Stability of equi. point

$$\frac{dx}{dt} = f(x) \quad \Big| \quad \frac{dx}{dT} = f(x;\alpha)$$

Equi. $\rightarrow$ rest

$x = x^*$ is an equi pt. iff $f(x^*) = 0$

It itself a sol^n of the given system. $\rightarrow$ Importance

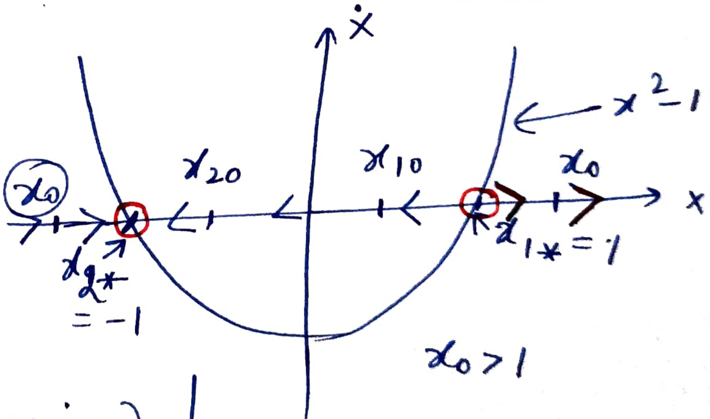
$$\frac{dx}{dt} = x^2 - 1 \Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1$$

$x_{1*} = 1, \quad x_{2*} = -1$

If we give a small perturbation to the equilibrium point, then whether the sol^n traj. coming back to it or going away.

$\Rightarrow$ If approaching to equi. pt. $\rightarrow$ stable

If not " " " $\rightarrow$ Not stable



At $x_0 > 1$ $\frac{dx}{dt} \geq 0$

$\Rightarrow$ Increasing fn $x(t) \uparrow (x_0)$

$-1 < x_{10}, x_{20} < 1$

$\Rightarrow -1 < x_0 < 1$

$\dot{x} < 0 \Rightarrow x(t) \downarrow$ (decreasing)

$x_0 < -1 \quad \dot{x} > 0 \Rightarrow x(t) \uparrow$

$$\Rightarrow \boxed{\begin{array}{ll} x_{1*} \text{ is} & \text{unstable} \\ x_{2*} \text{ is} & \text{stable} \end{array}}$$

from Geometrical Representation

This is called local stability because $\frac{dx}{dt}$ is defined for all real values of x but x_{2*} we can't make a statement that whatever be the initial cond^n, sol^n traj. will be approaching to x_{2*}.

It is approaching to x_{2*} if $x_0 < 1$ (initial cond^n)

Mathematical formulation:

$\dot{x} = f(x)$, $x = x_*$ is equi. pt.

Linear Analysis

$x(t) = x_* + \underbrace{y(t)}_{\text{perturbation}}$ $\quad |y(t)| << 1$

$\Rightarrow \dot{x}(t) = \dot{y}(t)$

$\Rightarrow \dot{y}(t) = f(x) = f(x_* + y(t))$ [$\because f(x_* + y(t))$ is smooth fn]

$= f(x_*) + f'(x_*)\,y(t) + \cdots$

$f(x_*) = 0$ & neglected higher order terms

$\Rightarrow \dot{y}(t) = f'(x_*)\,y(t) \equiv \alpha\, y_t$

$|y(t)| \to 0$ if $\alpha < 0$

$\Rightarrow x(t) \to x_*$ as $t \to \infty$ if $f'(x_*) < 0$ Stable

$|y(t)| \to \infty$ if $\alpha > 0$

$x(t) \nrightarrow x_*$ as $t \to \infty$ if $f'(x_*) > 0$ Unstable

If $f'(x_*) = 0 \Rightarrow$ Nothing can be said by linear stability analysis.

x_* is locally asymptotically stable if $f'(x_*) < 0$ &

unstable $f'(x_*) > 0$

We will check the previous example with this condition:

$\dot{x}(t) = x^2 - 1$, $x_{1*} = 1$, $x_{2*} = -1$

$f(x) = x^2 - 1$

$f'(x) = 2x$

$f'(x_{1*}) = 2 > 0 \to$ unstable $\Rightarrow$ x_{1*} is unstable

$f'(x_{2*}) = -2 < 0 \to$ stable $\Rightarrow$ x_{2*} is stable