CSC 2421H : Graphs, Matrices, and Optimization lecture 4:2018-10-1

Random Walk

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1 Remark from last class

If at step t, your distribution is given by P_t, then

$$P_{t+1} = AD^{-1}P_t \tag{1}$$

Where:
A: weighted adjacency matrix
D: diagonal weighted degree matrix

(1) is equivalent to the following expression:

$$P_{t+1}(x) = \sum_{y:(x,y)\in E} \frac{w(x,y)}{D(y)} * P_t(y) \tag{2}$$

2 Stationary state

Definition 1. If distance $\pi \in R^v$ is stationary for G if $AD^{-1}\pi = \pi$

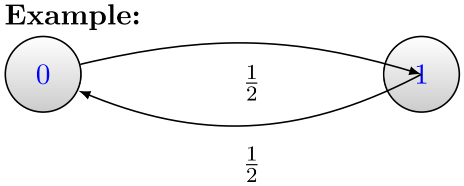
Lemma 1. *Any undriected graph has a stationary distance*

Proof. Given any undirected graph G, let $\pi = \frac{D*\vec{1}}{\vec{1}^T*D\vec{1}}$. This is the stationary distribution for G since $AD^{-1}\pi = \frac{1}{\vec{1}^T*D\vec{1}} * A * \vec{1} = \frac{1}{\vec{1}^T*D\vec{1}} * D * \vec{1} = \pi$ □

Remark:
If the graph is disconnected, then the stationary distribution must not unique

Claim: If G is connected, π is unique

Remark: Even if G is connected, it is not true that for any p_0, $p_t \to \pi$
Example:



3 Lazy Random Walk:

At each step,

$$\begin{cases} \text{with probability}\frac{1}{2} & \text{stay} \\ \text{with probability}\frac{1}{2} & \text{take a usual random step} \end{cases}$$

Lazy Random Walk Matrix $\mathbf{W} = \mathbf{1}_{\overline{2(I+AD^{-1})}}$

Definition 2. A symmetric matrix M is positive semi-definite(psd) if ∀ x, $x^T Mx \geq 0$

Theorem 2. *the follow statements are equivalent:*
(1) M is psd
(2) All eigenvalue of M are non-negative
(3) There exist an matrix A such toverrightarrow $M = A*A^T$

Lemma 3. *If M is psd, then for all matrix C,* $C^T MC$ *is psd*

Proof. ∀ x, $x^T C^T MCx = (Cx)^T M(Cx) \geq 0$ since M is psd

□

Notation:
M is psd: $M \succeq \mathbf{0}$

Lemma 4. $\mathscr{L} \succeq 0$ *where* $\mathscr{L}$ *is the laplacian matrix*

Remark: $\mathscr{L} \succeq \mathbf{0}$ **implies** $N \succeq \mathbf{0}$ **since N =** $D^{-\frac{1}{2}} \mathscr{L} D^{-\frac{1}{2}}$

Lemma 5. $\mathscr{L} \succeq 2D \Leftrightarrow N \succeq 2I \Leftrightarrow \lambda_i(N), \curlyvee_i \leq 2$

HW: If A $\succeq$ **B, then** $\lambda_i(A) \geq \lambda_i(B)$

4 Relation between W(Lazy random walk matrix) and N(Normalized Laplacian)

$$\begin{aligned} W &= \frac{1}{2} * (I + A\mathbf{D}^{-1}) \\ N &= D^{-\frac{1}{2}} \mathscr{L} D^{-\frac{1}{2}} \\ &= D^{-\frac{1}{2}} (D - A) D^{-\frac{1}{2}} \\ &= I - D^{-\frac{1}{2}} A D^{-\frac{1}{2}} \end{aligned} \tag{3}$$

Then,

$$
\begin{aligned}
W &= \frac{1}{2}I + \frac{1}{2}AD^{-1} \\
&= \frac{1}{2}I + \frac{1}{2}D^{\frac{1}{2}}D^{-\frac{1}{2}}AD^{-\frac{1}{2}}D^{-\frac{1}{2}} \\
&= \frac{1}{2}I + \frac{1}{2}D^{\frac{1}{2}}(I-N)D^{-\frac{1}{2}} \\
&= I - \frac{1}{2}D^{\frac{1}{2}}ND^{-\frac{1}{2}}
\end{aligned} \tag{4}
$$

Thus,

$$
W = I - \frac{1}{2}D^{\frac{1}{2}}ND^{-\frac{1}{2}} \tag{5}
$$

Lemma 6. *If*(Υ_i, ψ_i) *is an eigenpair for N, i.e* $N{*}\psi_i = \Upsilon_i\psi_i \Leftrightarrow (1-\frac{1}{2}\Upsilon_i, D^{\frac{1}{2}}\psi_i)$ *is an eigenpair for* W

Proof.
$$
\begin{aligned}
WD^{\frac{1}{2}} * \psi_i &= (I - \tfrac{1}{2} * D^{\frac{1}{2}}ND^{-\frac{1}{2}})D^{\frac{1}{2}}\psi_i \\
&= D^{\frac{1}{2}}\psi_i - \tfrac{1}{2}\,\Upsilon_i\, D^{\frac{1}{2}}\psi_i
\end{aligned}
$$
□

Corrolary: 0$\leq \lambda_i(W) \leq$ **1**

Warning: W is not symmetric. Thus, its eigenvector need not be orthogonal

5 Convergence:

State transition:

$$
p_{t+1} = Wp_t
$$

When t = 0:

$$
p_1 = Wp_0
$$

$$
D^{-\frac{1}{2}}p_0 = \sum_{i=1}^{n} a_i\psi_i \Leftrightarrow p_0 = \sum_{i=1}^{n} a_i D^{\frac{1}{2}}\psi_i
$$

$$
\Rightarrow p_1 = Wp_0 = \sum_{i=1}^{n} a_i(WD^{\frac{1}{2}}\psi_i) = \sum_{i=1}^{n} a_i(1-\frac{\Upsilon_i}{2})D^{\frac{1}{2}}\psi_i
$$

Iterating the proess above, we can obtain:

$$
p_t = \sum_{i=1}^{n} a_i(1-\frac{\Upsilon_i}{2})^t D^{\frac{1}{2}}\psi_i
$$

Claim: If G is connected $\Leftrightarrow \Upsilon_2 > \mathbf{0}$
Remark: the claim above implies the following:

$$\forall i \neq 1, 0 \leq 1 - \frac{\Upsilon_i}{2} < 1$$

5.1 Finding t given a ϵ

At arbitrary vertex u, we have the following:

$$\begin{aligned} \overrightarrow{1}_u^T P_t - \overrightarrow{1}_u^T \pi &= \overrightarrow{1}_u^T P_t - \frac{\overrightarrow{1}_u^T D \overrightarrow{1}}{\overrightarrow{1}^T D \overrightarrow{1}} \\ &= \sum_{i=1}^{n} a_i (1 - \frac{\Upsilon_i}{2})^t \overrightarrow{1}_u^T D^{\frac{1}{2}} \psi_i - \frac{\overrightarrow{1}_u^T D \overrightarrow{1}}{\overrightarrow{1}^T D \overrightarrow{1}} \end{aligned} \tag{6}$$

we know that:

$$\psi_1 = \frac{(D^{\frac{1}{2}} \overrightarrow{1})}{||D^{\frac{1}{2}} \overrightarrow{1}||} = \frac{(D^{\frac{1}{2}} \overrightarrow{1})}{\sqrt{\overrightarrow{1}^T D \overrightarrow{1}}} \tag{7}$$

Then,

$$\begin{aligned} \overrightarrow{1}_u^T D^{\frac{1}{2}} \psi_1 &= \overrightarrow{1}_u^T D^{\frac{1}{2}} \frac{(D^{\frac{1}{2}} \overrightarrow{1})}{||D^{\frac{1}{2}} \overrightarrow{1}||} \\ &= \frac{(\overrightarrow{1}_u^T D \overrightarrow{1})}{\sqrt{\overrightarrow{1}^T D \overrightarrow{1}}} \end{aligned} \tag{8}$$

Also,

$$\begin{aligned} D^{-\frac{1}{2}} p_0 &= \sum_{i=1}^{n} a_i \psi_i \\ \psi_1^T D^{-\frac{1}{2}} p_0 &= a_1 \end{aligned} \tag{9}$$

This gives us:

$$\begin{aligned} a_1 &= \frac{(\overrightarrow{1} D^{\frac{1}{2}}) D^{-\frac{1}{2}} p_0}{\sqrt{\overrightarrow{1}^T D \overrightarrow{1}}} \\ &= \frac{1}{\sqrt{\overrightarrow{1}^T D \overrightarrow{1}}} \end{aligned} \tag{10}$$

Now, we can further simplify (6):

$$\begin{aligned} \overrightarrow{1}_u^T p_t - \overrightarrow{1}_u^T \pi &= a_1 (1 - \frac{\Upsilon_1}{2})^t + \sum_{i=2}^{n} a_i (1 - \frac{\Upsilon_i}{2})^t \overrightarrow{1}_u^T D^{\frac{1}{2}} \psi_i - -\frac{\overrightarrow{1}_u^T D \overrightarrow{1}}{\overrightarrow{1}^T D \overrightarrow{1}} \\ &= \sum_{i=2}^{n} a_i (1 - \frac{\Upsilon_i}{2})^t \overrightarrow{1}_u^T D^{\frac{1}{2}} \psi_i \end{aligned} \tag{11}$$

Hence, we have the following

$$
\begin{aligned}
|\overrightarrow{1}_u^T p_t - \overrightarrow{1}_u^T \pi| &\leq \sum_{i=2}^{n} |a_i \overrightarrow{1}_u^T D^{\frac{1}{2}} \psi_i| (1 - \psi_i/2)^t \\
&\leq (1 - \frac{\Upsilon_2}{2})^t \sum_{i=2}^{n} |a_I \overrightarrow{1}_U^T D^{\frac{1}{2}} \psi_i| \\
&\leq (1 - \frac{\Upsilon_2}{2})^t \sqrt{\sum_{i=2}^{n} a_i^2 \sum_{i=2}^{n} (\overrightarrow{1}_u^T D^{\frac{1}{2}} \psi_i)^2}
\end{aligned}
\tag{12}
$$

Notice that:

$$
\begin{aligned}
\sum_{i=2}^{n} a_i^2 &\leq ||D^{-\frac{1}{2}} p_0||_2^2 \\
&\leq \overrightarrow{1}_v^T D^{-\frac{1}{2}} \overrightarrow{1}_v \\
&= \frac{1}{D(v)}
\end{aligned}
\tag{13}
$$

Claim:

$$
D^{\frac{1}{2}} \overrightarrow{1} = \sum_{i=1}^{n} (\overrightarrow{1}_u^T D^{\frac{1}{2}} \psi_i) \psi_j
$$

Proof.

$$
\begin{aligned}
D^{\frac{1}{2}} \overrightarrow{1} &= \sum_{i=1}^{n} B_i \psi_i \\
\psi_j^T D^{\frac{1}{2}} \overrightarrow{1}_u &= B_j \\
(\psi_j^T D^{\frac{1}{2}} \overrightarrow{1}_u)^T &= \overrightarrow{1}_u^T D^{\frac{1}{2}} \psi_j^T
\end{aligned}
$$

$$
\begin{aligned}
||D^{\frac{1}{2}} \overrightarrow{1}_u||_2^2 &= (D^{\frac{1}{2}} \overrightarrow{1}_u)^T (D^{\frac{1}{2}} \overrightarrow{1}_u) \\
&= (\sum_{i=1}^{n} B_i \psi_i)^T (\sum_{j=1}^{n} B_j \psi_j) \\
&= \sum_{i=1}^{n} B_i^2 \\
&= \sum_{i=1}^{n} (\overrightarrow{1}_u^T D^{\frac{1}{2}} \psi_i)^2
\end{aligned}
\tag{14}
$$

$\square$

Thus, (12) can be simplified as following:

$$
\begin{aligned}
|\overrightarrow{1}_u^T p_t - \overrightarrow{1}_u^T \pi| &\leq (1-\Upsilon_2)^t \sqrt{||D^{-\frac{1}{2}} p_0||_2^2 * ||D^{\frac{1}{2}} \overrightarrow{1}_u||_2^2} \\
&= ||D^{\frac{1}{2}} \overrightarrow{1}_u|| * ||D^{-\frac{1}{2}} p_0||(1-\frac{\Upsilon_2}{2})^t \\
\textbf{if you start at v, then} & \\
&= \sqrt{\frac{D_u}{D_v}}(1-\frac{\Upsilon_2}{2})^t \\
\textbf{Notice that} (1-x) \leq e^{-x} & \\
&\leq \sqrt{\frac{D_u}{D_v}} e^{-\Upsilon_2 t/2}
\end{aligned}
\tag{15}
$$

Theorem 7. *For*

$$t \geq \frac{2}{\Upsilon_2} log(\frac{n}{\epsilon})$$

then

$$|\overrightarrow{1}_u^T p_t - \overrightarrow{1}_u^T \pi| \leq \epsilon$$

Interpretation:

$$
\begin{aligned}
T(\frac{\epsilon}{2}) &= \frac{2}{\Upsilon_2} log(\frac{2n}{\epsilon}) \\
&= \Theta(\frac{1}{\Upsilon_2}) + T(\epsilon)
\end{aligned}
\tag{16}
$$

Application of the theorem on some examples
K_n**:complete graph with n vertices**
Lazy random walk mixes in $\Omega(logn)$ **steps**

$$
\begin{gathered}
\mathscr{L} K_n = nI - \overrightarrow{1}\overrightarrow{1}^T \\
\lambda_i(L) = \begin{cases} 0 & i = 1 \\ n-1 & O.W \end{cases} \\
\Upsilon_2(N_{K_n}) = 1
\end{gathered}
\tag{17}
$$

The theorem gives O(logn)
R_n**:n-ring**

$$
\begin{gathered}
\Upsilon_2(N_{K_n}) = \theta(\tfrac{1}{n^2}) \\
\mathscr{L} K_n = nI - \overrightarrow{1}\overrightarrow{1}^T \\
x_{i+1} = \begin{cases} x_i & \textbf{with probability} \frac{1}{2} \\ x_{i+1} & \textbf{with probability} \frac{1}{4} \\ x_{i-1} & \textbf{with probability} \frac{1}{4} \end{cases} \\
E[x_{i+1}|x_i] = x_i \\
E[x_{i+1}^2|x_i] = x_i^2 + \tfrac{1}{2}
\end{gathered}
\tag{18}
$$

The theorem gives O(n^2**logn)**