# Single Species Population Model:

$N(t)$: continuous fun, population density

$\frac{dN}{dt}$ : Rate of change of pop. density with time

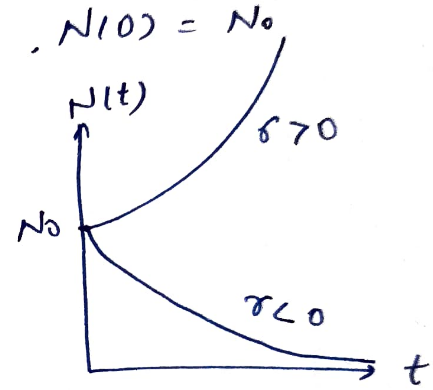
$\frac{1}{N}\frac{dN}{dt}$ : per capita growth rate

**Exponential Growth Model:** Growth has been controlled by birth & death of the individual. (No other consideration).

$b$ : per capita birth rate
$d$ : " " death rate

$$\frac{1}{N}\frac{dN}{dt} = \underbrace{b-d}_{r} \Rightarrow \frac{1}{N}\frac{dN}{dt} = r \Rightarrow \frac{dN}{dt} = rN$$

$$\Rightarrow \boxed{N = N_0 e^{rt}} \quad (1), \quad \text{At } t=0, \; N(0) = N_0$$




$r<0$ : $N(t) \to 0^+$ as $t \to \infty$

$r>0$ : $N(t) \to \infty$ as $t \to \infty$

In reality, it is not possible. →

**Assume:** Per capita Growth rate is NOT constant.
lets say it's linear

$$\frac{1}{N}\frac{dN}{dt} = a + bN$$

$$\frac{1}{N}\frac{dN}{dt} = r\left(1-\frac{N}{K}\right), \quad r, K > 0 \quad (2)$$

Per capita Growth rate is +ive if $N<K$
" " " " " -ive if $N>K$

$K \to$ ~~resource~~ (measure of population density that can be supported by existing resources)

$$\Rightarrow \frac{dN}{dt} = Nr\left(1-\frac{N}{K}\right) = rN - \frac{r}{K}N^2 \equiv f(N)$$

Equilibrium points : $\frac{dN}{dt} = 0 \Rightarrow N_{1*} = 0, \; N_{2*} = K$

$$f'(N) = r - \frac{2r}{K}N$$

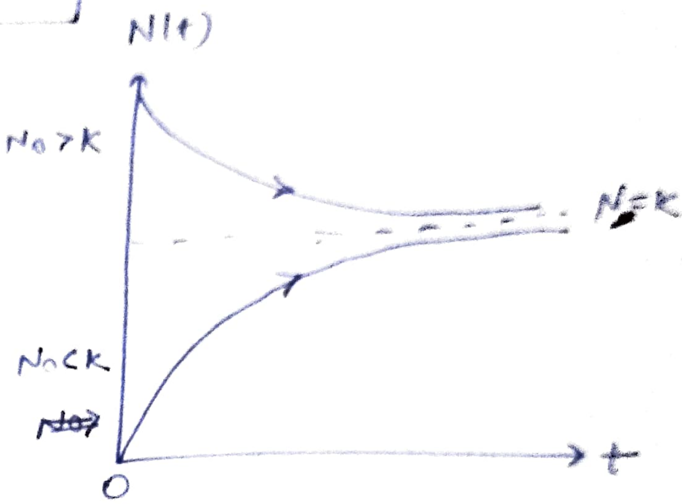
$f'(N_{1*}) = r, \quad f'(N_{2*}) = -r$

Linear stability theory says $N_{1*}$ is unstable
$N_{2*}$ is stable asymptotically

$\Rightarrow$ Single species population is surviving at its carrying capacity

$$\Rightarrow \boxed{N(t) = \frac{K}{1+\left(\frac{K}{N_0}-1\right)e^{-rt}}}$$ by soln of (2)

~~At $t \to \infty$~~ trajectory of $N(t)$




Non-dimensionalization of (2):

$N = \lambda x$

$t = \mu t'$

$$\frac{\lambda}{\mu}\frac{dx}{dt'} = r\lambda x\left(1-\frac{\lambda x}{K}\right)$$

$$\Rightarrow \frac{dx}{dt'} = r\mu x\left(1-\frac{\lambda x}{K}\right)$$

$\mu = \frac{1}{r}, \quad \lambda = K$

$$\Rightarrow \frac{dx}{dt'} = x(1-x)$$

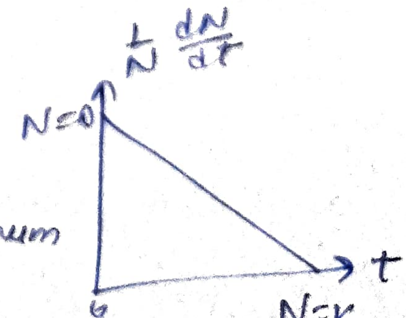
Replace ~~Remove~~ $t'$ with $t$ $\Rightarrow$ $\boxed{\frac{dx}{dt} = x(1-x)}$ Non-dimensional model of (2)

$x=0$ is unstable
$x=1$ is stable

$\Rightarrow N = Kx \Rightarrow x=0 \Rightarrow N_{1*} = 0$
$x=1 \Rightarrow N_{2*} = K$

$N=K$ is global attractor.

Limitations: $\frac{1}{N}\frac{dN}{dt} = r\left(1-\frac{N}{K}\right)$




$\Rightarrow$ Per capita growth rate is maximum when population is 0 i.e. $N=0$.
NOT TRUE for single species

$\Rightarrow$ We need a threshold value above which population can grow.

This was identified by Allee & so its called Allee's effect.

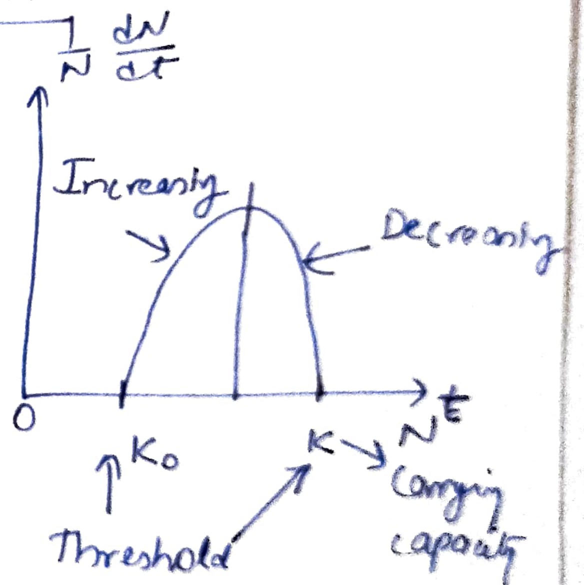
Modify your $\frac{1}{N}\frac{dN}{dt}$ such that it is +ive when population density is above some threshold value & it remains +ive upto its carrying capacity.

$$\Rightarrow \frac{1}{N}\frac{dN}{dt} = r\left(1-\frac{N}{K}\right)\left(\frac{N}{K_0}-1\right)$$ → Single species Population growth model with Allee effect.

$$\boxed{0 < K_0 < K}$$

$$\Rightarrow \frac{1}{N}\frac{dN}{dt} > 0 \quad \text{for} \quad K_0 < N < K$$

$< 0$ otherwise




In Increasing part, there is cooperative behaviour but in decreasing part there is competitive behaviour.

This is called strong Allee effect.

Dimensionless version:

$$\Rightarrow \frac{dx}{dt} = x(1-x)(x-\beta) \quad 0 < \beta < 1 = f(x)$$

$$f(x) = 0 \Rightarrow x_{1*} = 0, \quad x_{2*} = 1, \quad x_{3*} = \beta$$

$$f'(x) = (1-x)(x-\beta) - x(x-\beta) + x(1-x)$$

$f'(x_{1*}) = -\beta < 0 \rightarrow$ stable

$f'(x_{2*}) = \cancel{\beta}(\beta-1) < 0 \rightarrow$ stable

$f'(x_{3*}) = \beta(1-\beta) > 0 \rightarrow$ unstable

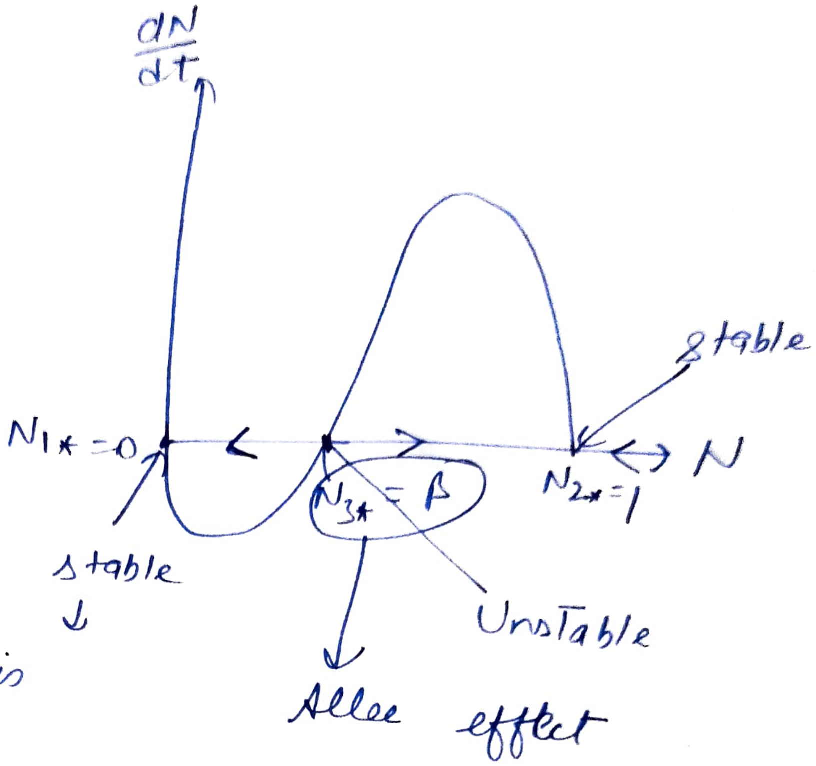
Sol$^n$ will be converging to either of the two.

$0 < N_0 < \beta$

$\Rightarrow \lim_{t \to \infty} N(t) = 0$

$N_0 > \beta$

$\lim_{t \to \infty} N(t) = 1$

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