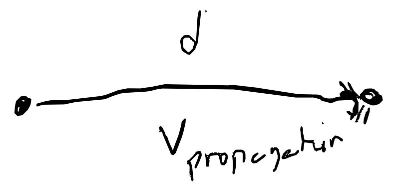
Principel of Relativity

- To describe a system, we need a frame of reference
- systems with no external forces are called inertial
- All laws of nature are identical in inertial frames
- in classical mech particles interact via potential energy. This requires instantaneous interactions.
  - No interactions are instantaneous
  - The distance between particle divided by the time for interaction to hit is "velocity of propagation"
  - This is the max of interactions
    - no motion can go faster than this
    - classical mech is the limit when $V_{propagation} \to \infty$
    - consequence: time is absolute. Δt same in all frames
- absolute time is contradiction when $V_{propagation} \not\to \infty$
  - evidence: velocity of propagation will differ between inertial frames
  - velocity of composit- motion is additive of velocities

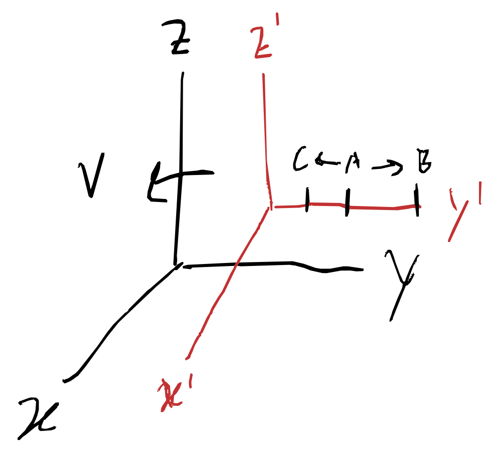


$$\Delta t = \frac{d}{V_{prop}}$$

$$\frac{d + V'\Delta t}{V'_{propagation}} = \frac{d}{V_{prop}}$$

$$1 + \frac{V'\Delta t}{d} = \frac{V'_{prop}}{V_{prop}}$$ ← Clearly, propagation is faster somehow...

However, in order for relativity to hold, V_{prop} must not vary. Otherwise, the motion of particles in one inertial frame compared to another will end up being different



In XY'Z', suppose a signal is sent from A→B and A→C simultaneously. B,C receive signal at the same time.

If time is absolute, XYZ must also think signal is received at the same time but in the view of XYZ, B is approaching A while C is distancing itself. Hence, in order for: A→B and A→C to occur simultaneously, signal velocity depends on direction. However, signal velocity is isotropic.

Intervals

- define an event.
  - where is it?
  - when is it?

$(t, x, y, z) \leftarrow$ world point

$\langle x(t), y(t), z(t) \rangle \leftarrow$ world line

Consider K, K' st Y, Z are at rest relative to Y', Z' while X, X' are moving relative to one another

Event 1: (signal emitted) (t_1, x_1, y_1, z_1) in K

Event 2: (signal receive) (t_2, x_2, y_2, z_2) in K

$$c(t_2 - t_1) = \sqrt{(z_2 - z_1)^2 + (y_2 - y_1)^2 + (x_2 - x_1)^2}$$

Now let's observe in K' system

signal emitted (t_1', x_1', y_1', z_1')

signal received (t_2', x_2', y_2', z_2')

Since this is about signal emission and reception, we again must agree

$$c(t_2' - t_1') = \sqrt{(z_2' - z_1')^2 + (y_2' - y_1')^2 + (x_2' - x_1')^2}$$

Define

$$s_{12} \equiv c^2(t_2 - t_1)^2 - (x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2$$

If $s_{12} = 0$ in one frame it must be zero in all frames.

In fact, for any 2 events, s_{12} must be invariant across frames. $\leftarrow$ let's show

Suppose we are considering an infinitesimal interval between events

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

In K_1 frame, at worst

$$ds^2 = a\, ds_1^2$$

a can only depend on absolute velocity of K_1 relative to K.

No dependence on coordinates
- space, time is homogenous,

- No dependence on direction
  - space is isotropic

consider K_1 and K_2 relative to K

$$ds^2 = a(v_1)\, ds_1^2 \qquad ds^2 = a(v_2)\, ds_2^2$$

$$ds_1^2 = \frac{a(v_2)}{a(v_1)}\, ds_2^2$$

also $ds_1^2 = a(v_{21})\, ds_2^2$

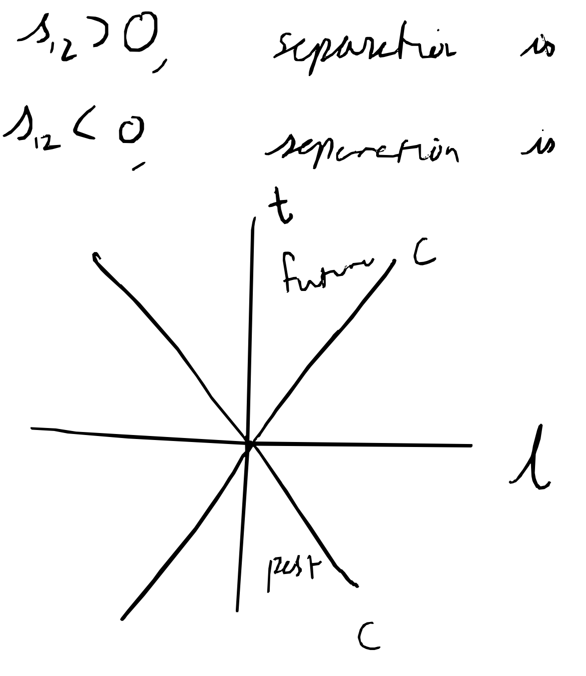
but v_{21} depends on direction, so in order for this to be true, a must be constant. The only constant satisfying all 3 is unity.

- A couple of consequences

$$s_{12} = s_{12}' = c^2 t_{21}^2 - l_{21}^2 = c^2 t_{21}'^2 - l_{21}'^2$$

If $s_{12} > 0$, separation is timelike

$s_{12} < 0$ separation is spacelike



Proper time

Consider a system at rest

$$ds^2 = c^2 dt^2$$

Compare this to the same interval in diff frame

$$c^2 dt^2 = c^2 dt'^2 - dx'^2 - dy'^2 - dz'^2$$

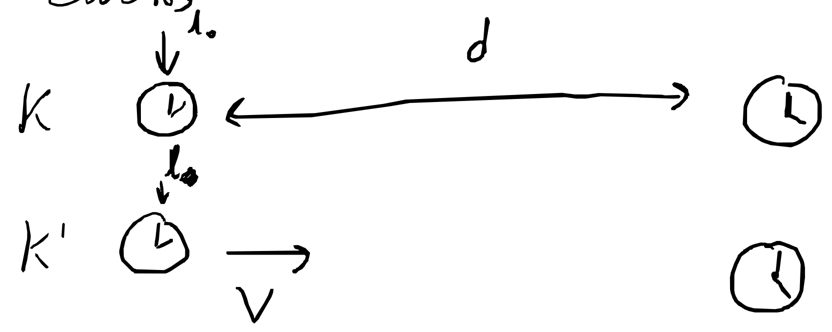
$$dt = dt' \sqrt{1 - \frac{dx'^2 + dy'^2 + dz'^2}{c^2 dt'^2}}$$

$$dt = \underset{\substack{\uparrow \\ \text{gets} \\ \text{longer}}}{dt'} \sqrt{1 - \underset{\substack{\uparrow \\ \text{gets} \\ \text{smaller}}}{\frac{v'^2}{c^2}}}$$

at rest is the shortest pure time interval between two events.

- However, doesn't the clock in K' think its at rest compared to K? Don't both clocks see the other as moving, relatively speaking?

Yes! ish. How can we compare clocks

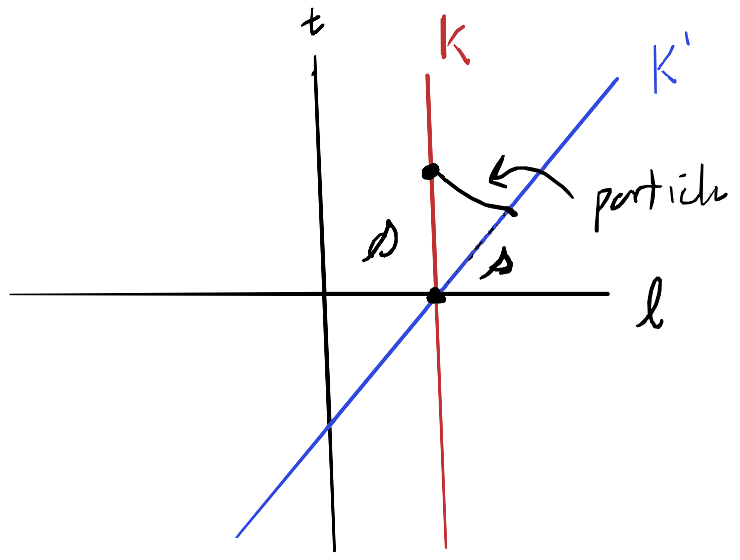


We need two clocks to compare K to K'. We must take then not be looking at the interval corresponding to l_0.

- Another way of saying it? In order to compare clock in K' to K, K' must not be inertial.

Said differently...

The world line view



particle in K' has to get back over here in order to be comparable. Now its interval isn't guaranteed to be the same.

Lorentz Transformations

Transform from x, y, z, t to x', y', z', t'

- In classical mechanics because time is absolute, transformation is straight forward.
- However, intervals are no longer guaranteed to be invariant

- To obtain the right transformation, we can look for transformations that conserve distances in 4D space.
- rotations and translations ← uninteresting

↑
focus here

6 possible rotations

$xy,\ xz,\ xt,\ yz,\ yt,\ zt$

consider the tx rotation

$c^2t^2 - x^2$ must be conserved.

Hence

$x = x'\cosh\psi + ct'\sinh\psi$

$ct = x'\sinh\psi + ct'\cosh\psi$

differs from typical rotation because of use of hyperbolic functions.

Consider the transformation assuming we have K' moving with respect to K. In particular, lets look at the movement of the origin of K'

$x = ct'\sinh\psi, \quad ct = ct'\cosh\psi$

$\frac{x}{ct} = \tanh\psi$

↑
V/c

$\tanh\psi = \frac{\sinh\psi}{\cosh\psi}$ and $\sinh^2\psi - \cosh^2\psi = 1$

$= \frac{\sinh\psi}{\sqrt{1-\sinh^2\psi}} = \frac{V}{c}$

$\sinh\psi = \frac{V/c}{\sqrt{1-\frac{V^2}{c^2}}}, \quad \cosh\psi = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}$

$$x = \frac{x' + Vt'}{\sqrt{1-\frac{V^2}{c^2}}}, \quad y = y', \quad z = z', \quad ct = \frac{t' + \frac{V}{c^2}x'}{\sqrt{1-\frac{V^2}{c^2}}}$$

From $K' \to K$ is easy as K moves with $-V$ with respect to K'

- Coordinates become imaginary if $V > c$, hence we can assume $V < c$

Consequences: Imagine there is a rod of length l in the K frame.

Given the transformation of x in the K' frame, can then be calced as

$x_1 = \frac{x_1' + Vt'}{\sqrt{1-\frac{V^2}{c^2}}}, \quad x_2 = \frac{x_2' + Vt'}{\sqrt{1-\frac{V^2}{c^2}}}$

$\Delta x' = x_2' - x_1'$

$\Delta x = \frac{x_2' - x_1'}{\sqrt{1-\frac{V^2}{c^2}}}$

$\Delta x' = \Delta x\sqrt{1-\frac{V^2}{c^2}}$ ← rod is shorter in moving frame

↑
proper length

Can also define proper time

$\Delta t' = \Delta t / \sqrt{1-\frac{V^2}{c^2}}$

Note: In general, lorentz transformation not commutative

Transformation of Velocities

Now we have $\frac{dx}{dt} = V_x$

$$dx = \frac{dx' + V dt'}{\sqrt{1 - V^2/c^2}} \qquad dt = \frac{dt' + \frac{V}{c^2} dx'}{\sqrt{1 - V^2/c^2}}$$

$$V = \frac{d\vec{r}}{dt}, \quad V' = \frac{dx'}{dt'}$$

$$V_x = \frac{dx' + V dt'}{dt' + \frac{V}{c^2} dx'} = \frac{V_x' + V}{1 + \frac{V}{c^2} V_x'}, \quad V_y = \frac{V_y'}{1 + \frac{V}{c^2} V_x'} \sqrt{1 - \frac{V^2}{c^2}}$$

$$V_z = \frac{V_z'}{1 + \frac{V}{c^2} V_x'} \sqrt{1 - V^2/c^2}$$

assume $V \ll c$,

$$v_x = v_x' + V\left(1 - \frac{v_x'^2}{c^2}\right), \quad v_y = v_y'\left(1 - v_x' \frac{V}{c^2}\right), \quad v_z = v_z' - v_z' v_x' \left(\frac{V}{c^2}\right)$$

$$\vec{v} = \vec{v}' + \vec{V} - \frac{1}{c^2}(\vec{V} \cdot \vec{v}')\, \vec{v}'$$

Consider the motion of a particle in the XY plane. in K system

$$v_x = v \cos\theta \qquad v_y = v \sin\theta$$

in K' system

$$v_x' = v' \cos\theta' \qquad v_y = v' \sin\theta'$$

can then write

$$\tan\theta = \frac{v' \sqrt{1 - \frac{V^2}{c^2}}\, \sin\theta'}{v' \cos\theta' + V}$$

consider deviation of light... $v = v' = c$

$$\tan\theta = \frac{\sqrt{1 - \frac{V^2}{c^2}}\, \sin\theta'}{\frac{V}{c} + \cos\theta'}$$

for $V \ll c$,

$$\Delta\theta = \frac{V}{c} \sin\theta'$$

Four-Vectors

- Coordinates of an event (ct, x, y, z) could be treated as a vector
- Call this vector $\vec{A}^i = (A^0, A^1, A^2, A^3)$

define this as the contravariant.

Can imagine a transformation st

$$A^0 = A_0, \quad A^1 = -A_1 \quad \dots$$

$\vec{A}_i$ is called covariant form

$$\vec{A}_i = \underset{\substack{\uparrow \\ \text{Minkowski metric}}}{M} \vec{A}^i$$

$$\|A\|^2 = (\vec{A}^i)^T \vec{A}_i$$

and more generally

$$\langle A, B \rangle = (A^i)^T M B^i = (\vec{A}^i)^T \vec{B}_i$$

$$(\underset{\substack{\uparrow \\ \text{time like}}}{A^0}, \overbrace{A^1, A^2, A^3}^{\text{spacelike}})$$

4 tensor

16 quantities A^{ik} which transform like the products of components of two 4 vectors

$$A_{00} = A^{00}, \quad A_{01} = -A^{01} \quad A_{11} = A^{11}$$

$$A_0{}^1 = A^{01} \quad A_1{}^0 = -A^{10}$$

A^{ik} ← contravariant A_{ik} ← covariant $A_i{}^k$, $A^i{}_k$ ← mixed

symmetric $A^{ik} = A^{ki}$

antisymmetric $A^{ik} = -A^{ki}$

Trace equals

$$A^i{}_i = A^0{}_0 + A^1{}_1 + A^2{}_2 + A^3{}_3$$

$$M = \underbrace{g_{ik}}_{\substack{\text{Metric} \\ \text{tensor}}} = g^{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

completely antisymmetric tensor

$$e^{iklm} = \pm 1, \quad e^{0123} = 1$$

$e^{iklm} = 1$ for even transitions of 0123

$e^{iklm} = -1$ for odd transitions

Consider the reflection of a coordinate system, components of vector change sign

- these are called <u>polar</u>

- Components of a vector that are written as the cross product of 2 polar vectors do not change sign. These are called <u>axial</u>

If $C = A \times B$

$$C_\alpha = \frac{1}{2} e_{\alpha\beta\gamma} C_{\beta\gamma}, \text{ where } C_{\beta\gamma} = A_\beta B_\gamma - A_\gamma B_\beta$$

Consider 4-gradient of scalar φ

$$\frac{\partial \varphi}{\partial x^i} = \left(\frac{1}{c}\frac{\partial \varphi}{\partial t}, \nabla \varphi\right)$$

$$\partial \varphi = \frac{\partial \varphi}{\partial x^i} dx^i$$

$\frac{\partial}{\partial x^i}$ ← covariant components of a 4-vector operator

Thus $\underbrace{\frac{\partial}{\partial x^i} A^i}_{\text{effectively, the divergence}}$ is a scalar

$\int dx^i$ ← integral over curve

$\int dx^i dx'^k - dx^k dx'^i$ ← integral over parallelogram formed by dx, dx'

$$\int dS^{ikl} = \int \begin{vmatrix} \partial x^i & \partial x'^i & \partial x''^i \\ \partial x^k & \partial x'^k & \partial x''^k \\ \partial x^l & \partial x'^l & \partial x''^l \end{vmatrix}$$ ← integral over hyper surface

More conveniently

$$\partial S^i = -\frac{1}{6} e^{iklm} \partial S_{klm}$$

$$\int dx^0 dx^1 dx^2 dx^3 = \int \underbrace{c\,dt\,\partial V}_{\partial \Omega}$$ ← integral over 4-volume

$$\partial S_i \to \partial \Omega \frac{\partial}{\partial x_i}$$

$$\oint A^i \partial S_i = \int \frac{\partial A^i}{\partial x^i} \partial \Omega$$

$$\partial f^*_{ik} = \partial S_i \frac{\partial}{\partial x^k} - \partial S_k \frac{\partial}{\partial x^i}$$

$$\underbrace{\frac{1}{2}\int A^{ik} \partial f^*_{ik} = \frac{1}{2}\int \partial S_i \frac{\partial A^{ik}}{\partial x^k} - \partial S_k \frac{\partial A^{ik}}{\partial x^i} = \int \partial S_i \frac{\partial A^{ik}}{\partial x^k}}_{\text{assuming } A^{ik} \text{ is antisymmetric}}$$

$$\underset{\substack{\uparrow \\ \text{int over curve}}}{\oint A_i dx^i} = \int df^{ki} \frac{\partial A_i}{\partial x^k} = \frac{1}{2} \int \underset{\substack{\uparrow \\ \text{int over surface}}}{df^{ik}} \left(\frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}\right)$$

4-dimensional velocities

$$u^i = \frac{\partial x^i}{\partial s}$$

$$\partial s = c\, dt \sqrt{1 - v^2/c^2}$$

$$u^1 = \frac{\partial x^1}{\partial s} = \frac{\partial x}{c\, dt \sqrt{1 - v^2/c^2}} = \frac{v_x}{c\sqrt{1 - v^2/c^2}}$$

$$u^i = \left(\frac{1}{\sqrt{1 - v^2/c^2}} , \frac{\vec{v}}{c\sqrt{1 - \frac{v^2}{c^2}}} \right) \leftarrow \text{dimensionless quantity}$$

$$u^i u_i = 1$$

$$w^i = \frac{\partial^2 x^i}{\partial s^2} = \frac{\partial u^i}{\partial s} \leftarrow \text{4-acceleration}$$

$$\frac{\partial}{\partial s}\left(u^i u_i\right) = \frac{\partial}{\partial s}(1)$$

$$u^i w_i + w^i u_i = 0$$

$$w^i u_i = u^i w_i$$

Relativistic Mechanics

Principal of Least Action

What is the action of a free particle?

- Integral value must be Lorentz invariant
- Integral must be differential of first order?

Hence

$$S = -\alpha \int_a^b ds \qquad a \to b \text{ along world line}$$

α — some constant

$$ds = L\,dt \qquad S = -\alpha \int_{t_1}^{t_2} L\,dt$$

recall that $ds = c\sqrt{1-\frac{v^2}{c^2}}\,dt$

Hence

$$S = -\alpha c \int \sqrt{1-v^2/c^2}\,dt$$

For free particle

$$L = -\alpha c\sqrt{1-v^2/c^2}$$

expand to first order in $\frac{v^2}{c^2}$ assuming $v \ll c$

$$\sqrt{1-x} \simeq 1 - \frac{x}{2}$$

$$L = -\alpha c + \frac{\alpha v^2}{2c^2}$$

$$L = -\alpha c + \frac{\alpha v^2}{2c}$$ If we ignore the constant,

We quickly see that α must be mc

Hence

$$L = -mc\sqrt{1-\frac{v^2}{c^2}}$$

Energy and Momentum

$p = \frac{\partial L}{\partial v}$ ← definition

$\vec{p} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}}$

$\frac{dp}{dt} = \frac{m\dot{\vec{v}}}{\sqrt{1-v^2/c^2}} + \frac{m\vec{v}\ \ \ (\vec{v}\cdot\dot{\vec{v}})/c^2}{\sqrt{1-\frac{v^2}{c^2}}^{\,3}}$ if force is perpendicular $\vec{v}\cdot\dot{\vec{v}} = 0$

$\frac{dp}{dt} = \frac{m\dot{v}}{\sqrt{1-v^2/c^2}}\left(\frac{1-\frac{v^2}{c^2}+\frac{v^2}{c^2}}{1-v^2/c^2}\right)$

$= \frac{m\dot{v}}{\sqrt{1-v^2/c^2}^{\,3}}$

$\mathcal{E} = \vec{p}\cdot\vec{v} - L$

For free particle

$\mathcal{E} = \frac{mv^2}{\sqrt{1-\frac{v^2}{c^2}}} + mc^2\sqrt{1-\frac{v^2}{c^2}} = mc^2\frac{\frac{v^2}{c^2}+1-\frac{v^2}{c^2}}{\sqrt{1-v^2/c^2}} = \frac{mc^2}{\sqrt{1-v^2/c^2}}$

at $v=0$ we uncover the rest energy $\mathcal{E} = mc^2$.

- This is about a system in free motion, not a particle.

m is total mass of the body.

However $\mathcal{E} = mc^2$ is about the whole body at rest. However, the components of the body can be moving.

Hence, $mc^2 = \sum m_a c^2$. Only energy is conserved not mass.

$\frac{\mathcal{E}^2}{c^2} = p^2 + m^2c^2 \qquad \mathcal{H} = c\sqrt{p^2+m^2/c^2}$

$\mathcal{H} = mc^2 + \frac{p^2}{2m}$ if $p \ll m/c$

$\vec{p} = \frac{\mathcal{E}\vec{v}}{c^2}$

If $v \to c$, $\mathcal{E}\to\infty$. Hence, $v \neq c$ if $m \neq 0$

If $m=0$

$p = \frac{\mathcal{E}}{c}$

$\delta S = -mc\,\delta\int_a^b ds = 0$

$ds = \sqrt{dx^i dx_i}$

$\delta S = -mc\int\frac{\delta dx^i dx_i + dx^i\delta dx_i}{2\sqrt{dx_i dx^i}} = -mc\int u_i\, d\delta x^i$

$\delta S = \underbrace{-mcu_i\delta x^i\Big|_a^b}_{0} + mc\int_a^b \delta x^i \frac{du_i}{ds}ds = 0$

Hence, $\frac{du}{ds} = 0$

$p_i = -\frac{\partial S}{\partial x_i}$ ← momentum four-vector

$p_i = (\mathcal{E}/c, -\vec{p})$

$p^i = (\mathcal{E}/c, \vec{p})$

$p^i = mcu^i$

$p_i p^i = m^2c^2$

$g^i = \frac{dp^i}{ds} = mc\frac{du^i}{ds}$

Hence, $g_i u^i = 0$

$g^i = \left(\frac{\vec{F}\cdot\vec{v}}{c^2\sqrt{1-\frac{v^2}{c^2}}}, \frac{\vec{F}}{\sqrt{1-v^2/c^2}}\right)$

$\frac{\partial S}{\partial x_i}\frac{\partial S}{\partial x^i} = m^2c^2$

To obtain classical, write

$S = S' - mc^2 t$

Transformation of Distribution Functions

distribution functions

$f(\vec{p})\,dp_x\,dp_y\,dp_z$

$d^3p = dp_x\,dp_y\,dp_z$ ← component of a 4-D hypersurface

- element of hypersurface is a 4-vector normal to the hypersurface. This surface defined by $p_\mu p^\mu = m^2c^2$

$p_0 = (\mathcal{E}/c)$

$dp_x\,dp_y\,dp_z$ ← just the 0th component because that's all that's left

$\dfrac{d^3p}{\mathcal{E}}$ is invariant, since they are both the 0th component of parallel vectors on the same surface

The number of particles, $f(\vec{p})\,dp_x\,dp_y\,dp_z$

$$\underbrace{f(\vec{p})\,\mathcal{E}}_{\text{also invariant}}\;\underbrace{\frac{dp_x\,dp_y\,dp_z}{\mathcal{E}}}_{\text{invariant}}$$

Hence

$$\frac{f(p)}{\mathcal{E}} = \frac{f'(p')}{\mathcal{E}'}$$ relates distribution between

K and K'

- Introduce spherical coordinates in momentum space

$p^2\,dp\,\underbrace{d\Omega}_{\text{solid angle}}$ $\qquad p\,dp = \dfrac{\mathcal{E}\,d\mathcal{E}}{c^2}$

$$\frac{p^2 dp^2\,do}{\mathcal{E}} = \frac{p\,d\mathcal{E}\,do}{c^2}$$ ← must be invariant

$f(\vec{r},\vec{p})\,dp_x\,dp_y\,dp_z\,dV$ ← phase volume

consider the phase space element

$$\frac{dp_x\,dp_y\,dp_z\,dV}{dp_x'\,dp_y'\,dp_z'\,dV'} = \frac{d\tau}{d\tau'}$$

Note that $\dfrac{d^3p}{\mathcal{E}} = \dfrac{d^3p'}{\mathcal{E}'} \Rightarrow \dfrac{\mathcal{E}}{\mathcal{E}'} = \dfrac{d^3p}{d^3p'} \Rightarrow \dfrac{1/\sqrt{1-v^2/c^2}}{1/\sqrt{1-v'^2/c^2}} = \dfrac{d^3p}{d^3p'}$

$dV = dV_0\sqrt{1-v^2/c^2}$ $\qquad dV' = dV_0\sqrt{1-v'^2/c^2}$

$\dfrac{d\tau}{d\tau'} = 1$, so phase space is invariant.

since $f\,d\tau$ is also invariant, f must be invariant under transformations

Decay of Particles

$Mc^2 = \mathcal{E}_{10} + \mathcal{E}_{20}$ ← conservation of energy

rest energy of one body

energies of constituent bodies.

$\mathcal{E}_{10} > m_1, \quad \mathcal{E}_{20} > m_2$ to occur simultaneously

$P_{10} + P_{20} = 0$ ← conservation of momentum

$$P_{10}^2 = P_{20}^2$$

recall $\mathcal{E}^2 = p^2 + mc^2$

$$\mathcal{E}_{10}^2 - m_1c^2 = \mathcal{E}_{20}^2 - m_2c^2$$

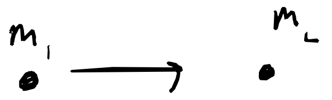
Can now write

$$\mathcal{E}_{10} = \mathcal{E}_{20}^2 - m_2c^2 + m_1c^2$$

$$\mathcal{E}_{10}^2 = (Mc^2 - \mathcal{E}_{10})^2 - m_2c^2 + m_1c^2$$

$$\mathcal{E}_{10} = \frac{Mc^2 - m_2c^2 + m_1c^2}{2M}$$

<u>Suppose</u>



in Laboratory frame

$\mathcal{E} = \mathcal{E}_1 + m_2c^2$ (for now, set $c = 1$)

$$\vec{P} = \vec{P}_1$$

$$V_{CoM} = \frac{\vec{P}}{\mathcal{E}} = \frac{\vec{P}_1}{\mathcal{E}_1 + m_2}$$

$$M_{cm}^2 = \mathcal{E}^2 - p^2 = (\mathcal{E}_1 + m_2)^2 - (\mathcal{E}_1 - m_1)^2$$

$$M_{cm}^3 = 2\mathcal{E}_1(m_1 + m_2) + m_2^2 - m_1^2$$

$$M_{cm}^2 = m_1(\mathcal{E}_1 - m_1) + m_2(\mathcal{E}_1 + m_2)$$

Invariant Cross Section

- Collisions characterized by an invariant quantity

Consider beam of particles moving $\vec{v}_1$ and another beam of particles $\vec{v}_2$ with densities n_1, n_2.

define V_{rel} to be the relative velocity of beam 1 from beam 2.

$$d\nu = \sigma v_{rel} n_1 n_2 \, dV dt$$

number of collisions in a volume in a fixed time frame

usual cross section

Hence $d\nu$ is an invariant quantity.

$dVdt$ is also invariant hence $A n_1 n_2$ must be invariant

$$n\,dV = n_0\, dV_0$$

$$n = \frac{n_0}{\sqrt{1-v^2}} \Rightarrow n = n_0 \, \mathcal{E}/m$$

Hence

$$A n_1 n_2 = A \mathcal{E}_1 \mathcal{E}_2$$

$$A \frac{\mathcal{E}_1 \mathcal{E}_2}{P_{1i} P_2^i} = A \frac{\mathcal{E}_1 \mathcal{E}_2}{\mathcal{E}_1 \mathcal{E}_2 - P_1 \cdot P_2} = inv$$

$$A = \sigma v_{rel} \frac{P_{1i} P_2^i}{\mathcal{E}_1 \mathcal{E}_2}$$ In arbitrary frame

In rest frame of beam 2 $\mathcal{E}_2 = m_2, \vec{P}_2 = 0$

and $A = \sigma v_{rel}$

$$P_{i1} P_2^i = \frac{m_1}{\sqrt{1-v_{rel}^2}} m_2$$

$$V_{rel} = \sqrt{1 - \left(\frac{m_1 m_2}{P_{1i} P_2^i}\right)^2}$$

re-expressing momentum in terms of velocities

$$V_{rel} = \frac{\sqrt{(\vec{v}_1 - \vec{v}_2)^2 - (\vec{v}_1 \times \vec{v}_2)^2}}{1 - \vec{v}_1 \cdot \vec{v}_2}$$

Elastic Collisions

Conservation of energy and momentum can be represented by

$$p_1^i + p_2^i = p_1'^i + p_2'^i$$

$$p_1^i + p_2^i - p_1'^i - p_2'^i = 0$$

$$p_1^i + p_2^i - p_1'^i = p_2'^i$$

$$m_1^2 + p_2^i p_{1i} - p_1'^i p_{1i} + p_1'^i p_{i2} + \cancel{m_2^2} - p_1 p_{i2} - p_1' p_{i1}' - p_2^i p_1'^i + m_1^2 = \cancel{m_2^2}$$

$$m_1^2 + p_1^i p_{i2} - p_1'^i p_{i1} - p_1'^i p_{i2} = 0$$

and similar for

$$p_1^i + p_2^i - p_2'^i = p_1'^i$$

Consider the case where we are in the frame where p_2^i is at rest.

In this limit...

$$p_1^i p_{i2} = \mathcal{E}_1 m_2$$

$$p_2^i p_1'^i = m_2 \mathcal{E}_1'$$

$$p_{i1} p_1'^i = \mathcal{E}_1 \mathcal{E}_1' - p_1 p_1' = \mathcal{E}_1 \mathcal{E}_1' - |p_1||p_1'| \cos\theta_1$$

solve for $\cos\theta_1$

$$\frac{\mathcal{E}_1\mathcal{E}_1' - p_{1i} p_1'^i}{|p||p'|} = \cos\theta_1$$

recall

$$m_1^2 + p_{1i} p_2^i - p_1'^i p_{1i} - p_{2i} p_1'^i = 0$$

$$\Rightarrow m_1^2 + m_2 \mathcal{E}_1 - p_{1i} p_1'^i - m_2 \mathcal{E}_1' = 0$$

$$\frac{\mathcal{E}_1 \mathcal{E}_1' - m_1^2 - m_2 \mathcal{E}_1 + m_2 \mathcal{E}_1'}{|p_1||p_1'|} = \cos\theta_1$$

Similarly can consider θ_2, the angle between p_{i1} and p_2'

$\mathcal{E}_1'$, $\mathcal{E}_2'$ in terms of θ_1, θ_2

$$\cos\theta_2 = \frac{(\mathcal{E}_1 + m_1)(\mathcal{E}_2' - m_2)}{p_1 p_2'}$$

$$\mathcal{E}_2' = m_2 \frac{(\mathcal{E}_1 + m_1)^2 + (\mathcal{E}_1^2 - m_1^2)\cos^2\theta_2}{(\mathcal{E}_1 + m_2)^2 - (\mathcal{E}_1^2 - m_1^2)\cos^2\theta_2}$$

If $m_1 > m_2$

$$\sin\theta_{1\max} = m_2/m_1$$

alternatively consider $m_1 = 0$

$$p_1 = \mathcal{E}_1, \quad p_1' = \mathcal{E}_1'$$

$$\mathcal{E}_1' = \frac{m_2}{1 - \cos\theta_1 + m_2/\mathcal{E}_1}$$

Now, consider COM system

$$p_{10} = -p_{20} = p_0$$

during collision, due to conservation of momentum, its just a rotation

χ be the angle of rotation

$$p_{1i} p_1'^i = \mathcal{E}_{10}\mathcal{E}_{10}' - p_{10} \cdot p_{10}'$$

$$= \mathcal{E}_{10}^2 - p_0^2 \cos\chi$$

$$= p_0^2(1 - \cos\chi) + m_1^2$$

Angular Momentum

For classical systems

$$M = \sum \vec{r} \times \vec{p}$$

$\uparrow$ radius $\quad \uparrow$ momentum

$$x^i$$

$\uparrow$

coordinates of one particle

x'^i is the coordinate of an infinitesimal rotation

$$x'^i - x^i = x_k \delta\Omega^{ik}$$

$\uparrow$ infinitesimal coefficients

Under a rotation $x_i x^i = x'_i x'^i$, eg norm must not change

Re-expressing $x'^i = x_k \delta\Omega^{ik} + x^i$

$$x_i x^i = (x_k \Omega^{ik} + x^i)(x_k \Omega_i^{\ k} + x_i)$$

$$\cancel{x_i x^i} = \cancel{x^i x_i} + 2x^i x_k \Omega_i^{\ k}$$

$$0 = x^i x^k \Omega_{ik}$$

Since $x^i x^k$ is a symmetric tensor

S_{ik} must be antisymmetric tensor

$$\delta\Omega_{ik} = -\delta\Omega_{ki}$$

Recall

$$\delta S = -\sum p^i \delta x_i \Big|_a^b \qquad \delta x_i = \delta\Omega_{ik} x^k$$

For rotation, we write

$$\delta S = -\delta\Omega_{ik} \sum p^i x^k$$

$$\delta S = -\delta\Omega_{ik}\left(\frac{1}{2}\sum(p^i x^k - p^k x^i)\Big|_a^b\right)$$

$$\sum(p^i x^k - p^k x^i)_b = \sum(p^i x^k - p^k x^i)_a$$

$$M^{ik} = \sum(x^i p^k - x^k p^i)$$

$$\vec{M} = \sum \vec{r} \times \vec{p}$$

$$M^{23} = M_x \qquad -M^{13} = M_y \qquad M^{12} = M_z$$

$M^{00} = 0$, ah, M is antisymmetric, so zero along the diagonals

$$M^{ik} = \left[c \cdot \sum\left(t\vec{p} - \frac{\mathcal{E}r}{c^2}\right), -M\right]$$

$$\sum tp - \frac{\mathcal{E}r}{c^2} = \text{const}$$

$$\frac{\sum \mathcal{E} r}{\sum \mathcal{E}} - \frac{c^2 \sum \vec{p}}{\sum \mathcal{E}} t = \text{const}$$

set $\mathcal{E} \approx mc^2$

$$R = \frac{\sum m r}{\sum m}$$

Charges in Electromagnetic Fields

Elementary Particle in theory of relativity

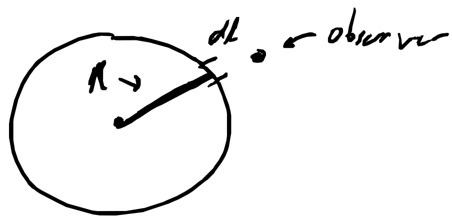
- Interaction of particle best described by a field
- because velocity of propagation of interactions, situation is changed.

- Process to understand

particle one interacts with field interacts with particle two

- Two types of fields, EM and gravitational

- Rigid bodies are impossible in relativistic mechanics
  - consider a disc



When disc spins, radial vector unimpacted, $\vec{V} \perp \vec{R}$, but, circumfrance shrinks when adding up dl as it spins. However, in the infinitesimally small rest frames, the circumfrence is unchanging. Hence, at relativistic speeds things cannot be rigid

Four-Potential of a Field

- Action of particle in field is characterized by two pieces
  - Action of a free particle
  - interaction of free particle
- Properties of particles interaction are completely described by a single parameter, the charge, q, of the particle in the field
  - Field characterized by Four-vector A_i the four potential, components are function of space-and-time

Contribution to action is as

$$\frac{e}{c}\int_a^b A_i\, dx^i \quad \Big\}\ \text{integrate along path}$$

Total action is then

$$S = -\int_a^b mc\, ds + \frac{e}{c} A_i\, dx^i$$

$$A^i = (\varphi, \vec{A})$$

Scalar field ↑ ↖ vector field

$$S = -\int_a^b mc\, ds + e\varphi\, dt - \frac{e}{c}\vec{A}\cdot d\vec{r}$$

define $d\vec{r}/dt = \vec{v}$

Integrate over t instead

$$S = -\int_{t_1}^{t_2} mc\frac{ds}{dt}\, dt + e\varphi\, dt - \frac{e}{c}\vec{A}\cdot\vec{v}\, dt$$

$$S = -\int_{t_1}^{t_2} mc^2\sqrt{1-\frac{v^2}{c^2}}\, dt + e\varphi\, dt - \frac{e}{c}\vec{A}\cdot\vec{v}\, dt$$

$$\mathcal{L} = -mc^2\sqrt{1-\frac{v^2}{c^2}} - e\varphi + \frac{e}{c}\vec{A}\cdot\vec{v}$$

$$\frac{\partial \mathcal{L}}{\partial \vec{v}} = \underset{\substack{\uparrow \\ \text{generalized} \\ \text{momentum}}}{\vec{P}} = \underbrace{+m\cancel{c^2}\frac{+\cancel{2}\vec{v}/\cancel{c^2}}{\cancel{2}\sqrt{1-\frac{v^2}{c^2}}}}_{\substack{\vec{p} \\ \uparrow \\ \text{ordinary} \\ \text{momentum}}} + \frac{e}{c}\vec{A}$$

$$\mathcal{H} = \mathbf{P}\cdot v - \mathcal{L}$$

$$\mathcal{H} = \frac{+mv^2}{\sqrt{1-\frac{v^2}{c^2}}} + \cancel{\frac{e}{c}\vec{A}\cdot\vec{v}} + mc^2\sqrt{1-\frac{v^2}{c^2}} + e\varphi - \cancel{\frac{e}{c}\vec{A}\cdot\vec{v}}$$

$$\mathcal{H} = +m\left(\frac{+c^2\left(1-\frac{v^2}{c^2}\right)+v^2}{\sqrt{1-\frac{v^2}{c^2}}}\right) + e\varphi$$

$$\mathcal{H} = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} + e\varphi$$

Need to replace $\vec{v}$ with $\mathbf{P}$

$$\mathcal{H} - e\varphi = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \qquad \frac{mv}{\sqrt{1-v^2/c^2}} = \mathbf{P} - \frac{e}{c}\vec{A}$$

$$\left(\frac{\mathcal{H}-e\varphi}{c}\right)^2 = \frac{m^2c^2}{1-\frac{v^2}{c^2}}$$

$$\left(\mathbf{P} - \frac{e}{c}A\right)^2 = \frac{m^2v^2}{1-\frac{v^2}{c^2}}$$

$$m^2c^2 + \left(\mathbf{P} - \frac{e}{c}A\right)^2 = \frac{m^2c^2\left(\cancel{1-\frac{v^2}{c^2}}\right) + m^2v^2}{1-v^2/c^2}$$

$$= \frac{m^2c^2}{1-v^2/c^2}$$

We can now write

$$\frac{\mathcal{H}-e\varphi}{c} = \sqrt{m^2c^2 + \left(\mathbf{P}-\frac{e}{c}A\right)^2}$$

$$\mathcal{H} = e\varphi + c\sqrt{m^2c^2 + \left(\mathbf{P}-\frac{e}{c}A\right)^2}$$

We can further write Hamilton-Jacobi equations

$$\mathcal{H} = -\frac{\partial S}{\partial t} \qquad \mathbf{P} = \frac{\partial S}{\partial r}$$

$$\left(\frac{\partial S}{\partial t} + e\varphi\right)^2 = c^2\left(m^2c^2 + \left(\frac{\partial S}{\partial r} - \frac{e}{c}\right)^2\right)$$

$$S(\vec{r}, t)$$

Equations of motions of a charge in a field

- Charge experiences force from field, but also exerts a force on the field.

- How can we figure this out?

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial v} = \frac{\partial \mathcal{L}}{\partial r}$$

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{e}{c}\nabla(A\cdot v) - e\nabla\phi$$

$$\nabla(a\cdot b) = (a\cdot\nabla)b + (b\cdot\nabla)a + b\times(\nabla\times a) + a\times(\nabla\times b)$$

Note that ∇v, $\nabla\times v = 0$ as v is held constant

Hence

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{e}{c}(v\cdot\nabla)A + v\times(\nabla\times A) - e\nabla\phi$$

$$\frac{\partial \mathcal{L}}{\partial v} = \underset{\substack{\uparrow\\ \text{momentum}\\ \text{of system}}}{\vec{P}} = (\underset{\substack{\uparrow\\ \text{momentum of particle}}}{\vec{p}} + \frac{e}{c}\vec{A})$$

$$\frac{d}{dt}\left(\vec{p} + \frac{e}{c}\vec{A}\right) = \frac{e}{c}(\vec{v}\cdot\nabla)A + \frac{e}{c}(\vec{v}\times(\nabla\times\vec{A})) - e\nabla\phi$$

$$\frac{d}{dt}A = \frac{\partial A}{\partial t} + (\nabla\cdot A)\frac{\partial r}{\partial t} \Rightarrow \frac{\partial A}{\partial t} + (v\cdot\nabla)\vec{A}$$

Hence, we can say

$$\frac{dp}{dt} = \underbrace{-\frac{e}{c}\frac{\partial A}{\partial t} - e\nabla\phi}_{\substack{E\text{ field}\\ \text{intensity}\\ \text{Does not depend}\\ \text{on particle's}\\ \text{motion}}} + \frac{e}{c}(\vec{v}\times\underbrace{(\nabla\times\vec{A})}_{\vec{B}\text{ field}})$$

$$\nabla\times A = \vec{B} \text{ (or } \vec{H})$$

$$E = -\frac{\partial A}{\partial t} - \nabla\phi$$

E is a polar vector,

H is axial vector

$$\frac{dE_{kin}}{dt} = \frac{d}{dt}\left(\frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}}\right) = \vec{v}\cdot\frac{d\vec{p}}{dt}$$

$$= eE\cdot v$$

(noting that $r\cdot(r\times B) = 0$ in general)

Gauge Invariance

To what extent are fields and potentials uniquely determined?

Field is characterized by its impact to the particle

Equations of motion doesn't know about potentials but field intensities E and H

- Given potentials $\vec{A}$ and φ these determine E and H. However E and H can correspond to different A, φ

Define

$$A_k' = A_k - \partial_{x^k} f$$ where f is a function of coordinates and time.

To action integral

$$\frac{e}{c} \partial_{x^k} f \, \partial x^k = d\left(\frac{e}{c} f\right)$$

just an additive component to Lagrangian, has no impact on

Introduce scalar and vector potentials

$$\underbrace{A' = A + \nabla f, \quad \varphi' = \varphi - \partial_t f}_{\text{gauge invariance (without changing } E, H)}$$

Constant EM field

$$H = \nabla \times A$$

$$E = -\nabla \phi$$

if A, ϕ have no dependence on time

However, ϕ can have an arbitrary additive constant with no impact to E, H.

For ease, define ϕ such that

$$\lim_{r \to \infty} \phi = 0$$

Similar constraint not imposed on A.

$$\mathcal{H} = v \cdot \frac{\partial L}{\partial v} - L = \mathcal{E}$$ in constant setting.

Thus

$$\mathcal{E} = \frac{m}{\sqrt{1-v^2}} + c\phi$$

- magnetic field does not impact energy

- If field is constant everywhere,

$$-E \cdot r = \phi$$

Uniform mag field is

$$A = \frac{1}{2}(H \times r)$$

Motion in a Constant E Field

Motion of a charge e in $E(\vec{r},t) = \vec{E}(\vec{r})$

$\vec{E} \parallel \hat{x}$

$$\dot{p}_x = e\vec{E} \quad \dot{p}_y = 0 \quad \dot{p}_z = 0$$

$$p_x = e\vec{E}t, \quad p_y = k_y \quad p_z = k_z$$

Kinetic energy of particle

$$\mathcal{E}_{kin} = \sqrt{m^2 + p^2}$$

$$= \sqrt{\underbrace{m^2 + p_0^2}_{\mathcal{E}_0} + (eEt)^2}$$

$$= \sqrt{\mathcal{E}_0^2 + (eEt)^2}$$

$$v = p/\mathcal{E}_{kin} \Rightarrow \quad \frac{dx}{dt} = \frac{(eEt)}{\sqrt{\mathcal{E}_0^2 + (eEt)^2}}$$

$$x = \frac{1}{eE}\sqrt{\mathcal{E}_0^2 + (eEt)^2}$$

$$\frac{dy}{dt} = \frac{p_0 c^2}{\sqrt{\mathcal{E}_0^2 + (eEt)^2}} \Rightarrow \quad y = \frac{p_0}{eE}\sinh\left(\frac{eEt}{\mathcal{E}_0}\right)$$

$$x = \frac{\mathcal{E}_0}{eE}\cosh\left(\frac{eEy}{p_0}\right)$$

Motion in a constant uniform $\vec{H}$ field

$\vec{H} \parallel \hat{z}$

$\dot{\vec{p}} = e\vec{v} \times \vec{H}$

$$\dot{\vec{P}} = e\begin{vmatrix} \hat{x} & \hat{y} & \hat{k} \\ v_x & v_y & v_z \\ 0 & 0 & H_z \end{vmatrix}$$

$$\frac{dp_x}{dt} = eV_y H_z, \quad \frac{\partial p_y}{\partial t} = -eV_x H_z$$

Recall that

$\vec{V} = \vec{P}/\mathcal{E}_{kin}$

$$\frac{dp_x}{dt} = \frac{eP_y}{\mathcal{E}_{kin}} H \qquad \frac{dp_y}{dt} = -\frac{eP_x H_z}{\mathcal{E}_{kin}}$$

Recall that H only interacts with particle perpendicular to its direction of motion. Hence, no work can be done, and so $\mathcal{E}_{kin}$ of particle must be fixed.

Now with this in mind, we can rewrite a little

$$\frac{dv_x}{dt} = \frac{ev_y}{\mathcal{E}_{kin}} H \qquad \frac{dv_y}{dt} = -\frac{ev_x H}{\mathcal{E}_{kin}}$$

$$\frac{d}{dt}(v_x + iv_y) = -\underbrace{\frac{eH}{\mathcal{E}_{kin}}}_{\omega}(i)(v_x + iv_y)$$

$$V_x + iV_y = V_0 e^{-i(\omega t + \alpha)}$$

Now, solving for x, y

$$x = x_0 + r\sin(\omega t + \alpha) \qquad y = y_0 + r\cos(\omega t + \alpha)$$

$$r = \frac{V_{0t}}{\omega}$$

$Z = Z_0 + V_{0z} t$ ← $\vec{H}$ field doesn't impact Z because $\hat{z} \parallel \vec{H}$

Now suppose mag field is uniform but varies slowly in time

Can be solved using adiabatic invariants

$$I = \frac{1}{2\pi}\oint P_t \cdot dr \quad \text{} \} \text{ along the period of motion}$$

$$I = \frac{1}{2\pi}\oint \left(\vec{P}_t + \frac{e}{c}\vec{A}\right) \cdot dr$$

$$I = \frac{1}{2\pi}\oint \vec{p}_t \cdot d\vec{r} + \frac{e}{2\pi c}\oint \vec{A} \cdot d\vec{r}$$

recall $\vec{p} \parallel d\vec{r}$, and by stokes theorem

$$I = \frac{1}{2\pi}(2\pi r p_t) + \frac{e}{2\pi c}(-H\pi r^2)$$

$$= rp_t - \frac{eHr^2}{2} = \frac{P_t^2}{2eH}$$

Motion of charge in constant $\vec{E}$ and M field

Consider limit where $v \ll c$

$H \parallel \hat{z}$

and E is in the ~~y~~ Z plane

Since $v \ll c$

$\vec{p} = m\vec{v}$

$m\dot{v} = e\vec{E} + e(\vec{v}\times\vec{H})$

$m\ddot{x} = e v_y H, \quad m\ddot{y} = eE_y - e v_z H, \quad m\ddot{z} = eE_z$

$z(t) = z_0 + \frac{eE_z}{2} t^2 + V_{0z} t$

$m\ddot{x} + im\ddot{y} = e v_y H + ieE_y - iev_z H$

$m(\ddot{x + iy}) = -ieH(\dot{x + iy}) + ieE_y$

$m\ddot{\xi} + ieH\dot{\xi} = ieE_y$

$m\dot{\xi} + ieH\xi = ieE_y t$

$m I \dot{\xi} + ieHI\xi = ieE_y t(I)$

$I\dot{\xi} + \frac{ieHI}{m}\xi = \frac{ieE_y t\, I}{m}$ — I is an integrating factor

$\int I\dot{\xi} + \frac{ieHI}{m}\xi \, dt = \int \frac{ieE_y t\, I}{m} dt$

$I = \exp\left(\frac{ieH}{m}t\right)$

We can solve this by integrating RHS by parts

$\dot{x} = a\cos(\omega t) + \frac{E_y}{H} \qquad \dot{y} = -a\sin(\omega t)$

$\bar{\dot{x}} = \frac{E_y}{H} \qquad \bar{\dot{y}} = 0$

$\uparrow$

electrical drift velocity

$\bar{v} = \frac{\vec{E}\times\vec{H}}{H^2}$

The Electromagnetic Field Tensor

$$\delta S = \delta \int_a^b -m\,ds - e A_i dx^i = 0$$

recall that 4 vector A_i is

$(\varphi, \vec{A})$ and $ds = \sqrt{dx_i dx^i}$

$$\delta S = \int -m \cdot \underbrace{\frac{\overset{u_i}{dx_i}\, d\delta x^i}{ds}} - e A_i\, d\delta x^i - e\,\delta A_i\, dx^i = 0$$

$$\int m u_i\, d\delta x^i = \cancel{m u_i \delta x^i}^{\,0} - \int m\,\delta x^i du_i$$

$$\delta S = \int m\,\delta x^i du_i + e\,\delta x^i dA_i - e\,\delta A_i\, dx^i = 0$$

$$\delta A_i = \frac{\partial A_i}{\partial x^k}\delta x^k \qquad dA_i = \frac{\partial A_i}{\partial x^k} dx^k$$

$$\delta S = \int m\,\delta x^i du_i + e\,\frac{\partial A_i}{\partial x^k} dx^k \delta x^i - \frac{\partial A_i}{\partial x^k}\delta x^k dx^i = 0$$

$$du_i = \frac{du_i}{ds} ds \qquad dx^i = u^i ds$$

$$\int \left(m\frac{du_i}{ds} + e\left(\frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i}\right)u^k\right)\delta x^i ds = 0$$

$$m\frac{du_i}{ds} = e\underbrace{\left(\frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}\right)}u^k$$

$$m\frac{du_i}{ds} = e\,F_{ik}\,u^k$$

Electromagnetic Field Tensor

Consider F_{ik}

$$\begin{pmatrix} \frac{\partial\varphi}{\partial t} & \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z} \\ -\frac{\partial\varphi}{\partial x} & -\frac{\partial A}{\partial t} & & \\ -\frac{\partial\varphi}{\partial y} & & \cdot & \\ -\frac{\partial\varphi}{\partial z} & & & \cdot \end{pmatrix} = (E, H)$$

$$\delta S = \left(-mcu_i + \frac{e}{c}A_i\right)\delta x^i$$

$$-\frac{\partial S}{\partial x^i} = mcu_i + eA_i = P_i + eA_i$$

$$\left(\mathcal{E}_{kin} + e\varphi,\ p + e\vec{A}\right)$$

Lorentz Transformation

$$\varphi = \frac{\varphi' + V A_x'}{\sqrt{1-V^2}} \qquad A_x = \frac{A_x' + V\varphi'}{\sqrt{1-V^2}}, \ \ldots$$

$$E_x = E_x' \qquad E_y = \frac{E_y' + V H_z'}{\sqrt{1-V^2}}. \qquad E_z = \frac{E_z' - V H_y'}{\sqrt{1-V^2}}$$

$$H_y = \frac{H_y' - V E_z'}{\sqrt{1-V^2}}, \qquad H_z = \frac{H_z' - V E_y'}{\sqrt{1-V^2}}$$

In $V << C$ limit

$$E = E' + H' \times V, \qquad H = H' - E' \times V$$

Invariants of the field

$F_{ik} F^{ik}$ = invariant

$e^{iklm} F_{ik} F_{lm}$ = invariant

$H^2 - E^2$ = invariant

$E \cdot H$ = invariant

The first pair of Maxwell's Eqns

Given $H = \nabla\times A$, $E = -\frac{\partial A}{\partial t} - \vec\nabla\varphi$

$$\nabla\times E = -\nabla\times\frac{\partial A}{\partial t} - \nabla\times\vec\nabla\varphi$$

$$= -\frac{\partial}{\partial t}(\nabla\times A) - \nabla\times\vec\nabla\varphi$$

$$= -\frac{\partial}{\partial t}H - \nabla\times\vec\nabla\varphi \leftarrow \text{isn't this zero...)}$$

yes...

$$\boxed{\nabla\times E = -\frac{\partial H}{\partial t}}$$

$$\nabla\cdot H = \nabla\cdot(\nabla\times A)$$

$$\boxed{\nabla\cdot H = 0}$$

First 2 maxwell's equations

$\nabla\cdot H = 0$

$$\Rightarrow \quad \int \nabla H\, dV = \oint \vec H\cdot d\vec f \quad \text{by Gauss's theorem}$$

$\oint$ ↑ integral over closed surface.

$\oint \vec H\cdot d\vec f = 0$ flux of mag field is 0

$$\int(\vec\nabla\times\vec E)\cdot d\vec f = \oint \vec E\cdot d\vec l$$

$$\oint \vec E\cdot d\vec l = \frac{\partial}{\partial t}\int H\cdot df$$

$$F_{ik} = \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k}$$

$$\frac{\partial F_{ik}}{\partial x^l} + \frac{\partial F_{kl}}{\partial x^i} + \frac{\partial F_{li}}{\partial x^k} = 0$$

$$\underbrace{e^{iklm}\frac{\partial F_{lm}}{\partial x^k}}_{\text{this is a contraction}} = 0$$

The Action Function of the EM field

$$S = S_f + S_m + S_{mf}$$ ← interaction between particles and field

action (S); action of field alone (S_f); action depending on property of particles (S_m)

$S_m = -\sum m \int ds$ (sum over particles)

$S_{mf} = -\sum e \int A_k dx^k$

e: charge of particles; A_k: potential at each coordinate

S_f is what?

Start from experimental observation

Principle of superposition

- fields of collective particles is described by sum of individual fields
- every solution to field equations can be summed to yield another solution
- Hence, field equations must be able to be mapped to a linear diff eq because linear diff eq must satisfy the same solutions.

- Thus under integral sign for action we must have a quadratic, in order to yield linear ODEs
- Potentials can't be a part of S_f because they are not uniquely defined
- The only remaining thing is F_{ik} so a quadratic in F_{ik} is

$$F_{ik} F^{ik}$$

$$S_f = \alpha \int F_{ik} F^{ik} \, dV dt$$

$$= \alpha \int (H^2 - E^2) \, dV dt$$

E has a term $\frac{\partial A}{\partial t}$

However $\left(\frac{\partial A}{\partial t}\right)^2$ must appear with positive sign. If $\left(\frac{\partial A}{\partial t}\right)^2$ appeared with negative sign, S_f can be made arbitrarily negative. As is written now, we have

$$\alpha \left(-\left(\frac{\partial A}{\partial t}\right)^2\right).$$

Hence, $\alpha < 0$ to ensure S_f must be positive

In gaussian $\alpha = -\frac{1}{16\pi}$

$$S_f = -\frac{1}{16\pi c} \int F_{ik} F^{ik} c \, dV dt$$

$$= \frac{1}{8\pi} \int (E^2 - H^2) \, dV dt$$

$$S = -\sum \int mc \, ds - \sum \int \frac{e}{c} A_k dx^k - \frac{1}{16\pi c} \int F_{ik} F^{ik} \underbrace{c \, dV dt}_{d\Omega}$$

The four-dimensional current vector

Instead of treating charges as points consider a continuous distribution of charges characterized by density ρ such that

ρdV = total charge in volume element

Note that at the most microscopic scale, charge is an attribute of points so

$$\underbrace{\int \rho\, dV = \sum e_a \delta(r - r_a)}_{\text{does not depend on coordinate system}}$$

$$de\, dx^i = \rho\, dV\, dx^i = \rho\, dV \frac{dx^i}{dt} dt$$

$\rho \frac{dx^i}{dt}$ is a four vector called the 4-current vector

current density vector given by space components

$$\vec{j} = \rho \frac{d\vec{x}}{dt}$$

time component is the spatial density

$$j^i = \rho \frac{dx^i}{dt} = (\rho, \vec{j})$$

$$\int \rho\, dV = \int j^0 dV = \int j^i dS_i$$

Replace charge with integral over density

$$-\int \rho A_i\, dx^i\, dV = -\int \rho \frac{dx^i}{dt} A_i\, dt\, dV$$

$$= -\int A_i j^i d\Omega$$

So action becomes

$$S = -\sum \int m\, ds - \int A_i j^i d\Omega - \frac{1}{16\pi} \int F_{ik} F^{ik} d\Omega$$

The Equation of Continuity

$$\underbrace{\frac{\partial}{\partial t}\int \rho \, dV}_{\text{rate of change at which charge changes within a volume (eg, flow)}}$$

Consider the surface of dV, $d\vec{f}$. Suppose charge density has velocity $\vec{v}$ through said surface?

Then $\rho\vec{v}\cdot d\vec{f}$ is the movement of charge along the surface.

If we assume charge is conserved, eg there are no source charges overall then

$$\underbrace{\frac{\partial}{\partial t}\int \rho \, dV}_{\text{rate of change of charge in volume}} = -\underbrace{\oint \rho\vec{v}\cdot d\vec{f}}_{\text{surface integral of flow}}$$

$$\oint \rho\vec{v}\cdot d\vec{f} = \int \nabla(\underbrace{\rho\vec{v}}_{\text{current density}})\, dV$$

Hence

$$\int \frac{\partial\rho}{\partial t} + (\nabla\cdot j) \quad dV = 0$$

In differential form, the above must hold for arbitrary sized volumes. Hence,

$$\frac{\partial\rho}{\partial t} + (\nabla\cdot j) = 0 \quad \text{everywhere.}$$

suppose

$$\rho = e\,\delta(r-r_0)$$

$$\vec{j} = e\vec{v}\,\delta(r-r_0)$$

$$\frac{\partial\rho}{\partial t} = \frac{\partial\rho}{\partial r_0}\frac{\partial r_0}{\partial t}$$

$$\frac{\partial\rho}{\partial r_0} = -\frac{\partial\rho}{\partial r}$$

$$\frac{\partial\rho}{\partial t} = -V\,\vec{\nabla}\rho = -\mathrm{div}(\rho\vec{v})$$

gives j

In four vector form

$$\partial_{x^i} j^i = 0$$

Can then write

$\int j^i \, dS_i$ where the integral is along hyperplanes perpendicular to x^0, ~~eg~~ the time axis

Note that charge is conserved, by continuity equations, so for all x^0, $\int j^i \, dS_i$ is const. We can then take the difference between $\int j^i \, dS_i$ at two units of time, writing it as

$$\underbrace{\oint j^i \, dS_i}_{\text{apply Gauss's theorem}} = 0$$

$$\int \nabla_i j^i \, \partial\Omega = 0$$

Recall that gauge invariance states that

E, H are defined only up to some $A + \nabla_i f^i$ some gradient, basically

In the action, this amounts to

$$\int A_i j^i \, \partial\Omega + \underbrace{\int j^i \, \nabla_i f^i \, \partial\Omega}$$

$$\int j^i \nabla_i f^i \, \partial\Omega = \int j^i \nabla_i f^i \, \partial\Omega + \int f \underbrace{\nabla_i j^i \, \partial\Omega}_{0}$$

$$= \int \nabla_i (f j^i)\, \partial\Omega$$

$$= \underbrace{\oint f j^i \, \partial S_i}_{\text{0 impact to action because surface and states along are fixed}}$$

The second pair of Maxwell Equations

- assume motion of charges is given, and vary only the potential to find field equations
- to find equations of motion assume field is given and vary trajectory of particles

$$\delta S = -\int \frac{1}{c}\left(\frac{1}{c} j^i \delta A_i + \frac{1}{8\pi} F^{ik}\delta F_{ik}\right) d\Omega = 0$$

substituting

$$F^{ik} = \frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i}$$

$$\delta S = -\int \frac{1}{c}\left(\frac{1}{c} j^i \delta A_i + \frac{1}{8\pi} F^{ik}\partial_{x^i}\delta A_k - \frac{1}{8\pi}F^{ik}\partial_{x^k}\delta A_i\right) d\Omega$$

$$\delta S = -\frac{1}{c}\int \left(\frac{1}{c}\left(j^i \delta A_i + \underbrace{\frac{1}{4\pi} F^{ik}\partial_{x^k}\delta A_i}_{\text{Integrate by parts}}\right)\right) d\Omega$$

$$\int F^{ik}\partial_{x^k}\delta A_i \, d\Omega = \underbrace{\int F^{ik}\delta A_i \, d\Omega}_{\substack{\text{Field out at } \infty \text{ must be } 0.\ \text{For time} \\ \text{integral, beginning and ending of} \\ \delta A = 0}} - \int \delta A_i \frac{\partial F^{ik}}{\partial x^k} d\Omega$$

$$\delta S = -\frac{1}{c}\int \frac{1}{c}\underbrace{\left(j^i \delta A_i + \frac{1}{4\pi}\delta A_i \partial_{x^k} F^{ik}\right)}_{\text{must be zero}} d\Omega$$

$$\frac{\partial F^{ik}}{\partial x^k} = -\frac{4\pi}{c} j^i$$

For $(j^0, \underbrace{j^1, j^2, j^3}_{\vec{j}})$

$$\nabla \times H = \frac{1}{c}\frac{\partial E}{\partial t} + \frac{4\pi}{c}\vec{j}$$

$$\nabla \cdot E = 4\pi\rho$$

$$\int \nabla \cdot E \, dV = \oint E \cdot dA \leftarrow \Phi \text{ the flux}$$

$$\oint E \cdot dA = 4\pi \int \rho \, dV$$

$$\nabla \times H \cdot dA = \oint H \cdot dl$$

$$\oint H \cdot dl = \underbrace{\frac{1}{c}\frac{\partial}{\partial t}\int E \cdot dA}_{\text{displacement current}} + \frac{4\pi}{c}\int j \cdot dA$$

$$\nabla \cdot (\nabla \times H) = \nabla \cdot \left(\frac{\partial E}{\partial t} + 4\pi j\right)$$

$$0 = \frac{\partial}{\partial t}(\nabla \cdot E) + 4\pi(\nabla \cdot j)$$

$$= \frac{\partial}{\partial t}(4\pi\rho) + 4\pi(\nabla \cdot j)$$

Thus in 4-d form (given the other equation)

$$\frac{\partial^2 F_{ik}}{\partial x^i \partial x^k} = -4\pi \, \nabla_i \cdot j^i$$

Energy Density and energy flux

$$\underbrace{E\cdot\frac{\partial E}{\partial t}+H\frac{\partial H}{\partial t}} = -4\pi\, j\cdot E - \underbrace{(H\cdot(\nabla\times E) - E\cdot(\nabla\times H))}_{\nabla\cdot(E\times H)}$$

$$\frac{1}{2}\frac{\partial}{\partial t}(E^2+H^2) = -4\pi\, j\cdot E - \nabla\cdot(E\times H)$$

$$S = \frac{1}{4\pi} E\times H \leftarrow \text{Poynting vector}$$

$$\frac{\partial}{\partial t}\int \frac{E^2+H^2}{8\pi}\,dV = -\int j\cdot E\,dV - \int \nabla\vec{S}\,dV$$

$$\frac{\partial}{\partial t}\int \frac{E^2+H^2}{8\pi}\,dV = -\int j\cdot E\,dV - \underbrace{\oint \vec{S}\cdot d\vec{A}}_{\text{Poynting vector is } 0 \text{ at } \infty}$$

$$\int j\cdot E\,dV = \sum \underbrace{e\vec{v}\cdot\vec{E}}_{\frac{\partial}{\partial t}\mathcal{E}_{kin}}$$

$$\frac{d}{dt}\underbrace{\left(\int \frac{E^2+H^2}{8\pi}\,dV + \sum \mathcal{E}_{kin}\right)}_{\text{conserved quantity}} = 0$$

energy of field and constituent particles

$\frac{E^2+H^2}{8\pi}$ is energy density of field

over integrals not at infinity

$$\frac{\partial}{\partial t}\left(\int \frac{E^2+H^2}{8\pi}\,dV + \underbrace{\sum \mathcal{E}_{kin}}_{\text{only particles within volume}}\right) = -\underbrace{\oint S\cdot df}_{\text{flux }\cancel{\text{density}}\text{ of field across surface}}$$

Energy-Momentum Tensor

- derive expression for energy of EM field, now need momentum of EM field

Consider action

$$S = \int \Lambda\left(q, \frac{\partial q}{\partial x^i}\right) dV\,dt = \int \Lambda\, d\Omega$$

Here q is the 4-potential of the EM field.

$$\int \Lambda\, dV \text{ is } L$$

so Λ is Lagrangian density

system is closed, as Λ has no explicit dependence on time or coordinates

$$\delta S = \int \left(\partial_q \Lambda\, \delta q + \partial_{\partial_{x^i} q} \Lambda\, \delta \partial_{x^i} q\right) d\Omega$$

$$= \int \Big(\partial_q \Lambda\, \delta q + \underbrace{\partial_{x^i}\left(\partial_{\partial_{x^i} q} \Lambda\, \delta q\right)}_{\substack{\text{Vanishes upon} \\ \text{integration over} \\ \text{all space} \\ \text{(Lagrangian density } 0 \text{ at } \infty)}} - \delta q\, \partial_{x^i} \partial_{\partial_{x^i} q} \Lambda\Big) d\Omega$$

$$= 0$$

Hence

$$\underbrace{\frac{\partial}{\partial x^i} \frac{\partial \Lambda}{\partial\left(\frac{\partial q}{\partial x^i}\right)}}_{\text{sum over repeated index, } i} - \frac{\partial \Lambda}{\partial q} = 0$$

Now, consider

$$\frac{\partial \Lambda}{\partial x^i} = \frac{\partial \Lambda}{\partial q} \frac{\partial q}{\partial x^i} + \frac{\partial \Lambda}{\partial\left(\frac{\partial q}{\partial x^k}\right)} \frac{\partial\left(\frac{\partial q}{\partial x^k}\right)}{\partial x^i}$$

$$\frac{\partial \Lambda}{\partial x^i} = \frac{\partial}{\partial x^k} \frac{\partial \Lambda}{\partial\left(\frac{\partial q}{\partial x^k}\right)} \frac{\partial q}{\partial x^i} + \frac{\partial \Lambda}{\partial\left(\frac{\partial q}{\partial x^k}\right)} \frac{\partial\left(\frac{\partial q}{\partial x^i}\right)}{\partial x^k}$$

$$= \frac{\partial}{\partial x^k}\left(\frac{\partial q}{\partial x^i} \frac{\partial \Lambda}{\partial\left(\frac{\partial q}{\partial x^k}\right)}\right)$$

$$\frac{\partial \Lambda}{\partial x^i} = \delta_i^k \frac{\partial \Lambda}{\partial x^k}$$

$$T_i^k = \frac{\partial q}{\partial x^i} \frac{\partial \Lambda}{\partial\left(\frac{\partial q}{\partial x^k}\right)} - \delta_i^k \Lambda$$

$$\frac{\partial T_i^k}{\partial x^k} = 0$$

$$T_i^k = \sum_l \frac{\partial q^{(l)}}{\partial x^i} \frac{\partial \Lambda}{\partial\left(\frac{\partial q^{(l)}}{\partial x^k}\right)} - \delta_i^k \Lambda$$

$$\nabla_k A^k = 0$$

$$P^i = \text{const} \int T^{ik}\, dS_k$$

$$P^0 = \text{const} \int T^{0k}\, dS_k = \text{const} \int T^{00}\, dV$$

$$T^{00} = \dot{q} \frac{\partial \Lambda}{\partial \dot{q}} - \Lambda \leftarrow \text{this is the Hamiltonian density}$$

$$P^i = \int T^{ik}\, dS_k$$

$$T^{ik} + \frac{\partial}{\partial x^l} \psi^{ikl} \text{ is equivalent}$$

for any $\psi^{ikl} = -\psi^{ilk}$

To define T^{ik} uniquely

$$M^{ik} = \int x^i dP^k - x^k dP^i = \int \underbrace{\left(x^i T^{kl} - x^k T^{il}\right)}_{\substack{\text{div } \nabla \text{ of this} \\ \text{equal to } 0 \text{ for} \\ \text{this to hold}}} dS_l$$

$$\nabla_l\left(x^i T^{kl} - x^k T^{il}\right) = 0$$

$$\delta_l^i T^{kl} - \delta_l^k T^{il} = 0$$

$$T^{ki} - T^{ik} = 0$$

Hence, T^{ik} must be symmetric

$$P^i = \int T^{i0}\, dV$$

T^{10}, T^{20}, T^{30} is momentum density

and T^{00} is energy density

$$\frac{\partial T^{00}}{\partial t} + \frac{\partial T^{0\alpha}}{\partial x^\alpha} = 0$$

$$\frac{\partial T^{\alpha 0}}{\partial t} + \frac{\partial T^{\alpha\beta}}{\partial x^\beta} = 0$$

$$\frac{\partial}{\partial t} \int T^{00}\, dV = -\oint T^{0\alpha}\, dA_\alpha$$

Energy-momentum tensor of EM field

$$\Lambda = -\frac{1}{16\pi} F_{kl} F^{kl}$$

$$T_i^k = \frac{\partial A_l}{\partial x^i} \frac{\partial \Lambda}{\partial\left(\frac{\partial A_l}{\partial x^k}\right)} - \delta_i^k \Lambda$$

$$\delta\Lambda = -\frac{1}{8\pi} F_{kl}\, \delta F^{kl}$$

$$= -\frac{1}{8\pi} F^{kl}\left(\delta \frac{\partial A_l}{\partial x^k} - \delta \frac{\partial A_k}{\partial x^l}\right)$$

$$= -\frac{1}{4\pi} F^{kl}\, \delta \frac{\partial A_l}{\partial x^k}$$

$$\frac{\partial \Lambda}{\partial\left(\frac{\partial A_l}{\partial x^k}\right)} = -\frac{1}{4\pi} F^{kl}$$

$$T_i^k = -\frac{1}{4\pi} \frac{\partial A_l}{\partial x^i} F^{kl} + \frac{1}{16\pi} \delta_i^k F_{lm} F^{lm}$$

$$T^{ik} = -\frac{1}{4\pi} \frac{\partial A^l}{\partial x_i} F_l^{\ k} + \frac{1}{16\pi} g^{ik} F_{lm} F^{lm}$$

To symmetrize, add

$$\frac{1}{4\pi} \frac{\partial A^i}{\partial x_l} F_l^{\ k}$$

$$\frac{1}{4\pi} \frac{\partial A^i}{\partial x_l} F_l^{\ k} = \frac{1}{4\pi} \frac{\partial}{\partial x^l}\left(A^i F^{kl}\right)$$

$$\frac{\partial A^l}{\partial x_i} - \frac{\partial A^i}{\partial x_l} = F^{il}$$

$$T^{ik} = \frac{1}{4\pi}\left(-F^{il} F_l^{\ k} + \frac{1}{4} g^{ik} F_{lm} F^{lm}\right)$$

$$T_i^i = 0$$

$T^{\alpha\beta}$ for $\alpha, \beta = 1, 2, 3$

$$\sigma_{\alpha\beta} = \underbrace{\frac{1}{4\pi}\left(E_\alpha E_\beta + H_\alpha H_\beta - \frac{1}{2}\delta_{\alpha\beta}(E^2 + H^2)\right)}_{\text{maxwell stress tensor}}$$

Now we need energy-momentum tensor of particles

mass density

$$\mu = \sum m_a\, \delta(r - r_a)$$

4 momentum-density

$$\mu u_i$$

$$T^{0\alpha} = \mu u^\alpha$$

$$T^{ik} = \mu \frac{dx^i}{ds} \frac{dx^k}{dt} = \mu u^i u^k \frac{ds}{dt}$$

$$\frac{\partial}{\partial x_k}\left(T_i^{(f)k} + T_i^{(p)k}\right) = 0$$

$$\frac{\partial T_i^{(f)k}}{\partial x^k} = \frac{1}{4\pi}\left(\frac{1}{2} F^{lm} \frac{\partial F_{lm}}{\partial x^i} - \frac{\partial F_{il}}{\partial x^k} F^{kl} - \frac{\partial F^{kl}}{\partial x^k} F_{il}\right)$$

$$\frac{\partial F^{kl}}{\partial x^k} = 4\pi j^l \qquad \frac{\partial F_{lm}}{\partial x^i} = -\frac{\partial F_{mi}}{\partial x^l} - \frac{\partial F_{il}}{\partial x^m}$$

$$\frac{\partial T_i^{(f)k}}{\partial x^k} = \frac{1}{4\pi}\left(-\frac{1}{2} \frac{\partial F_{mi}}{\partial x^l} F^{lm} - \frac{1}{2} \frac{\partial F_{il}}{\partial x^m} F^{lm} - \frac{\partial F_{il}}{\partial x^k} F^{kl} - \frac{4\pi}{c} F_{il}\, j^l\right)$$

permuting indices cancels first three terms

$$\frac{\partial T_i^{(f)k}}{\partial x_k} = -F_{ik}\, j^k$$

$$\frac{\partial T_i^{(p)k}}{\partial x^k} = \underbrace{c u_i \frac{\partial}{\partial x^k}\left(\overset{\text{mass current}}{\mu \frac{dx^k}{dt}}\right)}_{\text{by conservation of mass this is } 0} + \mu c \frac{dx^k}{dt} \frac{\partial u_i}{\partial x^k}$$

Virial Theorem

Sum of diagonal terms of EM tensor is 0
so sum T_i^i is of particle energy-momentum alone

$$T_i^i = T^{(p)\,i}_i = \mu u^i \frac{ds}{dt} = \mu \frac{ds}{dt} = \mu\sqrt{1-v^2}$$

$$T_i^i = \sum_a m_a \sqrt{1-v_a^2}\,\delta(r-r_a)$$

$$T_i^i = 0$$

$$\frac{\partial T^{\alpha 0}}{\partial t} + \frac{\partial T^{\alpha\beta}}{\partial x^\beta} = 0$$ } average this with respect to time

$\frac{\partial T^{\alpha 0}}{\partial t}$ averaged is 0 as this is a derivative of a bounded quantity

Hence

$$\frac{\partial}{\partial x^\beta}\overline{T_\alpha^\beta} = 0$$

$$\int x^\alpha \overline{\frac{\partial T_\alpha^\beta}{\partial x^\beta}}\, dV = -\int \frac{\partial x^\alpha}{\partial x^\beta}\overline{T_\alpha^\beta}\, dV$$

by Integration by parts

$$= \int \overline{T_\alpha^\alpha}\, dV = 0$$

$$\int \overline{T_i^i}\, dV = \int \overline{T_0^0}\, dV = \mathcal{E}$$

$$\mathcal{E} = \overline{\sum m_a \sqrt{1-v_a^2}}$$ ← virial theorem effectively

renormalize energy by removing contribution of E and H

$$\mathcal{E} \to \mathcal{E} - \sum_a \int \frac{E_a^2 + H_a^2}{8\pi}\, dV$$

$$\int T_\alpha^\alpha\, dV \to \int T_\alpha^\alpha\, dV + \sum \frac{E_a^2+H_a^2}{8\pi}\, dV$$

Energy-Momentum Tensor For Macroscopic Bodies

Flux of momentum through the element df of the surface of the body is just force acting on this surface element

$-\sigma_{\alpha\beta} df_\beta$ is α component of force acting on element

- Pascal's Law: pressure $\vec{p}$ applied to a given body is transmitted equally in all directions
- Find reference system where given element is at rest

$$\sigma_{\alpha\beta} df_\beta \text{ is therefore } -p\,df_\beta$$

$$\sigma_{\alpha\beta} = -p\,\delta_{\alpha\beta}$$

$T^{\alpha 0} \leftarrow$ momentum density is 0 as ~~vo~~lume element at rest

T^{00} is mass density of the body

Hence, in ref system

$$T^{ik} = \begin{bmatrix} \varepsilon & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{bmatrix}$$

for arbitrary system, just boost the above

$$T^{ik} = (p+\varepsilon) u^i u^k - p g^{ik}$$

$$T_i^{\ k} = (p+\varepsilon) u_i u^k - p\,\delta_i^k$$

$$W = \frac{\varepsilon + pV^2}{1-V^2} \qquad S = \frac{(p+\varepsilon)\vec{V}}{1-V^2}$$

$$\sigma_{\alpha\beta} = \frac{(p+\varepsilon) V_\alpha V_\beta}{1-V^2} - p\,\delta_{\alpha\beta}$$

Constant Electromagnetic Fields

- Coulomb's Law
  - For constant electric field (electrostatic)

Maxwell eqns

$$\nabla \times E = 0$$

$$\nabla \cdot E = 4\pi\rho$$

$$E = -\vec{\nabla}\varphi$$

$$\underbrace{\nabla \cdot (\vec{\nabla}\varphi) = -4\pi\rho}_{\text{poisson equation}}$$

If $\rho = 0$, Laplace equation

$$\underbrace{\nabla \cdot (\vec{\nabla}\varphi) = 0}$$

φ can not have a min or max anywhere because then, if there is a max or min, $\frac{\partial^2 \varphi}{\partial x}$, $\frac{\partial^2 \varphi}{\partial y}$, $\frac{\partial^2 \varphi}{\partial z}$ would all have the same sign, but then Laplace eqn cannot be satisfied.

consider a point charge

by isotropy of space, field must be directed along $\hat{r}$. Similarly, strength can only depend on $|r|$. distance from charge.

$$\vec{E} = \frac{e}{R^2}\hat{r}$$ where R distance from charge

$$= \frac{e\vec{R}}{R^3}$$

$$-\nabla\varphi = \vec{E}$$

so in this case

$$\varphi = \frac{e}{R}$$

by principle of superposition, for a collection of point charges, just add up potentials, add up fields

If a continuous density

$$\varphi = \int \frac{\rho}{R}\, dV$$

Green's function?

$$\nabla \cdot \vec{\nabla}\left(\frac{1}{R}\right) = -4\pi\,\delta(R)$$

Electrostatic energy of charges

energy of system of charges

$$U = \frac{1}{8\pi}\int E^2\, dV$$

$$E = -\vec{\nabla}\varphi$$

$$U = -\frac{1}{8\pi}\int E\cdot(\vec{\nabla}\varphi)\, dV = \underbrace{-\frac{1}{8\pi}\int \nabla\cdot(\vec{E}\varphi)\, dV}_{\text{Integrate by Gauss's theorem but goes to } 0 \text{ as } E\to 0 \text{ at } \infty} + \frac{1}{8\pi}\int \varphi(\nabla\cdot E)\, dV$$

$$U = +\frac{1}{8\pi}\int \varphi\,(\nabla\cdot E)\, dV$$

$$\nabla E = 4\pi\rho$$

$$U = \frac{1}{2}\int \varphi\rho\, dV$$

$$\rho = \sum e_a\,\delta(r - r_a)$$

so

$U = \frac{1}{2}\sum e_a\varphi_a$ where φ_a in the potential of the field

In the case of a single particle

$$U = \frac{e\varphi}{2}$$

However, $\varphi = \frac{e}{R}$

and at $R = 0$ $U \to \infty$

This is not possible, so some contradiction exists

- in relativity, every elementary particle must be point like
- basic electrodynamics has lots of observations
- it cannot be asked if the mass of an electron only contributes to its electric potential energy
- Classical electrodynamics only works down to finite length scales

EM self energy of electron should be similar to rest energy

$$\frac{e^2}{R_0} \sim mc^2$$

$$R_0 \sim \frac{mc^2}{e^2}$$

$$\varphi_a = \left.\sum \frac{e_b}{r_{ab}}\right\}\ \text{contribution from non self particles}$$

The field of a uniformly moving charge

Suppose K' is the frame at which a charge is at rest.

In K, the lab frame

$$\varphi = \frac{\varphi'}{\sqrt{1-v^2}} \quad \text{where } V \parallel x$$

so

$$\varphi = \frac{e}{R'\sqrt{1-v^2}}$$

$$\vec{R}' = \vec{x}' + \vec{y}' + \vec{z}'$$

$$x' = x - Vt \big/ \sqrt{1-v^2} \quad y' = y \quad z' = z$$

$$R' = \frac{(x-Vt)^2 + (1-v^2)(y^2+z^2)}{\sqrt{1-v^2}}$$

$$\varphi = \frac{e}{\sqrt{(x-Vt)^2 + (1-v^2)(y^2+z^2)}}, \quad E = (1-v^2)\frac{e\vec{R}}{(R^*)^3}$$

Vector potential $A' = 0$

$$A = A' + \varphi\frac{\vec{V}}{c} = \varphi\frac{\vec{V}}{c}$$

We can then write

$$H = V \times E = e\frac{V \times R}{R^3}$$

Motion in the Coulomb Field

Consider particle of mass m and charge e

$$\mathcal{E} = \sqrt{p^2 + m^2} + \frac{\alpha}{r}$$

$\alpha = ee'$ e' is charge of a particle generating a field

polar coordinates in plane of motion of the particle

constant angular momentum

$$p^2 = \left(\frac{M^2}{r^2}\right) + p_r^2$$

$$\mathcal{E} = \sqrt{p_r^2 + \left(\frac{M^2}{r^2}\right) + m^2} + \frac{\alpha}{r}$$

Can $r \to 0$?

Well if e and e' have different signs is sometimes possible.

If $|Mc| > \alpha$, first term $\to \infty$ as $r \to 0$ so not possible

If $|M_c| < \alpha$,

$r \to 0$ everything can remain finite. as contributions to ∞ $\left(p_r, \frac{M^2}{r^2}, \frac{\alpha}{r}\right)$ can cancel

Start from Hamilton-Jacobi equations in polar coordinates

$$\left(\frac{\partial S}{\partial t} + \frac{\alpha}{r}\right)^2 + \left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \varphi}\right)^2 + m^2 = 0$$

By separation of variables

$$S = -\mathcal{E}t + M\varphi + f(r) = 0$$

energy and angular momentum

these are constant.

$$S = -\mathcal{E}t + M\varphi + \int \sqrt{\left(\mathcal{E} - \frac{\alpha}{r}\right)^2 - \frac{M^2}{r^2} - m^2}\, dr$$

$\frac{\partial S}{\partial M} = \text{const}$ as this is a cyclic coordinate

If $M > |\alpha|$

$$\left(M^2 - \alpha^2\right)\frac{1}{r} = \sqrt{(M\mathcal{E})^2 - m^2(M^2 - \alpha^2)}\cos\left(\varphi\left(\sqrt{1 - \frac{\alpha^2}{M^2}}\right)\right) - \mathcal{E}\alpha$$

If $M < |\alpha|$

$$\left(\alpha^2 - M^2\right)\frac{1}{r} = \pm\sqrt{(M\mathcal{E})^2 + m^2(\alpha^2 - M^2)}\cosh\left(\varphi\sqrt{\frac{\alpha^2}{M^2} - 1}\right) + \mathcal{E}\alpha$$

If $M = \alpha$

$$\frac{2\mathcal{E}\alpha}{r} = \mathcal{E}^2 - m^2 - \varphi^2\left(\frac{\mathcal{E}\alpha}{M}\right)^2$$

Integration constant determined by reference line that measures φ

The Dipole Moment

$$\varphi = \sum \frac{e_a}{|\vec{R}_0 - \vec{r}_a|}$$

$R_0 >> r$

$$f(R_0 - r) = f(R_0) - \vec{r} \cdot \nabla f(R_0)$$

$$\varphi = \frac{\sum e_c}{R_0} - \sum e_c \vec{r}_c \cdot \nabla \frac{1}{R_0}$$

Define $\vec{d} = \sum e_a \vec{r}_c$

dipole moment

If $\sum e_c = 0$, then everything is independent of choice of origin

$$d = \sum e_c \vec{r}_c = \sum e_c^+ r_c^+ - \sum e_c^- r_c^-$$

$$= R_0^+ \sum e_c^+ - R_c^- \sum e_c^-$$

center of mass type thing

Can define charge centers if magnitude of charges + and − are equal $R_+ = R_+ - R_-$

$$\varphi = -\vec{d} \cdot \nabla \frac{1}{R_0} = \frac{d \cdot R_0}{R_0^3}$$

$$E = -\nabla \varphi = -\frac{1}{R_0^3} \nabla(\vec{d} \cdot \vec{R}_0) - \vec{d} \cdot \vec{R}_0 \nabla \left(\frac{1}{R_0}\right)^3$$

$$E = \frac{3(n \cdot d)n - d}{R_0^3}$$

$$E = (d \cdot \nabla) \nabla \left(\frac{1}{R_0}\right)$$

$$E_z = d \frac{3\cos^2\theta - 1}{R_0^3}, \quad E_x = d \frac{3 \sin\theta \cos\theta}{R_0^3}$$

$$E_R = d \frac{2\cos\theta}{R_0^3}, \quad E_\theta = -\frac{d \sin\theta}{R_0^3}$$

Multipole Moments

$$\varphi = \varphi^{(0)} + \varphi^{(1)} + \varphi^{(2)} + \dots$$

$$\varphi^{(n)} \sim \frac{1}{R_0^{n+1}}$$

$$\varphi^{(2)} = \frac{1}{2} \sum e x_\alpha x_\beta \frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left(\frac{1}{R_0}\right)$$

$$\Delta\left(\frac{1}{R_0}\right) = 0 \leftarrow \text{Laplace's equation}$$

$$\varphi^{(2)} = \frac{1}{2} \sum \left(e x_\alpha x_\beta - \frac{r^2}{3}\delta_{\alpha\beta}\right) \frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left(\frac{1}{R_0}\right)$$

$$D_{\alpha\beta} = \underbrace{\sum e\left(3x_\alpha x_\beta - r^2\delta_{\alpha\beta}\right)}_{\text{quadrupole moment}}$$

$$\varphi^{(2)} = \frac{D_{\alpha\beta}}{6} \frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left(\frac{1}{R_0}\right)$$

$$\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\left(\frac{1}{R_0}\right) = \frac{3x_\alpha x_\beta}{R_0^5} - \frac{\delta_{\alpha\beta}}{R_0^3}$$

$$\varphi^{(2)} = \frac{D_{\alpha\beta} n_\alpha n_\beta}{2R_0^3}$$

$$D_{xx} + D_{yy} = -D_{zz}$$

$$\varphi^{(2)} = \frac{D_{zz}}{4R_0^3}\left(3\cos^2\theta - 1\right)$$

l'th term of the expansion defines a tensor of rank l

expanded via spherical harmonics

$$\frac{1}{|R_0 - r|} = \sum_{l=0}^{\infty} \frac{r^l}{R_0^{l+1}} P_l(\cos\chi) \leftarrow \text{Legendre polynomials}$$

Can also express things in terms of $Y_m^{(l)}$

Systems of Charges in an External Field

$$U = \sum_c e_c \varphi(\vec{r}_a)$$

Assume external field changes slowly over the region of the system of the charges. Expand U in terms of r_c

$$U = U^{(0)} + U^{(1)} + U^{(2)} + \dots$$

$$U^{(0)} = \underset{\substack{\uparrow \\ \text{Value of potential of the origin}}}{\varphi_0} \sum e_a$$

$$U^{(1)} = (\nabla\varphi)_0 \cdot \underbrace{\sum e_a \vec{r}_a}_{d}$$

$$U^{(1)} = -d \cdot E_0$$

$$F = E_0 \sum e_a + [\nabla(d \cdot E)]_0$$

$F = (d \cdot \nabla) E$ if total charge goes to 0

Force determined by field intensity

$$\vec{K} = \sum (\vec{r}_a \times e_a \vec{E}_0) = \vec{d} \times \vec{E}_0$$

$$U = \frac{(\vec{d}_1 \cdot \vec{d}_2) R^2 - 3(\vec{d}_1 \cdot \vec{R})(\vec{d}_2 \cdot \vec{R})}{R^5}$$

$$U = e \frac{\vec{d} \cdot \vec{R}}{R^3}$$

$$U^{(2)} = \frac{1}{2} \sum e x_\alpha x_\beta \frac{\partial^2 \varphi_0}{\partial x_\alpha \partial x_\beta}$$

$$\frac{\partial^2 \varphi}{\partial x_\alpha^2} = \delta_{\alpha\beta} \frac{\partial^2 \varphi}{\partial x_\alpha \partial x_\beta} = 0$$

$$U^{(2)} = \frac{1}{2} \frac{\partial^2 \varphi_0}{\partial x_\alpha \partial x_\beta} \underbrace{\sum e \left(x_\alpha x_\beta - \frac{1}{3}\delta_{\alpha\beta} r^2\right)}_{\text{Quadrupole moment}}$$

In general

$$U^{(l)} = \sum_{m=-l}^{l} a_{lm} Q_m^{(l)}$$

Constant Magnetic Field

- consider mag field produced by charges producing finite motion

- consider average magnetic field, $\bar{H}$

Maxwell Equations,

$$\nabla \cdot H = 0, \quad \nabla \times H = \frac{\partial E}{\partial t} + 4\pi j$$

$$\nabla \cdot \bar{H} = 0$$

$$\nabla \times \bar{H} = \cancel{\frac{\partial E}{\partial t}}^{0} + 4\pi \bar{j}$$

average value of the derivative of any finite quantity is zero

$$\nabla \times \bar{A} = \bar{H}$$

$$\nabla(\nabla \cdot \bar{A}) - \Delta \bar{A} = 4\pi \bar{j}$$

Further impose $\nabla \cdot \bar{A} = 0$

Reduces

$$-\Delta \bar{A} = 4\pi \bar{j}$$

analogous to Poisson Equation

$$\bar{A} = \frac{1}{c} \int \frac{\bar{j}}{R} dV = \frac{1}{c} \sum \frac{e_a V_a}{R_a}$$

$$\bar{H} = \nabla \times \bar{A} = \frac{1}{c} \int \frac{\bar{j} \times R}{R^3} dV$$ assuming j is constant (Recall, this is all finite motion)

This is Biot-Savart's law

Magnetic Moments

- Consider average magnetic field in a system of charges in stationary motion
- denote radius vectors of charges, r_a

$$\bar{A} = \overline{\sum \frac{e_a v_a}{|R_0 - r_a|}}$$ ← expand in powers of r_a

$$\bar{A} = \frac{1}{R_0}\sum \overline{e\bar{v}} - \overline{\sum ev\left(r \cdot \nabla \frac{1}{R_0}\right)}$$

$\sum \overline{e\bar{v}} = \frac{d}{dt}\overline{\sum er}$ ← but this is finite motion so this is zero

$$\bar{A} = -\overline{\sum' ev\left(r \cdot \nabla \frac{1}{R_0}\right)}$$

$$= \frac{1}{R_0^3}\sum \overline{ev(r \cdot R_0)}$$

$v = \dot{r}$

$$\sum e(r \cdot R_0)\, v = \frac{1}{2}\frac{d}{dt}\sum er(r \cdot R_0) + \frac{1}{2}\sum e[v(r \cdot R_0) - r(v \cdot R_0)]$$

$$\bar{A} = \frac{1}{2R_0^3}\overline{\sum e\left[v(r \cdot R_0) - r(v \cdot R_0)\right]}$$

$\mu = \frac{1}{2}\sum e(\vec{r} \times \vec{v})$ ← magnetic moment

$$\bar{A} = \frac{\bar{\mu} \times R_0}{R_0^3} = \nabla \frac{1}{R_0} \times \bar{\mu}$$

$$\nabla \times (a \times b) = (b \cdot \nabla)a - (a \cdot \nabla)b + \vec{a}(\nabla \cdot b) - \vec{b}(\nabla \cdot \vec{a})$$

$$\bar{H} = \nabla \times \bar{A} = \nabla \times \left(\nabla \frac{1}{R_0} \times \bar{\mu}\right)$$

$$= \bar{\mu}\left(\nabla \cdot \frac{\vec{R}_0}{R_0^3}\right) - (\bar{\mu} \cdot \nabla)\frac{\vec{R}_0}{R_0^3}$$

$$\nabla \cdot \frac{\vec{R}_0}{R_0^3} = \vec{R}_0 \cdot \nabla \frac{1}{R_0^3} + \frac{1}{R_0^3}\nabla \cdot R_0 = 0$$

$$(\bar{\mu} \cdot \nabla)\frac{\vec{R}_0}{R_0^3} = \frac{1}{R_0^3}(\bar{\mu} \cdot \nabla)R_0 + R_0(\bar{\mu} \cdot \nabla)\frac{1}{R_0^3}$$

$$= \frac{\bar{\mu}}{R_0^3} - \frac{3\vec{R}_0(\bar{\mu} \cdot R_0)}{R_0^5}$$

$$\bar{H} = \frac{3n(\bar{\mu} \cdot n) - \bar{\mu}}{R_0^3}$$ ← same as $\bar{E}$ for dipole moment

if $v \ll c$

$$\mu = \frac{e}{2mc}\sum \vec{r} \times \vec{p}$$

Larmor's Theorem

Consider system of charges in an external constant uniform magnetic field

$$\bar{F} = \sum e \overline{v \times H} = \frac{d}{dt} \overline{\sum e\, r \times H} = 0$$

again, finite range argument.

$$\bar{K} = \sum \frac{e}{c} \overline{(r \times (v \times H))}$$

expressed by expanding the vector triple product

$$K = \sum \frac{e}{c}\left(v(r \cdot H) - H(v \cdot r)\right) = \sum \frac{e}{c}\left(v(r \cdot H) - \underbrace{\tfrac{1}{2} H \tfrac{d}{dt} \vec{r}^{\,2}}_{0 \text{ after averaging}}\right)$$

$$\bar{K} = \sum \frac{e}{c} \overline{v(r \cdot H)} = \frac{1}{2c} \sum e\left(\overline{v(r \cdot H)} - r\overline{(v \cdot H)}\right)$$

$$\bar{K} = \bar{\mu} \times H$$

$$\mathcal{L}_H = \sum \frac{e}{c} \vec{A} \cdot \vec{v} = \sum \frac{e}{2c} (H \times r) \cdot v = \sum \frac{e}{2c} (\vec{r} \times \vec{v}) \cdot \vec{H}$$

$$\mathcal{L}_H = \vec{\mu} \cdot \vec{H}$$

$$\mathcal{L}_E = \vec{d} \cdot \vec{E}$$

$v' = v + \Omega \times r$ ← transforming to a frame uniformly rotating around ~~an~~ axis

$$\mathcal{L} = \sum \frac{mv'^2}{2} - U$$

$$\mathcal{L} = \sum \frac{m(v + \Omega \times r)^2}{2} - U$$

$$\Omega = \frac{e}{2mc} H$$

If we can neglect H^2

$$\mathcal{L} = \sum \frac{mv^2}{2} + \frac{1}{2c} \sum e H \times r \cdot v - U$$

This reveals that a system with uniform magnetic field is equivalent to a system with no magnetic field ~~with~~ in a rotating frame.

$$\Omega = \frac{eH}{2mc}$$

$$\underbrace{\frac{d\bar{M}}{dt} = \bar{K} = \bar{\mu} \times H = -\Omega \times \bar{M}}_{\text{Larmor Precession}}$$

Electromagnetic Waves

The Wave Equation

consider EM field in a vacuum

eg $\rho = 0$, $\vec{j} = 0$

$$\nabla \times \vec{E} = -\frac{\partial \vec{H}}{\partial t} \qquad \nabla \cdot \vec{H} = 0$$

$$\nabla \times \vec{H} = \frac{\partial \vec{E}}{\partial t} \qquad \nabla \cdot E = 0$$

These equations have solutions that are non-zero

Hence EM fields can exist, even in the absence of charge

Fields must be time varying as

If $\frac{\partial H}{\partial t}, \frac{\partial E}{\partial t} = 0$ everything is stationary, and thus everything is zero

Okay, can take scalar potential

$$\phi = 0$$

$$\Rightarrow E = -\frac{\partial \vec{A}}{\partial t}, \quad H = \nabla \times A$$

$$\underbrace{\nabla \times (\nabla \times A) = -\Delta \vec{A} + \nabla(\nabla \cdot A) = -\frac{\partial^2 A}{\partial t^2}}$$

A not uniquely defined

$$\nabla \cdot A = 0$$

So

$$\left.\Delta A - \frac{\partial^2 A}{\partial t^2} = 0 \right\} \text{d'Alembert equation (wave equation)}$$

In 4-dimensional form

$$\frac{\partial F^{ik}}{\partial x^k} = 0$$

recall $F^{ik} = \frac{\partial A^k}{\partial x_i} - \frac{\partial A^i}{\partial x_k}$

$$\frac{\partial^2 A^k}{\partial x_i \partial x^k} - \frac{\partial^2 A^i}{\partial x_k \partial x^k} = 0$$

Imposing $\nabla \cdot A = 0 \Rightarrow \underbrace{\frac{\partial A^k}{\partial x^k} = 0}_{\text{Lorentz condition}}$

$$\left.\frac{\partial^2 A^i}{\partial x_k \partial x^k} = g^{kl}\frac{\partial^2 A^i}{\partial x^k \partial x^l} \right\} \text{wave eqn in 4-form}$$

Plane Waves

suppose we only have one spatial dimension and one time dimension

$$\frac{\partial^2 f}{\partial t^2} - \frac{\partial^2 f}{\partial x^2} = 0$$

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right) f = 0$$

define $\xi = t - x$, $\eta = t + x$

$$\underbrace{\frac{\partial^2 f}{\partial \xi \partial \eta} = 0}$$

solutions of form

$$f = f_1(\eta) + f_2(\xi)$$

$\uparrow$

things are wave like!

$$\nabla \cdot A = \frac{\partial A}{\partial x} = 0$$

$$\frac{\partial^2 A_x}{\partial t^2} = 0, \quad \text{so} \quad \underbrace{\frac{\partial A_x}{\partial t} = \text{const}}_{\text{must be zero}}$$

as A_x is freely zero, then

$$A \perp \hat{x}$$

$$E = -\frac{\partial A}{\partial t}, \quad H = \nabla \times A$$

$$H = \nabla \times A = \nabla\left(t - \frac{x}{c}\right) \times A' = -n \times A'$$

$$H = n \times E$$

$$E \perp H$$

consider the Poynting vector

$$S = \frac{1}{4\pi} \vec{E} \times \vec{H} = \frac{1}{4\pi} E \times (n \times E)$$

$$= \frac{E^2 \vec{n}}{4\pi}$$

W, energy density of wave

$$W = \frac{1}{8\pi}(E^2 + H^2) = \frac{E^2}{4\pi}$$

$$S = W\vec{n}$$

$$T^{xx} = -\sigma_{xx} = W$$

$$W = \frac{1}{1-V^2}\left(W' + 2V S'_x + V^2 \sigma'_{xx}\right)$$

$$S'_x = -W' \cos^2 \alpha'$$

Monochromatic Plane Waves

Important: when field is simply periodic function of time

- monochromatic wave

$$\frac{\partial^2 f}{\partial t^2} = -\omega^2 f$$

In a plane wave propagating along x axis

$$\vec{A} = \mathrm{Re}\{A_0 \exp(i\omega(t-x))\}$$

$\lambda = \frac{2\pi}{\omega}$ } wave length

$k = \omega\hat{n}$ ← vector in direction of wave

$$\vec{A} = \mathrm{Re}\{A_0 \exp(i(\vec{k}\cdot\vec{r} - \omega t))\}$$

Using Maxwell's eqns, given $\vec{A}$

$\vec{E} = ik\vec{A}$ $\qquad H = ik \times A$

$$\vec{E} = \mathrm{Re}\{E_0 \exp(i(\vec{k}\cdot\vec{r} - \omega t))\}$$

$$E_0^2 = |E_0|^2 \exp(-i2\alpha)$$

can define

$$E_0 = b\exp(-i\alpha)$$

$$b^2 = |E_0|^2$$

E is then

$$E = \mathrm{Re}\{b \exp(i(\vec{k}\cdot\vec{r} - \omega t - \alpha))\}$$

$$b = b_1 + i b_2$$

$|b|^2$ must be real. Thus, $b_1 \cdot b_2 = 0$

Choose x-axis to be the direction of the wave, y-axis to be the direction b_1

$$E_y = b_1 \cos(\omega t - k\cdot\vec{r} + \alpha)$$

$$E_z = \pm b_2 \sin(\omega t - \vec{k}\cdot\vec{r} + \alpha)$$

$$\left.\frac{E_y^2}{b_1^2} + \frac{E_z^2}{b_2^2} = 1 \right\}$$ the wave is elliptically polarized

If b_1 or b_2 are 0 is linearly polarized

$$k^i = (\omega, \vec{k})$$

$$k_i x^i = \omega t - \vec{k}\cdot\vec{r}$$

$$k^i k_i = 0$$

$$\vec{A} = A_0 \exp(-i k^i x_i)$$

$$T^{00} = T^{01} = T^{11} = W$$

$$T^{ik} = \frac{W}{\omega^2} k^i k^k$$

Doppler effect can be attained by Lorentz boosting k and applying boost result to ω

Spectral Resolution

Represent a wave as a superposition of monochromatic waves

$$f = \sum_{n=-\infty}^{n=\infty} f_n \exp(-i\omega_0 n t)$$

$$f_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\exp(in\omega_0 t)\,dt$$

f must be real, so

$$f_{-n} = f_n^*$$

$$\overline{f^2} = 2\sum_{n=0}^{\infty} |f_n|^2$$

$$f(t) = \int_{-\infty}^{\infty} f_\omega \exp(-i\omega t)\frac{d\omega}{2\pi}$$

Similar logic as above applies

Partially Polarized Light

- Consider waves that are approximately monochromatic

eg, wave contains frequency

$\omega \pm \Delta\omega$

$E = E_0(t)\exp(-i\omega t)$

$\uparrow$

slower variation than ω

E_0 determines the polarization of the wave, however E_0 changes, so $\vec{E}$ is partially polarized

- In the real world, polarization properties measured by intensities, eg quadratic quantities

$E_\alpha E_\beta = E_{0\alpha} E_{0\beta} \exp(2i\omega t) \leftarrow 0$ on averaging

Hence, $J_{\alpha\beta} = \overline{E_{0\alpha} E_{0\beta}^*}$

$E_0 \perp$ to wave

assume wave propagates along x axis

Then

$J_{\alpha\beta}$ has 4 non zero components

$J \equiv J_{\gamma\gamma} = \overline{E_0 \cdot E_0^*} \leftarrow$ energy flux density

$\rho_{\alpha\beta} = \frac{J_{\alpha\beta}}{J} \leftarrow$ polarization tensor

$\rho_{\alpha\beta}^\dagger = \rho_\beta$

For completely polarized light

$J_{\alpha\beta} = E_{0\alpha} E_{0\beta}^\dagger$

Components are from a constant vector

Hence, we can write $J_{\alpha\beta}$ with 2 identical row vectors, so

$|\rho_{\alpha\beta}| = 0$

For unpolarized light

$\rho_{\alpha\beta} = \frac{1}{2}\delta_{\alpha\beta} \ni |\rho_{\alpha\beta}| = \frac{1}{4}$

Hence, all polarizations between 0 and $\frac{1}{4}$

$|\rho_{\alpha\beta}| = \frac{1}{4}(1-P^2)$

$S_{\alpha\beta} = \frac{1}{2}(\rho_{\alpha\beta} + \rho_{\beta\alpha}) \leftarrow$ symmetric part, is real

$\frac{1}{2}(\rho_{\alpha\beta} - \rho_{\beta\alpha}) = -\frac{i}{2} e_{\alpha\beta} A \leftarrow$ antisymmetric part

$\rho_{\alpha\beta} = S_{\alpha\beta} - \frac{i}{2} e_{\alpha\beta} A$

$E_{02} = \pm i E_{01}$ circularly polarized

Write in terms of principal axes

For linearly polarized light,

$A = 0$,

$S_{\alpha\beta}$ can then be decomposed as the sum of independent linearly polarized waves (partially)

Suppose

$S_{\alpha\beta} = \lambda_1 |n_1\rangle\langle n_1| + \lambda_2 |n_2\rangle\langle n_2|$

$$S_{\alpha\beta} = \frac{1}{2}\begin{pmatrix} 1+l\cos 2\phi & l\sin 2\phi \\ l\sin 2\phi & 1-l\cos 2\phi \end{pmatrix}$$

where $l = \lambda_1 - \lambda_2$

polarization can then be expressed in terms of $l\cos 2\phi$, A, $l\sin 2\phi$

The fourier resolution of an electrostatic field

- Fields can also be formally expanded as plane waves

- However field produced by charges is inhomogeneous

$$\Delta \varphi = -4\pi e \delta(\vec{r})$$

rewrite φ

$$\varphi = \int_{-\infty}^{\infty} e^{ik\cdot r} \frac{d^3k}{(2\pi)^3}$$

$$\Delta \varphi = -\int k^2 e^{ik\cdot r} \frac{d^3k}{(2\pi)^3}$$

$$-k^2 \varphi_k = -4\pi e$$

$$\varphi_k = \frac{4\pi e}{k^2}$$

Recall that

$$E = -\nabla \varphi$$

$$= -\nabla \int \varphi_k e^{ik\cdot r} \frac{d^3k}{(2\pi)^3} = -\int ik \varphi_k e^{ik\cdot r} \frac{d^3k}{(2\pi)^3}$$

$$E_k = -i \frac{4\pi e \vec{k}}{k^2}$$

Characteristic Vibrations of the field

- consider EM field, no charges, in finite volume of space
  assume volume is parallelpiped side lengths A, B, C
- expand quantities characterizing the triple fourier series in the volume

Expansion is

$$\vec{A} = \sum \vec{A}_k \exp(i \vec{k} \cdot \vec{r})$$

$$k_x = \frac{2\pi n_x}{A}, \frac{2\pi n_y}{B}, \frac{2\pi n_z}{C}$$

A is real so

$$A_k^* = A_{-k}$$

$$k \cdot A_k = 0$$

from wave eqn

$$\ddot{A}_k + c^2 k^2 A_k = 0$$

We ask how many frequencies lie within a single unit of frequency actually exist

$$\Delta n_x = \frac{A}{2\pi} \Delta k_x$$

$$\Delta n = \frac{V}{(2\pi)^3} \Delta k_x \Delta k_y \Delta k_z$$

transform Δn to polar to get components through solid angle

(solid angle, 4π)

$$\Delta n = \frac{V}{(2\pi)^3} k^2 \Delta k \, \Delta O$$

$$\mathcal{E} = \frac{1}{8\pi} \int (E^2 + H^2) dV$$

$$E = -\frac{1}{c} \dot{A}$$

$$H = \nabla \times A = i \sum_k (k \times A_k) \exp(i \vec{k} \cdot \vec{r})$$

evaluations

$$\mathcal{E} = \frac{V}{8\pi c^2} \sum \left(\dot{A}_k \cdot \dot{A}_k^* + k^2 c^2 A_k \cdot A_k^* \right)$$

$$A_k = \sum_k a_k \exp(i k \cdot r) + a_k^* \exp(-i k \cdot r)$$

$$\dot{A}_k = -ick(a_k - a_{-k}^*)$$

$$\mathcal{E} = \sum_k \frac{k^2 V}{2\pi} a_k \cdot a_k^*$$

$$\frac{1}{4\pi c} \int (E \times H) dV = \sum_k \frac{\vec{k}}{k} \mathcal{E}_k / c$$

$$Q_k = \sqrt{\frac{V}{4\pi c^2}} (a_k + a_k^*)$$

$$P_k = -i\omega_k \sqrt{\frac{V}{4\pi c^2}} (a_k - a_k^*) = \dot{Q}_k$$

$$H = \sum_k \frac{1}{2} (P_k^2 + \omega_k^2 Q_k^2)$$

Chapter 7

Propagation of Light

Geometrical optics

- Plane waves have the property that direction and amplitude of propagation are the same everywhere. Not true in general
  - However reasonable assumption in small enough ΔV (over distances of order of the wavelength)
  - so-called wave-surface

- rays: curves whose tangents are parallel to the direction of propagation
- this is geometrical optics case where $\lambda \to 0$

Derive some eqns

f is any quantity describing field of wave

$$f = a e^{i(-k_i x^i + \alpha)}$$

$$f = a e^{i\psi}$$

phase, ψ, must be a large quantity (effectively $\lambda \to 0$)

expand ψ to first order (eikonal)

$$\psi = \psi_0 + r \cdot \frac{\partial \psi}{\partial r} + t \frac{\partial \psi}{\partial t}$$

$$k = \frac{\partial \psi}{\partial r} = \vec{\nabla} \psi, \quad \omega = -\frac{\partial \psi}{\partial t}$$

$$k_i = -\frac{\partial \psi}{\partial x^i}$$

$$\underbrace{\frac{\partial \psi}{\partial x^i}\frac{\partial \psi}{\partial x_i} = 0}_{\text{eikonal equations}}$$

recall the $k_i k^i = 0$ because everything has lightlike separation

$$\frac{\partial^2 f}{\partial x_i \partial x^i} = 0$$

substitute for $f = a e^{i\psi}$

$$\frac{\partial^2 f}{\partial x_i \partial x^i} = \frac{\partial^2 a}{\partial x_i \partial x^i} e^{i\psi} + 2i \frac{\partial a}{\partial x_i}\frac{\partial \psi}{\partial x^i} e^{i\psi} + if \frac{\partial^2 \psi}{\partial x_i \partial x^i}$$

$$\underbrace{- \frac{\partial \psi}{\partial x_i}\frac{\partial \psi}{\partial x^i} f}_{\text{this term dominates}} = 0$$

analogy between eikonal equations and mechanics of material particles

~~rect~~ recall

$$\underbrace{\vec{p} = \frac{\partial S}{\partial \vec{r}}}_{\text{like the wave vector}}, \quad \underbrace{\mathcal{H} = -\frac{\partial S}{\partial t}}_{\text{like the frequency}}$$

$k = \frac{\omega}{c}$ recall that $p = \frac{\mathcal{E}}{c}$

$$\dot{\vec{p}} = -\frac{\partial \mathcal{H}}{\partial r} \qquad v = \dot{r} = \frac{\partial \mathcal{H}}{\partial p}$$

$$\dot{k} = -\frac{\partial \omega}{\partial r} \qquad \dot{r} = \frac{\partial \omega}{\partial k}$$

$\omega = ck$, so $\dot{k} = 0$ $\qquad v = cn$

- consider a wave which is a superposition of monochromatic waves with frequencies in a small interval occupying a small region of space (wave packet)

$$p^i = A k^i$$

$$\vec{p} = A\vec{k}, \quad \mathcal{E} = A\omega$$

Hence, wave momentum and frequency transform as expected

$\delta \int L \, dt = 0$ impossible to introduce

$$L = p \cdot \frac{\partial \mathcal{H}}{\partial p} - \mathcal{H}$$

$$= \vec{k} \cdot \frac{\partial \omega}{\partial k} - \omega = 0 \qquad \text{since } \omega = ck$$

$$\psi = -\omega t + \psi_0(x, y, z)$$

since wave has constant frequency

$$\nabla \psi_0 \cdot \nabla \psi_0 = \frac{\omega^2}{c^2}$$

when energy is constant

$$\delta S = \delta \int \vec{p} \cdot d\vec{l} = 0$$

$$\delta \psi = \delta \int \vec{k} \cdot d\vec{l} = 0$$

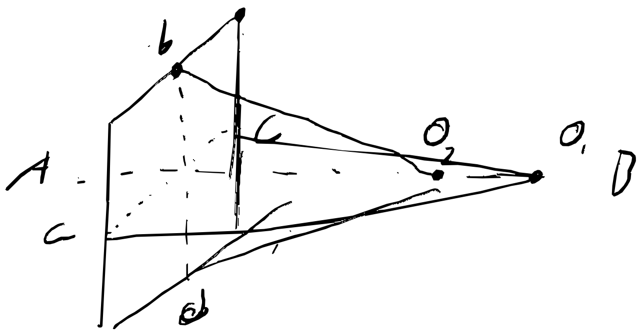
$$k = \frac{\omega}{c} \hat{n}$$

$$\delta \psi = \delta \int k \cdot dl = \delta \int dl = 0$$

No variation from straight lines, rays are rectilinear

Intensity

- Light wave can be considered a bundle of waves of light
  - rays themselves only determine orientation
  - distribution in space still unknown
- from differential geometry, it is known that every point has two principle radii of curvature



AB is a ray

- Fixed angular openings of the beams might have diff origins, so diff radii
  R_1, R_2

Intensity is inversely proportional to radius

$$I = \frac{\text{const}}{R_1 R_2}$$

Change of field along a ray is thus

$$f = \frac{\text{const}}{\sqrt{R_1 R_2}} \exp(ikR)$$

- caustics: centers of curvature
  - focus: if centers of curvature coincide
- rays are tangent to the caustic
- center of curvature surface may not lie on caustic - caustic is called imaginary

The angular Eikonal

- light will generally change directions when striking a transparent medium

- derive general laws relating change in direction to geometrical optics

- propagation of rays and motion of particles should obey similar rules

$$(\nabla \psi)^2 = 1$$ (assume we normalize eikonal by ω/c)

- this needs to be generalized to arbitrary bundle of rays

- introduce eikonal as function

$\psi(r, r')$ ← phase dif or optical path length

initial and of ray

suppose r or r' is fixed

Then

$$(\nabla_r \psi(r, r'))^2 = 1 \qquad (\nabla_{r'} \psi(r, r'))^2 = 1$$

fixed fixed

$$\vec{n}' = \frac{\partial \psi}{\partial \vec{r}'}, \quad \vec{n} = -\frac{\partial \psi}{\partial \vec{r}}$$ ← because one is beginning, other is end

Consider

$$d\psi = \frac{\partial \psi}{\partial r} \cdot r + \frac{\partial \psi}{\partial r'} \cdot r'$$

$$= -n \cdot dr + n' \cdot dr'$$

$$= -d(n \cdot r) + r \cdot dn + d(n' \cdot r') - r' \cdot dn'$$

$$\chi = n' \cdot r' - n \cdot r + \psi$$

$$\underbrace{d\chi = -r \cdot dn + r' \cdot dn'}_{\text{angular eikonal}}$$

Writing this out for x, y, z coordinates yields relationship between rays and the surface being struck.

Narrow Bundle of Rays

- special interest for bundles whose rays all pass through one point
- after passing through optical system, this is rarely true
- the most important case is transition of bundles into homocentric bundles is that of sufficiently narrow beams passing close to the optical axis
- axis of symmetry is the optical axis of the system

- Since bundle of rays nearly move along optical axis, $\vec{n}\ \vec{n}'$ along it

If this is along x axis

$$\frac{n_y}{n_x}, \frac{n_z}{n_x} << 1$$

If $n_x \sim 1$, n'_x is either nearly

-1 or $+1$

opposite direction — mirror

continues in dir it came from — lens

Expand angular eikonal to first order

- χ must be invariant to rotations about optical axis

assume optical axis $\parallel$ to $\hat{x}$ axis

· No terms prop to first order in n_y and n_z

$$\chi = \text{const} + \underbrace{\frac{g}{2}(n_y^2+n_z^2) + f(n_y n'_y + n_z n'_z) + \frac{h}{2}(n'^2_y + n'^2_z)}_{\text{constants}}$$

$n_y(x-g) - f n'_y = y \qquad f n_y + n'_y(x'+h) = y'$

$n_z(x-g) - f n'_z = z \qquad f n_z + n'_z(x'+h) = z'$

$$\frac{x-g}{f} = -\frac{f}{x'+h} = \frac{y}{y'} = \frac{z}{z'}$$

$(x-g)(x'+h) = -f^2$

$x = g \quad x' = -h$ are the principle foci

$x - g = X, \; x'+h = X' \quad Y' = y', \; Y = y, \; Z = z, \; Z' = z'$

$$\boxed{\frac{Y'}{Y}} = \frac{Z'}{Z} = \frac{f}{X} = -\frac{X'}{\boxed{f}}$$

Lateral magnification — focal length

$$\left|\frac{dX'}{dX}\right| = \frac{f^2}{X^2} = \left(\frac{Y'}{Y}\right)^2$$

bundle passing at $X = f$, passes through $X' = -f$

- these two points are principal points

thus in this case

$n_y = n'_y \quad n_z = n'_z$

$\xi' = X' + f \quad \xi = X - f$

$$\frac{1}{\xi} - \frac{1}{\xi'} = -\frac{1}{f}$$

$$x' = \frac{h}{g}x + \frac{f^2 - gh}{g}$$

$x' = \alpha x + \beta$

$$\underbrace{X' = \alpha^2 X \quad Y' = \pm\alpha Y \qquad Z' = \pm\alpha Z}$$

no magnification, but you get something telescopic

Image formation with broad bundle of rays

Field of Moving Charges

Retarded potentials

$$\frac{\partial F^{ik}}{\partial x^k} = -\frac{4\pi}{c} j^i$$

$$\frac{\partial A^i}{\partial x^i} = 0, \quad \frac{1}{c}\frac{\partial \varphi}{\partial t} + \nabla A = 0$$

3 form of

$$\frac{\partial^2 A^i}{\partial x_k \partial x^k} = \frac{4\pi}{c} j^i$$

$$\Delta A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -\frac{4\pi}{c} j$$

$$\Delta \varphi - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = -4\pi\rho$$

Inhomogeneous solutions are homogeneous part and not

If charge is moving write as follows

$$\Delta \varphi - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = -4\pi \underbrace{de(t)}_{\text{infinitesimal charge}} \delta(\vec{R})$$

suppose charge at origin

$$\Delta\varphi - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = 0 \qquad R \neq 0$$

Radially symmetric

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial \varphi}{\partial R}\right) - \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = 0$$

$$\varphi = \chi(R,t)/R$$

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\left(\frac{\partial \chi(R,t)}{\partial R}\left(\frac{1}{R}\right) - \frac{1}{R^2}\chi\right)\right) - \frac{1}{c^2}\frac{1}{R}\frac{\partial^2 \chi}{\partial t^2} = 0$$

$$\frac{1}{R^2}\frac{\partial}{\partial R}\left(R\frac{\partial \chi}{\partial R} - \chi\right) - \frac{1}{c^2}\frac{\partial^2 \chi}{\partial t^2}\frac{1}{R} = 0$$

$$\frac{1}{R^2}\left(\cancel{\frac{\partial \chi}{\partial R}} + R\frac{\partial^2 \chi}{\partial R^2} - \cancel{\frac{\partial \chi}{\partial R}}\right) - \frac{1}{c^2 R}\frac{\partial^2 \chi}{\partial t^2} = 0$$

$$\underbrace{\frac{\partial^2 \chi}{\partial R} - \frac{\partial^2 \chi}{c^2 \partial t^2} = 0}$$

radially spreading plane waves

$$\chi = f_1\left(t - \frac{R}{c}\right) + f_2\left(t + \frac{R}{c}\right)$$

We now must obtain the form of χ that gets everything right at the origin

as $R \to 0$, potential diverges so $\frac{\partial \chi}{\partial R}$ dominates

$$\Delta\varphi \sim -4\pi\, de(t)\, \delta(R)$$

double integrating over space yields

$$\varphi = \frac{de(t - R/c)}{R}$$

$$\left.\begin{aligned} \varphi(\vec{r}, t) &= \int \frac{1}{R}\rho\left(r', t - \frac{R}{c}\right) dV' + \varphi_0 \\ \vec{A} &= \frac{1}{c}\int \frac{\vec{j}_{t-R/c}}{R}\, dV + A_0 \end{aligned}\right\} \text{Retarded potentials}$$

It is sufficient to set initial conditions correct

- usually given condition at large distance

Lienard-Wiechert Potentials

$$\vec{r} = \vec{r}_0(t)$$

Field at point of observation

$P(x,y,z)$ at time t depends on motion at time t'

time of propagation of light signal is $t - t'$

$$R(t) = r - r_0(t)$$

Radius vector from charge to point $P(x,y,z)$

$$t' + \frac{R(t')}{c} = t$$

in system where charge is at rest

$$\varphi = \frac{e}{R(t')}, \quad \vec{A} = 0$$

$$\varphi = \frac{e}{c(t-t')}$$

4-vector rep of this system

$$A^i = e \frac{u^i}{R_k u^k}$$

4-velocity of charge

$$R_k = [c(t-t'), \vec{r} - \vec{r}_0]$$

$$R_k R^k = 0$$

Apply lorentz transform for arbitrary motion

$$\underbrace{\varphi = \frac{e}{R - \frac{\vec{v} \cdot \vec{R}}{c}}, \quad \vec{A} = \frac{e\vec{v}}{c(R - \frac{\vec{v} \cdot \vec{R}}{c})}}_{\text{Lienard-Wiechert potentials}}$$

$$\vec{E} = -\frac{1}{c}\frac{\partial A}{\partial t} - \vec{\nabla}\varphi \qquad H = \nabla \times A$$

We must differentiate φ and $\vec{A}$ with respect to coordinates x, y, z and time, t of observation

However, we have potentials in terms of t'

$$R(t') = c(t - t')$$

$$\frac{\partial R}{\partial t} = \frac{\partial R}{\partial t'}\frac{\partial t'}{\partial t} = -\frac{\vec{R} \cdot \vec{v}}{|R|}\frac{\partial t'}{\partial t} = c\left(1 - \frac{\partial t'}{\partial t}\right)$$

$$\left(c - \frac{\vec{R} \cdot \vec{v}}{|R|}\right)\frac{\partial t'}{\partial t} = c$$

$$\frac{\partial t'}{\partial t} = \frac{1}{1 - \frac{R \cdot v}{c|R|}}$$

$$\vec{\nabla}\left(R(t') = c(t - t')\right)$$

$$\nabla R(t') = -c\,\vec{\nabla} t'$$

$$\left(\frac{\partial R}{\partial t'}\vec{\nabla} t' + \frac{\vec{R}}{|R|}\right) = -c\,\vec{\nabla} t'$$

$$\nabla t' = -\frac{\vec{R}/|R|}{c - \frac{\vec{R} \cdot \vec{v}}{|R|}} = \frac{\vec{R}}{c|R| \cdot \vec{R} \cdot \vec{v}}$$

Can now calculate E, H

$$\vec{E} = e\frac{(1 - \frac{v^2}{c^2})}{(R - \frac{R \cdot v}{c})^3}\left(\vec{R} - \frac{\vec{v}}{c}|R|\right) + \frac{e\vec{R}}{c^2(R - \frac{R \cdot v}{c})^3} \times \left[\left(\vec{R} - \frac{\vec{v}}{c}R\right) \times \dot{v}\right]$$

$$H = \frac{1}{R}\vec{R} \times \vec{E}$$

$$\dot{v} = \frac{\partial v}{\partial t'}$$

Spectral Resolution of Retarded Potentials

Fields produced by moving charges can be expanded into monochromatic waves

$$\varphi_\omega \exp(-i\omega t), \quad A_\omega \exp(-i\omega t)$$

$$\varphi_\omega \exp(\cancel{-i\omega t}) = \int \rho_\omega \frac{\exp(-i\omega(t - R/c))}{R} dV$$

$$\varphi_\omega = \int \rho_\omega \frac{\exp(i\omega R/c)}{R} dV$$ } general solution to poisson equation

$$A_\omega = \int j_\omega \frac{\exp(i\omega R/c)}{cR} dV$$

For full solution we have to expand to fourier integral

$$\varphi_\omega = \iint_{-\infty}^{\infty} \frac{\rho}{R} \exp(i\omega t + ikR)\, dV\, dt$$

$$\rho = e\,\delta(r - r_0(t))$$

$$\varphi_\omega = \int \frac{e}{R(t)} \exp\left(i\omega t + i\omega \frac{R(t)}{c}\right) dt$$

$$A_\omega = \int \frac{e\, v(t)}{R(t)} \exp\left(i\omega t + i\omega \frac{R(t)}{c}\right) dt$$

$$v = \dot{r}$$

The Lagrangian terms to second order

- In classical mechanics, propagation of interactions is infinite
  - this lets everything be described by just positions and velocities
- field is an independent system because of the finite-ness of velocity of propagation of interactions
- if velocity is small, $\frac{v}{c}$ and $\frac{v^2}{c^2}$ can be neglected

$$\mathcal{L}^{(0)} = \frac{1}{2}\sum m_i v_i^2 - \sum_{i>j} \frac{q_i q_j}{R_{ij}} \leftarrow \text{zero'th approx}$$

External field

$$\mathcal{L}_a = -m_a c^2\sqrt{1-\frac{v_a^2}{c^2}} - q_a \varphi + \frac{q_a}{c} A \cdot v_a$$

- Choosing any one of the charges, express fields in terms of coordinates and velocities

φ to terms of order v^2/c^2, A to order $\frac{v}{c}$, rewrite Lagrangian

start from

$$\varphi = \int \frac{\rho_{t-R/c}}{R}\,dV + \frac{\partial}{\partial t}\int\left(t_0 - R/c\right)\frac{\rho(r', t_0 - \frac{R}{c})}{R}\,dV$$

$$+\frac{\partial}{\partial t}\int \frac{1}{2}\left(t_0 - \frac{R}{c}\right)^2 \frac{\rho(r', t_0 - \frac{R}{c})}{R}\,dV$$

$t_0 = 0$

total charge, charge is constant

$$\varphi = \int \frac{\rho}{R}\,dV - \frac{1}{c}\frac{\partial}{\partial t}\underbrace{\int \rho\, dV}_{0} + \frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\int R\rho\, dV$$

$$\varphi = \int \frac{\rho}{R}\,dV + \frac{1}{2c^2}\frac{\partial^2}{\partial t^2}\int R\rho\, dV$$

proceeding similarly with $\vec{A}$, however,

$$\vec{A} = \frac{1}{c}\int \frac{j(r', t - R/c)}{R}\,dV = \frac{1}{c}\int \frac{\vec{v}\rho}{R}\,dV$$

expansions of A will drive us beyond $\frac{1}{c^2}$ in the Lagrangian immediately

Consider the case of a single point charge

$$\varphi = \frac{e}{R} + \frac{e}{2c^2}\frac{\partial^2 R}{\partial t^2}, \qquad A = \frac{e\vec{v}}{Rc}$$

$$\varphi' = \varphi - \frac{1}{c}\frac{\partial f}{\partial t} \qquad \vec{A}' = \vec{A} + \vec{\nabla} f$$

if f is $\frac{e}{2c}\frac{\partial R}{\partial t}$

$$\varphi' = \frac{e}{R} \qquad \vec{A}' = \frac{e\vec{v}}{Rc} + \frac{e}{2c}\vec{\nabla}\frac{\partial R}{\partial t}$$

$$\vec{\nabla}\frac{\partial R}{\partial t} = \frac{\partial}{\partial t}(\vec{\nabla} R)$$

∇R is the unit vector $\hat{n}$ directed from charge e

$$A' = \frac{e\vec{v}}{Rc} + \frac{e}{2c}\dot{\hat{n}}$$

$$\dot{\hat{n}} = \frac{\partial}{\partial t}\left(\frac{\vec{R}}{R}\right) = \frac{\dot{\vec{R}}}{R} - \frac{\vec{R}\dot{R}}{R^2}$$

$$-\dot{\vec{R}} = \vec{V}$$

$$R^2 = \vec{R}^2$$

$$R\dot{R} = \vec{R}\cdot\dot{\vec{R}} = -\vec{R}\cdot\vec{v}$$

$$\dot{\hat{n}} = \frac{-\vec{v} + \hat{n}(\hat{n}\cdot\vec{v})}{R}$$

$$\varphi' = \frac{e}{R}, \quad \vec{A}' = \frac{e[\vec{v} + (v\cdot n)\hat{n}]}{2c}$$

Now writing out the full Lagrangian

$$\mathcal{L}_a = \frac{m_a v_a^2}{2} + \frac{m_a v_a^4}{8c^2} - e_a\sum \frac{e_b}{R_{ab}} + \frac{e_a}{2c^2}\sum \frac{e_b}{R_{ab}}\left[v_a\cdot v_b + (v_a\cdot n_{ab})(v_b\cdot n_{ab})\right]$$

n_{ab} is unit vector from a to b

$$\mathcal{L} = \sum_a \mathcal{L}_a$$

- to get the Hamiltonian, consider second and fourth terms. They are small changes
- We know small changes in $\mathcal{L}$ and $\mathcal{H}$ are equal and opposite in magnitude

replace, to first approximation

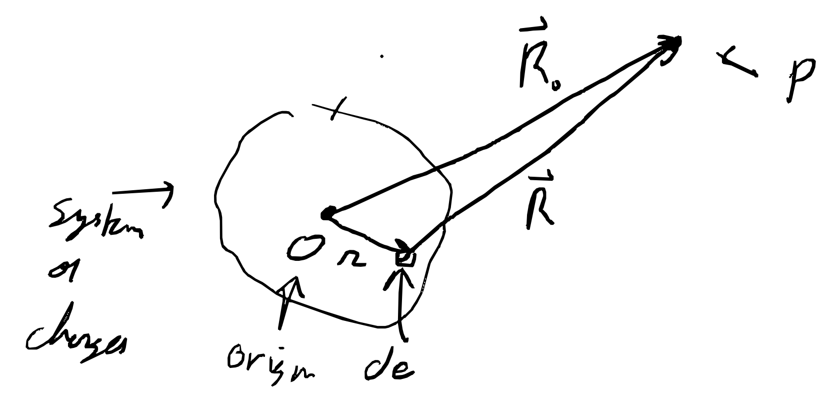
$$\frac{p_a}{m_a} = v_a$$

$$\mathcal{H}_a = \sum \frac{p_a^2}{2m_a} - \sum \frac{p_a^4}{8c^2 m_a^3} + \sum \frac{e_a e_b}{R_{ab}} - \sum_{a>b} \frac{e_a e_b}{2c^2 m_a m_b R_{ab}}\left[p_a\cdot p_b + (p_a\cdot n_{ab})(p_b\cdot n_{ab})\right]$$

Radiation of EM Waves

The field of a system of charges at large distances

- consider field produced by system of charges at distances large compared to system



$\hat{n} \parallel \vec{R}_o$

$\vec{R}_o = \vec{R} + \vec{r}$

$|R| \approx |R_o| - \vec{r} \cdot \hat{n}$ for

$t - R/c$, not possible to determine if $r \cdot \hat{n}$ can be neglected without investigating impact to ρ, $\vec{j}$

$$\varphi = \frac{1}{R_o} \int \rho\left(t - \frac{R_o}{c} + r \cdot \frac{n}{c}\right) dV$$

$$A = \frac{1}{cR_o} \int j\left(t - R_o/c + r \cdot n/c\right) dV$$

consider a small region of space in this style a plane wave

$\hat{E} = \hat{H} \times \hat{n}$

$H = \nabla \times A =$

so it is sufficient to calculate exclusively $\vec{A}$

In a plane wave

$H = \dot{A} \times n \left(\frac{1}{c}\right)$

knowing $\vec{A}$ yields $\vec{H}$ and $\vec{E}$

$\vec{E} = \frac{1}{c} (\dot{A} \times n) \times n$

$$\vec{A} = \frac{e\vec{v}(t')}{cR_o\left(1 - \frac{n \cdot v(t')}{c}\right)}$$

$$t' - \frac{r_o(t')}{c} \cdot n = t - R_o/c$$

Radiated EM waves carry off energy, calc'd by poyntons vector

$S = c\frac{H^2}{4\pi} \hat{n}$

intensity of radiation defined as the amount of energy through an element of solid angle times R_o^2

$$dI = S \cdot df\,(R_o^2) = \frac{cH^2R_o^2}{4\pi} do$$

$H \sim \frac{1}{R_o}$ so energy carried off is the same, for fixed $t - R_o/c$

$R = R_o - \vec{r} \cdot \hat{n}$

$$\vec{A}_\omega = \frac{e^{ikR_o}}{cR_o} \int j_\omega \exp(-ik \cdot r)\, dV$$

$H_\omega = i\vec{k} \times \vec{A}_\omega$

$E_\omega = \frac{ic}{\omega}(k \times A_\omega) \times k$

distinguish between fourier series and fourier integrals

- deal with fourier integral is the case of radiation accompanying collision of charged particles

- energy radiated at time of collision

$d\mathcal{E}_{n\omega}$ is the energy radiated through some element of solid angle for waves of frequency ω

$$\partial \mathcal{E}_{n\omega} = \frac{c}{2\pi} |H_\omega|^2 R_o^2\, do \left\{\frac{\partial \omega}{2\pi}\right\}$$ ← This is the part with frequency ω

intensity then

$$\partial I_n = \frac{c}{2\pi} |H_n|^2 R_o^2\, do$$

$$\vec{j}_\omega = \int_{-\infty}^{\infty} \vec{j} \exp(i\omega t)\, dt$$

going to point charge

$$A_\omega = \underbrace{\frac{\exp(ikR_o)}{cR_o} \int_{-\infty}^{\infty} q v(t) \exp(i(\omega t - k \cdot r_o(t)))\, dt}_{\text{Integrate by parts}}$$

$$A_\omega = \frac{\exp(ikR_o)}{cR_o} \int q \exp(i(\omega t - k \cdot \vec{r}_o))\, dr_o$$

then...

$$H_\omega = q \frac{i\omega \exp(ikR_o)}{c^2 R_o} \int \exp(i(\omega t - \vec{k} \cdot r_o))\, \hat{n} \times dr_o$$

expand over period of motion

$$H_n = q \frac{2\pi i n \exp(ikR_o)}{c^2 T^2 R_o} \oint \exp(i(n\omega_o t - k \cdot r_o))\, \hat{n} \times dr_o$$

Dipole Radiation

$\vec{r}\cdot\hat{n}/c$ for retarded potentials can be neglected in cases where charge distribution changes little during this time.

Let T denote order of magnitude of timescale of charge distribution change

- Radiation of system will contain periods of order T

a ~ order of magnitude of the dimensions of the system

$$\frac{a}{c} \ll T$$

cT is wavelength of radiation

$$a \ll cT$$

$$a \ll \lambda$$

dimensions must be small compared with radiated wavelength

Can read this off from

$$A_\omega = \int \frac{\exp(ikR_o)}{cR_o} j_\omega \exp(-ik\cdot r)\, dV$$

goes through values order a as $j = 0$ outside

Thus, $k\cdot r$ must be small and neglected for waves $ka \ll 1$

basically, because $\lambda \sim a$, $k\cdot r$ must be sizable to matter. Hence if k is small, ie λ large, it won't be true.

$$T \sim a/v, \quad \lambda \sim ca/v$$

so $v \ll c$, velocity of charges must not be relativistic

$$A = \frac{1}{cR_o}\int j_{t + \frac{r}{c}\cdot n - R_o}\, dV$$

this now is small

$$A = \frac{1}{cR_o}\int j_{t'}\, dV = \frac{1}{cR_o}\sum q\vec{v}$$

$$A = \frac{1}{cR_o}\frac{d}{dt}\sum q r = \frac{1}{cR_o}\dot{d}$$ ← dipole moment

$$H = \frac{1}{c^2R_o}\ddot{d}\times n$$

$$E = \frac{1}{c^2R_o}(\ddot{d}\times n)\times n$$

} dipole radiation. Only occurs if particles accelerate

$$dI = \frac{1}{4\pi c^3}(\ddot{d}\times n)^2\, do = \frac{\ddot{d}^2}{4\pi c^3}\sin^2\theta\, dO$$

where θ is angle between $\ddot{d}$ and $\hat{n}$

$$\int dI = \int_0^{\frac{\pi}{2}} \frac{\ddot{d}^2}{4\pi c^3}\sin^2\theta\, 2\pi\sin\theta\, d\theta = \frac{2}{3c^3}\ddot{d}^2$$

for a single charge

$$I = \frac{2q^2w^2}{3c^3}$$ ← accel of charge

suppose we have a closed system where charge to mass ratio is fixed

$$\ddot{d} = \sum q\ddot{r}_a = \sum \frac{q}{m} m_a r_a = \frac{q}{m}\sum m_a \ddot{r}_a$$

fixed charge to mass ratio

because $v \ll c$, this is fine to say this is the motion of the center of mass

$\ddot{d}$ is proportional to acceleration, which is 0 for closed systems.

$$d\mathcal{E}_\omega = \frac{4}{3c^3}(\ddot{d}_\omega)^2\frac{d\omega}{2\pi}$$

$$\ddot{d}_\omega e^{-i\omega t} = \frac{d^2}{dt^2}(d_\omega \exp(-i\omega t)) = -\omega^2 d_\omega \exp(-i\omega t)$$

$$d\mathcal{E}_\omega = \frac{4\omega^4}{3c^3}|d_\omega|^2\frac{d\omega}{2\pi}$$

Dipole Radiation during collisions

- usually, interested in radiation of a whole beam of particles

- if current density is 1, ie

  1 particle / unit Area / unit time

  number of particles which have impact parameter between ρ and $\rho + d\rho$ is $2\pi\rho\, d\rho$

- area of the ring

total radiation $= \int_0^\infty \underset{\substack{\uparrow \\ \text{single} \\ \text{particle} \\ \text{radiation}}}{\Delta \mathcal{E}} \; \underbrace{2\pi\rho\, d\rho}_{\text{number of particles}}$ ← integral over all ρ

$$\mathcal{X} = \int_0^\infty \Delta \mathcal{E}\, 2\pi\rho\, d\rho$$

determine radiation through solid angle per frequency similarly

average

$$dI = \frac{1}{4\pi c^3} \left(\ddot{d} \times n\right)^2 do$$

for all directions of $\ddot{d}$ in plane perpendicular to beam's motion

$$\left(\ddot{d} \times n\right)^2 = \underset{\substack{\uparrow \\ \text{constant}}}{(\ddot{d})^2} - \underbrace{(\ddot{d} \cdot n)^2}_{\text{change impacts this}}$$

because scattering is centrally symmetric

incident beam is parallel, scattering and radiation is axially symmetric.

The axis can be the x-axis

because of symmetry

$$\dot{d}_y = \dot{d}_z = 0$$

$\ddot{d}_x$ not impacted by averaging

$$\overline{\ddot{d}_x \ddot{d}_y} = 0 = \overline{\ddot{d}_x \ddot{d}_z}$$

$$\overline{\ddot{d}_y^{\,2}} = \overline{\ddot{d}_z^{\,2}} = \frac{1}{2}\left[\ddot{d}^2 - \ddot{d}_x^2\right]$$

$$\overline{\left(\ddot{d} \times n\right)^2} = \frac{1}{2}\left(\ddot{d}^2 + \ddot{d}_x^2\right) + \frac{1}{2}\left(\ddot{d}^2 - 3\ddot{d}_x^2\right)\cos^2\theta$$

$$d\mathcal{X}_n = \frac{do}{4\pi c^3}\left[\frac{2}{3}\int_0^\infty\int_{-\infty}^\infty \ddot{d}^2\, dt\, 2\pi\rho\, d\rho + \frac{1}{3}\iint\left(\ddot{d}^2 - 3\ddot{d}_x^2\right) dt\, 2\pi\rho\, d\rho\, \frac{3\cos^2\theta - 1}{2}\right]$$

$$\mathcal{X} = \frac{A}{c^3}$$

- symmetry about θ is specific to dipole radiation at first order v/c

- radiation in x-direction and $\hat{n}$-direction

- vector of E-field is $n \times (\ddot{d} \times n) = n(n \cdot \ddot{d}) - \ddot{d}$

  - projection onto xy-plane

    $$\left|\sin\theta\, \ddot{d}_z - \cos\theta\, \ddot{d}_y\right|$$

squaring $\vec{E}$ and averaging over all direction of vector $\ddot{d}$ in yz plane, we see the product of the projections of the field on the xy plane and perpendicular go to 0

Intensity of radiation with its electric vector perpendicular to xy plane

is $\frac{1}{2}\left(\ddot{d}^2 - \ddot{d}_x^2\right)$

effective radiation is

$$d\mathcal{X}_n^{\perp} = \frac{do}{8\pi c^3}\int_0^\infty\int_{-\infty}^\infty\left(\ddot{d}^2 - \ddot{d}_x^2\right) dt\, 2\pi\rho\, d\rho$$

$$d\mathcal{X}_n^{\parallel} + d\mathcal{X}_n^{\perp} = d\mathcal{X}_n$$

$$d\mathcal{X}_{n\,\omega} = \mathcal{F}\left(d\mathcal{X}_n\right)\frac{d\omega}{2\pi}$$

Radiations of Low Frequency in collisions

- Consider low frequency tail of the Bremsstrahlung
range of frequencies low compared to ω_0, the dominant mode of radiation

- velocities not guaranteed to be small

$$H_\omega = \int_{-\infty}^{\infty} H \exp(-i\omega t)\, dt$$

only dif from zero in a time interval of width $1/\omega_0$

Since $\omega \ll \omega_0$, $\omega t \ll 1$

So

$$H_\omega \approx \int_{-\infty}^{\infty} H\, dt$$

$$= \int_{-\infty}^{\infty} (\dot{A} \times n/c)\, dt$$

$$= \frac{1}{c} \underbrace{(A_2 - A_1)}_{\text{change in vector potential from collision}} \times \hat{n}$$

$$dE_{n,\omega} = \frac{R_0^2}{4\pi^2 c} \left[(\vec{A}_2 - \vec{A}_1) \times n \right]^2 do\, d\omega$$

$$dE_{n,\omega} = \frac{1}{4\pi^2 c^3} \left[\sum e \left[\frac{v_2 \times n}{1 - (\frac{1}{c}) \hat{n} \cdot \vec{v}_2} - \frac{v_1 \times \hat{n}}{1 (\frac{1}{c}) \hat{n} \cdot \vec{v}_1} \right] \right]^2 do\, d\omega$$

Velocity before
from expressions for retarded potentials
velo after

no ω dependence so everything is constant

$$dE_{n,\omega} = \frac{1}{4\pi^2 c^3} \left[\sum e (v_2 - v_1) \times n \right]^2$$

$$\frac{n \cdot v}{c} \ll 1$$

- Apply these to emissions of a new charged particle (β-particle from nucleus)

- instantaneous change in velocity from 0 to something. Because its instantaneous, all $\omega \ll \omega_0$ effectively, so assumptions are valid

Radiation In Case
of coulomb interaction

- consider relative motion of the particles.
- choose origin to be center of mass

$\vec{d} = e_1\vec{r}_1 + e_2\vec{r}_2$

$$= \frac{e_1 m_2 - e_2 m_1}{m_1 + m_2}\vec{r} = \underbrace{\mu}_{\text{reduced mass}}\left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)\vec{r}$$

$$1 + \varepsilon\cos\varphi = \frac{a(1-\varepsilon^2)}{r}$$

$$a = \frac{\alpha}{2|E|} \qquad \varepsilon = \sqrt{1 - \frac{2|E|M^2}{\mu\alpha^2}}$$ ← angular momentum $\mu r^2\dot{\varphi}$

time dependence can be expressed as

$$r = a(1 - \cos\varepsilon), \quad t = \sqrt{\frac{\mu a^3}{\alpha}}(\xi - \varepsilon\sin\xi)$$

$\xi \to 0$ to 2π is the period of one revolution.

$$T = \sqrt{\frac{\mu a^3}{\alpha}}(2\pi)$$

- Fourier components of the dipole moment

$$x = a(\cos\xi - \varepsilon) \qquad y = a\sqrt{1-\varepsilon^2}\sin\xi$$

$$\omega_0 t = \xi - \varepsilon\sin\xi$$

$$\omega_0 = 2\pi/T$$

Calc fourier components of velocity

$$\dot{x}_n = -i\omega_0 n x_n \qquad \dot{y}_n = -i\omega_0 n y_n$$

$$x_n = \frac{i}{\omega_0 n T}\int \dot{x}\exp(i\omega_0 n t)\,dt$$

$$\dot{x}\,dt = dx = -a\sin\xi\,d\xi$$

$$x_n = -\frac{ia}{2\pi n}\int_0^{2\pi}\exp(in(\xi - \varepsilon\sin\xi))\sin\xi\,d\xi$$

$$y_n = \frac{ia\sqrt{1-\varepsilon^2}}{2\pi n}\int_0^{2\pi}\exp(in(\xi - \varepsilon\sin\xi))\cos\xi\,d\xi$$

$$y_n = \frac{ia\sqrt{1-\varepsilon^2}}{2\pi n}\int_0^{2\pi}\exp(in(\xi - \varepsilon\sin\xi))\left(\cos\xi - \frac{1}{\varepsilon}\right)d\xi$$

$$+ \frac{ia\sqrt{1-\varepsilon^2}}{2\pi n\varepsilon}\int_0^{2\pi}\exp(in(\xi - \varepsilon\sin\xi))\,d\xi$$

$$u = \xi - \varepsilon\sin\xi$$

$$du = 1 - \varepsilon\cos\xi$$

$$\int_0^{2\pi}\exp(in(u))\frac{du}{\varepsilon}$$ ← symmetrically 0

$$y_n = \frac{ia\sqrt{1-\varepsilon^2}}{2\pi n\varepsilon}\int_0^{2\pi}\exp(in(\xi - \varepsilon\sin\xi))\,d\xi$$

from theory of Bessel functions

$$\frac{1}{2\pi}\int_0^{2\pi}\exp(i(n\xi - x\sin\xi))\,d\xi = \frac{1}{\pi}\int\cos(n\xi - x\sin\xi)\,d\xi$$

$$= J_n(x)$$

$$x_n = \frac{a}{n}J_n'(n\varepsilon) \qquad y_n = \frac{ia\sqrt{1-\varepsilon^2}}{n\varepsilon}J_n(n\varepsilon)$$

$$I_n = \frac{4\omega_0^4 n^4}{3c^3}\mu^2\left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)^2\left(|x_n|^2 + |y_n|^2\right)$$

$$I_n = \left(\frac{64 n^4 \varepsilon^4}{3c^3\alpha^2}\right)\left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)^2\left[J_n'^2(n\varepsilon) + \frac{(1-\varepsilon^2)}{\varepsilon^2}J_n^2(n\varepsilon)\right]$$

asymptotic formula of very high harmonics for motion in orbit close to parabola

$(n\to\infty,\ \varepsilon\to 1)$

$$J_n(n\varepsilon) = \frac{1}{\sqrt{\pi}}\left(\frac{2}{n}\right)^{1/3}\Phi\left[\left(\frac{n}{2}\right)^{2/3}(1-\varepsilon^2)\right]$$

$$1 - \varepsilon \ll 1, \quad n \gg 1$$

substituting $J_n(n\varepsilon)$ into I_n

$$I_n = \frac{64 \cdot 2^{2/3}}{3\pi}\left(\frac{n^{4/3}\varepsilon^4}{c^3\alpha^2}\right)\left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)^2\left[(1-\varepsilon^2)\Phi^2\left[\left(\frac{n}{2}\right)^{2/3}(1-\varepsilon^2)\right]\right.$$
$$\left. + \left(\frac{2}{n}\right)^{2/3}\Phi'^2\left[\left(\frac{n}{2}\right)^{2/3}(1-\varepsilon^2)\right]\right]$$

Can also be expressed in terms of MacDonald function K_ν

$$I_n = (1-\varepsilon^2)^2\frac{64}{9\pi^2}\frac{n^2\varepsilon^4}{c^3\alpha^2}\left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)^2\left[K_{1/3}^2\left[\frac{n}{3}(1-\varepsilon^2)^{3/2}\right]\right.$$
$$\left. + K_{2/3}^2\left[\frac{n}{3}(1-\varepsilon^2)^{3/2}\right]\right]$$

$$1 + \varepsilon\cos\varphi = \frac{a(\varepsilon^2 - 1)}{r}$$

$$a = \frac{\alpha}{2E} \qquad \varepsilon = \sqrt{1 + \frac{2EM^2}{\mu\alpha^2}}$$

$$r = a(\varepsilon\cosh\xi - 1), \quad t = \sqrt{\frac{\mu a^3}{\alpha}}(\varepsilon\sinh\xi - \xi)$$

$$\xi \in (-\infty, \infty)$$

$$x = a(\varepsilon - \cosh\xi) \qquad y = a\sqrt{\varepsilon^2 - 1}\sinh\xi$$

Fourier series

$$x_\omega = \frac{\pi a}{\omega}H_{i\nu}^{(1)\prime}(i\nu\varepsilon) \qquad y_\omega = \frac{-\pi a\sqrt{\varepsilon^2-1}}{\omega\varepsilon}H_{i\nu}^{(1)}(i\nu\varepsilon)$$

Hankel function of the first kind

$$\nu = \frac{\omega}{\sqrt{\frac{\alpha}{\mu a^3}}} = \frac{\omega\alpha}{\mu v_0^3}$$

v_0 is relative velocity of the particles at infinity

$$\int_{-\infty}^{\infty}\exp(p\xi - ix\sinh\xi)\,d\xi = i\pi H_p^{(1)}(ix)$$ ← from theory of Bessel functions, sub it in to get frequencies

$$d\mathcal{E}_\omega = \frac{4\omega^4\mu^2}{3c^3}\left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)^2\left(|x_\omega|^2 + |y_\omega|^2\right)\frac{d\omega}{2\pi}$$

$$d\mathcal{E}_\omega = \frac{\pi\mu^2\alpha^2\omega^3}{6c^3\varepsilon^2}\left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)^2\left[\left[H_{i\nu}^{(1)\prime}(i\nu\varepsilon)\right]^2 + \frac{\varepsilon^2-1}{\varepsilon^2}\left[H_{i\nu}^{(1)}(i\nu\varepsilon)\right]^2\right]d\omega$$

- of greater interest is the effective radiation during the scattering of parallel beams of particles

multiply $d\mathcal{E}_\omega$ by $2\pi\rho\,d\rho$

$$\int_0^\infty \rho\,d\rho \quad \to \int_1^\infty 2\pi a^2\varepsilon\,d\varepsilon$$

$$M = \mu\rho v_0 \qquad \mathcal{E} = \mu\frac{v_0^2}{2}$$

Integrate using the fact

$$z\left[Z_p'^2 + \left(\frac{p^2}{z^2} - 1\right)Z_p^2\right] = \frac{d}{dz}(z Z_p Z_p')$$

Z_p is solution to Bessel eqn of order p

$\varepsilon \to \infty$

$$d\mathcal{X}_\omega = \frac{4\pi^2\alpha^3\omega}{3c^3\mu v_0^5}\left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)^2\left|H_{i\nu}^{(1)}\right|\left|H_{i\nu}^{(1)}(i\nu)\right|d\omega$$

Consider high and low frequency cases

low frequencies $\nu \ll 1$, only region of large ξ is important

for large ξ, $\sinh\xi \gg \xi$ so

$$H_{i\nu}^{(1)}(i\nu) = -\frac{i}{\pi}\int_{-\infty}\exp(-i\nu\sinh\xi)\,d\xi = H_0^{(1)}(i\nu)$$

$$H_{i\nu}^{(1)\prime}(i\nu) \approx H_0^{(1)\prime}(i\nu)$$

$$iH_0^{(1)}(ix) \approx \frac{2}{\pi}\ln\left(\frac{2}{\gamma x}\right)$$

γ: $\exp(C)$, euler's constant

$$d\mathcal{X}_\omega = \frac{16\alpha^2}{3v_0^2c^3}\left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)^2\ln\left(\frac{2\mu v_0^3}{\gamma\omega\alpha}\right)d\omega$$

if $\nu \gg 1$ only ξ small matters

expand in powers of ξ

$$H_{i\nu}^{(1)}(i\nu) = -\frac{i}{\pi}\int_{-\infty}^{\infty}\exp\left(-\frac{i\nu\xi^3}{6}\right)d\xi = -\frac{2i}{\pi}\mathrm{Re}\left(\int_0^\infty\exp\left(-\frac{i\nu\xi^3}{6}\right)d\xi\right)$$

expressible as a Γ-function

$$H_{i\nu}^{(1)}(i\nu) \approx -\frac{i}{\pi\sqrt{3}}\left(\frac{6}{\nu}\right)^{1/3}\Gamma\left(\frac{1}{3}\right)$$

$$H_{i\nu}^{(1)\prime}(i\nu) \approx \frac{1}{\pi\sqrt{3}}\left(\frac{6}{\nu}\right)^{2/3}\Gamma\left(\frac{2}{3}\right)$$

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin\pi x}$$

at high frequencies

$$d\mathcal{X}_\omega = \frac{16\pi\alpha^2}{3^{3/2}v_0^2c^3}\left(\frac{e_1}{m_1} - \frac{e_2}{m_2}\right)^2 d\omega \qquad \omega \gg \frac{\mu v_0^3}{\alpha}$$

radiation from collision of two particles

$$-1 + \varepsilon\cos\varphi = \frac{a(\varepsilon^2 - 1)}{r}$$

$$x = a(\varepsilon + \cosh(\xi)) \qquad y = a\sqrt{\varepsilon^2 - 1}\sinh\xi$$

$$t = \sqrt{\frac{\mu a^3}{\alpha}}(\varepsilon\sinh\xi + \xi)$$

$$x_\omega = \frac{ia}{\omega}\int_{-\infty}^{\infty}\exp(i\nu(\varepsilon\sinh\xi + \xi))\sinh\xi\,d\xi$$

Quadrupole and Magnetic dipole radiation

consider radiation associated with succeeding terms in the expansion of the vector potential in powers of the ratio a/λ

a/λ small so these terms are small but important if dipole moment is 0, so dipole radiation is 0

$$\vec{A} = \frac{1}{cR_0}\int \vec{j}_{t' + \frac{r \cdot n}{c}}\, dV$$

$\vec{r} \cdot \frac{\hat{n}}{c}$ expand in

$$\vec{A} = \frac{1}{cR_0}\int j_{t'}\, dV + \frac{1}{c^2 R_0}\frac{\partial}{\partial t'}\int (\vec{r}\cdot\hat{n})\, j_{t'}\, dV$$

$j = \rho \vec{v}$ where $\rho = \sum e_i\, \delta(r - r_i)$

$$\vec{A} = \frac{\sum e_i \vec{v}_i}{cR_0} + \frac{1}{c^2 R_0}\frac{\partial}{\partial t}\left(\sum e_i v_i (r_i \cdot n)\right)$$

$$v(r\cdot n) = \frac{\partial r}{\partial t}(r \cdot n)$$

$$= \frac{1}{2}\frac{\partial}{\partial t} r(n\cdot r) + \frac{1}{2} v(n\cdot r) - \frac{1}{2} r(n\cdot v)$$

$$= \frac{1}{2}\frac{\partial}{\partial t}\vec{r}(n\cdot\vec{r}) + \frac{1}{2}(r\times v)\times n$$

$$A = \frac{\dot{d}}{cR_0} + \frac{1}{2c^2 R_0}\frac{\partial^2}{\partial t^2}\sum e r_i (n\cdot r) + \frac{1}{cR_0}(\dot{\vec{m}}\times n)$$

$$m = \frac{\sum e_i (r\times v)}{2c} \leftarrow \text{magnetic moment}$$

A can be rewritten as

$$\vec{A} = \frac{\dot{d}}{cR_0} + \frac{1}{6c^2 R_0}\frac{\partial^2}{\partial t^2}\sum e_i \underbrace{\left[3r(n\cdot r) - \hat{n} r^2\right]}_{n_\beta D_{\alpha\beta}} + \frac{1}{cR_0}\dot{\vec{m}}\times n$$

$$\vec{A} = \frac{\dot{d}}{cR_0} + \frac{1}{6c^2 R_0}\ddot{D} + \frac{1}{cR_0}\dot{m}\times n$$

Now...

$$\vec{H} = \frac{1}{c^2 R_0}\left[\ddot{d}\times n + \frac{1}{6c}\dddot{D}\times n + (\ddot{m}\times n)\times n\right]$$

$$\vec{E} = \frac{1}{c^2 R_0}\left[(\ddot{d}\times n)\times n + \frac{1}{6c}(\dddot{D}\times n)\times n + n\times\ddot{m}\right]$$

intensity dI in element of solid angle do integrated over solid angle

$$I = \underset{\text{dipole}}{\frac{2}{3c^3}\ddot{d}^2} + \underset{\text{quadrupole}}{\frac{1}{180c^5}\dddot{D}_{\alpha\beta}^2} + \underset{\text{magnetic moment}}{\frac{2}{3c^2}\ddot{m}^2}$$

The field of radiation at near distances

Formulas for dipole radiation were derived by us for the field at distances large compared with the wavelength

- Now assume system still big but distances are smaller

$$\vec{A} = \frac{1}{cR_0}\dot{d}$$ is valid, only relies on $R_0 \gg$ dim of system

However, $\vec{A}$ no longer plane-wave

$$\nabla\cdot\vec{A} + \frac{1}{c}\frac{\partial\varphi}{\partial t} = 0$$

$$\varphi = -\nabla\cdot\vec{d}/R_0$$

$\vec{d}$ at $t - R_0/c$

$$H = \frac{1}{c}\nabla\times\frac{\dot{\vec{d}}}{R_0}$$

$$E = \vec{\nabla}\,\nabla\cdot\frac{d}{R_0} - \frac{1}{c^2}\frac{\ddot{d}}{R_0}$$

$$\frac{d(t - R_0/c)}{R_0}$$ satisfies wave eqn

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\left(\frac{\vec{d}}{R_0}\right) = \Delta\left(\frac{\dot{d}}{R_0}\right)$$

$$\nabla\times(\nabla\times a) = \vec{\nabla}\,\nabla\cdot\vec{a} - \Delta a$$

$$\vec{E} = \nabla\times\left(\nabla\times\frac{d}{R_0}\right)$$

Now we need fourier components

$$d_\omega \exp\left(-i\omega t + i\omega\frac{R_0}{c}\right) = d_\omega\exp(-i\omega t + ikR_0)$$

$$H_\omega = -ik\,\nabla\times\left(d_\omega\frac{\exp(ikR_0)}{R_0}\right)$$

$$= ik\,d_\omega\times\nabla\frac{\exp(ikR_0)}{R_0}$$

$$= ik\,d_\omega\times\hat{n}\left(\frac{ik}{R_0} - \frac{1}{R_0^2}\right)\exp(ikR_0)$$

$$E_\omega = d_\omega\left(\frac{k^2}{R_0} + \frac{ik}{R_0^2} - \frac{1}{R^3}\right)\exp(ikR_0) + \hat{n}(\hat{n}\cdot d_\omega)\left(-\frac{k^2}{R_0} - \frac{3ik}{R_0^2} + \frac{3}{R_0^3}\right)\exp(ikR_0)$$

Radiation from a rapidly moving charge

Consider charge particle with $v \sim c$

- go to rest frame of particle

$$d\mathcal{E} = \frac{2e^2}{3c^3} w^2 dt$$

w is acceleration of particle

$$d\vec{P} = 0$$

$$dP^i = -\frac{2e^2}{3c^3} \frac{du^k}{ds} \frac{du_k}{ds} dx^i = -\frac{2e^2}{3c^3} \underbrace{\frac{du^k}{ds} \frac{du_k}{ds} u^i}_{\text{4-velocity}} ds$$

$$\Delta P^i = -\frac{2e^2}{3c^3} \int \frac{du^k}{ds} \frac{du_k}{ds} dx^i$$

$$mc \frac{du_k}{ds} = \frac{e}{c} F_{ke} u^l$$

$$\Delta p^i = -\frac{2e^4}{3m^2c^5} \int (F_{kl} u^l)(F^{km} u_m) dx^i$$

$$\Delta\mathcal{E} = \frac{2e^2}{3c^3} \int \frac{w^2 - \frac{(v \times w)^2}{c^2}}{\left(1 - \frac{v^2}{c^2}\right)^3} dt$$

$$\Delta\mathcal{E} = \frac{2e^4}{3m^2c^3} \int_{-\infty}^{\infty} \frac{[E + \frac{1}{c} v \times H] - \frac{1}{c^2}(E \cdot v)^2}{1 - v^2/c^2} dt$$

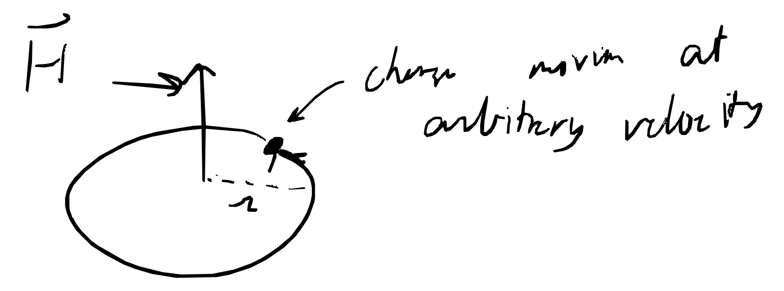
radiation usually dependent on energy of moving particle

$$E = \frac{e}{c^2 R} \frac{n \times ((n - \frac{v}{c}) \times w)}{\left(1 - \frac{(n \cdot v)}{c}\right)^3} \qquad H = n \times E$$

$$dI = \frac{e^2}{4\pi c^3} \left(\frac{2(n \cdot w)(v \cdot w)}{c\left(1 - \frac{v \cdot n}{c}\right)^5} + \frac{w^2}{\left(1 - \frac{v \cdot n}{c}\right)^4} - \frac{\left(1 - \frac{v^2}{c^2}\right)(n \cdot w)^2}{\left(1 - \frac{v \cdot n}{c}\right)^6} \right) d0$$

$$dt = \left(1 - \frac{n \cdot v}{c}\right) dt'$$

Synchotron radiation
(magnetic Bremsstrahlung)



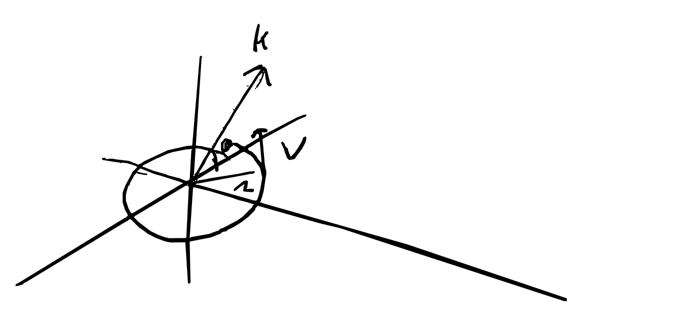
ω_H = cyclic frequency

$$r = \frac{mcv}{eH\sqrt{1-\frac{v^2}{c^2}}} \qquad \omega_H = \frac{v}{r} = \frac{eHv\sqrt{1-v^2/c^2}}{mc}$$

$$= eH\sqrt{1-v^2/c^2}/mc$$

$$I = \frac{2e^4H^2v^2}{3m^2c^5(1-v^2/c^2)}$$

$$I \propto p^2$$



angle between $\vec{k}$ and $\vec{v}$ $\cos\theta\cos\varphi$

$$\omega = \frac{e}{mc}\sqrt{1-\frac{v^2}{c^2}}\; v\times H$$

$$dI = do\,\frac{e^4H^2v^2}{8\pi^2m^2c^5}\left(1-\frac{v^2}{c^2}\right)\int_0^{2\pi}\frac{\left(1-\frac{v^2}{c^2}\right)\sin^2\theta + \left(\frac{v}{c}-\cos\theta\cos\varphi\right)^2}{\left(1-\frac{v}{c}\cos\theta\cos\varphi\right)^5}\,d\varphi$$

$$dI = do\,\frac{e^4H^2v^2\left(1-\frac{v^2}{c^2}\right)}{8\pi m^2c^5}\left[\frac{2+\frac{v^2}{c^2}\cos^2\theta}{\left(1-\frac{v^2}{c^2}\cos^2\theta\right)^{5/2}} - \frac{\left(1-\frac{v^2}{c^2}\right)\left(4+\frac{v^2}{c^2}\cos^2\theta\right)\cos^2\theta}{4\left(1-\frac{v^2}{c^2}\cos^2\theta\right)^{7/2}}\right]$$

$$\frac{(dI/do)_0}{(\partial I/do)_{\pi/2}} = \frac{4+3\frac{v^2}{c^2}}{8\left(1-\frac{v^2}{c^2}\right)^{5/2}}$$

Spectral distribution of radiation

$$\vec{A}_n = e\,\frac{\exp(ikR_0)}{cR_0T}\oint \exp(i(\omega_H nt - k\cdot r))\,d\vec{r}$$

$$x = r\cos(\omega_H t), \quad y = r\sin(\omega_H t), \quad \varphi = \omega_H t$$

$$\vec{k}\cdot\vec{r} = kr\cos\theta\sin\varphi = \frac{nv}{c}\cos\theta\sin\varphi$$

$$A_{xn} = -\frac{ev}{2\pi cR_0}\exp(ikR_0)\int_0^{2\pi}\exp\left(in\left(\varphi - \frac{v}{c}\cos\theta\sin\varphi\right)\right)\sin\varphi\,d\varphi$$

$$A_{xn} = -\frac{iev}{cR_0}\exp(ikR_0)\,J_n'(nv/c\,\cos\theta)$$

$$A_{yn} = -\frac{e}{R_0\cos\theta}\exp(ikR_0)\,J_n\left(\frac{nv}{c}\cos\theta\right)$$

$$\partial I_n = \frac{c}{2\pi}|H_n|^2R_0^2\,do = \frac{c}{2\pi}|k\times A_n|^2R_0^2\,do$$

$$\partial I_n = \frac{n^2e^4H^2}{2\pi c^3m^2}\left(1-\frac{v^2}{c^2}\right)\left[\tan^2\theta\cdot J_n\left(\frac{nv}{c}\cos\theta\right) + \frac{v^2}{c^2}J_n'^2\left(\frac{nv}{c}\cos\theta\right)\right]$$

$$I_n = \frac{2e^4H^2\left(1-\frac{v^2}{c^2}\right)}{m^2c^2v}\left[\frac{nv^2}{c^2}J_{2n}'\left(\frac{2nv}{c}\right) - n^2\left(1-\frac{v^2}{c^2}\right)\int_0^{v/c}J_n(2n\xi)\,d\xi\right]$$

Intensity of synchotron radiation is proportional to square of particle energy $\mathcal{E}$

$$I = \left(\frac{2e^4H^2}{3m^2c^3}\right)\left(\frac{\mathcal{E}}{mc^2}\right)^2$$

radiation mostly concentrated in the plane of orbits

$$\Delta\theta \sim \sqrt{1-\frac{v^2}{c^2}} = \frac{mc^2}{\mathcal{E}}$$

$$J_n(2n\xi) = \frac{1}{\sqrt{\pi n^{1/3}}}\,\Phi\left(n^{2/3}(1-\xi^2)\right)$$

$$I_n = -\frac{2e^4H^2\frac{mc^2}{\mathcal{E}}u^{1/2}}{\sqrt{\pi m^2c^3}}\left\{\Phi'(u) + \frac{u}{2}\int_n^{\infty}\Phi(u)\,du\right\}$$

$$u_n = n^{2/3}\left(\frac{mc^2}{\mathcal{E}}\right)^2$$

$$\Phi'(0) = -0.4587$$

$$I_n \approx 0.52\,\frac{e^4H^2}{m^2c^3}\left(\frac{mc^2}{\mathcal{E}}\right)^2u^{1/3} \qquad 1 \ll n \ll \left(\frac{\mathcal{E}}{mc^2}\right)^3$$

$$I_n = \frac{e^4H^2\left(\frac{mc^2}{\mathcal{E}}\right)^{5/2}n^{1/2}}{2\sqrt{\pi m^2c^3}}\exp\left(-\frac{2}{3}n\left(\frac{mc^2}{\mathcal{E}}\right)^3\right) \qquad n \gg \left(\frac{\mathcal{E}}{mc^2}\right)^3$$

$$n \sim \left(\frac{\mathcal{E}}{mc^2}\right)^3$$

$$\omega \sim \omega_H\left(\frac{\mathcal{E}}{mc^2}\right)^3 = \frac{eH}{mc}\left(\frac{\mathcal{E}}{mc^2}\right)^2$$

ω are large compared to ω_H

between neighboring frequencies

$$dI = I_n\,dn = I_n\frac{d\omega}{\omega_H}$$

$$dI = d\omega\,\frac{\sqrt{3}e^3H}{2\pi mc^2}F\left(\frac{\omega}{\omega_c}\right) \qquad F(\xi) = \xi\int_{\xi}^{\infty}K_{5/3}(\xi)\,d\xi$$

$$\omega_c = \frac{3eH}{2mc}\left(\frac{\mathcal{E}}{mc^2}\right)^2$$

$$V_{\parallel} = v\cos\chi$$

Radiation Damping

$$\varphi = \int \frac{1}{R}\rho\left(t-\frac{R}{c}\right)dV$$

$$\varphi = \varphi^{(0)} + \varphi^{(1)} + \varphi^{(2)} + \varphi^{(3)} + \dots$$

$$= \int \frac{1}{R}\rho(t)\,dV - \frac{\partial}{\partial t}\int \frac{R}{c}\frac{1}{R}\rho(t)\,dV + \frac{\partial^2}{\partial t^2}\int \frac{R^2}{2c^2 R}\rho(t)\,dV$$

$$- \frac{\partial^3}{\partial t^3}\int \frac{R^3}{6c^3}\frac{1}{R}\rho(t)\,dV + \dots$$

$$A^{(2)} = -\frac{1}{c^2}\frac{\partial}{\partial t}\int j\,dV$$

$$\varphi' = \varphi - \frac{1}{c}\frac{\partial f}{\partial t} \qquad A' = A + \nabla f$$

$$f = -\frac{1}{6c^2}\frac{\partial^2}{\partial t^2}\int R^2\rho\,dV$$

$$A'^{(2)} = -\frac{1}{c^2}\frac{\partial}{\partial t}\int j\,dV - \frac{1}{6c^2}\frac{\partial^2}{\partial t^2}\nabla\int R^2\rho\,dV$$

$$= -\frac{1}{c^2}\frac{\partial}{\partial t}\int j\,dV - \frac{1}{3c^2}\frac{\partial^2}{\partial t^2}\int \vec{R}\rho\,dV$$

$$-\frac{1}{c^2}\sum e\dot{v}$$

$$\vec{R} = \vec{R}_0 - \vec{r} \qquad \dot{R} = -\dot{\vec{r}} = -\vec{v}$$

$$A'^{(2)} = -\frac{2}{3c^2}\sum e\dot{v}$$

$H = \nabla \times A'^{(2)} = 0$ because $A'^{(2)}$ doesn't have explicit dependence of coordinates

$$\vec{E} = -\left(\frac{1}{c}\right)\dot{A}'^{(2)}$$

$$E = \frac{2}{3c^3}\dddot{d}$$

$$\vec{F} = e\vec{E} = \frac{2e}{3c^3}\dddot{d}$$

Work done is

$$\vec{F}\cdot\vec{v} = \frac{2}{3c^3}\dddot{d}\cdot\sum e\vec{v} = \frac{2}{3c^3}\dddot{d}\cdot\dot{d} = \frac{2}{3c^3}\frac{\partial}{\partial t}(\ddot{d}\cdot\dot{d}) - \frac{2}{3c^3}(\ddot{d})^2$$

averaging over time means first term vanishes (we are considering stationary motion)

$$\overline{\vec{F}\cdot\vec{v}} = -\frac{2}{3c^3}\overline{(\ddot{d})^2} \leftarrow \text{radiation damping}$$

$$M = \sum r\times p \qquad \dot{M} = \sum r\times\dot{p} \quad (\dot{r}\times p \propto v\times v = 0)$$

$$\dot{p} = f$$

$$\dot{M} = r\times f = \frac{2}{3c^3}\sum e\vec{r}\times\dddot{d} = \frac{2}{3c^3}\,d\times\dddot{d}$$

$$\overline{\frac{dM}{dt}} = -\frac{2}{3c^3}\overline{\dot{d}\times\ddot{d}}$$

$$\vec{f} = \frac{2e^2}{3c^3}\ddot{v} \leftarrow \text{for a single charge}$$

unsatisfactory, because this permits velocity to grow exponentially

this requires the subtraction of 2 ∞ so that doesn't work

other issues in classical EM

$$m\dot{v} = e\vec{E} + \frac{e}{c}(v\times H) + \frac{2}{3}\frac{e^2}{c^3}\ddot{v}$$

$$\ddot{v} = \frac{e\dot{E}}{m} + \frac{e}{mc}(\dot{v}\times H)$$

consider a system of reference in which the charge is at rest at a given moment

$$\dot{v} = \frac{e}{m}\vec{E}$$

$$\ddot{v} = \frac{e\dot{E}}{m} + \frac{e^2}{m^2c}(E\times H)$$

$$\vec{f} = \frac{2e^3}{3mc^3}\dot{E} + \frac{2e^4}{3m^2c^4}(E\times H)$$

ω is the frequency of motion then $\dot{E}$ is proportional to ωE

first term order $\frac{e^2\omega}{mc^3}E$ second order $\frac{e^4}{m^2c^4}EH$

$$\frac{e^2\omega}{mc^3} \ll 1 \quad \text{to avoid absurd accelerations}$$

$$\lambda \sim \frac{c}{\omega}$$

$$\lambda \gg \frac{e^2}{mc^2}$$ } what is the meaning of this limit

Radiation damping in relativistic case

$$mc\frac{du^i}{ds} = \frac{e}{c}F^{ik}u_k + g^i$$

$$g^i \to \vec{F}/c \quad \text{if} \quad v \ll c$$

$\frac{2e^2}{3c}\frac{d^2u^i}{ds^2}$ does this for the three components

However $g^i u_i$ must be 0 so we must add something such that if $v \to 0$, 3-space components $\to 0$

$$\frac{2e^2}{3c} \Big/ \frac{\partial^2 u^i}{\partial s^2} \qquad \propto u^i$$

Hence

$$g^i = \frac{2e^2}{3c}\left(\frac{d^2u^i}{ds^2} - u^i u^k \frac{d^2u_k}{ds^2}\right)$$

$$\frac{du^i}{ds} = \frac{e}{mc^2}F^{ik}u_k$$

$$\frac{d^2u^i}{ds^2} = \frac{e}{mc^2}\frac{\partial F^{ik}}{\partial x^l}u_k u^l + \frac{e^2}{m^2c^4}F^{ik}F_{kl}u^l$$

In making substitutions

$$g^i = \frac{2e^3}{mc^2}\frac{\partial F^{ik}}{\partial x^l}u_k u^l - \frac{2e^4}{3m^2c^5}F^{il}F_{kl}u^k + \frac{2e^4}{3m^2c^5}\left(F_{kl}u^l\right)\left(F^{km}u_m\right)u^i$$

four-force g^i must coincide with radiated four-momentum ΔP^i

$$-\int g^i ds = \frac{2e^2}{3c}\int u^k u^i \frac{\partial^2 u_k}{\partial s^2}ds$$

$$= \frac{2e^2}{3c^2}\left(u^k u^i \frac{\partial u_k}{\partial s}\Big|_0^\infty - \int \frac{du^k}{ds}\frac{du_k}{ds}\underbrace{u^i ds}_{dx^i}\right)$$

$$= -\frac{2e^2}{3c^2}\int \frac{du^k}{ds}\frac{du_k}{ds}dx^i$$

relate 4-vector g^i to three-dimensional

$$\vec{F} = \frac{2e^4}{3m^2c^5}\left(F_{kl}u^l\right)\left(F^{km}u_m\right)\hat{n} \qquad \hat{n} \parallel \vec{v}$$

$$F_x = -\frac{2e^4}{3m^2c^4}\left(\frac{(E_y - H_z)^2 + (E_z + H_y)^2}{1-\frac{v^2}{c^2}}\right)$$

Go back to the original situation. How do we prevent a blow up?

F is the order of magnitude of the field in system of reference where particle is going at velocity v

$F/\sqrt{1-\frac{v^2}{c^2}}$ in rest frame

$$\frac{e^3F}{m^2c^4\sqrt{1-\frac{v^2}{c^2}}} \ll 1$$

However external force eF ratio to damping could be

$$\frac{e^3F}{m^2c^4\sqrt{1-\frac{v^2}{c^2}}}$$

Loss of kinetic energy per unit length of the path

$$\frac{\partial \mathcal{E}_{kin}}{\partial x} = -k(x)\,\mathcal{E}_{kin}^2$$

$$\frac{1}{\mathcal{E}_{kin}} = \frac{1}{\mathcal{E}_0} + \int_{-\infty}^{x} k(x)\,dx'$$

$\mathcal{E}_0$ represents initial energy $\to \infty$

we set $\mathcal{E}_{crit}$

$$\mathcal{E}_{crit}^{-1} = \frac{2}{3m^2c^4}\left(\frac{e^2}{mc^2}\right)^2\int_{-\infty}^{\infty}\left[(E_y - H_z)^2 + (E_z + H_y)^2\right]dx$$

Spectral Resolution of the Radiation in the Ultrarelativistic case

$$\Delta\theta \sim \sqrt{1-\frac{v^2}{c^2}}$$

$\Delta\theta$ and α angle of deflection in passing through external field

α can be calculated as follows

- change in the transverse direction of the momentum is of order magnitude

$$eF \cdot t$$

$\to a/c$

dimensions of the field

$$p = \frac{mc}{\sqrt{1-v^2/c^2}}$$

$$\alpha \sim \frac{eFa/c}{\frac{mc}{\sqrt{1-\frac{v^2}{c^2}}}} = \frac{eFa\sqrt{1-v^2/c^2}}{mc^2}$$

$$\frac{\alpha}{\Delta\theta} \sim \frac{eFa}{mc^2}$$

$eF_a \gg mc^2$

total deflection is large compared to $\Delta\theta$

radiation in a given direction occurs mainly from portion of trajectory where velocity parallel to that direction

$$\omega \sim \frac{eF}{mc\left(1-\frac{v^2}{c^2}\right)}$$ is where radiation concentrated

$eF_a \ll mc^2$

total angle of deflection small compared to $\Delta\theta$

radiation is directed mainly into narrow angular range $\Delta\theta$ around the direction of motion

$$E_\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} E e^{i\omega t}\,dt$$

$$t' = t - \frac{R(t')}{c}$$

at large distances from a particle moving with an almost constant velocity $\vec{v}$

$$t' \cong t - \frac{R_0}{c} + \frac{\hat{n}\cdot r(t')}{c} \cong t - \frac{R_0}{c} + \frac{1}{c}\,n\cdot v t'$$

$$\vec{r} = \vec{r}(t') = \vec{v}t$$

$$dt = dt'\left(1-\frac{n\cdot v}{c}\right)$$

$$E_\omega = \frac{e}{c^2}\left(\frac{\exp(ikR_0)}{R_0\left(1-\frac{n\cdot v}{c}\right)^2}\right)\int_{-\infty}^{\infty}\hat{n}\times\left(\left(\hat{n}-\frac{\vec{v}}{c}\right)\times w(t')\right)\exp\left(i\omega t'\left(1-\frac{n\cdot v}{c}\right)\right)dt'$$

$$\omega' = \omega\left(1-\frac{n\cdot v}{c}\right)$$

$$E_\omega = \frac{e}{c^2}\frac{\exp(ikR_0)}{R_0}\left(\frac{\omega}{\omega'}\right)^2\hat{n}\times\left(\left(\hat{n}-\frac{\vec{v}}{c}\right)\times\vec{w}_{\omega'}\right)$$

energy radiated through solid angle

$$d\mathcal{E}_{\hat{n}\omega} = \frac{e^2}{2\pi c^3}\left(\frac{\omega}{\omega'}\right)^4\left|\hat{n}\times\left(\left(\hat{n}-\frac{\vec{v}}{c}\right)\times\vec{w}_{\omega'}\right)\right|^2 do\,\frac{d\omega}{2\pi}$$

$$af \sim \frac{e^4F^2c}{m^2c^4\left(1-v^2/c^2\right)}$$

dim of system

Lorentz friction

$$af \sim \frac{mc^2}{\sqrt{1-v^2/c^2}}$$

$$a \sim \frac{m^3c^6}{e^4F^2}\sqrt{1-v^2/c^2}$$

Scattering by free charges

If EM waves hit a system of charges, then all the charges will move.

If all the charges move, then they produce radiation

- scattering of the original wave

- scattering best characterized by the ratio of the amount of energy emitted in a given direction per unit time

ratio has dimensions of area

- called effective scattering cross section

$$d\sigma = \frac{\overline{dI}}{\bar{S}}$$

$\int d\sigma$ = total scattering cross sections

$\vec{E} = \vec{E}_0 \cos(\vec{k}\cdot\vec{r} - \omega t + \alpha)$

force acting on charge $e\vec{E}$ is the dominant force

$$\vec{E} = \vec{E}_0 \cos(\omega t - \alpha)$$

$$m\ddot{r} = e\vec{E}$$

$$\ddot{d} = \frac{e^2}{m} E$$

$$dI = \frac{e^4}{4\pi m^2 c^3} (E \times n')^2 \, do$$

$$S = \frac{c}{4\pi} E^2$$

$$d\sigma = \left(\frac{e^2}{mc^2}\right)^2 \sin^2\theta \, do$$

$$\sigma = \frac{8\pi}{3} \left(\frac{e^2}{mc^2}\right)^2$$

$$\sin^2\theta = 1 - (n\cdot e)^2 = 1 - n_\alpha n_\beta \overline{e_\alpha e_\beta}$$

$$\overline{e_\alpha e_\beta} = \frac{1}{2}\left(\delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2}\right)$$

$$\sin^2\theta = \frac{1}{2}\left(1 + \frac{(n\cdot k)^2}{k^2}\right) = \frac{1}{2}(1 + \cos^2(\theta))$$

θ angle between scattering and incident waves

$$d\sigma = \frac{1}{2}\left(\frac{e^2}{mc^2}\right)^2 (1 + \cos^2(\theta))\, do$$

Scattering leads to the occurrence of a certain force on the scattering particle

energy lost is $c\bar{W}\sigma$, where $\bar{W}$ is the average density

incident wave loses moment $\frac{\mathcal{E}}{c} = \frac{c\bar{W}\sigma}{c} = \bar{W}\sigma$

$\bar{F} = \sigma \bar{W} \hat{n}_0 \leftarrow$ force acting on particle per unit time is prop to momentum absorbed

$$\bar{F} = \frac{Ze^4}{3m^2c^4} \bar{E}^2 n_0$$

$$= \left(\frac{8\pi}{3}\right) \frac{e^2}{m^2c^4} \frac{\bar{E}^2}{4\pi} \hat{n}_0$$

Scattering of low-frequency waves

The scattering of a wave by a system of charges differs from the scattering of a single charge

- because if internal motion exists, frequency of scattered waves different from incident
- scattering is due to incoherent scattering, where frequency from internal motion impacts incident wave

$$\vec{j} = j_0 + j'$$

j_0 is current density without wave

j' is current due to wave

Can also write

$$\vec{A} = \vec{A}_0 + \vec{A}'$$

consider scattering of wave where

$\omega <<$ internal frequencies

$$A' = \frac{1}{cR_0} \int j'\left(t - \frac{R_0}{c} + \frac{r \cdot n}{c}\right) dt$$

expansion is fine as long as

$$\frac{r \cdot n}{c} << \frac{1}{\omega}$$

$$\vec{H}' = \frac{1}{c^2 R_0}\left(\ddot{d}' \times n' + (\ddot{m}' \times n') \times n'\right)$$

where d' and m' are dipole and magnetic moment produced by the radiation

$$\ddot{d}_\omega' = -\omega^2 d_\omega' \qquad \ddot{m}_\omega' = -\omega^2 m_\omega' \quad \text{satisfies wave eqn}$$

$$\vec{H}' = \frac{\omega^2}{c^2 R_0}\left(n' \times d_\omega' + n' \times (m_\omega' \times n)\right)$$

$$\vec{H}_\omega' \approx \frac{\omega^2}{c^2 R_0}(n' \times d_\omega') \quad \text{if } v << c, \ m_\omega \text{ drops out}$$

If $\Sigma e = 0$, then for $\omega \to 0$, d_ω' and m_ω' approach constant limits

$\omega << \frac{v}{a}$, scattered wave field $\sim \omega^2$

as d_ω, m_ω independent

Scattering of high frequency waves

Consider case where $\omega \gg \frac{v}{a}$

assume velocities $v \ll c$

- period of interaction of incident wave so short that you can treat the charges impact

v' acquired by charges in field is

$$m\frac{dv'}{dt} = e\vec{E} = eE_0 \exp(-i(\omega t - k\cdot r))$$

$\frac{\omega}{c}\hat{n}$ so small compared to first term

then

$$v' = -\frac{e}{i\omega m} E_0 \exp(-i(\omega t - k\cdot r))$$

$$A' = \frac{1}{cR_0}\sum (ev')_{t-\frac{R_0}{c}+\frac{r\cdot n'}{c}}$$

$$A' = -\frac{1}{icR_0\omega}\exp(-i\omega(t-\tfrac{R_0}{c}))E_0\sum\frac{e^2}{m}\exp(-iq\cdot r)$$

where $q = \vec{k}' - \vec{k}$

change in wave vector from incident to scattered

$$q = 2\frac{\omega}{c}\sin\left(\frac{\theta}{2}\right)$$

$$H' = \frac{E_0 \times n'}{c^2}\exp(-i\omega(t-\tfrac{R_0}{c}))\frac{e^2}{m}\sum\exp(-i\vec{q}\cdot\vec{r})$$

$$c\frac{|H'|^2}{8\pi}R_0^2\,do = \frac{e^4}{8\pi c^3 m^2}(n'\times E_0)^2\left|\sum\exp(-iq\cdot r)\right|^2 do$$

energy flow into solid angle element

divide by energy flux of incident wave $c|E_0|^2/8\pi$

$$d\sigma = \left(\frac{e^2}{mc^2}\right)^2\overline{\left|\sum\exp(-iq\cdot r)\right|^2}\sin^2\theta\, do$$

angle between incident and scattered

$$d\sigma = \left(\frac{Ze^2}{mc^2}\right)^2\sin^2\theta\, do$$

if $\lambda \ll ac/v$ and $\lambda \gg a$, since

$q\cdot r \ll 1$, $\exp(-iq\cdot r) \sim \exp(0) \sim 1$

so its just Z^2 where Z is number of electrons

$\lambda \ll a$

we must now care about

$$\underbrace{\exp(-i\,q\cdot(r_1 - r_2))}$$

rapidly oscillating, so average value is 0

so we are just left with variance, $\sim Z$ the number of electrons

$$d\sigma = Z\left(\frac{e^2}{mc^2}\right)^2\sin^2\theta\, do$$

to get coherent scattering isolate frequencies ω

part of the field obtained if we average sum $\sum\exp(-iq\cdot r)$ over time

$$d\sigma_{coh} = \left(\frac{e^2}{mc^2}\right)^2\left|\overline{\sum\exp(-iq\cdot r)}\right|\sin^2\theta\, do$$

$$e\,\overline{\sum\exp(-iq\cdot r)} = \int\rho(r)\exp(-iq\cdot r)\,dV = \rho_q$$

$$d\sigma_{coh} = \left(\frac{Zc^2}{mc^2}\right)\sin^2\theta\, d\theta$$

Particles in a Gravitational Field

Gravitational fields in non-relativistic mechanics

- grav fields assume all bodies move in same manner, independently of mass, provided initial condition held fixed

- property enables establishment of analogy between motion of a body in a gravitational field and the motion of a body not in external field but considered from non-inertial reference system

  - non-inertial reference frame equivalent to grav field in inertial reference system

- principle of equivalence

- however this is not actually true because as distance to mass $\to \infty$, grav field is supposed to be zero, but non-inertial frame will not

- The fields to which noninertial systems are equivalent vanish as soon as we transform to inertial frame

- this can be resolved by only considering a small region of space over which the field is roughly constant

Thus we can write Lagrangian as

$$L = \frac{mv^2}{2} - m\varphi$$ ← gravitational potential

$$\dot{v} = -\nabla\varphi$$

Gravitational Field in relativistic mechanics

$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$

In non-inertial frame, the above is not true

$x = x' \cos \Omega t - y' \sin \Omega t$

$y = x' \sin \Omega t + y' \cos \Omega t \qquad z = z'$

$ds^2 = [c^2 - \Omega^2(x'^2 + y'^2)]dt^2$
$\quad - dx'^2 - dy'^2 - dz'^2 + 2\Omega y' dx' dt - 2\Omega x' dy' dt$

$ds^2 = g_{ik} dx^i dx^k$ where g_{ik} are certain functions of space coordinates x^1, x^2, x^3

g_{ik} represents the geometric properties of curvilinear system

In inertial system, $\underbrace{g^{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}}_{\text{galilean metric (or Minkowski)}}$

gravitational field is a change in g_{ik}

- theory of gravitational fields based on relativity is called general theory of relativity?

- actual gravitational fields cannot be reduced to galilean metric
- This induces curvature in g_{ik}!

  $|g| < 0$ always for real space time
- in gravitational field, space-time has non euclidean geometry

consider

$K, \quad K'$

K ↑ inertial, K' ↖ rotation

circle in x, y plane in K is circle in x', y' plane in K'

- Measure length of circle and its diameter with yardstick in K frame

  suppose circumference is π, diameter $= 1$

In K' ... diameter doesn't lorentz contract, but circumference does

- In general case of an arbitrary gravitational field
  metric varies as function of time

- Now SR doesn't exist because you can't guarantee a frame where bodies are at rest relative to one another

- laws of nature must be written in 4-form

Curvilinear Coordinates

4-dimensional geometry in arbitrary curvilinear coordinates

$$x^i = f^i(x'^0, x'^1, x'^2, x'^3)$$

where f^i are certain functions

$$dx^i = \frac{\partial f^i}{\partial x'^k} dx'^k$$

$$A^i = \frac{\partial x^i}{\partial x'^k} A'^k \leftarrow \text{contravariant 4-vector}$$

a scalar, φ, transforms as

reversal-ish

$$\frac{\partial \varphi}{\partial x^i} = \frac{\partial \varphi}{\partial x'^k} \frac{\partial x'^k}{\partial x^i}$$

$$A_i = \frac{\partial x'^k}{\partial x^i} A'_k \leftarrow \text{covariant-4 vector}$$

$A^{ik} \leftarrow$ contravariant rank two tensor

$$A^{ik} = \frac{\partial x^i}{\partial x'^l} \frac{\partial x^k}{\partial x'^m} A'^{lm}$$

$A_{ik} \leftarrow$ covariant rank two tensor

$$A_{ik} = \frac{\partial x'^l}{\partial x^i} \frac{\partial x'^m}{\partial x^k} A'_{lm}$$

$$A^i_k = \frac{\partial x^i}{\partial x'^l} \frac{\partial x'^m}{\partial x^k} A'^l_m$$

$$A^i B_i = \frac{\cancel{\partial x^i}}{\partial x'^l} \frac{\partial x'^m}{\cancel{\partial x^i}} A'^l B'_m$$

$$= \frac{\partial x'^m}{\partial x'^l} A'^l B'_m = A'^l B'_l$$

transforms

$B'_m \to B'_l$

so $A^i B_i$ is invariant

$$A^k \delta^i_k = A^i$$

$$A^k \delta^i_k = A^i$$

$$ds^2 = g_{ik} dx^i dx^k$$

$$g_{ik} = g_{ki}$$

metric tensor

If

unit/identity tensor

$$A_{ik} B^{kl} = \delta^l_i$$

$$g_{ik} g^{kl} = \delta^l_i$$

transformation to one coordinate basis and then back

ie a legal g_{ik} must have an inverse, and its inverse must be the reciprocal and must be the contra-covariant dual

$$A^i = g^{ik} A_k \qquad A_i = g_{ik} A^k$$

Consider

is this right? shouldn't it be A_{lk}?

$$A^i_k = g^{il} A_{ik}$$

$$A^{ik} = g^{il} g^{km} A_{lm}$$

$$E^{iklm} = \frac{\partial x^i}{\partial x'^p} \frac{\partial x^k}{\partial x'^r} \frac{\partial x^l}{\partial x'^s} \frac{\partial x^m}{\partial x'^t} e^{prst}$$

$$= J e^{prst}$$

$$g^{ik} = \frac{\partial x^i}{\partial x'^l} \frac{\partial x^k}{\partial x'^m} g^{lm(0)}$$

$$|g^{ik}| = \frac{1}{g} \qquad |g^{lm(0)}| = -1$$

$$\frac{1}{g} = -1 \left| \frac{\partial x^i}{\partial x'^l} \frac{\partial x^k}{\partial x'^m} \right|$$

$$\frac{1}{g} = -J^2$$

$$J = \frac{1}{\sqrt{-g}}$$

$$E^{iklm} = \frac{1}{\sqrt{-g}} e^{iklm}$$

$$d\Omega' \to \sqrt{-g}\, d\Omega$$

$$dS_i \to d\Omega \frac{\partial}{\partial x^i}$$

Distances and time intervals

- triplet of space coordinates x^1, x^2, x^3 can be any quantities defining the position of bodies in space

x^0 - an arbitrary running clock

- Consider proper time τ
  - consider two infinitesimally separated events, occurring at the same point in space

$$ds = c\,d\tau$$

$$ds^2 = c^2 d\tau^2$$
$$= g_{00}(dx^0)^2$$

$$d\tau = \frac{1}{c}\sqrt{g_{00}}\,dx^0$$

$$\tau = \frac{1}{c}\int \sqrt{g_{00}}\,dx^0$$

$g_{00} > 0$ based on this

dl of spatial distance

Impossible in GR to set $dx^0 = 0$ because in a grav field, the proper time at different points in space have different dependences on x^0

to find dl

light signal directed from some point B to point A infinitesimally close and back.

The time observed at point B is twice the distance between B and A

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta + 2g_{0\alpha}dx^0 dx^\alpha + g_{00}(dx^0)^2$$

sum α, β over indices 1, 2, 3

solve $ds^2 = 0$ with respect to dx^0

$$dx^{0} = \frac{-g_{0\alpha}dx^\alpha \pm \sqrt{(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00})dx^\alpha dx^\beta}}{g_{00}}$$

$$dx^{0(2)} - dx^{0(1)} = \frac{2}{g_{00}}\sqrt{(g_{0\alpha}g_{0\beta} - g_{\alpha\beta}g_{00})dx^\alpha dx^\beta}$$

$$\frac{\sqrt{g_{00}}}{c}\cdot\left(dx^{(0)2} - dx^{(0)1}\right) = d\tau$$

$$dl = \frac{c}{2}\,d\tau$$

$\frac{\tau}{2}$ because there and back

$$dl^2 = \underbrace{\left(-g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}}\right)}_{\gamma_{\alpha\beta}} dx^\alpha dx^\beta$$

$$dl^2 = \gamma_{\alpha\beta}\,dx^\alpha dx^\beta$$

meaningless to integrate dl because world line itself is a choice, so proper distance not well defined in GR

for $\alpha, \beta = 1, 2, 3$

$$g^{ik}g_{kl} = \delta^i_l$$

$$-g^{\alpha\beta}\gamma_{\beta\gamma} = \delta^\alpha_\gamma$$

so

$$g^{\alpha\beta} = -\gamma^{\alpha\beta}$$

$$-g = g_{00}\gamma$$

$$g_\alpha = -\frac{g_{0\alpha}}{g_{00}}$$

$$g^\alpha = \gamma^{\alpha\beta}g_\beta = -g^{0\alpha}$$

$$g^{00} = \frac{1}{g_{00}} - g_\alpha g^\alpha$$

- simultaneity

$$x^0 + \Delta x^0 = x^0 + \frac{1}{2}\left(dx^{0(2)} + dx^{0(1)}\right)$$

$$\Delta x^0 = -\frac{g_{0\alpha}dx^\alpha}{g_{00}} \equiv g_\alpha dx^\alpha$$

Covariant Differentiation

In galilean coordinates

dA_i is a 4-vector

$\frac{\partial A_i}{\partial x^k}$ is a tensor

$A_i = \frac{\partial x'^k}{\partial x^i} A'_k$ in curvilinear

$$dA_i = \frac{\partial x'^k}{\partial x^i} dA'_k + A'_k d\left(\frac{\partial x'^k}{\partial x^i}\right)$$

$$= \frac{\partial x'^k}{\partial x^i} dA'_k + A'_k \frac{\partial^2 x'^k}{\partial x^i \partial x^l} dx^l$$

if this is zero

$$dA_i = \frac{\partial x'^k}{\partial x^i} dA'_k$$

Hence, dA_i does not transform like a vector in general

$\frac{\partial A_i}{\partial x^k}$ ← transform from galilean to curvilinear coordinates

In curvilinear coordinates, in order to obtain a differential of a vector which behaves like a vector

two vectors must be subtracted from one another <u>and</u> be co-located

- the translation of the two vectors must respect properties of difference of two

- under parallel translation of a vector, its components in galilean coordinates do not change

- curvilinear coordinates, then in general, the components of vector will change under translation

- thus, we consider this transport

  - consider arbitrary covariant vector

    if value at point x^i is A^i

    then at neighboring point $x^i + dx^i$ $A^i + dA^i$

    subject vector to parallel displacement the change in vector, call it δA^i

$$DA^i = \underbrace{dA^i}_{\text{infinitesimal transportation}} - \underbrace{\delta A^i}_{\text{parallel } \cancel{\text{transport}} \text{ displacement}}$$

$$\delta A^i = -\Gamma^i_{kl} A^k dx^l$$

for galilean coordinates, this is zero

Γ^i_{kl} is the collection of connection coefficients, Christoffel symbols

$$\Gamma_{i,kl} = g_{im} \Gamma^m_{kl}$$

$$\Gamma^{m}_{kl} = g^{im} \Gamma_{i,kl}$$

under parallel displacement, $A^i B_i$ unchanged

$$\delta(A^i B_i) = 0$$

$$\delta A^i B_i + A^i \delta B_i = 0$$

$$B^i \delta A_i = -A_i \delta B^i = \Gamma^i_{kl} B^k A_i dx^l$$

rewriting indices

$$B^i \delta A_i = \Gamma^k_{il} B^i A_k dx^l$$

Since B^i arbitrary

$$\delta A_i = \Gamma^k_{il} A_k dx^l$$

$$DA^i = \left(\frac{\partial A^i}{\partial x^l} + \Gamma^i_{kl} A^k\right) dx^l$$

$$DA_i = \underbrace{\left(\frac{\partial A_i}{\partial x^l} - \Gamma^k_{il} A_k\right)}_{\text{called the covariant derivative}} dx^l$$

denote covariant derivative as

$$A^i_{;l} = \frac{\partial A^i}{\partial x^l} + \Gamma^i_{kl} A^k$$

$$A_{i;l} = \frac{\partial A_i}{\partial x^l} - \Gamma^k_{il} A_k$$

$$\delta(A^i B^k) = A^i \delta B^k + B^k \delta A^i = -A^i \Gamma^k_{lm} B^l dx^m - B^k \Gamma^i_{lm} A^l dx^m$$

By virtue of linearity

$$\delta A^{ik} = -(A^{im}\Gamma^k_{ml} + A^{mk}\Gamma^i_{ml}) dx^l$$

$$DA^{ik} = dA^{ik} - \delta A^{ik} = A^{ik}_{;l} dx^l$$

$$A^{ik}_{;l} = \frac{\partial A^{ik}}{\partial x^l} + \Gamma^i_{ml} A^{mk} + \Gamma^k_{ml} A^{im}$$

$$A^i_{k;l} = \frac{\partial A^i_k}{\partial x^l} - \Gamma^m_{kl} A^i_m + \Gamma^i_{ml} A^m_k$$

$$A_{ik;l} = \frac{\partial A_{ik}}{\partial x^l} - \Gamma^m_{il} A_{mk} - \Gamma^m_{kl} A_{im}$$

$$A_i^{;k} = g^{kl} A_{i;l} \qquad A^{i;k} = g^{kl} A^i_{;l}$$

$$\Gamma^i_{kl} = \Gamma'^m_{np} \frac{\partial x^i}{\partial x'^m} \frac{\partial x'^n}{\partial x^k} \frac{\partial x'^p}{\partial x^l} + \frac{\partial^2 x'^m}{\partial x^k \partial x^l} \frac{\partial x^i}{\partial x'^m}$$

transforms like a tensor only for linear transformations

$$S^i_{kl} = \Gamma^i_{kl} - \Gamma^i_{lk} \leftarrow \text{symmetrized}$$

$$S^i_{kl} = S'^m_{np} \frac{\partial x^i}{\partial x'^m} \frac{\partial x'^n}{\partial x^k} \frac{\partial x'^p}{\partial x^l}$$

curvature tensor

must be zero

must be a galilean coordinate system at which Γ^i_{kl}, and hence $S^i_{kl} = 0$

S^i_{kl} is tensor, so if it vanishes in one frame, it vanishes everywhere

$$x'^i = x^i + \frac{1}{2}(\Gamma^i_{kl})_0 x^k x^l$$

$$\left(\frac{\partial^2 x'^m}{\partial x^l \partial x^k} \frac{\partial x^i}{\partial x'^m}\right)_0 = (\Gamma^i_{kl})_0$$

$$\left(\frac{\partial x'^i}{\partial x^k}\right)_0 = \delta^i_k$$

Relation of Christoffel symbols to the metric tensor

$$DA_i = g_{ik} DA^k$$

note that

$$A_i = g_{ik} A^k$$

$$DA_i = D(g_{ik} A^k) = g_{ik} DA^k + A^k Dg_{ik}$$

Hence

$$Dg_{ik} = 0$$

$$g_{ik;l} = 0$$

$$g_{ik;l} = \frac{\partial g_{ik}}{\partial x^l} - g_{mk}\Gamma^m_{il} - g_{im}\Gamma^m_{kl} = 0$$

$$= \frac{\partial g_{ik}}{\partial x^l} - \Gamma_{k,il} - \Gamma_{i,kl} = 0$$

$$\frac{\partial g_{ik}}{\partial x^l} = \Gamma_{k,il} + \Gamma_{i,kl}$$

$$\frac{\partial g_{li}}{\partial x^k} = \Gamma_{i,kl} + \Gamma_{l,ik}$$

$$-\frac{\partial g_{kl}}{\partial x^i} = -\Gamma_{l,ki} - \Gamma_{k,li}$$

$$\Gamma_{i,kl} = \frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^l} + \frac{\partial g_{il}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^i}\right)$$

$$\Gamma^i_{kl} = g^{im}\Gamma_{m,kl}$$

$$= \frac{g^{im}}{2}\left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m}\right)$$

consider $\quad dg = g g^{ik} dg_{ik}$

$$\Gamma^i_{ki} = \frac{1}{2} g^{im}\left(\frac{\partial g_{mk}}{\partial x^i} + \frac{\partial g_{mi}}{\partial x^k} - \frac{\partial g_{ki}}{\partial x^m}\right)$$

$$\Gamma^i_{ki} = \frac{g^{im}}{2}\frac{\partial g_{im}}{\partial x^k}$$

$$\Gamma^i_{ki} = \frac{1}{2g}\frac{\partial g}{\partial x^k} = \frac{\partial \ln\sqrt{-g}}{\partial x^k}$$

$$g^{kl}\Gamma^i_{kl} = g^{kl} g^{im}\left(\frac{\partial g_{mk}}{\partial x^l} - \frac{1}{2}\frac{\partial g_{kl}}{\partial x^m}\right)$$

$$g^{kl}\Gamma^i_{kl} = -\frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\, g^{ik})}{\partial x^k}$$

$$g_{il}\frac{\partial g^{lk}}{\partial x^m} = -g^{ik}\frac{\partial g_{il}}{\partial x^m}$$

$$\frac{\partial g^{ik}}{\partial x^l} = -\Gamma^i_{ml} g^{mk} - \Gamma^k_{ml} g^{im}$$

$$A^i_{;i} = \frac{\partial A^i}{\partial x^i} + \Gamma^i_{li} A^l = \frac{\partial A^i}{\partial x^i} + A^l \frac{\partial \ln\sqrt{-g}}{\partial x^l}$$

$$A^i_{;i} = \frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}A^i)}{\partial x^i}$$

$$A^{ik}_{;k} = \frac{\partial A^{ik}}{\partial x^k} + \Gamma^i_{mk}A^{mk} + \Gamma^k_{mk}A^{im}$$

$$A^{mk} = -A^{km}$$

$$\Gamma^i_{mk}A^{mk} = -\Gamma^i_{km}A^{km} = 0$$

$$A^{ik}_{;k} = \frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}A^{ik})}{\partial x^k}$$

$$A^k_{i;k} = \frac{\partial A^k_i}{\partial x^k} + \Gamma^k_{lk}A^l_i - \Gamma^l_{ik}A^k_l = \frac{1}{\sqrt{-g}}\frac{\partial(A^k_i\sqrt{-g})}{\partial x^k} - \Gamma^l_{ki}A^k_l$$

$$= \frac{1}{\sqrt{-g}}\frac{\partial(\sqrt{-g}A^k_i)}{\partial x^k} - \frac{1}{2}\frac{\partial g_{kl}}{\partial x^i}A^{kl}$$

$$A_{i;k} - A_{k;i} = \frac{\partial A_i}{\partial x^k} - \frac{\partial A_k}{\partial x^i}$$

Consider

$$\frac{\partial^2 \varphi}{\partial x_i \partial x^i} \qquad \varphi^{;i}_{;i}$$

$$\varphi_{;i} = \frac{\partial \varphi}{\partial x^i}$$

$$\varphi^{;i} = g^{ik}\frac{\partial \varphi}{\partial x^k}$$

$$\varphi^{;i}_{;i} = -\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}\left(\sqrt{-g}\, g^{ik}\frac{\partial \varphi}{\partial x^k}\right)$$

$$\oint A^i\sqrt{-g}\, dS_i = \int A^i_{;i}\sqrt{-g}\, d\Omega$$

Motion of a particle in a gravitational field

$$\delta S = -mc\,\delta\int ds = 0$$

In curvilinear coordinates this requires a geodesic

- not guaranteed to be a straight line

$$\frac{du^i}{ds} = 0 \qquad \Big\}\ \text{if galilean frame exists}$$

$$du^i = 0 \qquad u^i = \frac{dx^i}{ds}$$

because grav field there exists reference frame

$$Du^i = 0 \qquad \text{in general frame}$$

$$du^i + \Gamma^i_{kl}u^k dx^l = 0$$

$$\frac{du^i}{ds} + \Gamma^i_{kl}\frac{dx^k}{ds}\frac{dx^l}{ds} = 0$$

$$\frac{d^2u^i}{ds} + \underbrace{\Gamma^i_{kl}u^k u^l}_{} = 0$$

$$-m\Gamma^i_{kl}u^k u^l = \text{4-force}$$

$$p^i = mcu^i$$

$$p_i p^i = m^2c^2$$

$$-\frac{\partial S}{\partial x^i} = p_i \qquad g^{ik}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^k} - m^2c^2$$

geodesic not described by that eqn if it's light, as $ds \to 0$ in that limit

$$k^i = dx^i/d\lambda$$

$$dk^i = 0 \qquad \text{in SR}$$

$$Dk^i = 0 \qquad \text{in GR}$$

$$\frac{dk^i}{d\lambda} + \Gamma^i_{kl}k^k k^l = 0$$

$$k_i k^i = 0$$

sub $k^i = \dfrac{d\psi}{dk^i}$

$$g^{ik}\frac{d\psi}{dx^i}\frac{d\psi}{dx^k} = 0$$

$$L = -mc^2 + \frac{mv^2}{2} - m\varphi$$

$$S = \int L\,dt = -mc\int \underbrace{\left(c - \frac{v^2}{2c} + \frac{\varphi}{c}\right)dt}_{ds}$$

$$ds^2 = (c^2 + 2\varphi)dt^2 - dr^2 \qquad v\,dt = dr$$

$$g_{00} = 1 + \frac{2\varphi}{c^2}$$

The constant Gravitational field

- gravitational field is said to be constant if all the components in some frame of reference are independent of x^0
  - then x^0 becomes world time
    - not completely unique

      add x^0 to arbitrary function of space coordinates, g_{ik} will still not contain x^0

- Only field produced by single body can be constant
  - motion emerges if it is two bodies

- if body producing field is held fixed, we should have time reversal not impact ds
  - $g_{0\alpha} = 0$

- However, body doesn't have to be at rest for field to be constant
  - spinning axially symmetric, eg
  - the $g_{0\alpha}$ will change if time is flipped

$$\tau = \frac{1}{c}\sqrt{g_{00}}\,x^0$$

$\varphi < 0$

$\tau = \frac{x^0}{c}\left(1 + \frac{\varphi}{c^2}\right)$ if grav field is weak

gravity makes time move slowly

- If one of two identical clocks in grav field, the clock in the field will be slow

$$dl^2 = -g_{\alpha\beta}\,dx^\alpha dx^\beta$$

$$\Delta x^0 = -\int \frac{g_{0\alpha}\,dx^\alpha}{g_{00}}$$

$$\Delta x^0 = -\oint \frac{g_{0\alpha}\,dx^\alpha}{g_{00}}$$

$$\frac{\partial\psi}{\partial\tau} = \frac{\partial\psi}{\partial x^0}\frac{\partial x^0}{\partial\tau} = \frac{\partial\psi}{\partial x^0}\frac{c}{\sqrt{g_{00}}}$$

$$\omega = -\frac{\partial\psi}{\partial\tau}, \qquad \omega_0 = -c\frac{\partial\psi}{\partial x^0}$$

$$\omega = \frac{\omega_0}{\sqrt{g_{00}}}$$

$\varphi < 0$

$\omega = \omega_0\left(1 - \frac{\varphi}{c^2}\right)$ in weak grav fields

light frequency increases as it approaches, decreases as it leaves?

$$\frac{\omega}{1 - \frac{\varphi_1}{c^2}}\left(1 - \frac{\varphi_2}{c^2}\right) = \omega\left(1 + \frac{\varphi_1 - \varphi_2}{c^2}\right)$$

$\Delta\omega = \frac{\varphi_1 - \varphi_2}{c^2}\omega \leftarrow$ red shift, $\Delta\omega < 0$ or blue shift $\Delta\omega > 0$

$$-c\frac{\partial S}{\partial x^0}$$

$$\mathcal{E}_0 = mc^2 g_{00}\frac{dx^0}{ds} = mc^2 g_{00}\frac{dx^0}{\sqrt{g_{00}(dx^0)^2 - dl^2}}$$

$$v = \frac{dl}{d\tau} = \frac{c\,dl}{\sqrt{g_{00}}\,dx^0}$$

$$\mathcal{E}_0 = \frac{mc^2\sqrt{g_{00}}}{\sqrt{1 - v^2/c^2}}$$

$$(x^0 + dx^0) - \left(x^0 - \frac{g_{0\alpha}}{g_{00}}dx^\alpha\right) = dx^0 + \frac{g_{0\alpha}}{g_{00}}dx^\alpha$$

$$v^\alpha = \frac{c\,dx^\alpha}{\underset{h}{\sqrt{g_{00}}}\left(dx^0 + \underset{-g_\alpha}{\frac{g_{0\alpha}}{g_{00}}}dx^\alpha\right)}$$

$$v_\alpha = \gamma_{\alpha\beta}v^\beta \qquad v^2 = v_\alpha v^\alpha$$

$$ds^2 = h(dx^0 - g_\alpha dx^\alpha)^2\left(1 - \frac{v^2}{c^2}\right)$$

$$u^i = \frac{dx^i}{ds}$$

$$u^\alpha = \frac{v^\alpha}{c\sqrt{1 - \frac{v^2}{c^2}}} \qquad u^0 = \frac{1}{\sqrt{g_{00}}\sqrt{1 - \frac{v^2}{c^2}}} + \frac{g_\alpha v^\alpha}{c\sqrt{1 - v^2/c^2}}$$

$$\mathcal{E}_0 = mc^2 g_{0i}u^i = mc^2 h(u^0 - g_\alpha u^\alpha)$$

Rotation

As a special case, consider uniformly rotating reference system

$$ds^2 = c^2 dt'^2 - dr'^2 \; r'^2 d\varphi'^2 - dz'^2$$

cylindrical system

r, φ, z

assume z, z' coincide

$$\varphi' = \varphi + \Omega t, \quad r' = r, \; z' = z$$

$$ds^2 = c^2 dt^2 - dr^2 - r^2 (d\varphi + \Omega \, dt)^2 - dz^2$$

$$= c^2 dt^2 - dr^2 - r^2 d\varphi^2 - 2\Omega r^2 d\varphi dt - \Omega^2 r^2 dt^2 - dz^2$$

rotating system only valid for

$$c^2 - \Omega^2 r^2 > 0$$

else, g_{00} becomes negative, which isn't allowed

$$\Delta t = -\oint \frac{g_{0\alpha}}{g_{00}} dx^\alpha = \frac{1}{c^2} \oint \frac{\Omega r^2 d\varphi}{1 - \frac{\Omega^2 r^2}{c^2}}$$

← how much clocks differ by if they rotate vs not move

if $\frac{\Omega r}{c} \ll 1$

$$\Delta t = \frac{\Omega}{c^2} \int r^2 d\varphi = \pm \frac{2\Omega}{c^2} S$$

where S is the area of the contour we rotate over (in the plane perpendicular to axis of rotation)

- let us calculate to terms order v/c the time that elapses getting from point A to point B

$$t = \frac{L}{c} \pm \frac{2\Omega}{c^2} S$$

$$\frac{L}{t} \approx V_{light} \approx c + 2\Omega \frac{S}{L}$$

The equations of electrodynamics in the presence of a gravitational field

- EM field equations of the special theory of relativity can be easily generalized so they are applicable to arbitrary 4-d curvilinear coordinates

$$F_{ik} = \frac{\partial A_k}{\partial x^i} - \frac{\partial A_i}{\partial x^k} \quad \text{in SR}$$

$$F_{ik} = A_{k;i} - A_{i;k} \quad \text{in GR}$$

First pair of maxwell equations

$$\frac{\partial F_{ik}}{\partial x^l} + \frac{\partial F_{li}}{\partial x^k} + \frac{\partial F_{kl}}{\partial x^i} = 0$$

- in order to write second pair of equations

dx^1, dx^2, dx^3 is $\sqrt{\gamma}\, dV$ spatial volume element where γ determinant of spatial component

$$de = \rho\sqrt{\gamma}\, dV$$

$$de\, dx^i = \rho\sqrt{\gamma}\, dV\, dx^i$$

$$= \frac{\rho}{\sqrt{g_{00}}}\sqrt{-g}\, d\Omega\, \frac{dx^i}{dx^0}$$

$$-g = \gamma g_{00}$$

$$j^i = \frac{\rho c}{\sqrt{g_{00}}}\frac{dx^i}{dx^0}$$

$$\rho = \sum_a \frac{e_a}{\sqrt{\gamma}}\delta(r - r_a) \qquad j^l = \sum_a \frac{e_a c}{\sqrt{-g}}\delta(r - r_a)\frac{dx^i}{dx^0}$$

$$j^l{}_{;l} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^l}\left(\sqrt{-g}\, j^l\right) = 0$$

$$F^{ik}{}_{;k} = \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^k}\left(\sqrt{-g}\, F^{ik}\right) = -\frac{4\pi}{c} j$$

$$mc\frac{Du^i}{ds} = mc\left(\frac{du^i}{ds} + \Gamma^i_{kl} u^k u^l\right) = \frac{e}{c} F^{ik} u_k$$

Gravitational Field Equations

The curvature tensor

- parallel displacement of a vector
- infinitesimal displacement has no vector components changed

$x^i = x^i(s)$ where s is arc length measure

$u^i = \frac{dx^i}{ds}$ unit vector tangent to curve

If curve is geodesic

$Du^i = 0$

u^i is parallel displacement from x^i on geodesic to $x^i + dx^i$ on geodesic then it coincides with $u^i + du^i$

- when tangent to geodesic is transported along geodesic it remains tangent
- on the other hand, in parallel displacement, angle between the vector and its displacement is supposed to be 0
- hence, vector unchanged when displaced along geodesic

- this permits the following conclusion: parallel displacement over a closed contour in curvilinear coordinates doesn't bring things back to the same direction

$$\Delta A_k = \oint \delta A_k = \oint \Gamma^i_{kl} A_i dx^l$$

- Note that values A_i at points inside contour are not unique - depend on the path to get there
  - however, this non-uniqueness is secondorder

$$\frac{\partial A_i}{\partial x^l} = \Gamma^n_{il} A_n$$

$$\Delta A_k = \frac{1}{2}\left[\frac{\partial(\Gamma^i_{km} A_i)}{\partial x^l} - \frac{\partial(\Gamma^i_{kl} A_i)}{\partial x^m}\right] \Delta f^{lm}$$

Stokes Theorem

$$= \frac{1}{2}\left[\frac{\partial \Gamma^i_{km}}{\partial x^l} A_i - \frac{\partial \Gamma^i_{kl}}{\partial x^m} A_i + \Gamma^i_{km} \frac{\partial A_i}{\partial x^l} - \Gamma^i_{kl} \frac{\partial A_i}{\partial x^m}\right] \Delta f^{lm}$$

$$\Delta A_k = \frac{1}{2} R^i_{klm} A_i \Delta f^{lm}$$

$$R^i_{klm} = \frac{\partial \Gamma^i_{km}}{\partial x^l} - \frac{\partial \Gamma^i_{kl}}{\partial x^m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl}$$

Riemann Tensor or curvature tensor

under parallel displacement

$$0 = \Delta(A^k B_k) = \Delta A^k B_k + A^k \Delta B_k$$

$$= A^k \underbrace{\left[B_i \tfrac{1}{2} \Gamma^i_{klm} \Delta f^{lm}\right]}_{B_k \frac{1}{2} \Gamma^k_{ilm} \Delta f^{lm}} + \Delta A^k B_k$$

$$= B_k \underbrace{\left[\tfrac{1}{2} A^i \Gamma^k_{ilm} \Delta f^{lm} + \Delta A^k\right]}$$

arbitrary, so must be zero

$$\Delta A^k = \frac{1}{2} A^i \Gamma^k_{ilm} \Delta f^{lm}$$

$$A_{i;k;l} - A_{i;l;k} = A_m R^m_{ikl}$$

$$A^i_{;k;l} - A^i_{;l;k} = -A^m R^i_{mkl}$$

$$A_{ik;l;m} - A_{ik;m;l} = A_{in} R^n_{klm} + A_{nk} R^n_{ilm}$$

Properties of the curvature tensor

$$R_{iklm} = g_{in} R^{n}_{\ klm}$$

$$R_{iklm} = \frac{1}{2}\left(\frac{\partial^2 g_{im}}{\partial x^k \partial x^l} + \frac{\partial^2 g_{kl}}{\partial x^i \partial x^m} - \frac{\partial^2 g_{il}}{\partial x^k \partial x^m} - \frac{\partial^2 g_{km}}{\partial x^i \partial x^l}\right)$$
$$+ g_{np}\left(\Gamma^{n}_{kl}\Gamma^{p}_{im} - \Gamma^{n}_{km}\Gamma^{p}_{il}\right)$$

$$R_{iklm} = -R_{kilm} = -R_{ikml}$$

$$R_{iklm} = R_{lmik}$$

antisymmetric in pairs of i, k and in l, m

symmetric about transposition ik and l, m

$i = k$, or $l = m$, then $R_{iklm} = 0$

Bianchi identity

$$R^{n}_{\ ikl;m} + R^{n}_{\ imk;l} + R^{n}_{\ ilm;k} = 0$$

$$R^{n}_{\ ikl;m} = \frac{\partial R^{n}_{\ ikl}}{\partial x^m} = \frac{\partial^2 \Gamma^{n}_{il}}{\partial x^m \partial x^k} - \frac{\partial^2 \Gamma^{n}_{ik}}{\partial x^m \partial x^l}$$

$$\Gamma^{i}_{kl} = 0$$

$$R_{ik} = g^{lm} R_{limk} = R^{l}_{\ ilk}$$

↑

symmetric, Ricci tensor

$$R = g^{ik} R_{ik} = g^{il} g^{km} R_{iklm}$$

$$R^{l}_{\ m;l} = \frac{1}{2}\frac{\partial R}{\partial x^m}$$

definition of curvature tensor as given by the formulas above applies to a space of an arbitrary number of dimensions

- consider two-dimensional case

P_{abcd} ← curvature tensor

γ_{ab}

$$P = \frac{2P_{1212}}{\gamma} \qquad \gamma \equiv |\gamma_{ab}| = \gamma_{11}\gamma_{22} - (\gamma_{12})^2$$

$$\frac{P}{2} = K = \frac{1}{\rho_1 \rho_2}$$

$\underbrace{\rho_1, \rho_2}$ principal radii of curvature

same sign if centers of curvature on same side, opposite if opposite side of surface

curvature of 3-dimensional space

$P_{\alpha\beta\gamma\delta}$ and metric tensor $\gamma_{\alpha\beta}$

↑

6 independent components

$$\underbrace{P_{\alpha\beta} = g^{\gamma\delta} P_{\gamma\alpha\delta\beta}}$$

choose system of coordinates that is cartesian at a particular point

from suitable rotation, $P_{\alpha\beta}$ → principal axes.

4-space

6 possible coordinate pairs and

$\binom{6}{2}$ possible pairs of pairs, but Bianchi identity constrains some so 20 total components

6 of them vanished if galilean at a particular point

if $R_{ik} = 0$ constrain to 10, and with good choice of transformations, down to 4 coordinates

$$C_{iklm} = R_{iklm} - \frac{1}{2}R_{il}g_{km} + \frac{1}{2}R_{im}g_{kl} + \frac{1}{2}R_{kl}g_{im}$$
$$- \frac{1}{2}R_{km}g_{il} + \frac{1}{6}R(g_{il}g_{km} - g_{im}g_{kl})$$

Weyl tensor

assume metric in galilean form

$R_{ik} = 0$

$$A_{\alpha\beta} = R_{0\alpha0\beta} \qquad C_{\alpha\beta} = \frac{1}{4}e_{\alpha\gamma\delta}e_{\beta\lambda\mu}R_{\gamma\delta\lambda\mu}$$

$$B_{\alpha\beta} = \frac{1}{2}e_{\alpha\gamma\delta}R_{0\beta\gamma\delta}$$

$$B_{11} = R_{0123} \quad B_{21} = R_{0131} \quad B_{31} = R_{0112} \quad C_{11} = R_{2323}$$

$$R_{km} = g^{il}R_{iklm} = 0$$

$$A_{\alpha\alpha} = 0, \quad B_{\alpha\beta} = B_{\beta\alpha}, \quad A_{\alpha\beta} = -C_{\alpha\beta}$$

$$D_{\alpha\beta} = \frac{1}{2}(A_{\alpha\beta} + 2i B_{\alpha\beta} - C_{\alpha\beta}) = A_{\alpha\beta} + iB_{\alpha\beta}$$

$$D_{\alpha\beta} n_\beta = \lambda n_\alpha$$

↑

invariants of curvature tensor

since $D_{\alpha\alpha} = 0$, $\lambda_1 + \lambda_2 + \lambda_3 = 0$

- canonical Petrov types

-3 independent eigenvectors,

$$A_{\alpha\beta} = \begin{pmatrix} \lambda^{(1)'} & 0 & 0 \\ 0 & \lambda^{(2)'} & 0 \\ 0 & 0 & -\lambda^{(1)'}-\lambda^{(2)'} \end{pmatrix}$$

$$B_{\alpha\beta} = \begin{pmatrix} \lambda^{(1)''} & 0 & 0 \\ 0 & \lambda^{(2)''} & 0 \\ 0 & 0 & -\lambda^{(1)''}-\lambda^{(2)''} \end{pmatrix}$$

$$I_1 = \frac{1}{48}\left(R_{iklm}R^{iklm} - i R_{iklm}\overset{*}{R}{}^{iklm}\right)$$

$$I_2 = \frac{1}{96}\left(R_{iklm}R^{lmpr}R_{pr}^{\ \ ik} + i R_{iklm}R^{lmpr}\overset{*}{R}_{pr}^{\ \ ik}\right)$$

$$\overset{*}{R}_{iklm} = \frac{1}{2}E_{ikpr}R^{pr}_{\ \ lm}$$

The action function for the gravitational field

S_g ← action for gravitational field

$\int G\sqrt{-g}\,d\Omega$ taken over all space

any over time coordinate x^0

- to evaluate integral, start with fact that G cannot depend on derivatives no higher than second derivatives

- Field equations obtained by varying the action, G must contain derivatives of g_{ik} no higher than first order

- thus G depends on g_{ik} and Γ^i_{kl}

- impossible to construct invariant with g_{ik}, Γ^i_{kl} alone.
  - This is because with the right coordinate sends $\Gamma^i_{kl} \to 0$
- consider R scalar curvature

$\int R\sqrt{-g}\,d\Omega$ ← but depends on second derivatives

$$\int R\sqrt{-g}\,d\Omega = \int G\sqrt{-g}\,d\Omega + \underbrace{\int \frac{\partial(\sqrt{-g}\,w^i)}{\partial x^i}\,d\Omega}_{0 \text{ on variation}}$$

$$\delta\int R\sqrt{-g}\,d\Omega = \delta\int G\sqrt{-g}\,d\Omega$$

$$\delta S_g = -\frac{c^3}{16\pi k}\,\delta\int G\sqrt{-g}\,d\Omega = -\frac{c^3}{16\pi k}\,\delta\int R\sqrt{-g}\,d\Omega$$

where k is universal constant

$$\sqrt{-g}\,R = \sqrt{-g}\,g^{ik}R_{ik}$$

$$= \sqrt{-g}\left(g^{ik}\frac{\partial\Gamma^l_{ik}}{\partial x^l} - g^{ik}\frac{\partial\Gamma^l_{il}}{\partial x^k} + g^{ik}\Gamma^l_{ik}\Gamma^m_{lm} - g^{ik}\Gamma^m_{il}\Gamma^l_{km}\right)$$

$$\sqrt{-g}\,g^{ik}\frac{\partial\Gamma^l_{ik}}{\partial x^l} = \frac{\partial}{\partial x^l}\left(\sqrt{-g}\,g^{ik}\Gamma^l_{ik}\right) - \Gamma^l_{ik}\frac{\partial}{\partial x^l}\left(\sqrt{-g}\,g^{ik}\right)$$

$$G = g^{ik}\left(\Gamma^m_{il}\Gamma^l_{km} - \Gamma^l_{ik}\Gamma^m_{lm}\right)$$

$$g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial x^0} = 0$$

$$-\frac{1}{4}g^{\alpha\beta}g^{\gamma\delta}g^{00}\frac{\partial g_{\alpha\gamma}}{\partial x^0}\frac{\partial g_{\beta\delta}}{\partial x^0}$$

The Energy-Momentum tensor

$$S = \frac{1}{c}\int \Lambda \sqrt{-g}\, d\Omega$$

add suitable terms of form

$\frac{\partial}{\partial x^l}\psi_{ikl}$ where $\psi_{ikl} = -\psi_{ilk}$ to symmetrize

$$g'^{ik}(x'^l) = g^{lm}(x^l)\frac{\partial x'^i}{\partial x^l}\frac{\partial x'^k}{\partial x^m}$$

$$= g^{lm}\left(\delta^i_l + \frac{\partial \xi^i}{\partial x^l}\right)\left(\delta^k_m + \frac{\partial \xi^k}{\partial x^m}\right)$$

$$\approx g^{ik}x^l + g^{im}\frac{\partial \xi^k}{\partial x^m} + g^{kl}\frac{\partial \xi^i}{\partial x^l}$$

tensor g'^{ik} is a function of the x'^l while the tensor g^{ik} is function x^l

$$g'^{ik}x^l = (g'^{ik})x^l - \xi^l\frac{\partial g^{ik}}{\partial x^l} + g^{il}\frac{\partial \xi^k}{\partial x^l} + g^{kl}\frac{\partial \xi^i}{\partial x^l}$$

$\xi^{i;k} + \xi^{k;i}$

$g'^{ik} = g^{ik} + \delta g^{ik}$

$\delta g^{ik} = \xi^{i;k} + \xi^{k;i}$

$$\delta S = \frac{1}{c}\int\left(\frac{\partial\sqrt{-g}\,\Lambda}{\partial g^{ik}}\delta g^{ik} + \underbrace{\frac{\partial\sqrt{-g}}{\partial\frac{\partial g^{ik}}{\partial x^l}}\,\delta\frac{\partial g^{ik}}{\partial x^l}}_{\text{IBP}}\right)d\Omega$$

$$= \frac{1}{c}\int\underbrace{\left(\frac{\partial\sqrt{-g}\,\Lambda}{\partial g^{ik}} - \frac{\partial}{\partial x^l}\frac{\partial\sqrt{-g}\,\Lambda}{\partial\frac{\partial g^{ik}}{\partial x^l}}\right)}\delta g^{ik}\, d\Omega$$

$$\frac{1}{2}\sqrt{-g}\,T_{ik} = \frac{\partial\sqrt{-g}\,\Lambda}{\partial g^{ik}} - \frac{\partial}{\partial x^l}\frac{\partial\sqrt{-g}\,\Lambda}{\partial\frac{\partial g^{ik}}{\partial x^l}}$$

$$\delta S = \frac{1}{2c}\int T_{ik}\,\delta g^{ik} = -\frac{1}{2c}\int T^{ik}\delta g_{ik}\sqrt{-g}\, d\Omega$$

$$\delta S = \frac{1}{2c}\int T_{ik}(\xi^{i;k} + \xi^{k;i})\sqrt{-g}\, d\Omega$$

↑ symmetric

$$\delta S = \frac{1}{c}\int(T_{ik}\,\xi^{i;k})\sqrt{-g}\, d\Omega$$

$$= \underbrace{\frac{1}{c}\int(T_i^k\,\xi^i)_{;k}\sqrt{-g}\, d\Omega} - \int T^k_{i;k}\,\xi^i\sqrt{-g}\, d\Omega$$

$$\underbrace{\frac{1}{c}\int\frac{\partial}{\partial x^k}\left(\sqrt{-g}\,T_i^k\,\xi^i\right)d\Omega}$$

use gauss's theorem to convert to ~~integral~~ integral over hypersurface, but ξ^i disappears at surface, so send it to zero

$$\delta S = -\int T^k_{i;k}\,\xi^i\sqrt{-g}\, d\Omega = 0$$

↑ can be arbitrarily set, therefore

$$T^k_{i;k} = 0$$

$$\frac{\partial T_{ik}}{\partial x^k} = 0 \quad \text{valid in galilean}$$

coordinates

energy-momentum tensor

$$T_{ik} = \frac{1}{4\pi}\left(-F_{il}F_k^{\ l} + \frac{1}{4}F_{lm}F^{lm}g_{ik}\right)$$

$$T_{ik} = (p+\varepsilon)u_i u_k - p g_{ik}$$

Einstein Equations

$$\delta\int R\sqrt{-g}\,d\Omega = \delta\int g^{ik}R_{ik}\sqrt{-g}\,d\Omega$$

$$= \int\left(R_{ik}\sqrt{-g}\,\delta g^{ik} + R_{ik}g^{ik}\delta\sqrt{-g} + g^{ik}\sqrt{-g}\,\delta R_{ik}\right)d\Omega$$

$$\delta\sqrt{-g} = \frac{1}{2\sqrt{-g}}\,g_{ik}\delta g^{ik}$$

$$\delta\int R\sqrt{-g}\,d\Omega = \int\left(R_{ik} - \tfrac{1}{2}g_{ik}R^{ik}\right)\delta g^{ik}\sqrt{-g}\,d\Omega + \int g^{ik}\delta R_{ik}\sqrt{-g}\,d\Omega$$

δR_{ik} $\qquad$ Γ^{i}_{kl} not a tensor but

$\delta\Gamma^{i}_{kl}$ are tensors

use locally geodesic system of coordinates

$\Gamma^{i}_{kl} = 0$

$$g^{ik}\delta R_{ik} = g^{ik}\left(\frac{\partial}{\partial x^l}\delta\Gamma^{l}_{ik} - \frac{\partial}{\partial x^k}\delta\Gamma^{l}_{il}\right)$$

$$= g^{ik}\frac{\partial}{\partial x^l}\delta\Gamma^{l}_{ik} - \underbrace{g^{ik}\frac{\partial}{\partial x^k}\delta\Gamma^{l}_{il}}_{\text{swap } k \text{ and } l}$$

$$= \frac{\partial w^l}{\partial x^l}$$

$$w^l = g^{ik}\delta\Gamma^{l}_{ik} - g^{il}\delta\Gamma^{k}_{ik}$$

$$\int g^{ik}\delta R_{ik}\sqrt{-g}\,d\Omega = \underbrace{\int\frac{\partial(\sqrt{-g}\,w^l)}{\partial x^l}\,d\Omega}_{\text{gauss's theorem}}$$

field variations are zero on surface

$$\delta S_g = -\frac{c^3}{16\pi k}\int\left(R_{ik} - \tfrac{1}{2}g_{ik}R\right)\delta g^{ik}\sqrt{-g}\,d\Omega$$

If you start from

$$S_g = -\frac{c^3}{16\pi k}\int G\sqrt{-g}\,d\Omega$$

$$\delta S_g = -\frac{c^3}{16\pi k}\int\left(\frac{\partial(G\sqrt{-g})}{\partial g^{ik}} - \frac{\partial}{\partial x^l}\frac{\partial(G\sqrt{-g})}{\partial\frac{\partial g^{ik}}{\partial x^l}}\right)\delta g^{ik}\,d\Omega$$

Hence

$$R_{ik} - \tfrac{1}{2}g_{ik}R = \frac{1}{\sqrt{-g}}\left(\frac{\partial(G\sqrt{-g})}{\partial g^{ik}} - \frac{\partial}{\partial x^l}\frac{\partial(G\sqrt{-g})}{\partial\frac{\partial g^{ik}}{\partial x^l}}\right)$$

$$\delta S_m = \frac{1}{2c}\int T_{ik}\delta g^{ik}\sqrt{-g}\,d\Omega.$$

total action must be zero, so...

$$R_{ik} - \frac{g_{ik}R}{2} = \frac{8\pi k}{c^4}T_{ik}$$

$$R = -\frac{8\pi k}{c^4}T$$

$$R_{ik} = \frac{8\pi k}{c^4}\left(T_{ik} - \tfrac{1}{2}g_{ik}T\right)$$

Einstein eqns are nonlinear, so superposition not permitted

The Energy-Momentum pseudotensor of the gravitational field

In the absence of a gravitational field conservation $\frac{\partial T^{ik}}{\partial x^k} = 0$

If grav field is present

$$T^k_{i;k} = \frac{1}{\sqrt{-g}} \frac{\partial (T_i^k \sqrt{-g})}{\partial x^k} - \frac{1}{2} \frac{\partial g_{kl}}{\partial x^i} T^{kl} = 0$$

does not express a conservation law

4-momentum of matter + grav field is conserved

choose system of coordinates of such form such that at some particular point in space time such that first order derivatives of $g_{ik} \to 0$

this now reduces to

$$\frac{\partial (\sqrt{-g}\, T_i^k)}{\partial x^k} \neq 0$$

In contravariant components

$$\frac{\partial}{\partial x^k} T^{ik} = 0$$

$$T^{ik} = \frac{\partial}{\partial x^l} \eta^{ikl} \qquad \eta^{ikl} = -\eta^{ilk}$$

$$T^{ik} = \frac{c^4}{8\pi k}\left(R^{ik} - \frac{1}{2} g^{ik} R\right)$$

$$R^{ik} = \frac{1}{2} g^{im} g^{kp} g^{ln} \left(\frac{\partial^2 g_{lp}}{\partial x^m \partial x^n} + \frac{\partial^2 g_{mn}}{\partial x^l \partial x^p} - \frac{\partial^2 g_{ln}}{\partial x^m \partial x^p} - \frac{\partial^2 g_{mp}}{\partial x^l \partial x^n} \right)$$

$$T^{ik} = \frac{\partial}{\partial x^l}\left(\frac{c^4}{16\pi k} \left(\frac{1}{-g}\right) \frac{\partial}{\partial x^m} \left[(-g)(g^{ik} g^{lm} - g^{il} g^{km}) \right] \right)$$

$$h^{ikl} = \frac{\partial}{\partial x^m} \lambda^{iklm}$$

$$\lambda^{iklm} = \frac{c^4}{16\pi k} (-g)(g^{ik} g^{lm} - g^{il} g^{km})$$

$$\frac{\partial h^{ikl}}{\partial x^l} = -g\, T^{ik}$$

Now suppose $\frac{\partial g_{ik}}{\partial x^l} \neq 0$

$$\frac{\partial h^{ikl}}{\partial x^l} - (-g) T^{ik} = (-g) t^{ik}$$

$$\frac{\partial h^{ikl}}{\partial x^l} = -g (t^{ik} + T^{ik}) \leftarrow \text{symmetric}$$

Can express t^{ik} as

$$t^{ik} = \frac{c^4}{16\pi k}\Big(\left(2\Gamma^n_{lm}\Gamma^p_{np} - \Gamma^n_{lp}\Gamma^p_{mn} - \Gamma^n_{ln}\Gamma^p_{mp}\right)\left(g^{il} g^{km} - g^{ik} g^{lm}\right)$$
$$+ g^{il} g^{mn}\left(\Gamma^k_{lp}\Gamma^p_{mn} + \Gamma^k_{mn}\Gamma^p_{lp} - \Gamma^k_{np}\Gamma^p_{lm} - \Gamma^k_{lm}\Gamma^p_{np}\right)$$
$$+ g^{kl} g^{mn}\left(\Gamma^i_{lp}\Gamma^p_{mn} + \Gamma^i_{mn}\Gamma^p_{lp} - \Gamma^i_{np}\Gamma^p_{lm} - \Gamma^i_{lm}\Gamma^p_{np}\right)$$
$$+ g^{lm} g^{np}\left(\Gamma^i_{ln}\Gamma^k_{mp} - \Gamma^i_{lm}\Gamma^k_{np}\right)\Big)$$

$$\frac{\partial}{\partial x^k}(-g)(T^{ik} + t^{ik}) = 0$$

$$P^i = \frac{1}{c}\int (-g)(T^{ik} + t^{ik})\, dS_k$$

$\uparrow$ energy-momentum pseudotensor of grav field

choose integral over hypersurface for $x^0 = \text{const.}$

Then

$$P^i = \frac{1}{c}\int (-g)(T^{i0} + t^{i0})\, dV$$

there is also a result that gives conservation of angular momentum

$$M^{ik} = \int x^i dP^k - x^k dP^i$$

center of ~~mass~~ inertia can be expressed as

$$X^\alpha = \frac{\int x^\alpha (T^{00} + t^{00})(-g)\, dV}{\int (T^{00} + t^{00})(-g)\, dV}$$

express 4-momentum as integral over 3 momentum space

$$P^i = \frac{1}{c}\int \frac{\partial h^{ikl}}{\partial x^l} dS_k$$

$$P^i = \frac{1}{2c}\oint h^{ikl}\, df^*_{ikl}$$

$$P^i = \frac{1}{c}\oint h^{i0\alpha}\, df_\alpha$$

$$M^{ik} = \frac{1}{c}\int \left(x^i \frac{\partial^2 \lambda^{klmn}}{\partial x^m \partial x^n} - x^k \frac{\partial^2 \lambda^{ilmn}}{\partial x^m \partial x^n} \right) dS_l$$

Synchronous Reference System

- In order to synchronize clocks, $g_{0\alpha}$ must $= 0$
- If $g_{00} = 1$ then $x^0 = t$ is proper time everywhere

$g_{00} = 1$ and $g_{0\alpha} = 0$ is synchronous

$$ds^2 = dt^2 - \gamma_{\alpha\beta}\, dx^\alpha dx^\beta$$

$$\gamma_{\alpha\beta} = -g_{\alpha\beta}$$

$u^i = \frac{dx^i}{ds}$ In this reference system

$u^\alpha = 0$, $u^0 = 1$ automatically satisfies geodesic equations

$$\frac{du^i}{ds} + \Gamma^i_{kl} u^k u^l = \Gamma^i_{00} = 0$$

$n_i = \frac{\partial t}{\partial x^i}$, covariant components $n_0 = 1$, $n_\alpha = 0$

geometric construction of a synchronous reference in any space-time
- choose starting surface as space-like hypersurface, i.e. hypersurface whose normals are timelike
- family lines of geodesic lines normal to hyper surface
- Choose these timelines and determine the time coordinate as the length intervals of geodesic

$$g^{ik}\frac{\partial \tau}{\partial x^i}\frac{\partial \tau}{\partial x^k} = 1$$ (assume mass is 1)

$$\tau = f(\xi^\alpha, x^i) + A(\xi^\alpha)$$

$$\frac{\partial \tau}{\partial \xi^\alpha} = 0 \Rightarrow \frac{\partial f}{\partial \xi^\alpha} = -\frac{\partial A}{\partial \xi^\alpha}$$

$$\varkappa_{\alpha\beta} = \frac{\partial \gamma_{\alpha\beta}}{\partial t}$$

$$\varkappa^\alpha_\alpha = \gamma^{\alpha\beta}\frac{\partial \gamma_{\alpha\beta}}{\partial t} = \frac{\partial}{\partial t}\ln(\gamma)$$

$$\Gamma^0_{00} = \Gamma^\alpha_{00} = \Gamma^0_{0\alpha} = 0$$

$$\Gamma^0_{\alpha\beta} = \frac{1}{2}\varkappa_{\alpha\beta} \quad \Gamma^\alpha_{0\beta} = \frac{1}{2}\varkappa^\alpha_\beta \quad \Gamma^\alpha_{\beta\gamma} = \lambda^\alpha_{\beta\gamma}$$

$$R_{00} = -\frac{1}{2}\frac{\partial}{\partial t}\varkappa^\alpha_\alpha - \frac{1}{4}\varkappa^\beta_\alpha \varkappa^\alpha_\beta$$

$$R_{0\alpha} = \frac{1}{2}\left(\varkappa^\beta_{\alpha;\beta} - \varkappa^\beta_{\beta;\alpha}\right)$$

$$R_{\alpha\beta} = \frac{1}{2}\frac{\partial}{\partial t}\varkappa_{\alpha\beta} + \frac{1}{4}\left(\varkappa_{\alpha\beta}\varkappa^\gamma_\gamma - 2\varkappa^\gamma_\alpha \varkappa_{\beta\gamma}\right) + P_{\alpha\beta}$$

$P_{\alpha\beta}$ is Ricci tensor which is expressed in terms of $\gamma_{\alpha\beta}$, just like R_{ik} in terms of g_{ik}

$$R^0_0 = -\frac{1}{2}\frac{\partial}{\partial t}\varkappa^\alpha_\alpha - \frac{1}{4}\varkappa^\beta_\alpha \varkappa^\alpha_\beta = 8\pi k\left(T^0_0 - \frac{1}{2}T\right)$$

$$R^0_\alpha = \frac{1}{2}\left(\varkappa^\beta_{\alpha;\beta} - \varkappa^\beta_{\beta;\alpha}\right) = 8\pi k\, T^0_\alpha$$

$$R^\beta_\alpha = -P^\beta_\alpha - \frac{1}{2\sqrt{\gamma}}\frac{\partial}{\partial t}\left(\sqrt{\gamma}\,\varkappa^\beta_\alpha\right) = 8\pi k\left(T^\beta_\alpha - \frac{1}{2}\delta^\beta_\alpha T\right)$$

- synchronous reference systems are non-stationary

$-g = \gamma \to 0$ in finite length of time

$$T^0_0 - \frac{1}{2}T = \frac{1}{2}(\varepsilon + 3p) + \frac{(p+\varepsilon)v^2}{1-v^2}$$

$$-R^0_0 = \frac{1}{2}\frac{\partial}{\partial t}\varkappa^\alpha_\alpha + \frac{1}{4}\varkappa^\beta_\alpha \varkappa^\alpha_\beta \le 0$$ ← only equal in empty space

$$\varkappa^\alpha_\beta \varkappa^\beta_\alpha \ge \frac{1}{3}\left(\varkappa^\alpha_\alpha\right)^2$$

$$\frac{\partial}{\partial t}\varkappa^\alpha_\alpha + \frac{1}{6}\left(\varkappa^\alpha_\alpha\right)^2 \le 0$$

$$\frac{\partial}{\partial t}\left(\frac{1}{\varkappa^\alpha_\alpha}\right) \ge \frac{1}{6}$$

$\frac{1}{\varkappa^\alpha_\alpha}$ remains positive, decreasing as t decreases

$\varkappa^\alpha_\alpha = \frac{\partial \ln(\gamma)}{\partial t}$ so $\gamma \to 0$ at some rate

Tetrad Representation of Einstein Equations

Determination of components of the Ricci tensor (and the resulting formulation of the Einstein Equations) for a metric of some special form requires complex calculations

$$\underbrace{e^{i}_{(a)} e_{(b)i}}_{} = \eta_{ab}$$

four linearly independent coordinate vectors

$+ \; - - -$ $\quad \eta_{ab}$ is constant symmetric matrix with signature

$$\eta_{ab}\eta^{ab} = \delta^{a}_{b}$$

$$e^{(a)}_{i} e_{(b)i} = \delta_{a}^{b}$$

$$(e^{k}_{(a)} e^{(a)}_{i}) e^{i}_{(b)} = e^{k}_{(b)}$$

$$e^{i}_{(a)} e_{(c)i} = \eta_{ac}$$

$$e^{i}_{(a)} \eta^{bc} e_{(c)i} = \eta_{ac}\eta^{bc}$$

$$= \delta^{b}_{a}$$

$$e^{(b)}_{i} = \eta^{bc} e_{(c)i} \qquad e_{(b)i} = \eta_{bc} e^{(c)}_{i}$$

$$e^{(a)}_{i} = g_{il} e^{(a)l} \qquad e_{(a)k}$$

$$g_{ik} = e_{(a)i} e^{(a)}_{k}$$

$$\eta_{ab} = e^{(a)}_{i} e^{(b)}_{k}$$

$$ds^2 = \eta_{ab}(e^{(a)}_{i} dx^i)(e^{(b)}_{k} dx^k)$$

Tetrad components of A^i

$$A_{(a)} = e^{i}_{(a)} A_i$$

$$A^{(a)} = e^{(a)}_{i} A^i = \eta^{ab} A_{(b)}$$

$$\varphi_{,(a)} = e^{i}_{(a)} \frac{\partial \phi}{\partial x^i}$$

$$\gamma_{acb} = e_{(a)i;k} e^{i}_{(b)} e^{k}_{(c)}$$

$$\lambda_{abc} = \gamma_{abc} - \gamma_{acb}$$

$$= (e_{(a)i,k} - e_{(a)k,i}) e^{i}_{(b)} e^{k}_{(c)} = (e_{(a)i;k} - e_{(a)k;i}) e^{i}_{(b)} e^{k}_{(c)}$$

$$\gamma_{abc} = \frac{1}{2}(\lambda_{abc} + \lambda_{bca} - \lambda_{cab})$$

$$\gamma_{abc} = -\gamma_{bac} \qquad \lambda_{abc} = -\lambda_{acb}$$

$$e_{(a)i;k;l} - e_{(a)i;l;k} = e^{m}_{(a)} R_{mikl}$$

$$R_{(a)(b)(c)(d)} = (e_{(a)i;k;l} - e_{(a)i;l;k}) e^{i}_{(b)} e^{k}_{(c)} e^{l}_{(d)}$$

$$e_{(a)i;k} = \gamma_{abc} e^{(b)}_{i} e^{(c)}_{k}$$

$$R_{(a)(b)(c)(d)} = \gamma_{abc,d} - \gamma_{abd,c} + \gamma_{abf}(\gamma^{f}_{\;cd} - \gamma^{f}_{\;dc})$$

$$+ \gamma_{afc}\gamma^{f}_{\;bd} - \gamma_{afd}\gamma^{f}_{\;bc}$$

$$R_{(a)(b)} = -\frac{1}{2}(\lambda_{ab}{}^{c}{}_{,c} + \lambda_{ba}{}^{c}{}_{,c} + \lambda^{c}{}_{cb,a} + \lambda^{c}{}_{ca,b}$$

$$+ \lambda^{cd}{}_{a}\lambda_{cdb} + \lambda^{cd}{}_{b}\lambda_{dca} - \frac{1}{2}\lambda_{a}{}^{cd}\lambda_{bcd}$$

$$+ \lambda^{c}{}_{cd}\lambda_{ab}{}^{d} + \lambda^{c}{}_{cd}\lambda_{ba}{}^{d})$$

Field of Gravitating Bodies

- Newton's Law

  - Carry out equations in non-relativistic limit of assumption

$$g_{00} = 1 + \frac{2\varphi}{c^2}$$

$$T_i^k = \mu c^2 u_i u^k$$

$$u^\alpha = 0 \qquad u^0 = u_0 = 1$$

$$T_0^0 = \mu c^2$$

$$R_i^k = \frac{8\pi k}{c^4}\left(T_i^k - \frac{1}{2}\delta_i^k T\right)$$

$$R_0^0 = \frac{4\pi k}{c^2}\mu$$

terms containing products of the quantities T_{kl}^i are in every case quantities of second order

$x^0 = ct$ so derivatives containing x^0 are small

$$R_{00} = R_0^0 = \frac{\partial T_{00}^\alpha}{\partial x^\alpha}$$

$$T_{00}^\alpha = -\frac{1}{2}g^{\alpha\beta}\frac{\partial g_{00}}{\partial x^\beta} = \frac{1}{c^2}\frac{\partial\varphi}{\partial x^\alpha} \quad \leftarrow \text{sub this in}$$

$$R_0^0 = \frac{1}{c^2}\Delta\varphi$$

$$\Delta\varphi = 4\pi k\mu \quad \text{mass density}$$

$$\varphi = -k\int\frac{\mu\, dV}{R}$$

For single particle

$$\varphi = -\frac{km}{R}$$

$$F = -m'\frac{\partial\varphi}{\partial R} = -\frac{m' m k}{R^2}$$

$$U = \frac{1}{2}\int\mu\varphi\, dV$$

coordinate origin can always be selected as center of mass so dipole term always go to zero

$$\varphi = -k\left(\frac{M}{R_0} + \frac{1}{6}D_{\alpha\beta}\frac{\partial^2}{\partial x_\alpha \partial x_\beta}\frac{1}{R_0} + \ldots\right)$$

$$M = \int\mu\, dV$$

$$D_{\alpha\beta} = \int\mu\left(3x_\alpha x_\beta - r^2\delta_{\alpha\beta}\right)dV \quad \leftarrow \text{quadrupole moment tensor}$$

$$J_{\alpha\beta} = \int\mu\left(r^2\delta_{\alpha\beta} - x_\alpha x_\beta\right)dV \quad \leftarrow \text{inertia tensor}$$

$$D_{\alpha\beta} = J_{\gamma\gamma}\delta_{\alpha\beta} - 2J_{\alpha\beta}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \leftarrow \text{compute grav potential for a solid ellipsoid}$$

$$\varphi = -\mu k\int\frac{dV}{R}$$

for outside the ellipsoid

$$= -\pi\mu abck\int_\xi^\infty\left(1 - \frac{x^2}{a^2+s} - \frac{y^2}{b^2+s} - \frac{z^2}{c^2+s}\right)\frac{ds}{R_s}$$

$$R_s = \sqrt{(a^2+s)(b^2+s)(c^2+s)}$$

$$\xi = \frac{x^2}{a^2+\xi} + \frac{y^2}{b^2+\xi} + \frac{c^2}{c^2+\xi}$$

$$\varphi = -\pi\mu abck\int_0^\infty\left(1 - \left(\frac{x^2}{a^2+s} + \frac{y^2}{b^2+s} + \frac{z^2}{c^2+s}\right)\right)\frac{ds}{R_s}$$

$$U = \frac{3km^2}{8}\int_0^\infty\left[\frac{1}{5}\left(\frac{a^2}{a^2+s} + \frac{b^2}{b^2+s} + \frac{c^2}{c^2+s}\right) - 1\right]\frac{ds}{R_s}$$

$$= \frac{3km^2}{8}\int_0^\infty\left[\underbrace{\frac{2}{5}s\, d\left(\frac{1}{R_s}\right)}_{IBP} - \frac{2}{5}\frac{ds}{R_s}\right]$$

$$\frac{s}{R_s} - \int\frac{1}{R_s}ds$$

0 at either extreme

$$U = -\frac{3km^2}{10}\int_0^\infty\frac{ds}{R_s}$$

$$U = -\frac{3km^2}{5\sqrt{a^2-c^2}}\cos^{-1}\left(\frac{c}{a}\right) \quad \leftarrow \text{oblate ellipsoid } a = b > c$$

$$U = -\frac{3km^2}{5\sqrt{a^2-c^2}}\cosh^{-1}\left(\frac{a}{c}\right) \quad \leftarrow \text{prolate ellipsoid}$$

The centrally symmetric gravitational field

- consider a grav field possessing central symmetry

- field can only produce if the mass is centrally symmetric

- space-time metric is now the same for any fixed distance.

- However, in non-euclidean coordinates no clear notion of radius vector

$$ds^2 = h(r,t)\,dr^2 + k(r,t)(\sin^2\theta\, d\varphi^2 + d\theta^2) + l(r,t)\,dt^2 + a(r,t)\,dr\,dt$$

Most general form of space time interval.

Further,

$r \to f(r', t')$ $\quad t \to g(r', t')$ is allowed

given this, we can find a basis in which $dr\,dt$ term disappears and $k(r,t) \to -r^2$

$$ds^2 = \exp(v)\, c^2\, dt^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2) - \exp(\lambda)\, dt^2$$

Now, metric tensor

$$g_{ik} = \begin{pmatrix} \exp(v) & 0 & 0 & 0 \\ 0 & -\exp(\lambda) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix}$$

recall $g_{ik}g^{ik} = I$

Now christoffel symbols

$$\Gamma^1_{11} = \frac{\lambda'}{2} \qquad \Gamma^0_{10} = \frac{v'}{2} \qquad \Gamma^2_{33} = -\sin\theta\cos\theta$$

$$\Gamma^0_{11} = \frac{\dot\lambda}{2}\exp(\lambda - v) \qquad \Gamma^1_{22} = -r\exp(-\lambda)$$

$$\Gamma^1_{00} = \frac{v'}{2}\exp(v-\lambda) \qquad \Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r}$$

$$\Gamma^3_{23} = \cot\theta \qquad \Gamma^0_{00} = \dot v/2 \qquad \Gamma^1_{10} = \frac{\dot\lambda}{2}$$

$$\Gamma^1_{33} = -r\sin^2\theta\,\exp(-\lambda)$$

Now compute components of R^i_k

$$\frac{8\pi k}{c^4} T^1_1 = -\exp(-\lambda)\left(\frac{v'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2}$$

$$\frac{8\pi k}{c^4} T^2_2 = \frac{8\pi k}{c^4} T^3_3 = -\frac{1}{2}\exp(-\lambda)\left(v'' + \frac{v'^2}{2} + \frac{v' - \lambda'}{r} - \frac{v'\lambda'}{2}\right) + \frac{1}{2}\exp(-v)\left(\ddot\lambda + \frac{\dot\lambda^2}{2} - \frac{\dot\lambda \dot v}{2}\right)$$

$$\frac{8\pi k}{c^4} T^0_0 = -\exp(-\lambda)\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) + \frac{1}{r^2}$$

$$\frac{8\pi k}{c^4} T^1_0 = -\exp(-\lambda)\frac{\dot\lambda}{r}$$

$$\left.\begin{aligned} &\exp(-\lambda)\left(\frac{v'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 0 \\ &\dot\lambda = 0 \\ &\exp(-\lambda)\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 0 \end{aligned}\right\} \leftarrow \text{in a vacuum}$$

In a vacuum

$$\lambda' + v' = 0, \quad \dot\lambda = 0$$

$$\lambda + v = f(t)$$

because of the freedom to transform r, t can set $f(t) = 0$

$$e^{-\lambda} = e^{v} = 1 + \frac{\text{const}}{r}$$

$$ds^2 = \left(1 - \frac{r_g}{r}\right)c^2 dt^2 - r^2(\sin^2\theta\, d\varphi^2 + d\theta^2) - \frac{dr^2}{1 - \frac{r_g}{r}}$$

$$dl^2 = r^2(\sin^2\theta\, d\varphi^2 + d\theta^2) + \frac{dr}{1 - \frac{r_g}{r}}$$

$$\int_{r_1}^{r_2} \frac{dr}{\sqrt{1 - \frac{r_g}{r}}} > r_2 - r_1, \qquad g_{00} < 1$$

$$d\tau \leq dt$$

time slows for mass within finite distance of mass

approximate expression for ds^2 at large distances from origin

$$ds^2 = ds_0^2 - \underbrace{\frac{2km}{c^2 r}(dr^2 + c^2 dt^2)}_{\text{small correction to galilean metric}}$$

Some general considerations

$\lambda \to 0$ as r^2 as $r \to 0$ to prevent singularities

$$\lambda = -\ln\left(1 - \frac{8\pi k}{c^4 r}\int_0^r T^0_0\, r^2\, dr\right)$$

$$T^0_0 = \exp(-v)\, T_{00} \geq 0$$

$$\frac{\exp(-\lambda)}{r}(v' + \lambda') = \frac{8\pi k}{c^4}(T^0_0 - T^1_1)$$

$$= \frac{(\varepsilon + p)(1 + v^2/c^2)}{1 - v^2/c^2} \geq 0$$

Motion in a centrally symmetric field

$$g^{ik}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^k} - m^2c^2 = 0$$

$$\left(1-\frac{r_g}{r}\right)^{-1}\left(\frac{\partial S}{c\,dt}\right)^2 - \left(1-\frac{r_g}{r}\right)\left(\frac{\partial S}{\partial r}\right)^2 - \frac{1}{r^2}\left(\frac{\partial S}{\partial \varphi}\right)^2 - m^2c^2 = 0$$

$$r_g = \frac{2km'}{c^2}$$ gravitational radius of central body

$$S = -\mathcal{E}_0 t + M\varphi + S_r(r)$$

constants

$$S_r = \int\sqrt{\frac{\mathcal{E}_0^2}{c^2}\left(1-\frac{r_g}{r}\right)^{-2} - \left(m^2c^2+\frac{M^2}{r^2}\right)\left(1-\frac{r_g}{r}\right)^{-1}}\,dr$$

$$ct = \frac{\mathcal{E}_0}{mc^2}\int\frac{dr}{\left(1-\frac{r_g}{r}\right)\left[\left(\frac{\mathcal{E}_0}{mc^2}\right)^2 - \left(1+\frac{M^2}{m^2c^2r^2}\right)\left(1-\frac{r_g}{r}\right)\right]^{1/2}}$$

$\frac{\partial S}{\partial M} = \text{const}$ so

$$\varphi = \underbrace{\int\frac{M\,dr}{r^2\left(\sqrt{\frac{\mathcal{E}_0^2}{c^2} - \left(m^2c^2+\frac{M^2}{r^2}\right)\left(1-\frac{r_g}{r}\right)}\right)}}_{\text{elliptic integral}}$$

Velocity of planetary bodies with respect to sun is small compared to light

$\frac{r_g}{r}$ where r_g is gravitational radius of sun

rewrite

$$r(r-r_g) = r'^2 \qquad r - \frac{r_g}{2} \approx r'$$

expand S_r

$$S_r = \int\left[\left(2\mathcal{E}'m + \frac{\mathcal{E}'^2}{c^2}\right) + \frac{1}{r}\left(2m^2m'k + 4\mathcal{E}'m r_g\right) - \frac{1}{r^2}\left[M^2 - \frac{3m^2c^2r_g^2}{2}\right]\right]^{1/2}dr$$

dropped prime on r and brought in non-relativistic energy term

leads to shift in perihelion of the orbit

$$\Delta\varphi = -\frac{\partial}{\partial M}\Delta S_r$$

expanding S_r in terms of $\frac{1}{r^2}$

$$\Delta S_r = \Delta S_r^{(0)} - \frac{3m^2c^2r_g^2}{4M}\frac{\partial\Delta S_r^{(0)}}{\partial M}$$

closed ellipse so

$$-\frac{\partial}{\partial M}\Delta S_r^{(0)} = \Delta\varphi^{(0)} = 2\pi$$

$$\Delta\varphi = 2\pi + \frac{3\pi m^2c^2r_g^2}{2M^2} = 2\pi + \frac{6\pi k^2m^2m'^2}{c^2M^2}$$

$$\frac{M^2}{km'm^2} = a(1-e^2)$$

$$\delta\varphi = \frac{6\pi km'}{c^2a(1-e^2)}$$

$$g^{ik}\frac{\partial\psi}{\partial x^i}\frac{\partial\psi}{\partial x^k} = 0 \qquad \text{(eikonal equation)}$$

differs only from Hamilton-Jacobi equations by setting $m = 0$

$$\mathcal{E}_0 = -\left(\frac{\partial S}{\partial t}\right)$$

$$\omega_0 = -\left(\frac{\partial\psi}{\partial t}\right) \qquad M = \frac{\rho\,\omega_0}{c}$$

$$\varphi = \int\frac{dr}{r^2\sqrt{\frac{1}{\rho^2} - \frac{1}{r^2}\left(1-\frac{r_g}{r}\right)}}$$

$$r_g \to 0 \qquad r = \frac{\rho}{\cos\varphi}$$

$$\psi_r(r) = \frac{\omega_0}{c}\int\sqrt{\frac{r^2}{(r-r_g)^2} - \frac{\rho^2}{r(r-r_g)}}\,dr$$

$$\psi_r(r) = \frac{\omega_0}{c}\int\underbrace{\sqrt{1+\frac{2r_g}{r} - \frac{\rho^2}{r^2}}}_{\text{expand } r_g/r}\,dr$$

$$\psi_r = \psi_r^{(0)} + \frac{r_g\omega_0}{c}\int\frac{dr}{\sqrt{r^2-\rho^2}} = \psi_r^{(0)} + \frac{r_g\omega_0}{c}\cosh^{-1}\left(\frac{r}{\rho}\right)$$

$\Delta\psi_r$ as light comes from infinity to $r = \rho$ and back

$$\Delta\psi_r = \Delta\psi_r^{(0)} + \frac{2r_g\omega_0}{c}\cosh\left(\frac{R}{\rho}\right)$$

$$\Delta\varphi = -\frac{\partial\Delta\psi}{\partial M} = -\frac{\partial\Delta\psi_r^{(0)}}{\partial M} + \frac{2r_gR}{\rho\sqrt{R^2-\rho^2}}$$

$$\Delta\varphi = \pi + \frac{2r_g}{\rho}$$

$$\delta\varphi = \frac{2r_g}{\rho} = \frac{4km'}{c^2\rho}$$

Gravitational Collapse of a spherical body

In the Schwarzschild metric

$$g_{00} \to 0 \quad \text{and} \quad g_{11} \to \infty \quad \text{as } r \to r_g$$

Thus it must be impossible for bodies with mass to be smaller than a certain radius

- However, this is wrong

$$\det(g) = -r^4 \sin^2\theta \quad \text{has no singularity}$$

so $\det(g) < 0$ is not violated

- actually, impossible to set up rigid reference system for $r < r_g$

- Transform to

$$c\tau = \pm ct \pm \int \frac{f(r)\,dr}{1 - \frac{r_g}{r}}$$

$$R = ct + \int \frac{dr}{\left(1 - \frac{r_g}{r}\right) f(r)}$$

Now, in Schwarzschild metric

$$ds^2 = \frac{\left(1 - \frac{r_g}{r}\right)}{(1 - f^2)}\left(c^2 d\tau^2 - f^2 dR^2\right) - r^2\left(d\theta^2 + \sin^2\theta\, d\varphi\right)$$

eliminate singularity by choosing $f(r_g) \to 1$

$$f(r) = \sqrt{r_g/r}$$

Now $g_{\tau\tau} = 1$, so its a synchronous system

$$R - c\tau = \int \frac{(1 - f^2)}{\left(1 - \frac{r_g}{r}\right) f}\, dr = \int \sqrt{\frac{r}{r_g}}\, dr = \frac{2}{3}\sqrt{\frac{r^3}{r_g}}$$

$$r = \left(\frac{3}{2}(R - c\tau)\right)^{2/3} r_g^{1/3}$$

$$ds^2 = c^2 d\tau^2 - \frac{dR^2}{\left(\frac{3}{2r_g}(R - c\tau)\right)^{2/3}} - \left(\frac{3}{2}(R - c\tau)\right)^{4/3} r_g^{2/3}\left(\sin^2\theta\, d\varphi^2 + d\theta^2\right)$$

R is spacelike and τ is timelike everywhere

- timelines are geodesics

$$ds^2 = 0, \qquad c\frac{d\tau}{dR} = \pm \frac{1}{\left(\frac{3}{2r_g}(R - c\tau)\right)^{1/3}} = \pm\sqrt{\frac{r_g}{r}}$$

What this tells us is if $r < r_g$,

$c\frac{d\tau}{dR} > 1$, so particles at rest cannot be causally linked, as that would require them to communicate faster than the speed of light

Hence, particles must be moving, and in particular, must be moving towards the center

as $r \to r_g$ gravitational force becomes large but density, and hence, pressure, must be finite

so what does a fall look like in a Schwarzschild field

Consider purely radial motion

$$M = 0$$

$\mathcal{E}_0 = mc^2\sqrt{1 - \frac{r_g}{r_0}}$ ← distance r_0 from center

and $V_r = 0$

$$c(t - t_0) = \sqrt{1 - \frac{r_g}{r_0}} \int_r^{r_0} \frac{dr}{\left(1 - \frac{r_g}{r}\right)\sqrt{\frac{r_g}{r} - \frac{r_g}{r_0}}}$$

Integral diverges like

$$r_g \ln(r - r_g)$$

$$r - r_g \propto \exp\left(-\frac{ct}{r_g}\right)$$

$$V^2 = -\frac{g_{11}}{g_{00}}\left(\frac{\partial r}{\partial t}\right)^2$$

$$1 - \frac{v^2}{c^2} = \frac{1 - \frac{r_g}{r}}{1 - r_g/r_0}$$

$$c\tau = \int ds = \int\left[c^2 g_{00}\frac{\partial t^2}{dr^2} + g_{11}\right]^{1/2} dr$$

$$(\tau - \tau_0) = \frac{1}{c}\int_{r_0}^{r}\left(\frac{r_g}{r} - \frac{r_g}{r_0}\right)^{-\frac{1}{2}} dr$$

outside the sphere this appears to take infinite time

but inside, it is finite time

$$c\Delta t = \int_r^{r_0} \frac{dr}{1 - \frac{r_g}{r}} = r_0 - r + r_g \ln\left(\frac{r_0 - r_g}{r - r_g}\right)$$

$$\sqrt{g_{00}} = \sqrt{1 - r_g/r}$$

$$\omega \propto \left(1 - \frac{r_g}{r}\right)$$

Gravitational Collapse of Dustlike Sphere

- need solution of the Einstein equations for a gravitational field in a material medium

- In centrally symmetric case can solve if pressure of matter $p=0$

$$ds^2 = d\tau^2 - \exp(\lambda(\tau,R))dR^2 - r^2(\tau,R)(d\theta^2 + \sin^2\theta\, d\varphi^2)$$

$r(\tau,R)$ is the radius, defined $2\pi r$ is circumference of a circle

Ricci Tensor

$$-\exp(-\lambda)r'^2 + 2r\ddot{r} + \dot{r}^2 + 1 = 0$$

$$-\frac{\exp(-\lambda)}{r}(2r'' - r'\lambda') + \frac{\dot{r}\dot{\lambda}}{r} + \ddot{\lambda} + \frac{\dot{\lambda}^2}{2} + \frac{2\ddot{r}}{r} = 0$$

$$-\frac{\exp(-\lambda)}{r^2}(2rr'' + r'^2 - rr'\lambda') + \frac{1}{r^2}(r\dot{r}\dot{\lambda} + \dot{r}^2 + 1) = 8\pi k\varepsilon$$

$$2\dot{r}' - \dot{\lambda}r' = 0$$

$$\exp(\lambda) = \frac{r'^2}{1+f(R)}$$

$f(R)$ is arbitrary

$$2r\ddot{r} + \dot{r}^2 - f = 0$$

$$\dot{r}^2 = f(R) + \frac{F(R)}{r}$$ $F(R)$ is another arbitrary function

$$\tau = \pm\int \frac{dr}{\sqrt{f + \frac{F}{r}}}$$

$$r = \frac{F}{2f}(\cosh(\eta) - 1)$$ for $f>0$

$$\tau_0(R) - \tau = \frac{F}{2f^{3/2}}(\sinh(\eta) - \eta)$$

and swap in $-f$ if $f<0$

$$r = \left(\frac{9F}{4}\right)^{1/3}[\tau_0(R) - \tau]^{2/3}$$ for $f=0$

$$8\pi k\varepsilon = \frac{F'}{r'r^2}$$

$$m = 4\pi\int_0^{r(R_0)} \varepsilon r^2 dr = 4\pi\int^{R_0} \varepsilon r^2 r' dR$$

$$m = \frac{F(R_0)}{2k}, \quad r_g = F(R_0)$$

describes field of a point mass

$F = r_g \quad f=0 \quad \tau_0 = R$

$$r \approx \left(\frac{9F}{4}\right)^{1/3}(\tau_0 - \tau)^{2/3} \qquad \exp(\lambda/2) \approx \left(\frac{2F}{3}\right)^{1/3}\frac{\tau_0'}{\sqrt{1+f}}(\tau_0 - \tau)^{-1/3}$$

$$8\pi k\varepsilon = \frac{4}{3(\tau_0 - \tau)^2}$$

as $\tau \to \tau_0$ all distances vanish and matter density $\to \infty$

$$f = -\frac{1}{(R/r_g)^2 + 1} \qquad \tau_0 = \frac{\pi}{2}r_g(-f)^{-3/2}$$

$$\frac{r}{r_g} = \frac{1}{2}\left(\frac{R^2}{r_g^2} + 1\right)(1 - \cos\eta)$$

$$\frac{\tau}{r_g} = \frac{1}{2}\left(\frac{R^2}{r_g^2} + 1\right)^{3/2}(\pi - \eta + \sin\eta)$$

Gravitational collapse of non-spherical and rotating bodies

- Previous results applied to spherically symmetric
- gravitational collapse remains the same for bodies with small deviations from spherical symmetry
- Bodies whose deviations from central symmetric body is gravitationally unstable, instability will remain for small disturbances of symmetry
  - Let us determine weak disturbance of a spherical symmetry from rotation, not density distribution

$\vec{M}$ = described by a small off diagonal component in Schwarzschild metric

$$g_{03} = \frac{2kM}{r}\sin^2\theta$$

$$ds^2 = \left(1 - \frac{r_g r}{\rho^2}\right)dt^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2$$

$$- \left(r^2 + a^2 + \frac{r_g r a^2}{\rho^2}\sin^2\theta\right)\sin^2\theta\, d\varphi^2 + \frac{2 r_g r a}{\rho}\sin^2\theta\, d\varphi dt$$

where

$$\Delta = r^2 - r_g r + a^2 \qquad \rho^2 = r^2 + a^2\cos^2\theta$$

$$g_{00} \approx 1 - \frac{r_g}{r} \qquad g_{03} \approx \frac{r_g a}{r}\sin^2\theta$$

angular momentum $\to M = \frac{ma}{r_g}$ & if $a = 0$ metric becomes Schwarzschild metric

$$-g = \rho^4 \sin^4\theta$$

$$g^{ik}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^k} = \frac{1}{\Delta}\left(r^2 + a^2 + \frac{r_g r a^2}{\rho^2}\sin^2\theta\right)\left(\frac{\partial}{\partial t}\right)^2 - \frac{\Delta}{\rho^2}\left(\frac{\partial}{\partial r}\right)^2$$

$$- \frac{1}{\rho^2}\left(\frac{\partial}{\partial\theta}\right)^2 - \frac{1}{\Delta\sin^2\theta}\left(1 - \frac{r_g r}{\rho^2}\right)\left(\frac{\partial}{\partial\varphi}\right)^2 + \frac{2 r_g r a}{\rho^2\Delta}\frac{\partial}{\partial\varphi}\frac{\partial}{\partial t}$$

$$ds^2 = dt^2 - \frac{\rho^2}{r^2 + a^2}dr^2 - \rho^2 d\theta^2 - (r^2 + a^2)\sin^2\theta\, d\varphi^2$$

if $m = 0$, leads to

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \quad \text{where}$$

$$x = \sqrt{r^2 + a^2}\sin\theta\cos\varphi$$

$$y = \sqrt{r^2 + a^2}\sin\theta\sin\varphi$$

$$z = r\cos\theta$$

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2}$$

Kerr metric singularity is separated $g_{00} \to 0$ but at the same time as $g_{11} \to \infty$

$$r_0 = \frac{r_g}{2} + \sqrt{\left(\frac{r_g}{2}\right)^2 - a^2\cos^2\theta} \qquad g_{00} = 0$$

$$r_{hor} = \frac{r_g}{2} + \sqrt{\left(\frac{r_g}{2}\right)^2 - a^2} \qquad g_{11} = \infty$$

S_{hor} is a sphere, S_0 is an oblate figure of rotation

S_0 and S_{hor} touch at poles $\theta = 0, \pi$

S_0 and S_{hor} only exist when $a < \frac{r_g}{2}$

When $a > \frac{r_g}{2}$, non-causality can emerge

$$a_{max} = \frac{r_g}{2}, \quad M_{max} = \frac{m r_g}{2}$$

provides upper limit on angular momentum of collapse

$$r_0 = \frac{r_g}{2}(1 + \sin\theta) \qquad r_{hor} = \frac{r_g}{2}$$

S_{hor} in Kerr field!

$$g^{11}\left(\frac{\partial f}{\partial r}\right)^2 + g^{22}\left(\frac{\partial f}{\partial\theta}\right)^2 = \frac{1}{\rho^2}\left(\Delta\left(\frac{\partial f}{\partial r}\right)^2 + \left(\frac{\partial f}{\partial\theta}\right)^2\right) = 0$$

$$ds^2 = \left(g_{00} - \frac{g_{03}^2}{g_{33}}\right)dt^2 + g_{11}dr^2 + g_{22}d\theta^2 + g_{33}\left(d\varphi + \frac{g_{03}}{g_{33}}dt\right)^2$$

$$g_{00} - \frac{g_{03}^2}{g_{33}} = \frac{\Delta}{r^2 + a^2 + r_g r a^2\sin^2\theta/\rho^2}$$

$$-\frac{g_{03}}{g_{33}} = \frac{r_g a r}{\rho^2(r^2 + a^2) + r_g r a^2\sin^2\theta}$$

$$\mathcal{E}_0 = m(g_{00}u^0 + g_{03}u^3) = m\left(g_{00}\frac{\partial t}{\partial s} + g_{03}\frac{\partial\varphi}{\partial s}\right)$$

$$dl^2 = \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\Delta\sin^2\theta}{1 - \frac{r_g r}{\rho}}d\varphi^2$$

Gravitational fields at Large Distances from bodies

- consider stationary gravitational field at large distances from body producing it
  - determine first terms in expansion of $1/r$
- space time metric far from body is nearly galilean

$$g^{(0)}_{00}=0, \quad g^{(0)}_{0\alpha}=0, \quad g^{(0)}_{\alpha\beta}=-\delta_{\alpha\beta}$$

$$g_{ik}=g^{(0)}_{ik}+h_{ik}$$

$$h^{k}_{i}=g^{(0)kl}h_{il}$$

distinguish h^{ik} from the corrections to the contravariant components of g^{ik}

$$g_{il}g^{lk}=(g^{(0)}_{il}+h_{il})g^{lk}=\delta^{k}_{i}$$

$$g^{ik}=g^{ik(0)}-h^{ik}+h^{i}_{l}h^{lk}$$

$$|g|=g^{(0)}\left(1+h+\frac{1}{2}h^2-\frac{1}{2}h^{i}_{k}h^{k}_{i}\right)$$

$$h'_{ik}=h_{ik}-\frac{\partial \xi_i}{\partial x^k}-\frac{\partial \xi_k}{\partial x^i}$$

$$h^{(1)}_{00}=-\frac{r_g}{r} \quad h^{(1)}_{\alpha\beta}=-\frac{r_g}{r}n_\alpha n_\beta \quad h^{(1)}_{0\alpha}=0$$

$$r_g=\frac{2km}{c^2}$$

to second order

$$h^{(2)}_{00}=0, \quad h^{(2)}_{\alpha\beta}=-\left(\frac{r_g}{r}\right)^2 n_\alpha n_\beta$$

$$R_{iklm}=\frac{1}{2}\left(\frac{\partial^2 h_{im}}{\partial x^k \partial x^l}+\frac{\partial^2 h_{kl}}{\partial x^i \partial x^m}-\frac{\partial^2 h_{km}}{\partial x^i \partial x^l}-\frac{\partial^2 h_{il}}{\partial x^k \partial x^m}\right)$$

$$R_{ik}=g^{lm}R_{limk}\approx g^{(0)lm}R_{limk}$$

$$\frac{\partial \psi^k_i}{\partial x^k}=0, \quad \psi^k_i=h^k_i-\delta^k_i h$$ ← imposed on h

$$R_{ik}=-\frac{1}{2}g^{lm(0)}\frac{\partial^2 h_{ik}}{\partial x^l \partial x^m}$$

$$\Delta h_{ik}=0$$

$$\frac{\partial}{\partial x^\beta}\left(h^\beta_\alpha-\frac{1}{2}h\delta^\beta_\alpha\right)=0$$

$$\frac{\partial}{\partial x^\beta}h^\beta_0=0$$

$$\Delta \xi^i=0$$

$$h_{0\alpha}=\lambda_{\alpha\beta}\frac{\partial}{\partial x^\beta}\frac{1}{r}$$

$$\lambda_{\alpha\beta}\frac{\partial^2}{\partial x^\alpha \partial x^\beta}\frac{1}{r}=0$$

$$h^{(2)}_{0\alpha}=\frac{2k}{c^3}M_{\alpha\beta}\frac{\partial}{\partial x^\beta}\frac{1}{r}=-\frac{2k}{c^3}M_{\alpha\beta}\frac{n_\beta}{r^2}$$

$M_{\alpha\beta}$ only related to $h_{0\alpha}$

$$h^{\alpha 0\beta}=\frac{c^4}{16\pi k}\frac{\partial}{\partial x^\gamma}\left(g^{\alpha 0}g^{\beta\gamma}-g^{\gamma 0}g^{\alpha\beta}\right)$$

$$=-\frac{c^4}{16\pi k}\frac{\partial}{\partial x^\gamma}\left(h_{\alpha 0}\delta_{\beta\gamma}-h_{\gamma 0}\delta_{\alpha\beta}\right)$$

$$\frac{1}{c}\int\left(x^\alpha h^{\beta 0\gamma}-x^\beta h^{\alpha 0}{}_{\gamma}\right)df_\gamma$$

$$=-\frac{1}{4\pi}\int\left(n_\alpha n_\gamma M_{\beta\gamma}-n_\beta n_\gamma M_{\alpha\gamma}\right)do$$

$$=-\frac{1}{3}\left(\delta_{\alpha\gamma}M_{\beta\gamma}-\delta_{\beta\gamma}M_{\alpha\gamma}\right)=\frac{2}{3}M_{\alpha\beta}$$

$$\frac{1}{c}\int \lambda^{\alpha 0\gamma\beta}df_\gamma=-\frac{c^3}{16\pi k}\int\left(h_{\alpha 0}df_\beta-h_{\beta 0}df_\alpha\right)=\frac{1}{3}M_{\alpha\beta}$$

$$g=\frac{2k}{c^3 r^2}\vec{n}\times\vec{M}$$ } coriolis

$$\Omega=\frac{c}{2}\sqrt{g_{00}}\;\nabla\times g$$

$$\Omega=\frac{k}{c^2 r^3}\left[M-3n(M\cdot n)\right]$$

$$h^{\alpha 0\beta}=0$$

$$h^{00\alpha}=\frac{c^4}{16\pi k}\frac{\partial}{\partial x^\beta}\left(g^{00}g^{\alpha\beta}\right)=\frac{mc^2}{8\pi}\frac{\partial}{\partial x^\beta}\left(-\frac{\delta^{\alpha\beta}}{r}+\frac{x^\alpha x^\beta}{r^3}\right)$$

$$=\frac{mc^2}{4\pi}\frac{n^\alpha}{r^2}$$

$$\underbrace{P^\alpha=0, \quad P^0=mc}_{\text{equality of gravitational and inertial mass}}$$

$$R^0_0=\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{-g}\,g^{i0}\Gamma^\alpha_{0i}\right)$$

$$\int R^0_0\sqrt{-g}\,dV=\oint\sqrt{-g}\,g^{i0}\Gamma^\alpha_{0i}\,df_\alpha$$

$$\int R^0_0\sqrt{-g}\,dV=\frac{4\pi k}{c^2}m=\frac{4\pi k}{c^3}P^0$$

$$R^0_0=\frac{4\pi k}{c^4}\left(T^0_0-T^1_1-T^2_2-T^3_3\right)$$

$$P^0=mc=\frac{1}{c}\int\left(T^0_0-T^1_1-T^2_2-T^3_3\right)\sqrt{-g}\,dV$$

The equations of motion of a system of bodies in the second approximation

- system of bodies of mass in motion radiate gravitational waves
  - loss only appears in 5th approximation
- in first 4 approximations, energy doesn't dissipate

- It follows that a system of gravitating bodies can be described by a Lagrangian to terms of 5th order (assuming no EM fields)
- derive Lagrangian for gravity to second order

  - regard structures as pointlike

  - start with the determination in this approximation of the gravitational field, produced by bodies at distances large compared to dimensions, but small compared to λ, the wavelength of gravitational wave

$h^{(1)}_{ik}$ from previous section are the terms to $1/c^2$

$$h_\alpha^\beta = -\frac{2}{c^2}\phi\delta_\alpha^\beta$$

$$h_0^\alpha = 0 \quad h_0^0 = \frac{2}{c^2}\phi$$

where $\phi(\vec{r}) = -k\sum \frac{m_a}{|r-r_a|}$.

metric is now

$$ds^2 = \left(1+\frac{2}{c^2}\phi\right)c^2dt^2 - \left(1-\frac{2}{c^2}\phi\right)(dx^2+dy^2+dz^2)$$

To first order, ϕ appears on g_{00} and $g_{\alpha\beta}$

$$T^{ik} = \sum_a \frac{m_a c}{\sqrt{-g}}\frac{dx^i}{ds}\frac{dx^k}{dt}\delta(r-r_a)$$

$$T_{00} = \sum_a \frac{m_a c^3}{\sqrt{-g}}g_{00}^2\frac{dt}{ds}\delta(\vec{r}-\vec{r}_a)$$

In first approx

$$T_{00} = \sum_a m_a c^2\delta(\vec{r}-\vec{r}_a)$$

To next approx

$$T_{00} = \sum_a m_a c^2\left(1+\frac{5\phi_a}{c^2}+\frac{v_a^2}{2c^2}\right)\delta(r-r_a)$$

$$v^\alpha = \frac{dx^\alpha}{dt}, \quad \phi_a \text{ field}$$

$$T_{\alpha\beta} = \sum_a m_a v_{a\alpha}v_{a\beta}\delta(r-r_a)$$

$$T_{0\alpha} = -\sum_a m_a c\, v_{a\alpha}\delta(r-r_a)$$

$$R_{ik} = g^{lm}R_{limk}$$

$$R_{00} = \frac{1}{c}\frac{\partial}{\partial t}\left(\frac{\partial h_0^\alpha}{\partial x^\alpha}-\frac{1}{2c}\frac{\partial h_\alpha^\alpha}{\partial t}\right)+\frac{1}{2}\Delta h_{00}+\frac{1}{2}h^{\alpha\beta}\frac{\partial^2 h_{00}}{\partial x^\alpha\partial x^\beta}$$
$$-\frac{1}{4}\left(\frac{\partial h_{00}}{\partial x^\alpha}\right)^2-\frac{1}{4}\frac{\partial h_{00}}{\partial x^\beta}\left(2\frac{\partial h_\beta^\alpha}{\partial x^\alpha}-\frac{\partial h_\alpha^\alpha}{\partial x^\beta}\right)$$

$$\frac{\partial h_0^\alpha}{\partial x^\alpha}-\frac{1}{2c}\frac{\partial h_\alpha^\alpha}{\partial t} = 2$$

$$h_\alpha^\beta = -\frac{2}{c^2}\phi\delta_\alpha^\beta \qquad h_{00} = \frac{2}{c^2}\phi+O\left(\frac{1}{c^4}\right)$$

$$R_{00} = \frac{1}{2}\Delta h_{00}+\frac{2}{c^4}\phi\Delta\phi-\frac{2}{c^4}(\nabla\phi)^2$$

$$R_{0\alpha} = \frac{1}{2c}\frac{\partial^2 h_\alpha^\beta}{\partial t\,\partial x^\beta}+\frac{1}{2}\frac{\partial^2 h_0^\beta}{\partial x^\alpha\partial x^\beta}-\frac{1}{2c}\frac{\partial^2 h_\beta^\beta}{\partial t\,\partial x^\alpha}+\frac{1}{2}\Delta h_{0\alpha}$$

$$R_{0\alpha} = \frac{1}{2}\Delta h_{0\alpha}+\frac{1}{2c^3}\frac{\partial^2\phi}{\partial t\,\partial x^\alpha}$$

Write Einstein's equations

$$\Delta h_{00}+\frac{4}{c^4}\phi\Delta\phi-\frac{4}{c^4}(\nabla\phi)^2 = \frac{8\pi k}{c^4}\sum_a\left(1+\frac{5\phi_a}{c^2}+\frac{3v_a^2}{2c^2}\right)\delta(r-r_a)$$

$$\begin{cases} 4(\nabla\phi)^2 = 2\Delta(\phi)^2-4\phi\Delta\phi \\ \Delta\phi = 4\pi k\sum m_a\delta(r-r_a)\end{cases}$$

$$\Delta\left(h_{00}-\frac{2}{c^4}\phi^2\right) = \frac{8\pi k}{c^4}\sum m_a\left(1+\frac{\phi_a'}{c^2}+\frac{3v_a^2}{2c^2}\right)\delta(r-r_a)$$

$$\phi_a' = -k\sum_{b\neq a}\frac{m_b}{|r_a-r_b|}$$

$$\Delta\frac{1}{r} = -4\pi\delta(\vec{r})$$

$$h_{00} = \frac{2\phi}{c^2}+\frac{2\phi^2}{c^4}-\frac{2k}{c^4}\sum\frac{m_a\phi_a'}{|r-r_a|}-\frac{3k}{c^4}\sum_a\frac{m_a v_a^2}{|r-r_a|}$$

$$\Delta h_{0\alpha} = -\frac{16\pi k}{c^3}\sum_a m_a v_{a\alpha}\delta(r-r_a)-\frac{1}{c^3}\frac{\partial^2\phi}{\partial t\,\partial x^\alpha}$$

$$h_{0\alpha} = \frac{4k}{c^3}\sum_a\frac{m_a v_{a\alpha}}{|r-r_a|}-\frac{1}{c^3}\frac{\partial^2 f}{\partial t\,\partial x^\alpha}$$

$$\Delta f = \phi = -\sum_a\frac{km_a}{|r-r_a|}$$

if $\Delta r = \frac{2}{r}$ then $f = -\frac{k}{2}\sum_a m_a|r-r_a|$

$$h_{0\alpha} = \frac{k}{2c^3}\sum_a\frac{m_a}{|r-r_a|}\left[7v_{a\alpha}+(v_a\cdot n_a)n_{a\alpha}\right]$$

n_a is unit vector along $\vec{r}-\vec{r}_a$

$$L_a = -m_a c\frac{ds}{dt} = -m_a c^2\left(1+h_{00}+2h_{0\alpha}\frac{v_a^\alpha}{c}-\frac{v_a^2}{c^2}+h_{\alpha\beta}\frac{v_a^\alpha v_a^\beta}{c^2}\right)^{1/2}$$

$$L_a = \frac{m_a v_a^2}{2}+\frac{m_a v_a^4}{8c^2}-m_a c^2\left(\frac{h_{00}}{2}+h_{0\alpha}\frac{v_a^\alpha}{c}+\frac{1}{2c^2}h_{\alpha\beta}v_a^\alpha v_a^\beta\right.$$
$$\left.-\frac{h_{00}^2}{8}+\frac{h_{00}}{4c^2}v_a^2\right)$$

get total lagrangian by ensuring each individual Lagrangian force is correctly calculated

Gravitational Waves

Weak gravitational waves

- relativistic theory of gravitation permits finite velocity of propagation of interaction, resulting in the possibility of free gravitational fields, not from bodies - gravitational waves

can this mean repulsive gravitational waves

- Weak gravitational field in a vacuum

introduce h_{ik}

$$g_{ik} = g_{ik}^{(0)} + h_{ik}$$

$$g^{ik} = g^{ik(0)} - h^{ik}$$

$$|g| = |g^{(0)}|(1+h) \qquad h \equiv h_i^i$$

all raising and lowering tensor indices done with respect to unperturbed metric

$$h'_{ik} = h_{ik} - \frac{\partial \xi_i}{\partial x^k} - \frac{\partial \xi_k}{\partial x^i} \qquad x'^i = x^i + \xi^i \text{ with small } \xi^i$$

$$\frac{\partial \psi_i^k}{\partial x^k} = 0 \qquad \psi_i^k = h_i^k - \frac{1}{2}\delta_i^k h \quad \leftarrow \text{gauge arbitrariness}$$

$$R_{ik} = \frac{1}{2}\Box h_{ik}$$

$$\Box = -g^{lm(0)}\frac{\partial^2}{\partial x^l \partial x^m} = \Delta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

doesn't fix unique choice of reference frame
if h_{ik} satisfies h'_{ik} satisfies, so long as

$$\Box \xi^i = 0$$

$\Box h_i^k = 0 \leftarrow$ ordinary wave equation

we consider plane gravitational waves
field only changes along one direction in space

$$\Box h_i^k = \left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) h_i^k = 0$$

any $f(t \pm x/c)$ is a solution

If wave is propagating along x-axis in positive direction

auxiliary condition

$$\frac{\partial \psi_i^k}{\partial x^k} = \dot{\psi}_i^1 - \dot{\psi}_i^0 = 0$$

$$\psi_1^1 = \psi_1^0 \quad \psi_2^1 = \psi_2^0 \quad \psi_3^1 = \psi_3^0 \quad \psi_0^1 = \psi_0^0$$

with some transformation, everything but

$$h_{23} \text{ and } h_{22} = -h_{33}$$

gravitational wave is thus a transverse wave whose polarization is given by a 2×2 matrix with trace 0

for two independent polarizations, we can treat either $h_{23} = 0$ or $\frac{1}{2}(h_{22} - h_{33}) = 0$ distinguished by a rotation of $\pi/4$

energy-momentum tensor

$$\frac{1}{2} g^{il} g^{km} g_{np} g_{qr} g^{nr}{}_{,l} g^{pq}{}_{,m} = \frac{1}{2} h_q^{n,i} h_n^{q,k}$$

$$t^{ik} = \frac{c^4}{32\pi k} h_q^{n,i} h_n^{q,k}$$

$$-cgt^{0\alpha} \approx ct^{0\alpha}$$

$$ct^{01} = \frac{c^3}{16\pi k}\left[\dot{h}_{23}^2 + \frac{1}{4}(\dot{h}_{22} - \dot{h}_{33})^2\right]$$

Gravitational waves in curved space-time

- consider arbitrary unperturbed non galilean metric

$$g^{(0)}_{ik}$$

$$\Gamma^{i(1)}_{kl} = \frac{1}{2}\left(h^{i}_{k;l} + h^{i}_{l;k} - h_{kl}^{;i}\right)$$

$$R^{i(1)}_{klm} = \frac{1}{2}\left(h^{i}_{k;m;l} + h^{i}_{m;k;l} - h^{;i}_{km;l} - h^{i}_{k;l;m} - h^{i}_{l;k;m} + h^{;i}_{kl;m}\right)$$

and corrections to Ricci tensor

$$R^{(1)}_{ik} = R^{l(1)}_{ilk} = \frac{1}{2}\left(h^{l}_{i;k;l} + h^{l}_{k;i;l} - h^{;l}_{ik;l} - h_{;i;k}\right)$$

$$R^{k(0)}_{i} + R^{k(1)}_{i} = \left(R^{(0)}_{il} + R^{(1)}_{il}\right)\left(g^{kl(0)} - h^{kl}\right)$$

$$R^{k(1)}_{i} = g^{kl(0)} R^{(1)}_{il} - h^{kl} R^{(0)}_{il}$$

$$h^{l}_{i;k;l} + h^{l}_{k;i;l} - h^{;l}_{ik;l} - h_{;i;k} = 0$$ since for

Einstein eqns $R_{ik} = 0$, and for unperturbed metric, $R^{(0)}_{ik} = 0$ already held, so $R^{(1)}_{ik} = 0$ must now be preserved

cannot be easily simplified in general but in case of $\lambda \ll L$ and $\lambda/c \ll L/c$

Each differentiation of h_{ik} increases order by L/λ relative to unperturbed metrics

considering terms of order

$$h^{l}_{i;k;l} - h^{l}_{i;k;l} \approx h^{l}_{m} R^{m(0)}_{ikl} - h^{m}_{i} R^{l(0)}_{mkl}$$

is $(L/\lambda)^0$ order

Imposing on h_{ik} supplementary conditions

$\psi^{k}_{i;k} = 0$, we obtain

$$h_{ik}^{;l}{}_{;l} = 0$$

$$x'^{i} = x^{i} + \xi^{i}$$

$$\xi^{i;k}{}_{;k} = 0$$

$$h \equiv h^{i}_{i} = 0 \qquad \psi^{k}_{i} = h^{k}_{i}$$

$$h^{k}_{i;k} = 0 \qquad h = 0$$

so all transforms reduce to

$$\xi^{i}_{;i} = 0$$

pseudotensor t^{ik}, in addition to unperturbed terms of various order h_{ik}

- average over space with dimensions large compared to λ, small compared to L. Does not effect unperturbed metric, annihilates quantities linear in rapidly oscillating quantities h_{ik}

- What's left is

$$h_{ik;l} = \partial h_{ik}/\partial x^{l}$$

t^{ik} 4-divergences are dropped

by gauss's theorem order of magnitude in $1/\lambda$ reduced by 1

$$\langle h^{ln}_{,p}\, h^{p}_{l,n}\rangle = -\langle h^{ln}\, h^{p}_{l,p,n}\rangle = 0$$

$$\langle h^{il}_{,n}\, h^{k,n}_{l}\rangle = -\langle h^{il}\, h^{k,n}_{l,n}\rangle = 0$$

$$\langle t^{ik(2)}\rangle = \frac{c^4}{32\pi k}\langle h^{n,i}_{q}\, h^{q,k}_{n}\rangle$$

$$\langle t^{i(2)}_{i}\rangle^{0} = 0$$

gravity wave increases in significance for shorter wavelengths

$$R^{(0)}_{ik} = -\langle R^{(2)}_{ik}\rangle$$

$$\left\langle\left(R^{ik} - \frac{1}{2}g^{ik}R\right)^{(2)}\right\rangle = -\frac{8\pi k}{c^4}\langle t^{ik(2)}\rangle$$

$$\langle t^{ik(2)}\rangle = 0$$

$$\langle R^{(2)}_{ik}\rangle = -\frac{8\pi k}{c^4}\langle t^{(2)}_{ik}\rangle$$

$$R^{(0)}_{ik} = \frac{1}{4}\langle h^{n}_{q,i}\, h^{q}_{n,k}\rangle$$

Strong Gravitational Waves

Consider solution to Einstein equations which is a generalization of the weak, plane gravitational wave in flat space

Look for a reference frame where components of metric tensor are function of x^0

$$x^\sigma \to x^\sigma + \varphi^\sigma(x^0)$$

$$x^0 \to \varphi^0(x^0)$$

$\varphi^0, \varphi^\sigma$ arb

need to make $g_{0\sigma}$ disappear

need $|g_{\alpha\beta}| \neq 0$

$$g_{0\sigma} \to g_{0\sigma} + g_{\sigma\beta}\dot{\varphi}^\beta$$

if $|g_{\alpha\beta}| \neq 0$

$$g_{0\sigma} + g_{\sigma\beta}\varphi^\beta = 0$$

$$|g_{\alpha\beta}| = 0, \qquad g_{0\sigma} = 0$$

$$g_{01} = 1, \quad g_{00} = g_{02} = g_{03} = 0$$

$$dx^\alpha = 0, \quad dx^0 \neq 0 \qquad ds = 0$$

$$x^0 = \eta$$

$$ds^2 = 2dx^1 d\eta + g_{ab}(dx^a + g^a dx^1)(dx^b + g^b dx^1)$$

Then follows

$$g_{ac}\dot{g}^c = 0, \quad \dot{g}^c = 0$$

$$x^a + g^a x^1 \to x^a$$

$$ds^2 = 2\,dx^1 d\eta + g_{ab}(\eta)\,dx^a dx^b$$

The determinant $-g$ of the metric coincides with the determinant $|g_{ab}|$

non-zero Christoffel symbols

$$\Gamma^a_{b0} = \frac{1}{2}K^a_b, \quad \Gamma^1_{ab} = -\frac{1}{2}K_{ab}$$

$$K_{ab} = \dot{g}_{ab}, \quad K^b_c = g^{bc}K_{ac}$$

R_{00} only non-zero element of Ricci Tensor

$$R_{00} = -\frac{1}{2}\dot{K}^a_a - \frac{1}{4}K^b_a K^a_b$$

$$g_{22}(\eta),\ g_{23}(\eta),\ g_{33}(\eta)$$

$$g_{ab} = -\chi^2\gamma_{ab}, \quad |\gamma_{ab}| = 1$$

$$-g = |g_{ab}| = \chi^4$$

$$\ddot{\chi} + \frac{1}{8}(\dot{\gamma}_{ac}\gamma^{bc})(\dot{\gamma}_{bd}\gamma^{ad})\chi = 0 \qquad \leftarrow \text{generalization of weak wave}$$

$$\eta = \frac{(t+x)}{\sqrt{2}}, \quad x^1 = \frac{t-x}{\sqrt{2}}$$

$$\gamma_{ab} = \delta_{ab} + h_{ab}(\eta)$$

$$\frac{\partial\chi}{\partial t} = -\frac{1}{8}\int\left(\frac{\partial h_{ab}}{\partial t}\right)^2 dt < 0$$

$$ds^2 = 2\,d\eta\,dx^1 - \eta^2\left((dx^2)^2 + (dx^3)^2\right)$$

$$\eta x^2 = y \quad \eta x^3 = z \quad x^1 = \xi - \frac{y^2+z^2}{2\eta}$$

$$ds^2 = 2\eta\,d\xi - dy^2 - dz^2$$

Radiation of Gravitational Waves

- consider next a weak gravitational field, produced by bodies with speeds $v \ll c$

$\Box h_i^k = 0$ not correct, need terms from energy-momentum tensor

$$\frac{1}{2}\Box \psi_i^k = \frac{8\pi k}{c^4}\tau_i^k$$

intro

$$\psi_i^k = h_i^k - \frac{1}{2}\delta_i^k h$$

τ_i^k denote the auxiliary conditions from going from exact equations to weak field conditions

τ_0^0, τ_α^0 obtained directly from corresponding T_i^k

τ_β^α contains terms from T_β^α and $R_i^k - \frac{1}{2}\delta_i^k R$

$$\frac{\partial \psi_i^k}{\partial x^k} = 0 \quad \text{so}$$

$$\frac{\partial \tau_i^k}{\partial x^k} = 0$$

replaces $T_{i;k}^k = 0$

$$\psi_i^k = -\frac{4k}{c^4}\int (\tau_i^k)_{t-\frac{R}{c}}\frac{dV}{R}$$

for very long distances

$$\psi_i^k = -\frac{4k}{c^4 R_0}\int (\tau_i^k)_{t-\frac{R_0}{c}}\, dV$$

$$\frac{\partial \tau_{\alpha\gamma}}{\partial x^\gamma} - \frac{\partial \tau_{\alpha 0}}{\partial x^0} = 0$$

$$\frac{\partial \tau_{0\gamma}}{\partial x^\gamma} - \frac{\partial \tau_{00}}{\partial x^0} = 0$$

$$\frac{\partial}{\partial x^0}\int \tau_{\alpha 0} x^\beta dV = \int \frac{\partial \tau_{\alpha\gamma}}{\partial x^\gamma} x^\beta dV = \int \frac{\partial(\tau_{\alpha\gamma} x^\beta)}{\partial x^\gamma} dV - \int \tau_{\alpha\beta} dV$$

at infinity $\tau_{ik} = 0$

$$\int \tau_{\alpha\beta} dV = -\frac{1}{2}\frac{\partial}{\partial x^0}\int (\tau_{\alpha 0} x^\beta + \tau_{\beta 0} x^\alpha) dV$$

$$\frac{\partial}{\partial x^0}\int \tau_{00} x^\alpha x^\beta dV = -\int (\tau_{\alpha 0} x^\beta + \tau_{\beta 0} x^\alpha) dV$$

$$\int \tau_{\alpha\beta} dV = \frac{1}{2}\frac{\partial^2}{\partial x_0^2}\int \tau_{00} x^\alpha x^\beta dV$$

$$\tau_{00} = \mu c^2$$

$$\psi_{\alpha\beta} = -\frac{2k}{c^4 R_0}\frac{\partial^2}{\partial t^2}\int \mu x^\alpha x^\beta dV$$

$$h_{23} = \psi_{23} \qquad h_{22} - h_{33} = \psi_{22} - \psi_{33}$$

$$h_{23} = -\frac{2k}{3c^4 R_0}\dddot{D}_{23}$$

$$h_{22} - h_{33} = -\frac{2k}{3c^4 R_0}(\ddot{D}_{22} - \ddot{D}_{33})$$

$$D_{\alpha\beta} = \int \mu (3x^\alpha x^\beta - \delta_{\alpha\beta} r^2) dV$$

$$ct^{10} = \frac{k}{36\pi c^5 R_0^2}\left[\left(\frac{\dddot{D}_{22} - \dddot{D}_{33}}{2}\right)^2 + \dddot{D}_{23}^2\right]$$

energy flux along x^1 axis in the form

flux through solid angle is $R_0^2 do$

radiation of two independent polarizations

$e_{\alpha\beta}$ introduces three dimensional unit tensor of the plane gravitational wave

tensor is symmetric

$$e_{\alpha\alpha} = 0, \quad e_{\alpha\beta} n_\beta = 0, \quad e_{\alpha\beta} e_{\alpha\beta} = 1$$

$$dI = \frac{k}{72\pi c^5}(\dddot{D}_{\alpha\beta} e_{\alpha\beta})^2 do$$

expression depends implicitly on the direction $\hat{n}$ $\quad e_{\alpha\beta} n_\beta = 0$

$$\overline{e_{\alpha\beta} e_{\gamma\delta}} = \frac{1}{4}\Big(n_\alpha n_\beta n_\gamma n_\delta + (n_\alpha n_\beta \delta_{\gamma\delta} + n_\gamma n_\delta \delta_{\alpha\beta})$$
$$- (n_\alpha n_\gamma \delta_{\beta\delta} + n_\beta n_\gamma \delta_{\alpha\delta} + n_\alpha n_\delta \delta_{\beta\gamma} + n_\beta n_\delta \delta_{\alpha\gamma}) - \delta_{\alpha\beta}\delta_{\gamma\delta}$$
$$+ (\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\beta\gamma}\delta_{\alpha\delta})\Big)$$

$$dI = \frac{k}{36\pi c^5}\left(\frac{1}{4}(\dddot{D}_{\alpha\beta} n_\alpha n_\beta)^2 + \frac{1}{2}\dddot{D}_{\alpha\beta}^2 - \dddot{D}_{\alpha\beta}\dddot{D}_{\alpha\gamma} n_\beta n_\gamma\right) do$$

averaging yields

$$-\frac{d\mathcal{E}}{dt} = \frac{k}{45c^5}\dddot{D}_{\alpha\beta}^2$$

Relativistic Cosmology

Isotropic space

- General theory of relativity opens up new avenues of approach to the solution of problems related to the properties of the universe

- define action as

$$S_k = -\frac{c^3}{16\pi k}\int (G + 2\Lambda)\sqrt{-g}\, d\Omega$$

contains g_{ik} and Christoffel symbol

new constant

$$R_{ik} - \frac{1}{2} R g_{ik} = \frac{8\pi k}{c^4} T_{ik} + \Lambda g_{ik}$$

~~cosm~~ cosmological constant

- if cosmological constant is small, no significant effect, but, new types of cosmological solution will emerge
  - implies the existence of curvature not related to gravitational waves or matter

- Hence, let us progress without the cosmological constant

- stars are distributed throughout space in non-uniform manner
  - but on large scale, we can disregard local inhomogeneities

- current data, as of the writing of L & L indicates space is isotropic and homogeneous <u>at large enough scales</u>

-

$$dl^2 = \gamma_{\alpha\beta}\, x^\alpha x^\beta$$

curvature tensor

$$P^{\alpha}_{\beta\gamma\delta}$$

$$P_{\alpha\beta\gamma\delta} = \lambda(\gamma_{\alpha\gamma}\gamma_{\beta\delta} - \gamma_{\alpha\delta}\gamma_{\gamma\beta})$$

$$P_{\alpha\beta} = 2\lambda\gamma_{\alpha\beta} = P^{\gamma}_{\alpha\gamma\beta}$$

$$P = 6\lambda$$

so curvature given by just one constant.

consider hypersphere

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = a^2$$

$$dl^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$$

Consider x_1, x_2, x_3 to be spatial, x^4 can be eliminated

so

$$dl^2 = x_1^2 + x_2^2 + x_3^2 + \frac{(x_1 dx + x_2 dx_2 + x_3 dx_3)^2}{a^2 - x_1^2 - x_2^2 - x_3^2}$$

$$\gamma_{\alpha\beta} = \delta_{\alpha\beta} + \frac{x_\alpha x_\beta}{a^2}$$

$\lambda = \frac{1}{a^2}$, so a is radius of curvature

$$dl^2 = \frac{dr^2}{1 - \frac{r^2}{a^2}} + r^2(\sin^2\theta\, d\phi^2 + d\theta^2)$$

$$\int_0^r \frac{dr}{\sqrt{1 - r^2/a^2}} = a \sin^{-1}(r/a)$$

dl^2 can also be expressed in terms of χ where

$$r = a \sin\chi$$

$$dl^2 = a^2[d\chi^2 + \sin^2\chi\,(\sin^2\theta\, d\phi^2 + d\theta^2)]$$

$4\pi a^2 \sin^2\chi$ is surface of sphere, as χ goes from $0 \to \pi$

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{\pi} a^3 \sin^2\chi \sin\theta\, d\chi\, d\theta\, d\phi$$

$$V = 2\pi^2 a^3$$

- positive curvature is closed on itself
  electric charge must be zero and total momentum is zero

- for negative curvature
  possible if a is imaginary,
  replace $a \to ia$

$$\lambda = -\frac{1}{a^2}$$

$$dl^2 = \frac{dr^2}{1 + \frac{r^2}{a^2}} + r^2(\sin^2\theta\, d\phi^2 + d\theta^2)$$

$$dl^2 = a^2(d\chi^2 + \sinh^2\chi\,(\sin^2\theta\, d\phi^2 + d\theta^2))$$

The Closed Isotropic Model

Consider the co-moving reference system
- reference frame moving along with matter

$g_{0\alpha}$ of the metric tensor $=0$

$$ds^2 = g_{00}(dx^0)^2 - dl^2$$

g_{00} is $f(x_0)$
hence we can always choose
$g_{00}=1$

$$ds^2 = c^2dt^2 - dl^2$$

consider positive curvature

$$ds^2 = c^2dt^2 - a^2(t)\left(d\chi^2 + \sin^2\chi\,(d\theta^2+\sin^2\theta\, d\phi^2)\right)$$

$$c\,dt = a\,d\eta$$

$$ds^2 = a^2(\eta)\left(d\eta^2 - d\chi^2 - \sin^2\chi\,(d\theta^2+\sin^2\theta\, d\phi^2)\right)$$

x^0, x^1, x^2, x^3 are η, χ, θ, ϕ

$$g_{00}=a^2 \quad g_{11}=-a^2, \quad g_{22}=-a^2\sin\chi$$
$$g_{33}=-a^2\sin^2\chi\,\sin^2\theta$$

Christoffel is then

$$\Gamma^0_{00} = \frac{da/d\eta}{a}, \quad \Gamma^0_{\alpha\beta} = -\frac{da/d\eta}{a^3}\, g_{\alpha\beta}$$

$$\Gamma^\alpha_{0\beta} = \frac{da/d\eta}{a}\,\delta^\alpha_\beta \qquad \Gamma^0_{\alpha 0} = \Gamma^\alpha_{00} = 0$$

$$R^0_0 = \frac{3}{a^4}\left(a'^2 - aa''\right) \qquad a' = \frac{da}{d\eta}, \quad a'' = \frac{d^2a}{d\eta}$$

$$R^\beta_\alpha = -P^\beta_\alpha + \ldots = -\frac{2}{a^2}\delta^\beta_\alpha$$

Then

$$R^\beta_\alpha = -\frac{1}{a^4}\left(2a^2 + a'^2 + aa''\right)\delta^\beta_\alpha$$

$$T^0_0 = \mathcal{E}$$

$$R^0_0 - \frac{1}{2}R = \frac{8\pi k}{c^4}\, T^0_0$$

Thus

$$\frac{8\pi k}{c^4}\,\mathcal{E} = \frac{3}{a^4}\left(a^2 + a'^2\right)$$

two unknowns
$\mathcal{E}$ and a

Consider the entropy as constant

$$d\mathcal{E} = T\,dS - p\,dV \qquad \mathcal{E} = \mathcal{E}/V \text{ energy density definition}$$

$$d\mathcal{E} = -p\,dV$$

$$d\mathcal{E} = -(\mathcal{E}+p)\frac{dV}{V}$$

$$-\frac{d\mathcal{E}}{\mathcal{E}+p} = 3\,d(\ln(a))$$

$$3\ln(a) = -\int \frac{d\mathcal{E}}{\mathcal{E}+p} + \text{const}$$

$$\eta = \pm\int \frac{da}{a\sqrt{\frac{8\pi k}{3c^4}\mathcal{E}a^2 - 1}}$$

In some limit, we can consider the universe to be so diffuse that pressure, $p \approx 0$ and only energy density matters $\mathcal{E} = \mu c^2$

$\mu a^3 = \text{const}$ ← alternatively mass is conserved

The open Isotropic Model

$$ds^2 = c^2 dt^2 - a^2(t)\left(d\chi^2 + \sinh^2\chi\,(d\theta^2 + \sin^2\theta\, d\varphi^2)\right)$$

$$ds^2 = a^2(\eta)\left(d\eta^2 - d\chi^2 - \sinh^2\chi - \sinh^2\chi\,(d\theta^2 + \sin^2\theta\, d\varphi^2)\right)$$

$$c\,dt = a\,d\eta$$

convert to imaginary domain

$$3\ln a = -\int \frac{d\varepsilon}{\varepsilon + p} + \text{const}$$

$$\frac{8\pi k}{c^2}\varepsilon = \frac{3}{a^4}\left(a'^2 - a^2\right)$$

$$\eta = \pm\int \frac{da}{a\sqrt{\frac{8\pi k}{3c^4}\varepsilon a^2 + 1}}$$

for dust $\quad a = a_0(\cosh\eta - 1), \quad t = \frac{a_0}{c}(\sinh\eta - \eta)$

for large densities

$$p = \varepsilon/3$$

$$\varepsilon a^4 = \frac{3c^4 a_1^2}{8\pi k}$$

$$a = a_1 \sinh\eta \qquad t = \frac{a_1}{c}(\cosh\eta - 1)$$

$$a = \sqrt{2 a_1 c t}$$

$$ds^2 = c^2 dt^2 - b^2(t)\left(dx^2 + dy^2 + dz^2\right)$$

$$\frac{8\pi k}{c}\varepsilon = \frac{3}{b^2}\left(\frac{db}{c\,dt}\right)^2 \qquad 3\ln b = -\int \frac{d\varepsilon}{p + \varepsilon} + \text{const}$$

$$\mu b^3 = \text{const} \qquad b = \text{const}\; t^{2/3}$$

$$\varepsilon b^4 = \text{const} \qquad b = \text{const}\sqrt{t}$$

The Red Shift

- Mostly considered non-stationary metric
- the radius curvature of space is function of time
- change in radius leads to a change in all relative distances

dl is proportional to a

$\eta > 0$ in open model

$0 < \eta < \pi$ in closed model

- Consider the propagation of light in isotropic space

$ds = 0$

χ, θ, ϕ

line $\theta = \text{const}$, $\phi = \text{const}$

$d\theta = d\phi = 0$

$ds^2 = a^2(d\eta^2 - d\chi^2)$

$d\eta = \pm d\chi$

$\chi = \pm \eta + \text{const}$

plus sign for ray going out from origin
minus sign for ray coming to origin

- can finally arrive conjugate pole
- ray cannot return to its starting point

$\chi = \eta_0 - \eta$

for an observer located at the point $\chi = 0$, rays of light can only arrive at time $t(\eta_0)$ started from points located at distance $\chi = \eta_0$

- result which applies to the open model as well as the closed model

- visible region of the space is the section of 4-dimensional volume

$t(\eta - \chi)$ $\quad t(\eta)$

$4\pi a^2(\eta - \chi)$ $\quad a(\eta - \chi) = a(0) = 0$

$\varepsilon \to \infty$

$$M_{obs} = 4\pi \int_0^\eta \mu a^3 \sinh^2 \chi \, d\chi$$

$$M_{obs} = \frac{3c^2 a}{2k}\left(\sinh \eta \cosh \eta - \eta\right)$$

$$M_{obs} = \frac{M}{\pi}\left(\eta - \sin \eta \cos \eta\right)$$

As η increases 0 to π, this quantity increases from 0 to M