the r.v.

$$Y - X \leq 0 \quad \Rightarrow \quad X - Y \geq 0$$

Hence $$E(X-Y) \geq 0 \quad \Rightarrow \quad E(X) - E(Y) \geq 0$$

$$\Rightarrow \quad E(X) \geq E(Y) \quad \Rightarrow \quad E(Y) \leq E(X),$$

as desired.

Theorem 6·6. $|E(X)| \leq E|X|$, ...(6·26)

provided the expectations exist.

Proof. Since $X \leq |X|$, we have by Theorem 6·5(*b*)

$$E(X) \leq E|X| \quad ...(*)$$

Again since $-X \leq |X|$, we have by Theorem 6·5(*b*)

$$E(-X) \leq E|X|$$

$$\Rightarrow \quad -E(X) \leq E|X| \quad ...(**)$$

From (*) and (**), we get the desired result $|E(X)| \leq E|X|$.

Theorem 6·7. *If μ_r' exists, then μ_s' exists for all $1 \leq s \leq r$.*

Mathematically, if $E(X^r)$ exists, then $E(X^s)$ exists for all $1 \leq s \leq r$, i.e.,

$$E(X^r) < \infty \Rightarrow E(X^s) < \infty \quad \forall \quad 1 \leq s \leq r \quad ...(6·27)$$

Proof. $$\int_{-\infty}^{\infty} |x|^s \, dF(x) = \int_{-1}^{1} |x|^s \, dF(x) + \int_{|x|>1} |x|^s \, dF(x)$$

If $s < r$, then $|x|^s < |x|^r$ for $|x| > 1$.

$$\therefore \int_{-\infty}^{\infty} |x|^s \, dF(x) \leq \int_{-1}^{1} |x|^s \, dF(x) + \int_{|x|>1} |x|^r \, dF(x)$$

$$\leq \int_{-1}^{1} dF(x) + \int_{|x|>1} |x|^r \, dF(x),$$

since for $-1 < x < 1$, $|x|^s < 1$.

$$\therefore \int_{-\infty}^{\infty} |x|^s \, dF(x) \leq 1 + E|X|^r < \infty$$

$$\Rightarrow \quad E(x^s) \text{ exists} \ \forall \ 1 \leq s \leq r \qquad [\because E(X^r) \text{ exists}]$$

Remark. The above theorem states that if the moments of a specified order exist, then all the lower order moments automatically exist. However, the converse is not true, *i.e.*, we may have distributions for which all the moments of a specified order exist but no higher order moments exist. For example, for the r.v. with p.d.f.

$$p(x) = 2/x^3 \; ; \; x \geq 1$$
$$= 0 \; ; \; x < 1$$

we have :

$$E(X) = \int_{1}^{\infty} x \, p(x) \, dx = 2 \int_{1}^{\infty} x^{-2} \, dx = \left| \left(\frac{-2}{x} \right) \right|_{1}^{\infty} = 2$$

$$E(X^2) = \int_1^\infty x^2\, p(x)\, dx = 2\int_1^\infty \frac{1}{x}\, dx = \infty$$

Thus for the above distribution, 1st order moment (mean) exists but 2nd order moment (variance) does not exist.

As another illustration, consider a r.v. X with p.d.f.

$$p(x) = \frac{(r+1)\, a^{r+1}}{(x+a)^{r+2}}\ ; \ x \ge 0\ ; \quad a > 0$$

$$\mu_r' = E(X^r) = (r+1)\, a^{r+1} \int_0^\infty \frac{x^r}{(x+a)^{r+2}}\, dx$$

Put $x = ay$ and using Beta integral :

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} = \beta(m, n),$$

we shall get on simplification :

$$\mu_r' = (r+1)\, a^r \cdot \beta(r+1, 1) = a^r$$

However,

$$\mu_{r+1}' = E\left(X^{r+1}\right) = (r+1)\, a^{r+1} \int_0^\infty \frac{x^{r+1}}{(x+a)^{r+2}}\, dx \to \infty,$$

as the intergal is not convergent. Hence in this case only the moments up to rth order exist and higher order moments do not exist.

Theorem 6·8. *If X is a random variable, then*

$$V(aX+b) = a^2\, V(X), \qquad \text{...(6·28)}$$

where a and b are constants.

Proof. Let $\quad Y = aX + b$

Then $\quad E(Y) = a\,E(X) + b$

$\therefore \quad Y - E(Y) = a\{X - E(X)\}$

Squaring and taking expectation of both sides, we get

$$E\{Y - E(Y)\}^2 = a^2\, E\{X - E(X)\}^2$$

$\Rightarrow \quad V(Y) = a^2\, V(X) \quad \Rightarrow \quad V(aX+b) = a^2\, V(X),$

where $V(X)$ is written for variance of X.

Cor. *(i)* If $b = 0$, then $V(aX) = a^2\, V(X)$ ...(6·28*a*)

$\Rightarrow$ Variance is not independent of change of scale .

(ii) If $a = 0$, then $V(b) = 0$ ...(6·28*b*)

$\Rightarrow$ Variance of a constant in zero.

(iii) If $a = 1$, then $V(X+b) = V(X)$ ...(6·28*c*)

$\Rightarrow$ Variance is independent of change of origin.

6·6. Covariance: *If X and Y are two random variables, then covariance between them is defined as*

$$Cov\ (X, Y) = E\left[\{X - E(X)\}\{Y - E(Y)\}\right] \quad ...(6·29)$$

$$= E[XY - XE(Y) - YE(X) + E(X)E(Y)]$$

$$= E(XY) - E(Y)E(X) - E(X)E(Y) + E(X)E(Y)$$

$$= E(XY) - E(X)E(Y) \quad ...(6·29a)$$

If X and Y are independent then $E(XY) = E(X)E(Y)$ and hence in this case

$$Cov\ (X, Y) = E(X)E(Y) - E(X)E(Y) = 0 \quad ...(6·29b)$$

Remarks. 1. $Cov\ (aX, bY) = E\left[\{aX - E(aX)\}\{bY - E(bY)\}\right]$

$$= E\left[a\{X - E(X)\}b\{Y - E(Y)\}\right]$$

$$= ab\, E\left[\{X - E(X)\}\{Y - E(Y)\}\right]$$

$$= ab\ Cov\ (X, Y) \quad ...(6·30)$$

2. $$Cov\ (X + a,\ Y + b) = Cov\ (X, Y) \quad ...(6·30a)$$

3. $$Cov\left(\frac{X - \overline{X}}{\sigma_X},\ \frac{Y - \overline{Y}}{\sigma_Y}\right) = \frac{1}{\sigma_X\ \sigma_Y}\ Cov\ (X, Y) \quad ...(6·30b)$$

4. Similarly, we shall get :

$$Cov\ (aX + b,\ cY + d) = ac\ Cov\ (X, Y) \quad ...(6·30c)$$

$$Cov\ (X + Y, Z) = Cov\ (X, Z) + Cov\ (Y, Z) \quad ...(6·30d)$$

$$Cov\ (aX + bY,\ cX + dY) = ac\sigma_X^{\ 2} + bd\sigma_Y^{\ 2} + (ad + bc)\ Cov\ (X, Y) \quad ...(6·30e)$$

If X and Y are independent, $Cov\ (X, Y) = 0$; [c.f. (6·29*b*)]

However, the converse is not true.

(For details see Theorem 10·2)

6·6·1. Correlation Coefficient. The correlation coefficient (ρ_{XY}), between the variables X and Y is defined as :

$$\rho_{XY} = Correlation\ Coefficient\ (X, Y) = \frac{Cov\ (X, Y)}{\sigma_X\ \sigma_Y} \quad ...(6·30f)$$

For detailed discussion on correlation coefficient, see Chapter 10.

6·7. Variance of a Linear Combination of Random Variables

Theorem 6·9. *Let* $X_1, X_2, ..., X_n$ *be n random variables then*

$$V\left[\sum_{i=1}^{n} a_i\ X_i\right] = \sum_{i=1}^{n} a_i^{\ 2}\ V(X_i) + 2\ \sum_{i=1}^{n}\ \sum_{\substack{j=1 \\ i<j}}^{n} a_i\, a_j\ Cov\ (X_i, X_j) \quad ...(6·31)$$

Proof. Let $U = a_1 X_1 + a_2 X_2 + \ldots + a_n X_n$

$\therefore \quad E(U) = a_1 E(X_1) + a_2 E(X_2) + \ldots + a_n E(X_n)$

$\therefore \quad U - E(U) = a_1[X_1 - E(X_1)] + a_2[X_2 - E(X_2)] + \ldots + a_n[X_n - E(X_n)]$

Squaring and taking expectation of both sides, we get

$$E[U - E(U)]^2 = a_1^2 E[X_1 - E(X_1)]^2 + a_2^2 E[X_2 - E(X_2)]^2 + \dots + a_n^2 E[X_n - E(X_n)]^2 + 2 \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i<j}}^{n} a_i a_j E[\{X_i - E(X_i)\}\{X_j - E(X_j)\}]$$

$$\Rightarrow \quad V(U) = a_1^2 V(X_1) + a_2^2 V(X_2) + \dots + a_n^2 V(X_n) + 2 \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i<j}}^{n} a_i a_j \, Cov(X_i, X_j)$$

$$\Rightarrow V\left[\sum_{i=1}^{n} a_i X_i\right] = \sum_{i=1}^{n} a_i^2 V(X_i) + 2 \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i<j}}^{n} a_i a_j \, Cov(X_i, X_j)$$

Remarks. 1. If $a_i = 1$; $i = 1, 2, \dots, n$ then

$$V(X_1 + X_2 + \dots + X_n) = V(X_1) + V(X_2) + \dots + V(X_n) + 2 \sum_{i=1}^{n} \sum_{\substack{j=1 \\ i<j}}^{n} Cov(X_i, X_j) \quad \dots(6{\cdot}31a)$$

2. If $X_1, X_2, \dots, X_n$ are independent (pairwise) then $Cov(X_i, X_j) = 0$, $(i \neq j)$. Thus from (6·31) and (6·31a), we get

$$\left.\begin{aligned} V(a_1X_1 + a_2X_2 + \dots + a_nX_n) &= a_1^2V(X_1) + a_2^2V(X_2) + \dots + a_n^2V(X_n) \\ \text{and} \quad V(X_1 + X_2 + \dots + X_n) &= V(X_1) + V(X_2) + \dots + V(X_n) \end{aligned}\right\} \quad \dots(6{\cdot}31b)$$

3. If $a_1 = 1 = a_2$ and $a_3 = a_4 = \dots = a_n = 0$, then from (6·31), we get

$$V(X_1 + X_2) = V(X_1) + V(X_2) + 2\,Cov(X_1, X_2)$$

Again if $a_1 = 1$, $a_2 = -1$ and $a_3 = a_4 = \dots = a_n = 0$, then

$$V(X_1 - X_2) = V(X_1) + V(X_2) - 2\,Cov(X_1, X_2)$$

Thus we have

$$V(X_1 \pm X_2) = V(X_1) + V(X_2) \pm 2\,Cov(X_1, X_2) \quad \dots(6{\cdot}31c)$$

If X_1 and X_2 are independent, then $Cov(X_1, X_2) = 0$ and we get

$$V(X_1 \pm X_2) = V(X_1) + V(X_2) \quad \dots(6{\cdot}31d)$$

Theorem 6·10. *If X and Y are independent random variables then*

$$E[h(X) \cdot k(Y)] = E[h(X)]\, E[k(Y)] \quad \dots(6{\cdot}32)$$

where h (.) is a function of X alone and k (.) is a function of Y alone, provided expectations on both sides exist.

Proof. Let $f_X(x)$ and $g_Y(y)$ be the marginal p.d.f.'s of X and Y respectively. Since X and Y are independent, their joint p.d.f. $f_{XY}(x, y)$ is given by

$$f_{XY}(x, y) = f_X(x)\, f_Y(y) \quad \dots(*)$$

By definition, for continuous r.v.'s

$$E[h(X) \cdot k(Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x)\, k(y)\, f(x, y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x)\, k(y)\, f(x)\, g(y)\, dx\, dy$$

[From (*)]

Since $E[h(X)\; k(Y)]$ exists, the integral on the right hand side is absolutely convergent and hence by Fubini's theorem for integrable functions we can change the order of integration to get

$$E[h(X) \cdot k_1(Y)] = \left[\int_{-\infty}^{\infty} h(x)\, f(x)\, dx\right]\left[\int_{-\infty}^{\infty} k(y)\, g(y)\, dy\right]$$

$$= E[h(X)] \cdot E[k(Y)],$$

as desired.

Remark. The result can be proved for discrete random variables X and Y on replacing integration by summation over the given range of X and Y.

Theorem 6·11. Cauchy-Schwartz Inequality. *If X and Y are random variables taking real values, then*

$$[E(XY)]^2 \le E(X^2) \cdot E(Y^2) \qquad \text{...(6.33)}$$

Proof. Let us consider a real valued function of the real variable t, defined by

$$Z(t) = E(X + tY)^2$$

which is always non-negative, since $(X + tY)^2 \ge 0$, for all real X, Y and t.

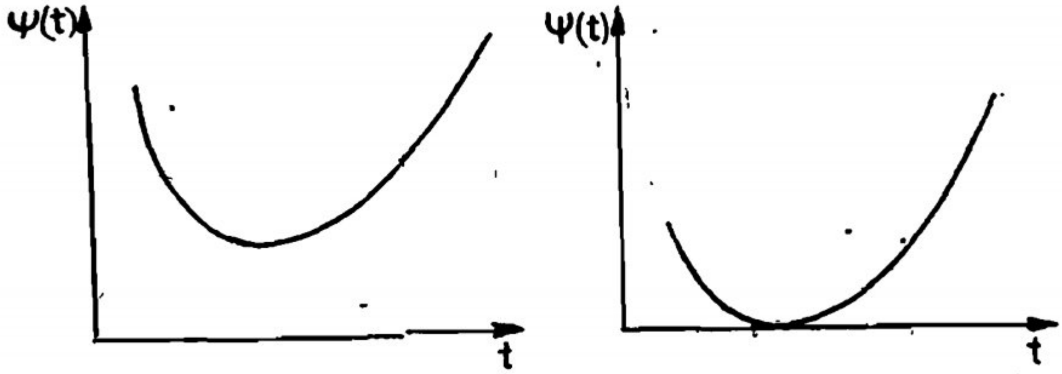
Thus $\quad Z(t) = E(X + tY)^2 \ge 0 \quad \forall \quad t.$

$\Rightarrow \quad Z(t) = E[X^2 + 2t\, XY + t^2 Y^2]$

$\qquad = E(X^2) + 2t \cdot E(XY) + t^2 E(Y^2) \ge 0$, for all t. $\qquad (*)$

Obviously, $Z(t)$ is a quadratic expression in 't'.

We know that the quadratic expression of the form :



3. If $M_X(t) = E(e^{tX})$ exists for all t and for some r.v. X, then

$$M_{X(u+v)} = E[e^{(u+v)X}] = E(e^{uX} . e^{vX})$$
$$\geq E(e^{uX}) . E(e^{vX})$$
$$= M_X(u) . M_X(v)$$
$$\therefore \quad M_X(u+v) \geq M_X(u) . M_X(v), \quad \text{for } u, v \geq 0.$$

Example 6·1. *Let X be a random variable with the following probability distribution :*

x	:	-3	6	9
$P_r(X = x)$	:	$1/6$	$1/2$	$1/3$

Find $E(X)$ and $E(X^2)$ and using the laws of expectation, evaluate $E(2X+1)^2$.

(Gauhati Univ. B.Sc., 1992)

Solution. $E(X) = \Sigma x . p(x)$

$$= (-3) \times \frac{1}{6} + 6 \times \frac{1}{2} + 9 \times \frac{1}{3} = \frac{11}{2}$$

$$E(X^2) = \Sigma x^2 p(x)$$
$$= 9 \times \frac{1}{6} + 36 \times \frac{1}{2} + 81 \times \frac{1}{3} = \frac{93}{2}$$

$$\therefore \quad E(2X+1)^2 = E[4X^2 + 4X + 1] = 4E(X^2) + 4E(X) + 1$$
$$= 4 \times \frac{93}{2} + 4 \times \frac{11}{2} + 1 = 209$$

Example 6·2. **(a)** *Find the expectation of the number on a die when thrown.*

(b) *Two unbiased dice are thrown. Find the expected values of the sum of numbers of points on them.*

Solution. *(a)* Let X be the random variable representing the number on a die when thrown. Then X can take any one of the values 1,2,3,..., 6 each with equal probability 1/6. Hence

$$E(X) = \frac{1}{6} \times 1 + \frac{1}{6} \times 2 + \frac{1}{6} \times 3 + \ldots + \frac{1}{6} \times 6$$
$$= \frac{1}{6}(1 + 2 + 3 + \ldots + 6) = \frac{1}{6} \times \frac{6 \times 7}{2} = \frac{7}{2} \qquad \ldots(*)$$

Remark. This does not mean that in a random throw of a dice, the player will get the number $(7/2) = 3{\cdot}5$. In fact, one can never get this (fractional) number in a throw of a dice. Rather, this implies that if the player tosses the dice for a "long" period, then on the average toss he will get $(7/2) = 3{\cdot}5$.

(b) The probability function of X (the sum of numbers obtained on two dice), is

Value of X : x	2	3	4	5	6	7	.....	11	12
Probability	1/36	2/36	3/36	4/36	5/36	6/36	.....	2/36	1/36

$$E(X) = \sum_i p_i x_i$$

$$= 2 \times \frac{1}{36} + 3 \times \frac{2}{36} + 4 \times \frac{3}{36} + 5 \times \frac{4}{36} + 6 \times \frac{5}{36} + 7 \times \frac{6}{36}$$
$$+ 8 \times \frac{5}{36} + 9 \times \frac{4}{36} + 10 \times \frac{3}{36} + 11 \times \frac{2}{36} + 12 \times \frac{1}{36}$$
$$= \frac{1}{36}(2 + 6 + 12 + 20 + 30 + 42 + 40 + 36 + 30 + 22 + 12)$$
$$= \frac{1}{36} \times 252 = 7$$

Aliter. Let X_i be the number obtained on the ith dice ($i = 1, 2$) when thrown. Then the sum of the number of points on two dice is given by

$$S = X_1 + X_2$$

$$\Rightarrow \quad E(S) = E(X_1) + E(X_2) = \frac{7}{2} + \frac{7}{2} = 7 \qquad \text{[On using (*)]}$$

Remark. This result can be generalised to the sum of points obtained in a random throw of n dice. Then

$$E(S) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} (7/2) = \frac{7n}{2}$$

Example 6·3. *A box contains 2^n tickets among which ${}^nC_i$ tickets bear the number i ; $i = 0, 1, 2 \ldots, n$. A group of m tickets is drawn. What is the expectation of the sum of their numbers ?*

Solution. Let X_i ; $i = 1, 2, \ldots, m$ be the variable representing the number on the ith ticket drawn. Then the sum 'S' of the numbers on the tickets drawn is given by

$$S = X_1 + X_2 + \ldots + X_m = \sum_{i=1}^{m} X_i$$

$$\therefore \quad E(S) = \sum_{i=1}^{m} E(X_i)$$

Now X_i is a random variable which can take any one of the possible values 0, 1, 2, ..., n with respective probabilities.

$${}^nC_0/2^n,\ {}^nC_1/2^n,\ {}^nC_2/2^n,\ \ldots,\ {}^nC_n/2^n,$$

$$\therefore \quad E(X_i) = \frac{1}{2^n}\left[1.{}^nC_1 + 2.{}^nC_2 + 3.{}^nC_3 + \ldots + n.{}^nC_n\right]$$
$$= \frac{1}{2^n}\left[1.n + 2.\frac{n(n-1)}{2!} + 3.\frac{n(n-1)(n-2)}{3!} + \ldots + n.1\right]$$
$$= \frac{n}{2^n}\left[1 + (n-1) + \frac{(n-1)(n-2)}{2!} + \ldots + 1\right]$$
$$= \frac{n}{2^n}\left[{}^{n-1}C_0 + {}^{n-1}C_1 + {}^{n-1}C_2 + \ldots + {}^{n-1}C_{n-1}\right]$$
$$= \frac{n}{2^n}.(1+1)^{n-1} = \frac{n}{2}$$

Hence $\quad E(S) = \sum_{i=1}^{m} (n/2) = \dfrac{m \cdot n}{2}$

Example 6·4. *In four tosses of a coin, let X be the number of heads. Tabulate the 16 possible outcomes with the corresponding values of X. By simple counting, derive the distribution of X and hence calculate the expected value of X.*

Solution. Let H represent a head, T a tail and X, the random variable denoting the number of heads.

S. No.	*Outcomes*	*No. of Heads (X)*	*S. No.*	*Outcomes*	*No. of Heads (X)*
1	*H H H H*	4	9	*H T H T*	2
2	*H H H T*	3	10	*T H T H*	2
3	*H H T H*	3	11	*T H H T*	2
4	*H T H H*	3	12	*H T T T*	1
5	*T H H H*	3	13	*T H T T*	1
6	*H H T T*	2	14	*T T H T*	1
7	*H T T H*	2	15	*T T T H*	1
8	*T T H H*	2	16	*T T T T*	0

The random variable X takes the values 0, 1, 2, 3 and 4. Since, from the above table, we find that the number of cases favourable to the coming of 0, 1, 2, 3 and 4 heads are 1, 4, 6, 4 and 1 respectively, we have

$$P(X=0) = \frac{1}{16},\quad P(X=1) = \frac{4}{16} = \frac{1}{4},\quad P(X=2) = \frac{6}{16} = \frac{3}{8},$$

$$P(X=3) = \frac{4}{16} = \frac{1}{4} \quad \text{and} \quad P(X=4) = \frac{1}{16}.$$

Thus the probability distribution of X can be summarised as follows :

x :	0	1	2	3	4
$p(x)$:	$\frac{1}{16}$	$\frac{1}{4}$	$\frac{3}{8}$	$\frac{1}{4}$	$\frac{1}{16}$

$$E(X) = \sum_{x=0}^{4} x\,p(x) = 1 \cdot \frac{1}{4} + 2 \cdot \frac{3}{8} + 3 \cdot \frac{1}{4} + 4 \cdot \frac{1}{16}$$

$$= \frac{1}{4} + \frac{3}{4} + \frac{3}{4} + \frac{1}{4} = 2.$$

Example 6·5. *A coin is tossed until a head appears. What is the expectation of the number of tosses required ?* **[Delhi Univ. B.Sc., Oct. 1989]**

Solution. Let X denote the number of tosses required to get the first head. Then X can materialise in the following ways :

$$\therefore \qquad E(X) = \sum_{x=1}^{\infty} x\,p(x)$$

Event	x	*Probability* $p(x)$
H	1	$1/2$
TH	2	$1/2 \times 1/2 = 1/4$
TTH	3	$1/2 \times 1/2 \times 1/2 = 1/8$
⋮	⋮	⋮

$$= 1 \times \frac{1}{2} + 2 \times \frac{1}{4} + 3 \times \frac{1}{8} + 4 \times \frac{1}{16} + \ldots \qquad \ldots(*)$$

This is an arithmetic-geometric series with ratio of GP being $r = 1/2$.

Let $\qquad S = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + 4 \cdot \frac{1}{16} + \ldots$

Then $\qquad \frac{1}{2} S = \qquad \frac{1}{4} + 2 \cdot \frac{1}{8} + 3 \cdot \frac{1}{16} + \ldots$

$\therefore \quad (1 - \frac{1}{2}) S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$

$\Rightarrow \qquad \frac{1}{2} S = \frac{1/2}{1 - (1/2)} = 1$

[Since the sum of an infinite G.P. with first term a and common ratio r (< 1) is $a/(1 - r)$]

$\Rightarrow \qquad S = 2$

Hence, substituting in (*), we get

$\qquad E(X) = 2$

Example 6·6. *What is the expectation of the number of failures preceding the first success in an infinite series of independent trials with constant probability p of success in each trial ?* **[Delhi Univ. B.Sc., Oct. 1991]**

Solution. Let the random variable X denote the number of failures preceding the first success. Then X can take the values 0, 1, 2, ..., ∞. We have

$p(x) = P(X = x) = P[x$ failures precede the first success $] = q^x p$

where $q = 1 - p$ is the probability of failure in a trial. Then by def.

$$E(X) = \sum_{x=0}^{\infty} x\, p(x) = \sum_{x=0}^{\infty} x \cdot q^x p = pq \sum_{x=1}^{\infty} x\, q^{x-1}$$

$$= pq\,[1 + 2q + 3q^2 + 4q^3 + \ldots]$$

Now $1 + 2q + 3q^2 + 4q^3 + \ldots$ is an infinite arithmetic-geometric series.

Let $\qquad S = 1 + 2q + 3q^2 + 4q^3 + \ldots$

$\qquad qS = \qquad q + 2q^2 + 3q^3 + \ldots$

$\therefore \qquad (1 - q) S = 1 + q + q^2 + q^3 + \ldots = \frac{1}{1 - q}$

$\Rightarrow \qquad S = \frac{1}{(1 - q)^2}$

$$\therefore \qquad 1 + 2q + 3q^2 + 4q^3 + \ldots = \frac{1}{(1-q)^2}$$

$$\text{Hence} \qquad E(X) = \frac{pq}{(1-q)^2} = \frac{pq}{p^2} = \frac{q}{p}$$

Example 6·7. *A box contains 'a' white and 'b' black balls. 'c' balls are drawn. Find the expected value of the number of white balls drawn.*

[Allahabad Univ. B.Sc., 1989; Indian Forest Service 1987]

Solution. Let a variable X_i, associated with ith draw, be defined as follows:

$X_i = 1$, if ith ball drawn is white

and $X_i = 0$, if ith ball drawn is black

Then the number 'S' of the white balls among 'c' balls drawn is given by

$$S = X_1 + X_2 + \ldots + X_c = \sum_{i=1}^{c} X_i \quad \Rightarrow \quad E(S) = \sum_{i=1}^{c} E(X_i)$$

Now $\quad P(X_i = 1) = P(\text{of drawing a white ball}) = \dfrac{a}{a+b}$

and $\quad P(X_i = 0) = P(\text{of drawing a black ball}) = \dfrac{b}{a+b}$

$$\therefore \qquad E(X_i) = 1 \,.\, P(X_i = 1) + 0 \,.\, P(X_i = 0) = \frac{a}{a+b}$$

$$\text{Hence} \qquad E(S) = \sum_{i=1}^{c} \left(\frac{a}{a+b} \right) = \frac{ca}{a+b}$$

Example 6·8. *Let variate X have the distribution*

$$P(X = 0) = P(X = 2) = p\,;\ P(X = 1) = 1 - 2p, \ \text{ for } 0 \le p \le \tfrac{1}{2}.$$

For what p is the Var (X) a maximum ?

[Delhi Univ. B.Sc. (Maths Hons.) 1987, 85]

Solution. Here the r.v. X takes the values 0, 1 and 2 with respective probabilities p, $1 - 2p$ and p, $0 \le p \le \frac{1}{2}$.

$$\therefore \qquad E(X) = 0 \times p + 1 \times (1 - 2p) + 2 \times p = 1$$

$$E(X^2) = 0 \times p + 1^2 \times (1 - 2p) + 2^2 \times p = 1 + 2p$$

$$\therefore \qquad Var(X) = E(X^2) - [E(X)]^2 = 2p\ ; \quad 0 \le p \le \tfrac{1}{2}$$

Obviously $Var(X)$ is maximum when $p = \frac{1}{2}$, and

$$[Var(X)]_{\max} = 2 \times \tfrac{1}{2} = 1$$

Example 6·9. $Var(X) = 0 \quad \Rightarrow \quad P[X = E(X)] = 1$. Comment.

Solution. $\quad Var(X) = E[X - E(X)]^2 = 0$

$\Rightarrow \qquad [X - E(X)]^2 = 0$, with probability 1

$\Rightarrow \qquad [X - E(X)] = 0$, with probability 1

$\Rightarrow \qquad P[X = E(X)] = 1$

Example 6·10. *Explain by means of an example that a probability distribution is not uniquely determined by its moments.*

Solution. Consider a r.v. X with p.d.f. [*c.f.* Log-Normal distribution § 8·2·15

$$f(x) = \frac{1}{\sqrt{2\pi}\,x} \cdot \exp\left[-\tfrac{1}{2}(\log x)^2\right] \; ; \; x > 0 \qquad \ldots(*)$$
$$= 0 \; ; \quad \text{otherwise}$$

Consider, another r. v. Y with p.d.f.

$$g(y) = \left[1 + a \sin(2\pi \log y)\right] f(y) = g_a(y), \text{(say)}, \quad y > 0 \quad \ldots(**)$$

which, for $-1 \le a \le 1$, represents a family of probability distributions.

$$E(Y^r) = \int_0^\infty y^r \left\{1 + a \sin(2\pi \log y)\right\} f(y)\, dy$$

$$= \int_0^\infty y^r f(y)\, dy + a \cdot \int_0^\infty y^r \cdot \sin(2\pi \log y) f(y)\, dy.$$

$$= EX^r + a \cdot \frac{1}{\sqrt{2\pi}} \int_0^\infty y^r \cdot \sin(2\pi \log y) \cdot \frac{1}{y} \exp\left[-\frac{1}{2}(\log y)^2\right] dy$$

$$= EX^r + \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{rz - z^2/2} \cdot \sin(2\pi z)\, dz$$

$$\left[\log y = z \;\Rightarrow\; y = e^z\right]$$

$$= EX^r + \frac{a \cdot e^{r^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-r)^2} \cdot \sin(2\pi z)\, dz$$

$$= EX^r + \frac{a \cdot e^{r^2/2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} \cdot \sin(2\pi y)\, dy$$

[$z - r = y \;\Rightarrow\; \sin(2\pi z) = \sin(2\pi r + 2\pi y) = \sin 2\pi y$, r being a positive integer].

$$= EX^r,$$

the value of the integral being zero, since the integrand is an odd function of y.

$\Rightarrow$ $E(Y^r)$ is independent of 'a' in (**).

Hence, $\{ g(y) = g_a(y);\ -1 \le a \le 1 \}$, represents a family of distributions, each different from the other, but having the same moments. This explains that the moments may not determine a distribution uniquely.

Example 6·11. *Starting from the origin, unit steps are taken to the right with probability p and to the left with probability q ($= 1 - p$). Assuming independent movements, find the mean and variance of the distance moved from origin after n steps (Random Walk Problem).*

Solution. Let us associate a variable X_i with the ith step defined as follows :

$X_i = +1$, if the ith step is towards the right,

$= -1$, if the ith step is towards the left.

Then $S = X_1 + X_2 + \ldots + X_n = \Sigma X_i$, represents the random distance moved from origin after n steps.

$$E(X_i) = 1 \times p + (-1) \times q = p - q$$

$$E(X_i^2) = 1^2 \times p + (-1)^2 \times q = p + q = 1$$

$$\therefore \quad Var(X_i) = E(X_i^2) - [E(X_i)]^2 = (q+p)^2 - (p-q)^2 = 4pq$$

$$\therefore \quad E(S_n) = \sum_{i=1}^{n} E(X_i) = n(p-q)$$

$$V(S_n) = \sum_{i=1}^{n} V(X_i) = 4npq$$

[$\because$ *Movements of steps are independent*].

Example 6·12. *Let r.v. X have a density function $f(\cdot)$, cumulative distribution function $F(\cdot)$, mean μ and variance σ^2. Define $Y = \alpha + \beta X$, where α and β are constants satisfying $-\infty < \alpha < \infty$ and $\beta > 0$.*

(a) Select α and β so that Y has mean 0 and variance 1.

(b) What is the correlation coefficient ρ_{XY} between X and Y ?

(c) Find the cumulative distribution function of Y in terms of α, β and $F(\cdot)$.

(d) If X is symmetrically distributed about μ, is Y necessarily symmetrically distributed about its mean?

Solution. *(a)* $E(X) = \mu$, $Var(X) = \sigma^2$. We want α and β s.t.

$$E(Y) = E(\alpha + \beta X) = \alpha + \beta\mu = 0 \qquad ...(1)$$

$$Var(Y) = Var(\alpha + \beta X) = \beta^2 . \sigma^2 = 1 \qquad ...(2)$$

Solving *(1)* and *(2)* we get :

$$\beta = 1/\sigma, \ (>0) \text{ and } \alpha = -\mu/\sigma \qquad ...(3)$$

(b) $Cov(X, Y) = E(XY) - E(X)E(Y) = E[X(\alpha + \beta X)]$

[$\because E(Y) = 0$]

$$= \alpha . E(X) + \beta . E(X^2) = \alpha\mu + \beta[\sigma^2 + \mu^2]$$

$$\therefore \quad \rho_{XY} = \frac{Cov(X, Y)}{\sigma_X \sigma_Y} = \frac{\alpha\mu + \beta[\sigma^2 + \mu^2]}{\sigma . 1} \qquad (\because \sigma_Y = 1)$$

$$= \frac{1}{\sigma^2}\left[-\mu^2 + \sigma^2 + \mu^2\right] = 1 \qquad [\textit{On using (3)}]$$

(c) Distribution function $G_Y(\cdot)$ of Y is given by :

$$G_Y(y) = P(Y \le y) = P[\alpha + \beta X \le y]$$

$$= P(X \le (y - \alpha)/\beta)$$

$$\Rightarrow \quad G_Y(y) = F_X\left(\frac{y - \alpha}{\beta}\right);$$

(d) We have : $Y = \alpha + \beta X = \frac{1}{\sigma}(X - \mu) = \beta(X - \mu)$ [*On using (3)*]

Since X is given to be symmetrically distributed about mean μ, $(X-\mu)$ and $-(X-\mu)$ have the same distribution.

Hence $Y = \beta(X-\mu)$ and $-Y = -\beta(X-\mu)$ have the same distribution. Since $E(Y) = 0$, we conclude that Y is symmetrically distributed about its mean.

Example 6·13. *Let X be a r.v. with mean μ and variance σ^2. Show that $E(X-b)^2$, as a function of b, is minimised when $b = \mu$.*

Solution.
$$\begin{aligned} E(X-b)^2 &= E[(X-\mu)+(\mu-b)]^2 \\ &= E(X-\mu)^2 + (\mu-b)^2 + 2(\mu-b)E(X-\mu) \\ &= Var(X) + (\mu-b)^2 \qquad [\because E(X-\mu)=0] \end{aligned}$$

$$\Rightarrow \qquad E(X-b)^2 \geq Var(X), \qquad \ldots(*)$$

since $(\mu-b)^2$, being the square of a real quantity is always non-negative.

The sign of equality holds in (*) iff

$$(\mu-b)^2 = 0 \quad \Rightarrow \quad \mu = b.$$

Hence $E(X-b)^2$ is minimised when $\mu = b$ and its minimum value is $E(X-\mu)^2 = \sigma_X{}^2$.

Remark. This result states that the sum of squares of deviations is minimum when taken about mean.

[Also see § 2·4, Property 3 of Arithmetic Mean]

Example 6·14. *Let $a_1, a_2, \ldots, a_n$ be arbitrary real numbers and $A_1, A_2, \ldots, A_n$ be events. Prove that*

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j\, P(A_i A_j) \geq 0$$

[Delhi Univ. B.A. (Spl. Course – Stat. Hons.), 1986]

Solution. Let us define the indicator variable :

$$X_i = I_{A_i} = 1 \quad \text{if } A_i \text{ occurs}$$
$$\qquad = 0 \quad \text{if } \bar{A}_i \text{ occurs}.$$

Then using (6·2b) :

$$E(X_i) = P(A_i)\,; \quad (i = 1, 2, \ldots, n) \qquad \ldots(i)$$

Also $\qquad X_i X_j = I_{A_i \cap A_j}$

$$\Rightarrow \qquad E(X_i X_j) = P(A_i A_j) \qquad \ldots(ii)$$

Consider, for real numbers $a_1, a_2, \ldots, a_n$, the expression $\left(\sum_{i=1}^{n} a_i X_i\right)^2$, which is always non-negative.

$$\Rightarrow \qquad \left(\sum_{i=1}^{n} a_i X_i\right)^2 \geq 0$$

$$\Rightarrow \qquad \left(\sum_{i=1}^{n} a_i X_i\right)\left(\sum_{j=1}^{n} a_j X_j\right) \geq 0$$

$$\Rightarrow \qquad \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j\, X_i X_j \geq 0, \qquad \ldots(iii)$$

for all a_i 's and a_j 's.

Since expected value of a non-negative quantity is always non-negative, on taking expectations of both sides in *(iii)* and using *(i)* and *(ii)* we get :

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \; E(X_i X_j) \geq 0 \;\Rightarrow\; \sum_{i=1}^{n} \sum_{j=1}^{n} a_i a_j \; P(A_i A_j) \geq 0.$$

Example 6·15. *In a sequence of Bernoulli trials, let X be the length of the run of either successes or failures starting with the first trial. Find E (X) and V (X).*

Solution. Let 'p' denote the probability of success. Then $q = 1 - p$ is the probability of failuure. $X = 1$ means that we can have any one of the possibilities SF and FS with respective probabilities pq and qp.

$$\therefore \qquad P(X = 1) = P(SF) + P(FS) = pq + qp = 2pq$$

Similarly

$$P(X = 2) = P(SSF) + P(FFS) = p^2 q + q^2 p$$

In general

$$P(X = r) = P[SSS...SF] + P[FFF...FS] = p^r . q + q^r . p$$

$$\therefore \qquad E(X) = \sum_{r=1}^{\infty} r P(X = r) = \sum_{r=1}^{\infty} r(p^r . q + q^r . p)$$

$$= pq \left[\sum_{r=1}^{\infty} r . p^{r-1} + \sum_{r=1}^{\infty} r . q^{r-1} \right]$$

$$= pq \left[(1 + 2p + 3p^2 + ...) + (1 + 2q + 3q^2 + ...) \right]$$

$$= pq \left[(1 - p)^{-2} + (1 - q)^{-2} \right] = pq \left[q^{-2} + p^{-2} \right]$$

(See Remark to Example 6·17)

$$= pq \left[\frac{1}{q^2} + \frac{1}{p^2} \right] = \frac{p}{q} + \frac{q}{p} .$$

$$V(X) = E(X^2) - \{E(X)\}^2 = E\{X(X - 1)\} + E(X) - \{E(X)\}^2$$

Now

$$E\{X(X-1)\} = \sum_{r=2}^{\infty} r(r-1) P(X = r) = \sum_{r=2}^{\infty} r(r-1)(p^r q + q^r p)$$

$$= \sum_{r=2}^{\infty} r(r-1) p^r q + \sum_{r=2}^{\infty} r(r-1) q^r p$$

$$= p^2 q \sum_{r=2}^{\infty} r(r-1) p^{r-2} + q^2 p \sum_{r=2}^{\infty} r(r-1) q^{r-2}$$

$$= 2p^2 q \sum_{r=2}^{\infty} \frac{r(r-1)}{2} p^{r-2} + 2q^2 p \sum_{r=2}^{\infty} \frac{r(r-1)}{2} q^{r-2}$$

$$= 2p^2 q (1 - p)^{-3} + 2q^2 p (1 - q)^{-3}$$

$$= 2\left(\frac{p^2}{q^2} + \frac{q^2}{p^2} \right)$$

$$V(X) = 2\left(\frac{p^2}{q^2}+\frac{q^2}{p^2}\right)+\left(\frac{p}{q}+\frac{q}{p}\right)-\left(\frac{p}{q}+\frac{q}{p}\right)^2$$

$$= \left(\frac{p}{q}-\frac{q}{p}\right)^2+\left(\frac{p}{q}+\frac{q}{p}\right)$$

Aliter. Proceed as in Example 6·17.

Example 6·16. *A deck of n numbered cards is thoroughly shuffled and the cards are inserted into n numbered cells one by one. If the card number 'i' falls in the cell 'i', we count it as a match, otherwise not. Find the mean and variance of total number of such matches.* **[Delhi Univ. B.Sc., (Stat. Hons.), 1988]**

Solution. Let us associate a random variable, X_i with the ith draw defined as follows :

$$X_i = \begin{cases} 1, & \text{if the } i\text{th card dealt has the number '}i\text{' on it} \\ 0, & \text{otherwise} \end{cases}$$

Then the total number of matches 'S' is given by

$$S = X_1+X_2+\ldots+X_n = \sum_{i=1}^{n} X_i$$

$$\therefore \quad E(S) = \sum_{i=1}^{n} E(X_i)$$

Now $\quad E(X_i) = 1.P(X_i=1)+0.P(X_i=0) = P(X_i=1) = \dfrac{1}{n}$

Hence $\quad E(S) = \displaystyle\sum_{i=1}^{n}\left(\frac{1}{n}\right) = n.\frac{1}{n} = 1$

$$V(S) = V(X_1+X_2+\ldots+X_n)$$

$$= \sum_{i=1}^{n} V(X_i) + 2\sum_{\substack{i,j=1 \\ i<j}}^{n}\sum \text{Cov}(X_i,X_j) \qquad \ldots(1)$$

Now $\quad V(X_i) = E(X_i^2) - \{E(X_i)\}^2$

$$= 1^2.P(X_i=1)+0^2.P(X_i=0)-\left(\frac{1}{n}\right)^2$$

$$= \frac{1}{n}-\frac{1}{n^2} = \frac{n-1}{n^2} \qquad \ldots(2)$$

$$\text{Cov}(X_i,X_j) = E(X_iX_j)-E(X_i)E(X_j) \qquad \ldots(3)$$

$$E(X_iX_j) = 1.P(X_iX_j=1)+0.P(X_iX_j=0)$$

$$= \frac{(n-2)!}{n!} = \frac{1}{n(n-1)},$$

since $X_iX_j = 1$ if and only if both card numbers i and j are in their respective matching places and there are $(n-2)!$ arrangements of the remaining cards that correspond to this event.

Substituting in (3), we get

$$\text{Cov}(X_i, X_j) = \frac{1}{n(n-1)} - \frac{1}{n}\cdot\frac{1}{n} = \frac{1}{n^2(n-1)} \qquad \ldots(4)$$

Substituting from (2) and (4) in (1), we have

$$V(S) = \sum_{i=1}^{n}\left(\frac{n-1}{n^2}\right) + 2\sum_{\substack{i=1\\i<j}}^{n}\sum_{j=1}^{n}\left[\frac{1}{n^2(n-1)}\right]$$

$$= n\left(\frac{n-1}{n^2}\right) + 2.{}^nC_2\,\frac{1}{n^2(n-1)} = \frac{n-1}{n} + \frac{1}{n} = 1$$

Example 6·17. *If t is any positive real number, show that the function defined by*

$$p(x) = e^{-t}(1-e^{-t})^{x-1} \qquad \ldots(*)$$

can represent a probability function of a random variable X assuming the values 1, 2, 3, ... Find the $E(X)$ and $Var(X)$ of the distribution.

[Nagpur Univ. B.Sc., 1988]

Solution. We have

$$e^t > 1,\ \forall\ t>0 \ \Rightarrow\ e^{-t}<1 \ \Rightarrow\ 1 - e^{-t} > 0$$

Also $\quad e^{-t} = \dfrac{1}{e^t} > 0,\ \forall\ t > 0$

Hence $\quad p(x) = e^{-t}(1-e^{-t})^{x-1} \geq 0 \quad \forall\ t>0, x = 1, 2, 3, \ldots$

Also $\quad \displaystyle\sum_{x=1}^{\infty} p(x) = e^{-t}\sum_{x=1}^{\infty}(1-e^{-t})^{x-1} = e^{-t}\sum_{x=1}^{\infty} a^{x-1}; \qquad [a = 1 - e^{-t}]$

$$= e^{-t}(1 + a + a^2 + a^3 + \ldots) = e^{-t}\times\frac{1}{(1-a)}$$

$$= e^{-t}[1-(1-e^{-t})]^{-1} = e^{-t}(e^{-t})^{-1} = 1$$

Hence $p(x)$ defined in (*) represents the probability function of a r.v. X.

$$E(X) = \Sigma x.p(x) = e^{-t}\sum_{x=1}^{\infty} x(1-e^{-t})^{x-1}$$

$$= e^{-t}\sum_{x=1}^{\infty} x.a^{x-1};\ [a = 1 - e^{-t}]$$

$$= e^{-t}(1 + 2a + 3a^2 + 4a^3 + \ldots) = e^{-t}(1-a)^{-2} \qquad \ldots(*)$$

$$= e^{-t}(e^{-t})^{-2} = e^t$$

$$E(X^2) = \Sigma x^2 p(x) = e^{-t}\sum_{x=1}^{\infty} x^2.a^{x-1}$$

$$= e^{-t}[1 + 4a + 9a^2 + 16a^3 + \ldots]$$

$$= e^{-t}(1+a)(1-a)^{-3} = e^{-t}(2-e^{-t})e^{3t}$$

Hence $\quad \text{Var}(X) = E(X^2) - [E(X)]^2 = e^{-t}(2-e^{-t})e^{3t} - e^{2t}$

$$= e^{2t}[(2-e^{-t})-1] = e^{2t}(1-e^{-t})$$

$$= e^t(e^t - 1)$$

Remark.

(*i*) Consider $\quad S = 1 + 2a + 3a^2 + 4a^3 + \ldots$ (Arithmetico-geometric series)

$$\Rightarrow \qquad aS = a + 2a^2 + 3a^3 + \ldots$$

$$\Rightarrow \qquad (1-a)S = 1 + a + a^2 + a^3 + \ldots = 1/(1-a) \Rightarrow S = (1-a)^{-2}$$

$$\sum_{x=1}^{\infty} x\, a^{x-1} = 1 + 2a + 3a^2 + 4a^3 + \ldots = (1-a)^{-2} \qquad \ldots(*)$$

(ii) Consider

$$S = 1 + 2^2 . a + 3^2 . a^2 + 4^2 . a^3 + 5^2 . a^4 + \ldots$$

$$\Rightarrow \qquad S = 1 + 4a + 9a^2 + 16a^3 + 25a^4 + \ldots$$

$$-3a S = -3a - 12a^2 - 27a^3 - 48a^4 - \ldots$$

$$+3a^2 S = + 3a^2 + 12a^3 + 27a^4 + \ldots$$

$$-a^3 S = - a^3 - 4a^4 - \ldots$$

Adding the above equations we get :

$$(1-a)^3 S = 1 + a \qquad \Rightarrow \qquad S = (1+a)(1-a)^{-3} \qquad \ldots(**)$$

$$\sum_{x=1}^{\infty} x^2 a^{x-1} = 1 + 4a + 9a^2 + 16a^3 + \ldots = (1+a)(1-a)^{-3}$$

The results in (*) and (**) are quite useful for numerical problems and should be committed to memory.

Example 6·18. *A man with n keys wants to open his door and tries the keys independently and at random. Find the mean and variance of the number of trials required to open the door (i) if unsuccessful keys are not eliminated from further selection, and (ii) if they are.* **[Rajasthan Univ. B.Sc.(Hons.), 1992]**

Solution. *(i)* Suppose the man gets the first success at the xth trial, *i.e.*, he is unable to open the door in the first $(x-1)$ trials. If unsuccessful keys are not eliminated then X is a random variable which can take the values 1, 2, 3,.... ad infinity.

Probability of success at the first trial = $1/n$

$\therefore$ Probability of failure at the first trial $= 1 - (1/n)$

If unsuccessful keys are not eliminated then the probability of success and consequently of failure is constant for each trial.

Hence $p(x)$ = Probability of 1st success at the xth trial

$$= \left(1 - \frac{1}{n}\right)^{x-1} . \frac{1}{n}$$

$$\therefore \qquad E(X) = \sum_{x=1}^{\infty} x\, p(x) = \sum_{x=1}^{\infty} x \left(1 - \frac{1}{n}\right)^{x-1} . \frac{1}{n}$$

$$= \frac{1}{n} \sum_{x=1}^{\infty} x A^{x-1}, \text{ where } A = 1 - \frac{1}{n}$$

$$E(X) = \frac{1}{n}[1 + 2A + 3A^2 + 4A^3 + \ldots] = \frac{1}{n}(1-A)^{-2}$$

[See (*), Example (6·17)]

$$\frac{1}{n}\left[1-\left(1-\frac{1}{n}\right)\right]^{-2} = n$$

$$E(X^2) = \sum_{x=1}^{\infty} x^2 p(x) = \sum_{x=1}^{\infty} x^2 \left(1-\frac{1}{n}\right)^{x-1} \cdot \frac{1}{n}$$

$$= \frac{1}{n} \sum_{x=1}^{\infty} x^2 A^{x-1}$$

$$= \frac{1}{n}[1 + 2^2 . A + 3^2 . A^2 + 4^2 . A^3 + ...]$$

$$= \frac{1}{n}(1+A)(1-A)^{-3} \qquad \text{[See (**), Example (6·17)]}$$

$$= \frac{1}{n}\left[1+\left(1-\frac{1}{n}\right)\right]\left[1-\left(1-\frac{1}{n}\right)\right]^{-3}$$

$$= (2n-1)\,n$$

Hence $\quad V(X) = E(X^2) - \{E(X)\}^2 = (2n-1)\,n - n^2 = n^2 - n = n(n-1)$

(ii) If unsuccessful keys are eliminated from further selection, then the random variable X will take the values from 1 to n. In this case, we have

Probability of success at the first trial = $1/n$

Probability of success at the 2nd trial = $1/(n-1)$

Probability of success at the 3rd trial = $1/(n-2)$

and so on.

Hence probability of 1st succes at 2nd trial = $\left(1-\frac{1}{n}\right)\frac{1}{n-1} = \frac{1}{n}$

Probability of first success at the third trial

$$= \left(1-\frac{1}{n}\right)\left(1-\frac{1}{n-1}\right)\cdot\frac{1}{n-2} = \frac{1}{n},$$

and so on. In general, we have

$$p(x) = \text{Probability of first success at the } x\text{th trial} = \frac{1}{n}$$

$$\therefore \quad E(X) = \sum_{x=1}^{\infty} x\,p(x) = \frac{1}{n}\sum_{x=1}^{n} x = \frac{n+1}{2}$$

$$E(X^2) = \sum_{x=1}^{n} x^2 p(x) = \frac{1}{n}\sum_{x=1}^{n} x^2 = \frac{(n+1)(2n+1)}{6}$$

Hence $\quad V(X) = E(X^2) - [E(X)]^2 = \dfrac{(n+1)(2n+1)}{6} - \left(\dfrac{n+1}{2}\right)^2$

$$= \frac{n+1}{12}\Big[2(2n+1) - 3(n+1)\Big] = \frac{n^2-1}{12}$$

Example 6·19. *In a lottery m tickets are drawn at a time out of n tickets numbered 1 to n. Find the expectation and the variance of the sum S of the numbers on the tickets drawn.* **[Delhi Univ. B.Sc. (Maths Hons.), 1987]**

Solution. Let X_i denote the score on the ith ticket drawn.

Then $$S = X_1 + X_2 + \dots + X_m = \sum_{i=1}^{m} X_i,$$

is the total score on the m tickets drawn.

$$\therefore \quad E(S) = \sum_{i=1}^{m} E(X_i)$$

Now each X_i is a random variable which assumes the values $1, 2, 3, \dots, n$ each with equal probability $1/n$.

$$\therefore \quad E(X_i) = \frac{1}{n}(1 + 2 + 3 + \dots + n) = \frac{(n+1)}{2}$$

Hence $$E(S) = \sum_{i=1}^{m}\left(\frac{n+1}{2}\right) = \frac{m(n+1)}{2}$$

$$V(S) = V(X_1 + X_2 + \dots + X_m)$$
$$= \sum_{i=1}^{m} V(X_i) + 2 \sum_{\substack{i,j \\ i<j}}\sum \text{Cov}(X_i, X_j)$$

$$E(X_i^2) = \frac{1}{n}\left(1^2 + 2^2 + 3^2 + \dots + n^2\right)$$
$$= \frac{1}{n} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6}$$

$$\therefore \quad V(X_i) = E(X_i^2) - [E(X_i)]^2$$
$$= \frac{(n+1)(2n+1)}{6} - \left(\frac{n+1}{2}\right)^2 = \frac{n^2-1}{12}$$

Also $$\text{Cov}(X_i, X_j) = E(X_i X_j) - E(X_i)E(X_j)$$

To find $E(X_i X_j)$ we note that the variables X_i and X_j can take the values as shown below :

X_i	X_j
1	$2, 3, \dots, n$
2	$1, 3, \dots, n$
$\vdots$	$\vdots$
n	$1, 2, \dots, (n-1)$

Thus the variable $X_i X_j$ can take $n(n-1)$ possible values and $P(X_i = l \cap X_j = k) = \dfrac{1}{n(n-1)}$, $k \neq l$. Hence

$$E(X_i X_j) = \frac{1}{n(n-1)} \begin{bmatrix} 1.2 + 1.3 + \ldots\ldots + 1.n \\ + 2.1 + 2.3 + \ldots\ldots + 2.n \\ + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ + n.1 + n.2 + \ldots + n.(n-1) \end{bmatrix}$$

$$= \frac{1}{n(n-1)} \begin{bmatrix} 1(1 + 2 + 3 + \ldots\ldots + n) - 1^2 \\ + 2(1 + 2 + 3 + \ldots\ldots + n) - 2^2 \\ + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ + n(1 + 2 + \ldots\ldots \overline{n-1} + n) - n^2 \end{bmatrix}$$

$$= \frac{1}{n(n-1)} \left[(1 + 2 + 3 + \ldots + n)^2 - (1^2 + 2^2 + \ldots + n^2) \right]$$

$$= \frac{1}{n(n-1)} \left[\left\{ \frac{n(n+1)}{2} \right\}^2 - \frac{n(n+1)(2n+1)}{6} \right]$$

$$= \frac{(n+1)(3n^2 - n - 2)}{12(n-1)}$$

$$\therefore \quad \text{Cov}(X_i, X_j) = \frac{(n+1)(3n^2 - n - 2)}{12(n-1)} - \left(\frac{n+1}{2} \right)^2$$

$$= \frac{(n+1)}{12(n-1)} \left[3n^2 - n - 2 - 3(n^2 - 1) \right]$$

$$= -\frac{(n+1)}{12}$$

Hence
$$V(S) = \sum_{i=1}^{m} \left(\frac{n^2 - 1}{12} \right) + 2 \sum_{i<j=1}^{m}\sum \left\{ -\frac{(n+1)}{12} \right\}$$

$$= \frac{m(n^2 - 1)}{12} + 2 \cdot \frac{m(m-1)}{2!} \left\{ \frac{-(n+1)}{12} \right\},$$

[since there are ${}^mC_2$ covariance terms in $\text{Cov}(X_i, X_j)$]

$$\therefore \quad V(S) = \frac{m(n+1)}{12} \Big[(n-1) - (m-1) \Big] = \frac{m(n+1)(n-m)}{12}$$

Example 6·20. *A die is thrown $(n + 2)$ times. After each throw a '+' is recorded for 4, 5 or 6 and '–' for 1, 2 or 3, the signs forming an ordered sequence. To each, except the first and the last sign, is attached a characteristic random variable which takes the value 1 if both the neighbouring signs differ from the one between them and 0 otherwise. If $X_1, X_2, \ldots, X_n$ are characteristic random variables, find the mean and variance of $X = \sum_{i=1}^{n} X_i$.*

Solution. $X = \sum_{i=1}^{n} X_i \quad \Rightarrow \quad E(X) = \sum_{i=1}^{n} E(X_i)$

Now $E(X_i) = 1.P(X_i = 1) + 0.P(X_i = 0) = P(X_i = 1)$

For $X_i = 1$, there are the following two mutually exclusive possibilities :

(i) $-\quad+\quad-$, (ii) $+\quad-\quad+$

and since the probability of each sign is $\frac{1}{2}$, we have by addition probability theorem:

$$P(X_i = 1) = P(i) + P(ii) = \left(\tfrac{1}{2}\right)^3 + \left(\tfrac{1}{2}\right)^3 = \tfrac{1}{4}$$

$$\therefore \quad E(X_i) = \frac{1}{4}$$

Hence $$E(X) = \sum_{i=1}^{n}\left(\frac{1}{4}\right) = \frac{n}{4},$$

$$V(X) = V(X_1 + V_2 + \ldots + X_n)$$

$$= \sum_{i=1}^{n} V(X_i) + 2\sum\sum_{i<j} \text{Cov}(X_i, X_j) \qquad \ldots(*)$$

Now $$E(X_i^2) = 1^2.P(X_i = 1) + 0^2.P(X_i = 0) = P(X_i = 1) = \frac{1}{4}$$

$$\therefore \quad V(X_i) = E(X_i^2) - [E(X_i)]^2 = \frac{1}{4} - \frac{1}{16} = \frac{3}{16}$$

Now $$E(X_i X_j) = 1.P(X_i = 1 \cap X_j = 1) + 0.P(X_i = 0 \cap X_j = 0)$$

$$4 + 0.P(X_i = 1 \cap X_j = 0) + 0.P(X_i = 0 \cap X_j = 1)$$

$$= P(X_i = 1 \cap X_j = 1)$$

Since there are the following two mutually exclusive possibilities for the event : $(X_i = 1 \cap X_j = 1)$,

(i) $-\quad+\quad-\quad+$

(ii) $+\quad-\quad+\quad-$, we have

$$P(X_i = 1 \cap X_j = 1) = P(i) + P(ii) = \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^4 = \frac{1}{8}$$

$$\therefore \quad \text{Cov}(X_i, X_j) = E(X_i X_j) - E(X_i)E(X_j)$$

$$= \frac{1}{8} - \frac{1}{4}\times\frac{1}{4} = \frac{1}{16}$$

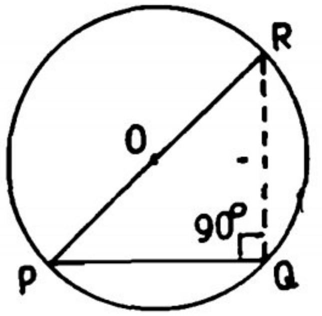
Hence $$V(X) = \sum_{i=1}^{n}(3/16) + 2\sum\sum_{i<j} \text{Cov}(X_i, X_j) \qquad \text{[From (*)]}$$

$$= \frac{3n}{16} + 2\,[\text{Cov}(X_1, X_2) + \text{Cov}(X_2, X_3) + \ldots + \text{Cov}(X_{n-1}, X_n)]$$

$$= \frac{3n}{16} + 2(n-1)\cdot\frac{1}{16} = \frac{5n-2}{16}$$

Example 6·21. *From a point on the circumference of a circle of radius 'a', a chord is drawn in a random direction, (all directions are equally likely). Show that the expected value of the length of the chord is $4a/\pi$ and that the variance of the* *length is* $2a^2(1 - 8/\pi^2)$. *Also show that the chance is 1/3 that the length of the chord will exceed the length of the side of an equilateral triangle inscribed in the circle.*

Solution. Let P be any point on the circumference of a circle of radius 'a' and centre 'O'. Let PQ be any chord drawn at random and let $\angle OPQ = \theta$. Obviously, θ ranges from $-\pi/2$ to $\pi/2$. Since all the directions are equally likely, the probability differential of θ is given by the rectangular distribution (*c.f.* Chapter 8) :



$$dF(\theta) = f(\theta)\, d(\theta) = \frac{d\theta}{\pi/2 - (-\pi/2)}$$
$$= \frac{d\theta}{\pi}, \quad \frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$$

Now, since $\angle PQR$ is a right angle, (angle in a semi-circle), we have

$$\frac{PQ}{PR} = \cos\theta \Rightarrow PQ = PR\cos\theta = 2a\cos\theta$$

$$E(PQ) = \int_{-\pi/2}^{\pi/2} (PQ) f(\theta)\, d(\theta) = \frac{2a}{\pi}\int_{-\pi/2}^{\pi/2} \cos\theta\, d\theta$$
$$= \frac{2a}{\pi}\Big|\sin\theta\Big|_{-\pi/2}^{\pi/2} = \frac{4a}{\pi}$$

$$E[(PQ)^2] = \int_{-\pi/2}^{\pi/2} [PQ]^2 f(\theta)\, d\theta = \frac{4a^2}{\pi}\int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta$$
$$= \frac{4a^2}{\pi}\int_0^{\pi/2} 2\cos^2\theta\, d\theta = \frac{4a^2}{\pi}\int_0^{\pi/2} (1 + \cos 2\theta)\, d\theta$$

(Since $\cos^2\theta$ is an even function of θ)

$$= \frac{4a^2}{\pi}\Big|\theta + \frac{\sin 2\theta}{2}\Big|_0^{\pi/2} = \frac{4a^2}{\pi}\cdot\frac{\pi}{2} = 2a^2$$

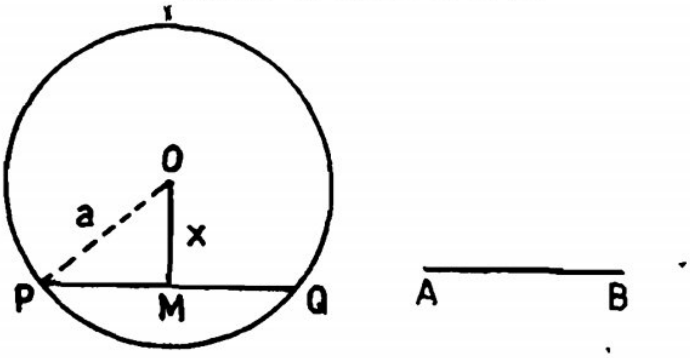
$$\therefore \quad V(PQ) = E[(PQ)^2] - [E(PQ)]^2 = 2a^2 - \frac{16a^2}{\pi^2} = 2a^2\left(1 - \frac{8}{\pi^2}\right)$$

We know that the length of the side of an equilateral triangle inscribed in a circle of radius 'a' is $a\sqrt{3}$. Hence

$$P(PQ > a\sqrt{3}) = P(2a\cos\theta > a\sqrt{3}) = P\left(\cos\theta > \frac{\sqrt{3}}{2}\right)$$
$$= P\left(|\theta| < \frac{\pi}{6}\right) = P\left(\frac{-\pi}{6} < \theta < \frac{\pi}{6}\right)$$
$$= \int_{-\pi/6}^{\pi/6} f(\theta)\, d\theta = \frac{1}{\pi}\int_{-\pi/6}^{\pi/6} 1 \,.\, d\theta = \frac{1}{\pi}\cdot\frac{\pi}{3} = \frac{1}{3}$$

Example 6·22. *A chord of a circle of radius 'a' is drawn parallel to a given straight line, all distances from the centre of the circle being equally likely. Show that the expected value of the length of the chord is $\pi a/2$ and that the variance of the length is $a^2(32 - 3\pi^2)/12$. Also show that the chance is 1/2 that the length of* *the chord will exceed the length of the side of an equilateral triangle inscribed in the circle.*

Solution. Let PQ be the chord of a circle with centre O and radius 'a' drawn at random parallel to the given straight line AB. Draw $OM \perp PQ$. Let $OM=x$. Obviously x ranges from $-a$ to a. Since all distances from the centre are equally



likely, the probability that a random value of x will lie in the small interval 'dx' is given by the rectangular distribution [c.f. Chapter 8]:

$$dF(x) = f(x)\,dx = \frac{dx}{a-(-a)} = \frac{dx}{2a}, \; -a \le x \le a$$

Length of the chord is

$$PQ = 2PM = 2\sqrt{a^2 - x^2}$$

Hence $E(PQ) = \int_{-a}^{a} PQ\, dF(x) = \frac{2}{2a}\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx$

$$= \frac{2}{a}\int_{0}^{a} \sqrt{a^2 - x^2}\, dx,$$

(since integrand is an even function of x).

$$= \frac{2}{a}\left| \frac{1}{2}x\sqrt{a^2 - x^2} + \frac{1}{2}a^2 \sin^{-1}\left(\frac{x}{a}\right)\right|_0^a$$

$$= \frac{2}{a}\cdot\frac{a^2}{2}\cdot\frac{\pi}{2} = \frac{\pi a}{2}$$

$$E[(PQ)^2] = \int_{-a}^{a} (PQ)^2\, dF(x) = \frac{4}{2a}\int_{-a}^{a} (a^2 - x^2)\, dx$$

$$= \frac{4}{a}\int_{0}^{a} (a^2 - x^2)\, dx = \frac{4}{a}\left| a^2 x - \frac{x^3}{3}\right|_0^a$$

$$= \frac{4}{a}\cdot\frac{2a^3}{3} = \frac{8a^2}{3}$$

Hence

$$\text{Var (length of chord)} = E[(PQ)^2] - [E(PQ)]^2 = \frac{8a^2}{3} - \frac{\pi^2 a^2}{4}$$

$$= \frac{a^2}{12}(32 - 3\pi^2)$$

The length of the chord is greater than the side of the equilateral triangle inscribed in the circle if

$$2\sqrt{a^2 - x^2} > a\sqrt{3} \quad \Rightarrow \quad 4(a^2 - x^2) > 3a^2$$

i.e., $$x^2 < a^2/4 \quad \Rightarrow \quad |x| < a/2$$

Hence the required probability is

$$P(|x| < a/2) = P\left(-\frac{a}{2} < X < \frac{a}{2}\right) = \int_{-a/2}^{a/2} dF(x)$$

$$= \frac{1}{2a}\int_{-a/2}^{a/2} 1.dx = \frac{1}{2}$$

Example 6·23. *Let $X_1, X_2, ..., X_n$ be a sequence of mutually independent random variables with common distribution. Suppose X_k assumes only positive integral values and $E(X_k) = a$, exists; $k = 1, 2, ..., n$. Let $S_n = X_1 + X_2 + \ldots + X_n$.*

(i) Show that $E\left(\frac{S_m}{S_n}\right) = \frac{m}{n}$, *for* $1 \le m \le n$

(ii) Show that $E\left(S_n^{-1}\right)$ *exists and*

$$E\left(\frac{S_m}{S_n}\right) = 1 + (m-n)\, a\, E(S_n^{-1}), \text{ for } 1 \le n \le m$$

(iii) Verify and use the inequality $x + x^{-1} \ge 2, (x > 0)$ *to show that*

$$E\left(\frac{S_m}{S_n}\right) \ge \frac{m}{n} \text{ for } m, n \ge 1$$

[Delhi Univ. M.Sc. (Stat.), 1988]

Solution. *(i)* We have

$$E\left[\frac{X_1 + X_2 + \ldots + X_n}{X_1 + X_2 + \ldots + X_n}\right] = E(1) = 1$$

$$\Rightarrow \quad E\left[\frac{X_1 + X_2 + \ldots + X_n}{S_n}\right] = 1$$

$$\Rightarrow \quad E\left(\frac{X_1}{S_n}\right) + E\left(\frac{X_2}{S_n}\right) + \ldots + E\left(\frac{X_n}{S_n}\right) = 1$$

Since X_i's, $(i = 1, 2, ..., n)$ are identically distributed random variables, (X_i/S_n), $(i = 1, 2, ..., n)$ are also identically distributed random variables.

$$\therefore \quad nE\left(\frac{X_i}{S_n}\right) = 1$$

$$\Rightarrow \quad E\left(\frac{X_i}{S_n}\right) = \frac{1}{n}\ ; \quad i = 1, 2, .., n \qquad \ldots(*)$$

Now

$$E\left(\frac{S_m}{S_n}\right) = E\left(\frac{X_1 + X_2 + \ldots + X_m}{S_n}\right) = E\left[\frac{X_1}{S_n} + \frac{X_2}{S_n} + \ldots + \frac{X_m}{S_n}\right]$$

$$= E\left(\frac{X_1}{S_n}\right) + E\left(\frac{X_2}{S_n}\right) + \dots + E\left(\frac{X_m}{S_n}\right)$$

$$= \frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n} \; [(m \text{ times})] \qquad [\textit{Using } (*)]$$

$$= \frac{m}{n}, (m < n)$$

(ii) Since X_i's assume only positive integral values, we have

$$n \le X_1 + X_2 + \dots + X_n < \infty$$

$$\Rightarrow \qquad \frac{1}{n} \ge \frac{1}{S_n} > 0 \quad \Rightarrow \quad 0 < S_n^{-1} \le \frac{1}{n}$$

Since S_n^{-1} lies between two finite quantities 0 and $\frac{1}{n}$, we get

$$0 < E(S_n^{-1}) \le \frac{1}{n}$$

Hence $E(S_n^{-1})$ exists.

$$E\left(\frac{S_m}{S_n}\right) = E\left[\frac{X_1 + X_2 + \dots + X_n + X_{n+1} + \dots + X_m}{S_n}\right], \; m \ge n$$

$$= E\left[1 + \frac{X_{n+1}}{S_n} + \dots + \frac{X_m}{S_n}\right]$$

$$= 1 + E\left(\frac{X_{n+1}}{S_n}\right) + \dots + E\left(\frac{X_m}{S_n}\right)$$

Since $X_{n+1}, X_{n+2}, \dots, X_m$ are independent of $S_n = X_1 + X_2 + \dots + X_n$, they are independent of S_n^{-1} also.

$$\therefore \quad E\left(\frac{S_m}{S_n}\right) = 1 + E(X_{n+1}) E(S_n^{-1}) + \dots + E(X_m) E(S_n^{-1})$$

$$= 1 + [E(X_{n+1}) + \dots + E(X_m)] E(S_n^{-1})$$

$$= 1 + (m - n)\, a\, E(S_n^{-1}), \; 1 \le n \le m$$

$$[\because E(X_i) = a \;\forall\; i]$$

(iii) Verification of $\quad x + \frac{1}{2} \ge 2, \; (x > 0)$.

$$x + \frac{1}{2} \ge 2$$

$$\Rightarrow \qquad x^2 + 1 \ge 2x \quad \text{(multiplication valid only if } x > 0)$$

$$\Rightarrow \qquad (x - 1)^2 \ge 0$$

which is always true for $x > 0$.

If $1 \le m \le n$, result follows from (*).

If $1 \le n \le m$, then using (**), we have to prove that

$$1 + (m-n)\, a\, E(S_n^{-1}) \geq \frac{m}{n}$$

$$\Rightarrow \qquad (m-n) a\, E(S_n^{-1}) \geq \frac{m-n}{n}$$

$$\Rightarrow \qquad E(S_n^{-1}) \geq \frac{1}{an} \qquad \ldots(***)$$

In (**), taking $x = \dfrac{S_n}{an} > 0$, we get

$$E(x) + E(x^{-1}) \geq 2$$

$$\Rightarrow \qquad E\left[\frac{S_n}{an}\right] + E\left[\frac{S_n}{an}\right]^{-1} \geq 2$$

$$\Rightarrow \qquad \frac{1}{an} . E(S_n) + an\, E(S_n^{-1}) \geq 2$$

$$\Rightarrow \qquad \frac{1}{an} . an + an\, E(S_n^{-1}) \geq 2$$

$$\Rightarrow \qquad an\, E(S_n^{-1}) \geq 1$$

$$\Rightarrow \qquad E(S_n^{-1}) \geq \frac{1}{an},$$

which was to be proved in (***).

Example 6·24. *Let X be a r.v. for which β_1 and β_2 exist. Then for any real k, prove that :*

$$\beta_2 \geq \beta_1 - (2k + k^2) \qquad \ldots(*)$$

Deduce that (i) $\beta_2 \geq \beta_1$, (ii) $\beta_2 \geq 1$. When is $\beta_2 = 1$?

Solution. Without any loss of generality we can take $E(X) = 0$. [If $E(X) \neq 0$, then we may start with the random variable $Y = X - E(X)$ so that $E(Y) = 0$.]

Consider the real valued function of the real variable t defined by :

$$Z(t) = E[X^2 + tX + k\mu_2]^2 \geq 0 \quad \forall \quad t,$$

where $\qquad \mu_r = EX^r, \qquad \ldots(i)$

is the rth moment of X about mean.

$$\therefore \quad Z(t) = E[X^4 + t^2 X^2 + k^2 \mu_2^2 + 2tX^3 + 2k\mu_2 X^2 + 2k\mu_2 tX]$$

$$= \mu_4 + t^2 \mu_2 + k^2 \mu_2^2 + 2t\mu_3 + 2k\mu_2^2$$

[Using (*i*) and $E(X) = 0$]

$$= t^2\mu_2 + 2t\mu_3 + \mu_4 + k^2\mu_2^2 + 2k\mu_2^2 \geq 0 \quad \text{for all } t. \qquad \ldots(ii)$$

Since $Z(t)$ is a quadratic form in t, $Z(t) \geq 0$ for all t iff its discriminant is ≤ 0, *i.e.*,

$$\text{iff} \qquad (2\mu_3)^2 - 4\mu_2[\mu_4 + k^2\mu_2^2 + 2k\mu_2^2] \leq 0$$

$$\Rightarrow \qquad \frac{\mu_3^2}{\mu_2^3} - \left[\frac{\mu_4}{\mu_2^2} + k^2 + 2k\right] \leq 0 \qquad [\text{Dividing by } 4\mu_2^3 > 0]$$

$$\Rightarrow \quad \beta_1 - \beta_2 - (2k + k^2) \le 0$$

$$\Rightarrow \quad \beta_2 \ge \beta_1 - (2k + k^2)$$

Deductions. *(i)* Taking $k = 0$ in (*) we get $\beta_2 \ge \beta_1$ ...(**)

(ii) Taking $k = -1$ in (*) we get : $\beta_2 \ge \beta_1 + 1$ ...(***)

a result, which is established differently in Example 6·26.

(iii) Since $\beta_1 = \mu_3^2/\mu_2^3$ is always non-negative, we get from (***) :

$$\beta_2 \ge 1 \qquad \text{...(****)}$$

Remark. The sign of equality holds in (****), *i.e.*, $\beta_2 = 1$ iff

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = 1 \quad \Rightarrow \quad \mu_4 = \mu_2^2$$

$$\Rightarrow \quad E[X - E(X)]^4 = \left[E(X - E(X))\right]^2$$

$$\Rightarrow \quad E(Y^2) - [E(Y)]^2 = 0, \qquad (Y = [X - E(X)]^2)$$

$$\Rightarrow \quad \text{Var}(Y) = 0$$

$$\Rightarrow \quad P[Y = E(Y)] = 1 \qquad [\textit{See Example } 6{\cdot}9]$$

$$\Rightarrow \quad P[(X - \mu)^2 = E(X - \mu)^2] = 1$$

$$\Rightarrow \quad P[(X - \mu)^2 = \sigma^2] = 1$$

$$\Rightarrow \quad P[(X - \mu) = \pm \sigma] = 1$$

$$\Rightarrow \quad P[X = \mu \pm \sigma] = 1$$

Thus X takes only two values $\mu + \sigma$ and $\mu - \sigma$ with respective probabilities p and q, (say).

$$\therefore \quad E(X) = p(\mu + \sigma) + q(\mu - \sigma) = \mu$$

$$\Rightarrow \quad p + q = 1 \quad \text{and} \quad (p - q)\sigma = 0$$

But since $\sigma \ne 0$, (since in this case β_2 is defined) we have :

$$p + q = 1 \quad \text{and} \quad p - q = 0 \quad \Rightarrow \quad p = q = \tfrac{1}{2}.$$

Hence $\beta_2 = 1$ iff the r.v. X assumes only two values, each with equal probability ½.

Example 6·25. *Let X and Y be two variates having finite means.*

Prove or disprove :

(a) $E[\text{Min}(X, Y)] \le \text{Min}[E(X), E(Y)]$

(b) $E[\text{Max}(X, Y)] \ge \text{Max}[E(X), E(Y)]$

(c) $E[\text{Min}(X, Y) + \text{Max}(X, Y)] = E(X) + E(Y)$

[Delhi Univ. B.A. (Stat. Hons.), Spl. Course, 1989]

Solution. We know that

$$\text{Min}(X, Y) = \frac{1}{2}(X + Y) - |X - Y| \qquad \text{...}(i)$$

and

$$\text{Max}(X, Y) = \frac{1}{2}(X + Y) + |X - Y| \qquad \text{...}(ii)$$

(a) $\therefore \quad E[\text{Min}(X, Y)] = \frac{1}{2} E(X+Y) - E|X-Y|$ ...(iii)

We have : $|E(X-Y)| \le E|X-Y|$

$\Rightarrow \quad -|E(X-Y)| \ge -E|X-Y|$

$\Rightarrow \quad E|X-Y| \le -|E(X-Y)| = -|E(X) - E(Y)|$ ...(*)

Substituting in (iii) we get :

$$E[\text{Min}(X, Y)] \le \frac{1}{2} E(X+Y) - E|X-Y|$$

$$\le \frac{1}{2} [E(X) + E(Y)] - |E(X) - E(Y)| \quad \text{...[From (*)]}$$

$\Rightarrow \quad E[\text{Min}(X, Y)] \le \text{Min}[E(X), E(Y)]$

(b) Similarly from (ii) we get :

$$E[\text{Max}(X, Y)] = \frac{1}{2} E(X+Y) + E|X+Y|$$

$$\ge \frac{1}{2} E(X+Y) + |E(X+Y)|$$

$$(\because |E(X+Y)| \le E|X+Y|)$$

$$= \frac{1}{2} [E(X) + E(Y)] + |E(X) + E(Y)|$$

$$= \text{Max}[E(X), E(Y)]$$

i.e., $\quad E[\text{Max}(X, Y)] \ge \text{Max}[E(X), E(Y)]$

(c) $\quad [\text{Min}(X, Y) + \text{Max}(X, Y)] = [X + Y]$

$\Rightarrow \quad E[\text{Min}(X, Y) + \text{Max}(X, Y)] = E(X+Y)$

$$= E(X) + E(Y),$$

as required.

Hence all the results in (a), (b) and (c) are true.

Example 6·26. *Use the relation $E(AX^a + BX^b + CX^c)^2 \ge 0$, X being a random variable with $E(X) = 0$, E denoting the mathematical expectation, to show that*

$$\begin{vmatrix} \mu_{2a} & \mu_{a+b} & \mu_{a+c} \\ \mu_{a+b} & \mu_{2b} & \mu_{b+c} \\ \mu_{a+c} & \mu_{b+c} & \mu_{2c} \end{vmatrix} \ge 0, \quad \text{...(*)}$$

μ_n denoting the nth moment about mean.

Hence or otherwise show that Pearson Beta- coefficients satisfy the inequality

$$\beta_2 - \beta_1 - 1 \ge 0.$$

Also deduce that $\beta_2 \ge 1$.

Solution. Since $E(X) = 0$, we get

$$E(X^r) = \mu_r \quad \text{...(**)}$$

We are given that

$$E(AX^a + BX^b + CX^c)^2 \ge 0$$

$\Rightarrow \quad E[A^2X^{2a} + B^2X^{2b} + C^2X^{2c} + 2ABX^{a+b} + 2ACX^{a+c} + 2BCX^{b+c}] \ge 0$

$\Rightarrow \quad A^2\mu_{2a} + B^2\mu_{2b} + C^2\mu_{2c} + 2AB\mu_{a+b} + 2AC\mu_{a+c} + 2BC\mu_{b+c} \ge 0$

[From (**)] ...(***)

for all values of A, B, C.

We know from matrix theory that the conditions for the quadratic form

$$a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy,$$

to be non-negative for all values of x, y and z are

$$a' \geq 0, \quad (ii) \begin{vmatrix} a' & h' \\ h' & b' \end{vmatrix} \geq 0, \text{ and } \quad (iii) \begin{vmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{vmatrix} \geq 0$$

Comparing with (***), we have

$$a' = \mu_{2a},\ b' = \mu_{2b},\ c' = \mu_{2c},\ f' = \mu_{b+c},\ g' = \mu_{a+c},\ h' = \mu_{a+b}$$

Substituting these values in condition *(iii)* above, we get the required result.

Taking $a = 0, b = 1$ and $c = 2$ in (*) and noting that $\mu_0 = 1$ and $\mu_1 = 0$, we get

$$\begin{vmatrix} 1 & 0 & \mu_2 \\ 0 & \mu_2 & \mu_3 \\ \mu_2 & \mu_3 & \mu_4 \end{vmatrix} \geq 0$$

$$\Rightarrow \qquad \mu_2 \mu_4 - \mu_3^2 + \mu_2 (-\mu_2^2) \geq 0$$

Dividing throughout by μ_2^3 (assuming that μ_2 is finite, for otherwise β_2 will ecome infinite), we get

$$\frac{\mu_4}{\mu_2^2} - \frac{\mu_3^2}{\mu_2^3} - 1 \geq 0$$

$$\Rightarrow \qquad \beta_2 - \beta_1 - 1 \geq 0$$

$$\Rightarrow \qquad \beta_2 \geq \beta_1 + 1$$

Further since $\beta_1 \geq 0$, we get $\beta_2 \geq 1$.

Example 6·27. *Let X be a non-negative random variable with distribution function F. Show that*

$$E(X) = \int_0^\infty [1 - F(x)]\, dx. \qquad \ldots(i)$$

Conjecture a corresponding expression for $E(X^2)$.

[Delhi Univ. M.Sc.(Stat). 1988]

Solution. Since $X \geq 0$, we have :

$$\text{R.H.S.} = \int_0^\infty [1 - P(X \leq x)]\, dx = \int_0^\infty \left[1 - \int_0^x f(u)\, du\right] dx,$$

where $f(\cdot)$ is the p.d.f. of r.v. X.

$$\therefore \qquad \text{R.H.S.} = \int_0^\infty \left[\int_x^\infty f(u)\, du\right] dx; \qquad \ldots(ii)$$

From the integral in bracket *(ii)*, we have, $u \geq x$ and since x ranges from 0 to ∞, u also range from 0 to ∞. Further $u \geq x \Rightarrow x \leq u$ and since x is non-negative, we have $0 \le x \le u$. [See Region R_1 in Remark 2 below]. Hence changing the order of integration in *(ii)*, [by Fubini's theorem for non-negative functions], we get

$$\text{R.H.S.} = \int_0^\infty \left[\int_0^u 1 \, . \, dx \right] f(u)\, du = \int_0^\infty u \, . \, f(u)\, du$$

$$= E(X) \qquad \textit{[Since } f(\cdot) \textit{ is p.d.f. of } X\textit{]}$$

Conjecture for $E(X^2)$. Consider the integral :

$$\int_0^\infty 2x\,[1 - F(x)]\, dx = \int_0^\infty 2x \left(\int_x^\infty f(u)\, du \right) dx$$

$$= \int_0^\infty \left(\int_0^u 2x\, dx \right) f(u)\, du,$$

(By Fubini's theorem for non-negative functions).

$$= \int_0^\infty u^2 \, . \, f(u)\, du = E(X^2)$$

Remarks. 1. If X is a non-negative r.v. then

$$\text{Var } X = EX^2 - [E(X)]^2 = \int_0^\infty 2x\,[1 - F(x)]\, dx - \mu_x^2 \qquad ...(iii)$$

2. If we do not restrict ourselves to non-negative random variables only, we have the following more generalised result.

If F denotes the distribution function of the random variable X then :

$$E(X) = \int_0^\infty [1 - F(x)]\, dx - \int_{-\infty}^0 F(x)\, dx, \qquad ...(iv)$$

provided the integrals exist finitely.

Proof of (iv). The first integral has already been evaluated in the above example, *i.e.,*

$$\int_0^\infty [1 - F(x)]\, dx = \int_0^\infty u \, . \, f(u)\, du \qquad ...(v)$$

Consider :

$$\int_{-\infty}^0 F(x)\, dx = \int_{-\infty}^0 P(X \le x)\, dx = \int_{-\infty}^0 \left(\int_{-\infty}^x f(u)\, du \right) dx$$

$$= \int_{-\infty}^0 \left(\int_u^0 1 \cdot dx \right) f(u)\, du$$

[Changing the order of integration in the Region R_2 where $u \le x$].

$$= \int_{-\infty}^{0} u \cdot f(u)\, du \qquad ...(vi)$$

Subtracting *(vi)* from *(v)*, we get :

$$\int_0^{\infty} [1 - F(x)]\, dx - \int_{-\infty}^{0} F(x)\, dx = \int_0^{\infty} u f(u)\, du + \int_{-\infty}^{0} u f(u)\, du$$

$$= \int_{-\infty}^{\infty} u f(u)\, du$$

$$= E(X), \qquad (\text{Since } f(\cdot) \text{ is p.d.f. of } X)$$

as desired.

In this generalised case,

$$\text{Var}(X) = \int_0^{\infty} 2x\, [1 - F_X(x) + F_X(-x)]\, dx - [E(X)]^2$$

3. The corresponding analogue of the above result for discrete random variable is given in the next Example 6·28.

Example 6·28. *If the possible values of a variate X are 0, 1, 2, 3, ... then*

$$E(X) = \sum_{n=0}^{\infty} P(X > n)$$

[Delhi Univ. B.Sc. (Maths Hons.), 1987]

Solution. Let $P(X = n) = p_n,\ n = 0, 1, 2, 3, \ldots \qquad ...(i)$

If $E(X)$ exists, then by definition :

$$E(X) = \sum_{n=0}^{\infty} n \cdot P(X = n) = \sum_{n=1}^{\infty} n \cdot p_n \qquad ...(ii)$$

Consider :

$$\begin{aligned}
\sum_{n=0}^{\infty} P(X > n) &= P(X > 0) + P(X > 1) + P(X > 2) + \ldots \\
&= P(X \geq 1) + P(X \geq 2) + P(X \geq 3) + \ldots \\
&= (p_1 + p_2 + p_3 + p_4 + \ldots\ldots) \\
&\quad + (p_2 + p_3 + p_4 + \ldots\ldots) \\
&\quad + (p_3 + p_4 + p_5 + \ldots\ldots) \\
&\quad + \ldots\ldots\ldots\ldots \\
&= p_1 + 2p_2 + 3p_3 + \ldots\ldots \\
&= \sum_{n=1}^{\infty} n\, p_n \\
&= E(X) \qquad [\text{From } (ii)]
\end{aligned}$$

As an illustration of this result, see Problem 24 in Exercise 6(a).

Aliter. R.H.S. $= \sum_{n=0}^{\infty} P(X > n) = \sum_{n=1}^{\infty} P(X \geq n)$

$$= \sum_{n=1}^{\infty} \left(\sum_{x=n}^{\infty} p(x) \right)$$

Since the series is convergent and $p(x) \geq 0 \ \forall \ x$, by Fubini's theorem, changing the order of summation we get :

$$\text{R.H.S.} = \sum_{x=1}^{\infty} \left(\sum_{n=1}^{x} p(x) \right) = \sum_{x=1}^{\infty} \left\{ p(x) \sum_{n=1}^{x} 1 \right\}$$

since $x \geq n \Rightarrow n \leq x$ and x assumes only positive integral values :

$$\therefore \qquad \text{R.H.S.} = \sum_{x=1}^{\infty} x\,p(x) = \sum_{x=0}^{\infty} x\,p(x) = E(X)$$

Example 6·29. *For any variates X and Y, show that*

$$\{E(X+Y)^2\}^{1/2} \leq \{E(X^2)\}^{1/2} + \{E(Y^2)\}^{1/2} \qquad ...(*)$$

Solution. Squaring both sides in (*), we have to prove

$$E(X+Y)^2 \leq \left[\{E(X^2)\}^{1/2} + \{E(Y^2)\}^{1/2}\right]^2$$

$$\Rightarrow \quad E(X^2) + E(Y^2) + 2E(XY) \leq E(X^2) + E(Y^2) + 2\sqrt{E(X^2)\,E(Y^2)}$$

$$\Rightarrow \qquad E(XY) \leq \sqrt{E(X^2)\,E(Y^2)}$$

$$\Rightarrow \qquad [E(XY)]^2 \leq E(X^2)\,.\,E(Y^2).$$

This is nothing but Cauchy-Schwartz inequality. [For proof see Theorem 6·11 page 6·13.]

Example 6·30. *Let X and Y be independent non-degenerate variates. Prove that*

$$Var(XY) = Var(X)\,.\,Var(Y)$$

iff $\qquad E(X) = 0, \ E(Y) = 0$

[Delhi Univ. B.Sc. (Maths Hons.), 1989]

Solution. We have

$$\text{Var}(XY) = E(XY)^2 - [E(XY)]^2 = E(X^2Y^2) - [E(XY)]^2$$

$$= E(X^2)\,E(Y^2) - [E(X)]^2\,[E(Y)]^2, \qquad ...(*)$$

since X and Y are independent.

$$\left.\begin{array}{ll} \text{If} & E(X) = 0 = E(Y) \\ \text{then} & \text{Var}(X) = E(X^2) \text{ and } \text{Var}(Y) = E(Y^2) \end{array}\right\} \qquad ...(**)$$

Substituting from (**) in (*), we get

$$\text{Var}(XY) = \text{Var}(X)\,.\,\text{Var}(Y),$$

as desired.

Only If. We have to prove that if

$$\text{Var}(XY) = \text{Var}(X)\,.\,\text{Var}(Y) \qquad ...(***)$$

then $\qquad E(X) = 0$ and $E(Y) = 0.$

Now (***) gives, [on using (*)]

$$E(X^2).E(Y^2) - [E(X)]^2.[E(Y)]^2 = \{E(X^2) - [E(X)]^2\} \times \{E(Y^2) - [E(Y)]^2\}$$

$$= E(X^2)E(Y^2) - E(X^2)[E(Y)]^2 - [E(X)]^2E(Y^2) + [E(X)]^2[E(Y)]^2$$

$$\Rightarrow E(X^2).[E(Y)]^2 - [E(X)]^2.[E(Y)]^2 + [E(X)]^2E(Y^2) - [E(X)]^2[E(Y)]^2 = 0$$

$$\Rightarrow [E(Y)]^2\{E(X^2) - [E(X)]^2\} + [E(X)]^2\{E(Y^2) - [E(Y)]^2\} = 0$$

$$\Rightarrow \quad [E(Y)]^2 \text{Var}(X) + [E(X)]^2 \text{Var}(Y) = 0 \qquad \ldots(****)$$

Since each of the quantities $[E(X)]^2$, $[E(Y)]^2$, Var (X) and Var (Y) is non-negative and since X and Y are given to be non-degenerate random variables such that Var $(X) > 0$ and Var $(Y) > 0$, (****) holds only if we have $E(X) = 0 = E(Y)$, as required.

EXERCISE 6(a)

1. *(a)* Define a random variable and its mathematical expectation.

(b) Show that the mathematical expectation of the sum of two random variables is the sum of their individual expectations and if two variables are independent, the mathematical expectation of their product is the product of their expectations.

Is the condition of independence necessary for the latter ? If not, what is the necessary condition ?

(c) If X is a random variable, prove that $|E(X)| \le E(|X|)$.

(d) If X and Y are two random variables such that $X \le Y$, prove that

$$E(X) \le E(Y).$$

(e) Prove that $E[(X-c)^2] = [\text{Var}(X)] + [E(X) - c]^2$, where c is a constant.

2. Prove that

(a) $E(aX + bY) = a\,E(X) + b\,E(Y)$.

where a and b are any constants.

(b) $E(a) = a$, a being a constant.

(c) $E[ag(X)] = aE[g(X)]$

(d) $E[g_1(X) + g_2(X) + \ldots + g_n(X)] = E[g_1(X)] + E[g_2(X)] + \ldots + E[g_n(X)]$

(e) $|E[g(X)]| \le E[|g(X)|]$.

(f) If $g(X) \ge 0$, everywhere then $E[g(X)] \ge 0$.

(g) If $g(X) \ge 0$ everywhere and $E[g(X)] = 0$, then $g(X) = 0$, *i.e.*, the random variable $g(X)$ has a one point distribution at $X = 0$.

3. Show that if X is non-negative random variable such that both $E(X)$ and $E(1/X)$ exist, then

$$E(1/X) \ge 1/E(X).$$

4. If X, Y are independent random variables with $E(X) = \alpha$, $E(X^2) = \beta$, and $E(Y^k) = a_k$; $k = 1, 2, 3, 4$, find $E(XY + Y^2)^2$.

5. *(a)* If X and Y are two independent random variables, show that

$$\text{Var}(aX + bY) = a^2\,\text{Var}(X) + b^2\,\text{Var}(Y).$$

(b) With usual notations, show that

$\text{Cov}(aX + bY, cX + dY) = ac \text{ Var}(X) + bd \text{ Var}(Y) + (ad + bc) \text{ Cov}(X, Y)$

(c) Show that

$$\text{Cov}\left(\sum_{i=1}^{n} a_i X_i, \sum_{i=1}^{n} b_j X_j\right) = \sum_{j=1}^{n} \sum_{i=1}^{n} a_i b_j \text{ Cov}(X_i, X_j)$$

6. (a) Define the indicator function $I_A(x)$ and show that $E(I_A(X)) = P(A)$.

(b) Prove that the probability function $P(X \in A)$ for set A and the distribution function $F_X(x)$, $(-\infty < x < \infty)$, can be regarded as expectations of some random variable.

Hint. Define the indicator functions :

$$I_A(x) = 1 \text{ if } x \in A \qquad I_y(x) = 1 \text{ if } x \le y$$
$$= 0 \text{ if } x \notin A \qquad = 0 \text{ if } x > y$$

Then we shall get :

$$E(I_A(X)] = P(X \in A) \text{ and } E(I_y(X)) = P(X \le y) = F_X(y)$$

7. (a) Let X be a continuous random variable with median m. Minimise $E|X - b|$, as an function of b.

Ans. $E|X - b|$ is minimum when $b = m =$ Median. This states that absolute sum of deviations of a given set of observations is minimum when taken about median. [See Example 5·19.]

(b) Let X be a random variable such that $E|X| < \infty$. Show that $E|X - C|$ is minimised if we choose C equal to the median of the distribution.

[Delhi Univ. B.Sc. (Maths Hons.), 1988]

8. If X and Y are symmetric, show that

$$E\left(\frac{X}{X+Y}\right) = \frac{1}{2}$$

Hint. $$1 = E\left[\frac{X+Y}{X+Y}\right] = E\left[\frac{X}{X+Y}\right] + E\left[\frac{Y}{X+Y}\right]$$

$$\Rightarrow \qquad 1 = 2E\left[\frac{X}{X+Y}\right] \qquad (\because X \text{ and } Y \text{ are symmetric.})$$

9. (a) If a r.v. X has a symmetric density about the point 'a' and if $E(X)$ exists, then

$$\text{Mean}(X) = \text{Median}(X) = a$$

Hint. Given $f(a - x) = f(a + x)$; $f(x)$ p.d.f. of X. prove that

$$E(X - a) = \int_{-\infty}^{\infty} (x - a) f(x)\, dx = \int_{-\infty}^{\infty} (x - a) f(x)\, dx + \int_{a}^{\infty} (x - a) f(x)\, dx = 0$$

(b) If X and Y are two random variables with finite variances, then show that

$$E^2(XY) \le E(X^2) \cdot E(Y^2) \qquad \ldots(*)$$

When does the equality sign hold in (*) ? **[Indian Civil Service, 1987]**

10. Let X be a non-negative arbitrary r.v. with distribution function F. show that

$$E(X) = \int_0^{\infty} [1 - F_X(x)]\, dx - \int_{-\infty}^{0} F_X(x)\, dx,$$

in the sense that, if either side exists, so does the other and the two are equal.

[Delhi Univ. B.Sc. (Maths Hons.), 1992]

11. Show that if Y and Z are independent random values of a variable X, the expected value of $(Y - Z)^2$ is twice the variance of the distribution of X.

[Allahab ad Univ. B.Sc., 1989]

Hint. $E(Y) = E(Z) = E(X) = \mu$, (say) ; $\sigma_y^2 = \sigma_z^2 = \sigma_x^2 = \sigma^2$, (say) . ...(*)

$$E(Y - Z)^2 = E(Y^2) + E(Z^2) - 2E(Y)E(Z)$$

($\because Y, Z$ are independent)

$$= (\sigma_y^2 + \mu_y^2) + (\sigma_z^2 + \mu_z^2) - 2\mu^2$$

$$= 2\sigma^2 = 2\sigma_x^2$$ [On using (*)]

12. Given the following table :

x	-3	-2	-1	0	1	2	3
$p(x)$	0·05	0·10	0·30	0	0·30	0·15	0·10

Compute (*i*) $E(X)$, (*ii*) $E(2X \pm 3)$, (*iii*) $E(4X + 5)$, (*iv*) $E(X^2)$

(*v*) $V(X)$, and (*vi*) $V(2X \pm 3)$.

13. (*a*) A and B throw with one die for a stake of Rs. 44 which is to be won by the playr who first throws a 6. If A has the first throw, what are their respective expectations?

Ans. Rs. 24, Rs. 20.

(*b*) A contractor has to choose between two jobs. The first promises a profit of Rs. 1,20,000 with a probability of 3/4 or a loss of Rs. 30,000 due to delays with a probability of 1/4 ; the second promises a profit of Rs. 1,80,000 with a probability of 1/2 or a loss of Rs. 45,000 with a probability of 1/2. Which job should the contractor choose so as to maximise his expected profit?

(*c*) A random variable X can assume any positive integral value n with a probability porportional to $1/3^n$. Find the expectation of X.

[Delhi Univ. B.Sc., Oct. 1987]

14. Three tickets are chosen at random without replacement from 100 tickets numbered 1, 2, ..., 100. Find the mathematical expectation of the sum of the numbers on the tickets drawn.

15. (*a*) Three urns contain respectavily 3 green and 2 white balls, 5 green and 6 white balls and 2 green and 4 white balls. One ball is drawn from each urn. Find the expected number of white balls drawn out.

Hint. Let us define the r.v.

$X_i = 1$, if the ball drawn from ith urn is white
$= 0$; otherwise

Then the number of white balls drawn is $S = X_1 + X_2 + X_3$.

$$E(S) = E(X_1) + E(X_2) + E(X_3) = 1 \times \frac{2}{5} + 1 \times \frac{6}{11} + 1 \times \frac{4}{6} = \frac{266}{165}.$$

(b) Urn A contains 5 cards numbered from 1 to 5 and urn B contains 4 cards numbered from 1 to 4. One card is drawn from each of these urns. Find the probability function of the number which appears on the cards drawn and its mathematical expectation.

Ans. 11/4.

16. *(a)* Thirteen cards are drawn from its pack simultaneously. If the values of aces are 1, face cards 10 and others according to denomination, find the expectation of the total score in all the 13 cards.

[Madurai Univ. B.Sc., Oct. 1990]

(b) Let X be a random variable with p.d.f. as given below :

x :	0	1	2	3
$p(x)$:	1/3	1/2	1/24	1/8

Find the expected value of $Y = (X - 1)^2$. **[Aligarh Univ. B.Sc. (Hons.), 1992]**

17. A player tosses 3 fair coins. He wins Rs. 8, if three heads occur; Rs. 3, if 2 heads occur and Re. 1, if one head occurs. If the game is to be fair, how much should he lose, if no heads occur? **[Punjab Univ. M.A. (Econ.), 1987]**

Hint. X : Player's prize in Rs.

x	8	3	1	a
No. of heads	3	2	1	0
$p(x)$	1/8	3/8	3/8	1/8

$E(X) = \Sigma x\, p(x) = \frac{1}{8}(8 + 9 + 3 + a)$

For the game to be fair, we have :

$E(X) = 0 \Rightarrow 20 + a = 0 \Rightarrow a = -20$

Hence the player loses Rs. 20, if no heads come up.

18. *(a)* A coin is tossed until a tail appears. What is the expectation of the number of tosses ?

Ans. 2.

(b) Find the expectation of *(i)* the sum, and *(ii)* the product, of number of points on n dice when thrown.

Ans. *(i)* $7n/2$, *(ii)* $(7/2)^n$

19. *(a)* Two cards are drawn at random from ten cards numbered 1 to 10. Find the expectation of the sum of points on two cards.

(b) An urn contains n cards marked from 1 to n. Two cards are drawn at a time. Find the mathematical expectation of the product of the numbers on the cards.

[Mysore Univ. B.Sc., 1991]

(c) In a lottery m tickets are drawn out of n tickets numbered from 1 to n. What is the expectation of the sum of the squares of numbers drawn ?

(d) A bag contains n white and 2 black balls. Balls are drawn one by one without replacement until a black is drawn. If 0, 1, 2, 3, ... white balls are drawn before the first black, a man is to receive $0^2, 1^2, 2^2, 3^2, \ldots$ rupees respectively. Find his expectation. **[Rajasthan Univ. B.Sc., 1992]**

(e) Find the expectation and variance of the number of successes in a series of independent trials, the probability of success in the ith trial being p_i $(i = 1, 2, \ldots, n)$. **[Nagarjuna Univ. B.Sc., 1991]**

20. Balls are taken one by one out of an urn containing w white and b black balls until the first white ball is drawn. Prove that the expectation of the number of black balls preceding the first white ball is $b/(w+1)$.

[Allahabad Univ. B.Sc. (Hons.), 1992]

21. (a) X and Y are independent variables with means 10 and 20, and variances 2 and 3 respectively. Find the variance of $3X + 4Y$.

Ans. 66.

(b) Obtain the variance of $Y = 2X_1 + 3X_2 + 4X_3$ where X_1, X_2 and X_3 are three random variables with means given by 3, 4, 5 respectively, variances by 10, 20, 30 respectively, and co-variances by $\sigma_{X_1X_2} = 0$, $\sigma_{X_2X_3} = 0$, $\sigma_{X_1X_3} = 5$, where $\sigma_{X_rX_s}$ stands for the co-variance of X_r and X_s.

22. (a) Suppose that X is a random variable for which $E(X) = 10$ and $\text{Var}(X) = 25$. Find the positive values of a and b such that $Y = aX - b$, has expectation 0 and variance 1.

Ans. $a = 1/5$, $b = 2$

(b) Let X_1 and X_2 be two stochastic random variables having variances k and 2 respectively. If the variance of $Y = 3X_2 - X_1$ is 25, find k.

(Poona Univ. B.Sc., 1990)

Ans. $k = 7$.

23. A bag contains $2n$ counters, of which half are marked with odd numbers and half with even numbers, the sum of all the numbers being S. A man is to draw two counters. If the sum of the numbers drawn is odd, he is to receive that number of rupees, if even he is to pay that number of rupees. Show that his expectation is $S/[n(2n-1)]$ rupees. **(I.F.S., 1989)**

24. A jar has n chips numbred 1, 2, ..., n. A person draws a chip, returns it, draws another, returns it, and so on, until a chip is drawn that has been drawn before. Let X be the number of drawings. Find $E(X)$.

[Delhi Univ. B.A. (Stat. Hons.), Spl. Course, 1986]

Hint. Obviously $P(X > 1) = 1$, because we must have at least two draws to get the chip which has been drawn before.

$$P(X > r) = P[\text{Distinct number on } i\text{th draw}; i = 1, 2, \ldots, r]$$

$$= \frac{n}{n} \times \frac{n-1}{n} \times \frac{n-2}{n} \times \dots \times \frac{n-(r-1)}{n}$$

$$P(X > r) \Rightarrow \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) \dots\dots \left(1 - \frac{r-1}{n}\right); r = 1, 2, 3, \dots \quad \dots(*)$$

Hence, using the result in Example 6·28

$$E(X) = \sum_{r=0}^{\infty} P(X > r)$$

$$= P(X > 0) + P(X > 1) + P(X > 2) + \dots$$

$$= 1 + 1 + \left(1 - \frac{1}{n}\right) + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) +$$

$$\dots + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\dots\left(1 - \frac{r-1}{n}\right) + \dots \quad \text{[Using (*)]}$$

25. A coin is tossed four times. Let X denote the number of times a head is followed immediately by a tail. Find the distribution, mean and variance of X.

Hint. $S = \{H, T\} \times \{H, t\} \times \{H, T\} \times \{H, T\}$

$= \{HHHH, HHHT, HHTH, HTHH, HTHT, \dots, TTTT\}$

X: 0, 1, 1, 1, 2, ..., 0

x	0	1	2
$p(x)$	5/16	10/16	1/16

$E(X) = 3/4,\ E(X^2) = 7/8$

$\text{Var } X = 7/8 - 9/16 = 5/16.$

26. An urn contains balls numbered 1, 2, 3. First a ball is drawn from the urn and then a fair coin is tossed the number of times as the number shown on the drawn ball. Find the expected number of heads.

[Delhi Univ. B.Sc. (Maths Hons.), 1984]

Hint. B_j : Event of drawing the ball numbered j.

$P(B_j) = 1/3;\ j = 1, 2, 3.$

X : No. of heads shown. X is a r.v. taking the values 0, 1, 2, and 3.

$$P(X = x) = \sum_{j=1}^{3} P(B_j) \cdot P(X = x \mid B_j) = \frac{1}{3} \sum_{j=1}^{3} P(X = x \mid B_j)$$

$$\therefore P(X = 0) = \frac{1}{3}[P(X = 0 \mid B_1) + P(X = 0 \mid B_2) + P(X = 0 \mid B_3)]$$

$$= \frac{1}{3}\left[\frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right] = \frac{7}{24}$$

$$P(X = 1) = \frac{1}{3}[P(X = 1 \mid B_1) + P(X = 1 \mid B_2) + P(X = 1 \mid B_3)]$$

$$= \frac{1}{3}\left[\frac{1}{2} + \frac{2}{4} + \frac{3}{8}\right]$$

e.g., $P(X = 0 \mid B_2) = P$[No head when two coins are tossed] $= 2/4$

$P(X = 1 \mid B_3) = P$[1 head when three coins are tossed] $= 3/8$

Similarly $P(X = 2) = \dfrac{1}{3}\left(0 + \dfrac{1}{4} + \dfrac{3}{8}\right) = \dfrac{5}{24}$

$$P(X=3)=\frac{1}{3}\left(0+0+\frac{1}{8}\right)=\frac{1}{24}$$

$$\therefore \quad E(X)=\sum_{x=0}^{3} x\,P(X=x)=\frac{11}{24}+\frac{10}{24}+\frac{3}{24}=1$$

27. An urn contains pN white and qN black balls, the total number of balls being N, $p+q=1$. Balls are drawn one by one (without being returned to the urn) until a certain number n of balls is reached.

Let $X_i = 1$, if the ith ball drawn is white.
$= 0$, if the ith ball drawn is black.

(*i*) Show that $E(X_i)=p$, $\text{Var}(X_i)=pq$.

(*ii*) Show that the co-variance between X_j and X_k is $-\dfrac{pq}{n-1}$, $(j \neq k)$

(*iii*) From (*i*) and (*ii*), obtain the variance of $S_n = X_1 + X_2 + \ldots + X_n$.

28. Two similar decks of n distinct cards each are put into random order and are matched against each other. Prove that the probability of having exactly r matches is given by

$$\frac{1}{r\,!}\sum_{k=0}^{n-r}\frac{(-1)^k}{k\,!},\ r=0,1,2,\ldots n$$

Prove further that the expected number of matches and its variance are equal and are independent of n.

29. (*a*) If X and Y are two independent random variables, such that $E(X)=\lambda_1$, $V(X)=\sigma_1^2$ and $E(Y)=\lambda_2$, $V(Y)=\sigma_2^2$, then prove that

$$V(XY)=\sigma_1^2\sigma_2^2+\lambda_1^2\sigma_2^2+\lambda_2^2\sigma_1^2$$ **[Gorakhpur Univ. B.Sc., 1992]**

(*b*) If X and Y are two independent random variables, show that

$$\frac{V(XY)}{[E(X)]^2\,[E(Y)]^2}=C_X^2\,C_Y^2+C_X^2+C_Y^2$$

where $$C_X=\frac{\sqrt{V(X)}}{E(X)},\ C_Y=\frac{\sqrt{V(Y)}}{E(Y)}$$

are the so-called coefficients of variation of X and Y? **[Patna Univ. B.Sc., 1991]**

30. A point P is taken at random in a line AB of length $2a$, all positions of the point being equally likely. Show that the expected value of the area of the rectangle $AP.\,PB$ is $2a^2/3$ and the probability of the area exceeding $1/2a^2$ is $1/\sqrt{2}$. **[Delhi Univ. B.Sc. (Maths Hons.), 1986]**

31. If X is a random variable with $E(X)=\mu$ satisfying $P(X \leq 0)=0$, show that $P(X > 2\mu) \leq 1/2$. **[Delhi Univ. B.Sc. (Maths Hons.), 1992]**

OBJECTIVE TYPE QUESTIONS

1. Fill in the blanks :

(*i*) Expected value of a random variable X exists if ....

(*ii*) If $E(X^r)$ exists then $E(X^s)$ also exists for ....

(*iii*) When X is a random variable, expectation of $(X-\text{constant})^2$ is minimum when the constant is ....

(iv) $E\,|X - A|$ is minimum when $A =$ ....

(v) Var (c) = ..., where c is a constant

(vi) Var $(X + c)$ = ..., where c is a constant

(vii) Var $(aX + b)$ = ..., where a and b are constants.

(viii) If X is a r.v. with mean μ and variance σ^2 then

$$E\left(\frac{X-\mu}{\sigma}\right) = \text{....},\ \text{Var}\left(\frac{X-\mu}{\sigma}\right) = \text{....}$$

(ix) $[E(XY)]^2$ .... $E(X^2) \cdot E(Y^2)$.

(x) $V(aX \pm bY) = \ldots$

where a and b are constants.

II. Mark the correct answer in the following :

(i) For two random variables X and Y, the relation

$$E(XY) = E(X)\,E(Y)$$

holds good

(a) if X and Y are statistically independent,

(b) for all X and Y,

(c) if X and Y are identical.

(ii) Var $(2X \pm 3)$ is

(a) 5 (b) 13 (c) 4, if Var $X = 1$.

(iii) $E(X - k)^2$ is minimum when

(a) $k < E(X)$, (b) $k > E(X)$, (c) $k = E(X)$.

III. Comment on the following :

If X and Y are mutually independent variables, then

(i) $E(XY + Y + 1) - E(X + 1)\,E(Y) = 0$

(ii) X and Y are independent if and only if

$$\text{Cov}(X, Y) = 0$$

(iii) For every univariable distribution :

(a) $V(cX) = c^2 V(X)$ (b) $E(c/X) = c/E(X)$

(iv) Expected value of a r.v. always exists.

IV. Mark true or false with reasons for your answers :

(a) Cov $(X, Y) = 0 \Rightarrow X$ and Y are independent.

(b) If Var $(X) >$ Var (Y), then $X + Y$ and $X - Y$ are dependent.

(c) If Var (X) = Var (Y) and if $2X + Y$ and $X - Y$ are independent, then X and Y are dependent.

(d) If Cov $(aX + bY, bX + aY) \neq ab$ Var $(X + Y)$, then X and Y are dependent.

6·8. Moments of Bivariate Probability Distributions. The mathematical expectation of a function $g(x, y)$ of two-dimensional random variable (X, Y) with p.d.f. $f(x, y)$ is given by

$$E[g(X, Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y) f(x, y)\, dx\, dy \qquad ...(6\cdot43)$$

(If X and Y are continuous variables)

$$= \sum_i \sum_j x_i y_j P(X = x_i \cap Y = y_j), \qquad ...(6\cdot43\ a)$$

(If X and Y are discrete variables)

provided the expectation exists.

In particular, the rth and sth product moment about origin of the random variables X and Y respectively is defined as

$$\mu_{rs}' = E(X^r Y^s) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^r y^s f(x, y)\, dx\, dy$$

or $$\mu_{rs}' = \sum_i \sum_j x_i^r y_j^s\ P(X = x_i \cap Y = y_j) \qquad ...(6\cdot44)$$

The joint rth central moment of X and sth central moment of Y is given by

$$\mu_{rs} = E\left[\left\{X - E(X)\right\}^r \left\{Y - E(Y)\right\}^s\right]$$

$$= E\left[(X - \mu_X)^r (Y - \mu_Y)^s\right], \ [E(X) = \mu_X, E(Y) = \mu_Y] \qquad ...(6\cdot45)$$

In particular

$$\mu_{00}' = 1 = \mu_{00}\ , \quad \mu_{10} = 0 = \mu_{01}$$
$$\mu_{10}' = E(X)\ , \quad \mu_{01}' = E(Y)$$
$$\mu_{20} = \sigma_X^2\ , \quad \mu_{02} = \sigma_Y^2 \text{ and } \mu_{11} = \text{Cov}(X, Y).$$

6·9. Conditional Expectation and Conditional Variance.

Discrete Case. The conditional expectation or mean value of a continuous function $g(X, Y)$ given that $Y = y_j$, is defined by

$$E\{g(X, Y) \mid Y = y_j\} = \sum_{i=1}^{\infty} g(x_i, y_j)\ P(X = x_i \mid Y = y_j)$$

$$= \frac{\sum_{i=1}^{\infty} g(x_i, y_j)\ P(X = x_i \cap Y = y_j)}{P(Y = y_j)} \qquad ...(6\cdot46)$$

i.e., $E[g(X, Y) \mid Y = y_j]$ is nothing but the expectation of the function $g(X, y_j)$ of X w.r.t. the conditional distribution of X when $Y = y_j$.

In particular, the conditional expectation of a discrete random variable X given $Y = y_j$ is

$$E(X \mid Y = y_j) = \sum_{i=1}^{\infty} x_i\ P(X = x_i \mid Y = y_j) \qquad ...(6\cdot47)$$

The conditional variance of X given $Y = y_j$ is likewise given by

$$V(X \mid Y = y_j) = E[\{X - E(X \mid Y = y_j)\}^2 \mid Y = y_j] \qquad ...(6\cdot47a)$$

The conditional expectation of $g(X, Y)$ and the conditional variance of Y given $X = x_i$ may also be defined in an exactly similar manner.

Continuous Case. The conditional expectation of $g(X, Y)$ on the hypothesis $Y = y$ is defined by

$$E\{g(X, Y) \mid Y = y\} = \int_{-\infty}^{\infty} g(x, y) f_{X|Y}(x \mid y)\, dx$$

$$= \frac{\int_{-\infty}^{\infty} g(x, y) f(x, y)\, dx}{f_Y(y)} \qquad \ldots(6\cdot48)$$

In particular, the conditional mean of X given $Y = y$ is defined by

$$E(X \mid Y = y) = \frac{\int_{-\infty}^{\infty} x f(x, y)\, dx}{f_Y(y)}$$

Similarly, we define

$$E(Y \mid X = x) = \frac{\int_{-\infty}^{\infty} y f(x, y)\, dy}{f_X(x)} \qquad \ldots(6\cdot48\ a)$$

The conditional variance of X may be defined as

$$V(X \mid Y = y) = E\left[\{X - E(X \mid Y = y)\}^2 \mid Y = y\right]$$

Similarly, we define

$$V(Y \mid X = x) = E\left[\{Y - E(Y \mid X = x)\}^2 \mid X = x\right] \qquad \ldots(6\cdot49)$$

Theorem 6·13. *The expected value of X is equal to the expectation of the conditional expectation of X given Y. Symbolically,*

$$E(X) = E[E(X \mid Y)] \qquad \ldots(6\cdot50)$$

[Calicut Univ. B.Sc. (Main Stat.), 1980]

Proof. $E[E(X \mid Y)] = E[\sum_i x_i\ P(X = x_i \mid Y = y_j)]$

$$= E\left[\sum_i x_i \frac{P(X = x_i \cap Y = y_j)}{P(Y = y_j)}\right]$$

$$= \sum_j \left[\sum_i \left\{x_i \frac{P(X = x_i \cap Y = y_j)}{P(Y = y_j)}\right\}\right] P(Y = y_j)$$

$$= \sum_j \sum_i x_i\, P(X = x_i \cap Y = y_i)$$

$$= \sum_i \left[x_i \{\sum_j P(X = x_i \cap Y = y_j)\}\right]$$

$$= \sum x_i P(X = x_i) = E(X).$$

Theorem 6·14. *The variance of X can be regarded as consisting of two parts, the expectation of the conditional variance and the variance of the conditional expectation.* Symbolically,

$$V(X) = E[V(X \mid Y)] + V[E(X \mid Y)] \qquad \ldots(6 \cdot 51)$$

Proof. $E[V(X \mid Y)] + V[E(X \mid Y)]$

$$= E\left[E(X^2 \mid Y) - \{E(X \mid Y)\}^2\right] + E[\{E(X \mid Y)\}^2] - [E\{E(X \mid Y)\}]^2$$

$$= E[E(X^2 \mid Y)] - E[\{E(X \mid Y)\}^2] + E[\{E(X \mid Y)\}^2] - [E\{E(X \mid Y)\}]^2$$

$$= E[E(X^2 \mid Y)] - [E(X)]^2 \qquad (c.f. \text{ Theorem } 6 \cdot 13)$$

$$= E[\sum_i x_i^2 P(X = x_i \mid Y = y_j)] - [E(X)]^2$$

$$= E\left[\sum_i x_i^2 \frac{P(X = x_i \cap Y = y_j)}{P(Y = y_j)}\right] - [E(X)]^2$$

$$= \sum_j \left\{\left[\sum_i x_i^2 \frac{P(X = x_i \cap Y = y_j)}{P(Y = y_j)}\right] P(Y = y_j)\right\} - [E(X)]^2$$

$$= \sum_i \left[x_i^2 \sum_j P(X = x_i \cap Y = y_j)\right] - [E(X)]^2$$

$$= \sum_i x_i^2 P(X = x_i) - [E(X)]^2$$

$$= E(X^2) - [E(X)]^2 = \text{Var}(X)$$

Hence the theorem.

Remarks The proofs of Theorems 6·13 and 6·14 for continous r.v.'s X and Y are left as an exercise to the reader.

Theorem 6·15. *Let A and B be two mutually exclusive events, then*

$$E(X \mid A \cup B) = \frac{P(A) E(X \mid A) + P(B) E(X \mid B)}{P(A \cup B)} \qquad \ldots(6 \cdot 52)$$

where by def.,

$$E(X \mid A) = \frac{1}{P(A)} \sum_{x_i \in A} x_i P(X = x_i)$$

Proof. $E(X \mid A \cup B) = \dfrac{1}{P(A \cup B)} \sum\limits_{x_i \in A \cup B} x_i \; P(X = x_i)$

Since A and B are mutually exclusive events,

$$\sum_{x_i \in A \cup B} x_i P(X = x_i) = \sum_{x_i \in A} x_i P(X = x_i) + \sum_{x_i \in B} x_i P(X = x_i)$$

$$\therefore \quad E(X \mid A \cup B) = \frac{1}{P(A \cup B)} [P(A) E(X \mid A) + P(B) E(X \mid B)]$$

Cor. $\qquad E(X) = P(A) E(X \mid A) + P(\overline{A}) E(X \mid \overline{A}) \qquad \ldots(6 \cdot 53)$

The corollary follows by putting $B = \overline{A}$ in the above Theorem.

Example 6·31. *Two ideal dice are thrown. Let X_1 be the score on the first die and X_2 the score on the second die. Let Y denote the maximum of X_1 and X_2, i.e., $Y = \max(X_1, X_2)$.*

(i) Write down the joint distribution of Y and X_1,

(ii) Find the mean and variance of Y and co-variance (Y, X_1).

Solution. Each of the random variables X_1 and X_2 can take six values 1, 2, 3, 4, 5, 6 each with probability 1/6, *i.e.*,

$$P(X_1 = i) = P(X_2 = i) = 1/6 \ ; \quad i = 1, 2, 3, 4, 5, 6 \qquad \ldots(i)$$

$$Y = \text{Max}(X_1, X_2).$$

Obviously

$$P(X_1 = i, Y = j) = 0\,, \text{ if } \; j < i = 1, 2, \ldots, 6$$

$$P(X_1 = i, Y = i) = P(X_1 = i, X_2 \leq i) = \sum_{j=1}^{i} P(X_1 = i, X_2 = j)$$

$$= \sum_{j=1}^{i} P(X_1 = i)\, P(X_2 = j)$$

$$(\because X_1, X_2 \text{ are independent.})$$

$$= \sum_{j=1}^{i} \left(\frac{1}{36}\right) = \frac{i}{36} \ ; \quad i = 1, 2, \ldots, 6.$$

$$P(X_1 = i, Y = j) = P(X_1 = i, X_2 = j) \ ; \quad j > i$$

$$= P(X_1 = i)\, P(X_2 = j) = \frac{1}{36} \ ; \quad j > i = 1, 2, \ldots, 6.$$

The joint probability table of X_1 and Y is given as follows:

X_1 \ Y	1	2	3	4	5	6	*Marginal Totals*
1	1/36	1/36	1/36	1/36	1/36	1/36	6/36
2	0	2/36	1/36	1/36	1/36	1/36	6/36
3	0	0	3/36	1/36	1/36	1/36	6/36
4	0	0	0	4/36	1/36	1/36	6/36
5	0	0	0	0	5/36	1/36	6/36
6	0	0	0	0	0	6/36	6/36
Marginal Totals	1/36	3/36	5/36	7/36	9/36	11/36	1

$$E(Y) = 1.\frac{1}{36} + 2.\frac{3}{36} + 3.\frac{5}{36} + 4.\frac{7}{36} + 5.\frac{9}{36} + 6.\frac{11}{36}$$

$$= \frac{1}{36}\Big[1 + 6 + 15 + 28 + 45 + 66\Big] = \frac{161}{36}$$

$$E(Y^2) = 1^2.\frac{1}{36} + 2^2.\frac{3}{36} + 3^2.\frac{5}{36} + 4^2.\frac{7}{36} + 5^2.\frac{9}{36} + 6^2.\frac{11}{36} = \frac{791}{36}$$

$$V(Y) = E(Y^2) - [E(Y)]^2 = \frac{791}{36} - \left(\frac{161}{36}\right)^2 = \frac{2555}{1296}$$

$$E(X_1) = \frac{6}{36}\left[1+2+3+4+5+6\right] = \frac{126}{36} = \frac{21}{6}$$

$$E(X_1 Y) = 1.\frac{1}{36} + 2.\frac{1}{36} + 3.\frac{1}{36} + 4.\frac{1}{36} + 5.\frac{1}{36} + 6.\frac{1}{36}$$
$$+ 4.\frac{2}{36} + 6.\frac{1}{36} + 8.\frac{1}{36} + 10.\frac{1}{36} + 12.\frac{1}{36}$$
$$+ 9.\frac{3}{36} + 12.\frac{1}{36} + 15.\frac{1}{36} + 18.\frac{1}{36}$$
$$+ 16.\frac{4}{36} + 20.\frac{1}{36} + 24.\frac{1}{36}$$
$$+ 25.\frac{5}{36} + 30.\frac{1}{36} + 36.\frac{6}{36}$$
$$= \frac{1}{36}\left[21+44+72+108+155+216\right] = \frac{1}{36} \times 616$$

$$\text{Cov}(X_1, Y) = E(X_1 Y) - E(X_1) E(Y)$$
$$= \frac{616}{36} - \frac{21}{6} \cdot \frac{161}{36} = \frac{3696 - 3381}{216} = \frac{315}{216}.$$

Example 6·32. *Let X and Y be two random variables each taking three values −1, 0 and 1, and having the joint probability distribution :*

(i) Show that X and Y have different expectations.

Y \ X	−1	0	1	Total
−1	0	·1	·1	·2
0	·2	·2	·2	·6
1	0	·1	·1	·2
Total	2	·4	·4	1 0

(ii) Prove that X and Y are uncorrelated.

(iii) Find Var X and Var Y.

(iv) Given that Y = 0, what is the conditional probability distribution of X ?

(v) Find V (Y | X = −1).

Solution. *(i)* $E(Y) = \Sigma p_i y_i = -1(\cdot 2) + 0(\cdot 6) + 1(\cdot 2) = 0$

$$E(X) = \Sigma p_i x_i = -1(\cdot 2) + 0(\cdot 4) + 1(\cdot 4) = \cdot 2$$

$\therefore$ $$E(X) \neq E(Y)$$

(ii) $$E(XY) = \Sigma p_{ij} y_i x_j$$
$$= (-1)(-1)(0) + 0(-1)(\cdot 1) + 1(-1)(\cdot 1)$$
$$+ 0(-1)(\cdot 2) + 0(0)(\cdot 2) + 0(1)(\cdot 2)$$
$$+ 1(-1)(0) + 1(0)(\cdot 1) + 1(1)(\cdot 1)$$
$$= -0\cdot 1 + 0\cdot 1 = 0$$

$\therefore$ $$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = 0$$

$\Rightarrow$ X and Y are uncorrelated (*c.f.* § 10·5)

(iii) $E(Y^2) = (-1)^2(\cdot2) + 0(\cdot6) + 1^2(\cdot2) = \cdot4$

$\therefore$ $V(Y) = E(Y^2) - [E(Y)]^2 = \cdot4$

$E(X^2) = (-1)^2(\cdot2) + 0(\cdot4) + 1^2(\cdot4) = \cdot2 + \cdot4 = \cdot6$

$V(X) = \cdot6 - \cdot04 = \cdot56$

(iv) $$P(X = -1 \mid Y = 0) = \frac{P(X = -1 \cap Y = 0)}{P(Y = 0)} = \frac{\cdot2}{\cdot6} = \frac{1}{3}$$

$$P(X = 0 \mid Y = 0) = \frac{P(X = 0 \cap Y = 0)}{P(Y = 0)} = \frac{\cdot2}{\cdot6} = \frac{1}{3}$$

$$P(X = 1 \mid Y = 0) = \frac{P(X = 1 \cap Y = 0)}{P(Y = 0)} = \frac{\cdot2}{\cdot6} = \frac{1}{3}$$

(v) $$V(Y \mid X = -1) = E(Y \mid X = -1)^2 - \{E(Y \mid X = -1)\}^2$$

$$E(Y \mid X = -1) = \sum_y y\, P(Y = y \mid X = -1) = (-1)0 + 0(\cdot2) + 1(0) = 0$$

$$E(Y \mid X = -1)^2 = \sum_y y^2\, P(Y = y \mid X = -1) = 1(0) + 0(\cdot2) + 1(0) = 0$$

$\therefore$ $V(Y \mid X = -1) = 0.$

Example 6·33. *Two tetrahedra with sides numbered 1 to 4 are tossed. Let X denote the number on the downturned face of the first tetrahedron and Y denote the larger of the downturned numbers. Investigate the following :*

(a) Joint density function of X, Y and marginals f_X and f_Y,

(b) $P\{X \le 2, Y \le 3\}$, *(c) $\rho(X, Y)$,* *(d) $E(Y \mid X = 2)$,*

(e) Construct joint density different from that in part (a) but possessing same marginals f_X and f_Y. **[Delhi Univ. B.A. (Stat. Hons.), Spl. Course, 1985]**

Hint. The sample space is $S = \{1, 2, 3, 4\} \times \{1, 2, 3, 4\}$ and each of the 16 sample points (outcomes) has probability $p = 1/16$ of occurrence.

Let X : Number on the first dice and Y : Larger of the numbers on the two dice. Then the above 16 sample points, in that order, give the following distribution of X and Y.

Sample Point	:	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(2, 1)	(2, 2)	(2, 3)	(2, 4)
X	:	1	1	1	1	2	2	2	2
Y	:	1	2	3	4	2	2	3	4
Sample Point	:	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(4, 1)	(4, 2)	(4, 3)	(4, 4)
X	:	3	3	3	3	4	4	4	4
Y	:	3	3	3	4	4	4	4	4

Since each sample point has probability $p = 1/16$, the joint density functions of X and Y and the marginal densities f_X and f_Y are given on page 6·61.

Here $p = 1/16$.

(b) $P(X \le 2, Y \le 3) = p + p + 2p + p + p = 6p = 3/8.$

(c) $\text{Var}(X) = EX^2 - [E(X)]^2 = \frac{15}{2} - \frac{25}{4} = \frac{5}{4}$ (Try it)

(a)

$y \backslash x$	1	2	3	4	Total (f_y)
1	p	0	0	0	p
2	p	$2p$	0	0	$3p$
3	p	p	$3p$	0	$5p$
4	p	p	p	$4p$	$7p$
Total (f_x)	$4p$	$4p$	$4p$	$4p$	$16p=1$

(e)

$y \backslash x$	1	2	3	4	Total (f_y)
1	p	0	0	0	p
2	p	$2p$	0	0	$3p$
3	p	$p+\epsilon$	$3p-\epsilon$	0	$5p$
4	p	$p-\epsilon$	$p+\epsilon$	$4p$	$7p$
Total (f_x)	$4p$	$4p$	$4p$	$4p$	1

$$\text{Var}(Y) = EY^2 - [E(Y)]^2 = \frac{85}{8} - \left(\frac{25}{8}\right)^2 = \frac{55}{64} \qquad \text{(Try it)}$$

$$\text{Cov}(X, Y) = E(XY) - E(X)E(Y) = \frac{135}{16} - \frac{5}{2} \times \frac{25}{8} = \frac{5}{8} \qquad \text{(Try it)}$$

$$\therefore \qquad \rho(X, Y) = \frac{5/8}{\sqrt{5/4 \times 55/64}} = \frac{2}{\sqrt{11}}$$

(d) $E(Y \mid X = 2) = \Sigma y \,.\, f(y \mid x = 2) = \Sigma y \,.\, \dfrac{f(x = 2 \cap y)}{f(x = 2)}$

$$= 4 \,.\, \Sigma y\, f(2, y) = 4\,[0 + 4p + 3p + 4p] = 44p = \frac{11}{4}$$

(e) Let $0 < \epsilon < p$. The joint density of X and Y given in (e) above is different from that in (a) but has the same marginals as in (a).

Example 6·34. *(a) Given two variates X_1 and X_2 with joint density function $f(x_1, x_2)$, prove that conditional mean of X_2 (given X_1) coincides with (unconditional) mean only if X_1 and X_2 are independent (stochastically).*

(b) Let $f(x_1, x_2) = 21x_1^2 x_2^3,\ 0 < x_1 < x_2 < 1,$ and zero elsewhere be the joint p.d.f. of X_1 and X_2. Find the conditional mean and variance of X_1 given $X_2 = x_2$, $0 < x_2 < 1$. **[Delhi Univ. M.A. (Eco.), 1986]**

Solution. (a) Conditional mean of X_2 given X_1 is given by :

$$E(X_2 \mid X_1 = x_1) = \int_{x_2} x_2 f(x_2 \mid x_1)\, dx_2 \qquad \ldots(*)$$

where $f(x_2 \mid x_1)$ is conditional p.d.f. of X_2 given $X_1 = x_1$.

But the joint p.d.f. of X_1 and X_2 is given by

$$f(x_1, x_2) = f_1(x_1) \,.\, f(x_2 \mid x_1)$$

$$\Rightarrow \qquad f(x_2 \mid x_1) = \frac{f(x_1, x_2)}{f_1(x_1)}$$

where $f_1(\cdot)$ is marginal p.d.f. of X_1.

Substituting in (*), we get

$$E(X_2 \mid X_1 = x_1) = \int_{x_2} \left[\frac{x_2 f(x_1, x_2)}{f_1(x_1)} \right] dx_2, \qquad ...(**)$$

Unconditional mean of X_2 is given by

$$E(X_2) = \int_{x_2} x_2 \; f_2(x_2) \, dx_2, \qquad ...(***)$$

where $f_2(\cdot)$ is marginal p.d.f. of X_2.

From (**) and (***), we conclude that the conditional mean of X_2 (given X_1) will coincide with unconditional mean of X_2 only if

$$\frac{f(x_1, x_2)}{f_1(x_1)} = f_2(x_2)$$

$$\Rightarrow \quad f(x_1, x_2) = f_1(x_1) \cdot f_2(x_2)$$

i.e., if X_1 and X_2 are (stochastically) independent.

(b) $\quad f(x_1, x_2) = 21\, x_1^2 x_2^3 \; ; \; 0 < x_1 < x_2 < 1$

$\qquad = 0, \qquad$ otherwise

Marginal p.d.f. of X_2 is given by

$$f_2(x_2) = \int_0^{x_2} f(x_1, x_2)\, dx_1 = 21\, x_2^3 \int_0^{x_2} x_1^2 \, dx_1$$

$$= 21\, x_2^3 \left| \frac{x_1^3}{3} \right|_0^{x_2} = 7\, x_2^6 \; ; \quad 0 < x_2 < 1$$

$\therefore$ Conditional p.d.f. of X_1 (given X_2) is given by

$$f_1(x_1 \mid x_2) = \frac{f(x_1, x_2)}{f_2(x_2)} = 3\, \frac{x_1^2}{x_2^3} \; ; \; 0 < x_1 < x_2 \, ; 0 < x_2 < 1$$

Conditional mean of X_1 is

$$E(X_1 \mid X_2 = x_2) = \int_0^{x_2} x_1 f_1(x_1 \mid x_2)\, dx_1 = \frac{3}{x_2^3} \int_0^{x_2} x_1^3 \, dx_1$$

$$= \frac{3}{x_2^3} \cdot \left| \frac{x_1^4}{4} \right|_0^{x_2} = \frac{3x_2}{4} \; ; \; 0 < x_2 < 1$$

Now

$$E(X_1^2 \mid X_2 = x_2) = \int_0^{x_2} x_1^2 f_1(x_1 \mid x_2)\, dx_1 = \frac{3}{x_2^3} \int_0^{x_2} x_1^4 \, dx_1$$

$$= \frac{3}{x_2^3} \cdot \frac{x_2^5}{5} = \frac{3}{5} x_2^2$$

$$\therefore \quad \text{Var}(X_1 \mid X_2 = x_2) = E(X_1^2 \mid X_2 = x_2) - \left[E(X_1 \mid X_2 = x_2) \right]^2$$

$$= \frac{3}{5} x_2^2 - \frac{9}{16} x_2^2 = \frac{3}{80} x_2^2 \; ; \quad 0 < x_2 < 1.$$

Example 6·35. *Two random variables X and Y have the followinig joint probability density function :*

$$f(x, y) = 2 - x - y\ ;\ 0 \le x \le 1,\ 0 \le y \le 1$$
$$= 0,\ \textit{otherwise}$$

Find *(i) Marginal probability density functions of X and Y.*
(ii) Conditional density functions.
(iii) Var (X) and Var (Y).
(iv) Co-variance between X and Y.

[Dibrugarh Univ. B.Sc. (Hons.), 1991]

Solution. *(i)* $f_X(x) = \int_{-\infty}^{+\infty} f(x, y)\, dy$

$$= \int_0^1 (2 - x - y)\, dy = \frac{3}{2} - x$$

$\therefore$ $$f_X(x) = \frac{3}{2} - x,\ 0 < x < 1$$
$$= 0,\ \text{otherwise}$$

Similarly $$f_Y(y) = \frac{3}{2} - y,\ 0 < y < 1$$
$$= 0,\ \text{otherwise}$$

(ii) $$f_{X|Y}(x \mid y) = \frac{f_{XY}(x, y)}{f_Y(y)} = \frac{(2 - x - y)}{(3/2 - y)},\ 0 < (x, y) < 1$$

$$f_{Y|X}(y \mid x) = \frac{f_{XY}(x, y)}{f_X(x)} = \frac{(2 - x - y)}{(3/2 - x)},\ 0 < (x, y) < 1$$

$$E(X) = \int_0^1 x f_X(x)\, dx = \int_0^1 x\left(\frac{3}{2} - x\right) dx = \frac{5}{12}$$

$$E(Y) = \int_0^1 y f_Y(y)\, dy = \int_0^1 y\left(\frac{3}{2} - y\right) dy = \frac{5}{12}$$

(iii) $$E(X^2) = \int_0^1 x^2\left(\frac{3}{2} - x\right) dx = \left[\frac{3}{6}x^3 - \frac{x^4}{4}\right]_0^1 = \frac{1}{4}$$

$$V(X) = E(X^2) - [E(X)]^2 = \frac{1}{4} - \frac{25}{144} = \frac{11}{144}$$

Similarly $$V(Y) = \frac{11}{144}$$

(iv) $$E(XY) = \int_0^1\int_0^1 xy(2 - x - y)\, dx\, dy$$

$$= \int_0^1 \left| 2\frac{x^2y}{2} - \frac{x^3y}{3} - \frac{x^2y^2}{2} \right|_0^1 dy$$

$$= \int_0^1 \left(\frac{2}{3}y - \frac{1}{2}y^2\right) dy$$

$$= \left| \frac{y^2}{3} - \frac{y^3}{6} \right|_0^1 = \frac{1}{6}$$

$$= \left| \frac{y^2}{3} - \frac{y^3}{6} \right|_0^1 = \frac{1}{6}$$

$$\therefore \qquad \text{Cov}(X, Y) = E(XY) - E(X)E(Y) = \frac{1}{6} - \frac{5}{12} \cdot \frac{5}{12} = -\frac{1}{144}.$$

Example 6·36. *Let $f(x, y) = 8xy, 0 < x < y < 1$; $f(x, y) = 0$, elsewhere. Find (a) $E(Y \mid X = x)$, (b) $E(XY \mid X = x)$, (c) $Var(Y \mid X = x)$.* **[Calcutta Univ. B.Sc. (Maths Hons.), 1988; Delhi Univ. B.Sc. (Maths Hons.), 1990]**

Solution. $f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$

$$= 8x \int_x^1 y\, dy$$

$$= 4x(1 - x^2), 0 < x < 1$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$

$$= 8y \int_0^y x\, dx$$

$$= 4y^3, 0 < y < 1$$

$$f_{X|Y}(x \mid y) = \frac{f(x, y)}{f_Y(y)} = \frac{2x}{y^2}, \quad f_{Y|X}(y \mid x) = \frac{2y}{1 - x^2}, 0 < x < y < 1.$$

(a) $$E(Y \mid X = x) = \int_x^1 y \left(\frac{2y}{1 - x^2} \right) dy = \frac{2}{3} \left(\frac{1 - x^3}{1 - x^2} \right) = \frac{2}{3} \left(\frac{1 + x + x^2}{1 + x} \right)$$

(b) $$E(XY \mid X = x) = x\, E(Y \mid X = x) = \frac{2}{3} \cdot \frac{x(1 + x + x^2)}{(1 + x)}$$

(c) $$E(Y^2 \mid X = x) = \int_x^1 y^2 \left(\frac{2y}{1 - x^2} \right) dy = \frac{1}{2} \left(\frac{1 - x^4}{1 - x^2} \right) = \frac{1 + x^2}{2}$$

$$\text{Var}(Y \mid X = x) = E(Y^2 \mid X = x) - [E(Y \mid X = x)]^2$$

$$= \frac{1 + x^2}{2} - \frac{4}{9} \cdot \frac{(1 + x + x^2)^2}{(1 + x)^2}$$

EXERCISE 6(b)

1. The joint probability distribution of X and Y is given by the following table:

X \ Y	1	3	9
2	1/8	1/24	1/12
4	1/4	1/4	0
6	1/8	1/24	1/12

(*i*) Find the marginal probability distribution of Y.

(ii) Find the conditional distribution of Y given that $X = 2$,

(iii) Find the covariance of X and Y, and

(iv) Are X and Y independent ?

2. A fair coin is tossed four times. Let X denote the number of heads occurring and let Y denote the longest string of heads occurring.

(i) Determine the joint distribution of X and Y, *and* (ii) Find Cov (X, Y).

Hint.

X \ Y	0	1	2	3	4	*Total*
0	1/16	0	0	0	0	1/16
1	0	4/16	0	0	0	4/16
2	0	3/16	3/16	0	0	6/16
3	0	0	2/16	2/16	0	4/16
4	0	0	0	0	1/16	1/16
Total	1/16	7/16	5/16	2/16	1/16	1

(ii) Cov $(X, Y) = 0{\cdot}85$.

3. X and Y are jointly discrete random variables with probability function

$$p(x, y) = 1/4 \text{ at } (x, y) = (-3, -5), (-1, -1), (1, 1), (3, 5)$$
$$= 0, \text{ otherwise}$$

Compute $E(X), E(Y), E(XY)$ and $E(X \mid Y)$. Are X and Y independent ?

4. X_1 and X_2 have a bivariate distribution given by

$$P(X_1 = x_1 \cap X_2 = x_2) = \frac{x_1 + 3x_2}{24}, \text{ where } (x_1, x_2) = (1, 1), (1, 2), (2, 1), (2, 2)$$

Find the conditional mean and variance of X_1, given $X_2 = 2$.

5. Two random variables X and Y have the following joint probability density function :

$$f(x, y) = k(4 - x - y)\,;\ 0 \le x \le 2\,;\ 0 \le y \le 2$$
$$= 0, \text{ otherwise}$$

Find (i) the constant k,

(ii) marginal density functions of X and Y,

(iii) conditional density functions, and

(iv) Var (X), Var (Y) and Cov (X, Y). (Poona Univ. B.Sc., Oct. 1991)

6. Let the joint probability density function of the random variables X and Y be

$$f(x, y) = 2(x + y - 3xy^2)\,;\ 0 < x < 1, 0 < y < 1$$
$$= 0, \text{ otherwise}$$

(i) Find the marginal distributions of X and Y.

(ii) Is $E(XY) = E(X)E(Y)$?

(*iii*) Find $E(X+Y)$ and $E(X-Y)$. **[Calicut Univ. B.Sc., Oct. 1990]**

7. (*a*) Let X and Y have the joint probability density function

$f(x, y) = 2, 0 < x < y < 1$

$= 0$, otherwise

Show that the conditional mean and variance of X given $Y = y$ are $y/2$ and $y^2/12$ respectively.

(*b*) If $f(x, y) = 2$; $0 < x < y, 0 < y < 1$

Find (*i*) $E(Y \mid X)$, (*ii*) $E(X \mid Y)$.

8. Give an example to show that $E(Y)$ may not exist though $E(XY)$ and $E(Y \mid X)$ may both exist? **[Delhi Univ. B.A. (Stat. Hons.) Spl. Course, 1985]**

Hint. Consider the joint p.d.f. :

$$f(x, y) = x \cdot e^{-x(1+y)}; x \geq 0, y \geq 0.$$
$$= 0, \text{ otherwise.}$$

Then we shall get :

$$f_X(x) = \int_0^\infty f(x, y)\, dy = e^{-x}; x \geq 0$$

$$f_Y(y) = \int_0^\infty f(x, y)\, dx = \frac{1}{(1+y)^2}; \ y \geq 0$$

$$f(y \mid x) = \frac{f(x, y)}{f_X(x)} = x e^{-xy}; \ y \geq 0$$

$$\therefore \quad E(Y) = \int_0^\infty y f(y) = \infty \quad \Rightarrow \quad E(Y) \text{ does not exist.}$$

$$E(XY) = \int_0^\infty \int_0^\infty xy \cdot f(x\, y)\, dx\, dy$$

$$E(Y \mid X = x) = \int_0^\infty y \cdot f(y \mid x)\, dy = \frac{1}{x}$$

$\Rightarrow$ Both $E(XY)$ and $(E(Y \mid X = x)$ exist, though $E(Y)$ does not exist.

9. Three coins are tossed. Let X denote the number of heads on the first two coins, Y denote the number of tails on the last two and Z denote the number of heads on the last two.

(*a*) Find the joint distribution of (*i*) X and Y, (*ii*) X and Z.

(*b*) Find the conditional distribution of Y given $X = 1$.

(*c*) Find $E(Z \mid X = 1)$.

(*d*) Find $\rho_{X,Y}$ and $\rho_{X,Z}$.

(*e*) Give a joint distribution that is not the joint distribution of X and Z in (*a*) and yet has the same marginals as $f(x, z)$ has in part (*a*).

[Delhi Univ. B.Sc. (Maths Hons.), 1989]

Hint. The sample space is $S' = \{H, T\} \times \{H, T\} \times \{H, T\}$

$= \{H, T\} \times \{HH, HT, TH, TT\}$

and each of the 8 sample points (outcomes) has the probability $p = 1/8$ of occurrence.

X : Number of heads on the 1st two coins.

Y : Number of tails on the last two coins.

Z : Number of heads on the last two coins.

Then the distribution of X, Y and Z is given below :

Sample Point :	*HHH*	*HHT*	*HTH*	*HTT*	*THH*	*THT*	*TTH*	*TTT*
Probability	p	p	p	p	p	p	p	p
X	2	2	1	1	1	1	0	0
Y	0	1	1	2	0	1	1	2
Z	2	1	1	0	2	1	1	0

Joint Distribution of X and Y

		y		*Total*
	0	1	2	(f_x)
x 0	0	1/8	1/8	1/4
1	1/8	2/8	1/8	1/2
2	1/8	1/8	0	1/4
Total (f_y)	1/4	1/2	1/4	1

Joint Distribution of X and Z

		z		*Total*
	0	1	2	(f_x)
x 0	1/8	1/8	0	1/4
1	1/8	2/8	1/8	1/2
2	0	1/8	1/8	1/4
Total (f_z)	1/4	1/2	1/4	1

(*b*) $$P(Y=0 \mid X=1) = \frac{P(Y=0, X=1)}{P(X=1)} = \frac{1/8}{1/2} = \frac{1}{4}$$

Similarly, $P(Y=1 \mid X=1) = \frac{2/8}{1/2} = \frac{1}{2}$; $P(Y=2 \mid X=1) = \frac{1/8}{1/2} = \frac{1}{4}$

(*c*) $$E(Z \mid X=1) = \Sigma z \,.\, P(Z \mid X=1) = 0 \times \frac{1/8}{1/2} + 1 \times \frac{2/8}{1/2} + 2 \times \frac{1/8}{1/2} = 1$$

(*d*) $$\rho_{XY} = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{-1/4}{\sqrt{1/2 \times 1/2}} = -\frac{1}{2}\,,$$

$$\rho_{XZ} = \frac{\text{Cov}(X, Z)}{\sigma_X \sigma_Z} = \frac{-1/4}{\sqrt{1/2 \times 1/2}} = -\frac{1}{2}$$

(*e*) Let $0 \le \varepsilon \le 1/8$. The joint probability distribution of (X, Z) given belc has the same marginals as in part (*a*).

	0	Z 1	2	Total (f_x)
X 0	1/8	1/8	0	1/4
1	1/8	$2/8 + \varepsilon$	$1/8 - \varepsilon$	1/2
2	0	$1/8 - \varepsilon$	$1/8 + \varepsilon$	1/4
Total (f_z)	1/4	1/2	1/4	1

10. Let $f_{XY}(x, y) = e^{-(x+y)}$; $0 < x < \infty, 0 < y < \infty$

Find :

(a) $P(X > 1)$ (d) m so that $P(X + Y < m) = 1/2$

(b) $P(1 < X + Y < 2)$ (e) $P(0 < X < 1 \mid Y = 2)$

(c) $P(X < Y \mid X < 2Y)$ (f) ρ_{XY}

Ans. $f_X(x) = e^{-x}$; $x \geq 0$; $f_Y(y) = e^{-y}$; $y \geq 0$

(a) $1/e$ (b) *Hint.* $X + Y$ is a Gamma variabte with parameter $n = 2$. [See Chapter 8] $(2/e - 3/e^2)$.

(c) $P(X < Y \mid X < 2Y) = \dfrac{P(X < Y \cap X < 2Y)}{P(X < 2Y)} = \dfrac{P(X < Y)}{P(X < 2Y)} = \dfrac{1/2}{2/3} = \dfrac{3}{4}$

(d) Use hint in (b). $e^{-m}(1 + m) = 1/2$; (e) $(e - 1)/e$

(f) $f_{XY}(x, y) = f_X(x) f_Y(y)$ $\Rightarrow$ X and Y are independent $\Rightarrow$ $\rho_{XY} = 0$.

11. The joint p.d.f. of X and Y is given by :

$$f(x, y) = 3(x + y) \ ; \ 0 \leq x \leq 1, \ 0 \leq y \leq 1 \ ; \ 0 \leq x + y \leq 1$$

Find : (a) Marginal density of X. (b) $P(X + Y < 1/2)$

(c) $E(Y \mid X = x)$ (d) Cov (X, Y)

Ans. (a) $f_X(x) = \dfrac{3}{2}(1 - x^2)$; $0 \leq x \leq 1$.

(b) $P(X + Y < 1/2) = \displaystyle\int_0^{0.5} \left[\int_0^{0.5 - x} 3(x + y)\, dy \right] dx = \frac{1}{8}$

(c) $\dfrac{(1 - x)(x + 2)}{3(1 + x)}$ (d) $\left[E(XY) = \displaystyle\int_0^1 \left[\int_0^{1-x} xy\, f(x, y) dy\, dx = \frac{1}{10} \right] \right]$

$$\text{Cov}(x, y) = E(XY) - E(X)\,E(Y) = \frac{1}{10} - \frac{3}{8} \times \frac{3}{8} = -\frac{13}{320}.$$

6·10. Moment Generating Function. The moment generating function (m.g.f.) of a random variable X (about origin) having the probability function $f(x)$ is given by

$$(6{\cdot}54)\ldots \quad \begin{cases} M_X(t) = E(e^{tX}) = \displaystyle\int e^{tx} f(x)\, dx, & \text{(for } \textit{continuous probability distribution}\text{)} \\ \quad = \displaystyle\sum_x e^{tx} f(x), & \text{(for } \textit{discrete probability distribution}\text{)} \end{cases}$$

the integration or summation being extended to the entire range of x, t being the real parameter and it is being assumed that the right-hand side of (6·54) is absolutely convergent for some positive number h such that $-h < t < h$. Thus

$$M_X(t) = E(e^{tX}) = E\left[1 + tX + \frac{t^2X^2}{2!} + \ldots + \frac{t^rX^r}{r!} + \ldots\right]$$

$$= 1 + t\,E(X) + \frac{t^2}{2!}E(X^2) + \ldots + \frac{t^r}{r!}E(X^r) + \ldots$$

$$= 1 + t\,\mu_1' + \frac{t^2}{2!}\mu_2' + \dots + \frac{t^r}{r!}\mu_r' + \dots \qquad \dots(6\cdot55)$$

where $\mu_r' = E(X^r) = \int x^r f(x)\,dx$, *for continuous distribution*

$= \sum_x x^r p(x)$, *for discrete distribution,*

is the rth moment of X about origin. Thus we see that the coefficient of $\frac{t^r}{r!}$ in $M_X(t)$ gives μ_r' (above origin). Since $M_X(t)$ generates moments, it is known as moment generating function.

Differentiating (6·55) w.r.t. t and then putting $t = 0$, we get

$$\left[\frac{d^r}{dt^r}\{M_X(t)\}\right]_{t=0} = \left[\frac{\mu_r'}{r!}\cdot r! + \mu'_{r+1}\,t + \mu'_{r+2}\cdot\frac{t^2}{2!} + \dots\right]_{t=0}$$

$$\Rightarrow \qquad \mu_r' = \left[\frac{d^r}{dt^r}\{M_X(t)\}\right]_{t=0} \qquad \dots(6\cdot56)$$

In general, the moment generating function of X about the point $X = a$ is defined as

$$M_X(t)\ (\text{about } X = a) = E\left[e^{t(X-a)}\right]$$

$$= E\left[1 + t(X-a) + \frac{t^2}{2!}(X-a)^2 + \dots + \frac{t^r}{r!}(X-a)^r + \dots\right]$$

$$= 1 + t\,\mu_1' + \frac{t^2}{2!}\mu_2' + \dots + \frac{t^r}{r!}\mu_r' + \dots \qquad \dots(6\cdot57)$$

where $\mu_r' = E\{(X-a)^r\}$, is the rth moment about the point $X = a$.

6·10·1. Some Limitations of Moment Generating Functions. Moment generating function suffers from some drawbacks which have restricted its use in Statistics. We give below some of the deficiencies of m.g.f.'s with illustrative examples.

1. *A random variable X may have no moments although its m.g.f. exists.* For example, let us consider a discrete r.v. with probability function

$$f(x) = \begin{cases} \dfrac{1}{x(x+1)}\ ; & x = 1, 2, \dots \\ 0, & \text{otherwise} \end{cases}$$

Here

$$E(X) = \sum_{x=1}^{\infty} x f(x) = \sum_{x=1}^{\infty} \frac{1}{(x+1)}$$

$$= \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

$$= \left[\sum_{x=1}^{\infty} \frac{1}{x}\right] - 1$$

Since $\sum_{x=1}^{\infty} \frac{1}{x}$ is a divergent series, $E(X)$ does not exist and consequently no moment of X exists. However, the m.g.f. of X is given by

$$M_X(t) = \sum_{x=1}^{\infty} e^{tx} . f(x) = \sum_{x=1}^{\infty} \frac{e^{tx}}{x(x+1)}$$

$$= \sum_{x=1}^{\infty} \frac{z^x}{x(x+1)}, \quad (z = e^t) \qquad ...(*)$$

$$= \frac{z}{1.2} + \frac{z^2}{2.3} + \frac{z^3}{3.4} + \frac{z^4}{4.5} + \ldots$$

$$= z\left[1 - \frac{1}{2}\right] + z^2\left[\frac{1}{2} - \frac{1}{3}\right] + z^3\left[\frac{1}{3} - \frac{1}{4}\right] + z^4\left[\frac{1}{4} - \frac{1}{5}\right] + \ldots$$

$$= \left[z + \frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \ldots\right] - \frac{z}{2} - \frac{z^2}{3} - \frac{z^3}{4} - \frac{z^4}{5} - \ldots$$

$$= -\log(1-z) - \frac{1}{z}\left[\frac{z^2}{2} + \frac{z^3}{3} + \frac{z^4}{4} + \ldots\right]$$

$$= -\log(1-z) + 1 + \frac{1}{z}\log(1-z), \; |z| < 1$$

$$= 1 + \left[\frac{1}{z} - 1\right]\log(1-z), \; |z| < 1$$

$$= 1 + (e^{-t} - 1)\log(1 - e^t), \; t < 0$$

$$[\because |z| < 1 \;\Rightarrow\; |e^t| < 1 \;\Rightarrow\; t < 0]$$

And $\quad M_X(t) = 1$, for $t = 0$, [From (*)]

while for $t > 0$, $M_X(t)$ does not exist.

2. *A random variable X can have m.g.f. and some (or all) moments, yet the m.g.f. does not generate the moments.* For example, consider a discrete r.v. with probability function

$$P(X = 2^x) = \frac{e^{-1}}{x!}\;;\; x = 0, 1, 2, \ldots$$

Here $\qquad E(X^r) = \sum_{\xi=0}^{\infty} (2^x)^r P(X = 2^x) = e^{-1} \sum_{\xi=0}^{\infty} \frac{(2^r)^x}{x!}$

$$= e^{-1} . \exp(2^r) = \exp(2^r - 1)$$

Hence all the moments of X exist.

The m.g.f. of X, if it exists is given by

$$M_X(t) = \sum_{x=0}^{\infty} \exp(t . 2^x)\left(\frac{e^{-1}}{x!}\right) = e^{-1} \sum_{x=0}^{\infty} \exp(t . 2^x)\frac{1}{x!}$$

By D'Alembert's ratio test, the series on the R.H.S. converges for $t \leq 0$ and diverges for $t > 0$. Hence $M_X(t)$ cannot be differentiated at $t = 0$ and has no Maclaurin's expansion and consequently it does not generate moments.

3. *A r.v. X can have all or some moments; but m.g.f. does not exist except perhaps at one point.*

For example, let X be a r.v. with probability function

$$P(X = \pm 2^x) = \frac{e^{-1}}{2x\,!} \; ; \; x = 0, 1, 2, \ldots$$
$$= 0, \text{ otherwise.}$$

Since the distribution is symmetric about the line $X = 0$, all moments of odd order about origin vanish, *i.e.*,

$$E(X^{2r+1}) = 0 \;\Rightarrow\; \mu_{2r+1} = 0$$

$$E(X^{2r}) = \sum_{x=0}^{\infty} (\pm 2^x)^{2r} \left(\frac{1}{2e\,x\,!} \right) = \frac{1}{e} \sum_{x=0}^{\infty} \frac{(2^{2r})^x}{x\,!}$$

$$= \frac{1}{e} \cdot \exp(2^{2r}) = \exp(2^{2r} - 1)$$

Thus all the moments of X exist. The m.g.f. of X, if it exists, is given by

$$M_X(t) = \sum_{x=0}^{\infty} \left[\left(e^{t.2^x} + e^{-t.2^x} \right) \frac{1}{2e\,x\,!} \right]$$

$$= e^{-1} \sum_{x=0}^{\infty} \left[\frac{\cos(t.2^x)}{x\,!} \right]$$

which converges only for $t = 0$.

As an illustration of a continuous probability distribution, consider Pareto distribution with p.d.f.

$$p(x) = \frac{\theta \cdot a^{\theta}}{x^{\theta+1}} \; ; \qquad x \geq a \; ; \; \theta > 1$$

$$E(X^r) = \theta \cdot a^{\theta} \int_a^{\infty} x^{r-\theta-1}\, dx = \theta \cdot a^{\theta} \cdot \left| \frac{x^{r-\theta}}{r-\theta} \right|_a^{\infty},$$

which is finite iff $r - \theta < 0 \;\Rightarrow\; \theta > r$ and then

$$E(X^r) = \theta\, a^{\theta} \left[0 - \frac{a^{r-\theta}}{r-\theta} \right] = \frac{\theta \cdot a^r}{\theta - r} \; ; \; \theta > r$$

However, the m.g.f. is given by :

$$M_X(t) = \theta \cdot a^{\theta} \int_a^{\infty} \frac{e^{tx}}{x^{\theta+1}}\, dx,$$

which does not exist, since e^{tx} dominates $x^{\theta+1}$ and $(e^{tx} / x^{\theta+1}) \to \infty$ as $x \to \infty$ and hence the integral is not convergent.

For more illustrations see Student's t-distribution and Snedecor's F-distributions, for which m.g.f.'s do not exist, though the moments of all orders exist. [c.f. Chapter 14, § 14·2·4 and 14·5·2.].

Remark. The reason that m.g.f. is a poor tool in comparison with characteristic function (c.f. § 6·12) is that the domain of the dummy parameter 't' of the m.g.f. depends on the distribution of the r.v. under consideration, while characteristic function exists for all real t, $(-\infty < t < \infty)$. If m.g.f. is valid for t lying in an interval containing zero, then m.g.f. can be expanded with perhaps some additional restrictions.

6·10·2. Theorems on Moment Generating Functions.

Theorem 6·17. $M_{cX}(t) = M_X(ct)$, c being a constant. ...(6·58)

Proof. By def.,

$$\text{L.H.S.} = M_{cX}(t) = E(e^{t.cX})$$
$$\text{R.H.S.} = M_X(ct) = E(e^{ctX}) = \text{L.H.S.}$$

Theorem 6·18. *The moment generating function of the sum of a number of independent random variables is equal to the product of their respective moment generating functions.*

Symbolically, if $X_1, X_2, ..., X_n$ are independent random variables, then the moment generating function of their sum $X_1 + X_2 + ... + X_n$ is given by

$$M_{X_1+X_2+...+X_n}(t) = M_{X_1}(t)\, M_{X_2}(t) \ldots M_{X_n}(t) \qquad ...(6·59)$$

Proof. By definition,

$$M_{X_1+X_2+...+X_n}(t) = E\left[e^{t(X_1+X_2+...+X_n)} \right]$$
$$= E\left[e^{tX_1} . e^{tX_2} \ldots e^{tX_n} \right]$$
$$= E(e^{tX_1})\, E(e^{tX_2}) \ldots E(e^{tX_n})$$
$$(\because X_1, X_2, ..., X_n \text{ are independent})$$
$$= M_{X_1}(t) . M_{X_2}(t) \ldots M_{X_n}(t)$$

Hence the theorem.

Theorem 6·19. *Effect of change of origin and scale on M.G.F.* Let us transform X to the new variable U by changing both the origin and scale in X as follows :

$$U = \frac{X - a}{h}, \text{ where } a \text{ and } h \text{ are constants}$$

M.G.F. of U (about origin) is given by

$$M_U(t) = E(e^{tU}) = E[\exp\{ t(x - a)/h \}]$$
$$= E[e^{tX/h} . e^{-at/h}] = e^{-at/h} E(e^{tX/h})$$
$$= e^{-at/h} E(e^{Xt/h}) = e^{-at/h} M_X(t/h) \qquad ...(6·60)$$

where $M_X(t)$ is the m.g.f. of X about origin.

In particular, if we take $a = E(X) = \mu$ (say) and $h = \sigma_X = \sigma$ (say), then

$$U = \frac{X - E(X)}{\sigma_X} = \frac{X - \mu}{\sigma} = Z \text{ (say)},$$

is known as a *standard variate*. Thus the m.g.f. of a standard variate Z is given by

$$M_Z(t) = e^{-\mu t/\sigma} M_X(t/\sigma) \qquad ...(6·61)$$

Remark. $E(Z) = E\left(\dfrac{X - \mu}{\sigma}\right) = \dfrac{1}{\sigma} E(X - \mu)$

$$= \frac{1}{\sigma}\{E(X) - \mu\} = \frac{1}{\sigma}(\mu - \mu) = 0$$

and $$V(Z) = V\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2} V(X-\mu) \quad \text{[c.f. Cor. (i) Theorem 6·8]}$$

$$= \frac{1}{\sigma^2} V(X) = \frac{1}{\sigma} \sigma^2 = 1 \quad \text{[c.f. Cor. (iii) Theorem 6·8]}$$

$\therefore$ $E(Z) = 0$ and $V(Z) =$ *i.e.*, the mean and variance of a standard variate are 0 and 1 respectively.

6·10·3. Uniqueness Theorem of Moment Generating Function. *The moment generating function of a distribution, if it exists, uniquelly determines the distribution.* This implies that corresponding to a given probability distribution, there is only one m.g.f. (provided it exists) and corresponding to a given m.g.f., there is only one probability distribution. Hence $M_X(t) = M_Y(t) \Rightarrow X$ and Y are identically distributed. [For detailed discussion, see Uniqueness Theorem of Characteristic Functions – Theorem 6·27, page 6·90]

6·11. Cumulants. Cumulants generating function $K(t)$ is defined as

$$K_X(t) = \log_e M_X(t), \quad \text{...(6·62)}$$

provided the right-hand side can be expanded as a convergent series in powers of t. Thus

$$K_X(t) = \kappa_1 t + \kappa_2 \frac{t^2}{2!} + \dots + \kappa_r \frac{t^r}{r!} + \dots = \log M_X(t)$$

$$= \log\left[1 + \mu_1' t + \mu_2' \frac{t^2}{2!} + \mu_3' \frac{t^3}{3!} + \dots + \mu_r' \frac{t^r}{r!} + \dots\right] \quad \text{...(6·62a)}$$

where $\kappa_r =$ coefficient of $\dfrac{t^r}{r!}$ in $K_X(t)$ is called the rth cumulant. Hence

$$\kappa_1 t + \kappa_2 \frac{t^2}{2!} + \kappa_3 \frac{t^3}{3!} + \kappa_4 \frac{t^4}{4!} + \dots$$

$$= \left[\left(\mu_1' t + \mu_2' \frac{t^2}{2!} + \mu_3' \frac{t^3}{3!} + \mu_4' \frac{t^4}{4!} + \dots\right)\right.$$

$$- \frac{1}{2}\left(\mu_1' t + \mu_2' \frac{t^2}{2!} + \mu_3' \frac{t^3}{3!} + \dots\right)^2$$

$$\left. + \frac{1}{3}\left(\mu_1' t + \mu_2' \frac{t^2}{2!} + \dots\right)^3 - \frac{1}{4}\left(\mu_1' t + \mu_2' \frac{t^2}{2!} + \dots\right)^4 + \dots\right]$$

Comparing the coefficients of like powers of 't' on both sides, we get the relationship between the moments and cumulants. Hence, we have

$$\kappa_1 = \mu_1' = \text{Mean}, \quad \frac{\kappa_2}{2!} = \frac{\mu_2'}{2!} - \frac{\mu_1'^2}{2!} \quad \Rightarrow \quad \kappa_2 = \mu_2' - \mu_1'^2 = \mu_2$$

$$\frac{\kappa_3}{3!} = \frac{\mu_3'}{3!} - \frac{1}{2}\frac{2\mu_1'\mu_2'}{2!} + \frac{\mu_1'^3}{3} \quad \Rightarrow \quad \kappa_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3 = \mu_3$$

Also

$$\frac{\kappa_4}{4\,!}=\frac{\mu_4'}{4}-\frac{1}{2}\left(\frac{\mu_2'^2}{4}+\frac{2\,\mu_1'\,\mu_3'}{3\,!}\right)+\frac{1}{3}\,\frac{3\,\mu_1'^2\,\mu_2'}{2}-\frac{\mu_1'^4}{4}$$

$$\Rightarrow\ \kappa_4=\mu_4'-3\,\mu_2'^2-4\,\mu_1'\,\mu_3'+12\,\mu_1'^2\,\mu_2'-6\,\mu_1'^4$$

$$=(\mu_4'-4\mu_3'\,\mu_1'+6\mu_2'\,\mu_1'^2-3\mu_1'^4)-3(\mu_2'^2-2\mu_2'\,\mu_1'^2+\mu_1'^4)$$

$$=\mu_4-3(\mu_2'-\mu_1'^2)^2=\mu_4-3\mu_2^2=\mu_4-3\kappa_2^2 \qquad (\because \mu_2=\kappa_2)$$

$$\Rightarrow \qquad \mu_4=\kappa_4+3\,\kappa_2^2$$

Hence we have obtained :

$$\left.\begin{aligned}&\text{Mean}=\kappa_1\\&\mu_2=\kappa_2=\text{Variance}\\&\mu_3=\kappa_3\\&\mu_4=\kappa_4+3\,\kappa_2^2\end{aligned}\right\} \qquad \text{...(6·62}b)$$

Remarks. 1. These results are of fundamental importance and should be committed to memory.

2. If we differentiate both sides of (6·62*a*) w.r.t. t 'r' times and then put $t=0$, we get

$$\kappa_r=\left[\frac{d^r}{dt^r}K_X(t)\right]_{t=0} \qquad \text{...(6·62}c)$$

6·11·1. Additive Property of Cumulants. *The rth cumulant of the sum of independent random variables is equal to the sum of the rth cumulants of the individual variables.* Symbolically,

$$\kappa_r(X_1+X_2+\ldots+X_n)=\kappa_r(X_1)+\kappa_r(X_2)+\ldots+\kappa_r(X_n), \qquad \text{...(6·63)}$$

where X_i ; $i=1,2,\ldots,n$ are independent random variables.

Proof. We have, since X_i 's are independent.

$$M_{X_1+X_2+\ldots+X_n}(t)=M_{X_1}(t)\,M_{X_2}(t)\ldots M_{X_n}(t)$$

Taking logarithm of both sides, we get

$$K_{X_1+X_2+\ldots+X_n}(t)=K_{X_1}(t)+K_{X_2}(t)+\ldots+K_{X_n}(t)$$

Differentiating both sides w.r.t. t 'r' times and putting $t=0$, we get

$$\left[\frac{d^r}{dt^r}K_{X_1+X_2+\ldots+X_n}(t)\right]_{t=0}=\left[\frac{d^r}{dt^r}K_{X_1}(t)\right]_{t=0}+\left[\frac{d^r}{dt^r}K_{X_2}(t)\right]_{t=0}+\ldots+\left[\frac{d^r}{dt^r}K_{X_n}(t)\right]_{t=0}$$

$$\Rightarrow \qquad \kappa_r(X_1+X_2+\ldots+X_n)=\kappa_r(X_1)+\kappa_r(X_2)+\ldots+\kappa_r(X_n),$$

which establishes the result.

6·11·2. Effect of Change of Origin and Scale on Cumulants. If we take

$$U=\frac{X-a}{h}, \text{ then } M_U(t)=\exp(-at/h)M_X(t/h)$$

$$\therefore \qquad K_U(t) = -\frac{at}{h} + K_X(t/h)$$

$$\kappa_1' + \kappa_1' t + \kappa_2' \frac{t^2}{2!} + \ldots + \kappa_r' \frac{t^r}{r!} + \ldots = -\frac{at}{h} + \kappa_1 (t/h) + \kappa_2 \frac{(t/h)^2}{2!} + \ldots + \kappa_r \frac{(t/h)^r}{r!} + \ldots$$

where κ_r' and κ_r are the rth cumulants of U and X respectively.

Comparing coefficients, we get

$$\kappa_1' = \frac{\kappa_1 - a}{h} \quad \text{and} \quad \kappa_r' = \frac{\kappa_r}{h^r} \; ; \; r = 2, 3, \ldots \qquad \ldots(6{\cdot}63a)$$

Thus we see that except the first cumulant, all cumulants are independent of change of origin. But the cumulants are not invariant of change of scale as the rth cumulant of U is $(1/h^r)$ times the rth cumulant of the distribiution of X.

Example 6·37. *Let the random variable X assume the value 'r' with the probability law :*

$$P(X = r) = q^{r-1} p \; ; \; r = 1, 2, 3, \ldots$$

Find the m.g.f. of X and hence its mean and variance.

[Calicut Univ. B.Sc., Oct. 1992]

Solution. $M_X(t) = E(e^{tX})$

$$= \sum_{r=1}^{\infty} e^{tr} \; q^{r-1} p = \frac{p}{q} \sum_{r=1}^{\infty} (qe^t)^r$$

$$= \frac{p}{q} qe^t \sum_{r=1}^{\infty} (qe^t)^{r-1} = pe^t \left[1 + qe^t + (qe^t)^2 + \ldots\right]$$

$$= \left(\frac{pe^t}{1 - qe^t}\right)$$

If dash (′) denotes the differentiation w.r.t. t, then we have

$$M_X'(t) = \frac{pe^t}{(1 - qe^t)^2}, \quad M_X''(t) = pe^t \frac{(1 + qe^t)}{(1 - qe^t)^3}$$

$$\therefore \qquad \mu_1' \text{ (about origin)} = M_X'(0) = \frac{p}{(1-q)^2} = \frac{1}{p}$$

$$\mu_2' \text{ (about origin)} = M_X''(0) = \frac{p(1+q)}{(1-q)^3} = \frac{1+q}{p^2}.$$

Hence $\qquad \text{mean} = \mu_1' \text{ (about origin)} = \dfrac{1}{p}$

and $\qquad \text{variance} = \mu_2 = \mu_2' - \mu_1'^2 = \dfrac{1+q}{p^2} - \dfrac{1}{p^2} = \dfrac{q}{p^2}$

Example 6·38. *The probability density function of the random variable X follows the following probability law :*

$$p(x) = \frac{1}{2\theta}\, exp\left(-\frac{|x-\theta|}{\theta}\right),\ -\infty < x < \infty$$

Find M.G.F. of X. Hence or otherwise find E (X) and V (X).

[Punjab Univ. M.A.(Eco.), 1991]

Solution. The moment generating function of X is

$$M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} \frac{1}{2\theta} \exp\left(-\frac{|x-\theta|}{\theta}\right) e^{tx}\, dx$$

$$= \int_{-\infty}^{\theta} \frac{1}{2\theta} \exp\left(-\frac{|\theta-x|}{\theta}\right) e^{tx}\, dx + \int_{\theta}^{\infty} \frac{1}{2\theta} \exp\left(-\frac{|x-\theta|}{\theta}\right) e^{tx}\, dx$$

For $x \in (-\infty, \theta)$, $x - \theta < 0 \Rightarrow \theta - x > 0$

$\therefore \qquad |x-\theta| = \theta - x \quad \forall\, x \in (-\infty, \infty)$

Similarly, $\quad |x-\theta| = x - \theta \quad \forall\, x \in (\theta, \infty)$

$$\therefore \quad M_X(t) = \frac{e^{-1}}{2\theta} \int_{-\infty}^{\theta} \exp\left[x\left(t+\frac{1}{\theta}\right)\right] dx + \frac{e}{2\theta} \int_{\theta}^{\infty} \exp\left[-x\left(\frac{1}{\theta}-t\right)\right] dx$$

$$= \frac{e^{-1}}{2\theta} \cdot \frac{1}{\left(t+\frac{1}{\theta}\right)} \cdot \exp\left[\theta\left(t+\frac{1}{\theta}\right)\right] + \frac{e}{2\theta} \cdot \frac{1}{\left(\frac{1}{\theta}-t\right)} \cdot \exp\left[-\theta\left(\frac{1}{\theta}-t\right)\right]$$

$$= \frac{e^{\theta t}}{2(\theta t+1)} + \frac{e^{\theta t}}{2(1-\theta t)} = \frac{e^{\theta t}}{1-\theta^2 t^2}$$

$$= e^{\theta t}(1-\theta^2 t^2)^{-1}$$

$$= \left[1 + \theta t + \frac{\theta^2 t^2}{2!} + \ldots\right] \left[1 + \theta^2 t^2 + \theta^4 t^4 + \ldots\right]$$

$$= 1 + \theta t + \frac{3\,\theta^2 t^2}{2!} + \ldots$$

$\therefore \qquad E(X) = \mu' = \text{Coefficient of } t \text{ in } M_X(t) = \theta$

$\qquad \mu_2' = \text{Coefficient of } \dfrac{t^2}{2!} \text{ in } M_X(t) = 3\,\theta^2$

Hence $\quad \text{Var}(X) = \mu_2' - \mu_1'^2 = 3\,\theta^2 - \theta^2 = 2\,\theta^2$

Example 6·39. *If the moments of a variate X are defined by*

$$E(X^r) = 0{\cdot}6\ ;\ r = 1, 2, 3, \dots$$

show that $P(X = 0) = 0{\cdot}4, P(X = 1) = 0{\cdot}6, P(X \geq 2) = 0.$

[Delhi Univ. B.Sc. (Maths Hons.), 1985]

Solution. The m.g.f. of variate X is :

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r' = 1 + \sum_{r=1}^{\infty} \frac{t^r}{r!}(0{\cdot}6)$$

$$= 0{\cdot}4 + 0{\cdot}6 \sum_{r=0}^{\infty} \frac{t^r}{r!} = 0{\cdot}4 + 0{\cdot}6\, e^t \qquad \dots(i)$$

But $$M_X(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} P(X = x)$$

$$= P(X = 0) + e^t . P(X = 1) + \sum_{x=2}^{\infty} e^{tx} . P(X = x) \qquad \dots(ii)$$

From *(i)* and *(ii)*, we get :

$$P(X = 0) = 0{\cdot}4\ ;\ P(X = 1) = 0{\cdot}6\ ;\ P(X \geq 2) = 0.$$

Remark. In fact *(i)* is the m.g.f. of Bernoulli variate X with $P(X = 0) = q = 0{\cdot}4$ and $P(X = 1) = p = 0{\cdot}6$ [See § 7·1·2] and $P(X \geq 2) = 0$.

Example 6·40. *Find the moment generating function of the random variable whose moments are*

$$\mu_r' = (r + 1)!\, 2^r$$

Solution. The m.g.f. is given by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu'_r = \sum_{r=0}^{\infty} \frac{t^r}{r!} (r + 1)!\, 2^r$$

$$= \sum_{r=0}^{\infty} (r + 1)(2t)^r$$

$$M_X(t) = 1 + 2 . (2t) + 3(2t)^2 + 4(2t)^3 + \dots$$

$$= (1 - 2t)^{-2}$$

Aliter. The R.H.S. is an arithmetic-geometric series with ratio $r = (2t)$

Let $$S = 1 + 2r + 3r^2 + 4r^3 + \dots$$

Then $$rS = r + 2r^2 + 3r^3 + \dots$$

$$\therefore \quad (1 - r)S = 1 + r + r^2 + \dots = \frac{1}{(1 - r)}$$

$$\Rightarrow \quad S = \frac{1}{(1 - r)^2} = (1 - r)^{-2} = (1 - 2t)^{-2}$$

Remark. This is the m.g.f. of Chi-square (χ^2) distribution with parameter (degrees of freedom) $n = 2$ [c.f. Chapter 13].

Example 6·41. *If μ_r' is the rth moment about origin, prove that*

$$\mu_r' = \sum_{j=1}^{r} \binom{r-1}{j-1} \mu_{r-j} \kappa_j ,$$

where κ_j is the jth cumulant.

Solution. Differentiating both sides of (6·62*a*) in § 6·11, page 6·72 w.r.t. t, we get

$$\kappa_1 + \kappa_2 t + \kappa_3 \frac{t^2}{2\,!} + \ldots + \kappa_r \frac{t^{r-1}}{(r-1)\,!} + \ldots$$

$$= \frac{\mu_1' + \mu_2' t + \mu_3' \frac{t^2}{2\,!} + \ldots + \mu_r' \frac{t^{r-1}}{(r-1)\,!} + \ldots}{1 + \mu_1' t + \mu_2' \frac{t^2}{2\,!} + \ldots + \mu_r' \frac{t^r}{r\,!} + \ldots}$$

$$\Rightarrow \quad \left[\kappa_1 + \kappa_2 t + \kappa_3 \frac{t^2}{2\,!} + \ldots + \kappa_r \frac{t^{r-1}}{(r-1)\,!} + \ldots\right]$$

$$\times \left[1 + \mu_1' t + \mu_2' \frac{t^2}{2\,!} + \ldots + \mu_r' \frac{t^r}{r\,!} + \ldots\right]$$

$$= \mu_1' + \mu_2' t + \mu_3' \frac{t^2}{2\,!} + \ldots + \mu_r' \frac{t^{r-1}}{(r-1)\,!} + \ldots$$

Comparing the coefficient of $\dfrac{t^{r-1}}{(r-1)\,!}$ on both sides, we get

$$\mu_r' = \kappa_1 \,.\, \mu_{r-1}' + (r-1)\, \kappa_2 \,.\, \mu_{r-2}' + \binom{r-1}{2} \kappa_3 \,.\, \mu_{r-3}' + \ldots + \kappa_r$$

$$= \binom{r-1}{0} \mu_{r-1}' \kappa_1 + \binom{r-1}{1} \mu_{r-2}' \kappa_2 + \binom{r-1}{2} \mu_{r-3}' \kappa_3 + \ldots + \binom{r-1}{r-1} \mu_0' \kappa_r$$

$$= \sum_{j=1}^{r} \binom{r-1}{j-1} \mu_{r-j}' \kappa_j,$$

which is the required result.

6·12. Characteristic Function. In some cases m.g.f. does not exist, since the integral $\int_{-\infty}^{\infty} e^{tx} f(x)\, dx$ or the series $\sum_x e^{tx} p(x)$ does not converge absolutely for real values of t for some distributions. For example, for the continuous probability distribution

$$dF(x) = C \frac{1}{(1+x^2)^m} dx \; ; m > 1, -\infty < x < \infty, \; .$$

the m.g.f. does not exist, since the integral

$$M_X(t) = C \int_{-\infty}^{\infty} e^{tx} \frac{1}{(1+x^2)^m} dx,$$

does not converge absolutely for finite positive values of m because the function e^{tx} dominates the function x^{2m} so that $e^{tx}/x^{2m} \to \infty$ as $x \to \infty$.

Again, for the discrete probability distribution

$$\left.\begin{aligned} f(x) &= \frac{6}{\pi^2 x^2}\ ;\ x = 1, 2, 3, \ldots \\ &= 0,\ \text{elsewhere} \end{aligned}\right\}$$

$$M_X(t) = \sum_x e^{tx} f(x) = \frac{6}{\pi^2} \sum_{x=1}^{\infty} \left(\frac{e^{tx}}{x^2}\right)$$

The series is not convergent (by D'Alembert's Ratio Test) for $t > 0$. Thus there does not exist a positive number h such that $M_X(t)$ exists for $-h < t < h$. Hence $M_X(t)$ does not exist in this case also.

A more serviceable function than the m.g.f. is what is known as characteristic function and is defined as

$$\phi_X(t) = E(e^{itX}) = \int e^{itx} f(x)\, dx$$

(for continuous probability distributions)

$$= \sum_x e^{itx} f(x)$$

(for discrete probability distributions) ...(6·64)

If $F_X(x)$ is the distsribution function of a continuous random variable X, then

$$\phi_X(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x) \qquad \text{...(6·64}a\text{)}$$

Obviously $\phi(t)$ is a complex valued function of real variable t. It may be noted that

$$|\phi(t)| = \left| \int e^{itx} f(x)\, dx \right| \le \int |e^{itx}|\, f(x)\, dx = \int f(x)\, dx = 1,$$

since $\quad |e^{itx}| = |\cos tx + i \sin tx|^{1/2} = (\cos^2 tx + \sin^2 tx)^{1/2} = 1$

Since $|\phi(t)| \le 1$, characteristic function $\phi_X(t)$ always exists.

Yet another advantage of characteristic function lies in the fact that it uniquely determines the distribution function, *i.e.*, if the characteristic function of a distribution is given, the distribution can be uniquely determined by the theorem, known as the Uniqueness Theorem of Characteristic Functions [c.f. Theorem 6·27 page 6·90].

6·12·1. Properties of Characteristic Functions. For all real 't', we have

(i) $\phi(0) = \int_{-\infty}^{\infty} dF(x) = 1$ ...(6·64b)

(ii) $|\phi(t)| \le 1 = \phi(0)$ ...(6·64c)

(iii) $\phi(t)$ is continuous everywhere, *i.e.*, $\phi(t)$ is a continuous function of t in $(-\infty, \infty)$. Rather $\phi(t)$ is uniformly continuous in 't'.

Proof. For $h \neq 0$, $|\phi_X(t+h) - \phi_X(t)| = \left| \int_{-\infty}^{\infty} \left[e^{i(t+h)x} - e^{itx} \right] dF(x) \right|$

$$\leq \int_{-\infty}^{\infty} \left| e^{itx} (e^{ihx} - 1) \right| dF(x)$$

$$= \int_{-\infty}^{\infty} \left| e^{ihx} - 1 \right| dF(x) \qquad ...(*)$$

The last integral does not depend on t. If it tends to zero as $h \to 0$ then $\phi_X(t)$ is uniformly continuous in 't'.

Now $\qquad \left| e^{ihx} - 1 \right| \leq \left| e^{ihx} \right| + 1 = 2$

$$\therefore \quad \int_{-\infty}^{\infty} \left| e^{ihx} - 1 \right| dF(x) \leq 2 \int_{-\infty}^{\infty} dF(x) = 2.$$

Hence by Dominated Convergence Theorem (D.C.T.), taking the limit inside the integral sign in (*), we get

$$\lim_{h \to 0} \left| \phi_X(t+h) - \phi_X(t) \right| \leq \int_{-\infty}^{\infty} \lim_{h \to 0} \left| e^{ihx} - 1 \right| dF(x) = 0$$

$$\Rightarrow \quad \lim_{h \to 0} \phi_X(t+h) = \phi_X(t) \ \forall \ t.$$

Hence $\phi_X(t)$ is uniformly continuous in 't'.

(*iv*) $\phi_X(-t)$ and $\phi_X(t)$ are conjugate functions, *i.e.*, $\phi_X(-t) = \overline{\phi_X(t)}$, where $\bar{a}$ is the complex conjugate of a.

Proof. $\qquad \phi_X(t) = E(e^{itX}) = E[\cos tX + i \sin tX]$

$$\begin{aligned} \Rightarrow \quad \overline{\phi_X(t)} &= E[\cos tX - i \sin tX] \\ &= E[\cos(-t)X + i \sin(-t)X] \\ &= E(e^{-itX}) = \phi_X(-t). \end{aligned}$$

6·12·2. Theorems on Characteristic Function.

Theorem 6·20. *If the distribution function of a r.v. X is symmetrical about zero, i.e.,*

$$1 - F(x) = F(-x) \quad \Rightarrow \quad f(-x) = f(x),$$

then $\phi_X(t)$ is real valued and even function of t.

Proof. By definition we have

$$\phi_X(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\, dx = \int_{-\infty}^{\infty} e^{-ity} f(-y)\, dy \qquad (x = -y)$$

$$= \int_{-\infty}^{\infty} e^{-ity} f(y)\, dy \qquad [\because f(-y) = f(y)]$$

$$= \phi_X(-t) \qquad ...(*)$$

$\Rightarrow \quad \phi_X(t)$ is an even function of t.

Also $\qquad \overline{\phi_X(t)} = \phi_X(-t) \qquad$ [c.f. Property (*iv*) § 6·12·1]

$\therefore \qquad \overline{\phi_X(t)} = \phi_X(-t) = \phi_X(t) \qquad$ (From *)

Hence $\phi_X(t)$ is a real valued function of t.

Theorem 6·21. *If X is some random variable with characteristic function $\phi_X(t)$, and if $\mu_r' = E(X^r)$ exists, then*

$$\mu_r' = (-i)^r \left[\frac{\partial^r}{\partial t^r} \phi(t) \right]_{t=0} \qquad \ldots(6{\cdot}65)$$

Proof. $\phi(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\,dx$

Differentiating (under the integral sign) 'r' times w.r.t. t, we get

$$\frac{\partial^r}{\partial t^r} \phi(t) = \int_{-\infty}^{\infty} (ix)^r \cdot e^{itx} f(x)\,dx = (i)^r \int_{-\infty}^{\infty} x^r e^{itx} f(x)\,dx$$

$$\therefore \quad \left[\frac{\partial^r}{\partial t^r} \phi(t) \right]_{t=0} = (i)^r \left[\int_{-\infty}^{\infty} x^r e^{itx} f(x)\,dx \right]_{t=0}$$

$$= (i)^r \int_{-\infty}^{\infty} x^r f(x)\,dx = i^r E(X^r) = i^r \mu_r'$$

$$\text{Hence } \mu_r' = \left(\frac{1}{i}\right)^r \left[\frac{\partial^r}{\partial t^r} \phi(t) \right]_{t=0} = (-i)^r \left[\frac{\partial^r}{\partial t^r} \phi(t) \right]_{t=0}$$

The theorems, *viz.*, 6·17, 6·18 and 6·19 on m.g.f. can be easily extended to the characteristic functions as given below.

Theorem 6·22. $\phi_{cX}(t) = \phi_X(ct)$, *c, being a constant.*

Theorem 6·23. *If X_1 and X_2 are independent random variables, then*

$$\phi_{X_1+X_2}(t) = \phi_{X_1}(t)\ \phi_{X_2}(t) \qquad \ldots(*)$$

More, generally for independent random variables X_i ; $i = 1, 2, \ldots, n$, we have

$$\phi_{X_1+X_2+\ldots+X_n}(t) = \phi_{X_1}(t)\ \phi_{X_2}(t)\ \ldots \phi_{X_n}(t)$$

Important Remark. Converse of (*) is not true, *i.e.*,

$\phi_{X_1+X_2}(t) = \phi_{X_1}(t)\ \phi_{X_2}(t)$ does not imply that X_1 and X_2 are independent.

For example, let X_1 be a standard Cauchy variate with p.d.f.

$$f(x) = \frac{1}{\pi(1+x^2)}, \quad -\infty < x < \infty$$

Then $\phi_{X_1}(t) = e^{-|t|}$ (c.f. Chapter 8)

Let $X_2 \equiv X_1$, *i.e.*, $P(X_1 = X_2) = 1$. $\ldots(**)$

Then $\phi_{X_2}(t) = e^{-|t|}$

Now $\phi_{X_1+X_2}(t) = \phi_{2X_1}(t) = \phi_{X_1}(2t) = e^{-2|t|}$

$= \phi_{X_1}(t)\ \phi_{X_2}(t)$

i.e., (*) is satisfied but obviously X_1 and X_2 are not independent, because of (**).

In fact, (*) will hold even if we take $X_1 = aX$ and $X_2 = bX$, a and b being real numbers so that X_1 and X_2 are connected by the relation :

$$\frac{X_1}{a} = X = \frac{X_2}{b} \quad \Rightarrow \quad aX_2 = bX_1 .$$

As another example let us consider the joint p.d.f. of two random variables X and Y given by

$$f(x, y) = \frac{1}{4a^2}[1 + xy(x^2 - y^2)]\ ; \ |x| \le a,\ |y| \le a,\ a > 0$$

$$= 0, \text{ elsewhere}$$

Then the marginal p.d.f.'s of X and Y are given by

$$g(x) = \int_{-a}^{a} f(x, y)\, dy = \frac{1}{2a}\ ;\ |x| \leq a \qquad \text{(on simplification)}$$

$$h(y) = \int_{-a}^{a} f(x, y)\, dx = \frac{1}{2a}\ ;\ |y| \leq a$$

Then

$$\phi_X(t) = \int_{-\infty}^{\infty} e^{itx} g(x)\, dx = \frac{1}{2a} \int_{-a}^{a} e^{itx}\, dx$$

$$= \frac{e^{iat} - e^{-iat}}{2ait} = \frac{\sin at}{at}$$

Similarly $\qquad \phi_Y(t) = \dfrac{\sin at}{at}$

$$\therefore \qquad \phi_X(t)\ \phi_Y(t) = \left(\frac{\sin at}{at}\right)^2 \qquad \ldots(*)$$

The p.d.f. $k(z)$ of the random variable $Z = X + Y$ is given by the convolution of p.d.f.'s of X and Y, *viz.*,

$$k(z) = \int f(u, z-u)\, du$$

$$= \frac{1}{4a^2} \int \left[1 + u(z-u)\left\{u^2 - (z-u)^2\right\}\right] du$$

$$= \frac{1}{4a^2} \int (1 + 3z^2u^2 - 2zu^3 - z^3u)\, du,$$

the limits of integration for u being in terms of z and are given by (left as an exercise to the reader)

$$-a \leq u \leq z + a\ ;\ u \leq 0$$

and

$$z - a \leq u \leq a\ ;\ u > 0$$

Thus

$$k(z) = \begin{cases} \dfrac{1}{4a^2} \displaystyle\int_{-a}^{z+a} (1 + 3z^2u^2 - 2zu^3 - z^3u)\, du = \dfrac{2a+z}{4a^2}\ ; -2a \leq z \leq 0 \\ \dfrac{1}{4a^2} \displaystyle\int_{z-a}^{a} (1 + 3z^2u^2 - 2zu^3 - z^3u)\, du = \dfrac{2a-z}{4a^2}\ ; 0 < z \leq 2a \\ 0, \text{ elsewhere} \end{cases}$$

Now

$$\phi_{X+Y}(t) = \phi_Z(t) = \int_{-2a}^{2a} e^{itz} k(z)\, dz$$

$$= \int_{-2a}^{0} \left(\frac{2a+z}{4a^2}\right) e^{itz}\, dz + \int_{0}^{2a} \left(\frac{2a-z}{4a^2}\right) e^{itz}\, dz$$

$$=\int_0^{2a}\left(e^{-itz}+e^{itz}\right)\left(\frac{2a-z}{4a^2}\right).\,dz$$

[Changing z to $-z$ in the first integral]

$$=\frac{1}{2a^2}\int_0^{2a}(2a-z)\cos tz\,dz$$

$$=\frac{2-2\cos 2at}{4a^2t^2}=\frac{1-\cos 2at}{2a^2t^2} \quad \text{(on simplification)}$$

$$=\left(\frac{\sin at}{at}\right)$$

$$=\phi_X(t)\,.\,\phi_Y(t) \quad \text{[From (*)]}$$

But $\qquad g(x)\,.\,h(y)\neq f(x,y)$

$\Rightarrow \qquad X$ and Y are not independent.

However

$$\phi_{X_1,X_2}(t_1,t_2)=E\left(e^{it_1X_1+it_2X_2}\right)=\phi_{X_1}(t)\,.\,\phi_{X_2}(t)$$

implies that X_1 and X_2 are independent.

(For proof see Theorem 6·28)

Theorem 6·24. *Effect of Change of Origin and Scale on Characteristic Function. If $U=\dfrac{X-a}{h}$, a and h being constants, then*

$$\phi_U(t)=e^{-iat/h}\,\phi_X(t/h)$$

In particular if we take $a=E(X)=\mu$ (say) and $h=\sigma_X=\sigma$ then the characteristic function of the standard variate

$$Z=\frac{X-E(X)}{\sigma_X}=\frac{X-\mu}{\sigma},$$

is given by $\qquad \phi_Z(t)=e^{-i\mu t/\sigma}\,\phi_X(t/\sigma) \qquad$ ...(6·66)

Definition. *A random variable X is said to be a Lattice variable or be lattice distributed, if for some $h>0$,*

$$P\left[\frac{X}{h}\text{ is an integer}\right]=1,$$

h is called a mesh.

Theorem 6·25. *If $|\phi_X(s)|=1$ for some $s\neq 0$, then for some real a, $X-a$ is a Lattice variable with mesh $h=2\pi/|s|$.*

Proof. Consider any fixed t. We can write

$\phi_X(t)=|\phi_X(t)|\;e^{iat}$, *(a dependent on t), since any complex number z can be written as* $z=|z|\,e^{i\theta}$.

$$\therefore \qquad |\phi_X(t)|=e^{-iat}\,\phi_X(t)=\phi_{(X-a)}(t)$$

$$=E\left[\cos t(X-a)+i\sin t(X-a)\right]=E\left[\cos t(X-a)\right]$$

since left-hand side being real, we must have $E[\sin t(X-a)]=0$.

$$\therefore \qquad 1-|\phi_X(t)|=E\left[1-\cos t(X-a)\right] \qquad \text{...(*)}$$

If $|\phi_X(s)| = 1$, $s \neq 0$ then for some a dependent on s, we have from (*)

$$E[1 - \cos s(X - a)] = 0 \qquad \ldots(**)$$

But since $1 - \cos s(X - a)$ is a non-negative random variable, (**)

$\Rightarrow \quad P[1 - \cos s(X - a) = 0] = 1$

$\Rightarrow \quad P[\cos s(X - a) = 1] = 1$

$\Rightarrow \quad P[s(X - a) = 2n\pi] = 1$

$\Rightarrow \quad P\left[(X - a) = \dfrac{2n\pi}{|s|}\right] = 1$, for some $n = 0, 1, 2, \ldots$

Thus $(X - a)$ is a Lattice variable with mesh $h = \dfrac{2\pi}{|s|}$.

6·12·3. Necessary and Sufficient Conditions for a Function $\phi(t)$ to be Characteristic Function. Properties *(i)* to *(iv)* in § 6·12·1 are merely the necessary conditions for a function $\phi(t)$ to be the characteristic function of an r.v. X. Thus if a function $\phi(t)$ does not satisfy any one of these four conditions, it cannot be the characteristic function of an r.v. X. For example, the function

$$\phi(t) = \log(1 + t),$$

cannot be the c.f. of r.v. X since $\phi(0) = \log 1 = 0 \neq 1$.

These conditions are, however, not sufficient. It has been shown (*c.f.* Methods of Mathematical Statistics by H. Cramer) that if $\phi(t)$ is near $t = 0$ of the form,

$$\phi(t) = 1 + 0(t^{2} + \delta),\ \delta > 0 \qquad \ldots(*)$$

where $0(t^r)$ divided by t^r tends to zero as $t \to 0$, then $\phi(t)$ cannot be the characteristic function unless it is identically equal to one. Thus, the functions

(i) $\phi(t) = e^{-t^4} = 1 + 0(t^4)$

(ii) $\phi(t) = \dfrac{1}{1 + t^4} = 1 + 0(t^4)$

being of the form (*) are not characteristic functions, though both satisfy all the necessary conditions.

We give below a set of sufficient but not necessary conditions, due to Polya for a function $\phi(t)$ to be the characteristic function :

$\phi(t)$ is a characteristic function if

(1) $\phi(0) = 1$,

(2) $\phi(t) = \phi(-t)$

(3) $\phi(t)$ is continuous

(4) $\phi(t)$ is convex for $t > 0$, *i.e.*, for $t_1, t_2 > 0$,

$$2\phi\left[\tfrac{1}{2}(t_1 + t_2)\right] \leq \phi(t_1) + \phi(t_2)$$

(5) $\lim\limits_{t \to \infty} \phi(t) = 0$.

Hence by Polya's conditions the functions $e^{-|t|}$ and $[1 + |t|]^{-1}$ are characteristic functions. However, Polya's conditions are only sufficient and not necessary for a characteristic function. For example, if $X \sim N(\mu, \sigma^2)$,

$$\phi(t) = e^{it\mu - t^2\sigma^2/2} \qquad [c.f. \S\ 8{\cdot}5]$$

and $\phi(-t) \neq \phi(t)$.

Various necessary and sufficient conditions are known, the simplest seems to be the following, due to Cramer.

"In order that a given, bounded and continuous function $\phi(t)$ should be the characteristic function of a distribution, it is necessary and sufficient that $\phi(0) = 1$ and that the function

$$\phi(x, A) = \int_0^A \int_0^A \phi(t-u)\, e^{(t-u)ix} dt\, du$$

is real and non-negative for all real x and all $A > 0$.

6·12·4. Multi-variate Characteristic Function. Then

$$X = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} \text{ and } t = \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_n \end{bmatrix}, \; t \text{ real}$$

be $z \times 1$ column vectors. Then characteristic function of X is defined as

$$\phi_X(t) = E(e^{it'X}) = E[e^{i(t_1X_1 + t_2X_2 + \dots + t_nX_n)}] \qquad \dots(6{\cdot}67)$$

We may also write it as

$$\phi_{X_1, X_2, \dots X_n}(t_1, t_2, \dots, t_n) \text{ or } \phi_X(t_1, t_2, \dots, t_n)$$

Some Properties.

(i) $\phi_X(0, 0, \dots 0) = 1$

(ii) $\phi_{-X}(t) = \overline{\phi_X(t)}$

(iii) $|\phi_X(t)| \leq 1$

(iv) $\phi_X(t)$ is uniformly continuous in n-dimensional Euclidian space.

(v) If $f_X(.)$ is p.d.f. of X,

$$\phi_X(t) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} f_X(x_1, x_2, \dots, x_n)\, e^{i\Sigma t_j x_j}\, dx_1\, dx_2 \dots dx_n$$

(vi) $\phi_X(t) = \phi_Y(t)$ for all t, then X and Y have the same distribuiton.

(vii) If $\displaystyle\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left| \phi_X(t_1, t_2, \dots t_n) \right| dt_1\, dt_2 \dots dt_n < \infty$,

then X is absolutely continuous and has a uniformly continuous p.d.f.

$$f_X(x_1, x_2, \dots, x_n) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\Sigma t_j x_j}\, \phi_X(t_1, t_2, \dots, t_n)\, dt_1\, dt_2 \dots dt_n$$

(viii) The random variables $X_1, X_2, \dots, X_n$ are (mutually) independent iff

$$\phi_{X_1, X_2, \dots X_n}(t_1, t_2, \dots, t_n) = \phi_{X_1}(t_1)\ \phi_{X_2}(t_2) \dots \phi_{X_n}(t_n)$$

Remark. *Multivariate Moment Generating Function.* Similarly, the m.g.f. of vector $X = (X_1, X_2, \dots, X_n)'$ is given by :

$$M_X(t) = E(e^{t'X}) = E(e^{t_1X_1 + t_2X_2 + \dots t_nX_n}) \qquad \dots(6\cdot68)$$

We may also write :

$$M_X(t) = M_{X_1, X_2 \dots X_n}(t_1, t_2, \dots, t_n) = E(e^{t_1X_1 + t_2X_2 + \dots t_nX_n})$$

In particular, for two variates X_1 and X_2

$$M_X(t) = M_{X_1, X_2}(t_1, t_2) = E(e^{t_1X_1 + t_2X_2}) = \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{t_1^r}{r!} \frac{t_2^s}{s!} E(X_1^r X_2^s), \qquad \dots(6\cdot69)$$

provided it exists for $-h_1 < t_1 < h_1$ and $-h_2 < t_2 < h_2$, where h_1 and h_2 are positive.

$$M_{X_1, X_2}(t_1, 0) = E(e^{t_1X_1}) = M_{X_1}(t_1) \qquad \dots(6\cdot69a)$$

$$M_{X_1, X_2}(0, t_2) = E(e^{t_2X_2}) = M_{X_2}(t_2) \qquad \dots(6\cdot69b)$$

If $M(t_1, t_2)$ exists, the moments of all orders of X and Y exist and are given by:

$$E(X_2^r) = \left[\frac{\partial^r M(t_1, t_2)}{\partial t_2^r}\right]_{t_1 = t_2 = 0} = \frac{\partial^r M(0, 0)}{\partial t_2^r} \qquad \dots(6\cdot70)$$

$$E(X_1^r) = \left[\frac{\partial^r M(t_1, t_2)}{\partial t_1^r}\right]_{t_1 = t_2 = 0} = \frac{\partial^r M(0, 0)}{\partial t_1^r} \qquad \dots(6\cdot70a)$$

$$E(X_1^r X_2^s) = \left[\frac{\partial^{r+s} M(t_1, t_2)}{\partial t_1^r \partial t_2^s}\right]_{t_1 = t_2 = 0} = \frac{\partial^{r+s} M(0, 0)}{\partial t_1^r \partial t_2^s} \qquad \dots(6\cdot70b)$$

Cumulant generating function of $X = (X_1, X_2)'$ is given by :

$$K_{X_1, X_2}(t_1, t_2) = \log M_{X_1, X_2}(t_1, t_2). \qquad \dots(6\cdot71)$$

Example 6·42. *For a distribution, the cumulants are given by*

$$\kappa_r = n[(r-1)!], \quad n > 0$$

Find the characteristic function. **(Delhi Univ. B.Sc. (Stat. Hons.), 1990)**

Solution. The cumulant generating function $K(t)$, if it exists, is given by

$$K(t) = \sum_{r=1}^{\infty} \frac{(it)^r}{r!} \kappa_r = \sum_{r=1}^{\infty} \frac{(it)^r}{r!} n\{(r-1)!\} = n \sum_{r=1}^{\infty} \frac{(it)^r}{r}$$

$$= n\left[it + \frac{(it)^2}{2} + \frac{(it)^3}{3} + \dots\right] = n[\log(1 - it)]$$

$$= -n \log(1 - it) = \log(1 - it)^{-n}$$

Also we have

$$K(t) = \log \phi(t) = \log(1 - it)^{-n}$$

$$\therefore \qquad \phi(t) = (1 - it)^{-n}$$

Remark. This is the characteristic function of the gamma distribution : (*c.f.* § 8·3·1)

$$f(x) = \frac{e^{-x}\, x^{n-1}}{\Gamma(n)}\,;\; n > 0,\ 0 < x < \infty.$$

Example 6·43. *The moments about origin of a distribution are given by*

$$\mu_r' = \frac{\Gamma(\nu + r)}{\Gamma(\nu)}.$$

Find the characteristic function.

(Madurai Kamaraj Univ. B.Sc., 1990)

Solution. We have

$$\phi(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r\,!}\,\mu_r' = \sum_{r=0}^{\infty} \frac{(it)^r}{r\,!}\cdot\frac{\Gamma(\nu + r)}{\Gamma(\nu)}$$

$$= \sum_{r=0}^{\infty} \frac{(it)^r}{r\,!}\cdot\frac{(\nu + r - 1)\,!}{(\nu - 1)\,!}$$

$$= \sum_{r=0}^{\infty} (it)^r\,.\,{}^{\nu + r - 1}C_r = \sum_{r=0}^{\infty} (-1)^r.\,{}^{-\nu}C_r\,(it)^r$$

$$[\because\ {}^{-\nu}C_r = (-1)^r.\,{}^{\nu + r - 1}C_r \;\Rightarrow\; (-1)^r\,.\,{}^{-\nu}C_r = {}^{\nu + r - 1}C_r]$$

$$\therefore \qquad \phi(t) = \sum_{r=0}^{\infty} {}^{-\nu}C_r\,(-it)^r = (1 - it)^{-\nu}$$

Example 6·44. *Show that*

$$e^{itx} = 1 + (e^{it} - 1)\,x^{(1)} + (e^{it} - 1)^2\,\frac{x^{(2)}}{2\,!} + \ldots + (e^{it} - 1)^r.\,\frac{x^{(r)}}{r\,!} + \ldots$$

where $x^{(r)} = x\,(x - 1)\,(x - 2)\,\ldots\,(x - r + 1)$. *Hence show that*

$$\mu_{(r)}' = [D^r\,\phi\,(t)]_{t=0},\ \textit{where}\ D = \frac{d}{d\,(e^{it})}\ \textit{and}\ \mu_{(r)}'\ \textit{is the}\ r^{th}\ \textit{factorial}$$

moment.

Solution. We have

$$\text{R.H.S.} = 1 + (e^{it} - 1)\,x^{(1)} + (e^{it} - 1)^2\,.\,\frac{x^{(2)}}{2\,!} + \ldots + (e^{it} - 1)^r\,.\,\frac{x^{(r)}}{r\,!} + \ldots$$

$$= 1 + (e^{it} - 1)\,({}^xC_1) + (e^{it} - 1)^2\,({}^xC_2) + (e^{it} - 1)^3\,({}^xC_3) + \ldots + (e^{it} - 1)^r\,({}^xC_r)$$

$$= [1 + (e^{it} - 1)]^x = e^{itx} = \text{L.H.S.}$$

By def.

$$\phi(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} \left[1 + (e^{it} - 1)x^{(1)} + (e^{it} - 1)^2 \cdot \frac{x^{(2)}}{2!} + \dots + (e^{it} - 1)^r \cdot \frac{x^{(r)}}{r!} + \dots \right] f(x)\, dx$$

$$= 1 + (e^{it} - 1) \int_{-\infty}^{\infty} x^{(1)} f(x)\, dx + \frac{(e^{it} - 1)^2}{2!} \int_{-\infty}^{\infty} x^{(2)} f(x)\, dx + \dots$$

$$+ \frac{(e^{it} - 1)^r}{r!} \int_{-\infty}^{\infty} x^{(r)} f(x)\, dx + \dots$$

$$\therefore \quad [D^r \phi(t)]_{t=0} = \left[\frac{d^r \phi(t)}{d(e^{it})^r} \right]_{t=0} = \int_{-\infty}^{\infty} x^{(r)} f(x)\, dx = \mu_{(r)}'$$

where $\mu_{(r)}'$ is the rth factorial moment.

Theorem 6·26. (Inversion Theorem). Lemma. *If $(a - h, a + h)$ is the continuity interval of the distribution function $F(x)$, then*

$$F(a + h) - F(a - h) = \lim_{T \to \infty} \frac{1}{\pi} \int_{-T}^{T} \frac{\sin ht}{t} e^{-ita} \phi(t)\, dt,$$

$\phi(t)$ being the characteristic function of the distribution.

Corollary. If $\phi(t)$ is absolutely integrable over R^1, *i.e.*, if

$$\int_{-\infty}^{\infty} |\phi(t)|\, dt < \infty,$$

then the derivative of $F(x)$ exists, which is bounded, continuous on R^1 and is given by

$$f(x) = F'(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi(t)\, dt, \qquad \dots(6·72)$$

for every $x \in R^1$.

Proof. In the above lemma replacing a by x and on dividing by $2h$, we have

$$\frac{F(x+h) - F(x-h)}{2h} = \frac{1}{2\pi} \cdot \lim_{T \to \infty} \int_{-T}^{T} \frac{\sin ht}{ht} e^{itx} \phi(t)\, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\sin ht}{ht} e^{-itx} \phi(t)\, dt$$

$$\therefore \quad \lim_{h \to 0} \frac{F(x+h) - F(x-h)}{2h} = \frac{1}{2\pi} \lim_{h \to 0} \int_{-\infty}^{\infty} \frac{\sin ht}{ht} \cdot e^{-itx} \phi(t)\, dt$$

Since $$\int_{-\infty}^{\infty} |\phi(t)|\, dt < \infty,$$

the integrand on the right hand side is bounded by an integrable function and hence by Dominated Convergence Theorem, we get

$$\lim_{h \to 0} \frac{F(x+h) - F(x-h)}{2h} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{h \to 0} \left(\frac{\sin ht}{ht} \right) \cdot e^{-itx} \phi(t)\, dt$$

By mean value theorem of differential calculus, we have

$$\lim_{h \to 0} \frac{F(x+h) - F(x-h)}{2h} = F'(x) = f(x),$$

where $f(.)$ is the p.d.f. corresponding to $\phi(t)$. Thus

$$f(x) = F'(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\,\phi(t)\,dt,$$

as desired.

Remark. Consider the function $\mathcal{F}_c$ defined by

$$\mathcal{F}_c = \int_{-c}^{c} e^{-itx}\,\phi(t)\,dt$$

Now if $F'(x) = f(x)$ exists, then

$$\lim_{c\to\infty} \frac{\mathcal{F}_c}{2c} = \lim_{c\to\infty} \frac{1}{2c}\int_{-c}^{c} \phi(t)\,e^{-itx}\,dt$$

$$= \lim_{c\to\infty} \left\{\frac{1}{2c}\,.\,2\pi f(x)\right\} = 0$$

Hence $\frac{\mathcal{F}_c}{2c} \to 0$ at all points where $F(x)$ is continuous. In other words, if the prbability distribution is continuous

$$\frac{\mathcal{F}_c}{2c} \to 0 \text{ as } c \to \infty$$

If, however, the frequency function is discontinuous, *i.e.*, distribution is discrete, consider one point of discontinuity say, the frequency f_j at $x = x_j$. Then the contribution of x_j to $\phi(t)$ is $f_j e^{itx_j}$ and hence its contributions to $\mathcal{F}_c$ will be

$$\int_{-c}^{c} f_j e^{itx_j}\, e^{-itx}\,dt$$

$$\therefore \quad \lim_{c\to\infty} \frac{\mathcal{F}_c}{2c} = \lim_{c\to\infty} \frac{1}{2c}\, f_j \int_{-c}^{c} e^{it(x_j - x)}\,dt$$

$$= \lim_{c\to\infty} \frac{1}{2c}\, f_j \left[\frac{e^{it(x_j-x)}}{i(x_j-x)}\right]_{-c}^{c}$$

$$= \begin{cases} 0 & \text{for } x \neq x_j \\ f_j & \text{for } x = x_j \end{cases}$$

Hence if $\mathcal{F}_c/2c \to 0$ at a point, there is no discontinuity in the distribution function at that point, but if it tends to a positive number f_j, the distribution is discontinuous at that point and the frequency is f_j. This gives us a criterion whether a given characteristic function represents a continuous distribution or not.

Theorem 6·27. Uniqueness Theorem of Characteristic Functions. *Characteristic function uniquely determines the distribution, i.e., a necessary and sufficient condition for two distributions with p.d.f.'s $f_1(.)$ and $f_2(.)$ to be identical is that their characteristic functions $\phi_1(t)$ and $\phi_2(t)$ are identical.*

Proof. If $f_1(\cdot) = f_2(\cdot)$, then from the definition of characteristic function, we get

$$\phi_1(t) = \int_{-\infty}^{\infty} e^{itx} f_1(x)\, dx = \int_{-\infty}^{\infty} e^{itx} f_2(x)\, dx = \phi_2(t)$$

Conversely if $\phi_1(t) = \phi_2(t)$, then from corollary to Theorem 6·26, we get

$$f_1(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\, \phi_1(t)\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\, \phi_2(t)\, dt = f_2(x)$$

Remark. This is one of the most fundamental theorems in the distribution theory. It implies that corresponding to a distribution there is only one characteristic function and corresponding to a given characteristic function, there is only one distribution. This *one to one correspondence between characteristic functions and the p.d.f.'s enables us to identify the form of the p.d.f. from that of characteristic function.*

Theorem 6·28. *Necessary and sufficient condition for the random variables X_1 and X_2 to be independent is that their joint characteristic function is equal to the product of their individual characteristic functions, i.e.,*

$$\phi_{X_1, X_2}(t_1, t_2) = \phi_{X_1}(t_1)\, \phi_{X_2}(t_2) \qquad \dots(*)$$

Proof. *(i) Condition is Necessary.* If X_1 and X_2 are independent then we have to show that (*) holds. By def.,

$$\begin{aligned}\phi_{X_1 X_2}(t_2, t_2) &= E\left(e^{it_1X_1 + it_2X_2}\right) = E\left(e^{it_1X_1} \cdot e^{it_2X_2}\right)\\ &= E\left(e^{it_1X_1}\right) E\left(e^{it_2X_2}\right) (\because X_1, X_2 \text{ are independent})\\ &= \phi_{X_1}(t_1)\, \phi_{X_2}(t_2),\end{aligned}$$

as required.

(ii) Condition is sufficient. We have to show that if (*) holds, then X_1 and X_2 are independent.

Let $f_{X_1, X_2}(x_1, x_2)$ be the joint p.d.f. of X_1 and X_2 and $f_1(x_1)$ and $f_2(x_2)$ be the marginal p.d.f.'s of X_1 and X_2 respectively. Then by definition (for continuous r.v.'s), we get

$$\phi_{X_1}(t_1) = \int_{-\infty}^{\infty} e^{it_1 x_1} f_1(x_1)\, dx_1$$

$$\phi_{X_2}(t_2) = \int_{-\infty}^{\infty} e^{it_2 x_2} f_2(x_2)\, dx_2$$

$$\begin{aligned}\therefore \quad \phi_{X_1}(t_1)\, \phi_{X_2}(t_2) &= \left[\int_{-\infty}^{\infty} e^{it_1 x_1} f_1(x_1)\, dx_1\right]\left[\int_{-\infty}^{\infty} e^{it_2 x_2} f_2(x_2)\, dx_2\right]\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(t_1 x_1 + t_2 x_2)} f_1(x_1) f_2(x_2)\, dx_1\, dx_2 \qquad \dots(**)\end{aligned}$$

by Fubini's theorem, since the integrand is bounded by an integrable function.

$$\therefore f(x) = \frac{1}{2\pi} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} \int_{-\infty}^{\infty} e^{-\xi^2/2} \frac{d\xi}{\sigma}$$

Hence

$$f(x) = \frac{1}{2\pi} \exp\left\{-\frac{x^2}{2\sigma^2}\right\} \frac{\sqrt{2\pi}}{\sigma} = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2\sigma^2}\right\}, \ -\infty < x < \infty$$

which is the p.d.f. of normal distribution.

Example 6·47. *Find the density function $f(x)$ corresponding to the characteristic function defined as follows :*

$$\phi(t) = \begin{cases} 1 - |t|, & |t| \le 1 \\ 0, & |t| > 1 \end{cases}$$

[Delhi Univ, B.Sc. (Maths Hons.), 1989]

Solution. By Inversion Theorem, the p.d.f. of X is given by :

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} \phi(t)\, dt$$

$$= \frac{1}{2\pi} \int_{-1}^{0} e^{-itx}(1+t)\, dt + \frac{1}{2\pi} \int_{0}^{1} e^{-itx}(1-t)\, dt \ *$$

Now

$$\int_{-1}^{0} e^{-itx}(1+t)\, dt = \left[\frac{e^{-itx}}{-ix}(1+t)\right]_{-1}^{0} + \frac{1}{ix} \int_{-1}^{0} e^{-itx}\, dt$$

$$= -\frac{1}{ix} + \frac{1}{ix}\left[\frac{e^{-itx}}{-ix}\right]_{-1}^{0}$$

$$= -\frac{1}{ix} + \frac{1}{(ix)^2}(e^{ix} - 1)$$

Similarly,

$$\int_{0}^{1} e^{-itx}(1-t)\, dt = \frac{1}{ix} + \frac{1}{(ix)^2}(e^{-ix} - 1)$$

$$\therefore \qquad f(x) = \frac{1}{2\pi}\left[\frac{1}{(ix)^2}\left\{e^{ix} - 1 + e^{-ix} - 1\right\}\right]$$

$$= \frac{1}{\pi x^2}\left[1 - \frac{e^{ix} + e^{-ix}}{2}\right] = \frac{1}{\pi} \cdot \frac{1 - \cos x}{x^2}, \ -\infty < x < \infty$$

EXERCISE 6 (c)

1. Define m.g.f. of a random variable. Hence or otherwise find the m.g.f. of:

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* $\because$ for $-1 < t < 0$, $|t| = -t$ and for $0 < t < 1$, $|t| = +t$

$$(i)\ Y = aX + b, \quad (ii)\ Y = \frac{X - m}{\sigma}.$$

[Sri Venkat Univ. B.Sc., Sept. 1990; Kerala Univ. B.Sc., Sept. 1992]

2. The random variable X takes the value n with probability $1/2^n$, $n = 1, 2, 3, \ldots$ Find the moment generating function of X and hence find the mean and variance of X.

3. Show that if $\overline{X}$ is mean of n independent random variables, then

$$M_{\overline{X}}(t) = \left[M_X\left(\frac{t}{n}\right)\right]^n$$

4. (*a*) Define moments and moment generating function (m.g.f.) of a random variable X. If $M(t)$ is the m.g.f. of a random variable X about the origin, show that the moment μ_r' is given by

$$\mu_r' = \left[\frac{d^r M(t)}{dt^r}\right]_{t=0}$$ **[Baroda Univ. B.Sc., 1992]**

(*b*) If μ_r' is the rth order moment about the origin and κ_j is the cumulant of jth order, prove that

$$\frac{\delta \mu_r'}{\delta \kappa_j} = \binom{r-1}{j-1} \mu'_{r-j}$$

(*c*) If μ_r' is the rth moment about the origin of a variable X and if $\mu_r' = r!$, find the m.g.f. of X.

5. (*a*) A random variable 'X' has probability function

$$p(x) = \frac{1}{2^x};\ x = 1, 2, 3, \ldots$$

Find the M.G.F., mean and variance.

(*b*) Show that the m.g.f. of r.v. X having the p.d.f.

$$f(x) = \tfrac{1}{3},\ -1 < x < 2$$
$$= 0, \text{ elsewhere,}$$

is $$M(t) = \frac{e^{2t} - e^{-t}}{3t},\ t \neq 0$$
$$= 1,\ t = 0$$ **[Gujarat Univ. B.Sc., Oct. 1991]**

(*c*) A random variable 'X' has the density function :

$$f(x) = \frac{1}{2\sqrt{x}},\ 0 < x < 1$$
$$= 0, \text{ elsewhere}$$

Obtain the moment generating function and hence the mean and variance.

6. X is a random variable and $p(x) = ab^x$, where a and b are positive, $a + b = 1$ and x taking the values 0, 1, 2, ... Find the moment generating function of X. Hence show that

$$m_2 = m_1(2m_1 + 1)$$

m_1 and m_2 being the first two moments.

7. Find the characteristic function of the following distributions and their variances :

(i) $dF(x) = ae^{-ax}\,dx,\ (a > 0, x > 0)$

(ii) $dF(x) = \frac{1}{2} e^{-|x|}\,dx,\ (-\infty < x < \infty)$

(iii) $P(X = j)$ lineu= $\binom{n}{j} p^j q^{n-j},\ (0 < p < 1, q = 1 - p, j = 0, 1, 2, ..., n)$.

8. Obtain the m.g.f. of the random variable X having p.d.f.,

$$f(x) = \begin{cases} x, & \text{for } 0 \le x < 1 \\ 2 - x, & \text{for } 1 \le x < 2 \\ 0, & \text{elsewhere} \end{cases}$$

Determine μ_1', μ_2, μ_3 and μ_4.

Ans. $\left(\frac{e^t - 1}{t}\right)^2$, $\mu_1' = 1$, $\mu_2' = 7$.

9. *(a)* Define cumulants and obtain the first four cumulants in terms of central moments.

(b) If X is a variable with zero mean and cumulants κ_r, show that the first two cumulants l_1 and l_2 of X^2 are given by $l_1 = \kappa_2$ and $l_2 = 2\kappa_2^2 + \kappa_4$.

10. Show that the rth cumulant for the distribution

$$f(x) = ce^{-cx}, \text{ where } c \text{ is positive and } 0 \le x < \infty$$

is $$\frac{1}{c^r} \cdot (r - 1)!$$

11. If X is a random variable with cumulants κ_r ; $r = 1, 2, ...$ Find the cumulants of

(i) cX, (ii) $c + X$, where c is a constant.

12. *(a)* Define the characteristic function of a random variable. Show that the characteristic function of the sum of two independent variables is equal to the product of their characteristic functions.

(b) If X is a random variable having cumulants κ_r ; $r = 1, 2, ...$ given by

$$\kappa_r = (r - 1)!\, pa^{-r}; p > 0, a > 0,$$

find the characteristic function of X.

(c) Prove that the characteristic function of a random variable X is real if and only if X has a symmetric distribution about 0.

13. Define $\phi(t)$, the characteristic function of a random variable. Find the characteristic function of a random variable X defined as follows :

$$f(x) = \begin{cases} 0, & x < 0 \\ 1, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$$

Ans. $e^{(it-1)}/it$

CHAPTER SEVEN

Theoretical Discrete Probability Distributions

7·0. Introduction. In the previous chapters we have discussed in detail the frequency distributions. In the present chapter we will discuss theoretical discrete distributions in which variables are distributed according to some definite probability law which can be expressed mathematically. The present study will also enable us to fit a mathematical model or a function of the form $y = p(x)$ to the observed data.

We have already defined distribution function, mathematical expectation, m.g.f., characteristic function and moments. This prepares us for a study of theoretical distributions. This chapter is devoted to the study of univariate (except for the multinomial) distributions like Binomial, Poisson, Negative binomial Geometric, Hypergeometric, Multinomial and Power-series distributions.

7·1. Bernoulli Distribution. A random variable X which takes two values 0 and 1, with probabilities q and p respectively, *i.e.*, $P(X = 1) = p$, $P(X = 0) = q$, $q = 1 - p$ is called a *Bernoulli variate* and is said to have a Bernoulli distribution.

Remark. Sometimes, the two values are +1, −1 instead of 1 and 0.

7·1·1. Moments of Bernoulli distribution. The r^{th} moment about origin is

$$\mu_r' = E(X^r) = 0^r \cdot q + 1^r \cdot p = p \; ; \; r = 1, 2, \ldots \qquad \ldots(7\cdot1)$$

$$\mu_1' = E(X) = p, \; \mu_2' = E(X^2) = p$$

$$\mu_2 = \text{Var}(X) = p - p^2 = pq$$

The m.g.f. of Bernoulli variate is given by :

$$M_X(t) = e^{0 \cdot t} \times P(X = 0) + e^{1 \cdot t} \cdot P(X = 1) = q + pe^t \qquad \ldots(7\cdot1a)$$

Remark. Degenerate Random Variable. Sometimes we may come across a variate X which is degenerate at a point 'c', say, so that : $P(X = c) = 1$ and $= 0$ otherwise, *i.e.*, the whole mass of the variable is concentrated at a single point 'c'.

$$\text{Since } P(X = c) = 1, \text{Var}(X) = 0.$$

Thus a degenerate r.v. X is characterised by Var $(X) = 0$.

M.g.f. of degenerate r.v. is given by

$$M_X(t) = E(e^{tX}) = e^{tc} P(X = c) = e^{ct} \qquad \ldots(7\cdot1b)$$

7·2. Binomial Distribution. Binomial distribution was discovered by James Bernoulli (1654-1705) in the year 1700 and was first published posthumously in 1713, eight years after his death). Let a random experiment be performed repeatedly and let the occurrence of an event in a trial be called a success and its non-occurrence a failure. Consider a set of n independent Bernoullian trials (n

being finite), in which the probability 'p' of success in any trial is constant for each trial. Then $q = 1 - p$, is the probability of failure in any trial.

The probability of x successes and consequently $(n - x)$ failures in n independent trials, in a specified order (say) $SSFSFFFS...FSF$ (where S represents success and F failure) is given by the compound probability theorem by the expression :

$$P(SSFSFFFS...FSF) = P(S)P(S)P(F)P(S)P(F)P(F)P(F)P(S) \times \ldots \times P(F)P(S)P(F)$$

$$= p \cdot p \cdot q \cdot p \cdot q \cdot q \cdot q \cdot p \ldots q \cdot p \cdot q$$

$$= \underbrace{p \cdot p \ldots p}_{\{x \text{ factors}\}} \quad \underbrace{q \cdot q \cdot q \ldots q}_{\{(n-x) \text{ factors}\}} = p^x q^{n-x}$$

But x successes in n trials can occur in $\binom{n}{x}$ ways and the probability for each of these ways is $p^x q^{n-x}$. Hence the probability of x successes in n trials in *any order* whatsoever is given by the addition theorem of probability by the expression:

$$\binom{n}{x} p^x q^{n-x}$$

The probability distribution of the number of successes, so obtained is called the *Binomial probability distribution*, for the obvious reason that the probabilities of 0, 1, 2, ..., n successes, *viz.*,

$q^n, \binom{n}{1} q^{n-1} p, \binom{n}{2} q^{n-2} p^2, \ldots, p^n$, are the successive terms of the binomial expansion $(q + p)^n$.

Definition. *A random variable X is said to follow binomial distribution if it assumes only non-negative values and its probability mass function is given by*

$$P(X = x) = p(x) = \begin{cases} \binom{n}{x} p^x q^{n-x} \; ; x = 0, 1, 2, \ldots, n \; ; q = 1 - p \\ 0, \textit{ otherwise} \end{cases} \qquad \ldots(7\cdot2)$$

The two independent constants n and p in the distribution are known as the *parameters* of the distribution. 'n' is also, sometimes, known as the degree of the binomial distribution.

Binomial distribution is a discrete distribution as X can take only the integral values, *viz.*, 0, 1, 2,..., n. Any variable which follows binomial distribution is known as *binomial variate*.

We shall use the notation $X \sim B(n, p)$ to denote that the random variable X follows binomial distribution with parameters n and p.

The probability $p(x)$ in (7·2) is also sometimes denoted by $b(x, n, p)$.

Remarks 1. This assignment of probabilities is permissible because

$$\sum_{x=0}^{n} p(x) = \sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x} = (q + p)^n = 1$$

2. Let us suppose that n trials constitute an experiment. Then if this experiment is repeated N times, the *frequency function* of the binomial distribution is given by

$$f(x) = Np(x) = N\binom{n}{x} p^x q^{n-x} ; x = 0, 1, 2, ..., n \qquad ...(7.3)$$

and the expected frequencies of 0, 1, 2, ..., n successes are the successive terms of the binomial expansion, $N(q+p)^n$, $q+p=1$.

3. Binomial distribution is important not only because of its wide applicability, but because it gives rise to many other probability distributions. Tables for $p(x)$ are available for various values of n and p.

4. **Physical conditions for Binomial Distribution.** We get the binomial distribution under the following experimental conditions.

(i) Each trial results in two mutually disjoint outcomes, termed as success and failure.

(ii) The number of trials 'n' is finite.

(iii) The trials are independent of each other.

(iv) The probability of success 'p' is constant for each trial.

The problems relating to tossing of a coin or throwing of dice or drawing cards from a pack of cards with replacement lead to binomial probabability distribution.

Example 7·1. *Ten coins are thrown simultaneously. Find the probability of getting at least seven heads.*

Solution. p = Probability of getting a head $= \frac{1}{2}$

q = Probability of not getting a head $= \frac{1}{2}$

The probability of getting x heads in a random throw of 10 coins is

$$p(x) = \binom{10}{x}\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{10-x} = \binom{10}{x}\left(\frac{1}{2}\right)^{10} ; x = 0, , 2, ..., 10 \quad ...$$

$\therefore$ Probability of getting at least seven heads is given by

$$P(X \geq 7) = p(7) + p(8) + p(9) + p(10)$$

$$= \left(\frac{1}{2}\right)^{10}\left\{\binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10}\right\}$$

$$= \frac{120 + 45 + 10 + 1}{1024} = \frac{176}{1024}.$$

Example 7·2. *A and B play a game in which their chances of winning are in the ratio 3 : 2. Find A's chance of winning at least three games out of the five games played.* **[Burdwan Univ. B.Sc. (Hons.), 1993]**

Solution. Let p be the probability that 'A' wins the game. Then we are given $p = 3/5 \Rightarrow q = 1 - p = 2/5$.

Hence, by binomial probability law, the probability that out of 5 games played, A wins 'r' games is given by :

$$P(X=r)=p(r)=\binom{5}{r}\cdot(3/5)^r\,(2/5)^{5-r}\;;\;r=0,1,2,\ldots,5$$

The required probability that 'A' wins at least three games is given by :

$$P(X\geq 3)=\sum_{r=3}^{5}\binom{5}{r}\frac{3^r\cdot 2^{5-r}}{5^5}$$

$$=\frac{3^3}{5^5}\left[\binom{5}{3}2^2+\binom{5}{4}.3\times 2+1.3^2\times 1\right]=\frac{27\times(40+30+9)}{3125}=0{\cdot}68$$

Example 7·3. *If m things are distributed among 'a' men and 'b' women, show that the probability that the number of things received by men is odd, is*

$$\frac{1}{2}\left[\frac{(b+a)^m-(b-a)^m}{(b+a)^m}\right]$$

(Nagpur Univ B.Sc., 1989, '93)

Solution. p = Probability that a thing is received by man $=\dfrac{a}{a+b}$, then

$q=1-p=1-\dfrac{a}{a+b}=\dfrac{b}{a+b}$, is the probability that a thing is received by woman.

The probability that out of m things exactly x are received by men and the rest by women, is given by

$$p(x)={}^mC_x\,p^x\,q^{m-x}\;;\;x=0,1,2,\ldots,m$$

The probability P that the number of things received by men is odd is given by

$$P=p(1)+p(3)+p(5)+\ldots={}^mC_1\cdot q^{m-1}\cdot p+{}^mC_3\cdot q^{m-3}\cdot p^3+{}^mC_5\cdot q^{m-5}\cdot p^5+\ldots$$

Now

$$(q+p)^m=q^m+{}^mC_1\cdot q^{m-1}\cdot p+{}^mC_2\cdot q^{m-2}\,p^2+{}^mC_3\cdot q^{m-3}\cdot p^3+{}^mC_4\cdot q^{m-4}\cdot p^4+\ldots$$

and

$$(q-p)^m=q^m-{}^mC_1\cdot q^{m-1}\cdot p+{}^mC_2\cdot q^{m-2}\cdot p^2-{}^mC_3\cdot q^{m-3}\cdot p^3+{}^mC_4\cdot q^{m-4}\cdot p^4-\ldots$$

$$\therefore\quad (q+p)^m-(q-p)^m=2\,[{}^mC_1\cdot q^{m-1}\cdot p+{}^mC_3\cdot q^{m-3}\cdot p^3+\ldots]=2P$$

But $\quad q+p=1$ and $q-p=\dfrac{b-a}{b+a}$

$$\therefore\quad 1-\left(\frac{b-a}{b+a}\right)^m=2P\;\Rightarrow\;P=\frac{1}{2}\left[\frac{(b+a)^m-(b-a)^m}{(b+a)^m}\right]$$

Example 7·4 *An irregular six faced die is thrown and the expectation that in 10 throws it will give five even numbers is twice the expectation that it will give four even numbers. How many times in 10,000 sets of 10 throws each, would you expect it to give no even number.* **(Gujarat Univ. B.Sc. 1988)**

Solution. Let p be the probability of getting an even number in a throw of a die. Then the probability of getting x even numbers in ten throws of a die is

$$P(X=x)=\binom{10}{x}p^x\,q^{10-x}\;;\;x=0,1,2\ldots 10$$

We are given that

$$P(X = 5) = 2\,P(X = 4)$$

i.e., $$\binom{10}{5} p^5 q^5 = 2 \binom{10}{4} p^4 q^6$$

$$\Rightarrow \quad \frac{10\,!\,p}{5\,!\,5\,!} = 2\,\frac{10\,!\,q}{4\,!\,6\,!}$$

$$\Rightarrow \quad \frac{p}{5} = \frac{2q}{6} = \frac{q}{3}$$

$\therefore\ 3p = 5q = 5(1-p) \quad \Rightarrow \quad 8p = 5 \quad \Rightarrow \quad p = 5/8$ and $q = 3/8$

$$\therefore \quad P(X = x) = \binom{10}{x}\left(\frac{5}{8}\right)^x \left(\frac{3}{8}\right)^{10-x}$$

Hence the required number of times that in 10,000 sets of 10 throws each, we get no even number

$$= 10{,}000 \times P(X = 0) = 10{,}000 \times \left(\frac{3}{8}\right)^{10} = 1 \text{ (approx.)}$$

Example 7·5 *In a precision bombing attack there is a 50% chance that any one bomb will strike the target. Two direct hits are required to destroy the target completely. How many bombs must be dropped to give a 99% chance or better of completely destroying the target ?* **[Gauhati Univ. M.A., 1992]**

Solution. We have :

p = Probability that the bomb strikes the target = 50% = $\frac{1}{2}$. Let n be the number of bombs which should be dropped to ensure 99% chance or better of completely destroying the target. This implies that "probability that out of n bombs, at least two strike the target, is greater than 0·99".

Let X be a r.v. representing the number of bombs striking the target. Then $X \sim B(n, p = \frac{1}{2})$ with

$$p(x) = P(X = x) = \binom{n}{x}\left(\frac{1}{2}\right)^x \cdot \left(\frac{1}{2}\right)^{n-x} = \binom{n}{x}\left(\frac{1}{2}\right)^n ; x = 0, 1, \ldots, n$$

We should have :

$$P(X \geq 2) \geq 0{\cdot}99$$

$$\Rightarrow \quad [1 - P(X \leq 1)] \geq 0{\cdot}99$$

$$\Rightarrow \quad [1 - \{p(0) + p(1)\}] \geq 0{\cdot}99$$

$$\Rightarrow \quad 1 - \left\{\binom{n}{0} + \binom{n}{1}\right\}\left(\frac{1}{2}\right)^n \geq 0{\cdot}99$$

$$\Rightarrow \quad 0{\cdot}01 \geq \frac{1+n}{2^n} \quad \Rightarrow \quad 2^n \times (0{\cdot}01) \geq 1 + n$$

$$\Rightarrow \quad 2^n \geq 100 + 100\,n \qquad \ldots(*)$$

By trial method, we find that the inequality (*) is satisfied by $n = 11$. Hence the minimum number of bombs needed to destroy the target completely is 11.

Example 7·6. *A department in a works has 10 machines which may need adjustment from time to time during the day. Three of these machines are old, each having a probability of 1/11 of needing adjustment during the day, and 7 are new, having corresponding probabilities of 1/21.*

Assuming that no machine needs adjustment twice on the same day, determine the probabilits that on a particular day :

(i) just 2 old and no new machines need adjustment.

(ii) If just 2 machines need adjustment, they are of the same type.

(Nagpur Univ. B.E., 1989)

Solution. Let p_1 = Probability that an old machine needs adjustment

$= 1/11$

$\therefore \quad q_1 = 1 - p_1 = 10/11$

and p_2 = Probability that a new machine needs adjustment $= 1/21$

$q_2 = 1 - p_2 = 20/21$

Then $P_1(r)$ = Probability that 'r' old machines need adjustment

$$= {}^3C_r\, p_1^r\, q_1^{3-r} = {}^3C_r\,(10/11)^{3-r}\,(1/11)^r$$

and $P_2(r)$ = Probability that 'r' new machine need adjustment

$$= {}^7C_r\, p_2^r\, q_2^{7-r} = {}^7C_r\,(1/21)^r\,(20/21)^{7-r}$$

(*i*) The probability that just two old machines and no new machine need adjustment is given (by the compound probability theorem) by the expression :

$$P_1(2)\cdot P_2(0) = {}^3C_2(1/11)^2\cdot(10/11)\cdot(20/21)^7 = 0{\cdot}016$$

(*ii*) Similarly the probability that just 2 new machines and no old machine need adjustment is

$$P_1(0)\cdot P_2(2) = (10/11)^3\cdot{}^7C_2\,(1/21)^2\cdot(20/21)^5 = 0{\cdot}028$$

The probability that "If just two machines need adjustment, they are of the same type" is the same as the probability that "either just 2 old and no new or just 2 new and no old machines need adjustment".

$\therefore$ Required probability $= 0{\cdot}016 + 0{\cdot}028 = 0{\cdot}044$

7·2·1 Moments. The first four moments about origin of binomial distribution are obtained as follows :

$$\mu_1' = E(X) = \sum_{x=0}^{n} x\binom{n}{x}p^x q^{n-x} = np\sum_{x=1}^{n}\binom{n-1}{x-1}p^{x-1}q^{n-x}\,*$$

$$= np(q+p)^{n-1} = np \qquad (\because q + p = 1)$$

Thus the mean of the binomial distribution is np.

$*\quad \binom{n}{x} = \frac{n}{x}\cdot\binom{n-1}{x-1} = \frac{n}{x}\cdot\frac{n-1}{x-1}\cdot\binom{n-2}{x-2}$

$= \frac{n}{x}\cdot\frac{n-1}{x-1}\cdot\frac{n-2}{x-2}\binom{n-3}{x-3}$, and so on.

$$\mu_2' = E(X^2) = \sum_{x=0}^{n} x^2 \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \{x(x-1) + x\} \frac{n(n-1)}{x(x-1)} \cdot \binom{n-2}{x-2} p^x q^{n-x}$$

$$= n(n-1)p^2 \left[\sum_{x=2}^{n} \binom{n-2}{x-2} p^{x-2} q^{n-x} \right] + np$$

$$= n(n-1)p^2 (q+p)^{n-2} + np = n(n-1)p^2 + np$$

$$\mu_3' = E(X^3) = \sum_{x=0}^{n} x^3 \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \{x(x-1)(x-2) + 3x(x-1) + x\} p^x q^{n-x}$$

$$= n(n-1)(n-2)p^3 \sum_{x=3}^{n} \binom{n-3}{x-3} p^{x-3} q^{n-x}$$

$$+ 3n(n-1)p^2 \sum_{x=2}^{n} \binom{n-2}{x-2} p^{x-2} q^{n-x} + np$$

$$= n(n-1)(n-2)p^3 (q+p)^{n-3} + 3n(n-1)p^2 (q+p)^{n-2} + np$$

$$= n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np$$

Similarly

$$x^4 = x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x$$

Let $x^4 = Ax(x-1)(x-2)(x-3) + Bx(x-1)(x-2) + Cx(x-1) + x$

By giving to x the values 1, 2 and 3 respectively, we find the values of arbitrary constants A, B and C. Therefore,

$$\mu_4' = E(X^4) = \sum_{x=0}^{n} x^4 \binom{n}{x} p^x p^{n-x}$$

$$= n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np$$

[On simplification]

Central Moments of Binomial Distribution :

$$\mu_2 = \mu_2' - \mu_1'^2 = n^2p^2 - np^2 + np - n^2p^2 = np(1-p) = npq$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$= \{n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np\} - 3\{n(n-1)p^2 + np\} np + 2(np)^3$$

$$= np[-3np^2 + 3np + 2p^2 - 3p + 1 - 3npq]$$

$$= np[3np(1-p) + 2p^2 - 3p + 1 - 3npq]$$

$$= np\,[2p^2 - 3p + 1] = np\,(2p^2 - 2p + q) = npq\,(1 - 2p)$$
$$= npq\,[q + p - 2p] = npq\,(q - p)$$

$$\mu_4 = \mu_4' - 4\mu_3'\,\mu_1' + 6\mu_2'\,\mu_1'^2 - 3\mu_1'^4 = npq\,[1 + 3\,(n-2)\,pq]$$

[On simplification]

Hence

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{n^2p^2q^2(q-p)^2}{n^3p^3q^3} = \frac{(q-p)^2}{npq} = \frac{(1-2p)^2}{npq} \quad ...(7\cdot4)$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{npq\,\{1 + 3\,(n-2)\,pq\}}{n^2p^2q^2} = \frac{1 + 3\,(n-2)\,pq}{npq} = 3 + \frac{1 - 6pq}{npq} \quad ...(7\cdot5)$$

$$\gamma_1 = \sqrt{\beta_1} = \frac{q-p}{\sqrt{npq}} = \frac{1-2p}{\sqrt{npq}},\ \gamma_2 = \beta_2 - 3 = \frac{1-6pq}{npq} \quad ...(7\cdot5\,a)$$

Example 7·7 . *Comment on the following :*
The mean of a binomial distribution is 3 and variance is 4'.

Solution. If the given binomial distribution has parameters n and p, then we are given

$$\text{Mean} = np = 3 \quad ...(*)$$

and $$\text{Variance} = npq = 4 \quad ...(**)$$

Dividing (**) by (*), we get $q = 4/3$,
which is impossible, since probability cannot exceed unity. Hence the given statement is wrong.

Example 7·8. *The mean and variance of binomial distribution are 4 and $\frac{4}{3}$ respectively. Find $P(X \geq 1)$.* **(Sardar Patel Univ. B.Sc. 1993)**

Solution. Let $X \sim B\ (n, p)$. Then we are given

$$\text{Mean} = E(X) = np = 4$$

and $$\text{Var}(X) = npq = \tfrac{4}{3} \quad ...(*)$$

Dividing, we get

$$q = \tfrac{1}{3} \quad \Rightarrow \quad p = \tfrac{2}{3}$$

Substituting in (*), we get

$$n = \frac{4}{p} = \frac{4 \times 3}{2} = 6.$$

$$P(X \geq 1) = 1 - P\,(X = 0) = 1 - q^n = 1 - (1/3)^6 = 1 - (1/729)$$
$$= 1 - 0\cdot00137 = 0\cdot99863$$

Example 7·9 *If $X \sim B\,(n, p)$, show that :*

$$E\left(\frac{X}{n} - p\right)^2 = \frac{pq}{n}\,;\ Cov\left(\frac{X}{n}, \frac{n-X}{n}\right) = -\frac{pq}{n}$$

(Delhi Univ. B.Sc., 1989)

Solution. Since $X \sim B(n, p)$, $E(X) = np$ and $\text{Var}(X) = npq$

$$\therefore \quad E\left(\frac{X}{n}\right) = \frac{1}{n} E(X) = p\,; \quad \text{Var}\left(\frac{X}{n}\right) = \frac{1}{n^2} \cdot \text{Var}(X) = \frac{pq}{n}$$

(i) $$E\left(\frac{X}{n} - p\right)^2 = E\left[\frac{X}{n} - E\left(\frac{X}{n}\right)\right]^2 = \text{Var}\left(\frac{X}{n}\right) = \frac{pq}{n}$$

(ii) $$\text{Cov}\left(\frac{X}{n}, \frac{n-X}{n}\right) = E\left[\left\{\frac{X}{n} - E\left(\frac{X}{n}\right)\right\}\left\{\frac{n-X}{n} - E\left(\frac{n-X}{n}\right)\right\}\right]$$

$$= E\left[\left(\frac{X}{n} - p\right)\left\{\left(1 - \frac{X}{n}\right) - (1-p)\right\}\right]$$

$$= E\left[\left(\frac{X}{n} - p\right)\left\{-\left(\frac{X}{n} - p\right)\right\}\right]$$

$$= -E\left(\frac{X}{n} - p\right)^2 = -\text{Var}\left(\frac{X}{n}\right) = -\frac{pq}{n}$$

7·2·2 Recurrence Relation for the moments of Binomial Distribution. *(Renovsky Formula)*

By def.,

$$\mu_r = E\{X - E(X)\}^r = \sum_{x=0}^{n} (x - np)^r \binom{n}{x} p^x q^{n-x}$$

Differentiating with respect to p, we get

$$\frac{d\mu_r}{dp} = \sum_{x=0}^{n} \binom{n}{x} \Big[-nr(x-np)^{r-1} p^x q^{n-x}$$

$$+ (x-np)^r \left\{x p^{x-1} q^{n-x} - (n-x) p^x q^{n-x-1}\right\}\Big]$$

$$= -nr \sum_{x=0}^{n} \binom{n}{x} (x-np)^{r-1} p^x q^{n-x}$$

$$+ \sum_{x=0}^{n} \binom{n}{x} (x-np)^r p^x q^{n-x} \left\{\frac{x}{p} - \frac{n-x}{q}\right\}$$

$$= -nr \sum_{x=0}^{n} (x-np)^{r-1} p(x) + \sum_{x=0}^{n} (x-np)^r p(x) \frac{(x-np)}{pq}$$

$$= -nr \sum_{x=0}^{n} (x-np)^{r-1} p(x) + \frac{1}{pq} \sum_{x=0}^{n} (x-np)^{r+1} p(x)$$

$$\therefore \quad \frac{d\mu_r}{dp} = -nr\,\mu_{r-1} + \frac{1}{pq}\,\mu_{r+1}$$

$$\Rightarrow \quad \mu_{r+1} = pq\left[nr\,\mu_{r-1} + \frac{d\mu_r}{dp}\right] \qquad ...(7·6)$$

Putting $r = 1, 2$ and 3 successively in (7·6), we get

$$\mu_2 = pq\left[n\mu_0 + \frac{d\mu_1}{dp}\right] = npq \qquad (\because \mu_0 = 1 \text{ and } \mu_1 = 0)$$

$$\mu_3 = pq\left[2n\mu_1 + \frac{d\mu_2}{dp}\right] = pq \cdot \frac{d(npq)}{dp} = npq\frac{d}{dp}\{p(1-p)\}$$

$$= npq\frac{d}{dp}(p - p^2) = npq(1-2p) = npq(q-p)$$

and
$$\mu_4 = pq\left[3n\mu_2 + \frac{d\mu_3}{dp}\right] = pq\left[3n \cdot npq + \frac{d}{dp}\{npq(q-p)\}\right]$$

$$= pq\left[3n^2pq + n\frac{d}{dp}\{p(1-p)(1-2p)\}\right]$$

$$= pq\left[3n^2pq + n\frac{d}{dp}(p - 3p^2 + 2p^3)\right]$$

$$= pq[3n^2pq + n(1 - 6p + 6p^2)] = pq[3n^2pq + n(1 - 6pq)]$$

$$= npq[3npq + 1 - 6pq] = npq[1 + 3pq(n-2)]$$

Example 7·10 *Show that the rth moment μ_r' about the origin of the binomial distribution of degree n is given by :*

$$\mu_r' = \left(p\frac{\partial}{\partial p}\right)^r (q+p)^n \qquad ...(*)$$

[Patna Univ. B.Sc. (Hons.), 1993]

Solution. We shall prove this result by using the principle of mathematical induction. We have

$$(q+p)^n = \sum_{x=0}^{n}\binom{n}{x}p^x q^{n-x} \Rightarrow \frac{\partial}{\partial p}(q+p)^n = \sum_{x=0}^{n}\binom{n}{x}q^{n-x}xp^{x-1}$$

$$\therefore p\frac{\partial}{\partial p}(q+p)^n = p\sum_{x=0}^{n}\binom{n}{x}q^{n-x}xp^{x-1} = \sum_{x=0}^{n}\binom{n}{x}p^x q^{n-x}x = \mu_1'$$

Thus the result (*) is true for $r = 1$.

Let us now assume that the result (*) is true for $r = k$, so that

$$\left(p\frac{\partial}{\partial p}\right)^k (q+p)^n = \mu_k' = \sum_{x=0}^{n}\binom{n}{x}p^x q^{n-x}x^k \qquad ...(**)$$

Differentiate (**) partially w.r. to p and multiply both sides by p to get :

$$p\left(\frac{\partial}{\partial p}\right)\left[\left(p\frac{\partial}{\partial p}\right)^k (q+p)^n\right] = \sum_{x=0}^{n}\binom{n}{x}p^x q^{n-x}x^{k+1} = E\left(X^{k+1}\right)$$

$$\Rightarrow \qquad \left(p\frac{\partial}{\partial p}\right)^{k+1}(q+p)^n = \mu_{k+1}'$$

Hence if the result (*) is true for $r = k$, it is also true for $r = k + 1$. It is already shown to be true for $k = 1$. Hence by the principle of mathematical induction, (*) is true for all positive integral values of r.

7·2·3. Factorial Moments of Binomial Distribution. The rth factorial moment of the Binomial distribution is:

$$\mu_{(r)}' = E\{X^{(r)}\} = \sum_{x=0}^{n} x^{(r)} p(x) = \sum_{x=0}^{n} x^{(r)} \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$= n^{(r)} p^r \sum_{x=r}^{n} \frac{(n-r)!}{(x-r)!(n-x)!} p^{x-r} q^{n-x} = n^{(r)} p^r (q+p)^{n-r}$$

$$= n^{(r)} p^r \qquad \ldots(7\cdot 7)$$

$$\mu_{(1)}' = E\{X^{(1)}\} = np = \text{Mean}$$

$$\mu_{(2)}' = E\{X^{(2)}\} = n^{(2)} p^2 = n(n-1)p^2$$

$$\mu_{(3)}' = E\{X^{(3)}\} = n^{(3)} p^3 = n(n-1)(n-2)p^3$$

Now $\mu_{(2)} = \mu_{(2)}' - \mu_{(1)}'^2 + \mu_{(1)}' = n^2p^2 - np^2 - n^2p^2 + np = npq$

$$\mu_{(3)} = \mu_{(3)}' - 3\mu_{(2)}'\mu_{(1)}' + 2\mu_{(1)}'^3 - 2\mu_{(1)}'$$

$$= n(n-1)(n-2)p^3 - 3n(n-1)p^2\, np + 2n^3p^3 - 2np = -2npq(1+p)$$

[On simplification]

7·2·4. Mean Deviation About Mean of Binomial Distribution. The mean deviation η about the mean np of the binomial distribution is given by

$$\eta = \sum_{x=0}^{n} |x-np|\, p(x) = \sum_{x=0}^{n} |x-np| \binom{n}{x} p^x q^{n-x},$$

(x being an integer)

$$= \sum_{x=0}^{np} -(x-np)\binom{n}{x} p^x q^{n-x} + \sum_{x=np}^{n} (x-np)\binom{n}{x} p^x q^{n-x}$$

$$= 2\sum_{x=np}^{n} (x-np)\binom{n}{x} p^x q^{n-x} \, *$$

$$= 2\sum_{\mu}^{n} (x-np)\binom{n}{x} p^x q^{n-x},$$

where μ is the greatest integer contained in $np+1$.

$$= 2\sum_{\mu}^{n} \left[\{xq-(n-x)p\}\binom{n}{x} p^x q^{n-x}\right]$$

$$= 2\sum_{\mu}^{n} \left[\frac{n!}{(x-1)!(n-x)!} p^x q^{n-x+1} - \frac{n!}{x!(n-x-1)!} p^{x+1} q^{n-x}\right]$$

---

* $\because \quad \sum_{x=0}^{n} x \binom{n}{x} p^x q^{n-x} = np$

$\Rightarrow \quad \sum_{x=0}^{n} (x-np)\binom{n}{x} p^x q^{n-x} = 0$

$$= 2 \sum_{x=\mu}^{n} \left[t_{x-1} - t_x \right], \text{ where } t_x = \frac{n!}{x!(n-x-1)!} p^{x+1} q^{n-x}$$

$$= 2 \left[t_{\mu-1} - t_n \right] = 2\, t_{\mu-1}$$

This is obtained by summing over x and using $t_n = o$

$$\therefore \eta = 2t_{\mu-1} = 2 \frac{n!}{(\mu-1)!(n-\mu)!} \cdot p^{\mu} q^{n-\mu+1}$$

$$= 2npq \binom{n-1}{\mu-1} p^{\mu-1} q^{n-\mu} \qquad \ldots(7\cdot8)$$

7·2·5. Mode of the Binomial Distribution. We have

$$\frac{p(x)}{p(x-1)} = \binom{n}{x} p^x q^{n-x} \Big/ \binom{n}{x-1} p^{x-1} q^{n-x+1}$$

$$= \frac{n!}{(n-x)!\,x!} p^x q^{n-x} \Big/ \frac{n!}{(x-1)!\,(n-x+1)!} p^{x-1} q^{n-x+1}$$

$$= \frac{(n-x+1)p}{xq} = \frac{xq + (n-x+1)p - xq}{xq}$$

$$= 1 + \frac{(n+1)p - x(p+q)}{xq} = 1 + \frac{(n+1)p - x}{xq} \qquad \ldots(7\cdot9)$$

Mode is the value of x for which $p(x)$ is maximum.

We discuss the following two cases :

Case 1. When $(n+1)p$ is not an integer

Let $(n+1)p = m + f$, where m is an integer and f is fractional such that $0 < f < 1$. Substituting in (7·9), we get

$$\frac{p(x)}{p(x-1)} = 1 + \frac{(m+f) - x}{xq} \qquad \ldots(*)$$

From (*), it is obvious that

$$\frac{p(x)}{p(x-1)} > 1 \text{ for } x = 0, 1, 2, \ldots, m$$

and
$$\frac{p(x)}{p(x-1)} < 1 \text{ for } x = m+1, m+2, \ldots, n$$

$$\Rightarrow \quad \frac{p(1)}{p(0)} > 1, \frac{p(2)}{p(1)} > 1, \ldots, \frac{p(m)}{p(m-1)} > 1,$$

and
$$\frac{p(m+1)}{p(m)} < 1, \frac{p(m+2)}{p(m+1)} < 1, \ldots, \frac{p(n)}{p(n-1)} < 1,$$

$$\therefore p(0) < p(1) < p(2) < \ldots < p(m-1) < p(m) > p(m+1) > p(m+2) > p(m+3) \ldots > p(n),$$

Thus in this case there exists unique modal value for binomial distribution and it is m, the integral part of $(n+1)p$.

Case II. *When* $(n + 1)p$ *is an integer.*

Let $(n + 1)p = m$ *(an integer).*

Substituting in (7·9), we get

$$\frac{p(x)}{p(x-1)} = 1 + \frac{m - x}{xq} \qquad ...(**)$$

From (**) it is obvious that

$$\frac{p(x)}{p(x-1)} \left\} \begin{array}{l} > 1 \text{ for } x = 1, 2, ..., m-1 \\ = 1 \text{ for } x = m \\ < 1 \text{ for } x = m+1, m+2, ..., n \end{array} \right.$$

Now proceeding as in case 1, we have :

$p(0) < p(1) < ... < p(m-1) = p(m) > p(m+1) > p(m+2) > ... > p(n)$

Thus in this case the distribution is bimodal and the two modal values are m and $m - 1$.

Example 7·11. *Determine the binomial distribution for which the mean is 4 and variance 3 and find its mode.* **(Madurai Kamraj Univ B.Sc. 1993)**

Solution, Let $X \sim B(n, p)$, then we are given that

$$E(X) = np = 4 \qquad ..(*)$$

and

$$\text{Var}(X) = npq = 3 \qquad ..(**)$$

Dividing (**) by (*), we get

$$q = \tfrac{3}{4} \quad \Rightarrow \quad p = 1 - q = \tfrac{1}{4}$$

Hence from (*), $\quad n = \frac{4}{p} = 16$

Thus the given binomial distribution has parameters $n = 16$ and $p = 1/4$.

Mode. We have $(n + 1)p = 4·25$, which is not an integer. Hence the unique mode of the binomial distribution is 4, the integral part of $(n + 1)p$.

Example 7·12. *Show that for* $p = 0·50$, *the binomial distribution has a maximum probability at* $X = \frac{1}{2}n$, *if* n *is even, and at* $X = \frac{1}{2}(n - 1)$ *as well as* $X = \frac{1}{2}(n + 1)$, *if* n *is odd.* **(Mysore Univ., B. Sc. 1991)**

Solution. Here we have to find the mode of the binomial distribution.

(i) Let n be even $= 2m$, (say), $m = 1, 2, ...$

$\therefore$ If $p = 0·5$, then $(n + 1)p = (2m + 1) \times \left(\frac{1}{2}\right) = m + 0·5$

Hence in this case, the distribution is unimodal, the unique mode being at $X = m = n/2$.

(ii) Let n be odd $= (2m + 1)$, say. Then

$(n + 1)p = (2m + 2) \times \frac{1}{2} = m + 1$ (Integer)

$$= \frac{n-1}{2} + 1 = \frac{n+1}{2}$$

Since $(n+1)p$ is an integer, the distribution is bimodal, the two modes being $\frac{1}{2}(n+1)$ and $\frac{1}{2}(n+1)-1 = \frac{1}{2}(n-1)$.

7·2·6. Moment Generating Function of Binomial Distribution. Let X be a variable following binomial distribution, then

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^{n} e^{tx}\binom{n}{x}p^x q^{n-x} = \sum_{x=0}^{n} (pe^t)^x q^{n-x}\binom{n}{x} = (q+pe^t)^n \qquad ...(7\cdot10)$$

M.G.F. about Mean of Binomial Distribution :

$$E\left\{e^{t(X-np)}\right\} = E(e^{tx}e^{-tnp}) = e^{-tnp}\cdot E(e^{tX}) = e^{-tnp}\cdot M_X(t)$$

$$= e^{-tnp}(q+pe^t)^n = (qe^{-pt}+pe^{tq})^n \qquad ...(7\cdot11)$$

$$= \left[q\left\{1-pt+\frac{p^2t^2}{2!}-\frac{p^3t^3}{3!}+\frac{p^4t^4}{4!}-\ldots\right\}\right.$$
$$\left.+\,p\left\{1+tq+\frac{t^2q^2}{2!}+\frac{t^3q^3}{3!}+\ldots\right\}\right]^n$$

$$= \left[1+\frac{t^2}{2!}pq+\frac{t^3}{3!}pq(q^2-p^2)+\frac{t^4}{4!}pq(q^3+p^3)+\ldots\right]^n$$

$$= \left[1+\left\{\frac{t^2}{2!}\cdot pq+\frac{t^3}{3!}\cdot pq(q-p)+\frac{t^4}{4!}pq(1-3pq)+\ldots\right\}\right]^n$$

$$= \left[1+\binom{n}{1}\left\{\frac{t^2}{2!}\cdot pq+\frac{t^3}{3!}pq(q-p)+\frac{t^4}{4!}pq(1-3pq)+\ldots\right\}\right.$$
$$\left.+\binom{n}{2}\left\{\frac{t^2}{2!}pq+\frac{t^3}{3!}pq(q-p)+\ldots\right\}^2+\ldots\right]$$

Now μ_2 = Coefficient of $\frac{t^2}{2!} = npq$

μ_3 = Coefficient of $\frac{t^3}{3!} = npq(q-p)$

μ_4 = Coefficient of $\frac{t^4}{4!} = npq(1-3pq)+3n(n-1)p^2q^2$

$= npq(1-3pq)+3n^2p^2q^2-3np^2q^2$

$= 3n^2p^2q^2+npq\,(1-6pq)$

Example 7·13 *X is binomially distributed with parameters n and p. What is the distrbution of $Y = n - X$?* **[Delhi Univ. B.Sc. (Maths Hons.), 1990]**

Solution. $X \sim B(n,p)$, represents the number of successes in n independent trials with constant probability p of success for each trial.

$\therefore$ $Y = n - X$, represents the number of failures in n independent trial with constant probability 'q' of failure for each trial. Hence $Y = n - X \sim B(n, q)$

Aliter Since $X \sim B(n, p)$; $M_X(t) = E(e^{tX}) = (q + pe^t)^n$

$$\therefore \quad M_Y(t) = E(e^{tY}) = E(e^{t(n-X)})$$
$$= e^{nt} \cdot E(e^{-tX}) = e^{nt} M_X(-t)$$
$$= e^{nt} \cdot (q + pe^{-t})^n$$
$$= \left[e^t \left(q + pe^{-t}\right)\right]^n = \left(p + qe^t\right)^n$$

Hence by uniqueness theorem of m.g.f., $Y = n - X \sim B(n, q)$

Example 7·14. *The m.g.f. of a r.v. X is* $\left(\frac{2}{3} + \frac{1}{3}e^t\right)^9$. *Show that :*

$$P(\mu - 2\sigma < X < \mu + 2\sigma) = \sum_{x=1}^{5} \binom{9}{x} \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{9-x}$$

[Delhi Univ. B.Sc. (Maths Hons.), 1989]

Solution. Since $M_X(t) = \left(\frac{2}{3} + \frac{1}{3}e^t\right)^9 = \left(q + pe^t\right)^n$,

by uniqueness theorem of m.g.f. $X \sim B\left(n = 9, p = \frac{1}{3}\right)$

Hence $E(X) = \mu_x = np = 3$; $\sigma_X^2 = npq = 9 \times \frac{1}{3} \times \frac{2}{3} = 2$

$\mu \pm 2\sigma = 3 \pm 2 \times \sqrt{2} = 3 \pm 2 \times 1{\cdot}4 = (0{\cdot}2,\ 5{\cdot}8)$

$$\therefore P(\mu - 2\sigma < X < \mu + 2\sigma) = P(0{\cdot}2 < X < 5{\cdot}8) = P(1 \le X \le 5)$$
$$= \sum_{x=1}^{5} p(x) = \sum_{x=1}^{5} {}^nC_x p^x q^{n-x}$$
$$= \sum_{x=1}^{5} {}^9C_x (1/3)^x (2/3)^{9-x}$$

7·2·7. Additive Property of Binomial Distribution. Let $X \sim B(n_1, p_1)$ and $Y \sim B(n_2, p_2)$ be independent random variables. Then

$$M_X(t) = (q_1 + p_1 e^t)^{n_1},\ M_Y(t) = (q_2 + p_2 e^t)^{n_2} \quad \ldots(*)$$

What is the distribution of $X + Y$?

We have

$$M_{X+Y}(t) = M_X(t) \cdot M_Y(t) \quad [\because X \text{ and } Y \text{ are independent}]$$
$$= (q_1 + p_1 e^t)^{n_1} \cdot (q_2 + p_2 e^t)^{n_2} \quad \ldots(**)$$

Since (**) cannot be expressed in the form $(q + p\,e^t)^n$, from uniqueness theorem of m.g.f.'s it follows that $X + Y$ is not a binomial variate. Hence, *in general the sum of two independent binomial variates is not a binomial variate.*

In other words, binomial distribution does not possess the additive or reproductive property.

However, if we take $p_1 = p_2 = p$ (say), then from (**), we get

$$M_{X+Y}(t) = (q + pe^t)^{n_1 + n_2},$$

which is the m.g.f. of a binomial variate with parameters $(n_1 + n_2, p)$. Hence, by uniqueness theorem of m.g.f.'s $X + Y \sim B(n_1 + n_2, p)$. Thus the *binomial distribution possesses the additive or reproductive property if* $p_1 = p_2$.

Generalisation. *If* X_i, $(i = 1, 2, ..., k)$ *are independent binomial variates with parameters* (n_i, p), $(i = 1, 2, ..., k)$ *then their sum* $\sum_{i=1}^{k} X_i \sim B\left(\sum_{i=1}^{k} n_i, p\right)$.

The proof is left as an exercise to the reader.

Example 7·15. *If the independent random variables X, Y are binomially distributed, respectively with* $n = 3$, $p = 1/3$, *and* $n = 5$, $p = 1/3$, *write down the probability that* $X + Y \geq 1$.

Solution. We are given

$$X \sim B(3, \tfrac{1}{3}) \text{ and } Y \sim B(5, \tfrac{1}{3}).$$

Since X and Y are independent binomial random variables, with $p_1 = p_2 = \frac{1}{3}$, by the additive property of binomial distribution, we get

$$X + Y \sim B(3 + 5, \tfrac{1}{3}), \text{ i.e., } X + Y \sim B(8, \tfrac{1}{3})$$

$$\therefore \quad P(X + Y = r) = {}^8C_r \left(\tfrac{1}{3}\right)^r \left(\tfrac{2}{3}\right)^{8-r} \qquad \text{...(*)}$$

$$\text{Hence } P(X + Y \geq 1) = 1 - P(X + Y < 1)$$
$$= 1 - P(X + Y = 0)$$
$$= 1 - \left(\tfrac{2}{3}\right)^8$$

7·2·8. Characteristic Function of Binomial Distribution.

$$\varphi_X(t) = E(e^{itX}) = \sum_{x=0}^{n} e^{itx} p(x) = \sum_{x=0}^{n} e^{itx} \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^{it})^x q^{n-x} = (q + pe^{it})^n \qquad \text{...(7·12)}$$

7·2·9. Cummulants of the Binomial Distribution. Cumulant generating function is

$$K_X(t) = \log M_X(t) = \log (q + pe^t)^n = n \log (q + pe^t)$$

$$= n \log \left[q + p\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \dots\right)\right]$$

$$= n \log \left[1 + p\left(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \dots\right)\right]$$

$$= n\left[p\left(t+\frac{t^2}{2!}+\frac{t^3}{3!}+\frac{t^4}{4!}+\ldots\right)-\frac{p^2}{2}\left(t+\frac{t^2}{2!}+\frac{t^3}{3!}+\ldots\right)^2\right.$$

$$\left.+\frac{p^3}{3}\left(t+\frac{t^2}{2!}+\frac{t^3}{3!}+\ldots\right)^3-\frac{p^4}{4}\left(t+\frac{t^2}{2!}+\frac{t^3}{3!}+\ldots\right)^4+\ldots\right]$$

Mean $= \kappa_1 =$ Coefficient of t in $K_X(t) = np$

$$\mu_2 = \kappa_2 = \text{Coefficient of } \frac{t^2}{2!} \text{ in } K_X(t) = n(p-p^2) = np(1-p) = npq$$

The coefficient of t^3 in $K_X(t)$

$$= n\left[\frac{p}{3!}-\frac{p^2}{2!}\cdot 2\cdot\frac{1}{2!}+\frac{p^3}{3}\right] = \frac{np}{3!}(1-3p+2p^2)$$

$$\therefore\ \kappa_3 = \text{Coefficient of } \frac{t^3}{3!} \text{ in } K_X(t) = np(1-3p+2p^2)$$

$$= np(1-p)(1-2p) = npq(1-p-p) = npq(q-p)$$

$$\therefore\qquad \mu_3 = \kappa_3 = npq(q-p)$$

The Coefficient of t^4 in $K_X(t)$

$$= n\left[\frac{p}{4!}-\frac{p^2}{2!}\left(\frac{2}{3!}+\frac{1}{4}\right)+\frac{p^3}{3}\cdot\frac{3}{2!}-\frac{p^4}{4}\right]$$

$$= \frac{np}{4!}[1-7p+12p^2-6p^3]$$

$$\therefore\ \kappa_4 = \text{Co efficient of } \frac{t^4}{4!} \text{ in } K_X(t) = np(1-p)(1-6p+6p^2)$$

$$= npq[1-6p(1-p)] = npq(1-6pq)$$

$$\therefore\ \mu_4 = \kappa_4 + 3\kappa_2^2 = npq(1-6pq) + 3n^2p^2q^2$$

$$= npq(1-6pq+3npq) = npq[1+3pq(n-2)]$$

7·2·10. Recurrence Relation for Cumulants of Binomial Distribution. By def.,

$$\kappa_r = \left[\frac{d^r}{dt^r}\log M_X(t)\right]_{t=0} = n\left[\frac{d^r}{dt^r}\log(q+pe^t)\right]$$

$$\frac{d\kappa_r}{dp} = n\left[\frac{d^r}{dt^r}\cdot\frac{d}{dp}\log(q+pe^t)\right]_{t=0} = n\left[\frac{d^r}{dt^r}\cdot\frac{(-1+e^t)}{q+pe^t}\right]_{t=0}$$

$$\kappa_{r+1} = n\left[\frac{d^{r+1}}{dt^{r+1}}\log(q+pe^t)\right]_{t=0}$$

$$= n\left[\frac{d^r}{dt^r}\cdot\frac{d}{dt}\log(q+pe^t)\right]_{t=0} = n\left[\frac{d^r}{dt^r}\left(\frac{pe^t}{q+pe^t}\right)\right]_{t=0}$$

$$= n \left[\frac{d^r}{dt^r} \left(1 - \frac{q}{q + pe^t} \right) \right]_{t=0} = - nq \left[\frac{d^r}{dt^r} \left(\frac{1}{q + pe^t} \right) \right]_{t=0}$$

Hence

$$\kappa_{r+1} - pq \frac{d\kappa_r}{dp} = - nq \left[\frac{d^r}{dt^r} \left(\frac{1}{q + pe^t} \right) \right]_{t=0} - npq \left[\frac{d^r}{dt^r} \left(\frac{e^t - 1}{q + pe^t} \right) \right]_{t=0}$$

$$= - nq \left[\frac{d^r}{dt^r} \left\{ \frac{1 + pe^t - p}{q + pe^t} \right\} \right]_{t=0}$$

$$= - nq \left[\frac{d^r}{dt^r} \left\{ \frac{q + pe^t}{q + pe^t} \right\} \right]_{t=0} = - nq \left[\frac{d^r}{dt^r} (1) \right]_{t=0} = 0$$

$$\therefore \qquad \kappa_{r+1} = pq \frac{d\kappa_r}{dp} \qquad \qquad ...(7{\cdot}13)$$

In particular,

$$\kappa_2 = pq \cdot \frac{d\kappa_1}{dp} = pq \cdot \frac{d}{dp}(np) = npq \qquad (\because \kappa_1 = \textbf{mean} = np)$$

$$\kappa_3 = pq \cdot \frac{d\kappa_2}{dp} = pq \cdot \frac{d(npq)}{dp} = npq(q - p)$$

$$\kappa_4 = pq \cdot \frac{d\kappa_3}{dp} = pq \cdot \frac{d}{dp} \{ npq (q - p) \}$$

$$= npq \frac{d}{dp} \{ p (1 - p)(1 - 2p) \}$$

$$= npq \cdot \frac{d}{dp} (p - 3p^2 + 2p^3) = npq(1 - 6p + 6p^2)$$

$$= npq [1 - 6p(1 - p)] = npq (1 - 6pq)$$

7·2·11. Probability Generating Function of Binomial Distribution

$$P(s) = \sum_{k=0}^{n} P(X = k)\, s^k = \sum_{k=0}^{n} \binom{n}{k} (ps)^k q^{n-k} = (ps + q)^n \qquad ...(7{\cdot}13\,a)$$

The fact that this generating function is nth power of $(q + ps)$ shows that $p(x) = \{ b(x; n, p) \}$ is the distribution of the sum $S_n = X_1 + X_2 + \dots + X_n$ of n random variables with the common generating function $(q + ps)$. Each variable X_i assumes the value 0 with probability q and 1 with probability p.

$$\text{Thus} \qquad \{ b(k; n, p) \} = \{ b(k; 1, p) \}^{n*} \qquad ...(7{\cdot}13b)$$

Let X and Y be two independent random variables having $b(k; m, p)$ and $b(k; n, p)$ as their distributions, then

$$P_X(s) = (q + ps)^m \text{ and } P_Y(s) = (q + ps)^n$$

$$\therefore \qquad P_{X+Y}(s) = (q + ps)^m (q + ps)^n = (q + ps)^{m+n}$$

$$\therefore \{ b(k; m, p) \} * \{ b(k; n, p) \} = \{ b(k; m + n, p) \} \qquad ...(7{\cdot}13c)$$

Also $\mu_{(1)}' = [\, n\,(q + ps)^{n-1} p\,]_{s=1} = np$

$\mu_{(2)}' = [\, n\,(n-1)\,(q + ps)^{n-2} p^2\,]_{s=1} = n\,(n-1)\,p^2$ and so on.

$$\mu_{(r)}' = [n\,(n-1)\ldots(n-r+1)\,(q+ps)^{n-r} p^r\,]_{s=1}$$
$$= n\,(n-1)\ldots(n-r+1)\,p^r$$

Example 7·16 *Show that*

$$E\left(\frac{1}{X+a}\right) = \int_0^1 t^{a-1}\,G(t)\,dt, \quad a > 0 \qquad \ldots(*)$$

where $G(t)$ is the probability generating function of X.

Find it when $X \sim B(n,p)$, and $a = 1$

[Delhi Univ. (Stat Hons.) Spl Course, 1988]

Solution. R.H.S. $= \int_0^1 t^{a-1}\cdot G(t)\,dt = \int_0^1 t^{a-1}\left(Et^X\right)dt$

$$= \int_0^1 \left\{t^{a-1}\left(\sum_x p_x t^x\right)\right\}dt = \sum_x \left[p_x \int_0^1 t^{x+a-1}\,dt\right]$$
$$= \sum_x p_x \cdot \frac{1}{(x+a)} = E\left(\frac{1}{X+a}\right)$$

If $X \; B(n,p)$, then $G(t) = \sum_{x=0}^{n} t^x p_x = (q+pt)^n \qquad \ldots(**)$

Hence taking $a = 1$ in (*) and using (**), we get :

$$E\left[\frac{1}{(X+a)}\right] = \int_0^1 (q+pt)^n\,dt = \left|\frac{(q+pt)^{n+1}}{(n+1)\,p}\right|_0^1 = \frac{1-q^{n+1}}{(n+1)\,p}$$

7·2·12. Recurrence Relation for the Probabilities of Binomial Distribution. *(Fitting of Binomial Distribution).*

We have

$$\frac{p\,(x+1)}{p\,(x)} = \frac{\binom{n}{x+1} p^{x+1} q^{n-x-1}}{\binom{n}{x} p^x q^{n-x}}$$
$$= \frac{n-x}{x+1}\cdot\frac{p}{q} \qquad \text{(On simplification)}$$

$$p\,(x+1) = \left\{\frac{n-x}{x+1}\cdot\frac{p}{q}\right\}p\,(x), \qquad \ldots(7\cdot 14)$$

which is the required recurrence formula.

This formula provides us a very convenient method of graduating the given data by a binomial distribution. The only probability we need to calculate is $p\,(0)$

which is given by $p(0) = q^n$, where q is estimated from the given data by equating the mean $\bar{x}$ of the distribution to np, the mean of the binomial distribution. Thus $\hat{p} = \bar{x}/n$.

The remaining probabilities, *viz.*, $p(1), p(2), \ldots$ can now be easily obtained from (7·14) as explained below :

$$p(1) = [p(x+1)]_{x=0} = \left(\frac{n-x}{x+1} \cdot \frac{p}{q}\right)_{x=0} p(0)$$

$$p(2) = [p(x+1)]_{x=1} = \left(\frac{n-x}{x+1} \cdot \frac{p}{q}\right)_{x=1} p(1)$$

$$p(3) = [p(x+1)]_{x=2} = \left(\frac{n-x}{x+1} \cdot \frac{p}{q}\right)_{x=2} p(2)$$

and so on.

Example 7·17. *Seven coins are tossed and number of heads noted. The experiment is repeated 128 times and the following distribution is obtained:*

No. of heads	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*Total*
Frequencies	*7*	*6*	*19*	*35*	*30*	*23*	*7*	*1*	*128*

Fit a Binomial distribution assuming

(i) The coin is unbaised,

(ii) The nature of the coin is not known.

(iii) Probability of a head for four coins is 0·5 and for the remaining three coins is 0·45.

Solution. In fitting Binomial distribution, first of all the mean and variance of the data are equated to np and npq respectively. Then the expected frequencies are calculated from these values of n and p. Here $n = 7$ and $N = 128$.

Case I. When the coin is unbaised

$$p = q = \tfrac{1}{2}, \ (p/q = 1)$$

Now
$$p(0) = q^n = (\tfrac{1}{2})^7 = (1/128)$$
$$f(0) = Nq^n = 128\,(\tfrac{1}{2})^7 = 1$$

Using the recurrence formula, the various probabilities, *viz.*, $p(1)$, $p(2), \ldots$ can be easily calculated as shown below.

x	$\dfrac{n-x}{x+1}$	$\dfrac{n-x}{x+1} \cdot \dfrac{p}{q}$	*Expected frequency* $f(x) = Np(x)$
0	7	7	$f(0) = Np(0) = 1$
1	3	3	$f(1) = 1 \times 7 = 7$

2	$\frac{5}{3}$	$\frac{5}{3}$	$f(2) = 7 \times 3 = 21$
3	1	1	$f(3) = 21 \times \frac{5}{3} = 35$
4	$\frac{3}{5}$	$\frac{3}{5}$	$f(4) = 35 \times 1 = 35$
5	$\frac{1}{3}$	$\frac{1}{3}$	$f(5) = 35 \times \frac{3}{5} = 21$
6	$\frac{1}{7}$	$\frac{1}{7}$	$f(6) = 21 \times \frac{1}{3} = 7$
7			$f(7) = 7 \times \frac{1}{7} = 1$

Case II. When the nature of the coin is not known, then

$$np = \frac{1}{N} \sum_{i=1}^{n} f_i x_i = \frac{433}{128} = 3\cdot3828\,;\ n = 7$$

$$\therefore \quad p = 0\cdot48326 \text{ and } q = 0\cdot51674, \ (p/q = 0\cdot93521)$$

$$f(0) = Nq^7 = 128\,(0\cdot5167)^7 = 1\cdot2593 \text{ (using logarithms)}$$

x	$\frac{n-x}{x+1}$	$\frac{n-x}{x+1}\cdot\frac{p}{q}$	Expected frequency $f(x) = Np(x)$
0	7	6·54647	$f(0) = Np(0) = 1\cdot2593 \simeq 1$
1	3	2·80563	$f(1) = 1\cdot2593 \times 6\cdot54647 = 8\cdot2438 \simeq 8$
2	$\frac{5}{3}$	1·55868	$f(2) = 2\cdot80563 \times 8\cdot2438 = 23\cdot129 \simeq 23$
3	1	0·93521	$f(3) = 1\cdot55868 \times 23\cdot129 = 36\cdot05 \simeq 36$
4	$\frac{3}{5}$	0·56113	$f(4) = 0\cdot93521 \times 36\cdot05 = 33\cdot715 \simeq 34$
5	$\frac{1}{3}$	0·31174	$f(5) = 0\cdot56113 \times 33\cdot715 = 18\cdot918 \simeq 19$
6	$\frac{1}{7}$	0·13360	$f(6) = 0\cdot31174 \times 18\cdot918 = 5\cdot897 \simeq 6$
7			$f(7) = 0\cdot13360 \times 5\cdot897 = 0\cdot788 \simeq 1$

The probability generating functions (p.g.f.), say $P_X(s)$ for the 4 coins and $P_Y(S)$ for the remaining 3 coins are given by

$$P_X(s) = (0\cdot50 + 0\cdot50\,s)^4,\ P_Y(s) = (0\cdot55 + 0\cdot45s)^3 \qquad \text{...[c.f. 7·13 (a)]}$$

Since all the throws are independent, the p.g.f. $P_{X+Y}(s)$ for the whole experiment is given by

$$P_{X+Y}(s) = P_X(s)\,P_Y(s) \qquad \text{...[c.f. 7·13 (b)]}$$
$$= (0\cdot 50 + 0\cdot 50\,s)^4\,(0\cdot 55 + 0\cdot 45\,s)^3$$
$$= (0\cdot 0625 + 0\cdot 25\,s + 0\cdot 375\,s^2 + 0\cdot 25\,s^3 + 0\cdot 0625\,s^4)$$
$$\times\,(0\cdot 166375 + 0\cdot 408375\,s + 0\cdot 334125\,s^2 + 0\cdot 091125\,s^3)$$

Now $f(x) = N \times$ coefficient of t^x in $P_{X+Y}(t)$

$\therefore \quad f(0) = 128 \times \cdot 0625 \times \cdot 16637 = 1\cdot 13310$

$f(1) = 128\,\{\cdot 25 + \cdot 166375 + \cdot 408375 \times \cdot 0625\} = 8\cdot 5910$

$f(2) = 128\,\{\cdot 28396\} = 36\cdot 3470 \qquad f(5) = 128\,\{\cdot 14602\} = 18\cdot 6934$

$f(3) = 128\,\{\cdot 184117\} = 23\cdot 5669 \qquad f(6) = 128\,\{\cdot 04366\} = 5\cdot 5889$

$f(4) = 128\,\{\cdot 260570\} = 33\cdot 3529 \qquad f(7) = 128\,\{\cdot 005695\} = \cdot 72896$

Example 7·18. *Let X and Y be independent binomial variates, each with parameters n and p. Find P (X − Y = k).* **(Calcutta Univ. B.Sc., 1993)**

Solution. Since each of the variables X and Y takes the values 0,1,2,.., n, $Z = X - Y$ takes on the values $-n, -(n-1), ..., -1, 0, 1 ..., n$

$$P(Z = k) = \sum_{r=0}^{n} P(X = k + r \cap Y = r)$$
$$= \sum_{r=0}^{n} P(X = k + r)\cdot P(Y = r) \quad (\because X \text{ and } Y \text{ are independent}).$$
$$= \sum_{r=0}^{n} \binom{n}{k+r} p^{k+r}\cdot q^{n-k-r} \binom{n}{r} p^r q^{n-r}$$
$$= \sum_{r=0}^{n} \binom{n}{k+r}\binom{n}{r} p^{2r+k}\, q^{2n-2r-k} \qquad \text{...(*)}$$

where $k = -n, -(n-1), ..., -2, -1, 0, 1, 2, ..., n$; and $q = 1 - p$.

In particular, we have :

$$P(Z = 0) = \sum_{r=0}^{n} \binom{n}{r}^2 \cdot p^{2r}\, q^{2n-2r}$$
$$P(Z = -n) = \sum_{r=0}^{n} \binom{n}{-n+r}\binom{n}{r} p^{2r-n}\, q^{3n-2r} = p^n q^n,$$

because we get the result when $r = n$ and for other values of $r < n$, $\binom{n}{-n+r}$ is not defined and hence taken as 0.

Example 7·19. *Find the m.g.f. of standard binomial variate $(X - np)/\sqrt{npq}$ and obtain its limiting form as $n \to \infty$. Also interpret the result.*

[Delhi Univ. B.Sc. (Stat. Hons.) 1990, 85]

Solution. We know that if $X \sim B(n, p)$, then

$$M_X(t) = (q + p\,e^t)^n$$

The m.g.f. of standard binomial variate.

$$Z = \frac{X - np}{\sqrt{npq}} = \frac{X - \mu}{\sigma}, \text{(say)}$$

where $\mu = np$ and $\sigma^2 = npq$, is given by

$$M_Z(t) = e^{-\mu t/\sigma} M_X(t/\sigma)$$

$$= e^{-npt/\sqrt{npq}} \cdot (q + p\,e^{t/\sqrt{npq}})^n \qquad \text{[From (**)]}$$

$$= \left[e^{-pt/\sqrt{npq}} (q + p\,e^{t/\sqrt{npq}})\right]^n$$

$$= \left[qe^{-pt/\sqrt{npq}} + pe^{qt/\sqrt{npq}}\right]^n$$

$$= \left[q\left\{1 - \frac{pt}{\sqrt{npq}} + \frac{p^2t^2}{2npq} + 0'(n^{-3/2})\right\}\right.$$

$$\left. + p\left\{1 + \frac{qt}{\sqrt{npq}} + \frac{q^2t^2}{2npq} + 0''(n^{-3/2}\right\}\right]^n$$

where $0'(n^{-3/2})$ and $0''(n^{-3/2})$ involve terms containing $n^{3/2}$ and higher powers of n in the denominator.

$$\therefore\ M_Z(t) = \left[(q + p) + \frac{t^2pq}{2npq}(p + q) + 0(n^{-3/2})\right]^n$$

$$= \left[1 + \frac{t^2}{2n} + 0(n^{-3/2})\right]^n$$

where $0(n^{-3/2})$ involves terms with $n^{3/2}$ and higher powers of n in the denominator.

$$\therefore\ \log M_Z(t) = n \log\left[1 + \frac{t^2}{2n} + 0(n^{-3/2})\right]$$

$$= n\left[\left\{\frac{t^2}{2n} + 0(n^{-3/2})\right\} - \frac{1}{2}\left\{\frac{t^2}{2n} + 0(n^{-3/2})\right\}^2 + \ldots\right]$$

$$= \frac{t^2}{2} + 0'''(n^{-1/2})$$

where $0'''(n^{-1/2})$ involve terms with $n^{1/2}$ and higher powers of n in the denominator. Proceeding to the limit as $n \to \infty$, we get

$$\lim_{n \to \infty} \log M_Z(t) = \frac{t^2}{2}$$

$$\Rightarrow \qquad \lim_{n \to \infty} M_Z(t) = \exp(t^2/2) \qquad \ldots(**)$$

Interpretation. (**) is the m.g.f. of standard normal variate [c.f. Remark to § 8·2·5]. Hence by uniqueness theorem of moment generating functions, standard binomial variate tends to standard normal variate as $n \to \infty$. In other words, binomial distribution tends to normal distribution as $n \to \infty$.

Example 7·20. *A drunk performs a random walk over positions 0, ± 1, ± 2, ..., as follows. He starts at 0. He takes successive one unit steps, going to the right with probability p and to the left with probability (1 − p). His steps are independent. Let X denote his position after n steps. Find the distribution of (X + n)/2 and find E (X).* **(I.I.T. B.Tech., Dec. 1991)**

Solution. With the ith step of the drunk, let us associate a variable X_i defined as follows :

$X_i = 1$, if he takes the step to the right

$= -1$ if he takes the step to the left

Then $X = X_1 + X_2 + \ldots + X_n$, gives the position of the drunkard after n steps.

Define $\quad Y_i = (X_i + 1)/2$

Then $\quad Y_i = (1 + 1)/2 = 1$, with probability p

$= (-1 + 1)/2 = 0$, with probability $1 - p = q$, (say).

Since the n steps of drunkard are independent, Y_i's, $(i = 1, 2, \ldots n)$ are i.i.d. Bernonlli variates with parameter p.

Hence $\displaystyle\sum_{i=1}^{n} Y_i \sim B(n, p)$

$$\Rightarrow \sum_{i=1}^{n} Y_i = \sum_{i=1}^{n} \left(\frac{X_i + 1}{2}\right) = \frac{1}{2}\left[\sum_{i=1}^{n} X_i + n\right] = \frac{X + n}{2} \sim B(n, p)$$

where $X = \displaystyle\sum_{i=1}^{n} X_i$, is the position of the drunkard after n steps.

Since $(X + n)/2 \sim B(n, p)$, we have

$$E\left[\frac{X + n}{2}\right] = np \Rightarrow \frac{1}{2} E(X + n) = np$$

$$\Rightarrow E(X) + n = 2np \Rightarrow E(X) = n(2p - 1)$$

Example 7·21. *Suppose that the r.v. X is uniformly distributed on (0,1)*

i.e., $f_X(x) = 1 ; 0 \le x \le 1.$ ...(*)

Assume that the conditional distributional $Y | X = x$ has a binomial distribution with parameters n and p = x, i.e.,

$$P(Y = y | X = x) = \binom{n}{y} x^y (1 - x)^{n - y};\ y = 0, 1, 2, \ldots, n \qquad (**)$$

Find (a) E (Y)

(b) Find the distribution of Y. **(Punjab P.C.S., 1990)**

Solution. (a) We are given that the conditional distribution of

$$Y | X = x \sim B(n, x) \qquad \ldots(i)$$

$$\therefore\ E(Y | X = x) = nx \qquad \ldots(ii)$$

We have :

$$E(Y) = E[E(Y|X)] = E[nX] = nE(X) \qquad \text{[On using (ii)]}$$

$$\text{Now } E(X) = \int_0^1 x f(x)\,dx = \int_0^1 x\,dx = \frac{1}{2}.$$

$$\therefore\ E(Y) = n \times (\tfrac{1}{2}) = \frac{1}{2} n$$

(b) We have : $f_{X,Y}(x, y) = f_X(x) \cdot f_{Y|X}(y|x)$

Since X has (continuous) uniform distribution on (0,1) marginal distribution of Y is given by.

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y)\,dx = \int_0^1 f_{Y|X}(y|x) \cdot f_X(x)\,dx$$

$$= \int_0^1 {}^nC_y \cdot x^y (1-x)^{n-y} \cdot 1 \cdot dx \qquad \text{[using (*) and (**)]}$$

$$= {}^nC_y \int_0^1 x^y (1-x)^{n-y}\,dx$$

$$= {}^nC_y \cdot \beta(y+1, n-y+1) = \frac{n!}{y!(n-y)!} \frac{\Gamma(y+1)\Gamma(n-y+1)}{\Gamma(n+2)}$$

$$= \frac{n!}{y!(n-y)!} \times \frac{y!(n-y)!}{(n+1)!}$$

$$= \frac{1}{n+1} \quad ; \qquad y = 0, 1, 2, \ldots, n$$

Since Y takes the values 0, 1, 2, ..., n each with equal probability $1/(n+1)$, Y has discrete uniform distribution.

Remark We could find $E(Y)$ on using the distribution of Y in (b).

$$E(Y) = \sum_{y=0}^{n} y\,p(y) = \frac{1}{n+1} \sum_{y=0}^{n} y$$

$$= \frac{1}{n+1}[0 + 1 + 2 + \ldots + n] = \frac{n}{2},$$

as in Part (a).

Example 7·22. *If $K(t)$ is the cumulative function about the origin of the Binomial Distribution of size n, show that*

$$\frac{d}{dt} K(t) = n\left\{1 + e^{-(z+t)}\right\}^{-1}, \text{ where } z = \log_e (p/q)$$

(b) *By expanding the R.H.S. in powers of t by Taylor's Theorem, show that*

$$\kappa_r = n \frac{d^{r-1} p}{dz^{r-1}}, \text{ where } \kappa_r \text{ is the rth cumulant.}$$

(c) *Hence or otherwise obtain the recurrence relation*

$$\kappa_{r+1} = pq \cdot \frac{d\kappa_r}{dp}, \; r > 1$$

[Baroda Univ. B.Sc. 1993; Delhi Univ. B.Sc. (Stat. Hons.) 1992]

(d) *Prove that* $\kappa_{r+1} = \dfrac{d\kappa_r}{dz}$, *where* $z = \log_e (p/q)$

Solution. For binomial distribution with parameters n and p, we have

$$K(t) = \log M(t) = n \log (q + pe^t)$$

(a) $$\frac{d}{dt} K(t) = \frac{npe^t}{q + pe^t} = n \left(1 + \frac{q}{p} e^{-t} \right)^{-1}$$

if $z = \log_e (p/q) \Rightarrow (p/q) = e^z \Rightarrow (q/p) = e^{-z}$, then

$$\frac{d}{dt} K(t) = n [1 + e^{-(z+t)}]^{-1} \qquad ...(*)$$

(b) $$\kappa_r = \left[\frac{d^r}{dt^r} \kappa(t) \right]_{t=0} = \left[\frac{d^{r-1}}{dt^{r-1}} \cdot \frac{d}{dt} \kappa(t) \right]_{t=0}$$

$$= n \left[\frac{d^{r-1}}{dt^{r-1}} \left\{ 1 + e^{-(z+t)} \right\}^{-1} \right]_{t=0} = n \left[\frac{d^{r-1}}{dt^{r-1}} \left(\frac{e^{z+t}}{1 + e^{z+t}} \right) \right]_{t=0} \qquad ...(**)$$

By summetry of the function $e^{z+t}/(1 + e^{z+t})$ in t and z we have

$$\frac{d}{dt} \left(\frac{e^{z+t}}{1 + e^{z+t}} \right) = \frac{d}{dz} \left(\frac{e^{z+t}}{1 + e^{z+t}} \right)$$

$$\Rightarrow \frac{d^{r-1}}{dt^{r-1}} \left(\frac{e^{z+t}}{1 + e^{z+t}} \right) = \frac{d^{r-1}}{dz^{r-1}} \left(\frac{e^{z+t}}{1 + e^{z+t}} \right)$$

Substituting in (**), we get

$$\kappa_r = n \left[\frac{d^{r-1}}{dz^{r-1}} \left(\frac{e^{z+t}}{1 + e^{z+t}} \right) \right]_{t=0} = n \frac{d^{r-1}}{dz^{r-1}} \left(\frac{e^z}{1 + e^z} \right)$$

$$= n \frac{d^{r-1}}{dz^{r-1}} (1 + e^{-z})^{-1} = n \frac{d^{r-1}}{dz^{r-1}} \left(1 + \frac{q}{p} \right)^{-1}$$

$$= n \frac{d^{r-1} p}{dz^{r-1}} \qquad ...(***)$$

(c) $$\frac{d\kappa_r}{dp} = n \frac{d}{dp} \left(\frac{d^{r-1} p}{dz^{r-1}} \right) = n \frac{d}{dz} \left(\frac{d^{r-1} p}{dz^{r-1}} \right) \frac{dz}{dp}$$

$$= n \frac{d^r p}{dz^r} \cdot \frac{1}{pq} \qquad [\because z = \log_e (p/q)]$$

$$= \frac{1}{pq} \cdot \kappa_{r+1} \qquad [\text{From } (***)]$$

(d) $\dfrac{d\kappa_r}{dz} = \dfrac{d\kappa_r}{dp}\cdot\dfrac{dp}{dz} = \dfrac{d\kappa_r}{dp}\Big/\dfrac{dz}{dp} = \dfrac{d\kappa_r}{dp}\Big/\dfrac{1}{pq} = pq\cdot\dfrac{d\kappa_r}{dp}$

$\therefore \qquad \dfrac{d\kappa_r}{dz} = \kappa_{r+1}$ [c.f. part (c)]

Example 7·23. *If* $b(r;n,p) = \binom{n}{r} p^r q^{n-r}$ *is the binomial probability in the usual notation and if*

$$B(k;n,p) = P(X \le k) = \sum_{r=0}^{k} b(r;n,p),$$

then prove that

$$B(k;n,p) = (n-k)\binom{n}{k}\int_0^q t^{n-k-1}(1-t)^k\,dt;\ q = 1-p$$

Solution. $B(k;n,p) = \sum_{r=0}^{k} b(r;n,p) = \sum_{r=0}^{k}\binom{n}{r}p^r q^{n-r}$

Differentiating w.r. to q and noting that $q = 1-p \Rightarrow \dfrac{dq}{dp} = -1$, we get:

$$\frac{d}{dq}\cdot B(k;n,p) = \sum_{r=0}^{k}\left[\binom{n}{r}\left\{r p^{r-1}(-1)\cdot q^{n-r} + p^r\cdot(n-r)q^{n-r-1}\right\}\right]$$

$$= \sum_{r=0}^{k}\left[\frac{n!\,(-r)}{r!\,(n-r)!}p^{r-1}q^{n-r} + \frac{n!\,(n-r)}{r!\,(n-r)!}p^r q^{n-r-1}\right]$$

$$= \sum_{r=0}^{k}\left[-\frac{n\,(n-1)!}{(r-1)!\,(n-r)!}p^{r-1}q^{n-r} + \frac{n\,(n-1)!}{r!\,(n-r-1)!}p^r q^{n-r-1}\right]$$

$$= \sum_{r=0}^{k}\left[n\cdot\binom{n-1}{r}p^r q^{n-r-1} - n\binom{n-1}{r-1}p^{r-1}q^{n-r}\right]$$

$$= \sum_{r=0}^{k}\left[n\left\{t_r - t_{r-1}\right\}\right] \qquad ...(**)$$

$$\text{where } t_r = \binom{n-1}{r}p^r q^{n-r-1} \qquad ...(***)$$

$$= n\left[\left(t_0 - t_{-1}\right) + \left(t_1 - t_0\right) + \left(t_2 - t_1\right) + \ldots + \left(t_k - t_{k-1}\right)\right]$$

$$= n\,t_k \qquad [\because t_{-1} = 0, \text{ From } (***)]$$

$$\therefore\ \frac{d}{dq}\cdot B(k,n,p) = n\binom{n-1}{k}p^k\cdot q^{n-k-1}, \quad p = 1-q$$

On integration, we get

$$B(k\,;n,p) = n \cdot \binom{n-1}{k} \int_0^q (1-u)^k \cdot u^{n-k-1}\, du.$$

$$\text{But } n \cdot \binom{n-1}{k} = \frac{n \cdot (n-1)\,!}{k\,!\,(n-1-k)\,!} = \frac{n\,!\,(n-k)}{k\,!\,(n-k)\,!} = (n-k)\binom{n}{k}$$

$$\therefore \qquad B(k\,;n,p) = (n-k)\binom{n}{k}\int_0^q (1-u)^k \cdot u^{n-k-1}\, du$$

as desired.

Remarks. 1. We further get :

$$\beta(k+1,\ n-k) = \frac{\Gamma(k+1)\ \Gamma(n-k)}{\Gamma(n+1)} = \frac{k\,!\,(n-k-1)\,!}{n\,!}$$

$$\Rightarrow \frac{1}{\beta(k+1, n-k)} = \frac{n\,!}{k\,!\,(n-k-1)\,!} = (n-k)\binom{n}{k}$$

Hence the result may be written as :

$$B(k\,;n,p) = P(X \le k) = \frac{1}{\beta(k+1,\ n-k)}\int_0^q (1-u)^k u^{n-k-1}\, du$$

This result is of great practical utility. It enables us to represent the cumulative Binomial Probabilities (which are generally quite tedious and time consuming to compute) in terms of Incomplete Beta Functions whbich are tabulated in Karl Pearson's Tables of the Incomplete Beta Functions.

2 Let us now work out the probability :

$$P(X \ge k) = \sum_{r=k}^{n} \binom{n}{r} p^r q^{n-r}$$

Differentiating w.r. to p, and proceeding similarly, we shall get :

$$\frac{d}{dp} P(X \ge k) = -n \sum_{r=k}^{n} \left(T_r - T_{r-1}\right) \qquad \text{(Try it)}$$

$$\text{where} \qquad T_r = \binom{n-1}{r} p^r q^{n-r-1}, \qquad (T_n = 0)$$

$$\therefore \frac{d}{dp} P(X \ge k) = n\,T_{k-1} = n\binom{n-1}{k-1} p^{k-1}(1-p)^{n-k} \quad (\because q = 1-p)$$

On integration, we shall get :

$$P(X \ge k) = n\binom{n-1}{k-1}\int_0^p u^{k-1}(1-u)^{n-k}\, du$$

$$P(X \ge k) = \frac{1}{\beta(k,\ n-k+1)}\int_0^p u^{k-1}(1-u)^{n-k}\, du$$

This is quite an important result and should be committed to memory. We shall use it in *'Order Statistics'*.

This result can be stated as follows :

If $X \sim B(n, p)$ and Y has Beta distribution with parameters k and $n - k + 1$ (c.f. Chapter 8), then

$$P(Y \le p) = P(X \ge k) = 1 - P(X \le k - 1)$$

$$\Rightarrow \qquad F_Y(p) = 1 - F_X(k - 1)$$

EXERCISE 7 (a)

1. (*a*) Describe the probability model from which the binomial distribution can be generated. Hence find the first four central moments.

(*b*) If p is the probability of 'success' at a single trial, obtain the probability of r 'successes' out of n independent trails. Determine the mode of the resulting distribution.

2. (*a*) Define the binomial distribution with parameters p and n, and give a situation in real life where the distribution is likely to be realized. Obtain the moment generating function of the binomial distribution and hence or otherwise obtain the mean, variance, skewness and kurtosis of the distribution.

(*b*) Obtain the Moment Generating Function of the Binomial Distribution. Derive from it the result that the sum of two binomial variates is a binomial variate if the variates are independent and have the same probability of success.

3. The mean and variance of a binomial variate X with parameters n and p are 16 and 8. Find

(*i*) $P(X = 0)$, (*ii*) $P(X = 1)$, (*iii*) $P(X \ge 2)$.

4. For a Binomial distribution the mean is 6 and the standard deviation is $\sqrt{2}$. Write out all the terms of the distribution.

Ans. $n = 9,\ p = 2/3,\ q = 1/3;\ p(r) = (1/3)^9 \cdot \binom{9}{r} 2^r;\ r = 0, 1, 2, \ldots, 9$

5. (*a*) A perfect cube is thrown a large number of times in sets of 8. The occurrence of a 2 or 4 is called a success. In what proportion of the sets would you expect 3 successes.

Ans 27·31%

(*b*) In eight throws of a die, 5 or 6 is considered a success. Find the mean number of successes and the standard deviation. (**Ans.** 2·66, 1·33)

(*c*) A man tosses a fair coin 10 times. Find the probability that he will have

(*i*) heads on the first five tosses and tails on the next five tosses

(*ii*) heads on tosses 1, 3, 5, 7, 9 and tails on tosses 2, 4, 6, 8, 10.

(*iii*) 5 heads and 5 tails

(*iv*) at least 5 heads

(*v*) not more than 5 heads. **[Madras Univ.B.Sc.(Main Stat) Nov. 1991]**

Ans. (*i*) $(1/2)^{10}$, (*ii*) $(1/2)^{10}$, (*iii*) ${}^{10}C_5\,(1/2)^{10}$

(*iv*) $\sum_{x=5}^{10} {}^{10}C_x \left(\frac{1}{2}\right)^{10}$ (*v*) $\sum_{x=0}^{5} {}^{10}C_x \left(\frac{1}{2}\right)^{10}$

6. (*a*) In 256 sets of twelve tosses of a fair coin, in how many cases may one expect eight heads and four tails ?

(Ans. 31) **(Delhi Univ. B.Sc. Oct. 1992)**

(*b*) In 100 sets of ten tosses of an unbaised coin, in how many cases should we expect

(*i*) Seven heads and three tails, (*ii*) at least seven heads ?

Ans. (*i*) 12, (*ii*) 17

7. (*a*) During war 1 ship out of 9 was sunk on an average in making a certain voyage. What was the probability that exactly 3 out of a convoy of 6 ships would arrive safely ? **(Madras Univ. B.Sc., 1992)**

Ans. ${}^6C_3\ (8/9)^3\ (1/9)^3$

(*b*) In the long run 3 vessels out of every 100 are sunk. If 10 vessels are out, what is the probability that

(*i*) exactly 6 will arrive safely, and

(*ii*) at least 6 will arrive safely ?

Hint. The probability 'p' that a vessel will arrive safely is

$$P = 97/100 = 0\cdot 97 \text{ and } q = 0\cdot 03$$

The probability that out of 10 vessels, x vessels will arrive safely is

$$p(x) = {}^{10}C_x p^x q^{10-x} = {}^{10}C_x (0\cdot 97)^x (0\cdot 03)^{10-x}$$

(*i*) Required probability $= p(6) = {}^{10}C_6 (0\cdot 97)^6 (0\cdot 03)^x$.

(*ii*) Required probability $= P\ (X \geq 6)$

8. (*a*) A student takes a true-false examination consisting of 10 questions. He is completely unprepared so he plans to guess each answer. The guesses are to be made at random. For example, he may toss a fair coin and use the outcome to determine his guess.

(*i*) Compute the probability that he guesses correctly at least five times.

(*ii*) Compute the probability that he guesses correctly at least 9 times.

(*iii*) What is the smallest n that the probability of guessing at least n correct answers is less than 1/2. **(Dibrugarh Univ. M.A., 1993)**

Ans. (*i*) 319/512 ; (*ii*) 11/1024 ; (*iii*) 6.

(*b*) A multiple choice test consists of 8 questions and 3 answers to each question, of which only one is correct. If a student answers each question by rolling a balanced die and checking the first answer if he gets 1 or 2, the second answer if he gets 3 or 4, and the third answer if he gets 5 or 6, find the probability of getting:

(*i*) exactly 3 correct answers,

(*ii*) no correct answer,

(*iii*) at least 6 correct answers. **[Gauhati Univ. M.A. (Econ.), 1993]**

9. (*a*) The incidence of occupational disease in an industry is such that the workers have a 20% chance of suffering from it. What is the probability that out of six workers chosen at random, four or more will suffer from the disease.

Ans. 52/3125

(*b*). (*a*) In a binomial distribution consisting of 5 independent trails, probabilities of 1 and 2 successes are 0·4096 and 0·2048 respectively. Find the parameter p of the distribution. (**Ans.** 0·2)

10. (*a*) With the usual notations, find p for a binomial random variable X if $n = 6$ and if $9P(X = 4) = P(X = 2)$. (**Ans.** 0·25)

(Mysore Univ. B.Sc. April 1992)

(*b*) X is a random variable following binomial distribution with mean 2·4 and variance 1·44. Find $P(X \geq 5)$, $P(1 < X \leq 4)$.

11. (*a*) In a certain town 20% of the population is literate, and assume that 200 investigators take a sample of ten individuals each to see whether they are literate. How many investigators would you expect to report that three people or less are literates in the sample ? **(Shivaji Univ. B.Sc., Oct. 1992)**

(*b*) A lot contains 1 per cent of defective items. What should be the number (n) of items in a random sample so that the probability of finding at least one defective in it, is at least 0·95 ? (**Ans.** 68)

12. (*a*) If on the average rain falls on ten days in every thirty days, find the probability

(*i*) that rain falls on at least three days of a given week,

(*ii*) that first three days of a given week will be dry and the remaining wet.

Ans. (*i*) $\sum_{x=3}^{7} {}^7C_x (1/3)^x (2/3)^{7-x}$, (*ii*) $(2/3)^3 \cdot (1/3)^4$.

(*b*) Suppose that weather records show that on the average 5 out of 31 days in October are rainy days. Assuming a binomial distribution with each day of October as an independent trial, find the probability that the next October will have at most three rainy days.

Ans. 0·2403

13. The probability of a man hitting a target is 1/4. (*i*) If he fires 7 times, what is the probability p of his hitting the target at least twice ? (*ii*) How many times must he fire so that the probability of his hitting the target at least once is greater than 2/3 ? [**Ans.** (*i*) 4547/8192, (*ii*) 4]

Hint. (*ii*) $p = \frac{1}{4}$, $q = \frac{3}{4}$ · We want n such that

$$1 - q^n > \frac{2}{3} \Rightarrow q^n < \frac{1}{3} \Rightarrow \left(\frac{3}{4}\right)^n < \frac{1}{3} \Rightarrow n = 4$$

14. (*a*) The probability of a man hitting a target is 1/3. How many times must be fire so that the probability of hitting the target at least once is more than 90%. **Ans.** 6. **(Shivaji Univ. B.Sc., 1991)**

(*b*) Eight mice are selected at random and they are divided into two groups of 4 each. Each mouse in group A is given a dose of certain poison '*a*' which is expected to kill one in four; each mouse in group B is given a dose of certain poison '*b*' which is expected to kill one or two. Show that nevertheless, there may be fewer deaths in group A and find the probability of this happening.

Ans. 525/4096

15 (*a*) A card is drawn and replaced in an ordinary deck of 52 cards. How many times must a card be drawn so that (*i*) there is at least an even chance of drawing a heart, (*ii*) the probability of drawing a heart is greater than 3/4 ?

Ans. (*i*) 3, (*ii*) 5

(*b*) Five coins are tossed. What is the variance of the number of heads per toss of the five coins:

(*i*) if each coins is unbiased,

(*ii*) if the probability of a head appearing is 0·75 for each coin, and

(*iii*) if four coins are unbaised and for the fifth the probability of a head appearing is 0·75 ?

Hint (*iii*) Use generating function. [See Ex. 7·17 (*iii*)]

16. An owner of a small hotel with five rooms is considering buying television sets to rent to room occupants. He expects that about half of his customers would be willing to rent sets, and finally he buys three sets. Assuming 100% occupancy at all times :

(*i*) What fraction of the evenings will there be more request than T.V. sets?

(*ii*) What is the probability that a customer who requests a television set will receive one?

(*iii*) If the owner's cost per set per day is C, what rent R must he charge in order to break even (neither gain nor lose) in the long run ?

Hint. (*i*) Let the random variable X denote the daily number of requests. Then required probability is

$$P(X \geq 4) = P(X = 4) + P(X = 5) = \binom{5}{4}\left(\frac{1}{2}\right)^5 + \binom{5}{5}\left(\frac{1}{2}\right)^5$$

(*ii*) The customer can get a T.V. in the following mutually exclusive ways,

(*a*) There are no other requests that night.

(*b*) There is one other request.

(*c*) There are two other requests.

(*d*) There are three other requests and his request precedes at least one of them.

(*e*) There are four other requests, and his request preceedes at least two of them.

The probability of the desired event

$$= (0\cdot 5)^4 \left\{1 + {}^4C_1 + {}^4C_2 + \frac{3}{4}\cdot {}^4C_3 + \frac{3}{5}\,{}^4C_4\right\}$$

(*iii*) Mean revenue

$$= (0\cdot5)^5 \cdot 0 + {}^5C_1\,(0\cdot5)^5 R + {}^5C_2\,(0\cdot5)^5\, 2R + \left\{{}^5C_3\,(0\cdot5)^5 + {}^5C_4\,(0\cdot5)^5 + {}^5C_5\,(0\cdot5)^5\right\} 3R$$

$$= \frac{73}{32} R$$

The break-even rental is the value of R for which

$$\frac{73}{32} R = 3C \quad\Rightarrow\quad R = 1\cdot 315\, C$$

17. A manufacturer claims that at most 10 per cent of his product is defective. To test this claim, 18 units are inspected and his claim is accepted if among these 18 units, at most 2 are defective. Find the probability that the manufacturer's claim will be accepted if the actual probability that a unit is defective is

(*a*) 0·05 (*b*) 0·10 (*c*) 0·15 and (*d*) 0·20.

Ans. (*a*) 0·9410 (*b*) 0·9326 (*c*) 0·4445 (*d*) 0·2715

18. (*a*) A set of 8 symmetrical coins was tossed 256 times and the frequencies of throws observed were as follows :

Number of heads :	0	1	2	3	4	5	6	7	8
Frequency of throws:	2	6	24	63	64	50	36	10	1

Fit a binomial distribution and find mean and standard deviation of fitted distribution.

(*b*) A set of 6 similar coins is tossed 640 times with the following results:

Number of heads :	0	1	2	3	4	5	6
Frequency :	7	64	140	210	132	75	12

Calculate the binomial frequencies on the assumption that the coins are symmetrical.

19. (*a*) The following data due to Weldon shows the results of throwing 12 dice 4096 times, a throw of 4, 5 or 6 being called a success (x).

x :	0	1	2	3	4	5	6	7	8	9	10	11	12	Total
f :	—	7	60	198	430	731	948	847	536	257	71	11	—	4096

Fit the binomial distribution and calculate the expected frequencies. Compare the actual mean and S.D. with those of the expected ones for the distribution.

Ans. Expected freq. : 1, 12, 66, 220 495 792, 924, 792, 495, 220, 66, 12, 0; mean = 6, variance = 1·71.

(*b*) In 103 litters of 4 mice, the number of litters which contained 0, 1, 2, 3, 4 females are recorded below :

Number of female mice	0	1	2	3	4	Total
Number of litters	8	32	34	24	5	103

(*i*) If the chance of obtaining a female in a single trial is assumed constant, estimate the constant but unknown probability.

(*ii*) If the size of the litter 4 had not been given, how could it be estimated from the data ?

20. X is random variable distributed according to the Binomial law :

$$b(x\,;n,p) = \binom{n}{x} p^x q^{n-x}\,;\; x = 0, 1, 2,..., n$$

Obtain the recurrence formula :

$$b(x+1\,;n,p) = \frac{n-x}{x+1}\cdot\frac{p}{q}\cdot b(x\,;n,p)$$

Use this as a reduction formula and get the theoretical frequencies when an unbaised coin is tossed 8 times and the experiment is repeated 256 times.

(Madras Univ. B. Sc. April 1992)

21. (*a*) By differentiating the following identity with respect to p and then multiplying by p,

$$\sum_{x=0}^{n} \binom{n}{x} p^x q^{n-x} = (q+p)^n, q = 1-p$$

prove that $\mu_1' = np$ and $\mu_2 = npq$.

22. (*a*) Let $X \sim b(x; n, p)$ and r be a non-negative integer. If the rth moment about the origin is denoted by $\mu_r' = E(X^r)$, prove that

$$\mu_{r+1}' = np\,\mu_r' + p(1-p)\frac{d\mu_r'}{dp}$$

[Delhi Univ. B.Sc. (Hons. Subs.), 1993, '88]

(*b*) Show that for the binomial distribution $B(n, p)$,

$$\mu_{r+1} = pq\left(nr\,\mu_{r-1} + \frac{d}{dp}\mu_r\right), \quad p + q = 1,$$

where symbols have their usual meanings.

[Delhi Univ. B.Sc. (Stat. Hons), 1989]

(c) If $X \sim B(n, p)$, obtain the recurrence relation for its central moments and hence find values of β_1 and β_2.

[Calcutta Univ. B.Sc. (Hons.), 1992]

23. (*a*) The following results were obtained when 100 batches of seeds were allowed to germinate on damp filter paper in a laboratory :

$$\beta_1 = \frac{1}{15} \text{ and } \beta_2 = \frac{89}{30}$$

Determine the binomial distribution and calculate the frequency for $X = 8$, considering $p > q$.

Hint. We have $\beta_1 = \dfrac{(q-p)^2}{npq} = \dfrac{1}{15}$ ...(*i*)

and $\beta_2 = 3 + \dfrac{1-6pq}{npq} = \dfrac{89}{30}$ ...(ii)

From (*i*) and (*ii*), we can find the value of n, p and q

(*b*) Between *a* Binomial distribution with $n = 5$ and $p = \frac{1}{2}$ and a distribution with frequency function

$$f(x) = 6x(1-x),\ 0 \le x \le 1;$$

determine which is more skewed.

24. (*a*) $x = r$ is the unique mode of Binomial Distribution having mean np and variance $np(1-p)$. Show that

$$(n+1)p - 1 < r < (n+1)p$$

Find the mode of the binomial distribution with $p = \frac{1}{2}$ and $n = 7$.

[Delhi Univ. B.Sc. (Stat. Hons.) 1991, '84]

Ans. 4, 3 (Bimodal).

(*b*) Show that if np be a whole number, the mean of the binomial distribution coincides with the greatest term.

(c) Compute the mode of a binomial distribution $b\,(7, \frac{1}{2})$.

[Delhi Univ. B.Sc. (Maths. Hons.), 1989]

Ans. 1, 2 (Bimodal).

(d) Define Bernoulli trials and state the binomial law of probability. Find the bounds for the most probable number of successes in a sequence of n Bernoulli trials.

One workers can manufacture 120 articles during a shift, another worker 140 articles, the probabilities of the articles being of a high quality are 0·94 and 0·80 respectively. Determine the most probable number of high quality articles manufactured by each worker. **[Calcutta Univ. B.Sc. (Maths. Hons.), 1988]**

25. Show that if two symmetrical binomial distributions $(p = q = \frac{1}{2})$ of degree n (and of the same number of observations) are so superimposed that the rth term of one coincides with the $(r + 1)$th term of the other, the distribution formed by adding superimposed terms is a symmetrical binomial of degree $(n + 1)$. **[Bhagalpur Univ. B.Sc., 1993]**

26. (a) Let X denote a binomially distributed random variable. Show that

$$E\left(\frac{X - np}{\sqrt{npq}}\right) = 0,\ E\left(\frac{X - np}{\sqrt{npq}}\right)^2 = 1, \text{ and}$$

$$E\left[\exp.\left\{t\left(\frac{X - np}{\sqrt{npq}}\right)\right\}\right] = \left[(1 - p)\exp\left\{-t\sqrt{\left(\frac{p}{nq}\right)}\right\} + p\exp\left\{t\sqrt{\left(\frac{q}{np}\right)}\right\}\right]$$

(b) Obtain the characteristic function of the standard binomial variate $(X - np)/\sqrt{npq}$, where X is the number of successes obtained in n independent trials, each with constant probability p of success, $q = 1 - p$. Obtain the limit of this function as $n \to \infty$. **[Delhi Univ. B.Sc.(Maths. Hons.), 1991]**

(c) If $X \sim B\,(n, p)$, prove that

$$\kappa_{r+1} = pq \cdot \frac{d}{dp}(\kappa_r),$$

where κ_r is the rth cumulant.

Hence deduce the values of κ_2 and κ_3.

[Delhi Univ. B.Sc. (Stat. Hons.), 1991, '87]

27. (a) If X and Y are two independent identically distributed binomial variates, obtain the probability that the absolute difference $|X - Y|$ equals a given value, say r.

(b) (i) If X and Y are independent binomial variates, with parameters p_1 and p_2 and indices n_1 and n_2 respectively, obtain the probability that $X + Y$ equals 'r'.

(ii) In the above if $p_1 = p_2$, what is the distribution of $X + Y$?

[Poona Univ. B.Sc., 1988]

(c) If X and Y are two independent binomial variates with parameters $n_1 = 6$, $p = 1/2$ and $n_2 = 4$, $p = 1/2$ respectively, evaluate,

(i) $P(X + Y = r)$, (ii) $P(X + Y \geq 3)$

(Gujarat Univ. B. Sc. Oct. 1992)

Hint $X + Y \sim B(6 + 4, 1/2) = B(10, 1/2)$

Ans. (i) $P(X + Y = r) = p(r) = {}^{10}C_r\,(1/2)^r$; $r = 0, 1, ..., 10$

(ii) $P(X + Y \geq 3) = 1 - [p(0) + p(1) + p(2)] = 0{\cdot}945$

(d) If X and Y are two independent binomial variates with parameters $(n_1 = 3, p = 0{\cdot}4)$ and $(n_2 = 4, p = 0{\cdot}4)$ respectively, find :

(i) $P(X = Y)$, (ii) $P(X + Y \leq 2)$, (iii) $P(X = 3 \mid X + Y = 4)$

Hint. $X + Y \sim B(3 + 4, 0{\cdot}4) = B(7, 0{\cdot}4)$

(i) $P(X = Y) = \sum_{r=0}^{3} P(X = r \cap Y = r) = \sum_{r=0}^{3} P(X = r)\,P(Y = r) = 0{\cdot}2871$

(ii) $P(X + Y \leq 2) = \sum_{r=0}^{2} \binom{7}{r} (0{\cdot}4)^r (0{\cdot}6)^{7-r} = 0{\cdot}420$

(iii) $P(X = 3 \mid X + Y = 4) = \dfrac{P(X = 3 \cap X + Y = 4)}{P(X + Y = 4)} = \dfrac{P(X = 3 \cap Y = 1)}{P(X + Y = 4)} = 0{\cdot}1141$

28. (a) Obtain the moment generating function of Binomial distribution with $n = 7$ and $p = 0{\cdot}6$. Find the first three moments of the distribution.

[Poona Univ. B. Sc. 1992)

Ans. $(0{\cdot}4 + 0{\cdot}6\,e^t)^7$: mean $= 4{\cdot}2$, $\mu_2 = 1{\cdot}68$, $\mu_3 = -0{\cdot}336$.

(b) Suppose that the m.g.f. of a random variable X is of the form

$$M_X(t) = (0{\cdot}4\,e^t + 0{\cdot}6)^8$$

What is the m.g.f. of the random variable $Y = 3X + 2$? Evaluate $E(X)$.

Ans. $E(X) = 3{\cdot}2$, $M_Y(t) = e^{2t}\,(0{\cdot}6 + 0{\cdot}4\,e^{3t})^8$

(c) Obtain the moment generating function of the binomial distribution. Hence or otherwise obtain the mean, variance and skewness of the distribution.

29. Show that the factorial moment generating function $w(t)$ of the binomial distribution $b(x ; n, p)$ is $(1 + pt)^n$. Hence or otherwise show that

$$\mu_{(r)}' = n^{(r)}\,p^r$$

Hint. Factorial moment generating function $\omega(t)$ is defined as

$$\omega(t) = E(1 + t)^X = \sum_x (1 + t)^x p(x) = \sum_x {}^nC_x \{p(1 + t)\}^x q^{n-x}$$

$$\mu_{(r)}' = \text{co efficient of } \frac{t^r}{r!} \text{ in } \omega(t) = {}^nC_r\, r!\, p^r = n^{(r)}\,p^r$$

30. Show that

(i) $$b(n, p ; k) = b(n, 1 - p ; n - k)$$

(ii) $$\sum_{k=r}^{n} b(n, p ; k) = 1 - \sum_{k=n-r+1}^{n} b(n, 1 - p ; k)$$

(iii) $$b(n + 1, p ; k) = p\,.\,b(n, p ; k - 1) + q\,.\,b(n, p ; k)$$

Hint. (*i*) $b(n, 1-p; n-k) = \binom{n}{n-k} (1-p)^{n-k} p^{n-(n-k)}$

(*ii*) $\sum_{k=r}^{n} b(n, p; k) = \sum_{k=r}^{n} b(n, 1-p; n-k) = \sum_{k=0}^{n-r} b(n, 1-p; k)$

31. For a binomial distribution, let

$$F_n(y) = \sum_{x=0}^{y} \binom{n}{x} p^x q^{n-x},$$

where $q = 1-p$,

prove that

(*i*) $F_{n+1}(y) = p\, F_n(y-1) + q\, F_n(y)$

(*ii*) $\text{Cov}(X, n-X) = -npq$ **(Bombay Univ. B.Sc., April 1990)**

32. (*a*) Random variable X follows binomial distribution with parameters $n = 40$ and $p = \frac{1}{4}$. Use Chebychev's inequality to find bounds for

(*i*) $P[\,|X-10| < 8]$; (*ii*) $P[\,|X-10| > 10]$

Compare these values with the actual values (**Hint** : Use Normal approximation for the Binomial). **(Madras Univ. B.Sc. (Main Stat.), 1988)**

Ans. (*i*) 113/128 (lower bound), (*ii*) 0.075 (upper bound).

(*b*) X follows binomial distribution with $n = 40$, $p = \frac{1}{2}$. Use Chebychev's lemma to

(*i*) find k such that

$P\{\,|X-20| > 10k\} \le 0{\cdot}25$, and

(*ii*) obtain a lower limit for $P\{|X-20| \le 5\}$.

[Delhi Univ. B.Sc. (Maths. Hons.), 1984]

Ans. (*i*) $2\sqrt{10}$, (*ii*) 3/5

(*c*) How many trials must be made of an event with binomial probability of success $\frac{1}{2}$ in each trial, in order to be assured with probability of at least 0·9 that the relative frequency of success will be between 0·48 and 0·52 ? **(Ans. 6250)**

Hint. Use Chebychev's Inequality.

33. (*a*) Show that if a coin is tossed n times, the probability of not more than k heads is :

$$\left[\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{k}\right] \left(\frac{1}{2}\right)^n$$

[South Gujarat Univ. B.Sc., 1988]

(*b*) If X has binomial distribution with parametes n and p, then prove that

$P[X \text{ is even}] = \frac{1}{2}[1 + (q-p)^n]$. **[Delhi Univ. B.Sc. (Stat. Hons.), 1988]**

34. If the probability of hitting a target is 1/5 and if 10 shots are fired, what is the conditional probability of the target being hit at least twice assuming that at least one hit is already scored ?

[Nagpur Univ. B.Sc., 1988, '93]

Hint. Let X donote the number of times a target is hit when 10 shots are fired. Then $X \sim B(10, 0\cdot2)$. The required probability is :

$$P(X \geq 2 \mid X \geq 1) = \frac{P[(X \geq 2) \cap (X \geq 1)]}{P(X \geq 1)} = \frac{P(X \geq 2)}{P(X \geq 1)}$$

$$= \frac{1 - [P(X = 0) + P(X = 1)]}{1 - [P(X = 0)]} = \frac{0\cdot625}{0\cdot893} = 0\cdot6999$$

35. (*a*) Let X be a $B(2, p)$ and Y be a $B(4, p)$. If $P(X \geq 1) = 5/9$, find $P(Y \geq 1)$ **[Kerala Univ. B. Sc., 1989]**

Hint. $P(X \geq 1) = 1 - P(X = 0) = 1 - q^2 = 5/9 \Rightarrow q = 2/3, p = 1/3.$

$P(Y \geq 1) = 1 - P(Y = 0) = 1 - q^4 = 65/81.$

36. Let B denote the number of boys in a family with five children. If p denotes the probability that a boy is there in a family, find the least value of p such that

$P(B = 0) > P(B = 1)$ **(Shivaji Univ. B. Sc., 1990)**

Ans. $q^5 > 5pq^4 \Rightarrow q > 5p \Rightarrow p < \frac{1}{6}.$

37. (*a*) Suppose $X \sim B(n, p)$. with $E(X) = 5$, $\text{Var}(X) = 4$. Find n and p. (**Ans.** $n = 25$, $p = 1/5$)

(*b*) Let $X \sim B(n, p)$. For what p is variance (X) maximised if we assume n is fixed.

Ans. $\text{Var}\, X = npq = n(p - p^2) = f(p)$, (say) ; $f'(p) = 0$, $f''(p) < 0$; $p = 1/2 = q$

38. (*a*) $X \sim B(n = 100, p = 0\cdot1)$. Find $P(X \leq \mu_x - 3\sigma_x)$

Ans. $\mu = 10$, $\sigma = 3$, $P(X \leq \mu_x - 3\sigma_x) = P(X \leq 1) = 10{\cdot}9 \times (0{\cdot}9)^{99}$

(*b*) If $X \sim B(25, 0\cdot2)$, find $P(X < \mu_x - 2\sigma_x)$

[Delhi Univ. B.A. (Stat. Hons.) Spl. Course 1989]

39. For one half of n events, the chance of success is p, and the chance of failure is q, whilst for the other half the chance of success is q, and the chance of failure is p. Show that the S.D. of the number of successes is the same as if the chance of success were p in all the cases i.e. $\sqrt{npq}$, but that the mean of the number of successes is $n/2$ and not np. **(Delhi Univ. B.A. 1992)**

Hint. $X \sim B(n/2, p)$ and $Y \sim B(n/2, q)$ are independent. Let $Z = X + Y$. Now prove that $\text{Var}(Z) = npq$ and $E(Z) = n/2$.

40. The discrete density of X is given by $f_X(x) = x/3$, for $x = 1, 2$ and $f_{Y|X}(y \mid x)$ is binomial with parameters x and $\frac{1}{2}$ i.e.,

$$F_{Y|X}(y \mid x) = P(Y = y \mid X = x) = \binom{x}{y} \cdot \left(\tfrac{1}{2}\right)^x;$$

for $y = 0, 1, ..., x$ and $x = 1, 2$.

(*a*) Find $E(X)$ and $\text{Var}(X)$; (*b*) Find $E(Y)$

(c) Find the joint distribution of X and Y.

Hint. Proceed as in Example 7·21.

Ans. (*a*) $E(X) = 5/3$, $\text{Var}(X) = 2/9$, (*b*) $E(Y) = 5/6$.

(*c*) $f(x, y) = \binom{x}{y} \cdot \left(\frac{x}{3}\right) \cdot \left(\frac{1}{2}\right)^x$; $n = 1, 2, ;$ $y = 0, 1, ..., x.$

41. Two dice are thrown n times. Let X denote the number of throws in which the number on the first dice exceeds the number on the second dice. What is the distribution of X ?

Ans. $X \sim B(n, p = 15/36)$

Hint. p is the probability that the number on the first dice exceeds the number on the second dice in a throw of two dice.

42. Let $X_1 \sim B(n, p_1)$ and $X_2 \sim B(n, p_2)$.

If $p_1 < p_2$, prove that :

$$P(X_1 \le k) \ge P(X_2 \le k) \text{ for } k = 0, 1, ..., n.$$

Hint. Use Example 7·23.

43. If $X \sim B(n, p)$, show that

$$P(X \le k) = \lambda \int_{p/q}^{\infty} \frac{y^k}{(1 + y)^{n+1}} \, dy$$

where $$\lambda^{-1} = \int_0^{\infty} \frac{y^k}{(1 + y)^{n+1}} \, dy = \beta(k + 1, n - k)$$

Hint. $\frac{d}{dq} P(X \le k) = n \binom{n-1}{k} \cdot p^k \cdot q^{n-k-1} = A_k$, (*say*)

[See Example 7·23]

Find $$\frac{d}{dq}(\text{RHS}) = \lambda \cdot \frac{d}{dq}\left(\int_{p/q}^{\infty} \frac{y^k}{(1 + y)^{n+1}} \, dy\right) = \lambda \frac{(p/q)^k}{[1 + (p/q)]^{n+1}} \left(\frac{1}{q^2}\right)$$

$$= \frac{1}{\beta(k + 1, n - k)} \cdot p^k \cdot q^{n-k-1} = A_k$$

(On simplification)

44. If $X \quad B(n, p)$ and Y has beta distribution with parameters k and $n - k + 1$, (See Chapter 8), then prove that

$P(Y \le p) = P(X \ge k)$ *i.e.*, $F_Y(p) = 1 - F_X(k - 1)$

45. If a fair coin is tossed an even number $2n$ times, show that the probability of obtaining more heads than tails is

$$\frac{1}{2}\left\{1 - {}^{2n}C_n \left(\frac{1}{2}\right)^{2n}\right\}$$

Hint. X : No. of heads; Y = No. of tails; No. of trials = $2n$

$P(X > Y) + P(X < Y) + P(X = Y) = 1$

$\Rightarrow \quad 2P(X > Y) = 1 - P(X = Y)$

[$\because$ By symmetry, $p = q = \frac{1}{2} \Rightarrow P(X > Y) = P(X < Y)$]

$$= 1 - {}^{2n}C_n\, p^n \cdot q^n = 1 - {}^{2n}C_n\, (\tfrac{1}{2})^{2n}$$

$$\Rightarrow \qquad P(X > Y) = \frac{1}{2}\left[1 - {}^{2n}C_n\, (\tfrac{1}{2})^{2n}\right]$$

7·3·0. Poisson Distribution (as a limiting case of Binomial Distribution). Poisson distribution was discovered by the French mathematician and physicist Simeon Denis Poisson (1781—1840) who published it in 1837. Poisson distribution is a limiting case of the binomial distribution under the following conditions:

(*i*) n, the number of trials is indefinitely large, *i.e.*, $n \to \infty$.

(*ii*) p, the constant probability of success for each trial is indefinitely small, *i.e.*, $p \to 0$.

(*iii*) $np = \lambda$, (say), is finite. Thus $p = \lambda/n$, $q = 1 - \lambda/n$, where λ is a positive real number.

The probability of x successes in a series of n independent trials is

$$b(x\,; n, p) = \binom{n}{x} p^x q^{n-x}\,; x = 0, 1, 2, ..., n \qquad ...(*)$$

We want the limiting form of (*) under the above conditions. Hence

$$\lim_{n \to \infty} b(x\,; n, p) = \lim_{n \to \infty} \frac{n\,!}{x\,!\,(n-x)\,!}\left(\frac{\lambda}{n}\right)^x \cdot \left[1 - \frac{\lambda}{n}\right]^{n-x}$$

Using Stirling's approximation for $n\,!$ as $n \to \infty$ *viz.*,

$$\lim_{n \to \infty} n\,! \approx \sqrt{2\pi}\, e^{-n}\, n^{n+(1/2)}, \text{ we get}$$

$$\lim_{n \to \infty} b(x; n, p) = \lim_{n \to \infty}\left[\frac{\sqrt{2\pi}\, e^{-n} \cdot n^{n+(1/2)}}{x\,!\sqrt{2\pi}\, e^{-(n-x)} \cdot (n-x)^{n-x+(1/2)}}\right]\left(\frac{\lambda}{n}\right)^x\left[1 - \frac{\lambda}{n}\right]^{n-x}$$

$$= \frac{\lambda^x}{e^x \cdot x\,!} \cdot \lim_{n \to \infty} \frac{n^{n-x+(1/2)}}{(n-x)^{n-x+(1/2)}} \cdot \left[1 - \frac{\lambda}{n}\right]^{n-x}$$

$$= \frac{\lambda^x}{e^x\, x\,!} \lim_{n \to \infty} \frac{\left(1 - \dfrac{\lambda}{n}\right)^{n-x}}{\left(1 - \dfrac{x}{n}\right)^{n-x+(1/2)}}$$

$$= \frac{\lambda^x}{e^x\, x\,!} \cdot \frac{\lim\limits_{n \to \infty}\left[1 - \dfrac{\lambda}{n}\right]^n \cdot \lim\limits_{n \to \infty}\left(1 - \dfrac{\lambda}{n}\right)^{-x}}{\lim\limits_{n \to \infty}\left[1 - \dfrac{x}{n}\right]^n \lim\limits_{n \to \infty}\left[1 - \dfrac{x}{n}\right]^{-x+(1/2)}}$$

But we know that

$$\left.\begin{array}{l}\lim\limits_{n \to \infty} \left(1 - \dfrac{\lambda}{n}\right)^n = e^{-\lambda}, \\ \\ \lim\limits_{n \to \infty} \left[1 - \dfrac{\lambda}{n}\right]^{\alpha} = 1, \ \alpha \text{ is not a function of } n\end{array}\right\} \quad ...(**)$$

and

Therefore

$$\lim_{n \to \infty} b(x; n, p) = \frac{\lambda^x}{e^x \cdot x!} \cdot \frac{e^{-\lambda} \cdot 1}{e^{-x} \cdot 1} = \frac{e^{-\lambda} \cdot \lambda^x}{x!}; x = 0, 1, 2, ..., \infty;$$

[Using (**)]

which is the required probability function of the Poisson distribution. 'λ' is known as the parameter of Poisson distribution.

Aliter. Poisson distribution can also be derived without using Stirling's approximation as follows :

$$b(x; n, p) = \binom{n}{x} p^x (1-p)^{n-x} = \binom{n}{x} \left[\frac{p}{1-p}\right]^x (1-p)^n$$

$$= \frac{n(n-1)(n-2)\ldots(n-x+1)}{x!} \cdot \frac{\left(\dfrac{\lambda}{n}\right)^x}{\left[1 - \dfrac{\lambda}{n}\right]^x} \left[1 - \frac{\lambda}{n}\right]^n$$

$$= \frac{\left[1 - \dfrac{1}{n}\right]\left[1 - \dfrac{2}{n}\right]\ldots\left[1 - \dfrac{x-1}{n}\right]}{x!\left[1 - \dfrac{\lambda}{n}\right]^x} \lambda^x \left[1 - \frac{\lambda}{n}\right]^n$$

$$\therefore \quad \lim_{n \to \infty} b(x; n, p) = \frac{e^{-\lambda}\lambda^x}{x!}; \ x = 0, 1, 2, \ldots \qquad \text{[From (**)]}$$

Definition. *A random variable X is said to follow a Poisson distribution if it assumes only non-negative values and its probability mass function is given by*

$$p(x, \lambda) = P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!}; \ x = 0, 1, 2, \ldots; \lambda > 0$$
$$= 0, \text{ otherwise} \qquad ...(7 \cdot 14)$$

Here λ is known as the parameter of the distribution.

We shall use the notation $X \sim P(\lambda)$ to denote that X is a Poisson variate with paramter λ.

Remarks 1. It should be noted that

$$\sum_{x=0}^{\infty} P(X = x) = e^{-\lambda} \sum_{x=0}^{\infty} \lambda^x / x\,! = e^{-\lambda} e^{\lambda} = 1$$

2. The corresponding distribution function is:

$$F(x) = P(X \leq x) = \sum_{r=0}^{x} p(r) = e^{-\lambda} \sum_{r=0}^{x} \lambda^r / r\,!\,;\, x = 0, 1, 2, \ldots$$

3. Poisson distribution occurs when there are events which do not occur as outcomes of a definite number of trials (unlike that in binomial) of an experiment but which occur at random points of time and space wherein our interest lies only in the number of occurrences of the event, not in its non-occurrences.

4. Following are some instances where Poisson distribution may be successfully employed·

(1) Number of deaths from a disease (not in the form of an epidemic) such as heart attack or cancer or due to snake bite.

(2) Number of suicides reported in a particular city.

(3) The number of defective material in a packing manufactured by a good concern.

(4) Number of faulty blades in a packet of 100.

(5) Number of air accidents in some unit of time.

(6) Number of printing mistakes at each page of the book.

(7) Number of telephone calls received at a particular telephone exchange in some unit of time or connections to wrong numbers in a telephone exchange.

(8) Number of cars passing a crossing per minute during the busy hours of a day.

(9) The number of fragments received by a surface area 't' from a fragment atom bomb.

(10) The emission of radioactive (alpha) particles.

7·3·1. The Poisson Process. The Poisson distribution may also be obtained independently (*i.e.,* without considering it as a limiting form of the Binomial distribution) as follows :

Let X_t be the number of telephone calls received in time interval 't' on a telephone switch board. Consider the following experimental conditions :

(1) The probability of getting a call in small time interval $(t, t + dt)$ is $\lambda\, dt$, where λ is a positive constant and dt denotes a small increment in time 't'.

(2) The probability of getting more than one call in this time interval is very small, *i.e.,* is of the order of $(dt)^2$ *i.e.,* $0\,[(dt)^2]$ such that

$$\lim_{dt \to 0} \frac{0\,(dt)^2}{dt} = 0$$

(3) The probability of any particular call in the time interval $(t, t + dt)$ is independent of the actual time t and also of all previous calls.

Under these conditions it can be shown that the probability of getting x calls in time 't', say, $P_x(t)$ is given by

$$P_x(t) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}; \; x = 0, 1, 2, ..., \infty$$

which is a Poisson distribution with parameter λt.

Proof: Let $P_x(t) = P$ {of getting x calls in a time interval of length 't'}. Also P {of at least one call during $(t, t + dt)$} $= \lambda\, dt + 0\,[(dt)^2]$

and P {of more than one call during $(t, t + dt)$} $= 0\,[(dt)^2]$.

The event of getting exactly x calls in time $t + dt$ can materialise in the following two mutually exclusive ways :

(*i*) x calls in $(0, t)$ and none during $(t, t + dt)$ and the probability of this event is $P_x(t)\left\{1 - [(\lambda\, dt + 0\,(dt)^2]\right\}$,

(*ii*) exactly $(x - 1)$ calls during $(0, t)$ and one call in $(t, t + dt)$ and the probability of this event is $P_{x-1}(t)\cdot(\lambda\, dt)$.

Hence by the addition theorem of probability, we get

$$P_x(t + dt) = P_x(t)\left\{1 - \lambda\, dt - 0\,(dt)^2\right\} + P_{x-1}(t)\,\lambda\, dt.$$

$$= P_x(t)(1 - \lambda\, dt) + P_{x-1}(t)\,\lambda\, dt + 0\,(dt)^2 P_x(t) \quad ...(1)$$

$$\Rightarrow \frac{P_x(t + dt) - P_x(t)}{dt} = -\lambda P_x(t) + \lambda P_{x-1}(t) + \frac{0\,(dt)^2}{dt} P_x(t)$$

Proceeding to the limit as $dt \to 0$, we get

$$\lim_{dt \to 0} \frac{P_x(t + dt) - P_x(t)}{dt} = -\lambda P_x(t) + \lambda P_{x-1}(t)$$

$$\therefore \qquad P_x'(t) = -\lambda P_x(t) + \lambda P_{x-1}(t), x \geq 1 \quad ...(2)$$

where ($'$) denotes differentiation w.r. to 't'.

For $x = 0, P_{x-1}(t) = P_{-1}(t) = P\left\{(-1) \text{ calls in time 't'}\right\} = 0$

Hence from (1), we get

$$P_0(t + dt) = P_0(t)\left\{1 - \lambda\, dt\right\} + 0\,(dt)^2$$

which on taking the limit $dt \to 0$, gives

$$P_0'(t) = -\lambda P_0(t) \Rightarrow \frac{P_0'(t)}{P_0(t)} = -\lambda$$

Integrating w.r. to. 't', we get

$$\log P_0(t) = -\lambda t + C,$$

where C is an arbitrary constant to be determined from the condition

$$P_0(0) = 1$$

Hence $\qquad \log 1 = C \Rightarrow C = 0$

$$\therefore \qquad \log P_0(t) = -\lambda t \Rightarrow P_0(t) = e^{-\lambda t}$$

Substituting this value of $P_0(t)$ in (2), we get, with $x = 1$

$$P_1'(t) = -\lambda P_1(t) + \lambda e^{-\lambda t}$$

$$\Rightarrow \qquad P_1'(t) + \lambda P_1(t) = \lambda e^{-\lambda t}$$

This is an ordinary linear differential equation whose integrating factor is $e^{\lambda t}$. Hence its solution is

$$e^{\lambda t} P_1(t) = \lambda \int e^{\lambda t} e^{-\lambda t} dt + C_1 = \lambda t + C_1,$$

where C_1 is an arbitrary constant to be determined from $P_1(0) = 0$, which gives $C_1 = 0$.

$$\therefore \qquad P_1(t) = e^{-\lambda t} \lambda t$$

Again substituting this in (2) with $x = 2$, we get

$$P_2'(t) + \lambda P_2(t) = \lambda e^{-\lambda t} \lambda t$$

Integrating factor of this equation is $e^{\lambda t}$ and its solution is

$$P_2(t) e^{\lambda t} = \lambda^2 \int t e^{-\lambda t} e^{\lambda t} dt + C_2 = \frac{\lambda^2 t^2}{2} + C_2$$

where C_2 is an arbitrary constant to be determined from $P_2(0) = 0$, which gives $C_2 = 0$. Hence

$$P_2(t) = e^{-\lambda t} \frac{(\lambda t)^2}{2}$$

Proceeding similarly step by step, we shall get

$$P_x(t) = \frac{e^{-\lambda t} (\lambda t)^x}{x!}; \; x = 0, 1, 2, ..., \infty.$$

7·3·2. Moments of the Poisson Distribution

$$\mu_1' = E(X) = \sum_{x=0}^{\infty} x\, p(x, \lambda)$$

$$= \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda} \lambda^x}{x!} = \lambda e^{-\lambda} \left[\sum_{x=1}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} \right]$$

$$= \lambda e^{-\lambda} \left(1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \dots \right) = \lambda e^{-\lambda} \cdot e^{\lambda} = \lambda$$

Hence the mean of the Poisson distribution is λ.

$$\mu_2' = E(X^2) = \sum_{x=0}^{\infty} x^2 p(x, \lambda) = \sum_{x=0}^{\infty} \{x(x-1) + x\} \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} x(x-1) \frac{\lambda^x}{x!} + \sum_{x=0}^{\infty} x \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= \lambda^2 e^{-\lambda} \left[\sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} \right] + \lambda = \lambda^2 e^{-\lambda} e^{\lambda} + \lambda = \lambda^2 + \lambda$$

$$\mu_3' = E(X^3) = \sum_{x=0}^{\infty} x^3 p(x, \lambda)$$

$$= \sum_{x=0}^{\infty} \{x(x-1)(x-2) + 3x(x-1) + x\} \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= \sum_{x=0}^{\infty} x(x-1)(x-2) \frac{e^{-\lambda} \lambda^x}{x!} + 3 \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda} \lambda^x}{x!} + \sum_{x=0}^{\infty} x \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= e^{-\lambda}\lambda^3\left[\sum_{x=3}^{\infty}\frac{\lambda^{x-3}}{(x-3)!}\right] + 3e^{-\lambda}\lambda^2\left[\sum_{x=2}^{\infty}\frac{\lambda^{x-2}}{(x-2)!}\right] + \lambda$$

$$= e^{-\lambda}\lambda^3 e^{\lambda} + 3e^{-\lambda}\lambda^2 e^{\lambda} + \lambda = \lambda^3 + 3\lambda^2 + \lambda$$

$$\mu_4' = E(X^4) = \sum_{x=0}^{\infty} x^4 \cdot p(x;\lambda)$$

$$= \sum_{x=0}^{\infty}\{x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + 7x(x-1) + x\}\frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda}\lambda^4\left[\sum_{x=4}^{\infty}\frac{\lambda^{x-4}}{(x-4)!}\right] + 6e^{-\lambda}\lambda^3\left[\sum_{x=3}^{\infty}\frac{\lambda^{x-3}}{(x-3)!}\right]$$

$$+ 7e^{-\lambda}\lambda^2\left[\sum_{x=2}^{\infty}\left(\frac{\lambda^{x-2}}{(x-2)!}\right)\right] + \lambda$$

$$= \lambda^4(e^{-\lambda}e^{+\lambda}) + 6\lambda^3(e^{-\lambda}e^{+\lambda}) + 7\lambda^2(e^{-\lambda}e^{+\lambda}) + \lambda = \lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda$$

The four central moments are now obtained as follows :

$$\mu_2 = \mu_2' - \mu_1'^2 = (\lambda^2 + \lambda) - \lambda^2 = \lambda$$

Thus the mean and the variance of the Poisson distribution are each equal to λ.

$$\mu_3 = \mu_3' - 3\mu_1'\mu_2' + 2\mu_1'^3 = (\lambda^3 + 3\lambda^2 + \lambda) - 3\lambda(\lambda^2 + \lambda) + 2\lambda^3 = \lambda.$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

$$= (\lambda^4 + 6\lambda^3 + 7\lambda^2 + \lambda) - 4\lambda(\lambda^3 + 3\lambda^2 + \lambda) + 6\lambda^2(\lambda^2 + \lambda) - 3\lambda^4 = 3\lambda^2 + \lambda$$

Co-efficients of skewness and kurtosis are given by

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\lambda^2}{\lambda^3} = \frac{1}{\lambda} \text{ and } \gamma_1 = \sqrt{\beta_1} = \frac{1}{\sqrt{\lambda}}$$

Also $$\beta_2 = \frac{\mu_4}{\mu_2^2} = 3 + \frac{1}{\lambda} \text{ and } \gamma_2 = \beta_2 - 3 = \frac{1}{\lambda} \qquad ...(7\cdot15)$$

Hence the Poisson distribution is always a skewed distribution.

Proceeding to the limit as $\lambda \to \infty$, we get

$$\beta_1 = 0 \text{ and } \beta_2 = 3$$

7·3·3. Mode of the Poisson Distribution

$$\frac{p(x)}{p(x-1)} = \frac{\dfrac{e^{-\lambda}\lambda^x}{x!}}{\dfrac{e^{-\lambda}\lambda^{x-1}}{(x-1)!}} = \frac{\lambda}{x} \qquad ..(7\cdot16)$$

We discuss the following cases :

Case I. *When λ is not an integer.*

Let us suppose that S is the intergral part of λ.

$$\frac{p(1)}{p(0)} > 1, \quad \dots, \frac{p(S-1)}{p(S-2)} > 1, \frac{p(S)}{p(S-1)} > 1,$$

and $$\frac{p(S+1)}{p(S)} < 1, \frac{p(S+2)}{p(S+1)} < 1, \dots$$

Combining the above expressions into a single expression, we get

$p(0) < p(1) < p(2) \dots < p(S-2) < p(S-1) < p(S) > p(S+1) > p(S+2) > \dots$, which shows that $p(S)$ is the maximum value. Hence in this case the distribution is unimodal and the integral part of λ is the unique modal value.

Case II. *When $\lambda = k$ (say) is an integer.* Here we have

$$\frac{p(1)}{p(0)} > 1, \frac{p(2)}{p(1)} > 1, \dots, \frac{p(k-1)}{p(k-2)} > 1$$

and $$\frac{p(k)}{p(k-1)} = 1, \frac{p(k+1)}{p(k)} < 1, \frac{p(k+2)}{p(k+1)} < 1, \dots$$

$\therefore\ p(0) < p(1) < p(2) < \dots < p(k-2) < p(k-1) = p(k) > p(k+1) > p(k+2) \dots$

In this case we have two maximum values, *viz.*, $p(k-1)$ and $p(k)$ and thus the distribution is bimodal and two modes are at $(k-1)$ and k, *i.e.*, at $(\lambda-1)$ and λ, (since $k = \lambda$).

7·3·4. Recurrence Relation for the Moments of the Poisson Distribution. By def.,

$$\mu_r = E\{X - E(X)\}^r = \sum_{x=0}^{\infty} (x-\lambda)^r p(x, \lambda)$$

$$= \sum_{x=0}^{\infty} (x-\lambda)^r \frac{e^{-\lambda}\lambda^x}{x!}$$

Differentiating with respect to λ, we get

$$\frac{d\mu_r}{d\lambda} = \sum_{x=0}^{\infty} r(x-\lambda)^{r-1}(-1)\frac{e^{-\lambda}\lambda^x}{x!} + \sum_{x=0}^{\infty} \frac{(x-\lambda)^r}{x!}\left\{x\lambda^{x-1}e^{-\lambda} - \lambda^x e^{-\lambda}\right\}$$

$$= -r\sum_{x=0}^{\infty} (x-\lambda)^{r-1} \cdot \frac{e^{-\lambda}\lambda^x}{x!} + \sum_{x=0}^{\infty} \frac{(x-\lambda)^r}{x!}\left\{\lambda^{x-1}e^{-\lambda}(x-\lambda)\right\}$$

$$= -r\sum_{x=0}^{\infty} (x-\lambda)^{r-1} \frac{e^{-\lambda}\lambda^x}{x!} + \frac{1}{\lambda}\sum_{x=0}^{\infty} (x-\lambda)^{r+1} \cdot \frac{e^{-\lambda}\lambda^x}{x!}$$

$$\therefore \qquad \frac{d\mu_r}{d\lambda} = -r\mu_{r-1} + \frac{1}{\lambda}\mu_{r+1}$$

$$\Rightarrow \qquad \mu_{r+1} = r\lambda\mu_{r-1} + \lambda\frac{d\mu_r}{d\lambda} \qquad \dots(7\cdot17)$$

Putting $r = 1, 2$ and 3 successively, we get

$$\mu_2 = r\mu_0 + \lambda\frac{d\mu_1}{d\lambda} = \lambda \qquad (\because \mu_0 = 1, \mu_1 = 0)$$

$$\mu_3 = 2\lambda\mu_1 + \lambda \frac{d\mu_2}{d\lambda} = \lambda, \quad \mu_4 = 3\lambda\mu_2 + \lambda\frac{d\mu_3}{d\lambda} = 3\lambda^2 + \lambda$$

7·3·5. Moment Generating Function of the Poisson Distribution

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx} \cdot \frac{e^{-\lambda}\lambda^x}{x!} = \sum_{x=0}^{\infty} \frac{e^{-\lambda}(\lambda e^t)^x}{x!}$$

$$= e^{-\lambda}\left\{1 + \lambda e^t + \frac{(\lambda e^t)^2}{2!} + \dots\right\} = e^{-\lambda} \cdot e^{\lambda e^t} = e^{\lambda(e^t - 1)} \qquad \text{...(7·18)}$$

7·3·6. Characteristic Function of the Poisson Distribution

$$\phi_X(t) = \sum_{x=0}^{\infty} e^{itx} \cdot p(x) = \sum_{x=0}^{\infty} e^{itx} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^{it})^x}{x!}$$

$$= e^{-\lambda}\, e^{\lambda e^{it}} = e^{\lambda(e^{it} - 1)} \qquad \text{...(7·19)}$$

7·3·7. Cumulants of the Poisson Distribution

$$K_X(t) = \log M_X(t) = \log\left[e^{\lambda(e^t - 1)}\right] = \lambda(e^t - 1)$$

$$= \lambda\left[\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots + \frac{t^r}{r!} + \dots\right) - 1\right]$$

$$= \lambda\left[t + \frac{t^2}{2!} + \frac{t^3}{3!} + \dots + \frac{t^r}{r!} + \dots\right]$$

κ_r = rth cumulant = co-efficient of $\dfrac{t^r}{r!}$ in $K_X(t) = \lambda$

$$\Rightarrow \qquad \kappa_r = \lambda;\ r = 1, 2, 3, \dots \qquad \text{...(7·19}a)$$

Hence all the cumulants of the Poisson distribution are equal, each being equal to λ. In particular, we have

Mean $= \kappa_1 = \lambda$, $\mu_2 = \kappa_2 = \lambda$, $\mu_3 = \kappa_3 = \lambda$ and $\mu_4 = \kappa_4 + 3\kappa_2^2 = \lambda + 3\lambda^2$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\lambda^2}{\lambda^3} = \frac{1}{\lambda} \text{ and } \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{\lambda + 3\lambda^2}{\lambda^2} = \frac{1}{\lambda} + 3$$

Remark. If m is the mean and σ is the s.d. of Poisson distribution with parameter λ, then

$$m\,\sigma\,\gamma_1\,\gamma_2 = \lambda \cdot \sqrt{\lambda} \cdot \sqrt{\beta_1}\,(\beta_2 - 3)$$

$$= \lambda \cdot \sqrt{\lambda} \cdot \frac{1}{\sqrt{\lambda}} \cdot \frac{1}{\lambda} = 1.$$

7·3·8. Additive or Reproductive Property of Independent Poisson Variates. *Sum of independent Poisson variates is also a Poisson variate. More elaborately, if X_i, $(i = 1, 2, \dots, n)$ are independent Poisson variates with param-*

ters λ_i ; $i = 1, 2, ..., n$ respectively, then $\sum_{i=1}^{n} X_i$ is also a Poisson variate with parameter $\sum_{i=1}^{n} \lambda_i$.

Proof. $M_{X_i}(t) = e^{\lambda_i(e^t - 1)}$; $i = 1, 2, ... n$

$$M_{X_1 + X_2 + \ldots + X_n}(t) = M_{X_1}(t)\, M_{X_2}(t) \ldots M_{X_n}(t),$$

[since X_i ; $i = 1, 2, ..., n$ are independent]

$$= e^{\lambda_1(e^t - 1)}\, e^{\lambda_2(e^t - 1)} \ldots e^{\lambda_n(e^t - 1)}$$

$$= e^{(\lambda_1 + \lambda_2 + \ldots + \lambda_n)(e^t - 1)}$$

which is the m.g.f. of a Poisson variate with parameter $\lambda_1 + \lambda_2 + \ldots + \lambda_n$. Hence by uniqueness theorem of m.g.f.'s, $\sum_{i=1}^{n} X_i$ is also a Poisson variate with parameter $\sum_{i=1}^{n} \lambda_i$.

Remarks 1. In fact, the converse of the above result is also true *i.e., If $X_1, X_2, ..., X_n$ are independent and $\sum_{i=1}^{n} X_i$ has a Poisson distribution, then each of the random variables $X_1, X_2, ..., X_n$ has a Poisson distribution.*

Let X_1 and X_2 be independent *r.v.'s* so that $X_1 \sim P(\lambda_1)$ and $X_1 + X_2 \sim P(\lambda_1 + \lambda_2)$. Then we want to prove that $X_2 \sim P(\lambda_2)$.

Proof. Since X_1 and X_2 are independent, we have

$$M_{X_1 + X_2}(t) = M_{X_1}(t)\, M_{X_2}(t)$$

$$\Rightarrow \quad e^{(\lambda_1 + \lambda_2)(e^t - 1)} = e^{\lambda_1(e^t - 1)} \cdot M_{X_2}(t)$$

$$\Rightarrow \quad M_{X_2}(t) = e^{\lambda_2(e^t - 1)}$$

$\Rightarrow$ $X_2 \sim P(\lambda_2)$, by uniqueness theorem of m.g.f.

2. The difference of two independent Poisson variates is not a Poisson variate.

$$M_{X_1 - X_2}(t) = M_{X_1 + (-X_2)}(t) = M_{X_1}(t) \cdot M_{(-X_2)}(t),$$

(since X_1 and X_2 are independent).

$$\therefore \quad M_{X_1 - X_2}(t) = M_{X_1}(t)\, M_{X_2}(-t) \qquad [\because M_{cX}(t) = M_X(ct)]$$

$$= e^{\lambda_1(e^t - 1)} \cdot e^{\lambda_2(e^{-t} - 1)} = e^{\lambda_1(e^t - 1) + \lambda_2(e^{-t} - 1)}$$

which cannot be put in the form $e^{\lambda(e^t - 1)}$ Hence $(X_1 - X_2)$ is not a Poisson variate.

Moreover the difference $(X_1 - X_2)$ cannot be a Poisson variate is evident from the fact that it may have positive as well as negative values, while a Poisson variate is always non-negative.

7·3·9. Probability Generating Function of Poisson Distribution

$$\text{P.G.F. of } X = \sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!} \cdot s^k = \sum_{k=0}^{\infty} e^{-\lambda} \frac{(\lambda s)^k}{k!} = e^{-\lambda} e^{\lambda s} = e^{\lambda(s-1)} \quad ...(7 \cdot 20)$$

Example 7·24. *A car hire firm has two cars which it fires out day by day. The number of demands for a car on each day is distributed as Poisson variate with mean 1·5. Calculate the proportion of days on which (i) neither car is used, and (ii) some demand is refused.* **[Meerut Univ. B.Sc. 1993]**

Solution. The proportion of days on which there are x demands for a car

$$= P\{\text{of } x \text{ demands in a day}\}$$

$$= \frac{e^{-1\cdot 5}(1\cdot 5)^x}{x!},$$

since the number of demands for a car on any day is a Poisson variate with mean 1·5. Thus

$$P(X = x) = \frac{e^{-1\cdot 5}(1\cdot 5)^x}{x!}; \quad x = 0, 1, 2, \ldots$$

(*i*) Proportion of days on which neither car is used is given by

$$P(X = 0) = e^{-1\cdot 5}$$

$$= \left[1 - 1\cdot 5 + \frac{(1\cdot 5)^2}{2!} - \frac{(1\cdot 5)^3}{3!} + \frac{(1\cdot 5)^4}{4!} - \ldots\right]$$

$$= 0\cdot 2231$$

(*ii*) Proportion of days on which some demand is refused is

$$P(X > 2) = 1 - P(X \le 2)$$

$$= 1 - [P(X = 0) + P(X = 1) + P(X = 2)]$$

$$= 1 - e^{-1\cdot 5}\left[1 + 1\cdot 5 + \frac{(1\cdot 5)^2}{2!}\right]$$

$$= 1 - 0\cdot 2231 \times 3\cdot 625 = 0\cdot 19126$$

Example 7·25. *A manufacturer of cotter pins knows that 5% of his product is defective. If he sells cotter pins in boxes of 100 and guarantees that not more than 10 pins will be defective, what is the approximate probability that a box will fail to meet the guaranteed quality?* **[Kanpur Univ. B.Sc. 1993]**

Solution. We are given $n = 100$.

Let p = Probability of a defective pin = 5% = 0·05

$\therefore$ λ = Mean number of defective pins in a box of 100

$= np = 100 \times 0\cdot 05 = 5$

Since 'p' is small, we may use Poisson distribution.

Probability of x defective pins in a box of 100 is

$$P(X = x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-5}5^x}{x!};\ x = 0, 1, 2, \ldots$$

Probability that a box will fail to meet the guaranteed quality is

$$P(X > 10) = 1 - P(X \le 10) = 1 - \sum_{x=0}^{10} \frac{e^{-5}5^x}{x!} = 1 - e^{-5} \sum_{x=0}^{10} \frac{5^x}{x!}$$

Example 7·26. *Six coins are tossed 6,400 times. Using the Poisson distribution, find the approximate probability of getting six heads r times.*

Solution. The probability of obtaining six heads in one throw of six coins (*a* single trial), is $p = (½)^6$, assuming that head and tail are equally probable.

$$\therefore \qquad \lambda = np = 6400 \times (½)^6 = 100.$$

Hence, using Poisson probability law, the required probability of getting 6 heads r times is given by :

$$P(X = r) = \frac{e^{-\lambda}\cdot\lambda^r}{r!} = \frac{e^{-100}\cdot(100)^r}{r!};\ r = 0, 1, 2, \ldots$$

Example 7·27. *In a book of 520 pages, 390 typo-graphical errors occur. Assuming Poisson law for the number of errors per page, find the probability that a random sample of 5 pages will contain no error.*

[Patna Univ. B.Sc. (Hons.), 1988]

Solution. The average number of typographical errors per page in the book is given by $\lambda = (390/520) = 0\cdot 75$

Hence using Poisson probability law, the probability of x errors per page is given by : $P(X = x) = \dfrac{e^{-\lambda}\lambda^x}{x!} = \dfrac{e^{-0\cdot 75}(0\cdot 75)^x}{x!};\ x = 0, 1, 2, \ldots$

The required probability that a random sample of 5 pages will contain no error is given by : $[P(X = 0)]^5 = \left(e^{-0\cdot 75}\right)^5 = e^{-3\cdot 75}$

Example 7·28. *Suppose that the number of telephone calls coming into a telephone exchange between 10 A.M. and 11 A.M. say, X_1 is a random variable with Poisson distribution with parameter 2. Similarly the number of calls arriving between 11 A.M. and 12 noon say, X_2 has a Poisson distribution with parameter 6. If X_1 and X_2 are independent, what is the probability that more than 5 calls come in between 10 A.M. and 12 noon ?* **[Calicut U. B. Sc. Oct. 1992]**

Solution. Let $X = X_1 + X_2$. By the additive property of Poisson distribution, X is also a Poisson variate with parameter (say) $\lambda = 2 + 6 = 8$

Hence the probability of x calls in-between 10 A.M. and 12 noon is given by $P(X = x) = \dfrac{e^{-\lambda}\lambda^x}{x!} = \dfrac{e^{-8}8^x}{x!};\ x = 0, 1, 2, \ldots$

Probability that more than 5 calls come in between 10 A.M. and 12 noon is given by

$$P(X > 5) = 1 - P(X \le 5) = 1 - \sum_{x=0}^{5} \frac{e^{-8}\, 8^x}{x\,!}$$

$$= 1 - 0 \cdot 1912 = 0 \cdot 8088$$

Example 7· 29. *A Poisson distribution has a double mode at $x = 1$ and $x = 2$. What is the probability that x will have one or the other of these two values?*

Solution. We have proved that if the Poisson distribution is bimodal, then the two modes are at the points $x = \lambda - 1$ and $x = \lambda$. Since we are given that the two modes are at the points $x = 1$ and $x = 2$, we find that $\lambda = 2$.

$$\therefore \qquad P(X = x) = \frac{e^{-\lambda}\lambda^x}{x\,!} = \frac{e^{-2}\, 2^x}{x\,!}\,;\; x = 0, 1, 2, \ldots$$

$$\Rightarrow \qquad P(X = 1) = e^{-2}\, 2$$

and $$P(X = 2) = \frac{e^{-2} \cdot 2^2}{2\,!} = e^{-2} \cdot 2$$

Required probability $= P(X = 1) + P(X = 2) = 2e^{-2} + 2e^{-2} = 0{\cdot}542$

Example 7· 30. *If X is a Poisson variate such that*

$$P(X = 2) = 9\, P(X = 4) + 90\, P(X = 6) \qquad \ldots(*)$$

Find (i) λ, the mean of X, (ii) β, the coefficient of skewness.

[Delhi Univ. B. Sc. (Maths. Hons.) 1992, '87]

Solution. If X is a Poisson variate with parameter λ, then

$$P(X = x) = \frac{e^{-\lambda} \cdot \lambda^x}{x\,!},\; x = 0, 1, 2, \ldots;\, \lambda > 0$$

Hence (*) gives

$$\frac{e^{-\lambda} \cdot \lambda^2}{2\,!} = e^{-\lambda}\left[9\, \frac{\lambda^4}{4\,!} + 90\, \frac{\lambda^6}{6\,!}\right]$$

$$= \frac{e^{-\lambda}\lambda^2}{8}\,[\,3\lambda^2 + \lambda^4\,]$$

$$\Rightarrow \qquad \lambda^4 + 3\lambda^2 - 4 = 0$$

Solving as a quadratic in λ^2, we get

$$\lambda^2 = \frac{-3 \pm \sqrt{9 + 16}}{2} = \frac{-3 \pm 5}{2}$$

Since $\lambda > 0$, we get $\lambda^2 = 1 \quad \Rightarrow \quad \lambda = 1$

Hence mean $= \lambda = 1$, and $\mu_2 =$ Variance $= \lambda = 1$

Also $\qquad \beta_1 =$ Coefficient of skewness $= \dfrac{1}{\lambda} = 1.$

Example 7· 31. *If X and Y are independent Poisson variates such that*

$$P(X = 1) = P(X = 2)$$

and $$P(Y = 2) = P(Y = 3) \qquad \ldots(*)$$

Find the varaince of $X - 2Y$.

Solution. Let $X \sim P(\lambda)$ and $Y \sim P(\mu)$.

Then we have

$$P(X = x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}, \; x = 0, 1, 2, \ldots ; \lambda > 0$$

and $$P(Y = y) = \frac{e^{-\mu} \cdot \mu^y}{y!}, \; y = 0, 1, 2, \ldots ; \mu > 0$$

Using (*), we get

$$\left.\begin{aligned} \lambda e^{-\lambda} &= \frac{\lambda^2 e^{-\lambda}}{2!} \\ \text{and} \quad \frac{\mu^2 e^{-\mu}}{2} &= \frac{\mu^3 e^{-\mu}}{3!} \end{aligned}\right\} \qquad \ldots(**)$$

Solving (**), we get

$$\lambda e^{-\lambda} [\lambda - 2] = 0 \text{ and } \mu^2 e^{-\mu} [\mu - 3] = 0$$

$\Rightarrow$ $\lambda = 2$ and $\mu = 3$, since $\lambda > 0, \mu > 0$.

Now $\text{Var}(X) = \lambda = 2$, and $\text{Var}(Y) = \mu = 3$ ...(***)

$\therefore$ $\text{Var}(X - 2Y) = 1^2 \text{ Var}(X) + (-2)^2 \cdot \text{Var } Y,$

covariance term vanishes since X and Y are independent.

Hence, on using (***), we get

$$\text{Var}(X - 2Y) = 2 + 4 \times 3 = 14$$

Example 7·32. *If X and Y are independent Poisson variates with means* λ_1 and λ_2 *respectively, find the probability that*

(*i*) $X + Y = k$, (*ii*) $X = Y$ [Delhi Univ. B. Sc. (Stat. Hons.), 1991]

Solution. We have

$$P(X = x) = \frac{e^{-\lambda_1} \cdot \lambda_1^x}{x!}, \; x = 0, 1, 2, 3, \ldots ; \lambda_1 > 0$$

and $$P(Y = y) = \frac{e^{-\lambda_2} \cdot \lambda_2^y}{y!}, \; y = 0, 1, 2, 3 \ldots ; \lambda_2 > 0$$

(*i*) $$P(X + Y = k) = \sum_{r=0}^{k} P(X = r \cap Y = k - r)$$

$$= \sum_{r=0}^{k} P(X = r) \, P(Y = k - r)$$

[$\because$ X and Y are independent]

$$= \sum_{r=0}^{k} \frac{e^{-\lambda_1} \lambda_1^r}{r!} \cdot \frac{e^{-\lambda_2} \cdot \lambda_2^{k-r}}{(k-r)!}$$

$$= e^{-(\lambda_1 + \lambda_2)} \sum_{r=0}^{k} \frac{\lambda_1^r \cdot \lambda_2^{k-r}}{r!\,(k-r)!}$$

$$= e^{-(\lambda_1+\lambda_2)}\left[\frac{\lambda_2^k}{k!}+\frac{\lambda_1\cdot\lambda_2^{k-1}}{1!(k-1)!}+\frac{\lambda_1^2\cdot\lambda_2^{k-2}}{2!(k-2)!}+\ldots+\frac{\lambda_1^k}{k!}\right]$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{k!}\left[\lambda_2^k + {}^kC_1\,\lambda_2^{k-1}\cdot\lambda_1 + {}^kC_2\cdot\lambda_2^{k-2}\cdot\lambda_1^2+\ldots+\lambda_1^k\right]$$

$$= \frac{e^{-(\lambda_1+\lambda_2)}}{k!}\times(\lambda_1+\lambda_2)^k,\ k=0,1,2,\ldots$$

which is the probability function of Poisson distribution with parameter $\lambda_1+\lambda_2$.

Aliter. Since $X \sim P(\lambda_1)$ and $Y \sim P(\lambda_2)$ are independent, by the additive property of Poisson distribution $X+Y \sim P(\lambda_1+\lambda_2)$. Hence

$$P(X+Y=k)=\frac{e^{-(\lambda_1+\lambda_2)}\times(\lambda_1+\lambda_2)^k}{k!};\ k=0,1,2,\ldots$$

(ii) $$P(X=Y)=\sum_{r=0}^{\infty}P(X=r\cap Y=r)$$

$$=\sum_{r=0}^{\infty}P(X=r)\,P(Y=r)$$

[$\because X$ and Y are independent]

$$=e^{-(\lambda_1+\lambda_2)}\sum_{r=0}^{\infty}\frac{(\lambda_1\lambda_2)^r}{(r!)^2}$$

Example 7·33. *Show that in a Poisson distribution with unit mean, mean deviation about mean is (2/e) times the standard deviation.*

[Patna Univ. B. Sc. (Stat. Hons.) 1992; Delhi Univ. B.Sc. (Stat. Hons.), 1993]

Solution. Here we are given $\lambda = 1$.

$$\therefore\quad P(X=x)=\frac{e^{-\lambda}\lambda^x}{x!}=\frac{e^{-1}\cdot 1}{x!}=\frac{e^{-1}}{x!};\ x=0,1,2,\ldots$$

Mean deviation about mean 1 is

$$E(|X-1|)=\sum_{x=0}^{\infty}|x-1|\,p(x)=e^{-1}\sum_{x=0}^{\infty}\frac{|x-1|}{x!}$$

$$=e^{-1}\left[1+\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\ldots\right]$$

We have $$\frac{n}{(n+1)!}=\frac{(n+1)-1}{(n+1)!}=\frac{1}{n!}-\frac{1}{(n+1)!}$$

$\therefore$ Mean deviation about mean

$$= e^{-1}\left[1 + \left(1 - \frac{1}{2\,!}\right) + \left(\frac{1}{2\,!} - \frac{1}{3\,!}\right) + \left(\frac{1}{3\,!} - \frac{1}{4\,!}\right) + \dots\right]$$

$$= e^{-1}(1 + 1) = \frac{2}{e} \times 1 = \frac{2}{e} \times \text{standard deviation},$$

since for the Poisson distribution, variance = mean = 1 (given).

Example 7·34. *Let $X_1, X_2, \dots, X_n$ be identically and independently distributed Bin (1, p) variates. Let $S_n = \sum_{j=1}^{n} X_j$ and $M_n(t)$ be the m.g.f. of S_n. Find $\lim_{n \to \infty} M_n(t)$, using $np = \lambda$ (const.)* **[Delhi Univ. B. Sc. (Maths Hons.), 1989]**

Solution. Since X_i, $i = 1, 2, \dots n$ are i.i.d. binomial variates $B(1, p)$,

$S_n = \sum_{j=1}^{n} X_j$, is a binomial $B(n, p)$ variate.

$$\therefore M_n(t) = M.g.f. \text{ of } S_n = \left(q + pe^t\right)^n = \left[1 + (e^t - 1)p\right]^n$$

If we take $np = \lambda \Rightarrow p = \lambda/n$ and let $n \to \infty$, we get

$$\lim_{n \to \infty} M_n(t) = \lim_{n \to \infty} \left[1 + \frac{(e^t - 1)\lambda}{n}\right]^n = \exp\left[\lambda\left(e^t - 1\right)\right],$$

which is the m.g.f. of Poisson distribution with parameter λ. Hence by uniqueness theorem of m.g.f., $S_n = \sum_{j=1}^{n} X_j \to P(\lambda)$, as $n \to \infty$, with $np = \lambda$ (fixed).

Example. 7·35. *(a) If X is a Poisson variate with mean m, show that the expectation of e^{-kX} is* $\exp\left[-m\left(1 - e^{-k}\right)\right]$. **[Nagpur Univ. B.Sc. 1993]**

Hence show that, if $\bar{X}$ is the arithmetic mean of n independent random variables $X_1, X_2, \dots, X_n$, each having Poisson distribution with parameter m, then $e^{-\bar{x}}$ as an estimate of e^{-m} is biased, although $\bar{X}$ is an unbaised estimate of m.

(b) If X is a Poisson variate with mean m, what would be the expectation of $e^{-kx}\,kX$, k being a constant.

Solution.

$$E\left(e^{-kX}\right) = \sum_{x=0}^{\infty} e^{-kx} p(x) = \sum_{x=0}^{\infty} e^{-kx} \cdot \frac{e^{-m} m^x}{x\,!} = e^{-m} \sum_{x=0}^{\infty} \frac{\left(me^{-k}\right)^x}{x\,!}$$

$$= e^{-m}\left[1 + me^{-k} + \frac{\left(me^{-k}\right)^2}{2\,!} + \dots\right]$$

$$= e^{-m} e^{me^{-k}} = e^{-m(1-e^{-k})} \qquad \text{...(*)}$$

We have

$$E(\bar{X}) = E\left(\frac{1}{n} \sum_{i=1}^{n} X_i\right) = \frac{1}{n} \sum_{i=1}^{n} E(X_i)$$

Since X_i; $i = 1, 2, ..., n$ is a Poisson variate with parameter m, $E(X_i) = m$.

$$\therefore E(\bar{X}) = \frac{1}{n} \sum_{i=1}^{n} m = \frac{1}{n} nm = m$$

Hence $\bar{X}$ is an unbaised estimate of m.

Now $$E(e^{-\bar{X}}) = E\left[\exp\left(\frac{-1}{n} \sum_{i=1}^{n} X_i\right)\right]$$
$$= E(e^{-X_1/n} . e^{-X_2/n} ... e^{-X_n/n}),$$
$$= E(e^{-X_1/n}) E(e^{-X_2/n}) ... E(e^{-X_n/n}),$$
(since $X_1, X_2, ..., X_n$ are independent)

$$\therefore \quad E(e^{-\bar{X}}) = \prod_{i=1}^{n} E(e^{-X_i/n}) \qquad \text{...(**)}$$

Using (*) with $k = 1/n$, we get

$$E(e^{-X_i/n}) = e^{-m(1-e^{-1/n})}, \text{(since } X_i \text{ is a Poisson variate with parameter } m)$$

$$\therefore \quad E(e^{-\bar{X}}) = \prod_{i=1}^{n} \left[\exp\left\{-m(1-e^{-1/n})\right\} = \exp\left\{-m(1-e^{-1/n})\right\}\right]^n$$
$$= \exp\left\{-mn(1-e^{-1/n})\right\} \neq e^{-m}$$

Hence $e^{-\bar{X}}$ is not an unbaised estimated of e^{-m}, though $\bar{X}$ is an unbiased estimate of m.

$$(b)\ E(e^{-kX} kX) = \sum_{x=0}^{\infty} e^{-kx} kx \cdot p(x) = k \sum_{x=1}^{\infty} e^{-kx}\, x\, \frac{e^{-m} m^x}{x!}$$
$$= ke^{-m} \sum_{x=1}^{\infty} \frac{(me^{-k})^x}{(x-1)!} = ke^{-m} me^{-k} \sum_{x=1}^{\infty} \frac{(me^{-k})^{x-1}}{(x-1)!}$$
$$= mke^{-m-k}\left\{1 + me^{-k} + \frac{(me^{-k})^2}{2!} + \dots\right\}$$
$$= mke^{-m-k} \cdot e^{me^{-k}} = mk \exp\left[\left\{m(e^{-k}-1)\right\} - k\right]$$

Example 7·36. *If X and Y are independent Poisson variates with means m_1 and m_2 respectively, prove that the probability that $X - Y$ has the value 'r' is the co-efficient of t^r in*

$$\exp\left\{m_1 t + m_2 t^{-1} - m_1 - m_2\right\}$$

[Delhi Univ. B.Sc. (Stat. Hons.), 1991, '89]

Solution. Since X and Y are independent Poisson variates with means m_1 and m_2 respectively,

$$\left\{\begin{array}{ll} & P(X = x) = \dfrac{e^{-m_1} m_1^x}{x!};\ x = 0, 1, 2, \ldots \infty \\ \text{and} & P(Y = y) = \dfrac{e^{-m_2} m_2^y}{y!};\ y = 0, 1, 2, \ldots \infty \end{array}\right\} \quad \ldots(1)$$

$$P(X - Y = r) = \sum_{s=0}^{\infty} P(X = r + s \cap Y = s) = \sum_{s=0}^{\infty} P(X = r + s)\, P(Y = s)$$

$$= \sum_{s=0}^{\infty} \frac{e^{-m_1} \cdot m_1^{r+s}}{(r+s)!} \cdot \frac{e^{-m_2} m_2^s}{s!} \qquad \ldots[\text{From (1)}]$$

$$= e^{-m_1 - m_2} \sum_{s=0}^{\infty} \frac{m_1^{r+s} m_2^s}{(r+s)!\, s!} \qquad \ldots(2)$$

We have $e^{m_1 t + m_2 t^{-1}} = e^{m_1 t} \times e^{m_2 t^{-1}}$

$$= \left\{1 + m_1 t + \frac{(m_1 t)^2}{2!} + \ldots + \frac{(m_1 t)^{r+s}}{(r+s)!} + \ldots\right\}$$

$$\times \left\{1 + m_2 t^{-1} + \frac{(m_2 t^{-1})^2}{2!} + \ldots + \frac{(m_2 t^{-1})^s}{s!} + \ldots\right\}$$

$\therefore$ Co-efficient of t^r in $e^{m_1 t + m_2 t^{-1}} = \displaystyle\sum_{s=0}^{\infty} \frac{m_1^{r+s} m_2^s}{(r+s)!\, s!}$

Hence from (2), we get

$$P(X - Y = r) = e^{-m_1 - m_2} \times \text{Coefficient of } t^r \text{ in } e^{m_1 t + m_2 t^{-1}},$$

$$= \text{Coefficient of } t^r \text{ in } e^{-m_1 - m_2 + m_1 t + m_2 t^{-1}}$$

which is the required result.

Example 7·37. *If X is a Poisson variate with mean m, show that $\dfrac{X - m}{\sqrt{m}}$ is a variable with mean zero and variance unity. Find the M.G.F. for this variable and show that it approaches $e^{t^2/2}$ as $m \to \infty$. Also interpret the result.*

[Delhi Univ. B. Sc. (Stat. Hons.), 1987]

Solution. Let $Y = \dfrac{X - m}{\sqrt{m}}$

$$\therefore \quad E(Y) = E\left(\frac{X - m}{\sqrt{m}}\right) = \frac{1}{\sqrt{m}} E(X - m) = 0$$

$$V(Y) = E\left(\frac{X-m}{\sqrt{m}}\right)^2 = \frac{1}{m}E(X-m)^2 = \frac{1}{m}\mu_2 = 1$$

$$M.G.F. \text{ of } Y = M_Y(t) = E(e^{tY}) = E\left[e^{t(X-m)/\sqrt{m}}\right]$$
$$= e^{-t\sqrt{m}}\left[E(e^{tX/\sqrt{m}})\right]$$
$$= e^{-t\sqrt{m}}\sum_{x=0}^{\infty}\frac{e^{-m}m^x}{x!}\cdot e^{tx/\sqrt{m}}$$
$$= e^{-t\sqrt{m}}\cdot e^{-m}\sum_{x=0}^{\infty}\frac{(me^{t/\sqrt{m}})^x}{x!}$$
$$= e^{-m-t\sqrt{m}}\left[1+\frac{me^{t/\sqrt{m}}}{1!}+\frac{(me^{t/\sqrt{m}})^2}{2!}+\ldots\right]$$
$$= e^{-m-t\sqrt{m}}\cdot\exp(me^{t/\sqrt{m}}) = \exp\left[-m-t\sqrt{m}+me^{t/\sqrt{m}}\right]$$
$$= \exp\left[-m-t\sqrt{m}+m\left(1+\frac{t}{\sqrt{m}}+\frac{t^2}{2!\,m}+\frac{t^3}{3!\,m^{3/2}}+\ldots\right)\right]$$
$$= \exp\left[\frac{1}{2}t^2+\frac{1}{3!}\frac{t^3}{\sqrt{m}}+\ldots\right]$$

Now proceeding to limit as $m \to \infty$, we get

$$\lim_{m\to\infty} M_Y(t) = e^{t^2/2} \qquad \ldots(*)$$

Interpretation. (*) is the m.g.f. of Standard Normal Variate [c.f. Remark to § 8·2·5]. Hence by uniqueness theorem of m.g.f.'s, standard Poisson variate tends to standard normal variate as $m \to \infty$. Hence Poisson distribution tends to Normal distribution for large values of parameter m.

Example 7·38. *Deduce the first four moments about the mean of the Poisson distribution from those of the Binomial distribution.*

Solution. The first four central moments of the binomial distribution are

$$\left\{\begin{array}{ll} \mu_1 = 0, & \text{Mean} = np \\ \mu_2 = npq, & \mu_3 = npq(q-p) \text{ and} \\ \mu_4 = npq(1-6pq) + 3n^2p^2q^2 & \end{array}\right\} \qquad \ldots(*)$$

Poisson distribution is a limiting form of the binomial distribution under the following conditions :

(*i*) $n \to \infty$, (*ii*) $p \to 0$, *i.e.*, $q \to 1$, and (*iii*) $np = \lambda$, (say), is finite.

Using these conditions, we get from (*) the moments of the Poisson distribution as

$$\mu_1 = 0$$
$$\text{Mean} = \lim (np) = \lambda$$
$$\mu_2 = \lim (npq) = \lim (np)\cdot\lim(q) = \lambda\cdot 1 = \lambda$$
$$\mu_3 = \lim[npq(q-p)] = \lambda\cdot 1\,(1-0) = \lambda$$
$$\mu_4 = \lim[npq(1-6pq) + 3(np)^2q^2]$$

$$= [\lambda \cdot 1 (1 - 6 \cdot 0 \cdot 1) + 3\lambda^2 \cdot 1] = \lambda + 3\lambda^2$$

Example 7·39. *If X is a Poisson variate with parameter m and Y is another discrete variable whose conditional distribution for a given X is given by*

$$P(Y = r \mid X = x) = \binom{x}{r} p^r (1-p)^{x-r} ;\ 0 < p < 1,\ r = 0, 1, 2, \ldots, x$$

then show that the unconditional distribution of Y is a Poisson distribution with parameter mp.

[Delhi Univ. B.Sc. (Stat. Hons.), 1993, Shivaji U.B.Sc. Nov. 1992]

Solution. We are given that

$$P(X = x) = \frac{e^{-m} m^x}{x!} ;\ x = 0, 1, 2, \ldots$$

and
$$P(Y = r \mid X = x) = \binom{x}{r} p^r (1-p)^{x-r} ; r \le x$$

$$\therefore\ P(X = x \cap Y = r) = P(X = x) P(Y = r \mid X = x)$$

$$= \frac{e^{-m} m^x}{x!} \binom{x}{r} p^r (1-p)^{x-r}$$

$\therefore\ P(Y = r) =$ The unconditional distribution of Y.

$$= \sum_{x=r}^{\infty} \left[\frac{e^{-m} m^x}{x!} \cdot \binom{x}{r} p^r (1-p)^{x-r} \right]$$

$$= e^{-m} \left[\sum_{x=r}^{\infty} \binom{x}{r} \frac{p^r m^x (1-p)^{x-r}}{x!} \right]$$

$$= e^{-m} \left[\sum_{x=r}^{\infty} \frac{m^x}{x!} \cdot \frac{x!}{r!(x-r)!} p^r (1-p)^{x-r} \right]$$

$$= \frac{e^{-m}}{r!} \left[\sum_{x=r}^{\infty} \frac{m^x}{(x-r)!} p^r (1-p)^{x-r} \right]$$

$$= \frac{e^{-m} (mp)^r}{r!} \left[\sum_{x=r}^{\infty} \frac{m^{x-r} (1-p)^{x-r}}{(x-r)!} \right]$$

$$= \frac{e^{-m} (mp)^r}{r!} \left[\sum_{x=r}^{\infty} \frac{\{m(1-p)\}^{x-r}}{(x-r)!} \right]$$

$$= \frac{e^{-m} (mp)^r}{r!} e^{m(1-p)} = \frac{e^{-mp} (mp)^r}{r!} ;\ r = 0, 1, 2, \ldots$$

Hence Y is a Poisson variate with parameter mp.

Example 7·40. *If X and Y are independent Poisson variates, show that the conditional distribution of X given X + Y, is binomial.*

[Madras Univ. B.Sc. Main 1992; Delhi Univ. B. Sc. (Maths Hons.), 1988]

Solution. Let X and Y be independent Poisson variates with parameters λ and μ respectively. Then $X + Y$ is also a Poisson variate with parameter $\lambda + \mu$.

$$P[X = r \mid (X + Y = n)] = \frac{P(X = r \cap X + Y = n)}{P(X + Y = n)} = \frac{P(X = r \cap Y = n - r)}{P(X + Y = n)}$$

$$= \frac{P(X = r)\,P(Y = n - r)}{P(X + Y = n)} \qquad \text{[since } X \text{ and } Y \text{ are indepdent]}$$

$$\therefore P[X = r \mid (X + Y = n)] = \frac{e^{-\lambda}\dfrac{\lambda^r}{r!}\cdot e^{-\mu}\dfrac{\mu^{n-r}}{(n-r)!}}{\dfrac{e^{-(\lambda+\mu)}\,(\lambda+\mu)^n}{n!}}$$

$$= \frac{n!}{r!\,(n-r)!}\left(\frac{\lambda}{\lambda+\mu}\right)^r\left(\frac{\mu}{\lambda+\mu}\right)^{n-r}$$

$$= \binom{n}{r} p^r q^{n-r}, \text{ where } p = \frac{\lambda}{\lambda+\mu},\ q = 1 - p$$

Hence the conditional distribution of X given $X + Y = n$, is a binomial distribution with parameters n and $p = \lambda/(\lambda + \mu)$.

Example 7·41. *If X is a Poisson variate with parameter m and μ_r is the rth central moment, prove that*

$$m\,[{}^rC_1\,\mu_{r-1} + {}^rC_2\,\mu_{r-2} + \ldots + {}^rC_r\,\mu_0] = \mu_{r+1}.$$

[Delhi Univ. B.Sc. (Stat. Hons.) 1990]

Solution Since $X \sim P(m)$, its probability function is given by

$$p(x) = \frac{e^{-m}\cdot m^x}{x!},\ x = 0, 1, 2, \ldots;\ m > 0$$

By definition,

$$\mu_{r+1} = E[X - E(X)]^{r+1} = E[X - m]^{r+1}$$

$$= \sum_{x=0}^{\infty} (x - m)^{r+1}\,p(x)$$

$$= \sum_{x=0}^{\infty} (x - m)^r (x - m)\,\frac{e^{-m}\cdot m^x}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{x(x - m)^r e^{-m} m^x}{x!} - m\sum_{x=0}^{\infty} (x - m)^r\cdot\frac{e^{-m} m^x}{x!}$$

$$= \sum_{x=1}^{\infty} \frac{(x-m)^r e^{-m} m^x}{(x-1)!} - m\mu_r$$

$$= \sum_{y=0}^{\infty} \frac{(y-m+1)^r \cdot e^{-m} \cdot m^{y+1}}{y!} - m\mu_r, \qquad (x-1=y)$$

$$= m \cdot \sum_{y=0}^{\infty} (y-m+1)^r \cdot p(y) - m\mu_r$$

$$= m \sum_{y=0}^{\infty} \Big[(y-m)^r + {}^rC_1 (y-m)^{r-1} + {}^rC_2 (y-m)^{r-2} + \ldots + {}^rC_{r-1}(y-m) + 1 \Big] p(y) - m\mu_r$$

$$= m[\mu_r + {}^rC_1 \mu_{r-1} + {}^rC_2 \mu_{r-2} + \ldots + {}^rC_r \mu_0] - m\mu_r$$

$$= m[{}^rC_1 \mu_{r-1} + {}^rC_2 \mu_{r-2} + \ldots + {}^rC_r \mu_0].$$

Example 7·42. *If X has a Poisson distribution with parameter λ, show that the distribution function of X is given by*

$$F(x) = \frac{1}{\Gamma(x+1)} \int_{\lambda}^{\infty} e^{-t} t^x \, dt \, ; \; x = 0, 1, 2, \ldots$$

[Delhi Univ. M. Sc. (Stat) 1986]

Solution. If X is a Poisson variate, then

$$P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!} \, ; \; x = 0, 1, 2, \ldots \qquad (*)$$

Consider the incomplete gamma integral;

$$I_x = \frac{1}{x!} \int_{\lambda}^{\infty} e^{-t} t^x \, dt \, ; \quad (x \text{ is a positive integer})$$

$$= \left| -\frac{e^{-t} t^x}{x!} \right|_{\lambda}^{\infty} + \frac{1}{(x-1)!} \int_{\lambda}^{\infty} e^{-t} t^{x-1} \, dt$$

$$= \frac{e^{-\lambda} \lambda^x}{x!} + I_{x-1} \qquad (**)$$

which is a reduction formula for I_x.

Repeated applications of (**) gives

$$I_x = \frac{e^{-\lambda} \lambda^x}{x!} + \frac{e^{-\lambda} \lambda^{x-1}}{(x-1)!} + \ldots + \frac{e^{-\lambda} \lambda}{1!} + I_0$$

But $I_0 = \int_{\lambda}^{\infty} e^{-t} \, dt = \left| -e^{-t} \right|_{\lambda}^{\infty} = e^{-\lambda}$

$$\therefore \quad I_x = e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2 e^{-\lambda}}{2!} + \ldots + \frac{\lambda^x}{x!} e^{-\lambda}$$

$$= P(X=0) + P(X=1) + \ldots + P(X=x) \qquad \text{[From (*)]}$$

$$= P(X \le x) = F(x)$$

where $F(\cdot)$ is the distribution function of the $r.v. X$.

$$\Rightarrow \qquad F(x) = \frac{1}{x!}\int_{\lambda}^{\infty} e^{-t}\, t^{x}\, dt = \frac{1}{\Gamma(x+1)}\int_{\lambda}^{\infty} e^{-t}\, t^{x}\, dt$$

($\because \Gamma(x+1) = x!$, since x is a positive integer.)

Remark. This result is of great practical utility. It enables us to represent the cumulative Poisson probabilities (which are generally tedious to compute numerically) in terms of incomplete gamma integral, the values of which are tabulated for different values of λ by Karl Pearson in his Tables of Incomplete Γ-Functions.

7·3·10. Recurrence Formula for the Probabilities of Poisson Distribution. *(Fitting of Poisson Distribution).* For a Poisson distribution with parameter λ, we have

$$p(x) = \frac{e^{-\lambda}\cdot\lambda^{x}}{x!}\,;\; x = 0, 1, 2, \ldots, \infty$$

and
$$P(x+1) = \frac{e^{-\lambda}\lambda^{x+1}}{(x+1)!}\,;\; x = 0, 1, 2, \ldots, \infty$$

$$\therefore \qquad \frac{p(x+1)}{p(x)} = \frac{\lambda}{(x+1)} \;\Rightarrow\; p(x+1) = \frac{\lambda}{x+1}\,p(x) \qquad \ldots(17\cdot20)$$

which is the required recurrence formula.

This formula provides us a very convenient method of graduating the given data by a Poisson distribution. The only probability we need to calculate is $p(0)$ which is given by $p(0) = e^{-\lambda}$, where λ is estimated from the given data. The other probabilities, *viz.*, $p(1), p(2)\ldots$. can now be easily obtained as explained below:

$$p(1) = [p(x+1)]_{x=0} = \left[\frac{\lambda}{x+1}\right]_{x=0} p(0),$$

$$p(2) = [p(x+1)]_{x=1} = \left[\frac{\lambda}{x+1}\right]_{x=1} p(1),$$

$$p(3) = [p(x+1)]_{x=2} = \left[\frac{\lambda}{x+1}\right]_{x=2} p(2),$$

and so on.

Example 7·43. *After correcting 50 pages of the proof of a book, the proof reader finds that there are, on the average, 2 errors per 5 pages. How many pages would one expect to find with 0, 1, 2, 3 and 4 errors, in 1000 pages of the first print of the book ? (Given that $e^{-0\cdot4} = 0\cdot6703$)*

[Nagpur Univ. M.A. (Eco.), 1989]

Solution. Let the random variable X denote the number of errors per page. Then the mean number of errors per page is given by :

$$\lambda = 2/5 = 0\cdot4$$

Using Poisson probability law, probability of x errors per page is given by:

$$P(X = x) = p(x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-0\cdot4}(0\cdot4)^x}{x!};\ x = 0, 1, 2, \ldots$$

Expected number of pages with x errors per page in a book of 1000 pages are :

$$1000 \times P(X = x) = 1000 \times \frac{e^{-0\cdot4}(0\cdot4)^x}{x!};\ x = 0, 1, 2, \ldots$$

Using the recurrence formula (17·20), various probabilities can be easily calculated as shown in the following table.

No. of errors per page (X)	*Probability* $p(x)$	*Expected number of pages* $1000\,p(x)$
0	$p(0) = e^{-0\cdot4} = 0\cdot6703$	$670\cdot3 \simeq 670$
1	$p(1) = \frac{0\cdot4}{0+1}\,p(0) = 0\cdot26812$	$268\cdot12 \simeq 268$
2	$p(2) = \frac{0\cdot4}{1+1}\,p(1) = 0\cdot053624$	$53\cdot624 \simeq 54$
3	$p(3) = \frac{0\cdot4}{2+1}\,p(2) = 0\cdot0071298$	$7\cdot1298 \simeq 7$
4	$p(4) = \frac{0\cdot4}{3+1}\,p(3) = 0\cdot00071298$	$0\cdot71298 \simeq 1$

Example 7·44. *Fit a Poisson distribution to the following data which gives the number of doddens in a sample of clover seeds.*

No. of doddens: (x)	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
Observed frequency: (f)	*56*	*156*	*132*	*92*	*37*	*22*	*4*	*0*	*1*

Solution.

$$\text{Mean} = \frac{1}{N}\sum fx = \frac{986}{500} = 1\cdot972$$

Taking the mean of the given distribution as the mean of the Poisson distribution we want to fit, we get $\lambda = 1\cdot972$,

and $$p(x) = \frac{e^{-\lambda}\cdot\lambda^x}{x!};\ x = 0, 1, 2, \ldots, \infty$$

$$p(0) = e^{-\lambda} = e^{-1\cdot972}$$

$$\therefore \log_{10} p(0) = -1{\cdot}972 \log_{10} e = -1{\cdot}972 \times 0{\cdot}43429$$
$$= -0{\cdot}856419 = \bar{1}{\cdot}143581$$
$$\therefore \qquad p(0) = 0{\cdot}1392$$

Using the recurrence formula (17·20) the various probabilities, *viz.*, $p(1), p(2), \ldots$, can be easily calculated as shown in the following table :

x	$\dfrac{\lambda}{x+1}$	$p(x)$	*Expected frequency* $N.p(x)$
0	1·972	0·13920	69·6000
1	0·986	0·27455	137·2512
2	0·657	0·27006	135·3296
3	0·493	0·17793	88·9566
4	0·394	0·10964	43·8556
5	0·328	0·03459	17·2966
6	0·281	0·01137	5·6846
7	0·247	0·00320	1·6013
8	0·219	0·00078	0·3942

Since frequencies are always integers, therefore by converting them to nearest integers, we get

Observed frequency :	56	156	132	92	37	22	4	0	1
Expected frequency :	70	137	135	89	44	17	6	2	0

Remark. In rounding the figures to the nearest integer it has to be kept in mind that the total of the observed and the expected frequencies should be same.

EXERCISE 7 (b)

1. (*a*) Derrive Poisson distribution as a limiting form of a binomial distribution. **[Madras Univ. B. E., Dec. 1991]**

Hence find β_1 and β_2 of the distribution.

Give some examples of the occurrence of Poisson distribution in different fields.

(*b*) State and prove the reproductive property of the Poisson distribution. Show that the mean and variance of the Poisson distribution are equal.

Find the mode of the Poisson distribution with mean value 5.

(*c*) Prove that under certain conditions to be stated by you, the number of telephone calls on a trunkline in a given interval of time has a Poisson distribution.

[Calcutta Univ. B.Sc. (Maths Hons.), 1989]

(*d*) Show that for a Poisson distribution, the coefficient of variation is the reciprocal of the standard deviation.

2. (*a*) If two independent variables X_1 and X_2 have Poisson distribution with means λ_1 and λ_2 respectively, then show that their sum $X_1 + X_2$ is a Poisson variate with mean $\lambda_1 + \lambda_2$.

Does the difference of two independent Poisson variates follow a Poisson distribution ? Give reasons. [Sri Venketeswara Univ. B.Sc., 1991]

(*b*) Prove that the sum of two independent Poisson variates is a Poisson variate. Is the result true for the difference also ? Give reasons.

[Delhi Univ. B.Sc. (Stat. Hons.) 1989]

(*c*) If $X_1, X_2, ..., X_k$ are independent random variables following the Poisson law with parameter $m_1, m_2, ..., m_k$ respectively, show that $\sum_{i=1}^{k} X_i$ follows the Poisson law with parameter $\sum_{i=1}^{k} m_i$ [Madras Univ. B. E., 1993]

3. (*a*) Prove the recurrence relation between the moments of Poisson distribution

$$\mu_{r+1} = \lambda \left(r\mu_{r-1} + \frac{d\mu_r}{d\lambda} \right), \text{ where } \mu_r = \sum_{j=0}^{\infty} \frac{e^{-\lambda}\lambda^j}{j!} (j-\lambda)^r$$

where μ_r is the *r*th moment about the mean λ. Hence obtain the skewness and kurtosis of Poisson distribution.

[Delhi Univ. B. Sc. (Stat. Hons.) 1989,' 86; Utkal Univ. B. Sc. 1993]

(*b*) Let X have a Poisson distribution with parameter $\lambda > 0$. If r is a non-negative integer and if $\mu_r' = E(X^r)$, prove that

$$\mu'_{r+1} = \lambda \left(\mu'_r + \frac{d\mu'_r}{d\lambda} \right)$$

[Madras Univ. B. Sc. Nov. 1988]

4. What do you understand by (*i*) cumulants, (*ii*) cumulative function. Obtain the cumulative function of a Poisson distribution with parameter λ. Hence or otherwise show that for a Poisson distribution with parameter λ, all the cumulants are λ.

5. For the Poisson distribution with parameter λ, show that the *r*th factorial moment $\mu'_{(r)}$ is given by $\mu'_{(r)} = \lambda^r$

Show further that $\mu_{(2)} = \lambda$, $\mu_{(3)} = -2\lambda$ and $\mu_{(4)} = 3\lambda(\lambda+2)$

6. (*a*) If X and Y are independent *r.v. s.*' so that $X \sim P(\lambda)$ and $X + Y \sim P(\lambda + \mu)$, find the distribution of Y. [Ans. $Y \sim P(\mu)$]

(*b*) If $X \sim P(\lambda)$, find

(*i*) Karl Pearson's coefficient of skewness

(*ii*) Moment measure of skewness.

Is Poisson distribution positively skewed or negatively skewed ?

7. (*a*) It is known that the probability that an item produced by a certain machine will be defective is 0·01. By applying Poisson's approximation, show that the probability that random sample of 100 items selected at random from the total output will contain no more than one defective item is $2/e$.

(*b*) The probability of success in a trail is known to be 10^{-4}. It is possible to repeat the trial independently any desired number of times. Do you think that the number of successes in a series of trials, if the number of trials in the series increases indefinitely, will tend to follow a Poisson distribution ? Give your reasons.

(*c*) The probability of getting no misprint in a page of a book is e^{-4}. What is the probability that a page contains more than 2 misprints ? [State the assumptions you make in solving this problem.] **[Bombay Univ. B.Sc., 1989]**

8. In a certain factory turning out optical lenses, there is a small chance 1/500 for any lens to be defective. The lenses are supplied in a packet of 10. Use Poisson distribution to calculate the approximate number of packets containing no defective, one defective, two defective and three defective lenses in a consignment of 20,000 packets.

Ans. 19604, 392, 4 and 0 packets.

9. Red blood cell deficiency may be determined by examining a specimen of the blood under a microscope. Suppose a certain small fixed volume contains on the average 20 red cells for normal persons. Using Poisson distribution, obtain the probability that a specimen from a normal person will contain less than 15 red cells.

Ans. $\sum_{x=0}^{14} \{e^{-20} (20)^x/x\,!\}$

10. Assuming that the chance of a traffic accident in a day in a street of Delhi is 0·001, on how many days out of a trial of 1,000 days can we expect :

(*i*) no accident

(*ii*) more than three accidents, if there are 1,000 such streets in the whole city ?

11. Patients arrive randomly and independently at a doctor's surgery from 8·0 A.M. at an average rate of one in five minutes.The waiting room holds 2 persons. What is the probability that the room will be full when the doctor arrives at 9·0 A.M. (Estimate the probability to an accuracy of 5 per cent.)

Ans. 53·84 %

12. An office switchboard receives telephone calls at the rate of 3 calls per minute on an average. What is the probability of receiving (*i*) no calls in a one-minute interval, (*ii*) at the most 3 calls in a 5-minute interval ?

Ans. (*i*) 0·0323, (*ii*) 0

13. A hospital switchboard receives an average of 4 emergency calls in a 10-minute interval. What is the probability that (*i*) there are at the most 2 emergency calls in a 10-minute interval, (*ii*) there are exactly 3 emergency calls in a 10-minute interval ?

Ans. (*i*) 13^{-4}, (*ii*) $(32/3)\, e^{-4}$

14. (*a*) A distributor of bean seeds determines from extensive tests that 5% of large batch of seeds will not germinate. He sells the seeds in packets of 200 and guarantees 90% germination. Determine the probability that a particular packet will violate the guarantee.

Ans. $1 - \sum_{r=0}^{10} (e^{-10}\, 10^r / r!)$

(*b*) In an automatic telephone exchange the probability that any one call is wrongly connected is 0·001. What is the minimum number of independent calls required to ensure a probability of 0·90, that at least one call is wrongly connected?

15. (*a*) Fit a Poisson distribution to the following data with respect to the number of red blood corpuscles (x) per cell :

x :	0	1	2	3	4	5
Number of cells f :	142	156	69	27	5	1

(*b*) Data was collected over a period of 10 years, showing number of deaths from horse kicks in each of the 20 army corps. From the 200 corps-years, the distribution of deaths was as follows :

No. of deaths :	0	1	2	3	4
Frequency :	122	60	15	2	1

Graduate the data by Poisson distribution and calculate the theoretical frequencies.

Given	e^{-m} :	0·6703	0·6065	0·5488	0·4966
	m :	0·4	0·5	0·6	0·7

(c) Fit a Poisson distribution to the following data and calculate the expected frequencies :-

x :	0	1	2	3	4	5	6	7	8
f :	71	112	117	57	27	11	3	1	1

16. (*a*) If X is the number of occurrences of the Poisson variate with mean λ ; show that : $P(X \geq n) - P(X \geq n+1) = P(X = n)$

(*b*) Suppose that X has a Poisson distribution. If

$$P(X = 2) = \tfrac{2}{3}\, P(X = 1).$$

Evaluate (*i*) $P(X = 0)$ and (*ii*) $P(X = 3)$ [Ans. (*i*) 0·264.]

(*c*) If X has a Poisson distribution such that

$P(X = 1) = P(X = 2)$, find $P(X = 4)$. [Ans 0·09]

(*c*) If a Poisson variate X is such that

$$P(X = 1) = 2\,P(X = 2),$$

find $P(X = 0)$, mean and the variance.

(*d*) If for a Poisson variate X, $E(X^2) = 6$, what is $E(X)$?

(*e*) If X and Y are independent Poisson variates having means 1 and 3 respectively, find the variance of $3X + Y$.

17. Show that for a Poisson distribution

(*i*) $M\sigma\gamma_1\gamma_2 = 1$, (*ii*) $\beta_1^{1/2}(\beta_2 - 3)\mu_1'\sigma = 1$

18. Show that the function which generates the central moments of the Poisson distribution with parameter λ is

$$M(t) = \exp\{\lambda(e^t - 1 - t)\}$$

Show that it satisfies the equation

$$\frac{dM(t)}{dt} = \lambda t\,M(t) + \lambda\,\frac{dM(t)}{d\lambda}$$

19. (*a*) The random variable X has p.d.f.

$$f(x) = e^{-\theta}\,\frac{\theta^x}{x!};\ x = 0, 1, 2, \ldots$$
$$= 0, \text{ elsewhere}$$

Find the m.g.f. of $Y = 2X - 1$ and Var (Y).

(*b*) Identify the distribution with the following mgf's :

$$M_X(t) = (0{\cdot}3 + 0{\cdot}7\,e^t)^{10}$$
$$M_Y(t) = \exp[3(e^t - 1)]$$

Ans. $X \sim B(10, 0{\cdot}7)$, $Y \sim P(3)$.

20. If X has Poisson distribution with parameter λ, then

$$P[X \text{ is even}] = \frac{1}{2}\left[1 + e^{-2\lambda}\right]$$

[Delhi Univ. B. Sc. (Stat. Hons.) 1991]

21. (*a*) The m.g.f. of a r.v. is X is $\exp[4(e^t - 1)]$. Show that

$P(\mu - 2\sigma < X < \mu + 2\sigma) = 0{\cdot}931$

Hint. $X \sim P(\lambda = 4)$;

Required Probability $= P(0 < X < 8) = P(1 \le X \le 7) = 0{\cdot}931$

(*b*) If $X \sim P(\lambda = 100)$, use Chebychev's inequality to determine a lower bound for $P(75 < X < 125)$ [**Ans.** $0{\cdot}84$]

22. If $X \sim P(m)$, show that $E|X - 1| = m - 1 + 2e^{-m}$

[Delhi Univ. B. Sc. (Maths. Hons.), 1983]

Hint. $E|X-1| = \sum_{x=0}^{\infty} |x-1|\,e^{-m}\,m^x/x! = e^{-m} + \sum_{x=2}^{\infty} \frac{(x-1)}{x!}\cdot e^{-m} m^x$

$$= e^{-m} + e^{-m}\cdot\sum_{x=2}^{\infty} m^x \left[\frac{1}{(x-1)!} - \frac{1}{x!}\right]$$

23. If $X \sim P(\lambda)$ and $Y|X = x \sim (B(x, p)$, then prove that $Y \sim P(\lambda p)$.

24. If the chances of 0, 1, 2, 3... events from one source are given by a Poisson distribution of mean m_1 and the chances of 0, 1, 2, 3,... events from another source by a Poisson distribution of mean m_2, show that the chances of 0, 1, 2, 3,... events from either source are given by

$$e^{-(m_1+m_2)}\left\{1, (m_1 + m_2), \frac{(m_1 + m_2)^2}{2!}, \ldots\right\}.$$

Show that the sum of any finite number of Poisson variates is itself a Poisson variate with mean equal to the sum of separate means.

25. X is a Poisson variate with mean λ.

Show that $E(X^2) = \lambda E(X + 1)$.

If $\lambda = 1$, show that $E|X - 1| = \dfrac{2}{e}$

26. Show that the mean deviation about mean for Poisson distribution

$$p(x) = \frac{e^{-m} m^x}{x!}; \; x = 0, 1, 2, \ldots$$

is $(2\mu) \cdot \dfrac{e^{-m} \cdot m^{\mu}}{\mu!}$

where μ is the greatest integer contained in $(m + 1)$.

[Delhi Univ. B. Sc. (Stat. Hons.), 1988,' 84]

27. Let X, Y be independent Poisson variates. The variance of $X + Y$ is 9 and

$$P(X = 3 | X + Y = 6) = 5/54$$

Find the mean of X. [Ans. $\frac{1}{2}(9 \pm 3\sqrt{3})$ *i.e.* 1·902 *or* 7·098]

28. If X is a Poisson variate with paramter m, show that

$$P(X < r) < \frac{m^r}{r!}; \; r = 0, 1, 2, \ldots$$

Deduce that $E(X) < e^m$. **[Delhi Univ. B.Sc. (Maths. Hons.), 1989]**

29. (*a*) The characteristic function of a variate X is

$$\varphi_X(t) = \left(\frac{1}{3} + \frac{2}{3}e^{it}\right)^6 \cdot \left[\exp\left\{-3(1 - e^{it})\right\}\right].$$

Recognise the variate.

[Burdwan Univ. B. Sc. (Maths. Hons.) 1989)

Hint. $X = U + V$, where $U \sim B\left(6, \frac{2}{3}\right)$ and $V \sim P(3)$ are independent *r.v.'s*

(*b*) Identify the variates X and Y where :

$$M_X(t) = (1/27)(1 + 2e^t)^3 \cdot \exp\left[3(e^t - 1)\right]$$

$$M_Y(t) = (1/32)(1 + e^t)^5 \cdot \exp\left[-2(1 - e^t)\right]$$

[Delhi Univ. B. Sc. (Stat. Hons.), 1987, 84]

Ans. $X = U + V$; $U \sim B(n = 3, p = 2/3)$ and $V \sim P(\lambda = 3)$ are independent.

$Y = U_1 + V_1$; $U_1 \sim B(n = 5, p = 1/2)$ and $V_1 \sim P(\lambda = 2)$ are independent.

30. If X and Y are correlated variates each having Poisson distribution, show that $X + Y$ cannot be a Poisson variate

[Delhi Univ. B. Sc. (Maths Hons.), 1988; Poona Univ. B.Sc., 1989]

Hint. Note that for Poisson variate mean and variance are equal. Let $X \sim P(\lambda)$, $Y \sim P(\mu)$; (X, Y) correlated.

$$\therefore E(X + Y) = E(X) + E(Y) = \lambda + \mu$$

$$\text{Var}(X + Y) = \text{Var}\,X + \text{Var}\,Y + 2\,\text{Cov}(X, Y)$$

$$= \lambda + \mu + 2\rho\sqrt{\lambda\mu}, \ (\rho \neq 0)$$

Since $E(X + Y) \neq \text{Var}(X + Y)$; $X + Y$ cannot be a Poisson variate.

31. Let X, Y, Z be independent Poisson variates with parameters a, b and c respectively. Obtain :

(i) m.g.f. of $X + 2Y + 3Z$,

(ii) Conditional expectation of X given $X + Y + Z = n$

(Indian Civil Services, 1985)

Hint. $M_{X + 2Y + 3Z}(t) = M_X(t) \cdot M_Y(2t) \cdot M_Z(3t)$

$$= \exp\left[a\left(e^t - 1\right) + b\left(e^{2t} - 1\right) + c\left(e^{3t} - 1\right)\right]$$

$$P(X = x \mid X + Y + Z = n) = \frac{P(X = x \cap X + Y + Z = n)}{P(X + Y + Z = n)}$$

$$= \frac{P(X = x)\,P(Y + Z = n - x)}{P(X + Y + Z = n)} \quad (\because X, Y, Z \text{ are indep.})$$

$$= \frac{e^{-a} \cdot a^x}{x!} \times \frac{e^{-(b+c)} \cdot (b + c)^{n-x}}{(n - x)!} \times \left[\frac{n!}{e^{-(a+b+c)} \cdot (a + b + c)^n}\right]$$

$$= \frac{n!}{x!\,(n - x)!}\left(\frac{a}{a + b + c}\right)^x \cdot \left(\frac{b + c}{a + b + c}\right)^n$$

$$\Rightarrow \quad X \mid (X + Y + Z = n) \sim B\{n, p = a/(a + b + c)\}$$

$$\Rightarrow \quad E[X \mid X + Y + Z = n] = np = \frac{na}{a + b + c}$$

32. The joint density of r.v.'s X and Y is :

$$f(x, y) = e^{-2} / [x!\,(y - x)!] ; \ y = 0, 1, 2, \ldots ; \ x = 0, 1, 2, \ldots, y.$$

Find the m.g.f. $M(t_1, t_2)$ of (X, Y) and correlation coefficient between X and Y. Show that the marginal distributions of X and Y are Poisson.

Hint. $$M(t_1, t_2) = \sum_{y=0}^{\infty} \sum_{x=0}^{y} e^{t_1 x + t_2 y} \times \left[\frac{e^{-2}}{x!(y-x)!} \right]$$

$$= e^{-2} \sum_{y=0}^{\infty} \left[\frac{e^{t_2 y}}{y!} \left\{ \sum_{x=0}^{y} {}^yC_x \cdot (e^{t_1})^x \right\} \right]$$

$$= e^{-2} \sum_{y=0}^{\infty} \left\{ \left[e^{t_2} \left(1 + e^{t_1} \right) \right]^y \Big/ y! \right\}$$

$$= e^{-2} \cdot \exp\left\{ e^{t_2} (1 + e^{t_1}) \right\}$$

$$M(t_1, 0) = \exp\left[2(e^{t_1} - 1) \right] \Rightarrow X \sim P(\lambda = 1)$$

$$M(0, t_2) = \exp\left[2(e^{t_2} - 1) \right] \Rightarrow Y \sim P(\mu = 2)$$

Observe $M(t_1, t_2) \neq M(t_1, 0) \times M(0, t_2) \Rightarrow X$ and Y are not independent.

$E(X) = 1$, $\text{Var}(X) = 1$; $E(Y) = 2 = \text{Var}\, Y$.

$$E(XY) = \left| \frac{\partial^2 M(t_1, t_2)}{\partial t_1 \partial t_2} \right|_{t_1 = t_2 = 0} = 3$$

$$\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{3 - 1 \times 2}{1 \times \sqrt{2}} = 1/\sqrt{2}.$$

33. The joint p.g.f. of the r.v.'s X and Y is given by:

$$P(s_1, s_2) = \exp\left[a(s_1 - 1) + b(s_2 - 1) + c(s_1 - 1)(s_2 - 1) \right],$$

a,b,c, are all positive. Find $\rho(X, Y)$

Hint. $P_X(s_1) = P(s_1, 1) = \exp[a(s_1 - 1)] \Rightarrow X \sim P(a)$

$P_Y(s_2) = P(1, s_2) = \exp[b(s_2 - 1)] \Rightarrow Y \sim P(b)$

$$E(XY) = \left(\frac{\partial^2 P(s_1, s_2)}{\partial s_1 \partial s_2} \right)_{s_1 = s_2 = 1} = c + ab.$$

$$\rho_{XY} = \frac{\text{Cov}(X, Y)}{\sigma_X \sigma_Y} = \frac{(c + ab) - ab}{\sqrt{a}\sqrt{b}} = \frac{c}{\sqrt{ab}}$$

34. An insurance company issues only two types of policy, household and motor. It has carried out an investigation into the experience of a group of policyholders who held one of each type of policy over a particular period and it has discovered that within that group and over that period the mean number of claims per household policy was 0·3 and the mean number of claims per motor policy was 0·8. Assume that the number of claims under each type of policy is independent of the number of claims under the other type of policy and that each can be represented by a Poisson distribution.

(*a*) If the number of claims per policyholder is the sum of the number of claims under each of his two policies, state with reasons how the number of claims per policyholder, within that group and over that period is distributed, and

(b) Calculate to the nearest whole number, the percentage of policyholders within that group and over that period who made more household claims than motor claims.

Hint. Household claim, $X \sim P(\cdot 3)$ and Motor claim, $Y \sim P(\cdot 8)$

$$\text{Required Probability} = P(X > Y) = \sum_{r=0}^{\infty}\left[\sum_{s=0}^{\infty} P(Y = r \cap X = r + s)\right]$$

$$= \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\left[P(Y=r)P(X=r+s)\right] = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\frac{e^{-\cdot 8}(\cdot 8)^r}{r!} \times \frac{e^{-\cdot 3}(\cdot 3)^{r+s}}{(r+s)!}$$

$$= e^{-\cdot 8}\, e^{-\cdot 3}\sum_{r=0}^{\infty}\left[\frac{(\cdot 8)^r}{r!}\sum_{s=0}^{\infty}\left\{\frac{(\cdot 3)^{r+s}}{(r+s)!}\right\}\right]$$

$$= \sum_{r=0}^{\infty}\left[\frac{e^{-\cdot 8}(\cdot 8)^r}{r!}\, e^{-\cdot 3}\left\{e^{\cdot 3} - \left(1 + \cdot 3 + \frac{(\cdot 3)^2}{2!} + \dots + \frac{(\cdot 3)^{r-1}}{(r-1)!}\right)\right\}\right]$$

$$= 1 - e^{-\cdot 8}e^{-\cdot 3}\left[\left\{\frac{\cdot 8}{1} + \frac{(\cdot 8)^2}{2!}(1 + \cdot 3) + \frac{(\cdot 8)^3}{3!}\left(1 + \cdot 3 + \frac{\cdot 09}{2}\right)\right.\right.$$

$$\left.\left. + \frac{(\cdot 8)^4}{4!}\left(1 + \cdot 3 + \frac{\cdot 09}{2} + \frac{\cdot 027}{3!}\right) + \dots\right]\right.$$

35. (i) An event occurs instantaneously and is equally likely to occur at any instant. There is no limit on the number of occurrneces that may happen in any interval of time, but the expected number in a given time interval is T. Prove that the probability of the event occurring exactly r times in an interval of the same duration is $(T^r e^{-T})/r\,!$.

(ii) An insurance company which writes only fire and accident business defines a major claim as one which costs at least Rs. 50,000 for an accident claim or Rs. 100,000 for a fire claim. Any excess over these amounts is paid by reinsurers and hence every major claim is recorded at a cost of Rs. 50,000 or Rs. 100,000 respectively. The company divides the year into equal monthly accounting periods and a report is produced of the recorded cost of major claims. The expected number of major accident claims is 0·2 per month and of major fire claims 0·5 per month. Calculate the probability that in a particular month the recorded cost of major claims is Rs. 2,00,000 or more.

36. (a) The number of aeroplanes arriving at an airport in a 30 minute interval obeys the Poisson law with mean 25. Use Chebychev's inequality to find the least chance, that the number of planes to arrive within a given 30 minutes interval will be between 15 and 35. **[Sri Venketeswara U. B.Sc. 1992]**

(b) Suppose that the number of motor cars arriving in a certain parking lot in any 15 minutes period obeys a Poisson probability law with mean 80. Use Chebychev's inequality to determine a lower bound for the probability that the

CHAPTER EIGHT

Theoretical Continuous Distributions

8·1. Rectangular (or Uniform Distribution. A random variable X is said to have a continuous uniform distribution over an interval (a, b) if its probability density function is constant $= k$ (say), over the entire range of X, *i.e.*,

$$f(x) = \begin{cases} k, a < x < b \\ 0, \text{ otnerwise} \end{cases}$$

Since total probability is always unity, we have

$$\int_a^b f(x)\, dx = 1 \Rightarrow k\int_a^b dx = 1 \quad i.e., \quad k = \frac{1}{b-a}$$

$$\therefore \qquad f(x) = \begin{cases} \dfrac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases} \qquad \ldots(8\cdot1)$$

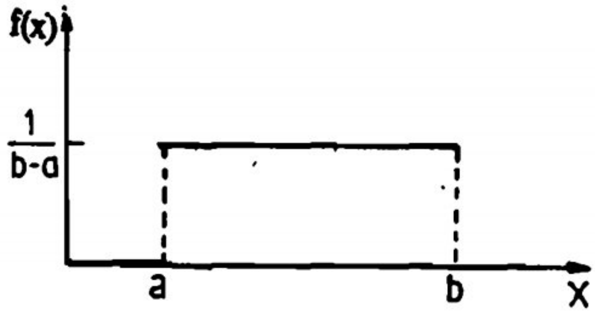
Remarks. 1. a and b, $(a < b)$ are the two parameters of the uniform distribution on (a, b).

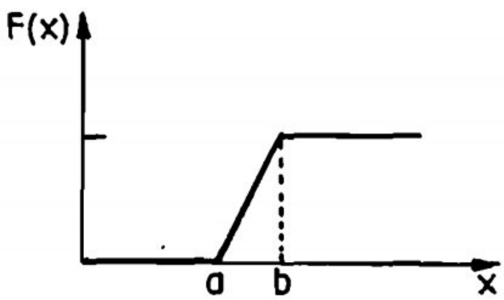
2. The distribution is also known as rectangular distribution, since the curve $y = f(x)$ describes a rectangle over the x-aixs and between the ordinates at $x = a$ and $x = b$.

3. The distribution function $F(x)$ is given by

$$F(x) = \begin{cases} 0, \text{ if } -\infty < x < a \\ \dfrac{x-a}{b-a}, \; a \le x \le b \\ 1, \quad b < x < \infty \end{cases} \qquad \ldots(8\cdot1\,a)$$

Since $F(x)$ is not continuous at $x = a$ and $x = b$, it is not differentiable at these points. Thus $\dfrac{d}{dx} F(x) = f(x) = \dfrac{1}{b-a} \neq 0$, exists everywhere except at the points $x = a$ and $x = b$. and consequently p.d.f. $f(x)$ is given by $(8\cdot1)$.





4. The graphs of uniform p.d.f. $f(x)$ and the corresponding distribution function $F(x)$ are given on page 8·1 :

5. For a rectangular or uniform variate X in $(-a, a)$, p.d.f. is given by

$$f(x) = \begin{cases} \dfrac{1}{2a}, & -a < x < a \\ 0, & \text{otherwise.} \end{cases}$$

8·1·1. Moments of Rectangular Distribution.

$$\mu'_r = \int_a^b x^r f(x)\,dx = \frac{1}{(b-a)}\int_a^b x^r\,dx = \frac{1}{(b-a)}\left[\frac{b^{r+1}-a^{r+1}}{r+1}\right] \quad \ldots(8\cdot2)$$

In particular

$$\text{Mean} = \mu_1' = \frac{1}{(b-a)}\left[\frac{b^2-a^2}{2}\right] = \frac{\boldsymbol{b+a}}{\boldsymbol{2}}$$

and
$$\mu_2' = \frac{1}{(b-a)}\left[\frac{b^3-a^3}{3}\right] = \frac{1}{3}(b^2+ab+a^2)$$

$$\therefore \qquad \mu_2 = \mu_2' - \mu_1'^2 = \frac{1}{3}(b^2+ab+a^2) - \left(\frac{b+a}{2}\right)^2 = \frac{1}{12}(b-a)^2$$

8·1·2. Moment Generating Function is given by

$$M_X(t) = \int_a^b e^{tx} f(x)\,dx = \frac{e^{bt}-e^{at}}{t(b-a)}$$

8·1·3. Characteristic Function is given by

$$\varphi_X(t) = \int_a^b e^{itx} f(x)\,dx = \frac{e^{ibt}-e^{iat}}{it(b-a)}$$

8·1·4. Mean Deviation about Mean, η is given by

$$\eta = E\,|X - \text{Mean}| = \int_a^b |x - \text{Mean}|\, f(x)\,dx$$

$$= \frac{1}{(b-a)}\int_a^b \left|x - \frac{a+b}{2}\right| dx$$

$$= \frac{1}{(b-a)}\int_{-(b-a)/2}^{(b-a)/2} |t|\,dt \qquad \left[t = x - \frac{a+b}{2}\right]$$

$$= \frac{1}{(b-a)}\cdot 2\int_0^{(b-a)/2} t\,dt = \frac{b-a}{4}$$

Example 8·1 *If X is uniformly distributed with mean 1 and variance 4/3, find $P(X < 0)$.* **[Delhi Univ. B.A. (Hons. Spl. Course-Statistics), 1989]**

Solution. Let $X \sim U[a, b]$, so that $p(x) = \frac{1}{b-a}$; $a < x < b$. We are given:

$$\text{Mean} = \frac{b+a}{2} = 1 \quad \Rightarrow \quad b + a = 2$$

$$\text{Var}(X) = \frac{1}{12}(b-a)^2 = \frac{4}{3} \quad \Rightarrow \quad (b-a)^2 = 16 \quad \Rightarrow \quad b - a = \pm 4$$

Solving, we get : $a = -1$ and $b = 3$; $(a < b)$.

$$\therefore \qquad p(x) = \frac{1}{4}\ ;\ -1 < x < 3$$

$$P(X < 0) = \int_{-1}^{0} p(x)\, dx = \frac{1}{4} \left| x \right|_{-1}^{0} = \frac{1}{4}$$

Example 8·2. *Subway trains on a certain line run every half hour between mid-night and six in the morning. What is the probability that a man entering the station at a random time during this period will have to wait at least twenty minutes?*

Solution. Let the *r.v.* X denote the waiting time (in minutes) for the next train. Under the assumption that a man arrives at the station at random, X is distributed uniformly on (0, 30), with *p.d.f.*,

$$f(x) = \begin{cases} \frac{1}{30}, & 0 < x < 30 \\ 0, & \text{otherwise} \end{cases}$$

The probability that he has to wait at least 20 minutes is

$$P(X \geq 20) = \int_{20}^{30} f(x)\, dx = \frac{1}{30}\int_{20}^{30} 1 \cdot dx = \frac{1}{30}(30 - 20) = \frac{1}{3}$$

Example 8·3. *If X has a uniform distribution in [0, 1], find the distribution (p.d.f.) of $-2 \log X$. Identify the distribution also.*

[Delhi Univ. B.Sc. (Stat. Hons.), 1989, '86]

Solution. Let $Y = -2 \log X$. Then the distribution function G of Y is

$$\begin{aligned} G_Y(y) &= P(Y \leq y) = P(-2 \log X \leq y) \\ &= P(\log X \geq -y/2) = P(X \geq e^{-y/2}) = 1 - P(X \leq e^{-y/2}) \\ &= 1 - \int_0^{e^{-y/2}} f(x)\, dx = 1 - \int_0^{e^{-y/2}} 1 \,.\, dx = 1 - e^{-y/2} \end{aligned}$$

$$g_Y(y) = \frac{d}{dy} G(y) = \frac{1}{2} e^{-y/2}, \; 0 < y < \infty \qquad \ldots(*)$$

[$\because$ as X ranges in (0, 1), $Y = -2 \log X$ ranges from 0 to ∞]

Remark. This example illustrates that if $X \sim U[0, 1]$, then $Y = -2 \log X$, has an exponential distribution with parameter $\theta = \frac{1}{2}$. [*c.f.* § 8·6] or $Y = -2 \log X$ has chi-square distribution with $n = 2$ degrees of freedom [*c.f.* Chapter 13, § 13·2].

Example 8·4. *Show that for the rectangular distribution :*

$$f(x) = \frac{1}{2a}, -a < x < a$$

the m.g.f. about origin is $\frac{1}{at}$ (*sinh at*) . *Also show that moments of even order are given by* $\mu_{2n} = \dfrac{a^{2n}}{(2n+1)}$

Solution. M.G.F. about origin is given by

$$M_X(t) = E(e^{tX}) = \int_{-a}^{a} e^{tx} f(x)\, dx = \frac{1}{2a} \int_{-a}^{a} e^{tx}\, dx$$

$$= \frac{1}{2a} \left| \frac{e^{tx}}{t} \right|_{-a}^{a} = \frac{1}{2at}(e^{at} - e^{-at}) = \frac{\sinh at}{at}$$

$$= \frac{1}{at}\left[at + \frac{(at)^3}{3!} + \frac{(at)^5}{5!} + \dots \right] = 1 + \frac{a^2 t^2}{3!} + \frac{a^4 t^4}{5!} + \dots$$

Since there are no terms with odd powers of t in $M(t)$, all moments of odd order about origin vanish, *i.e.*,

$$\mu'_{2n+1} \text{ (about origin)} = 0$$

In particular μ_1' (about origin) $= 0$, *i.e.*, mean $= 0$

Thus μ_r' (about origin) $= \mu_r$ (since mean is origin)

Hence $\mu_{2n+1} = 0$; $n = 0, 1, 2, \dots$

i.e., all moments of odd order about mean vanish. The moments of even order are given by

$$\mu_{2n} = \text{coefficient of } \frac{t^{2n}}{(2n)!} \text{ in } M(t) = \frac{a^{2n}}{(2n+1)}$$

Example 8·5. *If X_1 and X_2 are independent rectangular variates on [0, 1], find the distributions of*

(*i*) X_1/X_2, (*ii*) $X_1 X_2$, (*iii*) $X_1 + X_2$, and (*iv*) $X_1 - X_2$

Solution. We are given

$$f_{X_1}(x_1) = f_{X_2}(x_2) = 1 \ ; 0 < x_1 < 1, 0 < x_2 < 1$$

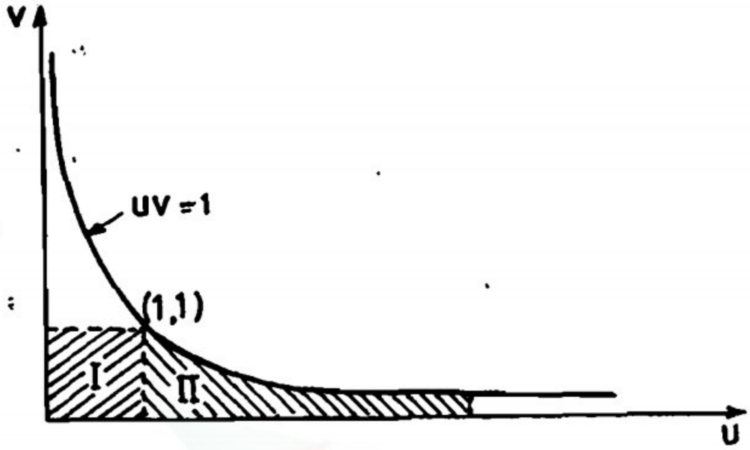
Since X_1 and X_2 are independent, their joint p.d.f. is

$$f(x_1, x_2) = f(x_1) f(x_2) = 1$$

(*i*) Let us transform to

$$u = \frac{x_1}{x_2}, \; v = x_2 \quad i.e., \quad x_1 = uv, \; x_2 = v$$

$$J = \frac{\partial (x_1, x_2)}{\partial (u, v)} = \begin{vmatrix} v & 0 \\ u & 1 \end{vmatrix} = v$$



$x_1 = 0$ maps to $u = 0, v = 0$

$x_1 = 1$ maps to $uv = 1$ (Rectangular hyperbola)

$x_2 = 0$ maps to $v = 0$ and $x_2 = 1$ maps to $v = 1$.

The joint p.d.f. of U and V becomes

$$g(u, v) = f(x_1, x_2) \, |J| = v \, ; 0 < u < \infty, 0 < v < \infty$$

To obtain the marginal distribution of U, we have to integrate out v.

In region (I) ,

$$g_1(u) = \int_0^1 v \, dv = \left| \frac{v^2}{2} \right|_0^1 = \tfrac{1}{2}, \; 0 \leq u \leq 1$$

In region (II) ,

$$g_1(u) = \int_0^{1/u} v \, dv = \left| \frac{v^2}{2} \right|_0^{1/u} = \frac{1}{2u^2}, \; 1 < u < \infty$$

Hence the distribution of $U = \dfrac{X_1}{X_2}$ is given by

$$g(u) = \tfrac{1}{2}, \; 0 \leq u \leq 1$$

$$= \frac{1}{2u^2}, \; 1 \leq u < \infty$$

(ii) Let $u = x_1 x_2$, $v = x_1$, $i.e.,$ $x_1 = v, x_2 = \dfrac{u}{v}$

$$J = \begin{vmatrix} 0 & 1 \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = -\frac{1}{v}$$

$x_1 = 0$ maps to $v = 0, x_1 = 1$ maps to $v = 1$

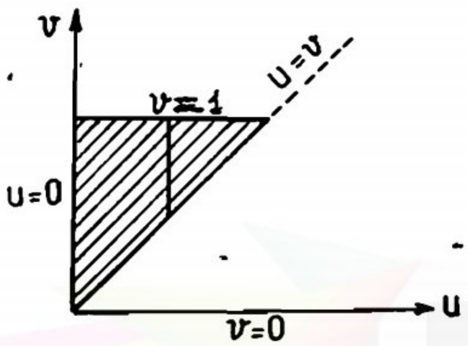
$x_2 = 0$ maps to $u = 0$, and $x_2 = 1$ maps to $u = v$

Moreover, $v = \dfrac{u}{x_2} \Rightarrow v \geq u$ (since $0 < x_2 < 1$),

The joint p.d.f. of U and V is

$$g(u, v) = f(x_1, x_2) \mid J \mid = \frac{1}{v} \; ; 0 < u < 1, 0 < v < 1$$

$$g(u) = \int_u^1 \frac{1}{v} \, dv = \left[\log v \right]_u^1 = -\log u, \; 0 < u < 1$$



(iii) and (iv). Let $u = x_1 + x_2$, $v = x_1 - x_2$

i.e., $x_1 = \dfrac{u+v}{2}$, $x_2 = \dfrac{u-v}{2}$

$x_1 = 0 \Rightarrow u + v = 0$ i.e., $v = -u$

$x_2 = 0 \Rightarrow u - v = 0$ i.e., $v = u$

$x_1 = 1 \Rightarrow u + v = 2$

$x_2 = 1 \Rightarrow u - v = 2$

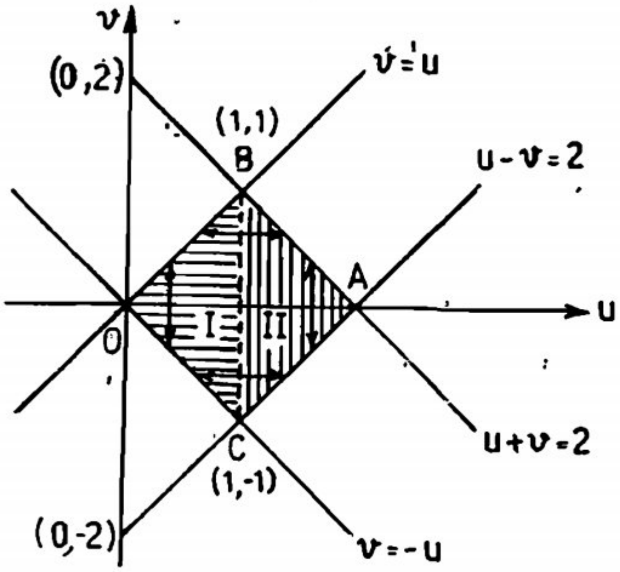
$$\text{and } J = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

$$\therefore \; g(u, v) = f(x_1, x_2) \mid J \mid = \tfrac{1}{2}, \; 0 < u < 2, -1 < v < 1$$

In region (I), (see figure below)

$$g_1(u) = \int_{-u}^{u} \tfrac{1}{2} \, dv = \tfrac{1}{2} \left| \, v \, \right|_{-u}^{u} = u$$

and in region (II),



$$g_2(u) = \int_{u-2}^{2-u} \tfrac{1}{2}\, dv = \tfrac{1}{2} \left| v \right|_{u-2}^{2-u} = 2-u$$

$$\therefore \quad g(u) = \begin{cases} u, & 0 < u < 1 \\ 2-u, & 1 < u < 2 \end{cases}$$

For the distribution of V, we split the region as: OAB and OAC

In region OAB:

$$h_1(v) = \int_{v}^{2-v} \tfrac{1}{2}\, du = \tfrac{1}{2}[2 - v - v] = 1 - v, \; 0 < v < 1$$

In region OAC:

$$h_2(v) = \int_{-v}^{2+v} \tfrac{1}{2}\, du = \tfrac{1}{2}[2(1+v)] = 1 + v, \; -1 < v < 0$$

Hence the distribution of $V = X_1 - X_2$ is given by

$$h(v) = \begin{cases} 1 - v, & 0 < v < 1 \\ 1 + v, & -1 < v < 0 \end{cases}$$

Example 8·6. *If X is a random variable with a continuous distribution function F, then $F(X)$ has a uniform distribution on $[0, 1]$.*

[Delhi Univ. B.Sc. (Stat. Hons.), 1992, 1987,' 85]

Solution. Since F is a distribution function, it is non-decreasing. Let $Y = F(X)$, then the distribution function G of Y is given by

$$G_Y(y) = P(Y \le y) = P[F(X) \le y] = P[X \le F^{-1}(y)],$$

the inverse exists, since F is non-decreasing and given to be continuous.

$$\therefore \quad G_Y(y) = F[F^{-1}(y)],$$

since F is the distribution function of X.

$$\therefore \quad G_Y(y) = y$$

Therefore the p.d.f. of $Y = F(X)$ is given by:

$$g_Y(y) = \frac{d}{dy}[G_Y(y)] = 1$$

Since F is a d.f., Y takes the values in the range $[0, 1]$.

Hence $g_Y(y) = 1, 0 \le y \le 1$

$\Rightarrow$ Y is a uniform variate on $[0, 1]$.

Remark. Suppose X is a random variable with p.d.f.,

$$f_X(x) = \begin{cases} e^{-x}, & x \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

then

$$F(x) = \begin{cases} 0, & \text{if } x < 0 \\ 1 - e^{-x}, & \text{if } x \ge 0 \end{cases}$$

Then by above result $F(X) = 1 - e^{-X}$ is uniformly distributed on $[0, 1]$.

Example 8·7. *If X and Y are independent rectangular variates for the range $-a$ to a each, then show that the sum $X + Y = U$, has the probability density*

$$\varphi(u) = \frac{2a + u}{4a^2}, \; -2a \le u \le 0$$

$$\varphi(u) = \frac{2a - u}{4a^2}, \; 0 \le u \le 2a$$

Solution. Since X and Y are independent rectangular variates, each in the interval $(-a, a)$, we have

$$f_1(x) = \begin{cases} \dfrac{1}{2a}, & -a < x < a \\ 0, & \text{elsewhere} \end{cases}$$

and
$$f_2(y) = \begin{cases} \dfrac{1}{2a}, & -a < y < a \\ 0, & \text{elsewhere} \end{cases}$$

Hence by compound probability theorem, the joint probability differential of X and Y is given by

$$dP(x, y) = f_1(x) f_2(y)\, dx\, dy = \frac{1}{4a^2}\, dx\, dy, \; -a < (x, y) < a$$

Let us define new variables U and V as follows :

$$u = x + y, \qquad v = x - y$$

$$\Rightarrow \qquad x = \frac{u+v}{2} \quad \text{and} \quad y = \frac{u-v}{2}$$

Jacobian of the transformation J is given by

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ \dfrac{1}{2} & -\dfrac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

Thus the probability differential of U and V becomes

$$dG(u, v) = \frac{1}{4a^2} | J |\, du\, dv = \frac{1}{8a^2}\, du\, dv \qquad \text{...(*)}$$

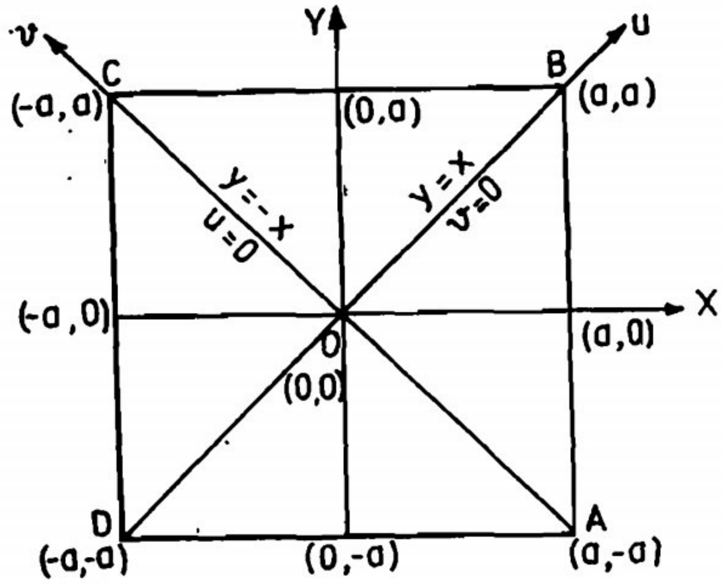
Integrating w.r.to. v over specified range, we can find the distribution of U.

Let us consider the region to the left of $v-$ axis, *i.e.*, to the left of the line AC. In this region, the values of v are bounded by the lines $x = -a$ and $y = -a$.

For fixed values of u,

$$x = -a \quad \Rightarrow \quad \frac{u+v}{2} = -a \quad \Rightarrow \quad v = -(u + 2a)$$

and
$$y = -a \quad \Rightarrow \quad \frac{u-v}{2} = -a \quad \Rightarrow \quad v = (u + 2a)$$



Thus integrating (*) w.r.to. v between the limits $-(u+2a)$ and $(u+2a)$, the distribution of U becomes

$$g_1(u)\,du = \int_{-(u+2a)}^{u+2a} \frac{1}{8a^2}\,.\,du\,dv = \frac{1}{8a^2}\,|\,v\,|_{-(u+2a)}^{u+2a}\; du = \frac{u+2a}{4a^2}\,du$$

In the region to the left of v-axis, *i.e.*, below the line AC, u varies from the points $(x=-a,\ y=-a)$ to the point $(x=0, y=0)$ and since $u=x+y$, in this region u lies between $(-a-a)$ and $(0+0)$, *i.e.*, between $-2a$ to 0.

$$\therefore \qquad g_1(u)\,du = \frac{u+2a}{4a^2}\,,\ -2a \le u \le 0$$

In the region to the right of v–axis, *i.e.*, above the line AC, the values of v are bounded by the lines $x=a$ and $y=a$ and for fixed values of u,

$$x=a \quad\Rightarrow\quad \frac{u+v}{2}=a \quad\Rightarrow\quad v=2a-u$$

$$y=a \quad\Rightarrow\quad \frac{u-v}{2}=a \quad\Rightarrow\quad v=-(2a-u)$$

In this region u varies from the point $(x=0, y=0)$ to the point $(x=a, y=a)$, i.e., $u=x+y$ varies from 0 to $2a$. Thus integrating (*) w.r.to. v between the limits $-(2a-u)$ to $(2a-u)$, we get the distribution of U as

$$g_1(u)\,du = \int_{-(2a-u)}^{2a-u} \frac{1}{8a^2}\,du\,dv = \frac{1}{8a^2}\,|\,v\,|_{-(2a-u)}^{2a-u}\; du$$

$$= \frac{2a-u}{4a^2}\,du,\ 0 \le u \le 2a$$

For an alternative and simpler solution, see Remark 5 to § 8·1·5, (Triangular Distribution).

Example. 8·8. *On the x-axis $(n+1)$ points are taken independently between the origin and $x=1$, all positions being equally likely. Show that probability* *that the $(k+1)$ th of these points, counted from the origin, lies in the interval $x - \frac{1}{2}\, dx$ to $x + \frac{1}{2}\, dx$ is*

$$\binom{n}{k}(n+1)\, x^k\ (1-x)^{n-k}\, dx$$

Verify that integral of this expression from $x = 0$ to $x = 1$ is unity.

Solution. Here X is given to be a random variable uniformly distributed on $[0, 1]$.

$$\therefore \qquad f_X(x) = 1,\ 0 \le x \le 1$$

Now $$P(0 < X < x) = \int_0^x f(x)\, dx = \int_0^x 1 \,.\, dx = x \qquad \ldots(1)$$

$$\therefore \qquad P(X > x) = 1 - P(X \le x) = 1 - x \qquad \ldots(2)$$

Also $$P\left(x - \frac{dx}{2} < X < x + \frac{dx}{2}\right) = \int_{x - \frac{dx}{2}}^{x + \frac{dx}{2}} f(x)\, dx = dx \qquad \ldots(3)$$

Required probability 'p' is given by

$p = P\,\{$out of $(n+1)$ points, k points lie in the closed interval $\left[0, x - \frac{dx}{2}\right]$

and out of the remaining $(n+1-k)$ points, $(n-k)$ points lie in

$\left[x + \frac{dx}{2}, 1\right]$ and one point lies in $\left(x - \frac{dx}{2}, x + \frac{dx}{2}\right)\Big\}$

$$= \left[\binom{n+1}{k} x^k\right] \times \left[\binom{n+1-k}{n-k} (1-x)^{n-k}\right] \times dx,$$

on using (1), (2) and (3) respectively.

$$\therefore \qquad p = \frac{(n+1)\,!}{k\,!\,(n+1-k)\,!} \,.\, x^k \,.\, \frac{(n+1-k)\,!}{(n-k)\,!} \,.\, (1-x)^{n-k}\, dx$$

$$= \binom{n}{k}(n+1)\, x^k\, (1-x)^{n-k}\, dx$$

To prove that the area of this expression from $x = 0$ to $x = 1$ is unity, use Beta-integral

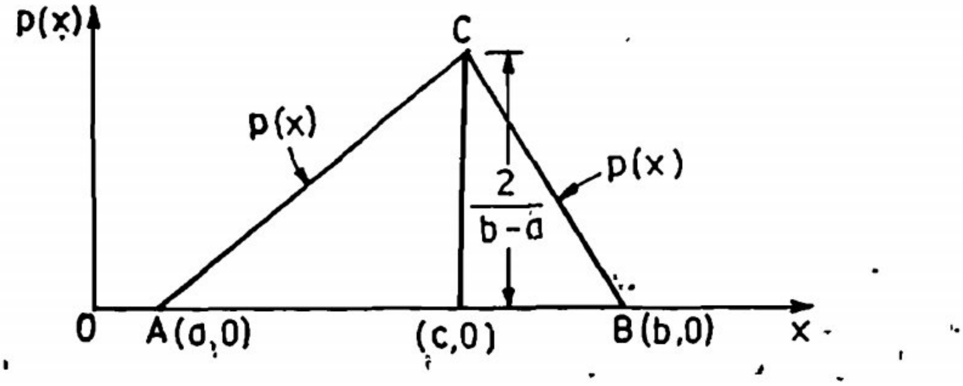
$$\int_0^1 x^{m-1}(1-x)^{n-1}\, dx = B(m, n) = \frac{\Gamma m\ \Gamma n}{\Gamma(m+n)}\,;\ m > 0, n > 0.$$

8·1·5. Triangular Distribution. A random variable X is said to have a triangular distribution in the interval (a, b), if its p.d.f. is given by:

$$f(x) = \begin{cases} 2\,(x-a)/\{(b-a)(c-a)\} & ;\ a < x \le c \\ 2\,(b-x)/\{(b-a)(b-c)\} & ;\ c < x < b \end{cases} \qquad \ldots(8\cdot 2a)$$

Remarks. 1. We write $X \sim$ Trg. (a, b), with peak at $x = c$. The graph of the **p.d.f. is shown in the diagram on page 8·11.**

2. The distribution is so called because the graph of its p.d.f. is a triangle with peak at $x = c$.



3. The m.g.f. of Trg (a, b) variate, with peak at $x = c$ is given by:

$$M_X(t) = \int_a^b e^{tx} f(x)\, dx = \left(\int_a^c + \int_c^b \right) e^{tx} f(x)\, dx.$$

$$= \frac{2}{(b-a)(c-a)} \int_a^c e^{tx}(x-a)\, dx + \frac{2}{(b-a)(b-c)} \int_c^b e^{tx} . (b-x)\, dx$$

$$= \frac{2}{t^2} \left\{ \frac{e^{at}}{(a-b)(a-c)} + \frac{e^{ct}}{(c-a)(c-b)} + \frac{e^{bt}}{(b-a)(b-c)} \right\} ; a < b < c$$

(On integration by parts) ...(8·2*b*)

4. In particular, taking $a = 0$, $c = 1$ and $b = 2$, in (8·2*a*), the p.d.f. of the Trg (0, 2) variate with peak at $x = 1$ is given by:

$$f(x) = \begin{cases} x\,; & 0 \le x \le 1 \\ 2 - x\,; & 1 \le x \le 2 \\ 0, & \text{otherwise} \end{cases} \qquad \text{...(8·2 c)}$$

and its m.g.f. is $\quad M_X(t) = (e^t - 1)^2 / t^2,$ ...(8·2*d*)

which is left as an exercise to the reader.

5. In particular, replacing a by $-2a$, b by $2a$ and c by 0, the p.d.f. of triangular distribution on the interval $(-2a, 2a)$ with peak at $x = 0$ is given by:

$$f(x) = \begin{cases} (2a + x)/4a^2\,; & -2a < x < 0 \\ (2a - x)/4a^2\,; & 0 < x < 2a \end{cases} \qquad \text{...(8·2e)}$$

The m.g.f. of (8·2*e*) is given by :

$$M_X(t) = \int_{-2a}^{2a} e^{tx} f(x)\, dx$$

$$= \frac{1}{4a^2} \left[\int_{-2a}^{0} e^{tx} . (2a + x)\, dx + \int_0^{2a} e^{tx} (2a - x)\, dx \right]$$

$$= \frac{1}{4a^2}\left[e^{tx}\left\{ \frac{2a+x}{t} - \frac{1}{t^2} \right\} \right]_{-2a}^{0} + \frac{1}{4a^2}\left[e^{tx}\left\{ \frac{2a-x}{t} + \frac{1}{t^2} \right\} \right]_{0}^{2a}$$

[On integrating by parts]

$$= \frac{1}{4a^2}\left[-\frac{2}{t^2} + \frac{1}{t^2}\left\{ e^{2at} + e^{-2at} \right\} \right]$$

$$= \frac{1}{4a^2 t^2}\left\{ e^{2at} + e^{-2at} - 2 \right\} = \left[\frac{e^{at} - e^{-at}}{2\,at} \right]^2 \qquad \ldots(8{\cdot}2\,f)$$

Aliter. We may obtain (8·2 *f*) directly from (8·2 *b*) on replacing *a* by $-2a$, *b* by $2a$ and *c* by 0.

Example 8·9. *If X and Y are i.i.d.* $U[-a, a]$ *variates, find the p.d.f. of* $Z = X + Y$ *and identify the distribution.*

Solution. Since X and Y are i.i.d. $U[-a, a]$, we have : [*c.f.* § 8·1·2.],

$$M_X(t) = M_Y(t) = (e^{at} - e^{-at})/(2\,at) \qquad \ldots(*)$$

$$M_{X+Y}(t) = M_X(t)\, M_Y(t) = \left[\frac{e^{at} - e^{-at}}{2at} \right]^2, \qquad \ldots(**)$$

since X and Y are independent.

But (**) is the m.g.f. of Trg $(-2a, 2a)$ variate with peak at $x = 0$

[*c.f.* Remark 5, equation (8·2*f*)]

Hence by uniqueness theorem of m.g.f., $Z = X + Y \sim$ Trg $(-2a, 2a)$ with p.d.f. as given in (8·2 *e*), Remark 5.

Aliter $\quad M_{X+Y}(t) = \dfrac{1}{4a^2 t^2}\left[e^{2at} - 2 + e^{-2at} \right]$ [From (**)]

$$= \frac{2}{t^2}\left[\frac{e^{-2at}}{(-2a-0)(-2a-2a)} + \frac{e^{0.t}}{(0+2a)(0-2a)} + \frac{e^{2at}}{(2a-0)(2a+2a)} \right]$$

which is of the form (8·2 *b*), [*c.f.* Remark 3], with *a* replaced by $-2a$ and *b* replaced by $2a$ and *c* by 0 . Hence $X + Y \sim$Trg $(-2a, 2a)$ with p.d.f. $p(x)$ given in (8·2*e*) .

Remarks 1. The distribution of $X + Y$ has also been obtained in Example 8·7.

2. Similarly we can find the distribution of $X - Y$.

$$M_{X-Y}(t) = M_X(t) \,.\, M_Y(-t) = \left[\frac{e^{at} - e^{-at}}{2at} \right]^2 \qquad \text{[From (*)]}$$

$\Rightarrow X - Y \sim$Trg $(-2a, 2a)$, with peak at $x = 0$.

EXERCISE 8 (a)

1. The bus company A schedules a north bound bus every 30 minutes at a certain bus-stop. A man comes to the stop at a random time. Let the random variable X count the number of minutes he has to wait for the next bus. Assume X has a uniform distribution over the interval (0, 30). This is how we interpret the statement that he enters the station at the random time].

(i) For each $k = 5, 10, 15, 20, 30$ compute the probability that he has to wait at least k minutes for the next bus.

(ii) A competitor, the bus company B is allowed to schedule a north bound bus every 30 minutes at the same station but at least 5 minutes must elapse between the arrivals of the competitive buses. Assume the passengers come at the bus stop at random times and always board the first bus that arrives. Show that the company B can arrange its schedule so that it receives five times as many passengers as that of its competitor.

2. (a) A random variable X has a uniform distribution over $(-3, 3)$, compute

(i) $P(X = 2), P(X < 2), P(|X| < 2)$ and $P(|X - 2| < 2)$

(ii) Find k for which $P(X > k) = 1/3$. **[Gorakhpur Univ. B.Sc. 1992)**

(b) Suppose that X is uniformly distributed over $(-\alpha, +\alpha)$, where $\alpha > 0$. Determine α so that

(i) $P(X > 1) = 1/3$, (ii) $P(X < 1/2) = 0{\cdot}3$ and

(ii) $P(|X| < 1) = P(|X| > 1)$.

Ans. (i) $\alpha = 3$, (ii) $\alpha = 5/6$, (iii) $\alpha = 2$.

(c) Calculate the coefficient of variation for the rectangular distribution in $(0, b)$ given that the probability law of the distribution is

$$P(X \le t) = \frac{t}{b}$$

(d) If X is uniformly distributed over $[1, 2]$, find z so that

$$P(X > z + \mu_x) = \frac{1}{4} \qquad \textbf{(Ans. } Z = \frac{1}{4}).$$

3 (a). If a random variable X has the density function $f(x)$, prove that

$$Y = \int_{-\infty}^{x} f(x)\,dx$$

has a rectangular distribution over (0, 1). If

$$f(x) = \tfrac{1}{2}(x - 1), \quad 1 \le x \le 3$$
$$= 0, \text{ otherwise}$$

determine what interval for Y will correspond to the interval

$$1{\cdot}1 \le X \le 2{\cdot}9.$$

Ans. $y = F(x) = (x-1)^2/4$; $1 \le x \le 3$; $0{\cdot}0025 \le y \le 0{\cdot}9025$

(b). Show that whatever be the distribution function $F(x)$ of a r.v. X,

$$P[a \le F(X) \le b] = b - a, \quad 0 \le (a, b) \le 1.$$

[Delhi Univ. B.Sc. (Stat. Hons), 1986]

Hint. $Y = F(X) \sim U[0, 1]$.

4. (a) For the rectantular distribution,

$$f(x) = \frac{1}{2a}, -a \le x \le a\ .$$
$$= 0, \text{ otherwise},$$

show that the moments of odd order are zero, and $\mu_{2r} = a^{2r}/(2r+1)$.

[Madurai Kamraj Univ. B.Sc. 1992]

(*b*) A distribution is given by

$$f(x)\, dx = \frac{1}{2a}\, dx, -a \le x \le a$$

Find the first four central moments and obtain β_1 and β_2.

[Delhi Univ B.Sc. Oct., 1992; Madras Univ. B.Sc., 1991]

(*c*) For a rectangular distribution

$$dP = k\, dx,\ 1 \le x \le 2,$$

show that Arithmetic mean > Geometric mean > Harmonic mean.

[Vikram Univ. B.Sc. 1993)

(*d*) If the random variable X follows the rectangular distribution with p.d.f.,

$$f(x) = 1/\theta, 0 \le x \le \theta\ ,$$

derive the first four moments and the skewness and kurtosis conffidents of the distribution.

(*e*) Let X and Y be independent variates which are uniformly distributed over the unit interval (0,1). Find the distribution function and the p.d.f. of random variable $Z = X + Y$. Is Z a uniformly distributed variable ? Give reasons.

[Delhi Univ. B.Sc. (Maths. Hons.), 1986]

5. Let X_1 and X_2 be independent random variables unifromly distributed over the interval (0, 1). Find

(*i*) $P(X_1 + X_2 < 0{\cdot}5)$, (*ii*) $P(X_1 - X_2 < 0{\cdot}5)$,

(*iii*) $P(X_1^2 + X_2^2 < 0{\cdot}5)$, (*iv*) $P(e^{-X_1} < 0{\cdot}5)$, and (*v*) $P(\cos \pi X_2 < 0{\cdot}5)$.

Ans. (*i*) 0·125, (*ii*) 0·875. (*iii*) 0·393, (*iv*) $1 - \log 2$, and (*v*) $2/3$.

6. A random variable X is uniformly distributed over (0, 1), find the probability density functions of

(*i*) $Y = X^2 + 1$, and (*ii*) $Z = 1/(X + 1)$.

7. (*a*) If the random variable X is uniformly distributed over $(0, \frac{1}{2}\pi)$, compute the expectation of the function $\sin X$. Also find the distribution of $Y = \sin X$, and show that the mean of this distribution is the same as the above expectation.

Ans. $2/\pi$, $f_Y(y) = 2/(\pi\sqrt{1-y^2})$, $0 < y < 1$.

(*b*) If $X \sim U[-\pi/2, \pi/2]$ distributed, find the p.d.f. of $Y = \tan X$.

[Delhi Univ. B.A. Hons. (Spl. Course-Statistics), 1989]

8. (*a*) Show that for the rectangular distribution :

$$dF = dx, 0 \le x < 1$$

μ'_1 (about origin) = 1/2, variance = 1/12 and mean deviation about mean =1/4. **[Madras Univ B.Sc. Sept. 1991; Delhi U. B.Sc. Sept. 1992]**

(*b*) Find the characteristic function of the random variable $Y = \log F(X)$ where $F(X)$ is the distribution function of a random variable X. Evaluate the rth moment of Y.

9. If $X \sim U[0, 1]$, find the distribution of $Y = 1/X$. Find $E(1/X)$, if it exists.

Ans. $g_Y(y) = 1/y^2$; $1 \le y < \infty$; $E(Y) = E(1/X)$ does not exist.

10. Let X be uniformly distributed on $[-1, 1]$. Find the distribution function and hence the p.d.f. of $Y = X^2$. **[Delhi Univ. B.Sc. (Maths. Hons.), 1988]**

11. Let $f_X(x) = 6x(1-x)$; $0 \le x \le 1$. Find y as a function of x such that Y has p.d.f.

$$g(y) = 3(1 - \sqrt{y}); \quad 0 \le y \le 1$$

[Delhi Univ. B.A. Hons. (Spl. Course-Statistics), 1988]

Hint.
$$F(x) = \int_0^x f(x)\,dx = 3x^2 - 2x^3 \sim U[0, 1]$$

$$G(y) = \int_0^y g(y)\,dy = 3y - 2y^{3/2} \sim U[0, 1]$$

Setting $F(x) = G(y)$, we get $y = x^2$.

12. The variates a and b are independently and uniformly distributed in the intervals [0,6] and [0, 9] respectively. Find the probability that $x^2 - ax + b = 0$ has two real roots.

Ans. $P(b \le a^2/4) = \int_{a=0}^{1} \int_{b=0}^{a^2/4} \frac{1}{6 \times 9}\, da\, db = 1/3$.

13. Find the probability that the roots of the equation $x^2 + 2bx + c = 0$ should be real, given that $b \sim U[-\alpha, \alpha]$ and $c \sim U[-\beta, \beta]$ are independent.

Ans. Probability $= P(b^2 \ge c) = 1 - P(b^2 \le c) = 1 - P(|b| \le \sqrt{c})$

$$= \int_{-\beta}^{\beta} \left(\int_{-\sqrt{c}}^{\sqrt{c}} \left(\frac{1}{2\alpha}\right)\left(\frac{1}{2\beta}\right) db \right) dc$$

14. If a, b, c are randomly chosen between 0 and 1, find the probability that the quadratic equation $ax^2 + bx + c = 0$ has real roots.

Ans. Probability $= P(b^2 \ge 4ac) = 1 - \int_0^1 \int_0^1 \int_{b=0}^{\sqrt{4ac}} 1\; da\, dc\, db = \frac{8}{9}$

15. (*a*) Suppose X has a rectangular distribution on $(-1, 1)$. Compute $P\left[\frac{|X - E(X)|}{\sigma_X} \ge 2\right]$ and compare it with the upper bound given by Chebyshev's inequality.

(*b*) Compare the upper bound of the probability,

$$P\{|X - E(X)| \geq 2\sqrt{V(X)}\},$$

obtained from Chebyshev's inequality with exact probability if X is uniformly distributed over $(-1, 3)$.

Ans. (*b*) Probability $\leq 1/4$, Exact Probability $= 0$

16. Two independent variates are each uniformly distributed within the range $-a$ to $+a$. Show that their sum X has a probability density given by

$$f(x) = \frac{2a+x}{4a^2}, \quad -2a \leq x \leq 0$$
$$= \frac{2a-x}{4a^2}, \quad 0 \leq x \leq 2a$$

Verify that the m.g.f. calculated from the value of $f(x)$ is equal to

$$\left(\frac{1}{at} \sinh at\right)^2$$

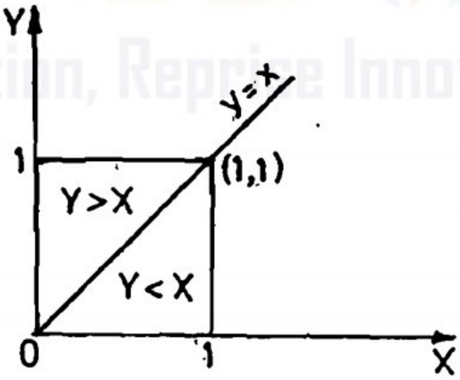
17. The random variables X and Y are independent and both have the uniform distribution on $[0, 1]$. Let $Z = |X - Y|$. Prove that, for real θ,

$$\varphi(Z, \theta) = 2\left[1 + i\theta - e^{i\theta}\right]/2.$$

Hence deduce the general expression for $E(Z^n)$.

Hint. $\varphi(\theta; |X - Y|) = \int_0^1 \int_0^1 e^{i\theta|x-y|} f(x, y)\, dx\, dy$

$$= 2\int_0^1 \left(\int_0^x e^{i\theta(x-y)}\, dy\right) dx$$



Ans. $2/[(n+1)(n+2)]$

18. If X and Y are independently and uniformly distributed random variables in the interval $(0, 1)$, show that the distribution of $X + Y$ is given by the density function

$$f(z) = \begin{cases} z & 0 \leq z < 1 \\ 2 - z & 1 \leq z \leq 2 \\ 0 & \text{elsewhere} \end{cases}$$

[**Hint.** See Triangular distribution]

19. Ship A makes radio signals to the base and the probability of the interval between consecutive signals is uniformly distributed between 4 hours and 24 hours and is zero outside this range. Ship B makes radio signals to the base and the probability of the interval between consecutive signals is uniformly distributed between 10 hours and 15 hours and is zero outside this range.

(*i*) Ship A has just signalled. What is the probability that it will make two further signals in the next 12 hours ?

(*ii*) **Ships** A and B have just signalled at the same time. What is the probability that Ship A will make at least two further signals before ship B next signals? **[Institute of Actuaries (London), April 1978]**

20. If $X \sim U[0, 1]$, prove that for $b<c$ fixed, $Y=(c-b)X+b$ is uniform on $[b, c]$.

8·2. Normal Distribution. The normal distribution was first discovered in 1733 by English mathematician De-Moivre, who obtained this continuous distribution as a limiting case of the binomial distribution and applied it to problems arising in the game of chance. It was also known to Laplace, no later than 1774 but through a historical error it was credited to Gauss, who first made reference to it in the beginning of 19th century (1809), as the distribution of errors in Astronomy. Gauss used the normal curve to describe the theory of accidental errors of measurements involved in the calculation of orbits of heavenly bodies. Throughout the eighteenth and nineteeth centuries, various efforts were made to establish the normal model as the underlying law ruling all continuous random variables. Thus, the name "*normal*". These efforts, however, failed because of false premises. The normal model has, nevertheless, become the most important probability model in statistical analysis.

Definition. *A random variable X is said to ahve a normal distribution with parameters* μ *(called "mean") and* σ^2 *(called "variance") if its density function is given by the probability law :*

$$f(x\,;\mu,\sigma)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left\{\frac{x-\mu}{\sigma}\right\}^2\right]$$

or $$f(x\,;\mu,\sigma)=\frac{1}{\sigma\sqrt{2\pi}}\,e^{-(x-\mu)^2/2\sigma^2}$$

$$-\infty<x<\infty,\ -\infty<\mu<\infty,\ \sigma>0 \quad ...(8\cdot3)$$

Remarks. 1. A random variable X with mean μ and variance σ^2 and following the normal law (8·3) is expressed by $X \sim N(\mu, \sigma^2)$

2. If $X \sim N(\mu, \sigma^2)$, then $Z=\dfrac{X-\mu}{\sigma}$, is a standard normal variate with

$$E(Z)=0 \text{ and } \mathrm{Var}(Z)=1$$

and we write $Z \sim N(0,1)$.

3. The p.d.f. of standard normal variate Z is given by

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} \cdot e^{-z^2/2}, -\infty < z < \infty$$

and the corresponding distribution function, denoted by $\Phi(z)$ is given by

$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \varphi(u)\, du$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2}\, du$$

We shall prove below two important results on the distribution function $\Phi(.)$ of standard normal variate.

Result 1. $\Phi(-z) = 1 - \Phi(z)$

Proof. $\Phi(-z) = P(Z \le -z) = P(Z \ge z)$ (By symmetry)

$$= 1 - P(Z \le z)$$

$$= 1 - \Phi(z)$$

Result 2. $P(a \le X \le b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$, where $X \sim N(\mu, \sigma^2)$

Proof. $P(a \le X \le b) = P\left(\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right)$; $\left(Z = \frac{X-\mu}{\sigma}\right)$

$$= P\left(Z \le \frac{b-\mu}{\sigma}\right) - P\left(Z \le \frac{a-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

4. The graph of $f(x)$ is a famous '*bell-shaped*' curve. The top of the bell is directly above the mean μ. For large values of σ, the curve tends to flatten out and for small values of σ, it has a sharp peak.

8·2·1. Normal Distribution as a Limiting form of Binomial Distribution. Normal distribution is another limiting form of the binomial distribution under the following conditions :

(*i*) n, the number of trials is indefinitely large, *i.e.*, $n \to \infty$ and

(*ii*) neither p nor q is very small.

The probability function of the binomial distribution with parameters n and p is given by

$$p(x) = \binom{n}{x} p^x q^{n-x} = \frac{n!}{x!\,(n-x)!}\, p^x q^{n-x}\, ; x = 0, 1, 2, \ldots, n \qquad \ldots(*)$$

Let us now consider the standard binomial variate :

$$Z = \frac{X - E(X)}{\sqrt{V(X)}} = \frac{X - np}{\sqrt{npq}}\, ; X = 0, 1, 2, \ldots, n \qquad \ldots(**)$$

When $X = 0, Z = \frac{-np}{\sqrt{npq}} = -\sqrt{np/q}$

and when $X = n$, $Z = \dfrac{n - np}{\sqrt{npq}} = \sqrt{nq/p}$

Thus in the limit as $n \to \infty$, Z takes the values from $-\infty$ to ∞. Hence the distribution of X will be a continuous distribution over the range $-\infty$ to ∞.

We want the limiting form of (*) under the above two conditions. Using Stirling's approximation to $r!$ for large r, *viz.*,

$$\lim_{r \to \infty} r! \simeq \sqrt{2\pi}\, e^{-r}\, r^{r + (1/2)},$$

we have in the limit as $n \to \infty$ and consequently $x \to \infty$,

$$\lim p(x) = \lim \left[\frac{\sqrt{2\pi}\, e^{-n}\, n^{n+\frac{1}{2}}\, p^x\, q^{n-x}}{\sqrt{2\pi}\, e^{-x}\, x^{x+\frac{1}{2}} \sqrt{2\pi}\, e^{-(n-x)}\, (n-x)^{n-x+\frac{1}{2}}} \right]$$

$$= \lim \left[\frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{npq}} \cdot \frac{(np)^{x+\frac{1}{2}}\, (nq)^{n-x+\frac{1}{2}}}{x^{x+\frac{1}{2}}\, (n-x)^{n-x+\frac{1}{2}}} \right]$$

$$= \lim \left[\frac{1}{\sqrt{2\pi}\sqrt{npq}} \left(\frac{np}{x} \right)^{x+\frac{1}{2}} \left(\frac{nq}{n-x} \right)^{n-x+\frac{1}{2}} \right] \quad \text{...(***)}$$

From (**), we have

$$X = np + Z\sqrt{npq} \quad \Rightarrow \quad \frac{X}{np} = 1 + Z\sqrt{q/(np)}$$

Also

$$n - X = n - np - Z\sqrt{npq} = nq - Z\sqrt{npq}$$

$$\therefore \quad \frac{n - X}{nq} = 1 - Z\sqrt{p/(nq)}\ . \text{ Also } dz = \frac{1}{\sqrt{npq}}\, dx.$$

Hence the probability differential of the distribution of Z, in the limit is given from (***) by

$$dG(z) = g(z)\, dz = \lim_{n \to \infty} \left[\frac{1}{\sqrt{2\pi}} \times \frac{1}{N} \right] dz \quad \text{...(8·4)}$$

where $\quad N = \left[\dfrac{x}{np} \right]^{x+\frac{1}{2}} \left[\dfrac{n-x}{nq} \right]^{n-x+\frac{1}{2}}$

$$\log N = (x + \tfrac{1}{2}) \log (x/np) + (n - x + \tfrac{1}{2}) \log \{ (n-x)/nq \},$$

$$= (np + z\sqrt{npq} + \tfrac{1}{2}) \log [1 + z\sqrt{(q/np)}]$$

$$+ (nq - z\sqrt{npq} + \tfrac{1}{2}) \log [1 - z\sqrt{(p/nq)}]$$

$$= (np + z\sqrt{npq} + \tfrac{1}{2}) [z \cdot \sqrt{(q/np)} - \tfrac{1}{2} z^2 (q/np) + \tfrac{1}{3} z^3 (q/np)^{3/2} - \ldots]$$

$$+ (nq - z\sqrt{npq} + \tfrac{1}{2})\left[-z\sqrt{(p/nq)} - \tfrac{1}{2}z^2(p/nq) - \tfrac{1}{3}z^3(p/nq)^{3/2} - \ldots\right]$$

$$= \left[\left\{z\sqrt{npq} - \tfrac{1}{2}qz^2 + \tfrac{1}{3}z^3\frac{q^{3/2}}{\sqrt{np}} + z^2q - \tfrac{1}{2}z^3\frac{q^{3/2}}{\sqrt{np}} + \tfrac{1}{2}z\sqrt{q/np} - \tfrac{1}{4}z^2\frac{q}{np} + \ldots\right\} + \left\{-z\sqrt{npq} - \tfrac{1}{2}z^2p - \tfrac{1}{3}z^3\frac{p^{3/2}}{\sqrt{nq}} + z^2p + \tfrac{1}{2}z^3\frac{p^{3/2}}{\sqrt{np}} - \tfrac{1}{2}z\sqrt{p/nq} - \tfrac{1}{4}z^2\frac{p}{np} + \ldots\right\}\right]$$

i.e.,

$$\log N = \left[-\frac{1}{2}z^2(p+q) + z^2(p+q) + \frac{z}{2\sqrt{n}}\left\{\sqrt{q/p} + \sqrt{p/q}\right\} + 0\left(n^{-1/2}\right)\right]$$

$$= \frac{z^2}{2} + 0(n^{-1/2}) \to \frac{z^2}{2} \text{ as } n \to \infty$$

$$\therefore \quad \lim_{n\to\infty} \log N = \frac{z^2}{2} \quad \Rightarrow \quad \lim_{n\to\infty} N = e^{z^2/2}$$

Substituting in (8·4), we get

$$dG(z) = g(z)\,dz = \frac{1}{\sqrt{2\pi}}e^{-z^2/2}\,dz, -\infty < z < \infty \qquad \ldots(8·4\,a)$$

Hence the probability function of Z is

$$g(z) = \frac{1}{\sqrt{2\pi}}e^{-z^2/2}, -\infty < z < \infty \qquad \ldots(8·4\,b)$$

This is the probability density function of the *normal distribution* with mean 0 and unit variance.

If X is normal variate with mean μ and s.d. σ then $Z = (X - \mu)/\sigma$ is standard normal variate. Jacobian of transformation is $1/\sigma$. Hence substituting in {8·4 (*b*)}, the p.d.f. of a normal variate X with $E(X) = \mu$, $\text{Var}(X) = \sigma^2$ is given by

$$f_X(x) = \begin{cases} \dfrac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}, & -\infty < x < \infty \\ 0, & \text{otherwise} \end{cases}$$

Remark. Normal distribution can also be obtained as a limiting case of Poisson Distribution with the parameter $\lambda \to \infty$.

8·2·2. Chief Characteristics of the Normal Distribution and Normal Probability Curve. The normal probability curve with mean μ and standard deviation σ is given by the equation

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}, -\infty < x < \infty$$

and has the following properties :

(*i*) The curve is bell shaped and symmetrical about the line $x = \mu$.

(*ii*) Mean, median and mode of the distribution coincide.

(*iii*) As x increases numerically, $f(x)$ decreases rapidly, the maximum probability occurring at the point $x = \mu$, and given by $[p(x)]_{max} = \dfrac{1}{\sigma\sqrt{2\pi}}$.

(*iv*) $\beta_1 = 0$ and $\beta_2 = 3$.

(*v*) $\mu_{2r+1} = 0, (r = 0, 1, 2, ...)$,
and $\mu_{2r} = 1.3.5 \ldots (2r-1)\sigma^{2r}, (r = 0, 1, 2, ...)$.

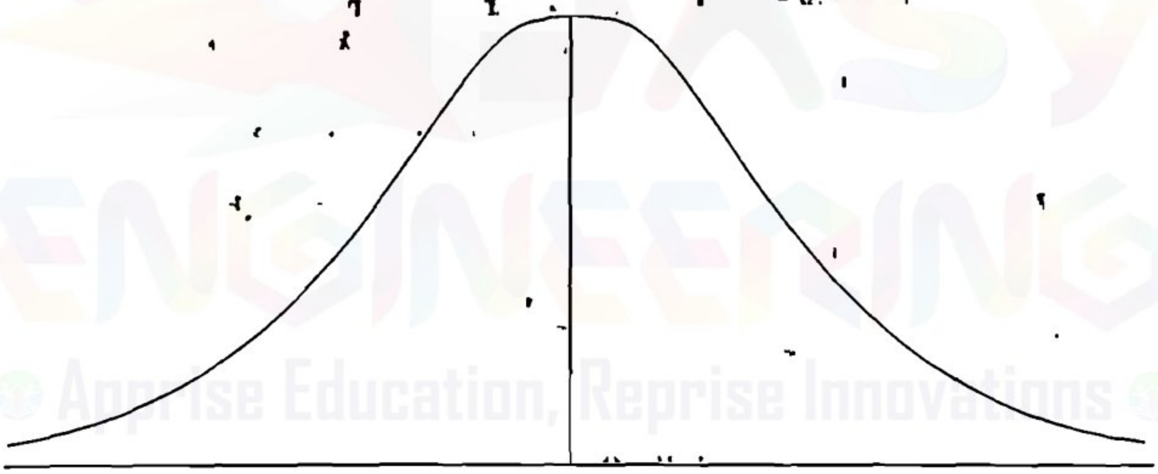
(*vi*) Since $f(x)$ being the probability, can never be negative, no portion of the curve lies below the x-axis.

(*vii*) Linear combination of independent normal variates is also a normal variate.

(*viii*) x-axis is an asymptote to the curve.

(*ix*) The points of inflexion of the curve are given by

$$\left[x = \mu \pm \sigma, f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2}\right]$$



(*Normal Probability Cuve*)

(*x*) Mean deviation about mean is $\sqrt{2/\pi}\,\sigma \approx \frac{4}{5}\sigma$ (approx.) Q.D. $= \dfrac{Q_3 - Q_1}{2} \simeq \dfrac{2}{3}\sigma$

We have (approximately)

$$Q.D. : M.D. : S.D. :: \frac{2}{3}\sigma : \frac{4}{5}\sigma : \sigma :: \frac{2}{3} : \frac{4}{5} : 1$$

$$\Rightarrow \quad Q.D. : M.D. : S.D. :: 10 : 12 : 15$$

(*xi*) *Area Property*

$$P(\mu - \sigma < X < \mu + \sigma) = 0.6826$$
$$P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9544$$
$$P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973$$

The following table gives the area under the normal probability curve for some important values of standard normal variate Z.

Distances from the mean ordinates in terms of $\pm\sigma$	*Area under the curve*
$Z = \pm 0{\cdot}745$	$50\% = 0{\cdot}50$
$Z = \pm 1{\cdot}00$	$68{\cdot}26\% = 0{\cdot}6826$
$Z = \pm 1{\cdot}96$	$95\% = 0{\cdot}95$
$Z = \pm 2{\cdot}0$	$95{\cdot}44\% = 0{\cdot}9544$
$Z = \pm 2{\cdot}58$	$99\% = 0{\cdot}99$
$Z = \pm 3{\cdot}0$	$99{\cdot}73\% = 0{\cdot}9973$

(*xii*) If X and Y are independent standard normal variates, then it can be easily proved that $U = X + Y$ and $V = X - Y$ are independently distributed, $U \sim N(0, 2)$ and $V \sim N(0, 2)$.

We state (without proof) the converse of this result which is due to D. Bernstein.

Bernstein's Theorem. If X and Y are independent and identically distributed random variables with finite varaince and if $U = X + Y$ and $V = X - Y$ are independent, then all r.v.'s X, Y, U and V are normally distributed.

(*xiii*) We state below another result which characterises the normal distribution.

If $X_1, X_2, ..., X_n$ are *i.i.d. r.v.'s* with finite variance, then the common distribution is normal if and only if :

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \quad \text{and} \quad s^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$$

$$\text{or} \qquad \sum_{i=1}^{n} X_i \text{ and } \sum_{i=1}^{n}(X_i - \bar{X})^2$$

are independent. **[For 'If part', see Theorem 13.5]**

In the following sequences we shall establish some of these properties.

8·2·3. Mode of Normal Distribution. Mode is the value of x for which $f(x)$ is maximum, *i.e.*, mode is the solution of

$$f'(x) = 0 \text{ and } f''(x) < 0$$

For normal distribution with mean μ and standard deviation σ,

$$\log f(x) = c - \frac{1}{2\sigma^2}(x-\mu)^2,$$

where $c = \log(1/\sqrt{2\pi}\,\sigma)$, is a constant.

Differentiating w.r.t. x, we get

$$\frac{1}{f(x)} \cdot f'(x) = -\frac{1}{\sigma^2}(x-\mu) \Rightarrow f'(x) = -\frac{1}{\sigma^2}(x-\mu) f(x)$$

and $$f''(x) = -\frac{1}{\sigma^2}\Big[\,1\,.\,f(x) + (x-\mu)\,f'(x)\,\Big] = -\frac{f(x)}{\sigma^2}\left[1 - \frac{(x-\mu)^2}{\sigma^2}\right] \qquad \ldots(8\cdot6)$$

Now $f'(x) = 0 \Rightarrow x - \mu = 0$ *i.e.*, $x = \mu$

At the point $x = \mu$, we have from (8·6)

$$f''(x) = -\frac{1}{\sigma^2}\,[\,f(x)\,]_{x=\mu} = -\frac{1}{\sigma^2}\cdot\frac{1}{\sigma\sqrt{2\pi}} < 0$$

Hence $x = \mu$, is the mode of the normal distribution.

8·2·4. Median of Normal Distribution. If M is the median of the normal distribution, we have

$$\int_{-\infty}^{M} f(x)\,dx = \frac{1}{2} \Rightarrow \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{M} \exp\left\{-(x-\mu)^2/2\sigma^2\right\}dx = \frac{1}{2}$$

$$\Rightarrow \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\mu} \exp\left\{-(x-\mu)^2/2\sigma^2\right\}dx$$

$$+\frac{1}{\sigma\sqrt{2\pi}}\int_{\mu}^{M} \exp\left\{-(x-\mu)^2/(2\sigma^2)\right\}dx = \frac{1}{2} \qquad \ldots(8\cdot7)$$

But $$\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\mu} \exp\left\{-(x-\mu)^2/2\sigma^2\right\}dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} \exp\,(-z^2/2)\,dz = \frac{1}{2}$$

$\therefore$ From (8·7), we get

$$\frac{1}{2} + \frac{1}{\sigma\sqrt{2\pi}}\int_{\mu}^{M} \exp\left\{-(x-\mu)^2/2\sigma^2\right\}dx = \frac{1}{2}$$

$$\Rightarrow \quad \frac{1}{\sigma\sqrt{2\pi}}\int_{\mu}^{M} \exp\left\{-(x-\mu)^2/2\sigma^2\right\}dx = 0 \quad \Rightarrow \quad \mu = M$$

Hence for the normal distribution, Mean = Median.

Remark. From § 8·2·3 and § 8·2·4, we find that for the normal distribution mean, median and mode coincide. Hence the distribution is *symmetrical.*

8·2·5. M.G.F. of Normal Distribution. The m.g.f. (about origin) is given by

$$M_X(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{tx}\exp\left\{-(x-\mu)^2/2\sigma^2\right\}dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\left\{t(\mu+\sigma z)\right\}\exp\,(-z^2/2)\,dz, \qquad \left[z = \frac{x-\mu}{\sigma}\right]$$

$$= e^{\mu t}\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp\left\{-\tfrac{1}{2}(z^2 - 2t\sigma z)\right\}dz$$

$$= e^{\mu t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left[-\tfrac{1}{2}\left\{(z-\sigma t)^2 - \sigma^2 t^2\right\}\right] dz$$

$$= e^{\mu t + t^2\sigma^2/2} \times \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{-\tfrac{1}{2}(z-\sigma t)^2\right\} dz$$

$$= e^{\mu t + t^2\sigma^2/2} \times \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-u^2/2)\, du$$

Hence $$M_X(t) = e^{\mu t + t^2\sigma^2/2} \qquad \text{...(8·8)}$$

Remark. *M.G.F. of Standard Normal Variate.* If $X \sim N(\mu, \sigma^2)$, then standard normal variate is given by

$$Z = (X-\mu)/\sigma$$

Now $$M_Z(t) = e^{-\mu t/\sigma}\, M_X(t/\sigma) = \exp(-\mu t/\sigma) \cdot \exp\left(\frac{\mu t}{\sigma} + \frac{t^2}{\sigma^2}\cdot\frac{\sigma^2}{2}\right)$$

$$= \exp(t^2/2) \qquad \text{...(8·8 a)}$$

8·2·6. Cumulant Generating Function (c.g.f.) of Normal Distribution. The c.g.f. of normal distribution is given by

$$K_X(t) = \log_e M_X(t) = \log_e\left(e^{\mu t + t^2\sigma^2/2}\right) = \mu t + \frac{t^2\sigma^2}{2}$$

$\therefore$ Mean $= \kappa_1 =$ Coefficient of t in $K_X(t) = \mu$

Variance $= \kappa_2 =$ Coefficient of $\frac{t^2}{2!}$ in $K_X(t) = \sigma^2$

and $\kappa_r =$ Coefficient of $\frac{t^r}{r!}$ in $K_X(t) = 0$; $r = 3, 4 \ldots$

Thus $\mu_3 = \kappa_3 = 0$ and $\mu_4 = \kappa_4 + 3\kappa_2^2 = 3\sigma^4$

Hence $$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0 \quad \text{and} \quad \beta_2 = \frac{\mu_4}{\mu_2^2} = 3 \qquad \text{...(8·9)}$$

8·2·7. Moments of Normal Distribution. Odd order moments about mean are given by

$$\mu_{2n+1} = \int_{-\infty}^{\infty} (x-\mu)^{2n+1} f(x)\, dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^{2n+1} \exp\left\{-(x-\mu)^2/2\sigma^2\right\} dx$$

$$\therefore \quad \mu_{2n+1} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z)^{2n+1} \exp(-z^2/2)\, dz \qquad \left[z = \frac{x-\mu}{\sigma}\right]$$

$$= \frac{\sigma^{2n+1}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2n+1} \exp(-z^2/2)\, dz = 0, \qquad \text{...(8·10)}$$

since the integrand $z^{2n+1} e^{-z^2/2}$ is an odd function of z.

Even order moments about mean are given by

$$\mu_{2n} = \int_{-\infty}^{\infty} (x-\mu)^{2n} f(x)\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z)^{2n} \exp(-z^2/2)\, dz$$

$$= \frac{\sigma^{2n}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2n} \exp(-z^2/2)\, dz$$

$$= \frac{\sigma^{2n}}{\sqrt{2\pi}} \cdot 2 \int_{0}^{\infty} z^{2n} \exp(-z^2/2)\, dz$$

(since integrand is an even function of z)

$$\therefore \quad \mu_{2n} = \frac{2\sigma^{2n}}{\sqrt{2\pi}} \int_0^{\infty} (2t)^n e^{-t} \frac{dt}{\sqrt{2t}} \qquad \left[\frac{z^2}{2} = t\right]$$

$$= \frac{2^n \cdot \sigma^{2n}}{\sqrt{\pi}} \int_0^{\infty} e^{-t}\, t^{(n+\frac{1}{2})-1}\, dt$$

$$\Rightarrow \quad \mu_{2n} = \frac{2^n \sigma^{2n}}{\sqrt{\pi}} \cdot \Gamma(n+\tfrac{1}{2})$$

Changing n to $(n-1)$, we get

$$\mu_{2n-2} = \frac{2^{n-1} \cdot \sigma^{2n-2}}{\sqrt{\pi}} \Gamma(n-\tfrac{1}{2})$$

$$\therefore \quad \frac{\mu_{2n}}{\mu_{2n-2}} = 2\sigma^2 \cdot \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n-\frac{1}{2})} = 2\sigma^2 (n-\tfrac{1}{2}) \; [\because \Gamma(r) = (r-1)\Gamma(r-1)]$$

$$\Rightarrow \quad \mu_{2n} = \sigma^2 (2n-1)\mu_{2n-2} \qquad \text{...(8·11)}$$

which gives the *recurrence relation* for the moments of normal distribution.

From (8·11), we have

$$\mu_{2n} = [(2n-1)\sigma^2][(2n-3)\sigma^2]\mu_{2n-4}$$

$$= [(2n-1)\sigma^2][2n-3)\sigma^2][(2n-5)\sigma^2]\mu_{2n-6}$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$= [(2n-1)\sigma^2][(2n-3)\sigma^2][(2n-5)\sigma^2]\ldots(3\sigma^2)(1\sigma^2)\cdot\mu_0$$

$$= 1.3.5\ldots(2n-1)\sigma^{2n} \qquad \text{...(8·12)}$$

From (8·10) and (8·12) we conclude that *for the normal distribution all odd order moments about mean vanish and the even order moments about mean are given by (8·12).*

Aliter. The above result can also be obtained quite conveniently as follows:

The m.g.f. (about mean) is given by

$$E\left[e^{t(X-\mu)}\right] = e^{-\mu t} E\left(e^{tX}\right) = e^{-\mu t} M_X(t)$$

where $M_X(t)$ is the m.g.f. (about origin).

$$\therefore \quad \text{m.g.f. (about mean)} = e^{-\mu t}\, e^{\mu t + t^2\sigma^2/2} = e^{t^2\sigma^2/2}$$

$$= \left[1 + (t^2\sigma^2/2) + \frac{(t^2\sigma^2/2)^2}{2\,!} + \frac{(t^2\sigma^2/2)^3}{3\,!} + \dots + \frac{(t^2\sigma^2/2)^n}{n\,!} + \dots\right] \quad \dots(8\cdot13)$$

The coefficient of $\dfrac{t^r}{r\,!}$ in (8·13) gives μ_r, the rth moment about mean. Since there is no term with odd powers of t in (8·13), all moments of odd order about mean vanish.

i.e., $\quad \mu_{2n+1} = 0\,;\ n = 0, 1, 2, \dots$

and $\quad \mu_{2n} =$ Coefficient of $\dfrac{t^{2n}}{(2n)\,!}$ in (8·13) $= \dfrac{\sigma^{2n} \times (2n)\,!}{2^n\, n\,!}$

$$= \frac{\sigma^{2n}}{2^n\, n\,!} \cdot \left[2n\,(2n-1)\,(2n-2)\,(2n-3) \dots 5.\,4.\,3.\,2.\,1.\right]$$

$$= \frac{\sigma^{2n}}{2^n\,.\,n\,!}\left[1.3.5 \dots (2n-1)\right]\left[2.4.6 \dots (2n-2)\,.\,2n\right]$$

$$= \frac{\sigma^{2n}}{2^n\,.\,n\,!}\left[1.3.5 \dots (2n-1)\right] 2^n \left[1.2.3 \dots n\right]$$

$$= 1.3.5 \dots (2n-1)\,\sigma^{2n}$$

Remark. In particular, we have from (8·10) and (8·12),

$$\mu_3 = 0 \text{ and } \mu_2 = 1\,.\,\sigma^2,\ \mu_4 = 1\cdot3\ \sigma^4$$

Hence $\quad \beta_1 = \dfrac{\mu_3^2}{\mu_2^3} = 0$ and $\beta_2 = \dfrac{\mu_4}{\mu_2^2} = \dfrac{3\,\sigma^4}{\sigma^4} = 3,$

the results which have already been obtained in (8·9).

8·2·8. A linear combination of independent normal variates is also a normal variate. Let X_i, $(i = 1, 2, \dots, n)$ be n independent normal variates with mean μ_i and variance σ_i^2 respectively. Then

$$M_{X_i}(t) = \exp\left\{\mu_i t + (t^2\sigma_i^2/2)\right\} \quad \dots(8\cdot14)$$

The m.g.f. of their linear combination $\sum\limits_{i=1}^{n} a_i X_i$, where $a_1, a_2, \dots, a_n$ are constants, is given by

$$M_{\Sigma\, a_i X_i}(t) = M_{a_1 X_1 + a_2 X_2 + \dots + a_n X_n}(t)$$

$$= M_{a_1 X_1}(t) \,.\, M_{a_2 X_2}(t) \ldots M_{a_n X_n}(t)$$

($\because$ X_i's are independent)

$$= M_{X_1}(a_1 t) \,.\, M_{X_2}(a_2 t) \ldots M_{X_n}(a_n t) \qquad \ldots(8\cdot15)$$

$[\because M_{cX}(t) = M_X(ct)]$

From (8·14), we have

$$M_{X_i}(a_i t) = e^{\mu_i a_i t + t^2 a_i^2 \sigma_i^2/2}$$

$\therefore$ (8·15), gives

$$M_{\sum_i a_i X_i}(t) = [e^{\mu_1 a_1 t + t^2 a_1^2 \sigma_1^2/2} \times e^{\mu_2 a_2 t + t^2 a_2^2 \sigma_2^2/2} \times \ldots \times e^{\mu_n a_n t + t^2 a_n^2 \sigma_n^2/2}]$$

$$= \exp\left[\left(\sum_{i=1}^{n} a_i \mu_i\right) t + t^2 \left(\sum_{i=1}^{n} a_i^2 \sigma_i^2\right)\Big/ 2\right],$$

which is the m.g.f. of a normal variate with mean $\sum_{i=1}^{n} a_i \mu_i$ and variance $\sum_{i=1}^{n} a_i^2 \sigma_i^2$. Hence by uniqueness theorem of m.g.f.,

$$\sum_{i=1}^{n} a_i X_i \sim N\left[\sum_{i=1}^{n} a_i \mu_i, \sum_{i=1}^{n} a_i^2 \sigma_i^2\right]. \qquad \ldots(8\cdot15\,a)$$

Remarks 1. If we take $a_1 = a_2 = 1, a_3 = a_4 = \ldots = 0$, then

$$X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$$

If we take $a_1 = 1, a_2 = -1, a_3 = a_4 = \ldots = 0$, then

$$X_1 - X_2 \sim N(\mu_1 - \mu_2, \sigma_1^2 + \sigma_2^2)$$

Thus we see that the sum as well as the difference of two independent normal variates is also a normal variate. This result provides a sharp contrast to the Poisson distribution, in which case though the sum of two independent Poisson variates is a Poisson variate, the difference is not a Poisson variate.

2. If we take

$a_1 = a_2 = \ldots = a_n = 1$, then we get $\qquad \ldots(8\cdot15\,b)$

$$\sum_{i=1}^{n} X_i \sim N\left[\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right]$$

i.e., the sum of independent normal variates is also a normal variate, which establishes the *additive property* of the normal distribution.

3. If X_i; $i = 1, 2, \ldots, n$ are identically and independently distributed as $N(\mu, \sigma^2)$ and if we take $a_1 = a_2 = \ldots = a_n = 1/n$,

then $$\frac{1}{n} \sum_{i=1}^{n} X_i \sim N\left\{\frac{1}{n} \sum_{i=1}^{n} \mu, \frac{1}{n^2} \sum_{i=1}^{n} \sigma^2\right\}$$

$$\Rightarrow \qquad \bar{X} \sim N(\mu, \sigma^2/n), \text{ where } \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$$

This leads to the following important conclusion :

If X_i, $(i = 1, 2, ..., n)$, are identically and independently distributed normal variates with mean μ and variance σ^2, then their mean $\bar{X}$ is also $N(\mu, \sigma^2/n)$.

8·2·9. Points of Inflexion of Normal Curve. At the point of inflexion of the normal curve, we should have

$$f''(x) = 0, \text{ and } f'''(x) \neq 0$$

For normal curve, we have from (8·6)

$$f''(x) = -\frac{f(x)}{\sigma^2}\left[1 - \frac{(x-\mu)^2}{\sigma^2}\right],$$

$$\therefore \qquad f''(x) = 0 \Rightarrow 1 - \frac{(x-\mu)^2}{\sigma^2} = 0 \Rightarrow x = \mu \pm \sigma$$

It can be easily verified that at the points $x = \mu \pm \sigma$, $f'''(x) \neq 0$.
Hence the points of inflexion of the normal curve are given by $x = \mu \pm \sigma$ and $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2}$ *i.e.*, they are equi-distant (at a distance σ) from the mean.

8·2·10. Mean Deviation from the Mean for Normal Distribution.

$$\text{M.D. (about mean)} = \int_{-\infty}^{\infty} |x-\mu| f(x)\, dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} |x-\mu| e^{-(x-\mu)^2/2\sigma^2} dx$$

$$= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |z| e^{-z^2/2} dz \qquad \left[\frac{x-\mu}{\sigma} = z\right]$$

$$= \frac{2\sigma}{\sqrt{2\pi}} \cdot \int_{0}^{\infty} |z| e^{-z^2/2} dz,$$

since the integrand $|z| e^{-z^2/2}$ is an even function of z.
Since in $[0, \infty]$, $|z| = z$, we have

$$\text{M.D. (about mean)} = \sqrt{2/\pi}\ \sigma \int_{0}^{\infty} z e^{-z^2/2} dz$$

$$= \sqrt{2/\pi}\ \sigma \int_{0}^{\infty} e^{-t} dt \qquad \left[\frac{z^2}{2} = t\right]$$

$$= \sqrt{2/\pi}\ \sigma \left| \frac{e^{-t}}{-1} \right|_0^\infty$$
$$= \sqrt{2/\pi}\ \sigma$$
$$= \frac{4}{5}\sigma \text{ (approx.)}$$

8·2·11. Area Property (Normal Probability Integral). If $X \sim N(\mu, \sigma^2)$, then the probability that random value of X will lie between $X = \mu$ and $X = x_1$ is given by

$$P(\mu < X < x_1) = \int_\mu^{x_1} f(x)\, dx = \frac{1}{\sigma\sqrt{2\pi}} \int_\mu^{x_1} e^{-(x-\mu)^2/(2\sigma^2)}\, dx$$

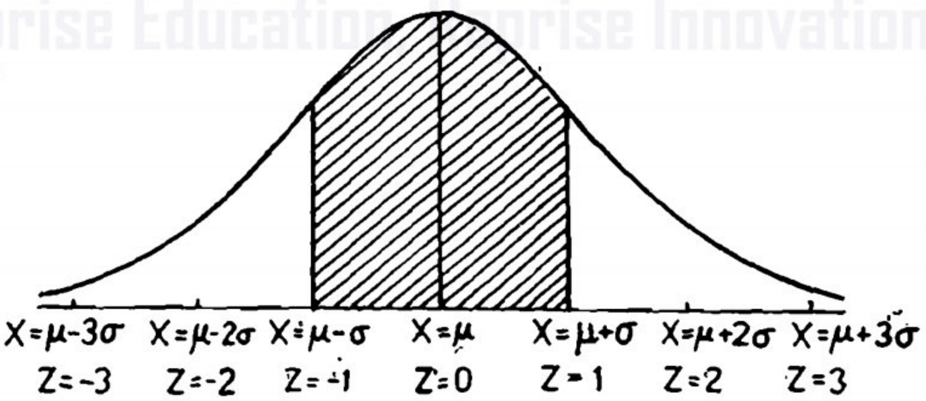
Put $\dfrac{X-\mu}{\sigma} = Z$, *i.e.*, $X - \mu = \sigma Z$

When $X = \mu, Z = 0$ and when $X = x_1, Z = \dfrac{x_1 - \mu}{\sigma} = z_1$, (say).

$$\therefore\ P(\mu < X < x_1) = P(0 < Z < z_1) = \frac{1}{\sqrt{2\pi}} \int_0^{z_1} e^{-z^2/2}\, dz = \int_0^{z_1} \varphi(z)\, dz$$

where $\varphi(z) = \dfrac{1}{\sqrt{2\pi}} e^{-z^2/2}$, is the probability function of standard normal variate.

The definite integral $\int_0^{z_1} \varphi(z)\, dz$ is known as *normal probability integral* and gives the area under standard normal curve between the ordinates at $Z = 0$ and $Z = z_1$. These areas have been tabulated for different values of z_1, at intervals of 0·01 [c.f. Appendix, Table IV].



In particular, the probability that a random value of X lies in the interval $(\mu - \sigma, \mu + \sigma)$ is given by

$$P(\mu - \sigma < X < \mu + \sigma) = \int_{\mu-\sigma}^{\mu+\sigma} f(x)\, dx$$

$$\Rightarrow \quad P(-1 < Z < 1) = \int_{-1}^{1} \varphi(z)\, dz \qquad \left[z = \frac{x-\mu}{\sigma} \right]$$

$$= 2\int_{0}^{1} \varphi(z)\, dz \qquad \text{(By symmetry)}$$

$$= 2 \times 0{\cdot}3413 = 0{\cdot}6826 \qquad \text{(From tables)} \quad ...(8{\cdot}17)$$

Similarly

$$P(\mu - 2\sigma < X < \mu + 2\sigma) = P(-2 < Z < 2) = \int_{-2}^{2} \varphi(z)\, dz$$

$$= 2\int_{0}^{2} \varphi(z)\, dz = 2 \times 0{\cdot}4772 = 0{\cdot}9544 \quad ...(8{\cdot}18)$$

and

$$P(\sigma - 3\sigma < X < \mu + 3\sigma) = P(-3 < Z < 3) = \int_{-3}^{3} \varphi(z)\, dz$$

$$= 2\int_{0}^{3} \varphi(z)\, dz = 2 \times 0{\cdot}49865 = 0{\cdot}9973 \quad ...(8{\cdot}19)$$

Thus the probability that a normal variate X lies outside the range $\mu \pm 3\sigma$ is given by

$$P(|X - \mu| > 3\sigma) = P(|Z| > 3) = 1 - P(-3 \le Z \le 3) = 0{\cdot}0027$$

Thus in all probability, we should expect a normal variate to lie within the range $\mu \pm 3\sigma$, though theoretically, it may range from $-\infty$ to ∞.

Remarks. 1. The total area under normal probability curve is unity, *i.e.*,

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{\infty} \varphi(z)\, dz = 1$$

2. Since in the normal probability tables, we are given the areas under standard normal curve, in numerical problems we shall deal with the standard normal variate Z rather than the variable X itself.

3. If we want to find area under normal curve, we will somehow or other try to convert the given area to the form $P(0 < Z < z_1)$, since the areas have been given in this form in the tables.

8·2·12. Error Function. If $X \sim N(0, \sigma^2)$, then

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}, \quad -\infty < x < \infty$$

If we take $h^2 = \dfrac{1}{2\sigma^2}$, then $f(x) = \dfrac{h}{\sqrt{\pi}} e^{-h^2 x^2}$

The probability that a random value of the variate lies in the range $\pm x$ is given by

$$P = \int_{-x}^{x} f(x)\, dx = \frac{h}{\sqrt{\pi}} \int_{-x}^{x} e^{-h^2x^2}\, dx$$

$$= \frac{2h}{\sqrt{\pi}} \int_{0}^{x} e^{-h^2x^2}\, dx = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-h^2x^2}\, (hdx) \quad ...(*)$$

Taking

$$\psi(y) = \frac{2}{\sqrt{\pi}} \int_{0}^{y} e^{-y^2}\, dy,$$ (*) may be re-written as

$$P = \psi(hx) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-h^2x^2}\, (hdx) \quad ...(**)$$

The function $\psi(y)$, known as the *error function*, is of fundamental importance in the *theory of errors* in Astronomy.

8·2·3. Importance of Normal Distribution. Normal distribution plays a very important role in statistical theory because of the following reasons :

(*i*) Most of the distributions occurring in practice, *e.g.*, Binomial, Poisson, Hypergeometric distributions, etc., can be approximated by normal distribution. Moreover, many of the sampling distributions, *e.g.*, Student's '*t*', Snedecor's F, Chi-square distributions, etc., tend to normality for large samples.

(*ii*) Even if a variable is not normally distributed, it can sometimes be brought to normal form by simple transformation of variable. For example, if the distribution of X is skewed, the distribution of $\sqrt{X}$ might come out to be normal [c.f. Variate Transformations at the end of this Chapter].

(*iii*) If $X \sim N(\mu, \sigma^2)$, then

$$P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973$$
$$\Rightarrow \quad P(-3 < Z < 3) = 0.9973$$
$$\Rightarrow \quad P(|Z| < 3) = 0.9973$$
$$\Rightarrow \quad P(|Z| > 3) = 0.0027$$

This property of the normal distribution forms the basis of entire *Large Sample* theory.

(*iv*) Many of the distributions of sample statistic (*e.g.*, the distributions of sample mean, sample variance, etc.) tend to normality for large samples and as such they can best be studied with the help of the normal curves.

(*v*) The entire theory of small sample tests, *viz.*, t, F, χ^2 tests etc., is based on the fundamental assumption that the parent populations from which the samples have been drawn follow normal distribution.

(*vi*) Theory of normal curves can be applied to the graduation of the curves which are not normal.

(*vii*) Normal distribution finds large applications in Statistical Quality Control in industry for setting control limits.

The following quotation due to Lipman rightly reveals the popularity and importance of normal distribution .

"Every body believes in the law of errors (the normal curve), the experimenters because they think it is a mathematical theorem, the mathematicians because they think it is experimental fact."

W J Youden of the National Bureau of Standards describes the importance of the Normal distribution artistically in the following words :

THE NORMAL
LAW OF ERRORS
STANDS OUT IN THE
EXPERIENCE OF MANKIND
AS ONE OF THE BROADEST
GENERALISATIONS OF NATURAL
PHILOSOPHY IT SERVES AS THE
GUIDING INSTRUMENT IN RESEARCHES.
IN THE PHYSICAL AND SOCIAL SCIENCES
AND IN MEDICINE, AGRICULTURE AND
ENGINEERING. IT IS AN INDISPENSABLE TOOL FOR
THE ANALYSIS AND THE INTERPRETATION OF THE
BASIC DATA OBTAINED BY OBSERVATION AND EXPERIMENT.

The above presentation, strikingly enough, gives the shape of the normal probability curve.

8·2 14. Fitting of Normal Distribution. In order to fit normal distribution to the given data we first calculate the mean μ, (say), and standard deviation σ, (say), from the given data. Then the normal curve fitted to the given data is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\{-(x-\mu)^2/2\sigma^2\}, \; -\infty < x < \infty$$

To calculate the expected normal frequencies we first find the standard normal variates corresponding to the lower limits of each of the class intervals. *i.e.*, we compute $z_i = \dfrac{x_i' - \mu}{\sigma}$, where x_i' is the lower limit of the ith class interval Then the areas under the normal curve to the left of the ordinate at $z = z_i$, say, $\varphi(z_i)$ are computed from the tables. Finally, the areas for the successive class intervals are obtained by subtraction, *viz.*, $\varphi(z_{i+1}) - \varphi(z_i)$, $(i = 1, 2, \ldots)$ and on multiplying these areas by N, we get the expected normal frequencies.

Example 8·10. *Obtain the equation of the normal curve that may be fitted to the following data :*

Class.	60–65	65–70	70–75	75–80	80–85	85–90	90–95	95–100
Frequency.	3	21	150	335	326	135	26	4

Also obtain the expected normal frequencies

Solution. For the given data, we have

$N = 1000$, $\mu = 79.945$ and $\sigma = 5.545$

Hence the equation of the normal curve fitted to the given data is

$$f(x) = \frac{1000}{\sqrt{2\pi} \times 5.545} \exp\left\{ -\frac{1}{2}\left(\frac{x - 79.945}{5.545} \right)^2 \right\}$$

Theoretical normal frequencies can be obtained as follows.

class	Lower class boundry (X')	$Z = \dfrac{X' - \mu}{\sigma}$	$\Phi(z) = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{z} e^{-z^2/2}\, dz$	$\Delta \varphi(z) = \psi_{z+1} - \psi_z$	Expected frequency $N \Delta \psi(z)$
Below 60	$-\infty$	$-\infty$	0	0.000112	0.12 ≅ 0
60–65	60	−3.663	0.000112	0.002914	2.914 ≅ 3
65–70	65	−2.745	0.003026	0.031044	31.044 ≅ 31
70–75	70	−1.826	0.034070	0.147870	147.870 ≅ 148
75–80	75	−0.908	0.181940	0.322050	322.050 ≅ 322
80–85	80	0.010	0.503990	0.319300	319.300 ≅ 319
85–90	85	0.928	0.823290	0.144072	144.072 ≅ 144
90–95	90	1.487	0.967362	0.029792	29.792 ≅ 30
95–100	95	2.675	0.997154	0.002733	2.733 ≅ 3
100 and over	100	3.683	0.999887		
Total					1000

Example 8·11. *For a certain normal distribution, the first moment about 10 is 40 and the fourth moment about 50 is 48. What is the arithmetic mean and standard deviation of the distribution ?*

[Delhi Univ. B.Sc. (Hons. Subs.), 1987; Allahabad Univ. B.Sc. 1990]

Solution. We know that if μ_1' is the first moment about the point $X = A$, then arithmetic mean is given by:

$$\text{Mean} = A + \mu_1'$$

We are given

μ_1' (about the point $X = 10$) $= 40 \Rightarrow$ Mean $= 10 + 40 = 50$

Also we are given

μ_4' (about the point $X = 50$) $= 48$, *i.e.*, $\mu_4 = 48$ ($\because$ Mean $= 50$)

But for a normal distribution with standard deviation σ,

$$\mu_4 = 3\sigma^4 \Rightarrow 3\sigma^4 = 48 \quad i.e., \quad \sigma = 2$$

Example 8·12. *X is normally distributed and the mean of X is 12 and S.D. is 4. (a) Find out the probability of the following :*

(*i*) $X \ge 20$, (*ii*) $X \le 20$, and (*iii*) $0 \le X \le 12$

(*b*) *Find* x', *when* $P(X > x') = 0.24$.

(*c*) *Find* x_0' *and* x_1', *when* $P(x_0' < X < x_1') = 0.50$ *and* $P(X > x_1') = 0.25$

Solution. (*a*) We have $\mu = 12$, $\sigma = 4$, *i.e.*, $X \sim N(12, 16)$.

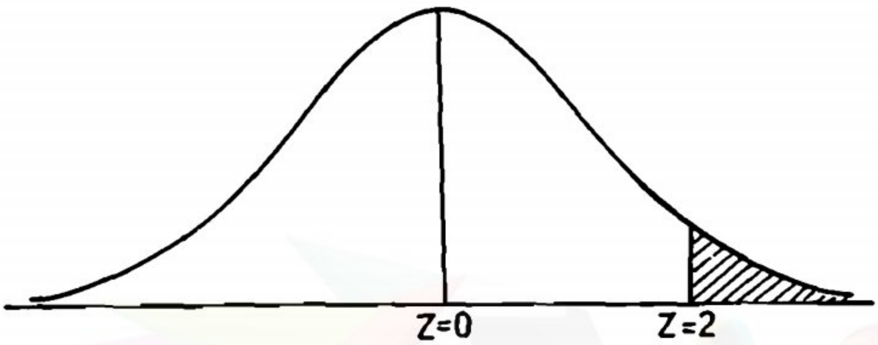
(*i*) $P(X \geq 20) = ?$

When $X = 20$, $Z = \dfrac{20-12}{4} = 2$

$\therefore \quad P(X \geq 20) = P(Z \geq 2) = 0{\cdot}5 - P(0 \leq Z \leq 2) = 0{\cdot}5 - 0{\cdot}4772 = 0{\cdot}0228$

(*ii*) $P(X \leq 20) = 1 - P(X \geq 20)$ ($\because$ Total probability = 1)

$= 1 - 0{\cdot}0228 = 0{\cdot}9772$



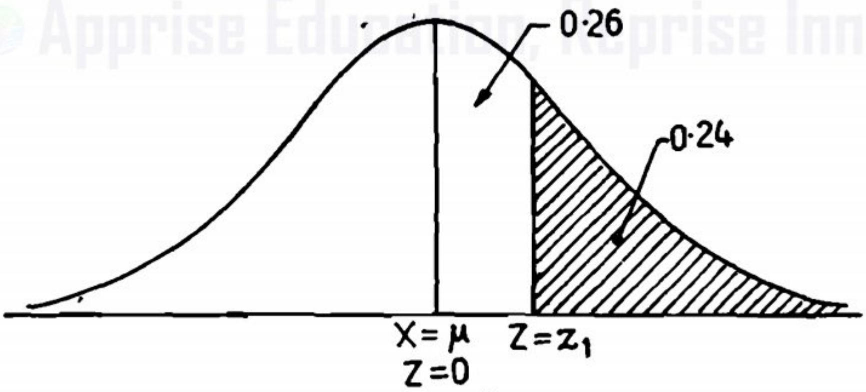
(*iii*) $P(0 \leq X \leq 12) = P(-3 \leq Z \leq 0)$ $\left(Z = \dfrac{X-12}{4}\right)$

$= P(0 \leq Z \leq 3) = 0{\cdot}49865$ (From symmetry)

(*b*) When $X = x'$, $Z = \dfrac{x'-12}{4} = z_1$ (say)

then, we are given

$P(X > x') = 0{\cdot}24 \quad \Rightarrow \quad P(Z > z_1) = 0{\cdot}24,$ *i.e.*, $P(0 < Z < z_1) = 0{\cdot}26$



$\therefore$ From normal tables,

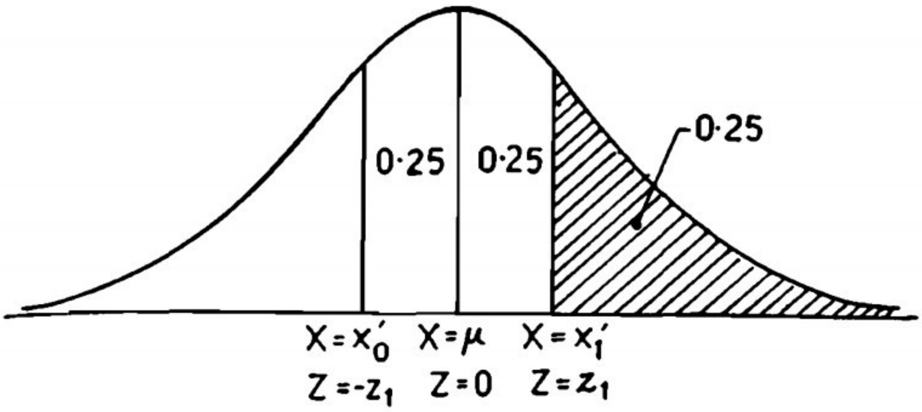
$z_1 = 0{\cdot}71$ (approx.)

Hence $\dfrac{x_1'-12}{4} = 0{\cdot}71 \quad \Rightarrow \quad x_1' = 12 + 4 \times 0{\cdot}71 = 14{\cdot}84$

(*c*) We are given

$P(x_0' < X < x_1') = 0{\cdot}50$ and $P(X > x_1') = 0{\cdot}25$ ...(*)

From (*), obviously the points x_0' and x_1' are located as shown in the figure.



When $X = x_1'$, $\quad Z = \dfrac{x_1' - 12}{4} = z_1$ (say)

and when $X = x_0'$, $\quad Z = \dfrac{x_0' - 12}{4} = -z_1$ (It is obvious from the figure)

We have

$$P(Z > z_1) = 0{\cdot}25 \quad \Rightarrow \quad P(0 < Z < z_1) = 0{\cdot}25$$

$\therefore \quad z_1 = 0{\cdot}67$ (From tables)

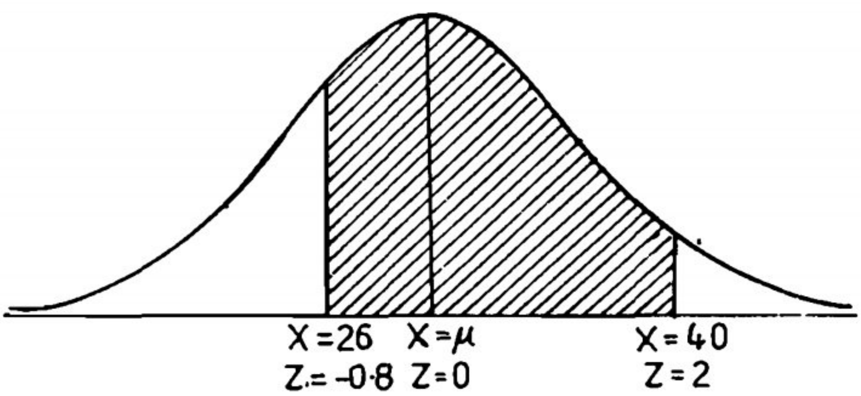
Hence $\quad \dfrac{x_1' - 12}{4} = 0{\cdot}67 \quad \Rightarrow \quad x_1' = 12 + 4 \times 0{\cdot}67 = 14{\cdot}68$

and $\quad \dfrac{x_0' - 12}{4} = -0{\cdot}67 \quad \Rightarrow \quad x_0' = 12 - 4 \times 0{\cdot}67 = 9{\cdot}32$

Example 8·13. *X is a normal variate with mean 30 and S.D. 5. Find the probabilities that*

(*i*) $26 \le X \le 40$, (*ii*) $X \ge 45$, and (*iii*) $|X - 30| > 5$.

Solution. Here $\mu = 30$ and $\sigma = 5$.

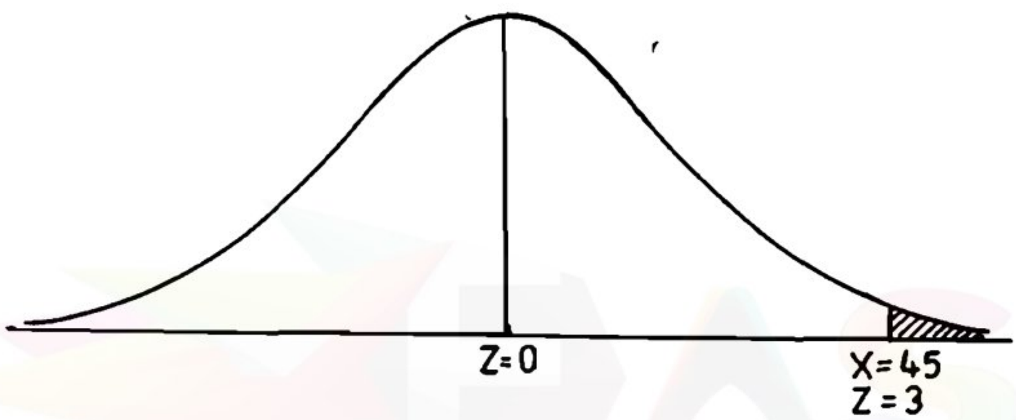


(*i*) When $X = 26$, $Z = \dfrac{X - \mu}{\sigma} = \dfrac{26 - 30}{5} = -0{\cdot}8$

and when $$X = 40,\ Z = \frac{40 - 30}{5} = 2$$

$$\begin{aligned}\therefore \quad P(26 \le X \le 40) &= P(-0.8 \le Z \le 2)\\ &= P(-0.8 \le Z \le 0) + P(0 \le Z \le 2)\\ &= P(-0.8 \le Z \le 0) + 0.4772 \quad &\text{(From tables)}\\ &= P(0 \le Z \le 0.8) + 0.4772 \quad &\text{(From symmetry)}\\ &= 0.2881 + 0.4772 = 0.7653\end{aligned}$$

$$P(X \ge 45) = ?$$



When $X = 45$, $\quad Z = \dfrac{45 - 30}{5} = 3$

$$\begin{aligned}\therefore \quad P(X \ge 45) &= P(Z \ge 3) = 0.5 - P(0 \le Z \le 3)\\ &= 0.5 - 0.49865 = 0.00135\end{aligned}$$

(*iii*) $$\begin{aligned}P(|X - 30| \le 5) &= P(25 \le X \le 35) = P(-1 \le Z \le 1)\\ &= 2\,P(0 \le Z \le 1) = 2 \times 0.3413 = 0.6826\end{aligned}$$

$$\begin{aligned}\therefore \quad P(|X - 30| > 5) &= 1 - P(|X - 30| \le 5)\\ &= 1 - 0.6826 = 0.3174\end{aligned}$$

Example 8·14. *The mean yield for one-acre plot is 662 kilos with a s.d. 32 kilos. Assuming normal distribution, how many one-acre plots in a batch of 1,000 plots would you expect to have yield (i) over 700 kilos, (ii) below 650 kilos, and (iii) what is the lowest yield of the best 100 plots?*

Solution. If the r.v. X denotes the yield (in kilos) for one-acre plot, then we are given that $X \sim N(\mu, \sigma^2)$, where $\mu = 662$ and $\sigma = 32$.

(*i*) The probability that a plot has a yield over 700 kilos is given by

$$P(X > 700) = P(Z > 1.19)\ ; \qquad Z = \frac{X - 662}{32}$$

$$\begin{aligned}&= 0.5 - P(0 \le Z \le 1.19)\\ &= 0.5 - 0.3830\\ &= 0.1170\end{aligned}$$

Hence in a batch of 1,000 plots, the expected number of plots with yield over 700 kilos is $1{,}000 \times 0.117 = 117$.

(*ii*) Required number of plots with yield below 650 kilos is given by

$$1000 \times P(X < 650) = 1000 \times P(Z < -0.38) \qquad \left[Z = \frac{650 - 662}{32} \right]$$

$$= 1000 \times P(Z > 0.38) \qquad \text{(By symmetry)}$$

$$= 1000 \times [0.5 - P(0 \le Z \le 0.38)]$$

$$= 1000 \times [0.5 - 0.1480] = 1000 \times 0.352$$

$$= 352$$

(*iii*) The lowest yield, say, x_1 of the best 100 plots is given by

$$P(X > x_1) = \frac{100}{1000} = 0.1$$

When $\quad X = x_1, \; Z = \dfrac{x_1 - \mu}{\sigma} = \dfrac{x_1 - 662}{32} = z_1$ (say) $\quad$ ...(*)

such that $\quad P(Z > z_1) = 0.1 \quad \Rightarrow \quad P(0 \le Z \le z_1) = 0.4$

$\Rightarrow \quad z_1 = 1.28$ (approx.) $\quad$ [From Normal Probability Tables]

Substituting in (*), we get

$$x_1 = 662 + 32 z_1 = 662 + 32 \times 1.28$$

$$= 662 + 40.96 = 702.96$$

Hence the best 100 plots have yield over 702·96 kilos.

Example 8·15. *There are six hundred Economics students in the post-graduate classes of a university, and the probability for any student to need a copy of a particular book from the university library on any day is 0·05. How many copies of the book should be kept in the university library so that the probability may be greater than 0·90 that none of the students needing a copy from the library has to come back disappointed? (Use normal approximation to the binomial distribution).* **[Delhi Univ. M.A. (Eco.), 1989]**

Solution. We are given :

$$n = 600, p = 0.05, \mu = np = 600 \times 0.05 = 30$$

$$\sigma^2 = npq = 600 \times 0.05 \times 0.95 = 28.5 \Rightarrow \sigma = \sqrt{28.5} = 5.3$$

We want x_1 such that

$$P(X < x_1) > 0.90$$

$$\Rightarrow \quad P(Z < z_1) > 0.90 \qquad \left[z_1 = \frac{x_1 - 30}{5.3} \right]$$

$$\Rightarrow \quad P(0 < Z < z_1) > 0.40$$

$$\Rightarrow \quad z_1 > 1.28 \qquad \text{[From Normal Probability Tables]}$$

$$\Rightarrow \quad \frac{x_1 - 30}{5.3} > 1.28 \quad \Rightarrow \quad x_1 > 30 + 5.3 \times 1.28$$

$$\Rightarrow \quad x_1 > 30 + 6.784 \quad \Rightarrow \quad x_1 > 36.784 \simeq 37$$

Hence the university library should keep at least 37 copies of the book.

Example 8·16. *The marks obtained by a number of students for a certain subject are assumed to be approximately normally distributed with mean value 65* *and with a standard deviation of 5. If 3 students are taken at random from this set what is the probability that exactly 2 of them will have marks over 70?*

Solution. Let the r.v. X denote the marks obtained by the given set of students in the given subject. Then we are given that $X \sim N(\mu, \sigma^2)$ where $\mu = 65$ and $\sigma = 5$. The probability 'p' that a randomly selected student from the given set gets marks over 70 is given by

$$p = P(X > 70)$$

When $\quad X = 70,\ Z = \dfrac{X - \mu}{\sigma} = \dfrac{70 - 65}{5} = 1.$

$$\therefore \quad p = P(X > 70) = P(Z > 1)$$
$$= 0{\cdot}5 - P(0 \le Z \le 1)$$
$$= 0{\cdot}5 - 0{\cdot}3413 = 0{\cdot}1587 \quad \text{[From Normal probability tables]}$$

Since this probability is same for each student of the set, the required probability that out of 3 students selected at random from the set, exactly 2 will have marks over 70, is given by the binomial probability law:

$${}^3C_2\, p^2 . (1 - p) = 3 \times (0{\cdot}1587)^2 \times (0{\cdot}8413) = 0{\cdot}06357$$

Example 8·17. (*a*) *If $\log_{10} X$ is normally distributed with meant 4 and variance 4, find the probability of*

$$1{\cdot}202 < X < 83180000$$

(Given $\log_{10}$ 1202 = 3·08, $\log_{10}$ 8318 = 3·92).

(b) $\log_{10} X$ is normally distributed with mean 7 and variance 3, $\log_{10} Y$ is normally distributed with mean 3 and variance unity. If the distributions of X and Y are independent, find the probability of $1{\cdot}202 < (X/Y) < 83180000$.

[Given $\log_{10}$ (1202) = 3·08, $\log_{10}$ (8318) = 3·92]

Solution. (*a*) Since $\log X$ is a non-decreasing function of X, we have

$$P(1{\cdot}202 < X < 83180000) = P(\log_{10} 1{\cdot}202 < \log_{10} X < \log_{10} 83180000)$$
$$= P(0{\cdot}08 < \log_{10} X < 7{\cdot}92)$$
$$= P(0{\cdot}08 < Y < 7{\cdot}92)$$

where $Y = \log_{10} X \sim N(4, 4)$ (given).

When $\quad Y = 0{\cdot}08,\ Z = \dfrac{0{\cdot}08 - 4}{2} = -1{\cdot}96$

and when $\quad Y = 7{\cdot}92,\ Z = \dfrac{7{\cdot}92 - 4}{2} = 1{\cdot}96$

$$\therefore \text{ Required probability } = P(0{\cdot}08 < Y < 7{\cdot}92)$$
$$= P(-1{\cdot}96 < Z < 1{\cdot}96) = 2P(0 < Z < 1{\cdot}96)$$
(By symmetry)
$$= 2 \times 0{\cdot}4750 = 0{\cdot}9500$$

(*b*) $P[\,1{\cdot}202 < (X/Y) < 83180000\,]$

$$= P[\log_{10} 1{\cdot}202 < \log_{10}(X/Y) < \log_{10} 83180000\,]$$
$$= (0{\cdot}08 < U < 7{\cdot}92)$$

where $\quad U = \log_{10}(X/Y) = \log_{10} X - \log_{10} Y$

Since $\log_{10} X \sim N(7, 3)$ and $\log_{10} Y \sim N(3, 1)$, are independent,

$$\log_{10} X - \log_{10} Y \sim N(7-3, 3+1) \qquad \text{(c.f. Remark 1, § 8·2·8)}$$

$$\Rightarrow \qquad U = (\log_{10} X - \log_{10} Y) \sim N(4, 4)$$

$\therefore$ Required probability is given by

$$p = P(0{\cdot}08 < U < 7{\cdot}92), \text{ where } U \sim N(4, 4)$$

$$= 0{\cdot}95 \qquad \text{[See part (a)]}$$

Example 8·18. *Two independent random variates X and Y are both normally distributed with means 1 and 2 and standard deviations 3 and 4 respectively. If $Z = X - Y$, write the probability density function of Z. Also state the median, s.d. and mean of the distribution of Z. Find Prob. $\{Z + 1 \le 0\}$.*

Solution. Since $X \sim N(1, 9)$ and $Y \sim N(2, 16)$ are independent, $Z = Y - Y \sim N(1-2, 9+16)$, *i.e.*, $Z = X - Y \sim N(-1, 25)$. Hence p.d.f. of Z is

$$p(z) = \frac{1}{5\sqrt{2\pi}} \exp\left[-\tfrac{1}{2}\left(\frac{z+1}{5}\right)^2\right]; -\infty < z < \infty.$$

For the distribution of Z,

$$\text{Median} = \text{Mean} = -1 \quad \text{and} \quad \text{s.d.} = \sqrt{25} = 5$$

$$P(Z+1 \le 0) = P(Z \le -1)$$

$$= P(U \le 0); \qquad \left[U = \frac{Z+1}{5} \sim N(0, 1)\right]$$

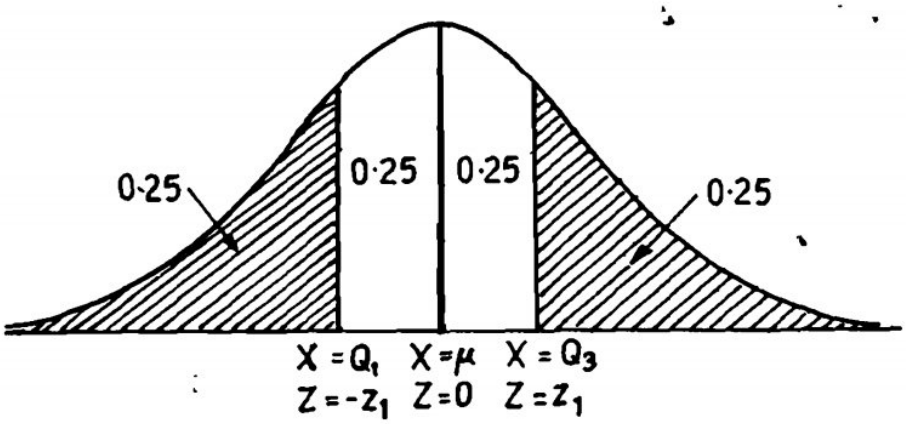
$$= 0{\cdot}5$$

Example 8·19. *Prove that for the normal distribution, the quartile deviation, the mean deviation and standard deviation are approximately 10 : 12 : 15.* **[Dibrugarh Univ. B.Sc. 1993]**

Solution. Let X be a $N(\mu, \sigma^2)$. If Q_1 and Q_3 are the first andthird quartiles respectively, then by definition

$$P(X < Q_1) = 0{\cdot}25 \quad \text{and} \quad P(X > Q_3) = 0{\cdot}25$$

The points Q_1 and Q_3 are located as shown in the figure given below.



When $\quad X = Q_3, \; Z = \dfrac{Q_3 - \mu}{\sigma} = z_1$, (say),

and when $\quad X = Q_1, \; Z = \dfrac{Q_1 - \mu}{\sigma} = -z_1$ (This is obvious from the figure)

Subtracting, we have

$$\frac{Q_3 - Q_1}{\sigma} = 2z_1$$

The quartile deviation is given by

$$Q.D. = \frac{Q_3 - Q_1}{2} = \sigma z_1$$

From the figure, obviously, we have

$P(0 < Z < z_1) = 0·25 \;\Rightarrow\; z_1 = 0·67$ (approx.) (From Normal Tables)

$\therefore \quad Q.D. = \sigma z_1 = 0·67\, \sigma \approx \frac{2}{3}\sigma$

For normal distribution mean deviation about mean (*c.f.* § 8·2·10) is given by

$$\text{M.D.} = \sqrt{2/\pi}\, \sigma \approx \frac{4}{5}\sigma$$

Hence Q.D. : M.D. : S.D. : : $\frac{2}{3}\sigma : \frac{4}{5}\sigma : \sigma :: \frac{2}{3} : \frac{4}{5} : 1 :: 10 : 12 : 15$

Example 8·20 (*a*). *In a distribution exactly normal, 7% of the items are under 35 and 89% are under 63. What are the mean and standard deviation of the distribution ?* **[Kerala Univ. B.Sc., May 1991]**

(*b*) *Of a large group of men, 5% are under 60 inches in height and 40% are between 60 and 65 inches. Assuming a normal distribution, find the mean height and standard deviation.* **[Nagpur Univ. B.Sc., 1992]**

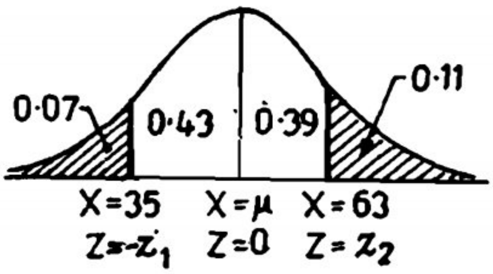
Solution. If $X \sim N(\mu, \sigma^2)$, then we are given

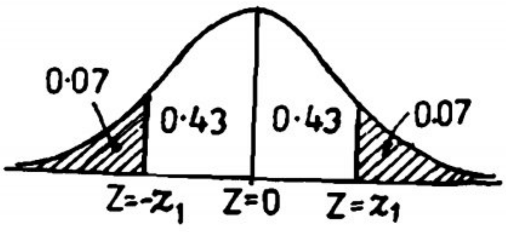
$$P(X < 63) = 0·89 \;\Rightarrow\; P(X > 63) = 0·11 \text{ and } P(X < 35) = 0·07$$

The points $X = 63$ and $X = 35$ are located as shown in Fig. (*i*) below.

Since the value $X = 35$ is located to the left of the ordinate at $X = \mu$, the corresponding value of Z is negative.

When $X = 35, \; Z = \dfrac{35 - \mu}{\sigma} = -z_1$, (say),





and when $X = 63$, $Z = \dfrac{63 - \mu}{\sigma} = z_2$, (say).

Thus we have, as is obvious from figures (*i*) and (*ii*)

$$P(0 < Z < z_2) = 0.39 \text{ and } P(0 < Z < z_1) = 0.43$$

Hence from normal tables, we have

$$z_2 = 1.23 \text{ and } z_1 = 1.48$$

$$\therefore \qquad \frac{63 - \mu}{\sigma} = 1.23 \text{ and } \frac{35 - \mu}{\sigma} = -1.48$$

Subtracting, we get

$$\frac{28}{\sigma} = 2.71 \Rightarrow \sigma = \frac{28}{2.71} = 10.33$$

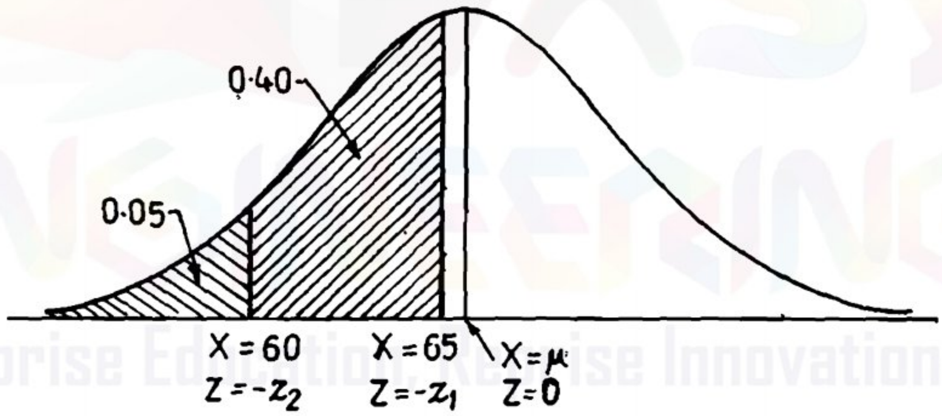
$$\therefore \qquad \mu = 35 + 1.48 \times 10.33 = 35 + 15.3 = 50.3$$

(*b*) We are given

$$P(X < 60) = 0.05 \text{ and } P(60 < X < 65) = 0.40$$

i.e., $\qquad P(X < 65) = 0.45$

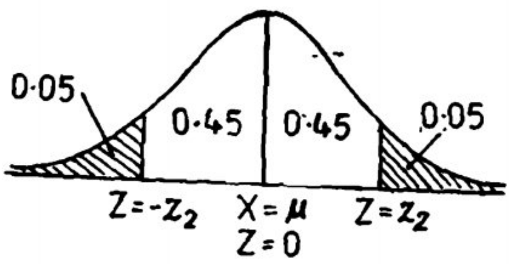
Since the total area to the left of the ordinate at $X = \mu$ is 0·5, both the points $X = 60$ and $X = 65$ are located to the left of $X = \mu$ and consequently the corresponding values of Z are negative.

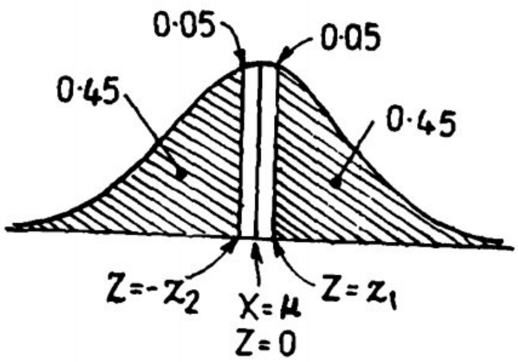


Let $X \sim N(\mu, \sigma^2)$.

When $X = 65$, $\qquad Z = \dfrac{65 - \mu}{\sigma} = -z_1$ (say),

and when $X = 60$, $\qquad Z = \dfrac{60 - \mu}{\sigma} = -z_2$ (say).





Thus we have

$$P(0 < Z < z_2) = 0{\cdot}45 \text{ and } P(0 < Z < z_1) = 0{\cdot}05$$

$\therefore$ $z_2 = 1{\cdot}645$ and $z_1 = 0{\cdot}13$ (approx.) (From Normal Tables)

Hence $\dfrac{60-\mu}{\sigma} = -1{\cdot}645$ ...(*) ; and $\dfrac{65-\mu}{\sigma} = -0{\cdot}13$ ...(**)

Dividing, we get $\dfrac{60-\mu}{65-\mu} = \dfrac{1{\cdot}645}{0{\cdot}13} \Rightarrow \mu = \dfrac{19825}{303} = 65{\cdot}42$

$\therefore$ From (*), we have $\sigma = \dfrac{60-65{\cdot}42}{-1{\cdot}645} = 3{\cdot}29$

Remarks. If we substitute the value of μ in (**), we get $\sigma = 3{\cdot}23$ which is only an approximate value since the value of $z_1 = 0{\cdot}13$, seen from the table, is not exact but only approximate. On the other hand, the value of $z_2 = 1{\cdot}645$ is exact and hence use of (*) for estimating σ gives better approximation.

Example 8·21 *If the skulls are classified as A, B and C according as the length-breadth index is under 75, between 75 and 80, or over 80, find approximately (assuming that the distribution is normal) the mean and standard deviation of a series in which A are 58%, B are 38% and C are 4%, being given that if*

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_0^t \exp(-x^2/2)\, dx\,,$$

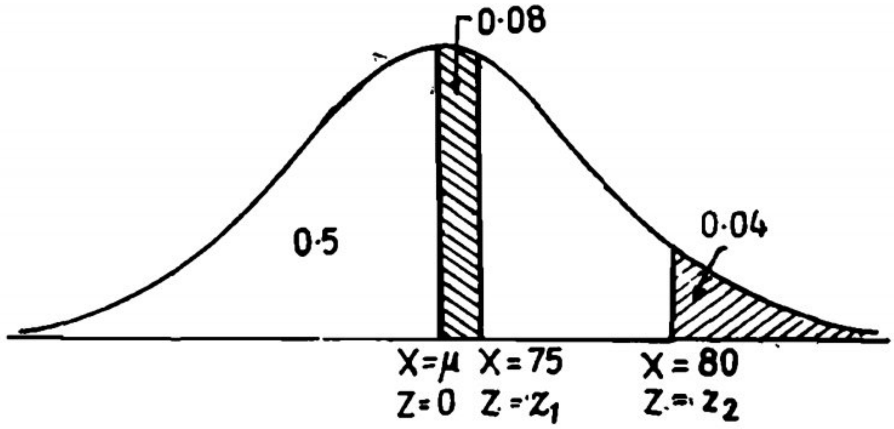
then $f(0{\cdot}20) = 0{\cdot}08$ and $f(1{\cdot}75) = 0{\cdot}46$

[Delhi Univ. B.Sc., 1989; Burdwan Univ. B.Sc., 1990]

Solution. Let the length-breadth index be denoted by the variable X, then we are given

$$P(X < 75) = 0{\cdot}58 \text{ and } P(X > 80) = 0{\cdot}04 \qquad ...(1)$$

Since $P(X < 75)$ represents the total area to the left of the ordinate at the point $X = 75$ and $P(X > 80)$ represents the total area to the right of the ordinate at the point $X = 80$, it is obvious from (1) that the points $X = 75$ and $X = 80$ are located at the positions shown in the figure below.



Now $\frac{1}{\sqrt{2\pi}} \int_0^t \exp(-x^2/2)\, dx$ represents the area under standard normal curve between the ordinates at $Z = 0$ and $Z = t$, Z being a $N(0, 1)$ variate.

$$\therefore \qquad f(t) = \frac{1}{\sqrt{2\pi}} \int_0^t \exp(-x^2/2)\, dx = P(0 < Z < t)$$

Hence $\quad f(0{\cdot}20) = P(0 < Z < 0{\cdot}20) = 0{\cdot}08 \qquad \text{...(2)}$

and $\quad f(1{\cdot}75) = P(0 < Z < 1{\cdot}75) = 0{\cdot}46$

Let μ and σ be the mean and standard deviation of the distribution. Then $X \sim N(\mu, \sigma^2)$.

When $X = 75$, $\qquad Z = \frac{75 - \mu}{\sigma} = z_1$ (say),

and when $X = 80$, $\qquad Z = \frac{80 - \mu}{\sigma} = z_2$ (say).

Thus from the figure, it is obvious that

$$P(X < 75) = 0{\cdot}58 \quad \Rightarrow \quad P(0 < Z < z_1) = 0{\cdot}08$$

$\therefore$ Using (2), we have

$$z_1 = \frac{75 - \mu}{\sigma} = 0{\cdot}20 \qquad \text{...(3)}$$

Also $\quad P(X > 80) = 0{\cdot}04 \quad \Rightarrow \quad P(0 < Z < z_2) = 0{\cdot}46$

$\therefore$ From (2), we get

$$z_2 = \frac{80 - \mu}{\sigma} = 1{\cdot}75 \qquad \text{...(4)}$$

Solving the equations (3) and (4), we get

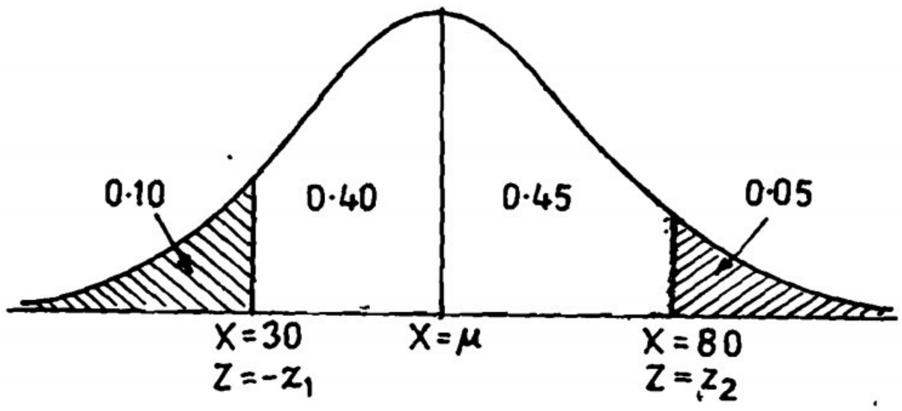
$$\mu = 74{\cdot}4 \text{ (approx.) and } \sigma = 3{\cdot}2 \text{ (approx.)}$$

Example 8·22. *In an examination it is laid down that a student passes if he secures 30 per cent or more marks. He is placed in the first, second or third division according as he secures 60% or more marks, between 45% and 60% marks and marks between 30% and 45% respectively. He gets distinction in case he secures 80% or more marks. It is noticed from the result that 10% of the students failed in the examination, whereas 5% of them obtained distinction. Calculate the percentage of students placed in the second division. (Assume normal distribution of marks.)* **[Aligarh Univ. B.Sc., 1991]**

Solution. Let the variable X denote the marks (out of 100) in the examination and let $X \sim N(\mu, \sigma^2)$. Then we are given

$$P(X < 30) = 0{\cdot}10 \text{ and } P(X \geq 80) = 0{\cdot}05$$

Thus from the figure on next page, we have



When $X = 30$, $\quad Z = \dfrac{30-\mu}{\sigma} = -z_1$ (say),

and when $X = 80$, $\quad Z = \dfrac{80-\mu}{\sigma} = z_2$ (say).

$\therefore \quad P(0 < Z < z_2) = 0.5 - 0.05 = 0.45$

and $\quad P(0 < Z < z_1) = P(-z_1 < Z < 0)$ (By symmetry)

$\quad = 0.50 - 0.10 = 0.40$

$\therefore$ From normal tables, we get

$$z_1 = 1.28 \quad \text{and} \quad z_2 = 1.64$$

Hence $\quad \dfrac{30-\mu}{\sigma} = -1.28$

$\Rightarrow \quad \dfrac{\mu-30}{\sigma} = 1.28 \quad$ and $\quad \dfrac{80-\mu}{\sigma} = 1.64$

Adding, we get

$$\frac{50}{\sigma} = 2.92 \Rightarrow \sigma = \frac{50}{2.92} = 17.12$$

$\therefore \quad \mu = 30 + 1.28 \times 17.12 = 30 + 21.9136 = 51.9136 \simeq 52$

The probability 'p' that a candidate is placed in the second division is equal to the probability that his score lies between 45 and 60, *i.e.*,

$$p = P(45 < X < 60) = P(-0.41 < Z < 0.47) \qquad \left[Z = \frac{X-52}{17.12} \right]$$

$$= P(-0.41 < Z < 0) + P(0 < Z < 0.47)$$

$$= P(0 < Z < 0.41) + P(0 < Z < 0.47) \qquad \text{(By symmetry)}$$

$$= 0.1591 + 0.1808 = 0.3399 = 0.34 \text{ (approx.)}$$

Therefore, 34% candiates got second division in the examination.

Example 8·23. *The local authorities in a certain city instal 10,000 electric lamps in the streets of the city. If these lamps have an average life of 1,000 burning hours with a standard deviation of 200 hours, assuming normality, what number of lamps might be expected to fail (i) in the first 800 burning hours? (ii) between 800 and 1,200 burning hours ? After what period of burning hours would you expect that (a) 10% of the lamps would fail? (b) 10% of the lamps would be still burning?*

[In a normal curve, the area between the ordinates corresponding to $\frac{X-\bar{X}}{\sigma}=0$ and $\frac{X-\bar{X}}{\sigma}=1$ *is* 0·34134 *and 80% of the area lies between the ordinates corresponding to* $\frac{X-\bar{X}}{\sigma}=\pm 1{\cdot}28$].

Solution. If the variable X denotes the life of a bulb in burning hours, then we are given that $X \sim N(\mu, \sigma^2)$, where $\mu = 1{,}000$ and $\sigma = 200$.

(*i*) The probability '*p*' that bulb fails in the first 800 burning hours is given by

$$p = P(X < 800) = P(Z < -1) = P(Z > 1) \qquad \left[Z = \frac{800-1000}{200}\right]$$

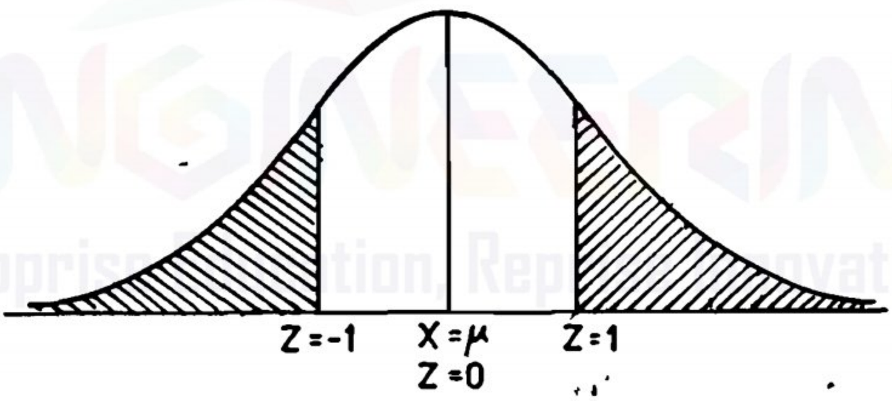
$$= 0{\cdot}5 - P(0 < Z < 1) = 0{\cdot}5 - 0{\cdot}3413 = 0{\cdot}1587$$

Therefore out of 10,000 bulbs, the number of bulbs which fail in the first 800 hours is

$$10{,}000 \times 0{\cdot}1587 = 1587$$

(*ii*) Required probability $= P(800 < X < 1200) = P(-1 < Z < 1)$

$$= 2\,P(0 < Z < 1) = 2 \times 0{\cdot}3413 = = 0{\cdot}6826$$



Hence the expected number of blubs with life between 800 and 1,200 hours of burning life is: $10{,}000 \times 0{\cdot}6826 = 6826$

(*a*) Let 10% of the bulbs fail after x_1 hours of burning life. Then we have to find x_1 such that $P(X < x_1) = 0{\cdot}10$

When $X = x_1$, $\qquad Z = \dfrac{x_1 - 1000}{200} = -z_1$ (say).

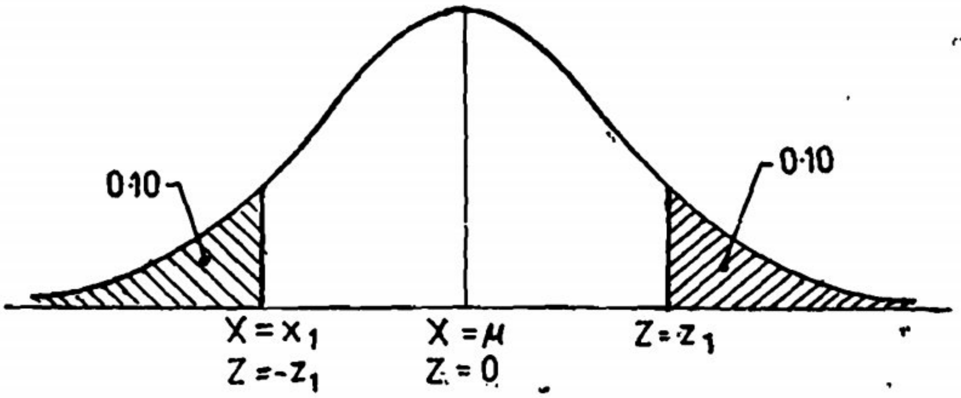
$\therefore \qquad P(Z < -z_1) = 0{\cdot}10 \Rightarrow P(Z > z_1) = 0{\cdot}10$

$\Rightarrow \qquad P(0 < Z < z_1) = 0{\cdot}40 \qquad ...(1)$

We are given that

$P(-1{\cdot}28 < Z < 1{\cdot}28) = 0{\cdot}80 \Rightarrow 2\,P(0 < Z < 1{\cdot}28) = 0{\cdot}80$

$\Rightarrow \qquad P(0 < Z < 1{\cdot}28) = 0{\cdot}40 \qquad ...(2)$



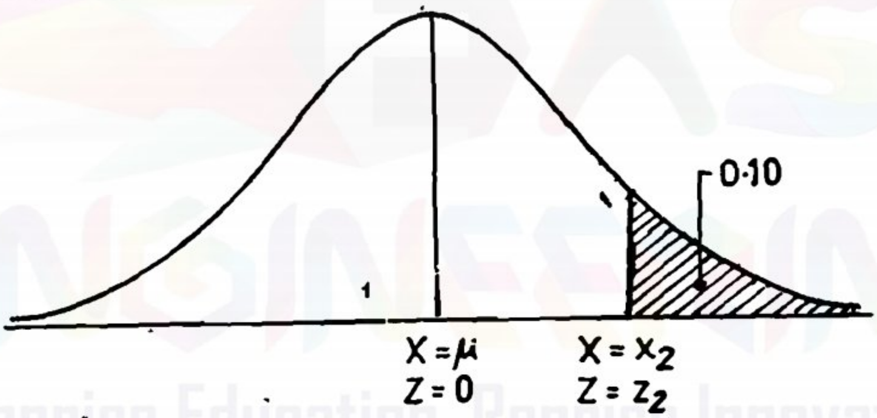
∴ From (1) and (2), we get

$$z_1 = 1{\cdot}28$$

Hence $\quad \dfrac{x_1 - 1000}{200} = -1{\cdot}28 \Rightarrow x_1 = 1000 - 256 = 744$

Thus after 744 hours of burning life, 10% of the blubs will fail.

(*b*) Let 10% of the blubs be still burning after, (say), x_2 hours of burning life. Then we have



$$P(X > x_2) = 0{\cdot}10 \Rightarrow P(Z > z_2) = 0{\cdot}10 \qquad \left[z_2 = \frac{x_2 - 1000}{200}\right]$$

$$\Rightarrow \qquad P(0 < Z < z_2) = 0{\cdot}40$$

$$\therefore \qquad z_2 = 1{\cdot}28 \qquad \text{[From (2)]}$$

i.e., $\quad \dfrac{x_2 - 1000}{200} = 1{\cdot}28 \Rightarrow x_2 = 1000 + 256 = 1256$

Hence after 1256 hours of burning life, 10% of the blubs will be still burning.

Example 8·24. *Let* $X \sim N(\mu, \sigma^2)$. *If* $\sigma^2 = \mu^2, (\mu > 0)$, *express* $P(X < -\mu \mid X < \mu)$ *in terms of cumulative distribution function of* $N(0, 1)$.

[Delhi Univ. B.Sc. (Maths. Hons.) 1988; (Stat. Hons.). 1993]

Solution.

$$P(X < -\mu \mid X < \mu) = \frac{P(X < -\mu \cap X < \mu)}{P(X < \mu)} = \frac{P(X < -\mu)}{P(X < \mu)}; \qquad (\because \mu > 0)$$

$$= \frac{P(Z < -2)}{P(Z < 0)} \qquad \left(Z = \frac{X-\mu}{\sigma} = \frac{X-\mu}{\mu} \right)$$

$$= \frac{P(Z > 2)}{(1/2)}; \qquad \text{(By symmetry)}$$

$$= 2[1 - P(Z \le 2)] = 2[1 - \Phi(2)]$$

where $\Phi(.)$ is the distribution function of standard normal variate.

Example 8·25 *Can X and $-X$ have the same distribution?*

If so, when? **[Delhi Univ. B.A., (Spl. Course Statistics), 1989]**

Solution. Yes; X and $-X$ can have the same distribution provided the p.d.f. $f(x)$ of X is symmetric about origin *i.e.,* if $f(-x) = f(x)$.

For example, X and $-X$ have the same distribution if :

(*i*) $X \sim N(0, 1)$

(*ii*) X has standard cauchy distribution [c.f. § 8·9]

$$f(x) = \frac{1}{\pi} \cdot \frac{1}{(1+x^2)}; \; -\infty < x < \infty$$

(*iii*) X has standard Laplace distribution [c.f. § 8·7]

$$p(x) = \tfrac{1}{2} e^{-|x|}; \; -\infty < x < \infty.$$

and so on. Obviously X and $Y = -X$ are not identical.

Remark. This example illustrates that if the r.v.'s. X and Y are identical, they have the same distributions. However if X and Y have the same distribution, it does not imply that they are identical.

Example 8·26. *If X, Y are independent normal variates with means 6, 7 and variances 9, 16 respectively, determine λ such that*

$$P(2X + Y \le \lambda) = P(4X - 3Y \ge 4\lambda)$$

[Delhi Univ. B.Sc. (Stat. Hons.), 1988; B.Sc., 1987]

Solution. Since X and Y are independent, by § 8·2·8 [c.f. equation (8·15*a*)], we have

$$U = 2X + Y \sim N(2 \times 6 + 7, 4 \times 9 + 16), \; i.e., \; U \sim N(19, 52)$$

$$V = 4X - 3Y \sim N(4 \times 6 - 3 \times 7, 16 \times 9 + 9 \times 16), \; i.e., \; V \sim N(3, 288)$$

and $\quad P(2X + Y \le \lambda) = P(U \le \lambda) = P\left(Z \le \frac{\lambda - 19}{\sqrt{52}}\right)$, where $Z \sim N(0, 1)$

and $\quad P(4X - 3Y \ge 4\lambda) = P(V \ge 4\lambda) = P\left(Z \ge \frac{4\lambda - 3}{12\sqrt{2}}\right)$, where $Z \sim N(0, 1)$

Now $\quad P(2X + Y \le \lambda) = P\{(4X - 3Y) \ge 4\lambda\}$

$$\Rightarrow \quad P\left(Z \le \frac{\lambda - 19}{\sqrt{52}}\right) = P\left(Z \ge \frac{4\lambda - 3}{12\sqrt{2}}\right)$$

$$\Rightarrow \quad \frac{\lambda - 19}{\sqrt{52}} = -\left(\frac{4\lambda - 3}{12\sqrt{2}}\right)$$

[Since $\quad P(Z \le a) = P(Z \ge b) \Rightarrow a = -b$,
because normal probability curve is symmetric about $Z = 0$].

$$\Rightarrow \qquad \frac{\lambda - 19}{\sqrt{13}} = \frac{3 - 4\lambda}{6\sqrt{2}}$$

$$\Rightarrow \qquad (6\sqrt{2} + 4\sqrt{13})\,\lambda = 114\sqrt{2} + 3\sqrt{13}$$

$$\Rightarrow \qquad \lambda = \frac{114\sqrt{2} + 3\sqrt{13}}{6\sqrt{2} + 4\sqrt{13}}$$

Example 8·27. *If X and Y are independent normal variates possessing a common mean* μ *such that*

$$P(2X + 4Y \le 10) + P(3X + Y \le 9) = 1$$
$$P(2X - 4Y \le 6) + P(Y - 3X \ge 1) = 1,$$

determine the values of μ and the ratio of the variances of X and Y.

Solution. Let $\text{Var}(X_1) = \sigma_1^2$ and $\text{Var}(Y) = \sigma_2^2$

Since $E(X) = E(Y) = \mu$, (Given) and X and Y are independent by § 8·2·8 [c.f. equation (8·15*a*)], we have

$$2X + 4Y \sim N(2\mu + 4\mu, 4\sigma_1^2 + 16\sigma_2^2), \text{ i.e., } N(6\mu, 4\sigma_1^2 + 16\sigma_2^2)$$

$$3X + Y \sim N(3\mu + \mu, 9\sigma_1^2 + \sigma_2^2), \text{ i.e., } N(4\mu, 9\sigma_1^2 + \sigma_2^2)$$

$$2X - 4Y \sim N(2\mu - 4\mu, 4\sigma_1^2 + 16\sigma_2^2), \text{ i.e., } N(-2\mu, 4\sigma_1^2 + 16\sigma_2^2)$$

$$Y - 3X \sim N(\mu - 3\mu, \sigma_2^2 + 9\sigma_1^2), \text{ i.e., } N(-2\mu, 9\sigma_1^2 + \sigma_2^2)$$

Let us further write :

$$4\sigma_1^2 + 16\sigma_2^2 = \alpha^2 \quad \text{and} \quad 9\sigma_1^2 + \sigma_2^2 = \beta^2 \qquad ...(1)$$

If Z denotes the Standard Normal Variate, *i.e,* if $Z \sim N(0, 1)$, we get

$$P(2X + 4Y \le 10) + P(3X + Y \le 9) = 1$$

$$\Rightarrow \qquad P\left(Z \le \frac{10 - 6\mu}{\alpha}\right) + P\left(Z \le \frac{9 - 4\mu}{\beta}\right) = 1$$

$$\Rightarrow \qquad P\left(Z \le \frac{10 - 6\mu}{\alpha}\right) = 1 - P\left(Z \le \frac{9 - 4\mu}{\beta}\right) = P\left(Z \ge \frac{9 - 4\mu}{\beta}\right)$$

$$\Rightarrow \qquad \frac{10 - 6\mu}{\alpha} = -\left(\frac{9 - 4\mu}{\beta}\right), \qquad ...(2)$$

(Since normal distribution is symmetric about $Z = 0$).

Similarly

$$P(2X - 4Y \le 6) + P(Y - 3X \ge 1) = 1$$

$$\Rightarrow \qquad P\left(Z \le \frac{6 + 2\mu}{\alpha}\right) + P\left(Z \ge \frac{1 + 2\mu}{\beta}\right) = 1$$

$$\Rightarrow \qquad P\left(Z \le \frac{6 + 2\mu}{\alpha}\right) = 1 - P\left(Z \ge \frac{1 + 2\mu}{\beta}\right) = P\left(Z \le \frac{1 + 2\mu}{\beta}\right)$$

$$\Rightarrow \qquad \frac{6 + 2\mu}{\alpha} = \frac{1 + 2\mu}{\beta} \qquad ...(3)$$

Solving (2) and (3), we get

$$\frac{\alpha}{\beta} = \frac{6 + 2\mu}{1 + 2\mu} = \frac{10 - 6\mu}{4\mu - 9} \qquad ...(4)$$

$\Rightarrow \quad (6+2\mu)(4\mu-9)=(10-6\mu)(1+2\mu)$

$\Rightarrow \quad 5\mu^2-2\mu-16=0$ (On simplification)

$$\Rightarrow \quad \mu=\frac{2\pm\sqrt{4+320}}{10}=\frac{2\pm 18}{10}$$

$\Rightarrow \quad \mu=2 \text{ or } -1\cdot 6$

Substituting $\mu=2$ in (4), we get.

$$\frac{\alpha}{\beta}=\frac{10}{5}=2, \quad i.e., \quad 4=\frac{\alpha^2}{\beta^2}$$

From (1), we get

$$4=\frac{4\sigma_1^2+16\sigma_2^2}{9\sigma_1^2+\sigma_2^2}=\frac{4+16\lambda}{9+\lambda} \qquad \left[\text{Taking } \lambda=\frac{\sigma_2^2}{\sigma_1^2}\right]$$

$$\Rightarrow \quad 4(9+\lambda)=4+16\lambda \Rightarrow \lambda=\frac{32}{12}=\frac{8}{3}$$

Again putting $\mu=-1\cdot 6$ in (4), we get

$$\left(\frac{14}{11}\right)^2=\frac{\alpha^2}{\beta^2}=\frac{4+16\lambda}{9+\lambda} \Rightarrow \lambda=\frac{1280}{1740}=\frac{64}{87}$$

Example 8·28. *If two normal universes A and B have the same total frequency but the standard deviation of universe A is k times that of the universe B, show that maximum frequency of universe A is 1/k times that of universe B.*

Solution. Let N be the same total frequency for each of the two universes A and B. If σ is the standard deviation of universe B, then the standard deviation of universe A is $k\sigma$. Let μ_1 and μ_2 be the means of the universes A and B respectively.

The frequency function of universe A is given by

$$f_A(x)=\frac{N}{k\sigma\sqrt{2\pi}}\exp\left\{-(x-\mu_1)^2/2k^2\sigma^2\right\}$$

and the frequency function of universe B is given by

$$f_B(x)=\frac{N}{\sigma\sqrt{2\pi}}\exp\left\{-(x-\mu_2)^2/2\sigma^2\right\}$$

Since, for a normal distribution, the maximum frequency occurs at the point $x=$ mean, we have

$$\begin{aligned}[f_A(x)]_{max} &= \text{Maximum frequency of universe } A\\ &=\left[f_A(x)\right]_{x=\mu_1}\\ &=\left[\frac{N}{k\sigma\sqrt{2\pi}}\exp\left\{-(x-\mu_1)^2/2k^2\sigma^2\right\}\right]_{x=\mu_1}=\frac{N}{k\sigma\sqrt{2\pi}}\end{aligned}$$

Similarly

$$[f_B(x)]_{max}=[f_B(x)]_{x=\mu_2}$$

$$= \left[\frac{N}{\sigma\sqrt{2\pi}} \exp\{-(x-\mu_2)^2/2\sigma^2\} \right]_{x=\mu_1} = \frac{N}{\sigma\sqrt{2\pi}}$$

$$\therefore \qquad \frac{[f_A(x)]_{\max}}{[f_B(x)]_{\max}} = \frac{1}{k}$$

EXERCISE 8 (b)

1. "If the Poisson and the Normal distributions are limiting cases of Binomial distribution, then there must be a limiting relation between the Poisson and the Normal distributions." Investigate the relation.

2. (*a*) Derive the mathematical form and properties of normal distribution Discuss the importance of normal distribution in Statistics.

(*b*) Mention the chief characteristics of Normal distribution and Normal probability curve. **[Delhi Univ. B.Sc. (Stat Hons.), 1989]**

3. (*a*) Explain, under what conditions and how the binomial distribution can be approximated to the normal distribution.

(*b*) For a normal distribution with mean 'μ' and standard deviation σ, show that the mean deviation from the mean 'μ' is equal to $\sigma\sqrt{(2/\pi)}$. What will be the mean deviation from median ?

(*c*) The distribution of a variable X is given by the law:

$$f(x) = \text{Constant} \exp\left[-\frac{1}{2}\left(\frac{x-100}{5}\right)^2\right], -\infty < x < \infty$$

Write down the value of :

(*i*) the constant,
(*ii*) the mean,
(*iii*) the median,
(*iv*) the mode,
(*v*) standard deviation,
(*vi*) the mean deviation,
(*vii*) the quartile deviation of the distribution.

(Gujarat Univ. B.Sc. April 1978)

Ans. (*i*) $\frac{1}{5\sqrt{2\pi}}$, (*ii*) 100, (*iii*) 100, (*iv*) 100, (*v*) 5 (*vi*) $\sqrt{(2/\pi)} \times 5 \simeq 4$, (*vii*) $\frac{2}{3} \times 5 = 3{\cdot}33$ **(approx .)**.

(*d*) Define Normal probability distribution. If the mean of a Normal population is μ and its variance σ^2, what are its (*i*) mode, (*ii*) Median, (*iii*) β_1 and β_2 ?

(*e*) For a normal distribution $N(\mu, \sigma^2)$:

(*i*) Show that the mean, the median and the mode coincide.

(*ii*) Find the recurrence relation between μ_{2n} and μ_{2n-2}.

(*iii*) State and prove additive property of normal variates.

(*iv*) Obtain the points of inflexion for the normal distribution $N(\mu, \sigma^2)$.

(*v*) Obtain mean deviation about mean.

[Delhi Univ. B.Sc. (Stat. Hons.), 1988]

(*f*) Show that any linear combination of n independent normal variates is also a normal variate. **[Delhi Univ. B.Sc. (Stat. Hons.), 1989]**

(*g*) Show that for the normal curve :

(*i*) The maximum occurs at the mean of the distribution, and

(*ii*) the points of inflexion lie at a distance of $\pm \sigma$ from the mean, where σ is the standard deviation. **[Delhi Univ. M.A. (Eco.), 1987]**

(*h*) Describe the steps involved in fitting a normal distribution to the given data and computing the expected frequencies.

(*i*) Explain how the normal probability integral

$$\int_0^{z_1} \varphi(z)\, dz ,$$

is used in computing normal probabilities.

4. Write a note on the salient features of a normal distribution. $N(\mu, \sigma^2)$ denotes the normal distribution of each of the random variables $X_1, X_2, X_3, \ldots, X_n$, where μ is the mean and σ^2 the variance. Prove the following :

(*i*) If $X_1, X_2, \ldots, X_n$ are independent, then $X_1 + X_2 + \ldots + X_n$ has the distribution $N(n\mu, n\sigma^2)$.

(*ii*) kX, where k is a constant has the distribution $N(k\mu, k^2\sigma^2)$.

(*iii*) $X + a$, where a is a constant has the distribution $N(\mu + a, \sigma^2)$

(*iv*) In (*i*) if $\bar{X} = \dfrac{X_1 + X_2 + \ldots + X_n}{n}$ then

$$\sqrt{n}\,\frac{(\bar{X} - \mu)}{\sigma} \text{ has the distribution } N(0, 1).$$

5. (*a*) Show that for a normal distribution with mean μ and variance σ^2, the central moments satisfy the relation

$$\mu_{2n} = (2n-1)\,\mu_{2n-2}\,\sigma^2 \;;\; \mu_{2n+1} = 0$$

[Delhi Univ. B.Sc. (Stat. Hons.), 1987]

Hence show that $\mu_{2n} = \dfrac{(2n)!}{n!}\left(\tfrac{1}{2}\sigma^2\right)^n$ and $\mu_{2n+1} = 0 \,;\, n = 1, 2, \ldots$

[Delhi Univ. B.Sc. (Stat Hons.) 1985]

(*b*) State the mathematical equation of a normal curve. Discuss its chief features.

(*c*) Find the moment generating function of the normal distribution (m, σ^2), and deduce that

$$\mu_{2n+1} = 0,$$
$$\mu_{2n} = 1 \cdot 3 \cdot 5 \ldots (2n-1)\,\sigma^{2n},$$

where μ_n denotes the nth central moment.

[Delhi Univ. B.Sc. (Stat. Hons.) 1990,' 82]

(*d*) Show that all central moments of a normal distribution can be expressed in terms of the standard deviation and obtain the expression in the general case.

[Aligarh Univ. B.Sc. 1992]

(*e*) The normal table gives the values of the integral:

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{1}{2}t^2\right) dt$$

for different values of x.

Explain how to use this table to obtain the proportion of observations of a normal variate with mean μ and S.D. σ, which lie above a given value 'a',

(*i*) where $a > \mu$, (*ii*) where $a < \mu$.

6. (*a*) If X_1 and X_2 and two independent normal variates with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2 respectively, show that the variables U and V where $U = X_1 + X_2$ and $V = X_1 - X_2$, are independent normal variates. Find the means and variances of U and V.

(*b*) If X_1 and X_2 are independent standard normal variates obtain the p.d.f. of $(X_1 - X_2)/\sqrt{2}$.

Ans. $U = (X_1 - X_2)/\sqrt{2} \sim N(0, 1)$

(*c*) Suppose $X_1 \sim N(0, 1)$ and $X_2 \sim N(0, 1)$ are independent r.v.'s.

(*i*) Find the joint distribution of $(X_1 + X_2)/\sqrt{2}$ and $(X_1 - X_2)/\sqrt{2}$.

(*ii*) Argue that $2X_1X_2$ and $X_2^2 - X_1^2$ have the same distribution.

Ans. (*i*) $U = (X_1 + X_2)/\sqrt{2}$ and $V = (X_1 - X_2)/\sqrt{2}$ are independent $N(0, 1)$ variates

(*ii*) **Hint.** $X_2^2 - X_1^2 = 2\left[\dfrac{X_2 + X_1}{\sqrt{2}}\right]\left[\dfrac{X_2 - X_1}{\sqrt{2}}\right] = 2(UV)$

$= 2 \times$ **[Product of two independent SNV 's]**

$2X_1X_2 = 2 \times$ **[Product of two independent SNV 's]**

Hence the result.

7. (*a*) Let X be normally distributed with mean 8 and s.d. 4. Find

(*i*) $P(5 \le X \le 10)$, (*ii*) $P(10 \le X \le 15)$, (*ii*) $P(X \ge 15)$, (*iv*) $P(X \le 5)$.

Ans. (*i*) 0·4649 (*ii*) 0·2684 (*iii*) 0·0401 (*vi*) 0·2266.

(*b*) The standard deviation of a certain group of 1,000 high school grades was 11% and the mean grade 78%. Assuming the distribution to be normal, find

(*i*) How many grades were above 90%?

(*ii*) What was the highest grade of the lowest 10?

(*iii*) What was the interquartile range ?

(*iv*) Within what limits did the middle 90% lie?

Ans. (*i*) 138, (*ii*) 52, (*iii*) $Q_1 = 70{\cdot}575$; $Q_3 = 85{\cdot}425$, and (*iv*) 60 % to 96·2%

(*c*) If X is normally distributed with mean 2 and variance 1, find $P(|X - 2| < 1)$. **Ans.** 0·6826 [or $\Phi(1) - \Phi(-1)$]

(*d*) If $X \sim N(\mu = 2, \sigma^2 = 2)$, find $P(|X - 1| \leq 2)$ in terms of distribution function of standard normal variate.

Ans. Probability $= P(-1 \leq X \leq 3) = \Phi(1/\sqrt{2}) - \Phi(-3/\sqrt{2})$

(*e*) If $X \sim N(30, 5^2)$ and $Y \sim N(15, 10^2)$, show that

$$P(26 \leq X \leq 40) = P(7 \leq Y \leq 35).$$

Hint. Each Probability $= P(-0{\cdot}8 \leq Z \leq 2)$ where $Z \sim N(0, 1)$

(*f*) If $X \sim N(30, 5^2)$, find the probabilities of

(*i*) $26 \leq X \leq 40$, (*ii*) $|X - 30| > 5$, (*iii*) $X \geq 42$, (*iv*) $X \leq 28$

[Bihar P.C.S., 1988]

Ans. (*i*) 0·7653, (*ii*) 0·3174, (*iii*) 0·0082, (*iv*) 0·3446

8. (*a*) In a normal population with mean 15·00 and standard deviation 3·5, it is known that 647 observations exceed 16·25. What is the total number of observations in the population? **(Sri Venkateswara Univ. B.Sc. April 1990)**

Hint. Let $X \sim N(\mu, \sigma^2)$ where $\mu = 15$ and $\sigma = 3{\cdot}5$.

If N is the total number of observations in the population,
then we have to find N such that

$$N \times P(X > 16{\cdot}25) = 647$$

(*b*) Assume the mean heights of soldiers to be 68·22 inches with a variance of 10·8 (in.)^2. How many soldiers in a regiment of 1,000 would you expect to be over 6 feet tall? (Given that the area under the standard normal curve between $X = 0$ and $X = 0{\cdot}35$ is 0·1368 and between $X = 0$ and $X = 1{\cdot}15$ is 0·3746).

Ans. 125 **[Osmania Univ. M.A., 1992]**

9. (*a*) If 100 true coins are thrown, how would you obtain an approximation for the probability of getting (*i*) 55 heads, (*ii*) 55 or more heads, using Tables of Area of normal probability function.

(*b*) Prove that Binomial distribution in certain cases becomes normal.

A six faced dice is thrown 720 times. Explain how an approximate value of the probability of the following events can be found out easily. (Finding out the numerical values of these probabilities is not necessary):

(*i*) 'six' comes for more than 130 times

(*ii*) chance of 'six' lies between 100 and 140.

10. (*a*) The number (X) of items of a certain kind demanded by customers follows the Poisson law with parameter 9. What stock of this item should a retailer keep in order to have a probability of 0·99 of meeting all demands made on him? Use normal approximation to the Poisson law.

(*b*) Show that the probability that the number of heads in 400 throws of a fair coin lies between 180 and 220 is $\simeq 2F(2) - 1$, where $F(x)$ denotes the standard normal distribution function.

11. In an intelligence test administered to 1,000 children, the average score is 42 and standard deviation 24.

(*i*) Find the number of children exceeding the score 60, and (*ii*) Find the number of children with score lying between 20 and 40. (Assume the normal distribution.) **Ans.** (*i*) 227 (*iii*) 289

12. The mean I.Q. (intelligence quotient) of a large number of children of age 14 was 100 and the standard deviation 16. Assuming that the distribution was normal, find

(*i*) What % of the children had I.Q. under 80?

(*ii*) Between what limits the I.Q.'s of the middle 40% of the children lay?

(*iii*) What % of the children had I.Q.'s within the range $\mu \pm 1.96\,\sigma$?

Ans. (*i*) 10·56%, (*ii*) 91·6, 108·4, (*iii*) 0·95

13. (*a*) In a university examination of a particular year, 60% of the students failed when mean of the marks was 50% and s.d. 5%. University decided to relax the conditions of passing by lowering the pass marks, to show its result 70%. Find the minimum marks for a student to pass, supposing the marks to be normally distributed and no change in the performance of students takes place.

Ans. 47·375.

(*b*) The width of a slot on a forging is normally distributed with mean 0·900 inch and standard deviation 0·004 inch. The specifications are 0·900 ± 0·005 inch. What percentage of forgings will be defective ?

Hint. Let X denote the width (in inches) of the slot. We want

$$\begin{aligned} & 100 \times P\,(X \text{ lies outside specification limits}) \\ &= 100\,[\,1 - P\,(X \text{ lies within specification limits})\,] \\ &= 100\,[\,1 - P\,(0{\cdot}895 < X < 0{\cdot}905)\,]\,. \end{aligned}$$

14. (*a*) The monthly incomes of a group of 10,000 persons were found to be normally distributed with mean Rs. 750 and s.d. Rs. 50. Show that of this group, about 95% had income exceeding Rs. 668 and only 5% had income exceeding Rs 832. What was the lowest income among the richest 100?

Ans. Rs. 866·3 .

(*b*) Given that X is normally distributed with mean 10 and

$$P\,(X > 12) = 0{\cdot}1587,$$

what is the probability that X will fall in the interval (9, 11)?

Take $\Phi\,(1) = 0{\cdot}8413$ and $\Phi\,(-\tfrac{1}{2}) = 0{\cdot}3085$

where $$\Phi\,(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\,(-u^2/2)\,du$$

Ans. 0·3830

(*c*) A normal distribution has mean 25 and variance 25. Find

(*i*) the limits which include the middle 50% of the area under the curve, and

(*ii*) the values of x corresponding to the points of inflexion of the curve..

Ans. (*i*) Limits which include the middle 50% of the area under the curve are:

$Q_1 = \mu - 0{\cdot}6745\,\sigma = 21{\cdot}7275$; $Q_3 = \mu + 0{\cdot}6745\,\sigma = 38{\cdot}2725$

(*ii*) (30, 20)

15. (*a*) In a distribution exactly normal 7% of the items are under 35 and 89% are under 63. What are the mean and standard deviation of the distribution?

Ans. $\mu = 50{\cdot}3, \quad \sigma = 10{\cdot}33$.

(*b*) In a normal distribution, 31% of the items are under 45 and 8% are over 64. Find the mean and variance of the distribution.

Given that area between mean-ordinates and ordinate at any σ distance from mean,

$$Z = \frac{X-\mu}{\sigma} : \quad 0{\cdot}496 \quad 1{\cdot}405$$

$$\text{Area} \quad : \quad 0{\cdot}19 \quad 0{\cdot}42$$

[Delhi Univ. B.Sc., 1987; Madras Univ. B.Sc., 1990]

Ans. $\mu = 50, \sigma = 10$

16. (*a*) A minimum height is to be prescribed for eligibility to government services such that 60% of the young men will have a fair chance of coming up to that standard. The heights of youngmen are normally distributed with mean 60·6" and s.d. 2·55". Determine the minimum specification.

Ans. 59·9".

Hint. We want x_1 s.t. $P(X > x_1) = 0{\cdot}6$

When $X = x_1$, $Z = \dfrac{x_1 - 60{\cdot}6}{2{\cdot}55} = -z_1$, (say) . ...(*)

[Note the negative sign, which is obvious from the diagram]

Obviously $P(0 < Z < z_1) = 0{\cdot}10 \Rightarrow z_1 = 0{\cdot}254$

Substituting in (*), we get

$$x_1 = 60{\cdot}6 - 2{\cdot}55 \times 0{\cdot}254 = 60{\cdot}6 - 0{\cdot}65 = 59{\cdot}95"$$

(*b*) The height measurements of 600 adult males are arranged in ascending order and it is observed that 180th and 450th entries are 64·2" and 67·8" respectively. Assuming that the sample of heights is drawn from a normal population, estimate the mean and s.d. of the distribution.

Ans. 67·78", 3".

17. (*a*) Marks secured by students in sections I and II of a class are independently normally distributed with means 50 and 60 respectively and variances 10 and 6 respectively. What is the probability that a randomly chosen student from section II scores more marks than a randomly chosen student from section I? What percentage of students are expected to secure first division (*i.e.*, 60 marks or more) in section I? Write down your results in terms of the standard normal distribution fucntion.

Hint. $X \sim N\,(50,\ 10)$, $Y \sim N\,(60,\ 6)$ are independent r.v.'s. $U = Y - X \sim N\,(10, 16)$. We want $P(Y > X) = P(U > 0)$.

(*b*) In an examination, the mean and standard deviation (s.d.) of marks in Mathematics and Chemistry are given below

	Mean	s.d.
Maths.	45	10
Chem.	50	15

Assuming the marks in the two subjects to be independent normal variates, obtain the probability that a student scores total marks lying between 100 and 130. [Full marks in each subject are 100]. Given that

$$F(0{\cdot}28) = 0{\cdot}1103,\ F(1{\cdot}94) = 0{\cdot}4738,$$

where
$$F(z) = \frac{1}{\sqrt{2\pi}} \int_0^z \exp(-\tfrac{1}{2}x^2)\,dx.$$

[Bhagalpur Univ. B.Sc., 1990]

18. (*a*) One thousand candidates in an examination were grouped into three classes I, II, III in descending order of merits. The numbers in the first two classes were 50 and 350 respectively. The highest and the lowest marks in class II were 60 and 50 respectively. Assuming the distribution to be normal, prove that the average mark is approximately 48·2 and standard deviation, approximately 7·1. The following data may be used:

The area A is measured from the mean zero to any ordinate X.

$\frac{X}{\sigma}$	A	$\frac{X}{\sigma}$	A
0·2	0·079	1·5	0·433
0·3	0·118	1·6	0·445
0·4	0·155	1·7	0·455

(*b*) In an examination marks obtained by the students in Mathematics, Physics and Chemistry are distributed normally about the means 50, 52 and 48 with S.D. 15, 12, 16 respectively. Find the probability of securing total marks of

(*i*) 180 or above, (*ii*) 90 or below.

$$\left[\frac{1}{\sqrt{2\pi}} \int_{1\cdot 2}^{\infty} \exp(-z^2/2)\,dz = 0{\cdot}1942,\ \frac{1}{\sqrt{2\pi}} \int_{2\cdot 4}^{\infty} \exp(-z^2/2)\,dz = 0{\cdot}0224\right]$$

Ans. 0·1942, 0·0224 **[Agra Univ. B.Sc., 1988]**

19. In a certain examination the percentage of passes and distinctions were 46 and 9 respectively. Estimate the average marks obtained by the candidates, the minimum pass and distinction marks being 40 and 75 respectively. (Assume the distribution of marks to be normal.) (**Ans.** $\mu = 36{\cdot}4$, $\sigma = 28{\cdot}2$)

Also determine what would have been the minimum qualifying marks for admission to a re-examination of the failed candidates, had it been desired that the best 25% of them should be given another opportunity of being examined.

Ans. 29.

20. The local authorities in a certain city installed 2,000 electric lamps in a street of the city. If the lamps have an average life of 1,000 burning hours with a S.D. of 200 hours,

(*i*) What number of the lamps might be expected to fail in the first 700 burning hours,

(*ii*) After what periods of burning hours would we expect that

(*a*) 10% of the lamps would have failed, and

(*b*) 10% of the lamps would be still burning?

Assume that lives of the lamps are normally distributed.

You are given that $F(1\cdot50) = 0\cdot933$, $F(1\cdot28) = \cdot900$,

where $$F(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\, dz$$

Ans. (*i*) 134, (*ii*) (*a*) 744, (*b*) 1256. **[Allahabad Univ. B.Sc., 1987]**

21. (*a*) The quartiles of a normal distribution are 8 and 14 respectively. Estimate the mean and standard deviation.

Ans. $\mu = 11$, $\sigma = 4\cdot4$.

(*b*) The third decile and the upper quartile of a normal distribution are 56 and 63 respectively. Find the mean and varaince of the distribution.

Ans. $\mu = 59\cdot1$, $\sigma = 5\cdot8$.

22. (*a*) 5,000 variates are normally distributed with mean 50 and probable error (semi-interquartile range) 13·49. Without using tables, find the values of the quartiles, median, mode standard deviation and mean deviation. Find also the value of the variate for which cumulative frequency is 1250.

[Meerut Univ. B.Sc., 1989]

Ans. $Q_1 = 36\cdot51$ $Q_3 = 63\cdot49$, $\sigma = 20$, M.D. $= 16$, $x_1 = 36\cdot51$.

(*b*) The following table gives frequencies of occurrence of a varaible X between certain limits :

Variable X	Frequency
Less than 40	30
40 or more but less than 50	33
50 and more	37

The distribution is exactly normal. Find the distribution and also obtain the frequency between $X = 50$ and $X = 60$. **[Kurukshetra Univ. M.A. (Eco.), 1990]**

Ans. Hint. $50 - \mu = 0\cdot33\,\sigma$; $40 - \mu = -0\cdot52\,\sigma$

$$\mu = 46\cdot12,\ \sigma = 11\cdot76$$

$N.P(50 < X < 60) = 100 \times 0\cdot2517 \simeq 25$

23. (*a*) Suppose that a doorway being constructed is to be used by a class of people whose heights are normally distributed with mean 70" and standard deviation 3". How long may the doorway be without causing more than 25% of the people to bump their heads? If the height of the doorway is fixed at 76", how many persons out of 5,000 are expected to bump their heads?

[For a normal distributionthe quartile deviation is 0·6745 times standard deviation. For a standard normal distribution $Z = \dfrac{X - \bar{X}}{\sigma}$, the area under the curve between $Z = 0$ and $Z = 2$ is 0 4762.]

(*b*) A normal population has a coefficient of variation 2% and 8% of the population lies above 120. Find the mean and S.D.

Ans. $\mu = 122, \sigma = 2{\cdot}44$

24. Steel rods are manufactured to be 3 inches in diameter but they are acceptable if they are inside the limits 2·99 inches and 3·01 inches. It is observed that 5% are rejected as oversize and 5% are rejected as undersize. Assuming that the diameters are normally distributed, find the standard deviation of the distribution. Hence calculate, what would be the proportion of rejects if the permissible limits were widened to 2·985 inches and 3·015 inches.

[**Hint.** Let X denote the diameter of the rods in inches and let $X \sim N(\mu, \sigma^2)$. Then we are given

$$P(X > 3{\cdot}01) = 0{\cdot}05 \quad \text{and} \quad P(X < 2{\cdot}99) = 0{\cdot}05$$

$$\Rightarrow \qquad \frac{3{\cdot}01 - \mu}{\sigma} = 1{\cdot}65 \quad \text{and} \quad \frac{2{\cdot}99 - \mu}{\sigma} = -1{\cdot}65$$

Solving we get $\mu = 3$ and $\sigma = \dfrac{1}{165}$

The probability that a random value of X lies within the rejection limits is

$$P(2{\cdot}985 < X < 3{\cdot}015) = P(-2{\cdot}475 < Z < 2{\cdot}475) = 2 \times P(0 < Z < 2{\cdot}475)$$
$$= 2 \times 0{\cdot}4933 = 0{\cdot}9866$$

Hence the probability that X lies outside the rejection limits is

$$1 - 0{\cdot}9866 = 0{\cdot}0134$$

Therefore, the proportion of the rejects outside the revised limits is 0·0134, *i.e.*, 1·34%].

25. Derive the moment generating function of a random variable which has a normal distribution with mean μ and variance σ^2. Hence or otherwise prove that a linear combination of independent normal variates is also normally distributed.

An investor has the choice of two of four investments X_1, X_2, X_3, X_4. The profits from these may be assumed to be independetnly distributed, and

the profit from X_1 is $N(2, 1)$,

the profit from X_2 is $N(3, 3)$,

the profit from X_3 is $N(1, \frac{1}{4})$.

the profit from X_4 is $N(2\frac{1}{2}, 4)$.

(**Profits are given in £ 1000 per annum).**

Which pair should he choose to maximise his probability of making a total annual profit of at least £ 2000? **(London Univ. B.Sc. 1977)**

26. (*a*) State the important properties of the normal distribution and obtain from the tables the inter-quartile range in terms of its mean μ and standard deviation σ.

Find the mean and standard deviation as well as the inter-quartile range of the following data. Compare the inter-quartile range with that obtained from mean and standard deviation on the assumption of normality.

X (central values) ...	0	1	2	3	4	5	6
f (frequency) ...	5	9	15	32	21	10	8

(*b*) The following table gives Baseball throws for a distance by 303 first year high school girls:

Distance in feet	Number of girls	Distance in feet	Number of girls
15 and under 25	1	85 and under 95	44
25 and under 35	2	95 and under 105	31
35 and under 45	7	105 and under 115	27
45 and under 55	25	115 and under 125	11
55 and under 65	33	125 and under 135	4
65 and under 75	53	135 and under 145	1
75 and under 85	64		

(*i*) Fit a normal distribution and find the theoretical frequencies for the classes of the above frequency distribution.

(*ii*) Find the expected number of girls throwing baseballs at a distance exceeding 105 feet on the basis that the data fit a normal distribution.

27. (*a*) The table given below shows the distribution of heights among freshmen in a college :

Height in inches	61	62	63	64	65	66	67	68
Frequency	4	20	23	75	114	186	212	252
Height in inches	69	70	71	72	73	74		
Frequency	218	175	149	46	18	8		

By comparing the proportion of cases lying between $\mu \pm (2/3)\,\sigma$, $\mu \pm \sigma$, $\mu \pm 2\,\sigma$ and $\mu \pm 3\,\sigma$, for this distribution and for a normal curve, state whether the distribution may be considered normal.

(*b*) Fit a normal distribution to the following data of heights in cms of 200 Indian adult males :

Height in (cms)	*Frequency*
144 — 150	3
150 — 156	12
156 — 162	23
162 — 168	52
168 — 174	61
174 — 180	39
180 — 186	10

(*c*) Two hundred and fifty-five metal rods were cut roughly six inches over size. Finally the lengths of the oversize amount were measured exactly and grouped with 1-inch intervals, there being in all 12 groups. The frequency distribution for the 255 lengths was

Central value : x	1	2	3	4	5	6
Frequency : f	2	10	19	25	40	44
x	7	8	9	10	11	12
f	41	28	25	15	5	1

Fit a normal distribution to the data by the method of ordinates and calculate the expected frequencies.

28. (*a*) Let $X \sim N(\mu, \sigma^2)$. Let

$\Phi(x) = P[X \le x]$,

calculate the probabilities of the following events in terms of Φ :

(*i*) $\alpha X + \beta \le t$, where α, β are finite constants.

(*ii*) $-X \ge t$

(*iii*) $|X| > t$ **[Poona Univ. B.E., 1991]**

(*b*) Determine C such that the following function becomes a distribution function:

$$F(x) = C \int_{-\infty}^{x} \exp\left[-\frac{(y-\mu)^2}{2\sigma^2}\right] dy$$

29. (*a*) Determine the constant C so that $C.e^{-2x^2+x}, -\infty < x < \infty$, is a density function. If the random variable X has the resulting density function, then find (*i*) the mean of X, (*ii*) the variance of X and (*iii*) $P(X \ge 1/4)$.

Ans. (*i*) 0·25 (*ii*) 0·25 (*iii*) 0·5

(*b*) If $f(x) = k \cdot \exp.\left\{-(9x^2 - 12x + 13)\right\}$, is the p.d.f. of a normal distribution (k, being a constant) find the mean and s.d. of the distribution.

(c) If X is a normal variate with p.d.f. $f(x) = 0{\cdot}03989 \exp(-0{\cdot}005\,x^2 + 0{\cdot}5\,x - 12{\cdot}5)$, express $f(x)$ in standard form and hence find the mean and variance of X. **[M.S. Baroda Univ. B.Sc., 1991]**

(d) Let the probability function of the normal distribution be

$$P(x) = ke^{-1/8\,x^2 + 2x}, \quad -\infty < x < \infty$$

Find k, μ and σ^2. **[Delhi Univ. B.Sc. (Stat. Hons.), 1985]**

(e) X_1, X_2, X_3, X_4 is a random sample from a normal distribution with mean 100 and variance 25 and $\bar{X} = \frac{1}{4}(X_1 + X_2 + X_3 + X_4)$.

State the distribution, expected value and variance of each of the following:

(i) $4\bar{X}$, (ii) $X_1 - 2X_2 + X_3 - 3X_4$,

(iii) $\frac{1}{25}\sum_{i=1}^{4}\{X_i - 100\}^2$ **[Bangalore Univ. B.Sc., 1989]**

Ans. (b) Mean $= 2/3$, $\sigma = \frac{1}{3\sqrt{2}}$

30. If X is a normal variate with mean 50 and s.d. 10, find $P(Y \le 3137)$, where $Y = X^2 + 1$,

$$\left[\frac{1}{\sqrt{2\pi}}\int_0^{0{\cdot}6} e^{-x^2/2}\,dx = 0{\cdot}2258\right]$$

[Delhi Univ. B.Sc. (Hons.), 1990]

Hint. Required Probability $= P(X^2 + 1 \le 3137) = P(-56 \le X \le 56)$].

Ans. 0·7258

31. Let X be normally distributed with mean μ and variance σ^2. Suppose σ^2 is some function of μ, say $\sigma^2 = h(\mu)$. Pick $h(.)$ so that $P(X \le 0)$ does not depend on μ for $\mu > 0$.

Ans. $P(X \le 0) = P(Z \le -\mu/\sqrt{h(\mu)}) = P(Z \le -1)$; independent of μ if we take $h(\mu) = \mu^2$.

32. (a) If X is a standard normal variate, find $E\,|X|$ [Ans. $\sqrt{2/\pi} \simeq 4/4$]

(b) X is a random variable normally distributed with mean zero and variance σ^2. Find $E\,|X|$ **[Delhi Univ. B.Sc. (Stat. Hons.) 1990]**

Hint. $E\,|X|$ = Mean Deviation about origin

= M.D. about mean ($\because$ Mean = 0)

Ans. $\sqrt{(2/\pi)}.\ \sigma \approx \frac{4}{5}\sigma$

32. (a) X is a normal variate with mean 1 and variance 4, Y is another normal variate independent of X with mean 2 and variance 3. What is the distribution of $X + 2Y$? **[Punjab Univ. B.Sc. (Hons.) 1993]**

Ans. $X + 2Y \sim N(5, 16)$

(*b*) If X is a normal variate with mean 1 and S.D. 0·6, obtain $P[X > 0]$, $P[|X - 1| \geq 0{\cdot}6]$ and $P[-1{\cdot}8 < X < 2{\cdot}0]$. What is the distribution of $4X + 5$?

34. (*a*) Let X and Y be two independent random variables each with a distribution which is $N(0, 1)$. Find the probability density function of $U = a_1 X + a_2 Y$, where a_1 and a_2 are constants.

(*b*) Show that if X_1, X_2 are mutually independent normal variates having means μ_1, μ_2 and standard deviations σ_1, σ_2 respectively, then $U = a_1 X_1 + a_2 X_2$ is also normally distributed.

34. (*c*) If X_i, $(i = 1, 2, ..., n)$ are independent $N(\mu_i, \sigma_i^2)$ variates, obtain the distribution of $\sum_{i=1}^{n} a_i X_i$

where a_i, $i = 1, 2, ..., n$ are constants. Hence deduce the distributions of :

(*i*) $X_1 + X_2$

(*ii*) $X_1 - X_2$

(*iii*) $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$; if X_i's are i.i.d. $N(\mu, \sigma^2)$.

How do the results in (*i*) and (*ii*) compare with those in Poisson distribution and result in (*iii*) compare with Cauchy distribution ?

[Delhi Univ. B.Sc. (Stat. Hons.), 1991]

Hint. For Cauchy distribution, see Remark 4, § 8·9·1 .

35. (*a*) If X is normal with mean 2 and standard deviation 3, describe the distribution of $Y = \frac{1}{2} X - 1$. Explain how you would find $P(Y \geq \frac{3}{2})$ from the tables.

Hint. (*a*) We are given that $X \sim N(\mu, \sigma^2)$ where $\mu = 2, \sigma = 3$. The distribution of the new variable $Y = aX + b$ is also normal with

$$\left.\begin{aligned} E(Y) &= E(aX + b) = aE(X) + b = a\mu + b \\ \text{and } \operatorname{Var}(Y) &= \operatorname{Var}(aX + b) = a^2 \operatorname{Var}(X) = a^2 \sigma^2 \end{aligned}\right\} \quad ...(*)$$

Hence $Y = \frac{1}{2} X - 1 \sim N(\mu_1, \sigma_1^2)$, where μ_1 and σ_1^2 are given by (*) with $a = \frac{1}{2}$ and $b = -1$, *i.e.*,

$$\mu_1 = \tfrac{1}{2} . 2 - 1 = 0 \,;\, \sigma_1^2 = (\tfrac{1}{2})^2 . 9 = \tfrac{9}{4}.$$

Thus $Y \sim N(\mu_1, \sigma_1^2)$, where $\mu_1 = 0, \sigma_1 = \frac{3}{2}$.

$P(Y \geq \frac{3}{2}) = P(Z \geq 1) = 0{\cdot}5 - P(0 < Z < 1) = 0{\cdot}5 - 0{\cdot}3413 = 0{\cdot}1587$.

(*b*) If X and Y are independent standard normal variables and if $Z = aX + bY + c$ where a, b and c are contants, what will be the distribution of Z? What is the mean, median and standard deviation of the distribution of Z?

Find $P(Z \leq 0{\cdot}1)$ if $a = 1, b = -1$ and $c = 0$. **(I.I.T. B. Tech. 1992)**

Hint. $Z \sim N(c, a^2 + b^2)$

If $a = 1, b = -1, c = 0$ then $Z = X - Y \sim N(0, 2)$

$$\therefore\ P(Z \le 0{\cdot}1) = P\left(U \le \frac{1}{14{\cdot}142}\right); \qquad U = \frac{Z-0}{\sqrt{2}} \sim N(0,1)$$

36. Let X be a random variable following normal distribution with mean μ and variance σ^2 and let r be a non-negative integer.

If $\mu_r' = E(X^r)$ and if $\mu_{2r} = [\,E(X-\mu)^{2r}\,]$, prove that

(*i*) $\mu'_{r+2} = 2\mu\,\mu'_{r+1} + (\sigma^2 - \mu^2)\,\mu_r' + \sigma^3 \dfrac{d\,\mu_r'}{d\sigma}$

(*ii*) $\mu_{2r+2} = \sigma^2\,\mu_{2r} + \sigma^3 \dfrac{d\,\mu_{2r}}{d\sigma}$ **[Madras Univ. B.Sc. (Main), Oct. 1989]**

Hint. (*i*)

$$\frac{d\,\mu_r'}{d\sigma} = -\int_{-\infty}^{\infty} \frac{x^r}{\sqrt{2\pi}\,\sigma^2} \exp\left\{-(x-\mu)^2/2\sigma^2\right\} dx + \int_{-\infty}^{\infty} \frac{x^r (x-\mu)^2}{\sqrt{2\pi}\,\sigma^4} \exp\left\{-(x-\mu)^2/2\sigma^2\right\} dx$$

$$= -\frac{\mu_r'}{\sigma} + \frac{\mu'_{r+2}}{\sigma^3} - \frac{2\mu\,\mu'_{r+1}}{\sigma^3} + \frac{\mu^2\,\mu_r'}{\sigma^3}$$

(*ii*)

$$\frac{d\,\mu_{2r}}{d\sigma} = -\int_{-\infty}^{\infty} \frac{(x-\mu)^{2r}}{\sqrt{2\pi}\,\sigma^2} \exp\left\{-(x-\mu)^2/2\sigma^2\right\} dx + \int_{-\infty}^{\infty} \frac{(x-\mu)^{2r+2}}{\sqrt{2\pi}\,\sigma^4} \exp\left\{-(x-\mu)^2/2\sigma^2\right\} dx = -\frac{\mu_{2r}}{\sigma} + \frac{\mu_{2r+2}}{\sigma^3}$$

37. Prove that if the independent random variables X and Y have the probability densities,

$$\frac{h}{\sqrt{\pi}}\, e^{-h^2x^2} \text{ and } \frac{k}{\sqrt{\pi}}\, e^{-k^2y^2}, -\infty < (x, y) < \infty$$

then the random variable $U = X + Y$ has the probability density,

$$\frac{l}{\sqrt{\pi}} \cdot e^{-l^2u^2}, -\infty < u < \infty$$

where $$\frac{1}{l^2} = \frac{1}{h^2} + \frac{1}{k^2}$$

38. If $\left[\sum\limits_{i=1}^{n} c_i\,\mu_i\right]^2 = 9 \sum\limits_{i=1}^{n} c_i^2\,\sigma_i^2$, find $P\left(0 \le Y \le 2 \sum\limits_{i=1}^{n} c_i\,\mu_i\right)$,

where $Y = \sum\limits_{i=1}^{n} c_i X_i$, X_i being a normal variate with mean μ_i and variance σ_i^2.

(Allahabad Univ. B.Sc., 1988)

Hint.

We know $Y = \sum_{i=1}^{n} c_i X_i \sim N(\mu, \sigma^2)$, where $\mu = \sum_{i=1}^{n} c_i \mu_i$ and $\sigma^2 = \sum_{i=1}^{n} c_i^2 \sigma_i^2$

Since $\left(\sum_i c_i \mu_i\right)^2 = 9\left(\sum_i c_i^2 \sigma_i^2\right)$, we have $\mu^2 = 9\sigma^2$ or $\dfrac{\mu}{\sigma} = 3$

If we write $Z = \dfrac{Y-\mu}{\sigma}$, then $Z \sim N(0, 1)$.

$$P\left(0 \le Y \le 2 \sum_{i=1}^{n} c_i \mu_i\right) = P(0 \le Y \le 2\mu) = P(-3 \le Z \le 3) = 0{\cdot}9973$$

39. (*a*). Find the mean deviation about mean for the normal distribution $N(\mu, \sigma^2)$.

(*b*) If $X \sim N(\mu, \sigma^2)$, find the mean and variance of

$$Y = \frac{1}{2}\left[(x-\mu)/\sigma\right]^2$$ **[Punjabi Univ. M.A. (Eco.), 1991]**

Ans. $E(Y) = 1/2$, $\text{Var}(Y) = 1/2$

Remark. Also see Example 8·30, on Gamma distribution.

(*c*) Derive normal distribution as a limiting case of binomial distribution, clearly stating the conditions involved. **[Delhi Univ. B.A. (Stat. Hons.), 1981]**

40. If $f(x)$ is the density function for the normal distribution with mean zero and standard deviation σ, then show that

$$\int_{-\infty}^{+\infty} [f(x)]^2\, dx = \frac{1}{2\sigma\sqrt{\pi}}$$

Hence show that if the normal distribution is grouped in intervals with total frequency N_1, and N_2 is the sum of the squares of the frequencies, an estimate of σ is $\dfrac{N_1^2}{2N_2\sqrt{\pi}}$ **(Gujarat Univ. B.Sc., 1992)**

Hint. $$\int_{-\infty}^{\infty} [f(x)]^2\, dx = \int_{-\infty}^{\infty} \left\{\frac{1}{\sqrt{2\pi}\,\sigma} \exp(-x^2/2\sigma^2)\right\}^2 dx$$

$$= \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty} e^{-x^2/\sigma^2}\, dx = \frac{1}{2\pi\sigma^2} \cdot \frac{\sqrt{\pi}}{(1/\sigma)}$$

$$= \frac{1}{2\sigma\sqrt{\pi}} \qquad \left(\because \int_{-\infty}^{\infty} e^{-a^2x^2}\, dx = \sqrt{\pi}/a\right)$$

$$N_2 = \int_{-\infty}^{\infty} \{N_1 f(x)\}^2\, dx = \frac{N_1^2}{2\sqrt{\pi}\,\sigma}$$

41. Obtain the normal distribution as a limiting case of Poisson distribution when the parameter $\lambda \to \infty$.

42. (*a*) If X is $N(0, 1)$, prove that the p.d.f. of $|X|$ is

$$h(x) = \begin{cases} \sqrt{2/\pi}\,\exp(-x^2/2), & x \ge 0 \\ 0, & \text{otherwise} \end{cases}$$

(*b*) Let $X \sim N(0, 1)$ and $Y \sim N(0, 1)$ be independent random variables. Show that $X + Y$ is independent of $X - Y$.

43. If $X \sim N(\mu, 9^2)$ and $Y \sim N(\mu, 12^2)$ are independent, and if $P(X + 2Y \le 3) = P(2X - Y \ge 4)$, determine μ.

[Calcutta Univ. B.Sc. (Maths Hons.), 1989]

44. If $X \sim N(0, 1)$ and $Y \sim N(0, 1)$, prove that

(*i*) $\text{Var}(\sin X) > \text{Var}(\cos X)$

(*ii*) $E|X - Y| \le \sqrt{8/\pi}$

[Delhi Univ. B.A. Hons. (Spl. Course-Statistics), 1988]

Hint. (*i*) $X \sim N(0, 1) \Rightarrow \varphi_X(t) = E[\cos tX + i \sin tX) = e^{-t^2/2}$.

$\Rightarrow E(\cos tX) = e^{-t^2/2}$ and $E(\sin tX) = 0$.

Taking $t = 1$ and 2, we get:

$$E(\cos X) = e^{-1/2};\ E(\cos 2X) = e^{-2};\ E(\sin X) = E(\sin 2X) = 0.$$

$$\text{Var}(\cos X) = E(\cos^2 X) - (E\cos X)^2 = E\left[\frac{1 + \cos 2X}{2}\right] - [E\cos X]^2$$

$$= \frac{1}{2}(1 - e^{-1})^2 \simeq 0{\cdot}99$$

Similarly $\text{Var}(\sin X) = E\left[\dfrac{1 - \cos 2X}{2}\right] - [E\sin X]^2 = \dfrac{1}{2}(1 - e^{-2}) \simeq 0{\cdot}43$

(*ii*) Use $|X - Y| \le |X| + |Y|$ and $E|X| = E|Y| = \sqrt{2/\pi}$

or $X - Y \sim N(0, \sigma^2 = 2)$; $\quad E|X - Y| = \sqrt{2/\pi}\,\sigma = \sqrt{4/\pi} < \sqrt{(8/\pi)}$

45. Let X and Y be independent $N(0, 1)$ variates. Let $X = R\cos\theta$, $Y = R\sin\theta$. Show that R and θ are independent variates.

[Delhi Univ. B.A. Hons. (Spl. Course Statistics), 1985]

46. If $X \sim N(0, 1)$, find p.d.f. of $|X|$. Hence or otherwise evaluate $E|X|$. **[Delhi Univ. B.Sc. (Maths. Hons.), 1980]**

Hint. Distribution function $G_Y(y)$ of $Y = |X|$ is given by:

$$G_Y(y) = P(Y \le y) = P(|X| \le y) = P(-y \le X \le y)$$
$$= P(X \le y) - P(X \le -y)$$
$$G_Y(y) = F_X(y) - F_X(-y).$$

where $F(.)$ is the distribution function of X. Differentiating, the p.d.f. of $Y = |X|$ is given by

$$g_Y(y) = f_X(y) + f_X(-y) = 2 f_X(y)$$

$$\Rightarrow \quad g_Y(y) = \sqrt{2/\pi}\,.\,e^{-y^2/2};\ y \ge 0 \qquad \text{[By symmetry, since } X \sim N(0, 1)\text{]}$$

8·2·15. The log-normal Distribution. The positive r.v. X is said to have a log-normal distribution if $\log_e X$ is normally distributed.

(b) Show that the mean value of the positive square root of a $\gamma(\lambda, n)$ variate is $\Gamma(n+\frac{1}{2}) / [\sqrt{\lambda}\,\Gamma(n)]$.

Hence prove that the mean deviation of a $N(\mu, \sigma^2)$ variate from its mean is $\sigma\sqrt{2/\pi}$

[Gauhati Univ. M.A. (Eco)., 1991; Delhi Univ. B.Sc. (Stat Hons.), 1989]

Hint. Proceed as in Example 8·31.

9. Show that the mean value of the positive square root of $\beta(\mu, \nu)$ variate is

$$\frac{\Gamma\left(\mu+\frac{1}{2}\right)\Gamma(\mu+\nu)}{\Gamma(\mu)\,\Gamma\left(\mu+\nu+\frac{1}{2}\right)}$$

10. (a) For the distribution :

$$dP(x) = \frac{1}{B(\mu, \nu)}\,\frac{x^{\mu-1}}{(1+x)^{\mu+\nu}}\,;\ 0<x<\infty, \nu>2$$

show that variance is $\dfrac{\mu(\mu+\nu-1)}{(\nu-1)^2(\nu-2)}$.

Find also the mode and μ_r' (about origin). Also show that harmonic mean is $(\mu-1)/\nu$.

(b) Find the arithmetic mean, harmonic mean and variance of a Beta distribution of first kind with parameter μ and ν. Verify that A.M. > H.M.

Also prove that if G is the geometric mean, then

$$\log G = \frac{1}{B(\mu, \nu)}\,\frac{\partial}{\partial \nu}\,\beta(\mu, \nu) = \frac{\partial}{\partial \nu}\left[\log\sqrt{2} - \log\Gamma(\mu+\nu)\right]$$

11. Given the Beta distribution in the following form :

$$p(x) = \frac{1}{B(\alpha+1, \lambda+1)}\,.\,x^{\alpha}(1-x)^{\lambda}\,;\ \alpha>-1, \lambda>-1, 0\le x\le 1$$

find its variance.

Also find the distribution of (i) $\dfrac{1}{X}$, (ii) $\dfrac{1-X}{X}$.

12. If X is a normal variate with mean μ and standard deviation σ, find the mean and variance of Y defined by

$$Y = \tfrac{1}{2}\left(\frac{X-\mu}{\sigma}\right)^2$$ **(Meerut Univ. B. Sc., 1993)**

8·6. The Exponential Distribution. A continuous random variable X assuming non-negative values is said to have an exponential distribution with parameter $\theta > 0$, if its p.d.f. is given by

$$f(x) = \begin{cases} \theta\,.\,e^{-\theta x}, & x \ge 0 \\ 0, & \text{otherwise} \end{cases} \qquad \ldots(8\cdot24)$$

The cumulative distribution function $F(x)$ is given by

$$F(x) = \int_0^x f(u)\, du = \theta \int_0^x \exp(-\theta u)\, du$$

$$F(x) = \begin{cases} 1 - \exp(-\theta x), & x \geq 0 \\ 0, & \text{otherwise} \end{cases} \qquad \text{...[8·24 (a)]}$$

8·6·1. Moment Generating Function of Exponential Distribution

$$M_X(t) = E(e^{tX}) = \theta \int_0^\infty e^{tx}\, e^{-\theta x}\, dx$$

$$= \theta \int_0^\infty \exp\{-(\theta - t)x\}\, dx = \frac{\theta}{(\theta - t)}, \quad \theta > t$$

$$= \left(1 - \frac{t}{\theta}\right)^{-1} = \sum_{r=0}^{\infty} \left(\frac{t}{\theta}\right)^r$$

$\therefore$ $$\mu_r' = E(X^r) = \text{Coefficient of } \frac{t^r}{r!} \text{ in } M_X(t)$$

$$= \frac{r!}{\theta^r}\,;\ r = 1, 2, \ldots$$

$\therefore$ $$Mean = \mu_1' = \frac{1}{\theta}$$

$$Variance = \mu_2 = \mu_2' - \mu_1'^2 = \frac{2}{\theta^2} - \frac{1}{\theta^2} = \frac{1}{\theta^2}$$

Theorem. If $X_1, X_2, \ldots, X_n$ are independent random variables, X_i having an exponential distribution with parameter $\theta_i\,;\ i = 1, 2, \ldots, n$; then $Z = \min(X_1, X_2, \ldots, X_n)$ has exponential distribution with parameter $\sum_{i=1}^{n} \theta_i$.

[Delhi Univ. B.Sc. (Stat. Hons.), 1986]

Proof. $G_Z(z) = P(Z \leq z) = 1 - P(Z > z)$

$$= 1 - P[\min(X_1, X_2, \ldots, X_n) > z]$$

$$= 1 - P[X_i > z;\ i = 1, 2, \ldots, n]$$

$$= 1 - \prod_{i=1}^{n} P(X_i > z) = 1 - \prod_{i=1}^{n} [1 - P(X_i \leq z)]$$

$$= 1 - \prod_{i=1}^{n} [1 - F_{X_i}(z)]$$

where F is the distribution function of X_i.

$$= 1 - \prod_{i=1}^{n} \left[1 - \left(1 - e^{-\theta_i z}\right)\right] \qquad \text{[c.f. 8·24 (a)]}$$

$$= \begin{cases} 1 - \exp\left\{ \left(-\sum_{i=1}^{n} \theta_i\right) z \right\}, & z > 0 \\ 0, & \text{otherwise} \end{cases}$$

$$\therefore \quad g_Z(z) = \begin{cases} \left(\sum_{i=1}^{n} \theta_i\right) \exp\left\{ \left(-\sum_{i=1}^{n} \theta_i\right) z \right\}, & z > 0 \\ 0, & \text{otherwise} \end{cases}$$

$\Rightarrow$ $Z = \min(X_1, X_2, ..., X_n)$ is an exponential variate with **parameter** $\sum_{i=1}^{n} \theta_i$.

Cor. If X_i ; $i = 1, 2, ..., n$ are identically distributed, following exponential distribution with parameter θ, then $Z = \min(X_1, X_2, ..., X_n)$ is also exponentially distributed with parameter $n\theta$.

Example 8·40. *Show that the exponential distribution "lacks memory", i.e., if X has an exponential distribution, then for every constant $a \geq 0$, one has $P(Y \leq x \mid X \geq a) = P(X \leq x)$ for all x, where $Y = X - a$.*

[Delhi Univ. B.Sc. (Stat. Hons.), 1989; Calicut Univ. B.Sc. (Main Stat.), 1991]

Solution. The p.d.f. of the exponential distribution with parameter θ is

$$f(x) = \theta e^{-\theta x} ; \theta > 0, 0 < x < \infty$$

We have

$$P(Y \leq x \cap X \geq a) = P(X - a \leq x \cap X \geq a) \qquad (\because Y = X - a)$$

$$= P(X \leq a + x \cap X \geq a) = P(a \leq X \leq a + x)$$

$$= \theta \int_a^{a+x} e^{-\theta x} dx = e^{-a\theta}(1 - e^{-\theta x})$$

and $$P(X \geq a) = \theta \int_a^{\infty} e^{-\theta x} dx = e^{-a\theta}$$

$$\therefore \quad P(Y \leq x \mid X \geq a) = \frac{P(Y \leq x \cap X \geq a)}{P(X \geq a)} = 1 - e^{-\theta x} \qquad ...(*)$$

Also $$P(X \leq x) = \theta \int_0^x e^{-\theta x} dx = 1 - e^{-\theta x} \qquad ...(**)$$

From (*) and (**), we get

$$P(Y \leq x \mid X \geq a) = P(X \leq x)$$

i.e., exponential distribution lacks memory.

Example 8·41. *X and Y are independent with a common p.d.f. (exponential):*

$$f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Find a p.d.f. for $X - Y$. **[Delhi Univ. B.Sc. (Stat. Hons.), 1988, '85]**

Solution. Since X and Y are independent and identically distributed (*i.i.d.*), their joint *p.d.f.* is given by

$$fxy(x, y) = \begin{cases} e^{-(x+y)}; x > 0, y > 0 \\ 0 \end{cases}$$

otherwise

$$\text{Let} \quad \begin{cases} u = x - y \\ v = y \end{cases} \Rightarrow \begin{cases} x = u + v \\ y = v \end{cases} \quad ...(1)$$

$$J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

Thus the joint p.d.f. of U and V becomes

$$g(u, v) = e^{-(u+2v)}; v > 0, -\infty < u < \infty$$

(1) $\Rightarrow$ $u = x - v \Rightarrow v = x - u$

Thus $v > -u$ if $-\infty < u < 0$

and $v > 0$ if $u > 0$

For $-\infty < u < 0$,

$$g(u) = \int_{-u}^{\infty} g(u, v)\, dv = \int_{-u}^{\infty} e^{-(u+2v)}\, dv = e^{-u} \left| \frac{e^{-2v}}{-2} \right|_{-u}^{\infty} = \frac{1}{2} e^{u}$$

and for u > 0,

$$g(u) = \int^{\infty} g(u, v)\, dv = e^{-u} \left| \frac{e^{-v}}{-2} \right|_{-u}^{\infty} = \frac{1}{2} e^{-u^2}$$

Hence the p.d.f of $U = X - Y$ is given by

$$g(u) = \begin{cases} \frac{1}{2} e^{u}, -\infty < u < 0 \\ \frac{1}{2} e^{-u}, u > 0 \end{cases}$$

These results can be combined to give

$$g(u) = \frac{1}{2} e^{-|u|}, -\infty < u < \infty$$

which is the p.d.f. of standard Laplace distribution (*c.f.* § 8·7).

Aliter.

$$M_X(t) = \int_0^{\infty} e^{tx} f(x)\, dx = \int_0^{\infty} e^{-(1-t)x}\, dx = \left| \frac{e^{-(1-t)x}}{-(1-t)} \right|_0^{\infty} = \frac{1}{1-t}, t < 1$$

$\therefore$ Characteristic function of X is

$$\varphi_X(t) = \frac{1}{1-it} = \varphi_Y(t),$$

(since X and Y are identically distributed.)

$\therefore$ $\varphi_{X-Y}(t) = \varphi_{X+(-Y)}(t) = \varphi_{X}(t)\, \varphi_{-Y}(t)$ $\quad$ ($\because$ X, Y are independent)

$$= \varphi_X(t)\,.\,\varphi_Y(-t) = \frac{1}{(1-it)(1+it)} = \frac{1}{1+t^2}$$

which is the characteristic function of the Laplace distribution, (c.f. § 8·7)

$$g(u) = \frac{1}{2} e^{-|u|}, -\infty < u < \infty \qquad \text{...(*)}$$

Hence by the uniqueness theorem of characteristic functions, $U = X - Y$ has the p.d.f. given in (*).

8·7. Laplace (Double Exponential) Distribution. A continuous random variable X is said to follow standard Laplace distribution if its p.d.f. is given by

$$f(x) = \frac{1}{2} e^{-|x|}, -\infty < x < \infty \qquad \text{...(8·25)}$$

Characteristic function is given by

$$\varphi_X(t) = \int_{-\infty}^{\infty} e^{itx} f(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} e^{itx}\,.\,e^{-|x|}\,dx$$

$$= \frac{1}{2}\left[\int_{-\infty}^{\infty} \cos tx\,.\,e^{-|x|}\,dx + i\int_{-\infty}^{\infty} \sin tx\,.\,e^{-|x|}\,dx\right]$$

$$= \frac{1}{2}\,.\,2\int_0^{\infty} \cos tx\,.\,e^{-|x|}\,dx,$$

Since the integrands in the first and second integrals are even and odd function of x respectively.

$$\therefore \qquad \varphi_X(t) = \int_0^{\infty} e^{-x} \cos tx\,dx$$

$$= 1 - t^2 \int_0^{\infty} e^{-x} \cos tx\,dx \quad \text{(on integration by parts)}$$

$$= 1 - t^2\,\varphi_X(t)$$

$$\Rightarrow \qquad \varphi_X(t) = \frac{1}{1+t^2} \qquad \text{...(8·25 } a)$$

The mean of this distribution is zero, standard deviation is $\sqrt{2}$ and mean deviation about mean is 1.

Remark. Generalised Laplace Distribution. A continuous $r.v.$ X is said to have Laplace distribution with two parameters λ and μ if its p.d.f. is given by

$$f(x) = \frac{1}{2\lambda} \exp\left[-|x-\mu|/\lambda\right], -\infty < x < \infty\,;\, \lambda > 0 \qquad \text{...(8·26)}$$

Taking $U = \dfrac{X-\mu}{\lambda}$, in (8·26) we obtain the p.d.f. of standard Laplace variate given in (8·25).

Moments. The rth moment about origin is given by

$$\mu'_r = E(X^r) = \frac{1}{2\lambda} \int_{-\infty}^{\infty} x^r \exp\left(\frac{-|x-\mu|}{\lambda}\right) dx$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} (z\lambda + \mu)^r \exp(-|z|)\, dz, \qquad \left[z = \frac{x-\mu}{\lambda}\right]$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \left[\sum_{k=0}^{r} \binom{r}{k} (z\lambda)^k \mu^{r-k}\right] \exp(-|z|)\, dz$$

$$= \frac{1}{2} \sum_{k=0}^{r} \left[\binom{r}{k} \lambda^k \mu^{r-k} \int_{-\infty}^{\infty} z^k \exp(-|z|)\, dz\right]$$

$$= \frac{1}{2} \sum_{k=0}^{r} \left[\binom{r}{k} \lambda^k \mu^{r-k} \left\{\int_{-\infty}^{0} z^k e^{(-|z|)}\, dz + \int_{0}^{\infty} z^k e^{-|z|}\, dz\right\}\right]$$

$$= \frac{1}{2} \sum_{k=0}^{r} \left[\binom{r}{k} \lambda^k \mu^{r-k} \left\{(-1)^k \int_{0}^{\infty} e^{-z} z^k\, dz + \int_{0}^{\infty} e^{-z} z^k\, dz\right\}\right]$$

$$= \frac{1}{2} \sum_{k=0}^{r} \left[\binom{r}{k} \lambda^k \mu^{r-k}\, \Gamma(k+1)\, \{(-1)^k + 1\}\right]$$

$$\Rightarrow \qquad \mu'_r = \frac{1}{2} \sum_{k=0}^{r} \left[\binom{r}{k} \lambda^k \mu^{r-k}\, k!\, \{1 + (-1)^k\}\right] \qquad \ldots(8\cdot26a)$$

$\therefore$ Mean $= \mu'_r = \mu_1' = \mu$ and $\mu_2' = \mu^2 + 2\lambda^2$

$\therefore$ $\sigma_X^2 = \mu_2' - \mu_1'^2 + 2\lambda^2.$

Similarly we can obtain higher order moments from (8·26 a) and hence the values of β_1 and β_2 can be obtained.

The characteristic function of (8·26) can be obtained exactly similarly as we obtained the characteristic function of standard Cauchy distribution, c.f.§ 8.9.

8·8. Weibul Distribution. A random variable X has a Weibul distribution with three parameters c (> 0), α(> 0) and μ if the r.v.

$$Y = \left(\frac{X-\mu}{\alpha}\right)^c \qquad \ldots(i)$$

has the exponential distribution with p.d.f.

$$p_Y(y) = e^{-y},\ y > 0 \qquad \ldots(ii)$$

The p.d.f. of X is given by

$$p_X(x) = c\,\alpha^{-1}\left(\frac{x-\mu}{\alpha}\right)^{c-1} \exp\left[-\left(\frac{x-\mu}{\alpha}\right)^{c}\right],\; x > \mu,\; c > 0 \qquad ...(iii)$$

The standard Weibul distribution is obtained on taking $\alpha = 1$ and $\mu = 0$, so that the p.d.f. of *standard Weibul distribution* which depends only on a single parameter c is given by

$$p_X(x) = c\,x^{c-1}\,.\exp(-x^c)\;;\; x > 0,\, c > 0 \qquad ...(iv)$$

8·8·1. Moments of Standard Weibul Distribution (iv)

For standard Weibul distribution, $(\alpha = 1, \mu = 0)$, from (i), we get $Y = X^c$ which has the exponential distribution (ii). We have

$$\mu'_r = E(X^r) = E(Y^{1/c})^r = E(Y^{r/c})$$

$$= \int_0^\infty e^{-y}\,.\,y^{r/c}\,dy \qquad [\because Y \text{ has p.d.f. } (ii)]$$

$$\Rightarrow \qquad \mu'_r = \Gamma\left(\frac{r}{c}+1\right) \qquad ...(v)$$

$$\therefore \qquad \text{Mean} = E(X) = \Gamma\left(\frac{1}{c}+1\right)$$

and $\quad \text{Var}(X) = E(X^2) = [E(X)]^2$

$$= \Gamma\left(\frac{2}{c}+1\right) - \left[\Gamma\left(\frac{1}{c}+1\right)\right]^2$$

Similarly, we can obtain expressions for higher order moments and hence for β_1 and β_2. For large c, the mean is approximated by

$$E(X) \simeq 1 - \frac{\gamma}{c} + \frac{1}{2c^2}\left(\frac{\pi^2}{6}+\gamma^2\right)$$

$$= 1 - 0{\cdot}57722\;c^{-1} + 0{\cdot}98905\;c^{-2}$$

where $\gamma = 0{\cdot}57722$ is Euler's constant.

The distribution is named after Waloddi Weibul, a Swedish physicist, who used it in 1939 to represent the distribution of the breaking strength of materials. Kao, J.H.K. (1958-59) advocated the use of this distribution in reliability studies and quality control work. It is also used as a tolerance distribution in the analysis of quantum response data.

8·8·2. Characterisation of Weibul Distribution. Dubey, S.D. (1968) has obtained the following result :

"Let $X_i\,(i = 1, 2, ..., n)$ be i.i.d. random variables. Then min $(X_1, X_2, ..., X_n)$ has a Weibul distribution if and only if the common distribution of X_i's is a Weibul distribution".

Proof. Let X_i, $(i = 1, 2, ..., n)$ be *i.i.d. r.v.* each with Weibul distribution (iii) and let $Y = \min(X_1, X_2, ..., X_n)$. Then

$$P(Y > y) = P[\,\min(X_1, X_2, ..., X_n) > y\,]$$

$$= P\left[\bigcap_{i=1}^{n} X_i > y\right]$$

$$= \prod_{i=1}^{n} P(X_i > y) = \left[P(X_i > y)\right]^n \qquad \ldots(*)$$

since X_i's are *i.i.d. r.v.'s*.

Now $$P(X_i > y) = \int_y^{\infty} c\,\alpha^{-1}\left(\frac{x-\mu}{\alpha}\right)^{c-1} \cdot \exp\left[-\left(\frac{x-\mu}{\alpha}\right)^c\right] dx$$

$$= \int_{\left(\frac{y-\mu}{\alpha}\right)^c}^{\infty} e^{-t}\,dt \qquad \left[t = \left(\frac{x-\mu}{\alpha}\right)^c\right]$$

$$= \exp\left[-\left(\frac{y-\mu}{\alpha}\right)^c\right]$$

Substituting in (*), we get

$$P(Y > y) = \left[\exp\left\{-\left(\frac{y-\mu}{\alpha}\right)^c\right\}\right]^n$$

$$= \exp\left[-n\left(\frac{y-\mu}{\alpha}\right)^c\right]$$

$$= \exp\left[-\left\{\frac{n^{1/c}(y-\mu)}{\alpha}\right\}^c\right]$$

This implies that Y has the same Weibul distribution as X_i's with the difference that the parameter α is replaced by $\alpha\, n^{-1/c}$.

8·8·3. Logistic Distribution. A continuous *r.v.* X is said to have a Logistic distribution with parameters α and β, if its distribution function is of the form:

$$F_X(x) = [\,1 + \exp\{-(x-\alpha)/\beta\}\,]^{-1}, \; \beta > 0 \qquad \ldots(8{\cdot}26\,b)$$

$$= \frac{1}{2}\left[1 + \tanh\left\{\frac{1}{2}(x-\alpha)/\beta\right\}\right] \; ; \; \beta > 0 \qquad \ldots(8{\cdot}26\,c)$$

(See Remark 1 on page 8·94).

The p.d.f. of Logistic distribution with parameters α and β (> 0) is given by

$$f(x) = \frac{d}{dx}(F(x))$$

$$= \frac{1}{\beta}[\,1 + \exp\{-(x-\alpha)/\beta\}\,]^{-2}\,[\exp\{-(x-\alpha)/\beta\}\,] \qquad \ldots(8{\cdot}26\,d)$$

$$= \frac{1}{4\beta}\,\text{sech}^2\left\{\frac{1}{2}(x-\alpha)/\beta\right\} \qquad \ldots(8{\cdot}26\,e)$$

The p.d.f. of standard Logistic variate $Y = (X-\alpha)/\beta$, is given by:

$$g_Y(y) = f(x) \cdot \left| \frac{dx}{dy} \right|$$

$$= e^{-y}\left(1+e^{-y}\right)^{-2} \; ; -\infty < y < \infty \qquad \text{...(8·26 f)}$$

$$= \frac{1}{4} \operatorname{sech}^2\left(\frac{1}{2} y\right) ; -\infty < y < \infty \qquad \text{...(8·26 g)}$$

The distribution function of Y is :

$$G_Y(y) = (1+e^{-y})^{-1} \; ; \; -\infty < y < \infty \qquad \text{...(8·26 h)}$$

Logistic distribution is extensively used as growth function in population and demographic studies and in time series analysis. Theoretically, Logistic distribution can be obtained as :

(*i*) The limiting distribution (as $n \to \infty$) of the standardised mid range, (average of the smallest and the largest sample observations), in random samples of size n

(*ii*) A mixture of extreme value distributions.

Moment Generating Function. The m.g.f. of standard Logistic variate Y is given by:

$$\mu_Y(t) = E\left(e^{tY}\right) = \int_{-\infty}^{\infty} e^{ty} \cdot g(y)\, dy$$

$$= \int_{-\infty}^{\infty} e^{ty} \cdot e^{-y} (1+e^{-y})^{-2}\, dy$$

$$= \int_{-\infty}^{\infty} e^{ty} e^{-y} \left(\frac{1+e^y}{e^y}\right)^{-2} dy$$

$$= \int_{-\infty}^{\infty} e^{ty} \cdot e^y (1+e^y)^{-2}\, dy$$

Put $\quad z = (1+e^y)^{-1} \Rightarrow e^y = \dfrac{1}{z} - 1 = \dfrac{1-z}{z}$

$$\therefore M_Y(t) = \int_{1}^{0} \left(\frac{1-z}{z}\right)^t \cdot (-dz) = \int_0^1 z^{-t}(1-z)^t\, dz$$

$$= \beta(1-t,\ 1+t), \quad 1-t > 0$$

$$= \Gamma(1-t)\,\Gamma(1+t)/\Gamma 2$$

$$= \Gamma(1-t)\ \Gamma(1+t)$$

$$= \pi t \ \text{cosec}\ \pi t \quad ; t < 1 \qquad \text{...(8·26 i)}$$

$$= 1 + \frac{\pi^2 t^2}{6} + \frac{7}{360}\pi^4 t^4 + \qquad \text{...(*)}$$

(See Remark 2 below.)

$\therefore \ E(Y) = \text{Coefficient of } t \text{ in } (*) = 0$

$\Rightarrow \qquad \text{Mean} = 0$

$\therefore \qquad \mu_2 = E(Y^2) = \text{Coefficient of } \dfrac{t^2}{2!} \text{ in } (*) = \dfrac{\pi^2}{3}$

$\mu_3 = E(Y^3) = 0$

$\mu_4 = E(Y^4) = \text{Coefficient of } \dfrac{t^4}{4!} \text{ in } (*) = \dfrac{7}{15}\pi^4$

Hence for standard Logistic distribution :

Mean = 0 , Variance = $\mu_2 = \pi^2/3$,

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0, \quad \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{7 \times 9}{15} = 4{\cdot}2$$

Remarks 1. We have :

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1 - e^{-2x}}{1 + e^{-2x}}$$

$$\Rightarrow \qquad 1 + \tanh x = \frac{2}{1 + e^{-2x}} \quad \Rightarrow \quad \frac{1}{2}[\,1 + \tanh x\,] = (1 + e^{-2x})^{-1}$$

2. $\qquad x \operatorname{cosec} x = 1 + \dfrac{x^2}{6} + \dfrac{7}{360}x^4 + \ldots$

Proof. $x \operatorname{cosec} x = \dfrac{x}{\sin x} = \dfrac{x}{\left[x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} = \ldots\right]}$

$$= \left[1 - \left(\frac{x^2}{6} - \frac{x^4}{120} + \frac{x^6}{7!}\ldots\right)\right]^{-1}$$

$$= 1 + \left(\frac{x^2}{6} - \frac{x^4}{120} + \ldots\right) + \left(\frac{x^2}{6} - \frac{x^4}{120} + \ldots\right)^2 + \ldots$$

$$= 1 + \frac{x^2}{6} + x^4\left(\frac{1}{36} - \frac{1}{120}\right) + \ldots$$

$$= 1 + \frac{x^2}{6} + \frac{7}{360}x^4 + \ldots$$

3. We have:

$$g(y) = e^{-y}\left(1 + \frac{1}{e^y}\right)^{-2} = e^y\left(1 + e^y\right)^{-2} = g(-y)$$

$\Rightarrow$ The probability curve of Y is symmetric about the line $y = 0$.

Since p.d.f. $g(y)$ is symmetric about origin $(y = 0)$, all odd order moments about origin are zero *i.e.*,

$$\mu'_{2r+1} = E\left(Y^{2r+1}\right) = 0, \quad r = 0, 1, 2, \ldots$$

In particular

Mean $= \mu_1' = 0$

$\therefore \qquad \mu_r' = r$th moment about origin

$$= r\text{th moment about mean}$$
$$= \mu_r$$

$$\Rightarrow \qquad \mu_{2r+1} = \mu'_{2r+1} = 0$$

i.e., all odd order moments about mean of the standard logistic distribution are zero.

In particular $\mu_3 = 0 \;\Rightarrow\; \beta_1 = 0$

4. The mean and variance of the logistic Variable (X) with parameters α and β are given by :

$$E(X) = E(\alpha + \beta Y) \qquad \left(\because\; Y = \frac{X-\alpha}{\beta}\right)$$
$$= \alpha + \beta E(Y)$$
$$= \alpha$$
$$\text{Var } X = \text{Var }(\alpha + \beta Y) = \beta^2 \text{ Var }(Y) = \beta^2 \pi^2/3.$$

5. We have :

$$G(y) = (1 + e^{-y})^{-1} = \left(\frac{1+e^y}{e^y}\right)^{-1} = \frac{e^y}{1+e^y}$$

$$\Rightarrow \qquad 1 - G(y) = 1 - \frac{e^y}{1+e^y} = \frac{1}{1+e^y}$$

$$\therefore\; G(y) \cdot [\,1 - G(y)\,] = \frac{e^y}{(1+e^y)^2} = g(y) \qquad \ldots(8{\cdot}26\,j)$$

(*c.f.* Remark 3)

Also $$\frac{G(y)}{1-G(y)} = e^y \;\Rightarrow\; y = \log_e\left[\frac{G(y)}{1-G(y)}\right] \qquad \ldots(8{\cdot}26\,k)$$

6. Mean deviation for the standard Logistic distribution is

$$2\left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots\right] = 2\sum_{i=1}^{\infty}\left[\frac{(-1)^{i-1}}{i}\right] = 2\log_e 2$$

Proof is left as an exercise to the reader.

EXERCISE 8(e)

1. (*a*) Show that for the exponential distribution

$$p(x) = y_0 \cdot e^{-x/\sigma}, \quad 0 \le x < \infty; \quad \sigma > 0,$$

mean and variance are equal. Also obtain the interquartile range of the distribution.

[Delhi Univ. B.Sc. (Stat. Hons.), 1985, 1982]

(*b*) Suppose that during rainy season on a tropical island the length of the shower has an exponential distribution, with parameter $\lambda = 2$, time being measured in minutes. What is the probability that a shower will last more than three minutes? If a shower has already lasted for 2 minutes, what is the probability that it will last for at least one more minute ? **[Madras Univ. B.Sc. (Main Stat) 1988]**

2. (*a*) If $X_1, X_2, \ldots, X_n$ are independent random variables having exponential distribution with parameter λ, obtain the distribution of $Y = \sum_{i=1}^{n} X_i$.

(*b*) Obtain the moment generating function and the cumulant generating function of the distribution with p.d.f.

$$f(x) = \frac{1}{\sigma} e^{-x/\sigma} \, ; 0 < x < \infty, \sigma > 0$$

[Madras Univ. B.Sc. (Main Stat.) Oct. 1992]

Hence or otherwise obtain the values of the contants $\beta_1, \beta_2, \gamma_1$ and γ_2.

(*c*) A continuous random variable X has the probability density function $f(x)$ given by

$$f(x) \begin{array}{ll} = A e^{-x/5} & x > 0 \\ = 0, & \text{otherwise} \end{array}$$

Find the value of A and show that for any two positive numbers s and t,

$$P[X > s + t \mid X > s] = P[X > t].$$

3. If X_1 and X_2 are independent and identically distributed each with frequency function $e^{-x}, x > 0$, find the frequency function of $X_1 + X_2$.

(*b*) If $X_1, X_2, \ldots, X_n$ are independent r.v.'s X_i having an exponential distribution with parameter θ_i, $(i = 1, 2, \ldots, n)$, then prove that $Z = \min(X_1, X_2, \ldots, X_n)$ has an exponential distribution with parameter $\sum_{i=1}^{n} \theta_i$

[Delhi Univ. B.Sc. (Stat. Hons.), 1990, '88, '86]

4. Let X and Y have common p.d.f. $\alpha e^{-\alpha x}, 0 < x < \infty, \alpha > 0$. Find the p.d.f. of

(*i*) X^3, (*ii*) $3 + 2X$, (*iii*) $X - Y$, and (*iv*) $|X - Y|$

Ans. (*i*) $\frac{\alpha}{3} x^{-2/3} \exp(-\alpha x^{1/3})$, (*ii*) $\frac{\alpha}{2} e^{-\alpha(x-3)/2}, x > 3$

(*iii*) $\frac{\alpha}{2} e^{-\alpha|x|}$, all x, and (*iv*) $\alpha e^{-\alpha x}, x > 0$.

5. (*a*) X and Y are independent random variables each exponentially distributed with the same parameter θ. find p.d.f. for $\frac{X}{X+Y}$ and identify its distribution. **[Delhi Univ. B.Sc. (Stat. Hons.), 1989]**

(*b*) The density functions of the independent random variables X and Y are:

$$f_X(x) = \lambda e^{-\lambda x}, \quad x > 0, \lambda > 0 \qquad\Big|\qquad f_Y(y) = \lambda e^{-\lambda y}, \quad y > 0, \lambda > 0$$
$$= 0, \quad x \le 0 \qquad\Big|\qquad = 0, \quad \text{otherwise}$$

Find the density function of the random variable $Z = X/Y$.

6. (*a*) For the distribution given by the density function

$$f(x) = \tfrac{1}{2} e^{-|x|}, -\infty < x < \infty,$$

obtain the moment generating function.

(*b*) Find the characteristic function of standard Laplace distribution and hence find its mean and standard deviation.

[Delhi Univ. B.Sc. (Stat. Hons.), 1990]

7. (*a*) If X has exponential distribution with mean 2, find $P(X < 1) \mid X < 2)$

Ans. $P(X < 1)$ $[P(X < 2) = (1 - e^{-\theta})/(1 - e^{2\theta})$ where $\theta = 1/2$.

[Delhi Univ. B.A. (Spl. Hons. Course-Statistics), 1989]

(*b*) If $X \sim$ Expo (λ) with $P(X \leq 1) = P(X > 1)$,

...(*)

find Var X. **[Delhi Univ. B.Sc. (Maths Hons.), 1985]**

Hint. $P(X \leq 1) + P(X > 1) = 1 \Rightarrow P(X \leq 1) = 1/2$ [Using (*)]

Ans. Var $(X) = 1/\lambda^2 = 1/(\log e^{-})^2$

8. (*a*) Show that $Y = -(1/\lambda) \log F(X)$ is Expo (λ)

[Delhi Univ. B.A. Hons. (Spl. Course-Statistics), 1985]

Hint.
$$\mu_Y(t) = E(e^{tY}) = E \exp\left[-\frac{t}{\lambda} \log F(x)\right]$$
$$= E[F(X)^{-t/\lambda}] = E[Z^{-t/\lambda}] \text{ where } Z = F(X) \sim U[0, 1]$$

(*b*) If X_1, X_2, X_3 and X_4 are i.i.d. $N(0, 1)$ variates, show that $Y = X_1 X_2 - X_3 X_4$, has p.d.f.

$$f(y) = \frac{1}{2} \exp[-|y|], -\infty < y <$$

[Indian Civil Services, 1984]

Hint. Show that $\varphi_Y(t) = 1/(1 + t^2) \Rightarrow Y$ has Standard Laplace distribution.

Use :
$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(ax^2 + 2hxy + by^2)}\, dx\, dy = \frac{\pi}{\sqrt{ab - h^2}}$$

9. 200 electric light bulbs were tested and the average life time of the bulbs was found to be 25 hours. Using the summary given below, test the hypothesis that the lifetime is exponentially distributed.

Lifetime in hours :	0–20	20–40	40–60	60–80	80–100
Number of bulbs :	104	56	24	12	4

[You are given that an exponential distribution with parameters $\alpha > 0$ has the probability density function:

$$p(x) = \alpha e^{-\alpha x}, \quad (x \geq 0)$$
$$= 0, \quad (x < 0)$$

[Institute of Actuaries (London), April 1978]

10. Find the first four cumulants of the Laplace distribution defined by