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# Notes on Analysis of Activation Functions

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**Ue-Yu Pen\***  
Chicago, IL  
ueyupen@gmail.com

**Vedant Sachdeva†**  
Chicago, IL 60615  
sachdved@gmail.com

## Abstract

Here, we explore the solutions to regression functions, with varying degrees of non-linearity and depth to explore how and when certain correlations are captured by MLPs.

## 1 Intro

Typically, multi-layer perceptron (MLP) networks are characterized by two parameters - depth and width. Depth refers to the number of layers in a network, while the width of a given layer typically refers to the number of nodes in said layer. Layers are parametrized by their weights,  $W^{(j)}$ , where  $j$  indicates that we are examining the  $j$ th layer.  $W^{(j)}$  is typically drawn from  $\mathbb{R}^{d^{(j)} \times d^{(j+1)}}$ , where  $d^{(j)}$  is the width of the  $j$ th layer. Previous works have shown that with non-linear activation functions and MLP architectures, a vast family of functions can be arbitrarily approximated. However, how such functions correlate to the moments of the data being leveraged is not clearly explained. This document will detail how non-linearities uncover the higher order correlations in data. For the purposes of this note, we will focus exclusively on mean-square-error objective functions.

## 2 Linear Activation Functions

In MLPs with linear activation functions, we observe that the output of the  $j$ th layer is simply given by

$$x^{(j+1)} = W^{(j)}x^{(j)} + b^{(j)}. \quad (1)$$

For simplicity, instead of including a bias term, we will append a node to each layer with value, 1, independent of input. This value will provide an effective bias to each layer. We can then express the output of the  $j$ th layer from the initial layer by writing:

$$x^{(j+1)} = \prod_{i=1}^{j+1} W^{(i)} x^{(0)} \quad (2)$$

Consequently, any network of arbitrary depth and width with linear activation functions can be re-written as a single layered MLP. Thus, when optimizing our objective function, we are simply optimizing:

$$\mathcal{L} = \min_W \sum_i \|y_i - Wx_i\|^2. \quad (3)$$

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\*No funding :(

†Also unfunded :(

We optimize this objective function by computing a derivative with respect to each element of  $W$  and equating to 0. This yields a set of simultaneous equations of the form:

$$\frac{\partial \mathcal{L}}{\partial W^{(k)}} = \sum_i (-2y_i x_i^{(k)} + \sum_{m \neq k} W^m x_i^m x_i^k + W^k x_i^k x_i^k) \quad (4)$$

Thus, the optimal weight vector can be expressed as:

$$W^k = \frac{\text{Cov}(Y, X^k) - \sum_{m \neq k} \text{Cov}(X^m, X^k)}{\text{Cov}(X^k, X^k)} \quad (5)$$

As we can see, one outcome will be that if much of the covariance in  $Y$  with  $X_k$  can be explained by  $X_k$  covarying with another feature in  $X$ , the net weight will be constrained, but the individual weights can become unbounded. Further, such a model cannot consider further than second order moments in the dataset.

### 3 Linear Activation Functions with Arbitrary Depth

Please read the instructions below carefully and follow them faithfully. **Important:** This year the checklist will be submitted separately from the main paper in OpenReview, please review it well ahead of the submission deadline: <https://neurips.cc/public/guides/PaperChecklist>.

#### 3.1 Style

Papers to be submitted to NeurIPS 2023 must be prepared according to the instructions presented here. Papers may only be up to **nine** pages long, including figures. Additional pages *containing only acknowledgments and references* are allowed. Papers that exceed the page limit will not be reviewed, or in any other way considered for presentation at the conference.

The margins in 2023 are the same as those in previous years.

Authors are required to use the NeurIPS L<sup>A</sup>T<sub>E</sub>X style files obtainable at the NeurIPS website as indicated below. Please make sure you use the current files and not previous versions. Tweaking the style files may be grounds for rejection.

#### 3.2 Retrieval of style files

The style files for NeurIPS and other conference information are available on the website at

<http://www.neurips.cc/>

The file `neurips_2023.pdf` contains these instructions and illustrates the various formatting requirements your NeurIPS paper must satisfy.

The only supported style file for NeurIPS 2023 is `neurips_2023.sty`, rewritten for L<sup>A</sup>T<sub>E</sub>X 2<sub>ε</sub>. **Previous style files for L<sup>A</sup>T<sub>E</sub>X 2.09, Microsoft Word, and RTF are no longer supported!**

The L<sup>A</sup>T<sub>E</sub>X style file contains three optional arguments: `final`, which creates a camera-ready copy, `preprint`, which creates a preprint for submission to, e.g., arXiv, and `nonatbib`, which will not load the `natbib` package for you in case of package clash.

**Preprint option** If you wish to post a preprint of your work online, e.g., on arXiv, using the NeurIPS style, please use the `preprint` option. This will create a nonanonymized version of your work with the text “Preprint. Work in progress.” in the footer. This version may be distributed as you see fit, as long as you do not say which conference it was submitted to. Please **do not** use the `final` option, which should **only** be used for papers accepted to NeurIPS.

At submission time, please omit the `final` and `preprint` options. This will anonymize your submission and add line numbers to aid review. Please *do not* refer to these line numbers in your paper as they will be removed during generation of camera-ready copies.

The file `neurips_2023.tex` may be used as a “shell” for writing your paper. All you have to do is replace the author, title, abstract, and text of the paper with your own.

The formatting instructions contained in these style files are summarized in Sections 4, 5, and 6 below.

## 4 General formatting instructions

The text must be confined within a rectangle 5.5 inches (33 picas) wide and 9 inches (54 picas) long. The left margin is 1.5 inch (9 picas). Use 10 point type with a vertical spacing (leading) of 11 points. Times New Roman is the preferred typeface throughout, and will be selected for you by default. Paragraphs are separated by  $\frac{1}{2}$  line space (5.5 points), with no indentation.

The paper title should be 17 point, initial caps/lower case, bold, centered between two horizontal rules. The top rule should be 4 points thick and the bottom rule should be 1 point thick. Allow  $\frac{1}{4}$  inch space above and below the title to rules. All pages should start at 1 inch (6 picas) from the top of the page.

For the final version, authors’ names are set in boldface, and each name is centered above the corresponding address. The lead author’s name is to be listed first (left-most), and the co-authors’ names (if different address) are set to follow. If there is only one co-author, list both author and co-author side by side.

Please pay special attention to the instructions in Section 6 regarding figures, tables, acknowledgments, and references.

## 5 Headings: first level

All headings should be lower case (except for first word and proper nouns), flush left, and bold.

First-level headings should be in 12-point type.

### 5.1 Headings: second level

Second-level headings should be in 10-point type.

#### 5.1.1 Headings: third level

Third-level headings should be in 10-point type.

**Paragraphs** There is also a `\paragraph` command available, which sets the heading in bold, flush left, and inline with the text, with the heading followed by 1 em of space.

## 6 Citations, figures, tables, references

These instructions apply to everyone.

### 6.1 Citations within the text

The `natbib` package will be loaded for you by default. Citations may be author/year or numeric, as long as you maintain internal consistency. As to the format of the references themselves, any style is acceptable as long as it is used consistently.

The documentation for `natbib` may be found at

<http://mirrors.ctan.org/macros/latex/contrib/natbib/natnotes.pdf>

Of note is the command `\citet`, which produces citations appropriate for use in inline text. For example,

```
\citet{hasselmo} investigated\dots
```



Figure 1: Sample figure caption.

produces

Hasselmo, et al. (1995) investigated...

If you wish to load the `natbib` package with options, you may add the following before loading the `neurips_2023` package:

```
\PassOptionsToPackage{options}{natbib}
```

If `natbib` clashes with another package you load, you can add the optional argument `nonatbib` when loading the style file:

```
\usepackage[nonatbib]{neurips_2023}
```

As submission is double blind, refer to your own published work in the third person. That is, use “In the previous work of Jones et al. [4],” not “In our previous work [4].” If you cite your other papers that are not widely available (e.g., a journal paper under review), use anonymous author names in the citation, e.g., an author of the form “A. Anonymous” and include a copy of the anonymized paper in the supplementary material.

## 6.2 Footnotes

Footnotes should be used sparingly. If you do require a footnote, indicate footnotes with a number<sup>3</sup> in the text. Place the footnotes at the bottom of the page on which they appear. Precede the footnote with a horizontal rule of 2 inches (12 picas).

Note that footnotes are properly typeset *after* punctuation marks.<sup>4</sup>

## 6.3 Figures

All artwork must be neat, clean, and legible. Lines should be dark enough for purposes of reproduction. The figure number and caption always appear after the figure. Place one line space before the figure caption and one line space after the figure. The figure caption should be lower case (except for first word and proper nouns); figures are numbered consecutively.

You may use color figures. However, it is best for the figure captions and the paper body to be legible if the paper is printed in either black/white or in color.

## 6.4 Tables

All tables must be centered, neat, clean and legible. The table number and title always appear before the table. See Table 1.

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<sup>3</sup>Sample of the first footnote.

<sup>4</sup>As in this example.

Table 1: Sample table title

Part		
Name	Description	Size ( $\mu\text{m}$ )
Dendrite	Input terminal	$\sim 100$
Axon	Output terminal	$\sim 10$
Soma	Cell body	up to $10^6$

Place one line space before the table title, one line space after the table title, and one line space after the table. The table title must be lower case (except for first word and proper nouns); tables are numbered consecutively.

Note that publication-quality tables *do not contain vertical rules*. We strongly suggest the use of the `booktabs` package, which allows for typesetting high-quality, professional tables:

<https://www.ctan.org/pkg/booktabs>

This package was used to typeset Table 1.

## 6.5 Math

Note that display math in bare TeX commands will not create correct line numbers for submission. Please use LaTeX (or AMSTeX) commands for unnumbered display math. (You really shouldn't be using  $\$$  anyway; see <https://tex.stackexchange.com/questions/503/why-is-preferable-to> and <https://tex.stackexchange.com/questions/40492/what-are-the-differences-between-align-equation-and-displaymath> for more information.)

## 6.6 Final instructions

Do not change any aspects of the formatting parameters in the style files. In particular, do not modify the width or length of the rectangle the text should fit into, and do not change font sizes (except perhaps in the **References** section; see below). Please note that pages should be numbered.

## 7 Preparing PDF files

Please prepare submission files with paper size "US Letter," and not, for example, "A4."

Fonts were the main cause of problems in the past years. Your PDF file must only contain Type 1 or Embedded TrueType fonts. Here are a few instructions to achieve this.

- You should directly generate PDF files using `pdflatex`.
- You can check which fonts a PDF file uses. In Acrobat Reader, select the menu Files>Document Properties>Fonts and select Show All Fonts. You can also use the program `pdf fonts` which comes with `xpdf` and is available out-of-the-box on most Linux machines.
- `xfig` "patterned" shapes are implemented with bitmap fonts. Use "solid" shapes instead.
- The `\bbold` package almost always uses bitmap fonts. You should use the equivalent AMS Fonts:

```
\usepackage{amsfonts}
```

followed by, e.g., `\mathbb{R}`, `\mathbb{N}`, or `\mathbb{C}` for  $\mathbb{R}$ ,  $\mathbb{N}$  or  $\mathbb{C}$ . You can also use the following workaround for reals, natural and complex:

```
\newcommand{\RR}{\mathbb{R}} %real numbers
\newcommand{\Nat}{\mathbb{N}} %natural numbers
\newcommand{\CC}{\mathbb{C}} %complex numbers
```

Note that `amsfonts` is automatically loaded by the `amssymb` package.

If your file contains type 3 fonts or non embedded TrueType fonts, we will ask you to fix it.

## 7.1 Margins in L<sup>A</sup>T<sub>E</sub>X

Most of the margin problems come from figures positioned by hand using `\special` or other commands. We suggest using the command `\includegraphics` from the `graphicx` package. Always specify the figure width as a multiple of the line width as in the example below:

```
\usepackage[pdftex]{graphicx} ...  
\includegraphics[width=0.8\linewidth]{myfile.pdf}
```

See Section 4.4 in the graphics bundle documentation (<http://mirrors.ctan.org/macros/latex/required/graphics/grfguide.pdf>)

A number of width problems arise when L<sup>A</sup>T<sub>E</sub>X cannot properly hyphenate a line. Please give LaTeX hyphenation hints using the `\-` command when necessary.

## Acknowledgments and Disclosure of Funding

Use unnumbered first level headings for the acknowledgments. All acknowledgments go at the end of the paper before the list of references. Moreover, you are required to declare funding (financial activities supporting the submitted work) and competing interests (related financial activities outside the submitted work). More information about this disclosure can be found at: <https://neurips.cc/Conferences/2023/PaperInformation/FundingDisclosure>.

Do **not** include this section in the anonymized submission, only in the final paper. You can use the `ack` environment provided in the style file to automatically hide this section in the anonymized submission.

## 8 Supplementary Material

Authors may wish to optionally include extra information (complete proofs, additional experiments and plots) in the appendix. All such materials should be part of the supplemental material (submitted separately) and should NOT be included in the main submission.

## References

References follow the acknowledgments in the camera-ready paper. Use unnumbered first-level heading for the references. Any choice of citation style is acceptable as long as you are consistent. It is permissible to reduce the font size to `small` (9 point) when listing the references. Note that the Reference section does not count towards the page limit.

- [1] Alexander, J.A. & Mozer, M.C. (1995) Template-based algorithms for connectionist rule extraction. In G. Tesauro, D.S. Touretzky and T.K. Leen (eds.), *Advances in Neural Information Processing Systems 7*, pp. 609–616. Cambridge, MA: MIT Press.
- [2] Bower, J.M. & Beeman, D. (1995) *The Book of GENESIS: Exploring Realistic Neural Models with the GEneral NEural Simulation System*. New York: TELOS/Springer-Verlag.
- [3] Hasselmo, M.E., Schnell, E. & Barkai, E. (1995) Dynamics of learning and recall at excitatory recurrent synapses and cholinergic modulation in rat hippocampal region CA3. *Journal of Neuroscience* **15**(7):5249-5262.