Principle of Relativity

1)

$$A^{ik} = A^{ki}$$

A^{ik} transforms like the product of 2 4-vectors. Hence

$$A^{00} = \left(\frac{A'^0 + \frac{v}{c}A'^1}{\sqrt{1-\frac{v^2}{c^2}}}\right)\left(\frac{A'^0 + A'^1\frac{v}{c}}{\sqrt{1-v^2/c^2}}\right)$$

$$= \left(A'^{00} + \frac{2v}{c}A'^{10} + \frac{v^2}{c^2}A'^{11}\right)\left(\frac{1}{1-v^2/c^2}\right)$$

$$A^{10} = \frac{A'^1 + \frac{v}{c}A'^0}{\sqrt{1-v^2/c^2}}\;\frac{A'^0 + \frac{v}{c}A'^1}{\sqrt{1-v^2/c^2}} = \frac{\left(A'^{10} + \frac{v}{c}(A'^{11} + A'^{00}) + \frac{v^2}{c^2}A'^{10}\right)}{1-v^2/c^2}$$

$$A^{11} = \frac{A'^{11} + 2\frac{v}{c}A'^{10} + \frac{v^2}{c^2}A'^{00}}{1-v^2/c^2}$$

$$A^{02} = \frac{A'^{02} + \frac{v}{c}A'^{12}}{\sqrt{1-v^2/c^2}}$$

2) $A^{00} = 0,\ A^{11} = 0,\ A^{22} = 0,\ A^{33} = 0$

$$A^{01} = \frac{A'^0 + \frac{v}{c}A'^1}{\sqrt{1-\frac{v^2}{c^2}}}\;\frac{A'^1 + \frac{v}{c}A'^0}{\sqrt{1-v^2/c^2}}$$

$$= A'^{01} - v^2/c^2\, A'^{01}$$

$$= A'^{01}$$

$$A^{02} = \frac{A'^{02} + \frac{v}{c}A'^{12}}{\sqrt{1-v^2/c^2}}$$

$$A^{12} = \frac{A'^{12} + \frac{v}{c}A'^{02}}{\sqrt{1-v^2/c^2}}$$

3) In proper reference frame, $v = 0$

$$\frac{du^i}{ds} = \left(\frac{d}{c\sqrt{1-v^2/c^2}\,dt}1,\ \frac{d}{c\sqrt{1-\frac{v^2}{c^2}}}\frac{v}{c\sqrt{1-v^2/c^2}},\ 0, 0\right)$$

$$= \left(0, \frac{a}{c^2}, 0, 0\right)$$

$w^i w_i = -\frac{a^2}{c^4}$ ← invariant across all frames.

We can now consider a fixed frame wrt motion

$$\frac{du^i}{ds}\frac{du_i}{ds} = -\frac{a^2}{c^4}$$

$$\frac{d}{ds}(u^i)\frac{d}{ds}(u_i) = -\frac{a^2}{c^4}$$

$$\frac{d}{ds}\left(\gamma, \gamma\frac{v}{c}, 0, 0\right) = \frac{\gamma}{c}\left(\dot\gamma, \dot\gamma\frac{v}{c} + \gamma\frac{\dot v}{c}, 0, 0\right)$$

$$= \frac{\dot\gamma}{c}\left(\gamma, \gamma\frac{v}{c}, 0, 0\right) + \frac{\gamma}{c}\left(0, \gamma\frac{\dot v}{c}, 0, 0\right)$$

$$\left(\frac{\dot\gamma}{c}\underbrace{\left(\gamma, \gamma\frac{v}{c}\right)}_{u^i} + \frac{\gamma}{c}\left(0, \gamma\frac{\dot v}{c}\right)\right)\left(\frac{\dot\gamma}{c}\underbrace{\left(\gamma, -\gamma\frac{v}{c}\right)}_{u_i} + \frac{\gamma}{c}\left(0, -\gamma\frac{\dot v}{c}\right)\right) = -a^2/c^{4\,2}$$

$$\left(\dot\gamma u^i + \gamma\left(0, \gamma\frac{\dot v}{c}\right)\right)\frac{du_i}{ds} = -a^2/c^2$$

$$\gamma\left(0, \gamma\frac{\dot v}{c}\right)\frac{du_i}{ds} = -\frac{a^2}{c^2}$$

$$\gamma\left(0, \gamma\frac{\dot v}{c}\right)\left(\dot\gamma\left(\gamma, -\gamma\frac{v}{c}\right) + \gamma\left(0, -\gamma\frac{\dot v}{c}\right)\right) = -\frac{a^2}{c^2}$$

$$-\gamma^3\dot\gamma\frac{v\dot v}{c^2} - \gamma^4\frac{\dot v^2}{c^2} = -\frac{a^2}{c^2}$$

Note that

$$\dot\gamma = \gamma^3 v\dot v/c^2$$

$$-\gamma^6 v^2\dot v^2/c^2 - \gamma^4\dot v^2 = -a^2$$

$$\gamma^4\dot v^2\left(-\gamma^2\frac{v^2}{c^2} - 1\right) = -a^2$$

$$\frac{d}{dt}(\gamma v) = \frac{\gamma^3 v^2\dot v}{c^2} + \gamma\dot v$$

$$= \gamma\dot v\left(\gamma^2\frac{v^2}{c^2} + 1\right)$$

$$= \gamma\dot v\left(\frac{v^2/c^2}{1-\frac{v^2}{c^2}} + \frac{1-v^2/c^2}{1-v^2/c^2}\right)$$

$$= \gamma^3\dot v$$

$$\gamma^6\dot v^2 = a^2$$

$$\gamma^3\dot v = a$$

$$\frac{d}{dt}(\gamma v) = a$$

$$\gamma v = at$$

Solve for v

$$\frac{v^2}{1-\frac{v^2}{c^2}} = a^2t^2$$

$$v^2 = a^2t^2 - \frac{v^2a^2t^2}{c^2}$$

$$v^2\left(1 + \frac{a^2t^2}{c^2}\right) = a^2t^2$$

$$v = \frac{at}{\sqrt{1+\frac{a^2t^2}{c^2}}}$$

$$x = \int v\,dt$$

$$x = \int\frac{at\,dt}{\sqrt{1+\frac{a^2t^2}{c^2}}}$$

$$c\tan\theta = at$$

$$a\,dt = c\sec^2\theta\,d\theta$$

$$x = \int\frac{c\sec^2\theta\,d\theta\;\frac{c}{a}\tan\theta}{\sqrt{1+\tan^2\theta}}$$

$$\frac{+\sin\theta}{\cos^2\theta}$$

$$x = \int\frac{c^2}{a}\sec\theta\tan\theta\,d\theta$$

$$x = \frac{c^2}{a}\sec\theta$$

$$x = \frac{c^2}{a}\sec\left(\tan^{-1}\left(\frac{at}{c}\right)\right)$$

Relativistic Mechanics

1)

Since particle has mass m it must follow

$$E = m c^2 \sqrt{1-v^2/c^2}$$

Further, in lab frame

$$p_x = 0$$

$$E = mc^2$$

$$\Gamma = \frac{1}{\sqrt{1-v^2/c^2}} \quad P = m\Gamma v \quad (P_x = 0)$$

$$E = \frac{m c^2}{\sqrt{1-v^2/c^2}} \quad \Rightarrow \left(\frac{E}{c^2}\right)^2 = \frac{m^2}{1-v^2/c^2}$$

$$P_x = \frac{m v}{\sqrt{1-v^2/c^2}} \quad \Rightarrow \left(\frac{P_x}{c}\right)^2 = \frac{m^2 v^2/c^2}{1-v^2/c^2}$$

$$\Rightarrow \left(\frac{E}{c^2}\right)^2 - \left(\frac{P_x}{c}\right)^2 = \frac{m^2}{1-v^2/c^2}$$

$\Rightarrow$ $\frac{E}{c^2}, \frac{P_x}{c}$ are the components of a 4-vector, called the 4-momentum. Now, in some other inertial frame, we can write conservation of momentum

$$E^{(1)} + E^{(2)} = E^{(3)} + E^{(4)}$$

$$P_x^{(1)} + P_x^{(2)} = P_x^{(3)} + P_x^{(4)}$$

$$P_y^{(1)} + P_y^{(2)} = P_y^{(3)} + P_y^{(4)}$$

$$P_z^{(1)} + P_z^{(2)} = P_z^{(3)} + P_z^{(4)}$$

$$\left(\frac{E^{(1)}}{c^2}\right)^2 - \left(\frac{P_x^{(1)}}{c}\right)^2 = \frac{m^2}{1-v^2/c^2}$$

$$\left(\frac{E^{(2)}}{c^2}\right)^2 - \left(\frac{P_x^{(2)}}{c}\right)^2 = \frac{m^2}{1-v^2/c^2}$$

$$\left(\frac{E^{(3)}}{c^2}\right)^2 - \left(\frac{P_x^{(3)}}{c}\right)^2 = \frac{m^2}{1-v^2/c^2}$$

$$\left(\frac{E^{(4)}}{c^2}\right)^2 - \left(\frac{P_x^{(4)}}{c}\right)^2 = \frac{m^2}{1-v^2/c^2}$$

Can solve the rest of the momenta

2) The energy of particle this collision is $E = mc^2$

We know a photon from nucleus is emitted. So

$$\Delta E = \frac{1}{2} m v_0^2 + \Delta E \Rightarrow \Delta E > 0$$

Then check that

$$E = \frac{E_0 + \Delta E}{\sqrt{1-v_0^2/c^2}}$$

$$\Rightarrow E - \frac{E_0 + \Delta E}{\sqrt{1-v_0^2/c^2}}$$

$$\Delta E = \frac{E_0 + \Delta E}{\sqrt{1-v_0^2/c^2}} - E_0$$

3) The mass of atom is the difference of the nucleus, m_N and the binding energy ΔE

$$m_{atom} = m_N - \frac{\Delta E}{c^2}$$

4)

In rest frame

$$E = E_0 \quad \text{(E = energy in rest frame)}$$

$$P^{(1)} = (E, 0, 0, 0)$$

$$P^{(2)} = (E, 0, 0, 0)$$

$$P^1 = \frac{E + v_0 m c}{\sqrt{1-v_0^2/c^2}} \quad P^2 = \frac{E - v_0 m c}{\sqrt{1-v_0^2/c^2}}$$

$$P^3 = \frac{E + v_0 m c}{\sqrt{1-v_0^2/c^2}} \quad P^4 = \frac{E - v_0 m c}{\sqrt{1-v_0^2/c^2}}$$

$$P^{(3)} = (P_0, 0, 0, 0) \quad P^{(4)} = (P_0, 0, 0, 0)$$

$$\Rightarrow P_0 = \frac{E + v_0 m c}{\sqrt{1-v_0^2/c^2}}$$

$$\Rightarrow P_0 = \frac{E - v_0 m c}{\sqrt{1-v_0^2/c^2}}$$

$$\Rightarrow E + v_0 m c = E - v_0 m c$$

$$\Rightarrow E = \frac{v_0 m c}{2}$$

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5)

$$\Delta E = \frac{1}{2} v_0^2 m$$

$$\Rightarrow E = \frac{v_0 m c}{2}$$

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6)

$$\Delta E = \frac{1}{2} v_0^2 m$$

$$\Rightarrow E = \frac{v_0 m c}{2}$$

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$$\Rightarrow E = \frac{v_0 m c}{2}$$

7) Relation of momentum of particle and the rest mass energy can be found

$$\text{Let } \Theta = \frac{v_0 m c}{\sqrt{1-v_0^2/c^2}} \quad (\text{Let } \Theta = \frac{v_0 m c}{\sqrt{1-v_0^2/c^2}})$$

$$\Rightarrow \frac{E + \Theta}{\sqrt{1-v_0^2/c^2}} = \frac{E - \Theta}{\sqrt{1-v_0^2/c^2}}

Charge in an EM field

What is the acceleration of a particle in an EM field

Need $\frac{dv}{dt}$

$$P = E_0 \frac{v^2}{c^2}$$

$$\frac{dE}{dt} = \frac{dP}{dt} = v \cdot F_{EM} \frac{dv}{dt}$$

effective mass of the particle

$$\frac{d}{dt}\left(\frac{m_0 v}{\sqrt{1-v^2}}\right) = \frac{m_0}{\sqrt{1-v^2}}\frac{dv}{dt} + \frac{m_0 v^2}{(1-v^2)^{3/2}}\frac{dv}{dt}$$

$$\frac{d}{dt}\left(\frac{E}{c^2}\right) = \frac{E}{c^2}\frac{v}{\sqrt{1-v^2}}\frac{dv}{dt}$$

$$\left(\frac{E}{c^2} + \frac{e(\phi_{EM} - e\vec{A}\cdot\vec{v})}{\sqrt{1-v^2}}\right)\frac{dv}{dt}$$

Given the lorentz and minimum force in field in a combined EM field in relativistic mechanics

Variational principle states

$$\delta \int P \cdot ds = 0$$

$$\delta \int \left(\frac{m_0 c}{\sqrt{1-v^2}} + \frac{e}{c}\vec{A}\cdot ds\right) = 0$$

$$\delta \int (\gamma m_0 c + \frac{e}{c}\vec{A}\cdot ds) = 0$$

Note that $\gamma = \frac{1}{\sqrt{1-v^2}}$

$$\delta\left(\int \sqrt{(c-ev)^2 - m^2 c^4}\, dt + \frac{e}{c}\int \vec{A}\cdot ds\right) = 0$$

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charged spatial oscillator with proper frequency ω_0 in a weak uniform mag field $\vec{B}$

Recall that

$$\mathcal{L} = \mathcal{L}_{EM} - U_{Lorentz}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i}$$

Recall that E_{kin} is fixed in uniform constant mag field (velocity may vary, $\vec{B}$ fixed)

$$\ddot{\vec{x}} + \omega_0^2 \vec{x} = \frac{eB}{m}\dot{\vec{y}}$$

$$\ddot{\vec{y}} + \omega_0^2 \vec{y} = -\frac{eB}{m}\dot{\vec{x}}$$

$$\ddot{\vec{z}} + \omega_0^2 \vec{z} = 0$$

$$\vec{z}(t) = \vec{z}_0 \exp(-i\omega_0 t)$$

$$\dot{\vec{x}} + \omega_0^2 \vec{x} = \frac{eB}{m}\dot{\vec{y}} = \frac{eB}{m}\dot{\vec{x}}$$

$$(\dot{\vec{x}} + i\vec{x})^{\cdot} + \omega_0^2 (\vec{x} + i\vec{x}) = \frac{eB}{m}(\dot{\vec{x}} + i\vec{x})$$

$$\vec{\xi} = \vec{x} + i\vec{y}$$

$$\dot{\vec{\xi}} + \omega_0^2 \vec{\xi} = -\frac{eB}{m}\vec{\xi}$$

$$\vec{\xi} = \exp(-i\omega t)$$

$$-\omega^2 + \omega_0^2 = \frac{eB}{m}\omega \Rightarrow \omega^2 + \frac{eB}{m}\omega - \omega_0^2 = 0$$

$$\omega = \frac{-\frac{eB}{m} \pm \sqrt{\left(\frac{eB}{m}\right)^2 + 4\omega_0^2}}{2}$$

Relativistic motion of charge in a uniform EM field

Recall that the Lagrangian yields the following

$$\dot{P} = e\vec{E} + e(v \times H)$$

Recall that in a relativistic setting

$$E_{kin} = \sqrt{m^2 + p^2}$$

$$= \underbrace{\sqrt{m^2 + (eAH)^2}}_{\text{initial kinetic energy of system}}$$

$$\dot{P}_1 = e v_2 H$$

$$\dot{P}_2 = -e v_1 H$$

$$\dot{P}_3 = eE$$

$$\dot{P}_4 = eEt$$

Further

$$\vec{v} = \frac{P}{E_{kin}} \qquad \dot{P} = \dot{\vec{v}} E_{kin} + v \dot{E}_{kin}$$

$$\dot{\vec{v}} = \frac{eEt}{\sqrt{E_{kin}^2 + P^2}} \qquad \dot{P} = E_{kin}$$

$$\dot{v}_1 E_{kin} + v_1 \dot{E}_{kin} + v_2 \dot{E}_{kin} + v_3 \dot{E}_{kin} = v_1 eEt (v_1 + v_2 + v_3)$$

$$(v_1 + v_2 + v_3)\, E_{kin} = (v_1 + v_2 + v_3)^2 eEt - \dot{E}_{kin}$$

$$\ln\left(\frac{v_1 + v_2 + v_3}{v_1 + v_2 + v_3}\right) = \ln\left(\frac{E_{kin}}{E_{kin}}\right) = \frac{1}{2}\ln\left(eEt + \sqrt{E_{kin}^2 + P^2}\right)$$

$$v_1 + v_2 + v_3 = \exp\left(-\frac{1}{2}\ln\left(eEt + \sqrt{E_{kin}^2 + P^2}\right)\right)$$

Now we can solve for x and y trivially

All sets (the L and L^{-1} sets)

$$(P_x + iP_y) = \exp(-i\, v_{tot} t)$$

$$(P_x + iP_y)^{-1} = \frac{-ieEt}{E_{kin}}(P_x + iP_y)$$

$$\ln(P_x + iP_y) = \int \frac{-ieEt}{E_{kin}}\, dt$$

$$= -i\varphi$$

$$P_x + iP_y = C \exp(-i\varphi)$$

where $d\varphi = -\frac{eEt}{E_{kin}}\, dt$

Note that this yield

$$\frac{E_{kin}}{eE} \sin(\frac{eEt}{E_{kin}}) = t$$

$$P_x + iP_y = E_{kin}(v_x + iv_y) = E_{kin}(\dot{\vec{x}} + i\dot{\vec{y}})$$

$$= \frac{d(x+iy)}{d\varphi}(eH)$$

$$x = \frac{cE_{kin}}{eH}\sin\varphi \qquad y = \frac{cE_{kin}}{eH}\cos(\varphi)$$

2. Alternative relativistic motion of $H \perp E$ and $\|H\| = \|E\|$

$$\|H\| = \|E\|$$

$$\dot{P} = eE + e(v \times H)$$

$$\dot{P}_x = eE(1 + v_y H) = eE(1 + v_y)$$

$$\dot{P}_y = -e v_x H = -e v_x E$$

$$\dot{P}_z = 0 \Rightarrow P_z = \text{const}$$

$$\frac{dE_{kin}}{dt} = eE v_x \qquad E_{kin} \| P_z = \text{const}$$

$$E_{kin} = eEx$$

$$v = \frac{P}{E}$$

$$\dot{x} eEx + (\dot{x})^2 eEx = \dot{P}_x$$

$$\dot{v}_x E_{kin} + v_x \dot{E}_{kin} = \dot{E}_{kin} \Rightarrow \dot{E}_x(v_x^2 + 1) = v_x \dot{E}_{kin}$$

recall that

$$E_{kin}^2 = E^2 + P_x^2 + P_z^2$$

$$E = E^2 + P_x^2$$

$$\frac{d}{dt}(E_{kin} - P_z) = \frac{P_x \dot{P}_x}{E}$$

$$\int eEx\, dx = \int P_x\, dP_x$$

$$2\sqrt{eEx} = \sqrt{P_x^2} = P_x$$

3) Velocity of drift of $\vec{B}$ induced under influence of a non-relativistic charge particle in quasi-homogeneous magnetic field

$$a = \underbrace{f_1(t)}_{\text{force of linear oscillator}} + \underbrace{f_2(t)}_{\text{force of nonlinear oscillator}}$$

$$m\ddot{\vec{R}} = \frac{e}{c}\dot{\vec{R}} \times H + \frac{e}{c}\dot{R} \times (\vec{S} \cdot \nabla) H$$

$$F = -\frac{e}{c}\dot{\vec{R}} \times (\vec{S} \cdot \nabla) H$$

assume constant orbit

$$\vec{S} = \omega \vec{S}_0 \times \hat{n} \qquad S = \frac{V_0}{c}$$

I don't understand this qm at all...

Relativistic motion of electron where $E \| H$

Suppose we have fields where

$$E \| H \| V \perp \hat{z}$$

One moment does E' change when we consider movement perpendicular to E' objects, we now know

$$E' = \gamma(E + v \times H) - \gamma(\gamma - 1)(E \cdot \hat{v})\hat{v}$$

$$H' = \gamma(H - v \times E) - \gamma(\gamma - 1)(H \cdot \hat{v})\hat{v}$$

I can invert the lorentz transform by setting $v \to -v$

I further note that for any $V \| E', H'$ the will work, so I only consider

$$E \times H = \gamma^2(E \times H - (v \times H) \times H - E \times (v \times E) - (v \times H)(E \cdot v))$$

$$= \frac{\gamma^2}{1-v^2}\left(E^{-1} \times H^{-1}\right)$$

$$E^2 = \left((E^{-1})^2 + E^{-1}(v \times H) + v^2 E^2\right)\gamma^2$$

$$H^2 = \left((H^{-1})^2 - 2H^{-1}(v \times E) + v^2 E^2\right)\gamma^2$$

$$E^2 + H^2 = \gamma^2\left((E^{-1})^2 + H^{-1}\right)\left(1 + v^2\right)$$

$$E \times H = \gamma^2 v\left(E^2 + H^2\right) = \gamma^2(1 + v^2)$$

$$\frac{E \times H}{E^2 + H^2} = \frac{v}{1 + v^2}$$

The EM Field Equations

Transformations of energy density, energy flux density and stress tensor under Lorentz boost

Recall that

$$T = \begin{bmatrix} W & S_x & S_y & S_z \\ S_x & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\ S_y & -\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz} \\ S_z & -\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz} \end{bmatrix}$$

assuming $V \parallel \hat{x}$

$$W' = \frac{1}{1-V^2}\left(W - 2VS_x - V^2\sigma_{xx}\right)$$

$$S_x' = \frac{1}{1-V^2}\left(S_x(1+V^2) + VW + V\sigma_{xx}\right)$$

$$S_y' = \frac{1}{\sqrt{1-V^2}}\left(S_y + V\sigma_{xy}\right)$$

$$S_z' = \frac{1}{\sqrt{1-V^2}}\left(S_z + V\sigma_{xz}\right)$$

$$\sigma_{xx}' = \frac{1}{1-V^2}\left(+\sigma_{xx} - 2VS_x + V^2W\right)$$

$$\sigma_{xy}' = \frac{1}{\sqrt{1-V^2}}\left(\sigma_{xy} + VS_y\right)$$

Determine the force between two charged masses at velocity $\vec{v}$

$F = e_1 E_1 + e_2 v \times H_1$

consider frame where both particles are at rest

$V = 0$

consider

$\vec{r}'$, E' $\frac{e_1 r_1}{|R|}$ in the rest frame

in frame of interest

$r = \frac{\vec{R}}{R\sqrt{1-v^2}}$

$x' = \frac{x - vt}{\sqrt{1-v^2}}$ $y' = y$, $z' = z$

$R'^2 = \frac{(x-vt)^2 + (1-v^2)(y^2+z^2)}{1-v^2}$ $R'^2 = (x-vt)^2 + (1-v^2)(y^2+z^2)$

$\varphi = \frac{e}{\sqrt{(x-vt)^2 + (1-v^2)(y^2+z^2)}}$ $A = \frac{e\vec{v}}{\sqrt{(x-vt)^2 + (1-v^2)(y^2+z^2)}}$

$E_1' = \frac{e_1 \vec{R}'}{R'^3}$

$E_{1x} = E_{1x}' = \frac{e_1 x'}{R'^3} = \frac{e(x-vt)}{(\sqrt{(x-vt)^2 + (1-v^2)(y^2+z^2)})^{3/2}/(1-v^2)^{3/2}}$

$= \frac{e(x-vt)}{R^3}(1-v^2)$

$E_y = \frac{E_y'}{\sqrt{1-v^2}} = \frac{e_1 y}{R^3}(1-v^2)$

$E = (1-v^2)\frac{e_1 R}{(R^2)^{3/2}}$

$H = V \times E$

$F = e_1 \quad E + e_2 V \times (V \times E)$

$= e_1 \quad E' + e_2 (\vec{V}(V \cdot E) - \vec{E} V^2)$

$F = e_1(1-v^2)E + e_2 (\vec{V}) (V \cdot E)$

let θ be angle between V, E, $V \cdot E$

$F = e_1 (1-v^2)\frac{e_1 R}{(1-v^2 \sin^2\theta)^{3/2}} \quad \frac{(1-v^2)}{R^2} + e_2 V\left(v \cdot \frac{e_1 R}{(1-v^2 \sin^2\theta)^{3/2}}\right)$

$F = \frac{e_1(1-v^2)^2 \sin\theta}{(1-v^2 \sin^2\theta)^{3/2}} \quad F_x = \frac{e_1 e_2 (1-v^2)^2 \cos\theta}{(1-v^2 \sin^2\theta)^{3/2} R^2} + \frac{e_2 e_1 v^2 (1-v^2)\cos\theta}{R^2 (1-v^2 \sin^2\theta)^{3/2}}$

$F_x = \frac{e_1 e_2 \cos\theta (1-v^2)}{R^2 (1-v^2 \sin^2\theta)^{3/2}}$

Determine Angle of Deflection of charge passing through repulsive Coulomb field

$\chi = \pi - 2\theta_0$

where θ_0 is val of θ at $r = r_0$

For $\alpha < 0$ infinite orbits only

when $E > m$ and $M > |\alpha|$

$0 = \sqrt{(M\alpha)^2 - m^2(M^2 - m^2)} \cos\left(\sqrt{1 - \frac{\alpha^2}{M^2}}\right) - \alpha$

$\frac{E\alpha}{\sqrt{(M\alpha)^2 - m^2(M^2 - \alpha^2)}} = \cos\left(\theta\sqrt{1 - \frac{\alpha^2}{M^2}}\right)$

$\theta_0 = \frac{1}{\sqrt{1 - \frac{\alpha^2}{M^2}}} \cos^{-1}\left(\frac{E\alpha}{\sqrt{(M\alpha)^2 - m^2(M^2 - \alpha^2)}}\right)$

$\varepsilon = \frac{m}{\sqrt{1-v^2}}$

$\theta_0 = \frac{1}{\sqrt{1 - \frac{\alpha^2}{M^2}}} \cos^{-1}\left(\frac{m\alpha}{\sqrt{1-v^2}\sqrt{M^2\left(\frac{m^2}{1-v^2}\right) - m^2(M^2 - \alpha^2)}}\right)$

$\theta_0 = \frac{1}{\sqrt{1 - \frac{\alpha^2}{M^2}}} \cos^{-1}\left(\frac{\alpha}{\sqrt{M^2 - (1-v^2)(M^2 - m^2)}}\right)$

$\theta_0 = \frac{1}{\sqrt{1 - \frac{\alpha^2}{M^2}}} \cos^{-1}\left(\frac{\alpha}{\sqrt{m^2 v^2 + M^2 - v^2 M^2}}\right)$

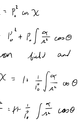
$\theta_0 = \frac{1}{\sqrt{1 - \frac{\alpha^2}{M^2}}} \tan^{-1}\left(\frac{\alpha}{\sqrt{M^2 - m^2}}\right)$

$\theta_0 = \frac{M}{\sqrt{M^2 - m^2}} \tan^{-1}\left(\frac{\alpha}{\sqrt{M^2 - m^2}}\right)$

Determine scattering cross section in Coulomb field (at small angles)

$d\sigma = 2\pi\rho\, d\rho = 2\pi\rho \frac{d\rho}{d\chi} d\chi$

ρ is impact parameter



$P_{\infty} = P_0 + \int_{-\infty}^{\infty} dt$

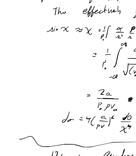
$P_{\infty}' P_0' = P_0' \cos\chi$

$P_{\infty}' P_0' = P_0^2 + P_0 \int_{-\infty}^{\infty} \frac{\alpha}{r^2} \cos\theta\, dt$ where θ is angle between field and P_0

$\cos\chi = 1 + \frac{1}{P_0^2} \int_{-\infty}^{\infty} \frac{\alpha}{r^2} \cos\theta\, dt$

$1 - \frac{\chi^2}{2} = 1 + \frac{1}{P_0^2} \int_{-\infty}^{\infty} \frac{\alpha}{r^2} \cos\theta\, dt$

$\chi^2 = \frac{2}{P_0^2} \int_{-\infty}^{\infty} \frac{\alpha}{r^2} \cos\theta\, dt$



Because, this is small angle scattering, we can consider interactions only subsequent to the force deflecting the particle upwards. This is effectively $\frac{\pi}{2}$ $\frac{L}{2}$

We can then assume to say $\sin\chi$ is sine to the sum of the upward deflection.

The effectively yields

$\sin\chi \approx \chi = \frac{1}{P_0} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\alpha}{r^2} \cos\theta\, dt$

$= \frac{1}{P_0} \int_{-\infty}^{\infty} \frac{\alpha\, r}{(r^2 \dot{\varphi})^2} dt$

$= \frac{2\alpha}{P_0 r_0}$

$d\sigma = \pi\left(\frac{\alpha}{p_0 v}\right)^2 \frac{d\rho}{\chi^2}$

Determine Quadrupole moment of an ellipsoid with uniform charge density

$D_{ij} = \iiint \rho\, (3x_i x_j - (x_1^2 + x_2^2 + x_3^2)\delta_{ij})\, dx_1 dx_2 dx_3$

$D_{xx} = \rho \iiint (2x^2 - y^2 - z^2)\, dx_1 dx_2 dx_3$

Should be given compared to spherical

$x \to cx$, $y \to by$, $z \to cz$

$D_{xx} = \rho \iiint (2(\frac{x}{a})^2 - \frac{y^2}{b^2} - \frac{z^2}{c^2})\, dV$

$x = r\sin\theta\cos\varphi$

$y = r\sin\theta\sin\varphi$

$z = r\cos\theta$

$D_{xx} = \rho \int_0^1 \int_0^{\pi} \int_0^{2\pi} r^2 \left(\frac{2\sin^2\theta\cos^2\varphi}{a^2} - \frac{\sin^2\theta\sin^2\varphi}{b^2} - \frac{\cos^2\theta}{c^2}\right) r^2 \sin\theta\, d\varphi\, d\theta\, dr$

$D_{xx} = \rho \int_0^1 \int_0^{\pi} r^4 \left(\frac{2\sin^3\theta\cos^2\varphi}{a^2} - \frac{\sin^3\theta\sin^2\varphi}{b^2} - \frac{\cos\theta\sin\theta}{c^2}\right) d\varphi\, d\theta\, dr$

$= \rho\left(\frac{2}{5\pi}\right)\left(\frac{4}{3}\right)(\pi) - \rho\frac{2\pi}{5}\frac{\pi}{3} - \frac{2\pi}{5}\frac{(2\pi)}{3}$

Find value of momentum vector and angular momentum for a system of 2 charges in the non-relativistic limit

Choose origin at COM

$\vec{R} = \frac{m_1 r_1 + m_2 r_2}{m_1 + m_2} = 0 \quad r_2 = \frac{m_1 r_1}{m_2}$

$P_{tot} = m_1 \dot{r}_1 + m_2 \dot{r}_2 = \frac{d}{dt}(m_1 r_1 + m_2 \underset{\text{always } 0}{\vec{R}}) = 0$

so $P_1 = -P_2 = \vec{P}$

$\vec{L} = \frac{1}{2c}\left(e_1 \frac{r_1}{c} \times \vec{P}_1 + e_2 \frac{r_2}{c} \times P_2\right)$

we need $r_1 \times P_1 + r_2 \times P_2 = \vec{M}$

$(r_1 - r_2) \times P = \vec{M}$

$\frac{m_1 + m_2}{m_1} r_1 \times P = \vec{M}$

$\vec{M} = \frac{1}{2c}\left(\frac{e_1}{c}(r_1 \times P_1) + \frac{e_2}{c}\left(\frac{m}{m_2}(r_1 \times P_1)\right)\right)$

$\vec{L} = \frac{1}{2c}\left(\frac{e_1}{c}\left(\frac{m_2}{m_1}\right)\vec{M} + \frac{e_2}{c}\left(\frac{m_1}{m_2}\right)\left(\frac{m_2}{m_1 + m}\vec{M}\right)\right)$

$\vec{L} = \frac{1}{2c}\frac{m_1 m_2}{m_1 + m_2}\left(\frac{e_1}{m_1^2} + \frac{e_2}{m_2^2}\right)\vec{M}$

Determine the force exerted on the wire from that an incident electromagnetic

$E = \nabla U$

$\nabla U = W \cdot \hat{n}$

$E = W n(\theta) + K W n'(\theta) n'$ $\quad n' = \cos\theta\, \hat{n}$

$F_N = W \cos\theta \cos\theta + K W \sin\theta \cos\theta$

$= W(\cos^2\theta) + K W \sin\theta \cos\theta$

$F_T = W \sin\theta \cos\theta - K W \cos\theta \cos\theta$

$= W(-\sin\theta) \cos^2\theta$

Use Hamilton-Jacobi method to find the motion of a charge in an EM field given by a plane wave with vector potential

$A(t, x)$

Recall

$(\nabla - eA)^2 = m^2 (\tau - eA)^2$

$W = \frac{\partial S}{\partial \tau}$ $\quad P = \nabla S$

$\tau = 0$

$\left(\frac{\partial S}{\partial t}\right)^2 - (\nabla S - eA(t, x))^2 = m^2$

This yields the following differential equation

$\left(\frac{\partial S}{\partial t}\right)^2 - \left(\frac{\partial S}{\partial x_1} - eA_1(t, x)\right)^2 - \left(\frac{\partial S}{\partial x_2}\right)^2 - \left(\frac{\partial S}{\partial x_3}\right)^2 = m^2$

$= e^2 (A_1^2 + A_2^2 + A_3^2)$

where $A^1 = A_1, A_2, A_3$

$\frac{\partial S}{\partial t} = \frac{\partial S}{\partial x_1} + \frac{\partial S}{\partial x_2} + \frac{\partial S}{\partial x_3} + e A_1(t, x) - e A_1(t, x) + m^2$

$\frac{\partial S}{\partial x_1} = e A_1(t, x) \Rightarrow \frac{\partial S}{\partial x_1} = e A_1(t, x) + m^2$

$A = A_{\text{incident}} + A_{\text{scattered}}$

$A_1 = E_0 e^{i\omega t}$ and then

$(E_0 e^{i\omega t})^2 = m^2 (\tau - eA)^2$

Then

$A_{\text{incident}} = 0$

$\left(\frac{\partial S}{\partial t}\right)^2 + e A_1(t, x)\, \frac{\partial S}{\partial x_1} + e A_1(t, x) = 0$

Let

$\xi = k_1 x$ and that $k_2 x = 0$

$\frac{\partial S}{\partial t} = \frac{\partial S}{\partial(\omega)} = \frac{1}{k} \frac{\partial S}{\partial x_1}$, $\quad A = F(\xi)$

$k_1 \frac{\partial S}{\partial \xi} = \omega$

$\Rightarrow \left(\frac{\partial S}{\partial \xi}\right)^2 + e A_1(t, x) \frac{\partial S}{\partial \xi} = \left(k_1 \frac{\partial S}{\partial \xi}\right)^2 + F^2 \omega_1^2 (t, x)$

$\omega^2 (k_1 A^2 + k_1^2 A^2) + k_1 \omega \left(\frac{\partial S}{\partial \xi}\right) A + \left(\frac{\partial S}{\partial \xi}\right)^2 = F^2 \omega_1^2$

can finally set $k_2 x = 0$ as well

$2k_1 \frac{\partial S}{\partial \xi} = \omega^2 (k_1 A^2 + k_1^2 A^2)$

since A is only a t, x function, we can rewrite $k_2 x^2 = t, x$ like $k_1 (t, 0, 0)$

Call this chunk y & should not be rewrite

$F^2 \omega_1^2 = t_1^2 \cdot \frac{\omega^2}{4} \cdot k_1^2 \cdot \omega^2$

The action is now

$S = -k_1 \xi - \frac{t}{2} (\omega) + \frac{\omega^2 k_1^2}{32} t + \frac{1}{2} \int \sqrt{F^2 \omega_1^2}\, d\xi$

$= \frac{\omega}{2} \int \sqrt{\omega_1^2}$

Solve for S by method of Lagrange Multiplier

$\frac{\partial S}{\partial t} = 0 = \frac{\omega}{2} + \frac{\omega^2 k_1^2}{32} t \quad \xi(t) = \frac{\omega}{2} \int \sqrt{k_1 A^2 + \frac{k_1^2}{2} A^2}\, d\xi$

$\frac{\partial S}{\partial \xi} = 0 = -\xi \cdot t - \frac{k_1}{2} \xi - \frac{\omega}{2} \int \xi \, d\xi$

solving for x, y, z

$\xi = \xi x$

$x = \frac{1}{2} k_1 \xi - \frac{\omega}{2} \int \sqrt{k_1 A^2}$ $\quad y = \frac{1}{2} k_2 \xi - \frac{\omega}{2} \int k_2 A$

$x = \frac{1}{2} \left(\frac{\omega^2 k_1^2}{32}\right) \cdot \xi \quad z = \frac{\omega}{2} \int \sqrt{k_1 A^2 + \frac{k_1^2}{2} A^2}\, d\xi$

Determine polarization in terms of E_0 (left direction was matching)

$\vec{E} = \hat{\rho}_R \left[E_1 \cos(\omega t + \omega(t - x)) \right]$

Can write E_0 as

$E_0 = E_0 \exp(i\omega) + E_0 \exp(-i\omega)$ $\quad$ Use the left

Note that

$E_1 x^2 = k_1 A_1^2 + k_1^2 A_1^2 + k_2^2$

$E_1 x^2 = \frac{1}{2} \cdot A_1^2 \cdot k_1$

Given by we may can step

$\frac{E_0}{\exp(-i\omega)} + \frac{E_0}{\exp(i\omega)} \Rightarrow$ then left defines 2 fields

$\lim_{\omega \to 0} \frac{\omega \sin(\omega t) \cos\theta}{\omega^2 \cos^2\theta}$

Determine the nature of a charge in a monochromatic linearly polarized wave

Assume polarization is along y axis of the form

$E_0 = E \times E_0 \cos(\omega t - \frac{\omega x}{c})$

Note that $-\nabla \times E$

$A = A_0 = \frac{c}{\omega} E_0 \sin(\omega t - \frac{\omega x}{c})$

Recall that

$\gamma^2 = m^2 c^2 + \frac{p_x^2}{c^2} c^2 = m^2 c^2 + \frac{e^2 E_0^2}{\omega^2} c^2$

$\gamma = \frac{\omega}{\sqrt{2}} \left[\frac{c}{\omega} E_0 \sin^2(\omega t - \frac{\omega x}{c}) - \frac{c}{\omega} E_0 \right]$

$\omega^2 = \frac{\omega}{c^2} \cdot \frac{c}{\omega} E_0 \sin(\omega t - \frac{\omega x}{c})$

$\xi = \frac{\omega}{\sqrt{2}} \left[\frac{c}{\omega} E_0 \sin^2(\omega t - \frac{\omega x}{c}) - \frac{c}{\omega} E_0 \right]$

Repeat for circularly polarized field

Same as above with

$E_y = E_0 \cos\omega t \quad E_z = E_0 \sin\omega t$

Positive ambiturning partially polarized light was reflected and refracted parts

Consider Energy flux density tensor $\mathcal{T}$

Result that normal (unpolarized) is proportional to identity so

$\mathcal{T} = \frac{c}{8\pi} \begin{pmatrix} \mathcal{T}^{00} & 0 \\ 0 & \mathcal{T}^{ij} \end{pmatrix}$ $\quad$ or polarization tensor

ρ_{ij}

For determining the determinant of ρ_{ij}

Lets find the closer into the form of the density parameters

Consider

$\| J - \frac{1}{2} \mathcal{T}^{00} \mathcal{T} \| = 0$

$\left\| \mathcal{T} \begin{pmatrix} \frac{c}{8\pi} & \frac{c}{8\pi} E_x \\ \frac{c}{8\pi} E_x & \frac{c}{8\pi} E_y \end{pmatrix} - \frac{c}{2} \begin{pmatrix} \mathcal{T}^{00} & 0 \\ 0 & \mathcal{T}^{ij} \end{pmatrix} \right\| = 0$

$\left\| \begin{pmatrix} \frac{(\mathcal{T}^{00} - \mathcal{T}^{ij})}{2} & \frac{c}{8\pi} E_x \\ \frac{c}{8\pi} E_x & \frac{(\mathcal{T}^{00} - \mathcal{T}^{ij})}{2} \end{pmatrix} \right\| = 0$

$\left[\frac{(\mathcal{T}^{00} - \mathcal{T}^{ij})}{2} \right]^2 - E_x^2 \left[\frac{(\mathcal{T}^{00} - \mathcal{T}^{ij})}{2} \right]^2 = E_x^2 (\mathcal{T}^{00} - E_x^2 (\mathcal{T}^{ij})^2 = 0$

$\mathcal{T}^{00} = \frac{2(\mathcal{T}^{11} + \sqrt{(\mathcal{T}^{11})^2 - 4(\mathcal{T}^{11} - \mathcal{T}^{22})})}{2}$

$\mathcal{T}^{00} = \| \mathcal{T} (1 \pm \mathcal{P}^{11}$

$= \| \mathcal{T} (1 \pm \mathcal{P})$ $\quad$ as above with + solution has no meaning

$\mathcal{T}^{00} = \mathcal{T} (1 - \mathcal{P})$

Then $\mathcal{T}^{00} = \mathcal{T} \mathcal{P}$

Represent an arbitrary partially polarized wave as the sum of 2 elliptically polarized waves

Consider the arbitrary polarization tensor

$\rho_{ij} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{pmatrix}$

Note that this can be rewritten as

$\rho_{ij} = \lambda_1 |\alpha_1\rangle\langle\alpha_1| + \lambda_2 |\alpha_2\rangle\langle\alpha_2|$

Now another product corresponds to each ellipse

Lets begin with the eigenvalue relations

$\begin{vmatrix} 1 - \xi_2^2 \cdot \lambda & \xi_1 - \xi_2 \\ \xi_2 + \xi_1 & 1 - \xi_2 \cdot \lambda \end{vmatrix} = 0$

$1 - \xi_2^2 - 2\lambda + \lambda^2 = \xi_1^2 - \xi_2^2 \lambda^2 = 0$

$\lambda^2 - 2\lambda + 1 - \xi_1^2 = 0$

$\lambda^2 - 2\lambda + 1 - P^2 = 0$

$\lambda = \frac{2 \pm \sqrt{4 - 4(1 - P^2)}}{2}$

$\lambda = 1 \pm P$

We can now find the 2 eigenvectors

$\begin{pmatrix} 1 - \xi_2^2 \cdot \lambda & \xi_1 - \xi_2 \\ \xi_2 + \xi_1 & 1 - \xi_2 \cdot \lambda \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = (1 \pm P) \begin{pmatrix} a \\ b \end{pmatrix}$

$(1 \pm \xi_2) a + (\xi_1 - \xi_2) b = (1 \pm P) a$

$(P - \xi_2) a = (\xi_1 - \xi_2) b$

$\begin{pmatrix} 1 \\ \frac{\xi_1 - \xi_2}{P - \xi_2} \end{pmatrix}$ is one eigenvector

Find perpendicular for the other

Find rotation parameter subject to a rotation by angle θ

$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{pmatrix} 1 + \xi_2 & \xi_1 - \xi_2 \\ \xi_1 + \xi_2 & 1 - \xi_2 \end{pmatrix}$

$\begin{pmatrix} \cos\theta(1 + \xi_2) + \sin\theta(\xi_1 + \xi_2) & \cos\theta(\xi_1 - \xi_2) + \sin\theta(1 - \xi_2) \\ -\sin\theta(1 + \xi_2) + \cos\theta(\xi_1 + \xi_2) & -\sin\theta(\xi_1 - \xi_2) + \cos\theta(1 - \xi_2) \end{pmatrix}$

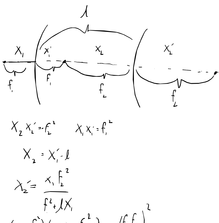
$\cos\theta(1 + \xi_2) + \sin\theta(\xi_1 + \xi_2) = \sin\theta(1 + \xi_2')$

$\xi_1' + \xi_2' = -\sin\theta(1 + \xi_2) + \cos\theta(\xi_1 + \xi_2)$

Propagation of Light

Find focal distance for image formation with the aid of circuits symmetric optical system when optical axes coincide

$X_1 X_1' = -f_1^2 \quad , \quad X_2 X_2' = -f_2^2$



$X_2 X_2' = -f_2^2 \qquad X_1 X_1' = f_1^2$

$X_2 = X_1' - l$

$$X_2' = \frac{X_1 f_2^2}{f_1^2 + l X_1}$$

$$\left(X_1 + \frac{f_1^2}{l}\right)\left(X_2' - \frac{f_2^2}{l}\right) = -\left(\frac{f_1 f_2}{l}\right)^2$$

focal length

Focal length for charged particle of e magnetic ~~beam~~ in the form of a longitudinal homogeneous field of length l -

$$(\nabla \phi - \frac{e}{c} A)^2 = p^2$$

$$p^2 = \frac{E^2}{c^2} - m^2 c^2 \text{ - constant}$$

$$(\nabla \phi - \frac{e}{c} H \times r)^2 = p^2$$

$$\left(\frac{\partial \phi}{\partial z}\right)^2 + \left(\frac{\partial \phi}{\partial r}\right)^2 + \frac{e^2}{4c^2} H^2 r^2 = p^2$$

solve for ϕ in terms of z and r.

If we have a narrow beam around the optical axis

$$\phi = pz + \frac{1}{2}\sigma(z) r^2$$

$$(p + \frac{1}{2}\sigma'(z) r^2)^2 + (\sigma'(z) r \phi)^2 + \frac{e^2}{4c^2} H^2 r^2 = p^2$$

$$r^2 \sim 0$$

$$p\sigma'(z) + \sigma^2(z) + \frac{e^2}{4c^2} H^2 = 0$$

so $\sigma(z) \sim \frac{k}{z-f}$ outside of the field

Inside the field

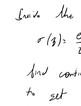
$$\sigma(z) = \frac{eH}{2c} \cot\left(\frac{eH}{2cp} z + C\right)$$

find constants when $z=0$ $z=l$ to get to matching of the form

$$(x-f)(x_2+l) = -f^2$$

$$f = \frac{2cp}{eH \sin\left(\frac{eHl}{2cp}\right)}$$

Determine the order of magnitude of the smallest width of light beam produced from a parallel beam at distance l from diaphragm



$$\theta_y = \frac{1}{k a_y} = \frac{\lambda}{d}$$

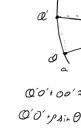
width $= d\theta \left(\frac{\lambda}{d}\right) \lambda$

when does this minimize

$$|\lambda - \frac{\lambda}{d^2} l| = 0$$

$$d \sim \sqrt{\lambda l}$$

Determine intensity of light in neighborhood of the point where the ray of light is tangent to the caustic



$$CO'O'O \approx CO \quad \text{for small const } \theta$$

$$CO'O' \cdot p \sin\theta \approx CO - OO' \cdot p \sin\theta$$

curvature

$$\approx CO - p\theta p \sin\theta$$

$$\approx CO - p\theta^2$$

$$CO'P \approx CO' - x \sin\theta$$

$$\approx CO' - x\theta$$

$$R = D - x\theta - \frac{1}{2} p\theta^3$$

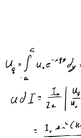
const

$$U_y = \int_{-\infty}^{\infty} \exp\left(-ikx\theta - \frac{ikx}{c}\theta^3\right) d\theta$$

$$U_y = 2\int_0^{\infty} \cos\left(kx\theta + \frac{kx}{c}\theta^3\right) d\theta$$

$$I \sim |U_y|^2$$

Calculate the Fraunhofer diffraction of a plane wave normally incident on an infinite slit (of width $2a$) with parallel sides cut in opaque screen

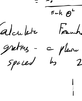


$$U_q = \int_{-a}^{a} u_0 e^{-iqy} dy = \frac{2u_0}{q} \sin(qa)$$

$$\text{wd } I = \frac{I_0}{2\pi} \left|\frac{U_q}{u_0}\right|^2 \frac{d_y}{2\pi} = \frac{I_0}{4\pi} \frac{4u_0^2 \sin^2 qa}{q^2} \frac{dq}{2\pi}$$

$$= \frac{I_0 \sin^2(ka\theta)}{\pi k \theta} d$$

Calculate Fraunhofer diffraction by a diffraction grating - a plane with identical slits width $2a$ spaced by $2b$. N of them



$$U_q' = \int_{-a}^{2a} u_0 \exp(-iq\theta) d\theta = -\frac{u_0}{iq} \underbrace{\left(\exp(-2iqa) - 1\right)}_{\text{shift } (a \to 2a)}$$

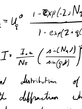
$$U_q = \sum_{n=1}^{N} U_q'$$

$$U_q = \sum_{n=1}^{N} u_0 \exp(-2ib(n-1))$$

$$U_q = U_q^0 \frac{1 - \exp(-2iN_q(n-1))}{1 - \exp(2i(n-1))}$$

$$I \sim \frac{I_0}{\pi} \left(\frac{\sin(N_q)}{\sin(qb)}\right)^2 \left(\frac{\sin(qa)}{qa}\right)^2 dq$$

Find distribution of intensity over direction for the diffraction that is incident normal to the plane of circular aperture of radius a



$$U_q = U_0 \int_0^{2\pi} \int_0^{a} \exp(-iq\, r\cos\varphi)\, r\, dr\, d\varphi = 2\pi u_0 \int_0^{a} J_0(qr)\, r\, dr$$

Derive Liénard-Wiechert potentials by integration expressions for retarded potentials

$$\varphi = \iiint \frac{\rho(\vec{r}', t - \frac{R}{c})}{R} dV' + \varphi_0 \leftarrow \text{not part of retarded potential}$$

$$\vec{R} = \vec{r} - \vec{r}' \qquad dV' = dx'dy'dz'$$

R is distance from volume element to field point

consider ρ as a point charge

$$\rho(\vec{r}', t - \tfrac{R}{c}) = e\delta[\vec{r}' - \vec{r}_0(\tau)]$$

$$\varphi = \iint \frac{e\,\delta[\vec{r}' - \vec{r}_0(\tau)]}{|\vec{r} - \vec{r}'|} \delta\left(\tau - t + \tfrac{1}{c}(\vec{r} - \vec{r}')\right) d\tau\, dV'$$

$$\varphi = e \int \frac{\delta(\tau - t + \frac{1}{c}(\vec{r} - \vec{r}_0(\tau)))}{|\vec{r} - \vec{r}_0(\tau)|} d\tau$$

$$\tau' = t - \tfrac{1}{c}(\vec{r} - \vec{r}_0(\tau))$$

Charge moving in uniform rectilinear motion

$$\rho = e\delta(\vec{r} - \underset{\text{const}}{\vec{v}}t)$$

$$\Delta\varphi - \frac{1}{c^2}\ddot{\varphi} = -4\pi e\delta(\vec{r} - \vec{v}t)$$

$$\varphi = \int \varphi_\omega e^{i\omega R/c} d\omega$$

$$\varphi_\omega = \int \varphi\, e^{-i\omega R/c} dV$$

$$\Delta\varphi_\omega = -(k_x^2 + k_y^2 + k_z^2)\int \varphi \exp(-ik\vec{r}) dV$$

$$= -k^2 \int \varphi \exp(-ik\vec{r}) dV$$

$$-k^2\varphi_k - \frac{1}{c^2}\ddot{\varphi}_k = -4\pi e \exp(-ik\cdot vt)$$

$$\varphi_k = A \exp(-ik\cdot vt)$$

$$+k^2 A \exp(-ik\cdot vt) + \frac{A}{c^2}(-ik\cdot v)^2 \exp(-ik\cdot vt) = 4\pi e \exp(-ik\cdot vt)$$

$$A\left(k^2 - \frac{(k\cdot v)^2}{c^2}\right) = 4\pi e$$

$$\varphi_k = \frac{4\pi e}{k^2 - \frac{(k\cdot v)^2}{c^2}}$$

$$\Delta A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = -\frac{4\pi}{c}\vec{j}$$

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot j$$

$$e\frac{\partial}{\partial t}\delta(\vec{r} - \vec{v}t) = -\nabla j$$

$$ev\,\nabla\cdot\delta(\vec{r} - \vec{v}t) = -\nabla\cdot j$$

$$j = -ev\,\delta(\vec{r} - \vec{v}t)$$

$$\Delta A - \frac{1}{c^2}\frac{\partial^2 A}{\partial t^2} = 4\pi ev\,\delta(\vec{r} - \vec{v}t)$$

$$-k^2 A_k - \frac{\ddot{A}_k}{c^2} = -4\pi ev \exp(-ik\cdot vt)$$

$$A_k = B\exp(-ik\cdot vt)$$

$$-k^2 B + \frac{(k\cdot v)^2}{c^2}B = 4\pi ev$$

$$B = \frac{+4\pi e\vec{v}}{k^2 - \left(\frac{k\cdot v}{c}\right)^2}$$

$$A_k = \frac{4\pi e\vec{v}}{k^2 - \left(\frac{k\cdot v}{c}\right)^2}\exp(-i\vec{k}\cdot\vec{v}t)$$

$$E = -\nabla\varphi - \frac{\partial A}{\partial t}$$

$$E_k = ik\varphi_k + i(k\cdot v)A_k$$

$$H = \nabla\times A$$

$$H_k = ik\times A_k$$

Determine (correctly to second order) the centre of inertia of a system of interacting particles

$$\vec{R} = \frac{\sum \underbrace{\xi_a r_a}_{\text{kinetic energy}} + \int \underbrace{W_2 r dV}_{\text{field energy}}}{\sum \xi_a + \int W dV}$$

$$\int W_2 r dV = \frac{1}{8\pi}\int (E^2 + H^2) r dV \quad \text{proportional to } 1/c^2, \text{ basically } 0$$

$$\int E^2 r dV = \int (\nabla\varphi)^2 r dV$$

$$(\nabla\varphi)^2 = \nabla\varphi\,\nabla\varphi$$

$$\nabla(\varphi^2) = \nabla\varphi\,\nabla\varphi + \varphi\nabla^2\varphi$$

$$\int E^2 r dV = \underbrace{\int \nabla(\varphi^2 r) dV}_{\text{IBP}} - \int \varphi\nabla^2\varphi\, r dV$$

$$\underbrace{\frac{\nabla\varphi^2}{2}\vec{r}\Big|_V}_{0 \text{ because of boundaries}} - \underbrace{\vec{r}\int\frac{\nabla\varphi^2}{2}dV}_{\text{also } 0 \text{ because of boundaries}}$$

$$\int E^2 r dV = -\int \varphi\,\nabla^2\varphi\, r dV$$

recall $\nabla^2\varphi = -4\pi\rho$

$$\int E^2 r dV = 4\pi\int \rho r \rho\, dV$$

sum over infinitesimal charges

$$\int W r dV = \frac{1}{2}\sum \varphi_a r_a q$$

$$\vec{R} = \frac{\sum r_a\left(m_a c^2 + \frac{p_a^2}{2m_a} + \varphi_a q_a\right)}{\sum \xi_a + \sum \varphi_a q_a}$$

energy from perturbations of all other charges at charge q

Write the Hamiltonian in second approximation for system of two particles, omitting motion of system as a whole

$$\vec{P}_1 + \vec{P}_2 = 0$$

$$\mathcal{H} = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - \frac{p_1^4}{8c^2m_1^3} - \frac{p_2^4}{8c^2m_2^3} + \frac{q_1 q_2}{r_1 - r_2}$$

$$- \frac{q_1 q_2}{2c^2 m_1 m_2 |r_1 - r_2|}\left[P_1\cdot P_2 + (P_1\cdot\frac{r_1 - r_2}{|r_1 - r_2|})(P_2\cdot\frac{r_1 - r_2}{|r_1 - r_2|})\right]$$

The Constant Gravitational Field

Derive equations of motion from principle of least action

$$\delta(ds^2) = 2ds\,\delta ds = 0$$

$$\delta(ds^2) = \delta(g_{ik}dx^i dx^k)$$

$$= \delta g_{ik}\,dx^i dx^k + g_{ik}\,\delta dx^i dx^k + g_{ik}\,dx^i \delta dx^k$$

$$= \delta g_{ik}\,dx^i dx^k + 2g_{ik}\,dx^i \delta dx^k$$

$$\delta ds = \frac{\delta g_{ik}}{2}\frac{dx^i}{ds}\frac{dx^k}{ds}ds + g_{ik}\frac{dx^i}{ds}\delta\frac{dx^k}{ds}ds$$

$$\mathcal{L} = mc\int \frac{1}{2}\left(u^i u^k \frac{\partial g_{ik}}{\partial x^l}\delta x^l + \underbrace{2g_{ik}\frac{dx^i}{ds}\frac{d\delta x^k}{ds}}_{\text{by parts}}\right)ds$$

$$\int g_{ik}\frac{dx^i}{ds}\frac{d\delta x^k}{ds}ds = \delta x^k\, g_{ik}\frac{dx^i}{ds}\Big|_{ini}^{fin} - \int\frac{d}{ds}\left(g_{ik}\frac{dx^i}{ds}\right)\delta x^k ds$$

$$\mathcal{L} = mc\int\left(\frac{1}{2}u^i u^k\frac{\partial g_{ik}}{\partial x^l}\delta x^l ds + mc\frac{d}{ds}\left(g_{ik}\frac{dx^i}{ds}\right)\delta x^k ds\right)$$

swap $k \leftrightarrow l$

$$0 = \frac{1}{2}u^i u^k\frac{\partial g_{ik}}{\partial x^l} - \frac{d g_{il}}{ds}u^i - g_{il}\frac{du^i}{ds}$$

$$\frac{d g_{il}}{ds}u^i = \frac{\partial g_{il}}{\partial x^k}u^k u^i$$

$$0 = \frac{1}{2}u^i u^k\frac{\partial g_{ik}}{\partial x^l} - g_{il}\frac{du^i}{ds} - u^i u^k\left(\frac{1}{2}\right)\left(\frac{\partial g_{il}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^i}\right)$$

$$0 = g_{il}\frac{du^i}{ds} - \frac{1}{2}u^i u^k\left(\frac{\partial g_{il}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^i} - \frac{\partial g_{ik}}{\partial x^l}\right)$$

$$0 = g_{il}\frac{du^i}{ds} - \Gamma_{l,ik}\,u^i u^k$$

Determine force action in a particle in a gravitational field

$$\Gamma^\alpha_{00} = \frac{1}{2}h^{i\alpha}$$

$$\Gamma^\alpha_{0\beta} = \frac{1}{2}\left(\partial_\beta h^\alpha_\mu - \partial_\mu h^{\beta\alpha}\right) - \frac{1}{2}\partial_\mu h^{i\alpha}$$

$$\Gamma^\alpha_{\beta\gamma} = \lambda^\alpha_{\beta\gamma} + \frac{1}{2}\left(\partial_\beta\left(\delta^\alpha_\gamma - \delta^{i\alpha}_{i\gamma}\right) + \partial_\gamma\left(\delta^\alpha_\beta - \delta^{i\alpha}_{i\beta}\right)\right)$$

$$+\frac{1}{2}\partial_\mu h^{i\alpha}$$

$$\frac{du^\alpha}{ds} = -\Gamma^\alpha_{00}(u^0)^2 - 2\Gamma^\alpha_{0\beta}u^0 u^\beta - \Gamma^\alpha_{\beta\gamma}u^\beta u^\gamma$$

$$\frac{d}{ds}\frac{V^\alpha}{c\sqrt{1-\frac{v^2}{c^2}}} = -\frac{h^{i\alpha}}{2(c(1-\frac{v^2}{c^2}))} - \frac{\sqrt{1}\left(\partial_\beta\delta^\alpha_\gamma - \partial_\gamma\delta^\alpha_\beta\right)V^\beta}{c(1-\frac{v^2}{c^2})} - \frac{\lambda^\alpha_{\beta\gamma}v^\beta v^\gamma}{c(1-\frac{v^2}{c^2})}$$

$$F^\alpha = c\sqrt{1-\frac{v^2}{c^2}}\frac{D\xi^\alpha}{ds} = c\sqrt{1-\frac{v^2}{c^2}}\frac{d}{ds} + \frac{m v^\alpha}{\sqrt{1-v^2/c^2}} + \lambda^\alpha_{\beta\gamma}\frac{m v^\beta v^\gamma}{\sqrt{1-v^2/c^2}}$$

$$F_\alpha = \frac{m c^2}{\sqrt{1-v^2/c^2}}\left(-\nabla h\sqrt{1} + \frac{v}{c}\times(\nabla\times g)\right)$$

Derive Fermat's principle for the propagation of a ray in a constant gravitational field

$$\delta\int k_\alpha dx^\alpha = 0$$

$$k_0 = -\frac{\partial\varphi}{\partial x^0} = \frac{\omega_1}{c}$$

$$\frac{\omega_1}{c} = k_0 = g_0\,k^i = g_{00}k^0 + g_{0\alpha}k^\alpha = h\left(k^0 g_{\alpha}k^\alpha\right)$$

$$k_i k^i = 0 = g_{ik}k^i k^k$$

$$h\left(k^0 g_\alpha k^\alpha\right)^2 - \gamma_{\alpha\beta}k^\alpha k^\beta = 0$$

$$\frac{1}{h}\left(\frac{\omega_1}{c}\right)^2 - \gamma_{\alpha\beta}k^\alpha k^\beta = 0$$

$$k^\alpha = \frac{\omega_1}{c\sqrt{h}}\frac{dx^\alpha}{dl}$$

$$k^\alpha = g^{\alpha i}k_i = g^{\alpha 0}k_0 + g^{\alpha\beta}k_\beta = -g^\alpha\frac{\omega_1}{c} - g^{\alpha\beta}k_\beta$$

$$k_\alpha = -\gamma_{\alpha\beta}\left(k^\beta + \frac{\omega_1}{c}g^\beta\right) = -\frac{\omega_1}{c}\left(\frac{\gamma_{\alpha\beta}}{\sqrt{h}}\frac{dx^\beta}{dl} + g_\alpha\right)$$

$$\delta\left(\int\frac{dl}{\sqrt{h}} + g_\alpha dx^\alpha\right) = 0$$

Calculate element of spatial distance in a rotating coordinate system

$$dl^2 = \gamma_{\alpha\beta}\,dx^\alpha dx^\beta$$

$$\gamma_{\alpha\beta} = \left(-g_{\alpha\beta} + \frac{g_{0\alpha}g_{0\beta}}{g_{00}}\right)$$

consider $ds^2 = (c^2 - \Omega^2 r^2)dt^2 - dr^2 - 2\Omega r^2\,d\varphi\,dt$
$- dz^2 - r^2 d\varphi^2 - dz^2$

$$g_{\mu\nu} = \begin{pmatrix} c^2-\Omega^2r^2 & 0 & +\Omega r^2 & 0 \\ 0 & -1 & 0 & 0 \\ +\Omega r^2 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$dl^2 = dr^2 + dz^2\left(r^2 + \frac{\Omega^2 r^4}{c^2-\Omega^2 r^2}\right) + dz^2$$

$$dl^2 = dr^2 + d\varphi^2\left(\frac{c^2r^2 - \Omega^2r^4 + \Omega^2r^4}{c^2-\Omega^2r^2}\right) + dz^2$$

$$dl^2 = dr^2 + r^2 d\varphi^2\left(\frac{1}{1-\frac{\Omega^2r^2}{c^2}}\right) + dz^2$$

Write Maxwell equations in a given gravitational field in a three dimensional form, with three vectors $\vec{E}$, $\vec{D}$ and antisymmetric 3-tensors, $B_{\alpha\beta}$, $H_{\alpha\beta}$

$$E_\alpha = F_{0\alpha} \qquad B_{\alpha\beta} = F_{\alpha\beta}$$

$$D^\alpha = \sqrt{g_{00}}\,F^{0\alpha} \qquad H^{\alpha\beta} = \sqrt{g_{00}}\,F^{\alpha\beta}$$

$$F_{0\alpha} = g_{0l}g_{\alpha m}F^{lm} \qquad F^{\alpha\beta} = g^{\alpha l}g^{\beta m}F_{lm}$$

$$\gamma_{\alpha\beta} = -g_{\alpha\beta} + h g_{0\alpha}g_{\beta}$$

$$D_\alpha = \frac{E_\alpha}{\sqrt{h}} + g^\beta H_{\alpha\beta}$$

$$B^{\alpha\beta} = \frac{H^{\alpha\beta}}{\sqrt{h}} + g^\beta E^\alpha - g^\alpha E^\beta$$

$$B^{\alpha\beta} = -\frac{1}{2\sqrt{\gamma}}\,\epsilon^{\alpha\beta\gamma}B_{\beta\gamma}$$

$$H_\alpha = -\frac{1}{2}\sqrt{\gamma}\,\epsilon_{\alpha\beta\gamma}H^{\beta\gamma}$$

$$\vec{D} = \frac{\vec{E}}{\sqrt{h}} + H g \qquad \vec{B} = \frac{H}{\sqrt{h}} + g\times E$$

$$\frac{\partial B_{\alpha\beta}}{\partial x^\gamma} + \frac{\partial B_{\gamma\alpha}}{\partial x^\beta} + \frac{\partial B_{\beta\gamma}}{\partial x^\alpha} = 0$$

$$\frac{\partial B_{\alpha\beta}}{\partial x^0} + \frac{\partial E_\alpha}{\partial x^\beta} - \frac{\partial E_\beta}{\partial x^\alpha} = 0$$

$$\nabla\cdot B = 0 \qquad \nabla\times E = -\frac{1}{c\sqrt{\gamma}}\frac{\partial}{\partial t}\left(\sqrt{\gamma}\,\vec{B}\right)$$

$$\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^\alpha}\left(\sqrt{\gamma}\,D^\alpha\right) = 4\pi\rho$$

$$\frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^\beta}\left(\sqrt{\gamma}\,H^{\alpha\beta}\right) + \frac{1}{\sqrt{\gamma}}\frac{\partial}{\partial x^0}\left(\sqrt{\gamma}\,D^\alpha\right) = -4\pi j^\alpha = \frac{j_0^\alpha}{\partial x^0}$$

Alternative derivation of equations of motion for a test particle moving through a gravitational field.

Consider

$$U^i = \frac{dx^i}{ds}$$

define

$$\frac{\partial U^i}{\partial x^j} = \frac{\partial U^i}{\partial x^j} \frac{dx^j}{ds} = \frac{\partial^2 x^i}{\partial x^j \partial s} = \frac{\partial U^j}{\partial x^i}$$

$$U_{i;k} = \frac{\partial U_i}{\partial x^k} + \Gamma^l_{ik} U^k$$

$$= \frac{\partial^2 x^l}{\partial x^i \partial x^k} + \Gamma^l_{ik} \frac{\partial x^k}{\partial s}$$

$$U^i_{;k} = \frac{\partial U^i}{\partial x^k} + \Gamma^i_{kl} U^l$$

$$U^i_{;k} U^k = \frac{\partial U^i}{\partial x^k} U^k + \Gamma^i_{kl} U^k U^l$$

$$\Downarrow$$

$$U^i_{;k} U^k = \frac{\partial U^i}{\partial s} + \Gamma^i_{kl} U^k U^l$$

$$\frac{\partial U^i}{\partial s} = \frac{\partial^2 x^i}{\partial s^2} = \frac{\partial}{\partial s} \frac{dx^i}{ds}$$

Now we will show that this is not the 4-vol relation acceleration?

$$\frac{D^* V^i}{DS} = \frac{D}{DS}\left(U^k V^i_{;k}\right)$$

$$= \frac{D}{DS}\left(U^k U^j_{;k}\right)$$

$$= \frac{D}{DS} U^k \left(U^i_{;k}\right) + U^k \frac{D}{DS} U^i_{;k}$$

$$+ U^k U^j_{;k} U^i_{;j} + U^k U^i_{;k;j} U^j$$

$$= U^k U^j_{;k} U^i_{;j} + U^k U^i_{;k;j} U^j$$

$$U^i_{;k;j} U^k = U^k R^i_{mkj} U^m_{;j;k}$$

$$= U^k U^j_{;k} U^i_{;j} - U^k U^j U^m R^i_{mkj} + U^k U^j U^i_{;j;k}$$

$$= U^k U^j_{;k} U^i_{;j} - U^k U^j U^m R^i_{mkj}$$

$$+ U^j U^i_{;j;k} U^k - U^k U^j U^m R^i_{mkj}$$

$$\frac{D^* V^i}{DS} = \frac{DV^i}{DS} - R^i_{mkj} U^k U^j V^m$$

Write Maxwell equations in the absence of charges for the 4-potential Lorentz gauge

$$A^i_{;k} = 0 \quad \text{in absence of charge}$$

$$F^{ik}_{;k} = 0$$

$$F_{ik}^{;k} = \underbrace{A_{k;i}^{;k}}_{\text{swap the}} - A_{i;k}^{;k} = 0$$

$$A_{k;i}^{;k} = A_{k;k}^{;k} - A^m R^k_{mki}$$

$$A_{k;i}^{;k} = A^k_{;ik} - A^m_{;k} R^k_{mki}$$

$$+ A^k A_{;i}$$

$$A_{;ik}^{;k} = A_{;i} A^k$$

Express curvature tensor R_{pqrs} of three-dimensional space in terms of the second rank tensor R_{pq}

$$R_{pqrs} = A_{ar} T_{pq} - A_{as} T_{pq} + A_{pq} T_{rs} - A_{pr} T_{qs}$$

A_{pq} relationship to R_{pq} can be controlled as T_{pq}

$$R_{pq} = A_{ar} T_{pq} - A_{as} T_{pq} + A_{pq} T_{rs} - A_{pr} T_{qs}$$

$$R_{qp} = A_{ar} T_{qp} - A_{as} T_{qp} + A_{qp} T_{rs} - A_{qr} T_{ps}$$

$$R_{pq} - R_{qp} = A_{ar} T_{pq} - A_{as} T_{pq} + A_{pq} T_{rs} - A_{pr} T_{qs} - A_{ar} T_{qp} + A_{as} T_{qp} - A_{qp} T_{rs} + A_{qr} T_{ps}$$

Calculate the components of the tensor R_{pqrs} and R_{pq} in a metric in which $g_{ik} = 0$ for $i \neq k$

$$g_{11} = e^{2\Gamma_1} \qquad g_{22} = 1 \qquad g_{33} = -1$$

$$R_{1212}: \; e^{2\Gamma_1}\left[T_{11} - T_{22} + T_{11} T_{22} - T_{12} T_{12}\right]$$
$$- T_{33}$$

$$R_{1313}: \; e^{2\Gamma_1}\left[T_{11} - T_{33} + T_{11} T_{33} - T_{13} T_{13}\right]$$
$$+ e^{2\Gamma_1}\left[T_{11} - T_{22} + T_{11} T_{22} - T_{12} T_{12}\right]$$
$$- e^{2\Gamma_1} T_{33} \quad \underset{\text{not}}{\text{used}} \quad R_{1212} = e^{2\Gamma_1} R_{1313}$$

Consider the possible cases of solution to canonical form of a symmetric tensor of second rank in a pseudo-Euclidean space

$$A_{ik} n^k = \lambda n_i$$

$$|A_{ik} - \lambda g_{ik}| = 0$$

consists of the tensor

$$A^{ik} \quad \text{not} \quad \text{in} \quad \text{pseudo-unitary}$$

Why?

$$n^i A_{ik} n^k = (\lambda^{(1)} n_i) n^i + \lambda^{(2)} (n^{(1)} n_i) = 0$$

unless $\lambda^{(1)} = \lambda^{(2)}$ in which case, we can freely set it

A_{ik} can be expressed in terms of its basis set

$$A_{ik} = \sum_{\alpha=1}^{4} \frac{n_i n_k}{\beta_\alpha^2}$$

With the equation for covariant gravitational field expressed all the equations of differentiation with respect to the space coordinate as covariant derivatives in a space with the metric tensor

$$g_{ik} = h_{ik}$$

In future our equation for 3 dimensional

We require tensorial invariance

$$x^i \to x^i + f(x^k) \qquad g^{ik} \to g^{ik}$$

$$\partial_{ik} \to \partial_{ik} - \frac{\partial f}{\partial x^k}$$

$$h_{ik} \to \partial_{ik} - \frac{\partial h_{ik}}{\partial x^k} - \frac{\partial h_{ik}}{\partial x^k}$$

can not $\partial_{ik} \to \partial_{ik} + \partial_{ik} = 0$

$$\Gamma^{ik}_{;i} = \frac{1}{2} g^{ik} h_{ik}$$

$$\Gamma^{ik}_{;i} = \frac{1}{2} h_{;i}$$

$$\Gamma^{ik}_{;k} = \frac{1}{2} h_{ik} - \frac{1}{2} g^{ik} h^{;k}$$

$$\Gamma^{ik}_{;k} = \frac{1}{2} h_{ik} - \frac{1}{2} g^{ik} h^{;k}$$

$$\Gamma^{ik}_{;k} = \frac{1}{2} h_{ik} - \frac{1}{2} g^{ik} h^{;k}$$

$$\Gamma^{ik}_{;k} = \frac{1}{2} h_{ik} - \frac{1}{2} g^{ik} h^{;k}$$

$$\Gamma^{ik}_{;k} = \frac{1}{2} h_{ik} - \frac{1}{2} g^{ik} h^{;k}$$

$$\Gamma^{ik}_{;k} = \frac{1}{2} h_{ik} - \frac{1}{2} g^{ik} h^{;k}$$

$$\Gamma^{ik}_{;k} = \frac{1}{2} h_{ik} - \frac{1}{2} g^{ik} h^{;k}$$

$$\Gamma^{ik}_{;k} = \frac{1}{2} h_{ik} - \frac{1}{2} g^{ik} h^{;k}$$

$$\Gamma^{ik}_{;k} = \frac{1}{2} h_{ik} - \frac{1}{2} g^{ik} h^{;k}$$

$$\frac{1}{2} h_{ik} \frac{1}{\sqrt{h}} \left(\sqrt{h}\right)^{;k} = \frac{1}{2} h_{ik} g^{ik} \frac{\partial \sqrt{h}}{\partial x^k} = \frac{\partial \sqrt{h}}{\partial x^k} \left(\frac{\partial \sqrt{h}}{\partial x^k}\right)^2$$

$$\frac{1}{2} h_{ik} \frac{1}{\sqrt{h}} \left(\sqrt{h}\right)^{;k} = \frac{1}{2} h_{ik} g^{ik} \frac{\partial \sqrt{h}}{\partial x^k} = \frac{\partial \sqrt{h}}{\partial x^k} \left(\frac{\partial \sqrt{h}}{\partial x^k}\right)^2$$

$$R^{ik} \Gamma^{ik}_{;k} = \frac{1}{2} \Gamma^{ik}_{;k} h_{ik} - \frac{1}{2} \left(\frac{\partial \sqrt{h}}{\partial x^k}\right)^2 \left(\frac{\partial \sqrt{h}}{\partial x^k}\right)^2 - \frac{1}{2} \left(\frac{\partial \sqrt{h}}{\partial x^k}\right)^2$$

Find the expression for the total 4-momentum of matter plus gravitational field

$$A = \int \Lambda \sqrt{-g}\, dV dt \quad \text{to obtain}$$

identity which is conserved

$$P_i = \frac{1}{c} \int \left(\Lambda \sqrt{-g}\, g^{ik} + \left(\frac{\partial \Lambda}{\partial g^{ik}} \frac{\partial \sqrt{-g}}{\partial x^k} - \frac{\partial \Lambda}{\partial g^{ik}} \Lambda\right)\right) dV_3$$

gravitational g^{ik} differences from g^{ik} we can take and $\sqrt{-g}$ from differentiation

Integral on the $\sqrt{-g}\, g^{ik}$

$$P_i = \frac{1}{c} \int \left(\sqrt{-g}\, g^{ik} + \frac{c^3}{16\pi} \left(\left(\sqrt{-g}\, h^{ik} - \frac{\partial h^{ik}}{\partial x^k} \frac{\partial (\sqrt{-g})}{\partial x^k}\right) - \frac{\partial h^{ik}}{\partial x^k} \frac{\partial (\sqrt{-g})}{\partial x^k}\right)\right) dV_3$$

$$P_i = \frac{1}{c} \int \left(\sqrt{-g}\, g^{ik} + \frac{c^3}{16\pi} \left[\sqrt{-g}\, h^{ik} + \frac{\partial h^{ik}}{\partial x^k} \frac{\partial (\sqrt{-g})}{\partial x^k}\right]\right) dV_3$$

Find form of the solution to gravitational field equations in a vacuum in the vicinity of a point that a spherically not symmetric or time

consider

$$T_{ik} = \underbrace{\sigma_{ik}}_{\text{function of spatial coordinates}} e^{i\omega t} + \ldots$$

Background tensor

T^{ik} tensor σ_{ik} is independent to σ^{ik}

$$T^{ik} = \sigma^{ik} e^{i\omega t} = \left(\sigma^{ik} + \left(\sigma^{ik}\right)^* e^{-i\omega t}\right)$$

$$R^{ik}_{;i} = C + i \sigma^{ik}_{;i} R^{ik}_{;i} = 0$$

$$R^{ik}_{;i} = \left(R_{ik}, R_{ik}\right) + \left(C - i \sigma^{ik}_{;i} R^{ik}_{;i}\right) \quad \sigma^{ik}_{;i} = \left(\sigma^{ik}_{;i}\right)^* R^{ik}_{;i}$$

$$= \left(R^{ik}_{;i}, R^{ik}_{;i}\right) + \left(C - \sigma^{ik}_{;i} R^{ik}_{;i}\right)$$

$$R^{ik} = \sigma^{ik} - \frac{1}{2} R^{ik}_{;i} R^{ik}_{;i} + \frac{1}{2} \sigma^{ik}_{;i} R^{ik}_{;i} - C = 0$$

$$R^{ik} = \sigma^{ik} - \frac{1}{2} R^{ik}_{;i} R^{ik}_{;i} = 0$$

$$R^{ik}_{;i} = 0$$

Calculate components of curvature tensor R_{ijkl} in synchronous reference system

$$R_{0202}: \; -R_{0202} + \frac{1}{2}\left(X_{02;0} X_{02;0} - X_{02;0} X_{02;0}\right)$$

$$R_{0202}: \; \left(X_{02;0} - X_{02;0}\right)$$

$$R_{0202}: \; \frac{1}{2} X_{02;0} - \frac{1}{2} X_{02;0} X_{02;0}$$

For small transform from one reference system to another from one synchronous reference system to another

$$\xi_i = \left(\xi_0(x^0, x^1, x^2), x^1 - x^0, x^2\right) \quad \text{generates} \quad \xi_{0;0} \quad \text{as transform independent of } t \text{ to } \frac{\partial \xi_0}{\partial t} = 1$$

To maintain $\xi_{0;0} = 0$

$$\frac{\partial \xi_1}{\partial x^0} = \frac{\partial \xi_0}{\partial x^1} \qquad \frac{\partial \xi_1}{\partial t} = \frac{\partial \xi_0}{\partial t}$$

$$\frac{\partial \xi_0}{\partial x^0} = \xi_{0;0} \frac{\partial \xi_0}{\partial t} = 0 \quad \text{to hold the condition}$$

$$\xi_0 = \int \xi_{0;0} \left(\frac{\partial \xi_0}{\partial t}\right) dt$$

define up to a constant in the space coordinate

Obtain the equations for the hydrogen atom in the case of two fixed nuclei in a field

Equation expressed in terms of the coordinates of the electron and the nuclei. Separation of variables. Separation parabolic coordinates and spherical polar coordinates

$$\psi = \frac{\xi_1 \xi_2}{\sqrt{\xi_1 + \xi_2}} \frac{\eta_1 \eta_2}{\sqrt{\eta_1 + \eta_2}} \frac{\varphi_1 \varphi_2}{\sqrt{\varphi_1 + \varphi_2}}$$

$$(\xi_1 + \xi_2)^2 \xi_1 \xi_2 \frac{\partial}{\partial \xi_1} \left(\xi_1 \frac{\partial \psi}{\partial \xi_1} \right) + (\eta_1 + \eta_2)^2 \eta_1 \eta_2 \frac{\partial}{\partial \eta_1} \left(\eta_1 \frac{\partial \psi}{\partial \eta_1} \right) + \frac{(\xi_1 + \xi_2)(\eta_1 + \eta_2)}{\varphi_1 \varphi_2} \frac{\partial^2 \psi}{\partial \varphi_1 \partial \varphi_2} - \frac{1}{4} \psi \left(\frac{\xi_1 \eta_2}{\xi_1 + \xi_2} + \frac{\xi_2 \eta_1}{\eta_1 + \eta_2} \right) = 0$$

$$\frac{1}{\xi_1 \eta_1} \frac{\partial}{\partial \xi_1} \left(\xi_1 \frac{\partial \psi}{\partial \xi_1} \right) + \frac{1}{\xi_1 \eta_1} \frac{\partial}{\partial \eta_1} \left(\eta_1 \frac{\partial \psi}{\partial \eta_1} \right) + \frac{1}{\xi_1 \eta_1 \varphi_1 \varphi_2} \frac{\partial^2 \psi}{\partial \varphi_1 \partial \varphi_2} - \frac{1}{4} \psi \left(\frac{\xi_1}{\eta_1} + \frac{\xi_2}{\eta_2} \right) = 0$$

$$\frac{\partial^2 \psi}{\partial \varphi_1 \partial \varphi_2} = 0$$

Find the measure of the volume element

$$dx = \begin{pmatrix} \frac{\partial x}{\partial \xi_1} & \frac{\partial x}{\partial \eta_1} & \frac{\partial x}{\partial \varphi_1} \\ \frac{\partial y}{\partial \xi_1} & \frac{\partial y}{\partial \eta_1} & \frac{\partial y}{\partial \varphi_1} \\ \frac{\partial z}{\partial \xi_1} & \frac{\partial z}{\partial \eta_1} & \frac{\partial z}{\partial \varphi_1} \end{pmatrix} \begin{pmatrix} d\xi_1 \\ d\eta_1 \\ d\varphi_1 \end{pmatrix}$$

$$\det dx = \begin{vmatrix} \frac{\partial x}{\partial \xi_1} & \frac{\partial x}{\partial \eta_1} & \frac{\partial x}{\partial \varphi_1} \\ \frac{\partial y}{\partial \xi_1} & \frac{\partial y}{\partial \eta_1} & \frac{\partial y}{\partial \varphi_1} \\ \frac{\partial z}{\partial \xi_1} & \frac{\partial z}{\partial \eta_1} & \frac{\partial z}{\partial \varphi_1} \end{vmatrix} d\xi_1 d\eta_1 d\varphi_1

Gravitational Waves

Determine the curvature tensor in a weak plane gravitational wave

Recall that only h_{11}, h_{12}, h_{13} are non-zero and $h_{22} = -h_{33}$ AND

$$R_{jklm} = \frac{1}{2}\left(\frac{\partial^2 h_{jm}}{\partial x^k \partial x^l} + \frac{\partial^2 h_{kl}}{\partial x^j \partial x^m} - \frac{\partial^2 h_{jl}}{\partial x^k \partial x^m} - \frac{\partial^2 h_{km}}{\partial x^j \partial x^l}\right)$$

Recall we have a plane wave propagating in $x^1 = x$ direction only, so no dependence of h on y, z, as h_{12}, h_{13} .. change in $x - ct$ direction. Note that this means

$$\frac{1}{c}\frac{\partial}{\partial t} = -\frac{\partial}{\partial x}$$

$$R_{2002} = \frac{1}{c}\left(\frac{\partial^2 h_{22}}{\partial t^2} - \frac{\partial^2 h_{00}}{\partial x^2}\right) = -\frac{1}{2}\frac{\partial^2 h_{22}}{\partial x^2}$$

$$R_{3003} = -R_{2002}$$

$$R_{0220} = R_{1001}, \quad R_{0330} = R_{1003}$$

$$R_{0202} = -R_{1001}, \quad R_{0303} = -R_{1003}$$

$$R_{1203} = \frac{1}{2}\frac{\partial^2 h_{23}}{\partial t^2}$$

When is

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 + f(t-x, y, z)(dt - dx)^2$$

an exact solution of the Einstein equation

$$ds^2 = -dy^2 - dz^2 + 2\,du\,dv + 2f(u, y, z)\,du^2$$

Ricci tensor is easiest calculation

$$u = (t-x)/\sqrt{2} \qquad v = (t+x)/\sqrt{2}$$

$$ds^2 = -dy^2 - dz^2 + 2\,du\,dv + 2f(u, y, z)\,du^2$$

$$g_{22} = g_{33} = -1, \quad g_{uv} = 1, \quad g_{uu} = 2f$$

$$g_{uv} = -2f, \quad g^{uv} = 1, \quad |g| = -1$$

$$R_{yuyu} = \frac{\partial^2 f}{\partial y^2}, \quad R_{zuzu} = \frac{\partial^2 f}{\partial z^2}, \quad R_{yuzu} = \frac{\partial^2 f}{\partial y \partial z}$$

$$R_{uu} = \Delta f$$

$\Uparrow$

Laplacian in y, z coordinates

so condition is $\Delta f = 0$

Two bodies, attraction each other according to Newton's Law, now in circular orbits. Determine average intensity of the radiation and direction of polarization

$$r_1 = \frac{m_2}{m_1 + m_2} r, \quad r_2 = -\frac{m_1}{m_1 + m_2} r, \quad \vec{r} = \vec{r}_1 - \vec{r}_2$$

$$D_{xx} = \mu r^2 (3\cos^2\psi - 1), \quad D_{yy} = \mu r^2 (3\sin^2\psi - 1)$$

$$D_{xy} = 3\mu r^2 \cos\psi \sin\psi, \quad D_{zz} = -\mu r^2$$

$\mu = \frac{m_1 m_2}{(m_1 + m_2)}$ $\quad \psi$ is polar angle of vector r in xy plane

$r = \text{const}$ for circular motion

$$\dot{\psi} = \underbrace{r^{3/2}\sqrt{k(m_1+m_2)}}_{\text{Kepler's law}} = \omega$$

Polarization $\quad e_{\theta\phi} = \frac{1}{\sqrt{2}}$, $\quad$ or $e_{\theta\theta} = -e_{\phi\phi} = \frac{1}{\sqrt{2}}$

project $D_{\alpha\beta}$ on e_θ and e_ϕ

$$I = I_1 + I_2$$

$$\frac{dI_1}{d\Omega} = \frac{k\mu^2\omega^6 r^4}{2\pi c^5} 4\cos^2\theta, \quad \frac{dI_2}{d\Omega} = \frac{k\mu^2\omega^6 r^4}{2\pi c^5}(1 + \cos\theta)^2$$

$$\frac{dI}{d\Omega} = \frac{k\mu^2\omega^6 r^4}{2\pi c^5}(1 + 6\cos^2\theta + \cos^4\theta)$$

integrating over all directions

$$-\frac{d\mathcal{E}}{dt} = I = \frac{32 k\mu^2\omega^6 r^4}{5c^5} = \frac{32 k^4 m_1^2 m_2^2 (m_1+m_2)}{5c^5 r^5}, \quad \frac{P_1}{I_2} = \frac{5}{32}$$

Loss of energy

$$\mathcal{E} = -\frac{k m_1 m_2}{2r}$$

$$i = \frac{2r}{k m_1 m_2}\frac{d\mathcal{E}}{dt} = -\frac{(4k^3 m_1 m_2 (m_1+m_2)}{5c^5 r^3}$$

Find the average of the radiated energy radiated in the form of gravitational waves by a system of two bodies moving in elliptical orbit

$$\underbrace{r}_{\text{distance}} = \underbrace{\frac{a(1-e^2)}{1 + e\cos\psi}}_{\text{semi-major ellipse}}, \quad e \text{ eccentricity}, \quad \psi \text{ polar angle}$$

$$\frac{d\psi}{dt} = \frac{1}{r^2}\left(k(m_1+m_2)\,a(1-e^2)\right)^{1/2}$$

$$-\frac{d\mathcal{E}}{dt} = \frac{8k^4 m_1^2 m_2^2 (m_1+m_2)}{15c^5 a^5 (1-e^2)^5}\left(1 + e\cos\psi\right)^4\left[12(1+e\cos\psi)^2 + e^2\sin^2\psi\right]$$

average after period so average over ψ yields

$$-\frac{d\mathcal{E}}{dt} = \frac{32 k^4 m_1^2 m_2^2 (m_1+m_2)}{15c^5 a^5 (1-e^2)^5}\left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right)$$

Determine the time-averaged rate of loss of angular momentum from a system of bodies in stationary motion and emitting gravitational waves

represent average rate of loss as work of the frictional forces

$$\vec{f}$$

$$\frac{d\mathcal{E}}{dt} = \sum \vec{f}\cdot\vec{v}$$

$$\frac{d\vec{M}_\alpha}{dt} = \sum_\gamma \overline{(x \times f)_\gamma} = \sum e_{\alpha\beta\gamma}\,\overline{x_\beta f_\gamma}$$

$$\frac{d\mathcal{E}}{dt} = -\frac{k}{45c^5}\,\overline{\ddot{D}_{\alpha\beta}\ddot{D}_{\alpha\beta}} = -\frac{k}{45c^5}\,\overline{\dot{D}_{\alpha\beta}\ddot{D}^{(c)}_{\alpha\beta}}$$

total time derivatives vanish

$$\dot{D}_{\alpha\beta} = \sum m\left(3x_\alpha v_\beta + 3x_\beta v_\alpha - 2x\cdot v\,\delta_{\alpha\beta}\right)$$

$$f = -\frac{2k}{15c^5}\ddot{D}^{(c)}_{\alpha\beta} m x_\beta$$

$$\frac{d\vec{M}_\alpha}{dt} = \frac{2k}{45c^5} e_{\alpha\beta\gamma}\,\overline{\ddot{D}^{(c)}_{\beta\delta}\dot{D}_{\gamma\delta}} = -\frac{2k}{45c^5} e_{\alpha\beta\gamma}\,\overline{\dot{D}_{\beta\delta}\ddot{D}_{\gamma\delta}}$$

For the above system, find average loss of angular momentum per unit time

similar to above, but with elliptical motion

$$-\frac{d\vec{M}_z}{dt} = \frac{32 k^{7/2} m_1^2 m_2^2 \sqrt{m_1+m_2}}{5c^5 a^{7/2}}\frac{1}{(1-e^2)^2}\left(1 + \frac{7}{8}e^2\right)$$

Relativistic Cosmology

Transform the element of length

$$dl^2 = \frac{dr^2}{1-\frac{r^2}{a^2}} + r^2(\sin^2\theta\, d\varphi^2 + d\theta^2)$$

to a form in which it is proportional to its euclidean expression

$$r = \frac{r_1}{1+\frac{r_1^2}{4a^2}}$$

$$dr = \frac{dr_1}{1+\frac{r_1^2}{4a^2}} - \frac{r_1^2}{\left(1+\frac{r_1^2}{4a^2}\right)^2}\frac{dr_1}{2a^2}$$

$$dr = \frac{1}{\left(1+\frac{r_1}{4a^2}\right)^2}\left(dr_1 + \frac{r_1}{4a^2}dr_1 - \frac{r_1^2}{2a^2}dr_1\right)$$

with a little mathematics

$$dl^2 = \frac{1}{\left(1+\frac{r_1^2}{4a^2}\right)^2}\left(dr_1^2 + r_1^2\sin^2\theta\, d\varphi^2 + r_1^2 d\theta^2\right)$$

Find the general form near the singular point for the metric in which the expansion of the space proceeds "quasihomogeneously", i.e. so that all components $\gamma_{\alpha\beta} = -g_{\alpha\beta}$ tend to zero according to the same law. The space is filled with the equation of state $p = \varepsilon/3$

$$\gamma_{\alpha\beta} = t a_{\alpha\beta} + t^2 b_{\alpha\beta} + \dots$$

$$\gamma^{\alpha\beta} = \frac{1}{t}a^{\alpha\beta} - b^{\alpha\beta} \quad \leftarrow \text{reciprocal tensor}$$

$a^{\alpha\beta}$ is reciprocal to $a_{\alpha\beta}$

$$a^{\alpha\gamma}a^{\beta\delta}b_{\gamma\delta}$$

$$-\frac{3}{4t^2} + \frac{1}{2t}b = \frac{8\pi k}{3}\varepsilon(-4u_0^2+1)$$

$$\frac{1}{2}\left(b_{;\alpha} - b^{\beta}_{\alpha;\beta}\right) = -\frac{32\pi k}{3}\varepsilon u_\alpha u_0$$

$$1 = u_i u^i \approx u_0^2 - \frac{1}{t}u_\alpha u_\beta a^{\alpha\beta}$$

$$8\pi k\varepsilon = \frac{3}{4t^2} - \frac{b}{2t} \qquad u_\alpha = -\frac{t^2}{2}\left(b_{;\alpha} - b^{\beta}_{\alpha;\beta}\right)$$

$$p^{\beta}_{\alpha} + \frac{3}{4}b^{\beta}_{\alpha} + \frac{5}{12}\delta^{\beta}_{\alpha}b = 0$$

$$b^{\beta}_{\alpha} = -\frac{4}{3}p^{\beta}_{\alpha} + \frac{5}{18}\delta^{\beta}_{\alpha}p$$

$$p = a^{\beta\gamma}p_{\beta}$$

$$p^{\beta}_{\alpha;\beta} - \frac{1}{2}p_{;\alpha} = 0$$

$$b^{\beta}_{\alpha;\beta} = \frac{7}{9}b_{;\alpha}$$

$$u_\alpha = -\frac{t^2}{9}b_{;\alpha}$$

Find the first two terms in the expansion of the apparent brightness of a galaxy as a function of its red shift: the absolute brightness of a galaxy varies according to an exponential law. $I_{abs} \propto \exp(\alpha t)$

Dependence of distance χ of the apparent brightness of a galaxy at instant η is given by formula

$$I = \text{const}\cdot\exp(\alpha(t(\eta-\chi)-t(\eta)))\,\frac{a^2(\eta-\chi)}{a^4(\eta)\sin^2(\chi)}$$

$$z = \frac{\lambda-\lambda_0}{\lambda_0} = \frac{\omega_0-\omega}{\omega} = \frac{a(\eta)-a(\eta-\chi)}{a(\eta-\chi)}$$

$$I = \text{const}\,\frac{1}{z^2}\left(1-\left(1-\frac{q}{2}+\frac{\alpha}{H}\right)z\right)$$

$$q = \frac{2}{1+\cos\eta} = \frac{\mu}{\mu_k} < 1$$

$$q = \frac{2}{1+\cosh\eta} = \frac{\mu}{\mu_k} < 1$$

Find the leading terms in the expansion of the number of galaxies contained inside "sphere" of a given radius, as a function of the red shift at the boundary of the sphere

$$N = \text{const}\int_0^{\chi}\sin^2(\chi)\,d\chi = \text{const}\;\chi^3$$

$$N = \text{const}\cdot z^3\left[1-\frac{3}{4}(2+q)z\right]$$