Lanchester Combat Model:

→ British Engineer in Royal Air force F.W. Lanchester (1914) developed this model based on World War I aircraft engagements to explain why concentration of forces was useful in modern warfare.

He discovered a way to model battle field casualties using system of diff eq^n

Lanchester's Linear Law: (for ancient combat).

One soldier could only ever fight exactly one other soldier at a time. If each soldier kills & is killed by exactly one other, then the no. of soldiers remaining at the end of the battle is simply the difference between the larger army & the smaller, assuming identical weapons.

Lanchester's Square Law: (for modern combat) With fireman engaging each other directly with aimed shooting from a distance, they can attack multiple targets and receive fire from multiple direction. The rate of attrition now depends only on the no. of weapons shooting. He determined that power of such a force is proportional to not ~~only~~ the no. of units it has but the square of the number of units.

This law specifies the casualties a shooting force will inflict over a period of time, relative to those inflicted by the opposing force. It is only useful to predict outcomes & casualties by attrition. It does not apply to whole army but works where each unit (soldier, ship etc) can kill only one equi. unit at a time.

Assumptions: If two armies fight with $x(t)$ & $y(t)$ troops at each side, the rate at which soldier in one army are put out of action is proportional to the troop strength of their enemy.

$$\therefore \begin{cases} \frac{dx}{dt} = -a\,y(t), & x(0) = x_0 \\ \frac{dy}{dt} = -b\,x(t), & y(0) = y_0 \end{cases}$$

$a, b > 0 \Rightarrow$ called as fighting effectiveness coeff., x_0, y_0 are initial troop strength.

$$\Rightarrow \boxed{b\left[x_0^2 - x^2\right] = a\left[y_0^2 - y^2\right]} \quad \text{State Eq}^n$$

$\sqrt{ab} \Rightarrow$ Battle intensity

$\sqrt{\frac{a}{b}} \Rightarrow$ Relative effectiveness

Qs Answered By Square law State Eq^n:

1) Who will win?
2) What force ratio is required to gain victory?
3) How many survivor will the winners have?
   → Basic assumption is that the other side is annihilated (not true in real world battle)
4) How long will the battle last?
5) How do force levels change over time?
6) How do changes in A, B, x_0, y_0 affect the outcome of battle?
7) Is concentration of forces a good tactic?

Who wins a fight - to - the - finish

To determine who will wins, each side must have victory conditions', i.e. 'battle termination model'. Assume both sides fight to annihilation.

One of the 3 outcomes at a time t_f,

1) X wins $x(t_f) > 0$, $y(t_f) = 0$
2) Y wins $y(t_f) > 0$, $x(t_f) = 0$
3) Draw $x(t_f) = y(t_f) = 0$

Also, a square law battle will be won by X if & only if $\frac{x_0}{y_0} > \sqrt{\frac{a}{b}}$

How many survivors are there when X wins a ~~battle~~ fight - to finish

$$x_f = \sqrt{x_0^2 - \left(\frac{a}{b}\right) y_0^2}$$

When X wins, how long does it take:

$$t(x_f) = \frac{1}{2\sqrt{ab}} \log \left[\frac{1 + \frac{y_0}{x_0}\sqrt{\frac{a}{b}}}{1 - \frac{y_0}{x_0}\sqrt{\frac{a}{b}}} \right]$$

Q1 A battle is modeled by

$$x' = -4y \quad , \quad x(0) = 150$$
$$y' = -x \quad , \quad y(0) = 90$$

1) Write the solⁿ in parametric form?
2) Who wins & when? State the losses at each side.

Ans: Take Laplace Transform

$$sX(s) - x(0) + 4Y(s) = 0$$
$$sY(s) - y(0) + X(s) = 0$$

$$\Rightarrow sX(s) = 150 - 4Y(s), \quad sY(s) = 90 - X(s)$$

$$s^2 X(s) = 150s - 4(90 - X(s))$$

$$\Rightarrow X(s) = \frac{150s - 360}{s^2 - 4}$$

$$Y(s) = \frac{90s - 150}{s^2 - 4}$$

$$\therefore x(t) = -15e^{2t} + 165e^{-2t}$$

$$y(t) = 90\cosh 2t - 75\sinh 2t$$
$$= \frac{15}{2}e^{2t} + \frac{15}{2}e^{-2t}$$

$\Rightarrow$ 'y wins'

t_{win} : $x(t) = 0 \Rightarrow 15e^{2t} = 165e^{-2t}$
$$\Rightarrow e^{4t} = 11$$
$$\Rightarrow 4t = \lg 11 \Rightarrow t_{win} = \frac{\lg 11}{4}$$

No. of survivors:

$$y(t_{win}) = \frac{15}{2}e^{2\left(\frac{\lg 11}{4}\right)} + \frac{15}{2}e^{-2\left(\frac{\lg 11}{4}\right)}$$

$$\simeq 49.749$$
$$\approx 50 \text{ survies}$$