GRAPH THEORY ASSIGNMENT - 1

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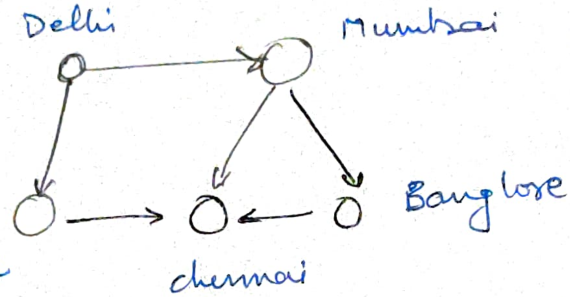
Qus:1 The given situation can be modelled into the graph below :-



have each node represent a city and each ~~eg~~ edge $\overrightarrow{(u, v)}$ represents a highway from city u to v, the weights on the edges represents the no. of lanes ~~of~~ on the highway

Qu: 2 (i) $(6, 6, 5, 4, 3, 3, 2)$

here $\Sigma deg \Rightarrow 6+6+5+4+3+3+2$
$\neq$ even

which means that the degree cannot be graphical..

Qu: 2 (ii) $(6, 6, 5, 4, 3, 3, 1)$

here we can see that Σ deg. = odd so,

using havel hakimi theorem

$\hookrightarrow$ $(6, 6, 5, 4, 3, 3, 1)$

$\downarrow$

$(5, 4, 3, 2, 2, 0)$

$\downarrow$

$(3, 1, 1, -1)$ $\longleftarrow$ one element becomes negative and so, the sequence cannot be graphical.

Qu: 3 The degree sequence $(0, 0, 0)$ is one such example :-

(i)

$$A = \begin{bmatrix} 0 & 1 & 1 & 2 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 2 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

Since the vercites $(1, 4)$ has more than one edge between them, so therefore graph is not simple.

(ii) degree of vertices

$1 \longrightarrow 4$
$2 \longrightarrow 2$
$3 \longrightarrow 3$
$4 \longrightarrow 3$
$5 \longrightarrow 2$

(iii) Sum of degree $\Rightarrow 4+2+3+3+2 \Rightarrow 14$

and $|E| \Rightarrow 14/2 \Rightarrow 7$

Qus: 5 let $d_1, d_2, d_3 \ldots d_n$ be the degree of vertices.

Now, we know that $d_1 + d_2 + d_3 + \ldots = 2m$

$$\sum d_i = 2m$$

also, $d_1 + d_2 + \ldots d_n \geq \delta + \delta + \ldots$

$$d_1 + d_2 + \ldots + d_n \geq n\delta$$

$$2m \geq n\delta$$

$$\frac{2m}{n} \geq \delta$$

or $$\delta \leq \frac{2m}{n}$$

Since, $d_1 + d_2 + d_3 + \ldots + d_n \leq n\Delta$

So; $\sum d_i \leq n\Delta$

and $2m \leq n\Delta$

$$\Delta \geq \frac{2m}{n}$$

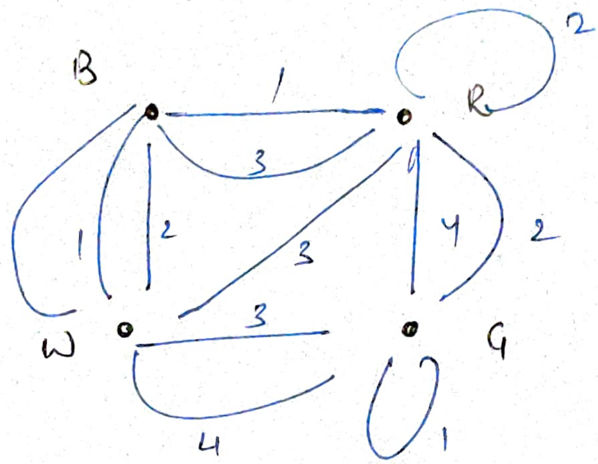
which mean:

$$\boxed{\delta \leq \frac{2m}{n} \leq \Delta}$$ Hence proved

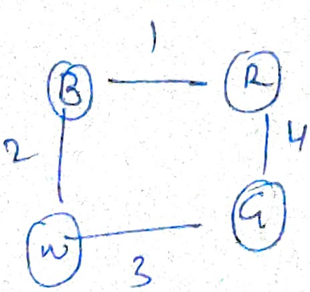
P.T.O

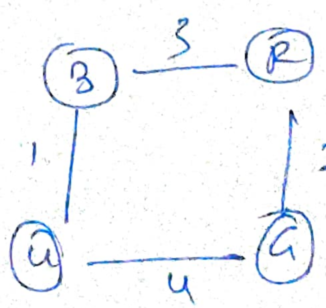
Qu: lets draw a graph with 4 vertices R, W, B & G where edges u-v represent color v is opposite to v, and the edge, label

B	G	B	G
G W G	R W R	R W W	W W B
R	R	R	R
B	B	G	G



Now, in order to have a stack of 4 dices with no two colors appearing twice on either side, we should be able to find two edge disjoint subgraphs of the above graph where each sub-graph contain each of the 4 vertices with different edge labels, also the degree of each ~~each~~ vertex should be 2.





(8) let, the number of vertices in each of the k components, of a graph G be $n_1, n_2, n_3 \ldots n_k$, thus we have,

$$n_1 + n_2 + n_3 + \cdots + n_k = n$$

$$n_i \geq 1$$

$$\sum n_i = n$$

$$\sum_{i=1}^{k} (n_i - 1) = n - k \quad \text{—— (1)}$$

Squaring on both side in eq (1)

$$\left(\sum_{i=1}^{k} (n_i - 1)\right)^2 = n^2 + k^2 - 2nk$$

$$\sum n_i^2 - 2n_i + k + \text{non-negative cross terms}$$

$$= n^2 + k^2 - 2nk$$

$$\text{or} \quad \sum n_i^2 \leq n^2 + k^2 - 2nk - k + 2n$$

$$\leq n^2 - (k-1)(2n-k)$$

now; maximum no. of edges in the ith components of G $= \frac{1}{2} n_i (n_i - 1)$

$\therefore$ ~~mini~~ no. of edges $= \frac{1}{2} \sum_{i=1}^{k} (n_i - 1) n_i$

$$= \frac{1}{2} \left(\sum_{i=1}^{k} n_i^2\right) - \frac{n_i}{2}$$

and

$$\leq \frac{1}{2} \left(n^2 - (k-1)(2n-k) \right] - \frac{n}{2}$$

$$= (n-k)(n-k+1)/2$$

(9) Q let's say that the partite graph would be divided into two pieces having p & q vertices.

now every vertex from set 1 can have at most q edges.

→ Sum of degree in set 1 = $p*q$

and in set 2 = $q*p$

Total = $2qp$

→ The graph can have at most pq edges, also $p+q = n$

maximum edges = $p(n-p)$

using calculus we can deduce that

$f(p) = p(n-p)$ on $[0, n]$ will be maximum.

at $p = n/2$ (at $f'(p) = 0$ at $p = n/2$)

$$f(p)_{max} = \frac{n^2}{4}$$

or $m \leq p^2/4$

(10) Q let $\{v_1, v_2, v_3 \ldots v_n\}$ & $\{e_1, e_2, \ldots e_m\}$ be the vertex set & edges set of the graph G_1.

let G_2 be a complete graph having $\{v_1', v_2', v_3' \ldots v_n'\}$

let f be a function such that

$f(v_1) = v_1'$

$f(v_2) = v_2'$

⋮

$f(v_n) = v_n'$

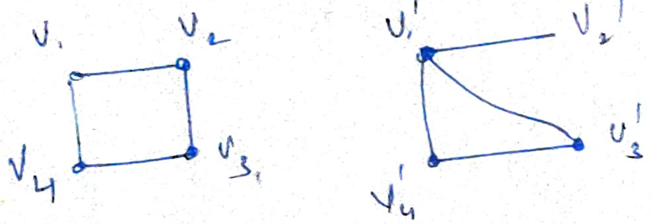
now, let H be a sub-graph of G_2 such that ~~of~~ it contains all the vertices from G_2.

Now, in H_0, we only take edge $\{v_j', v_k'\}$ from G_2 if edge $\{v_j, v_k\}$ (where $i, j \in [1, n]$)

in H each edge $\{v_j', v_k'\}$ corresponds to $\{v_j, v_k\}$

Hence, H is isomorphic G_1.

Qu 11 Consider the graph G_1 and G_2



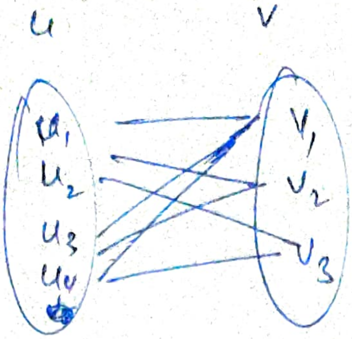
Here both, G_1 & G_2 have 4 vertices & 4 edges but they're not isomorphic as the one of vertex in G_2 is of degree 3 i.e v_1' but there is no vertex in G_1 having degree 2.

(12) Qus let's assume that if a known b then b also known a. Now, let's mode the given situation using as a graph. let the vertex set $\{1, 2, 3, 4, 3, 6, 7\}$ represent each of the person and edges person known exactly 3 persons in the group therefore, total degree $\Rightarrow 7 \times 3 = 21$

so, using hand shaking lemma, such a graph not possible ~~le &~~ therefore the given situation is not possible.

Ques: 14

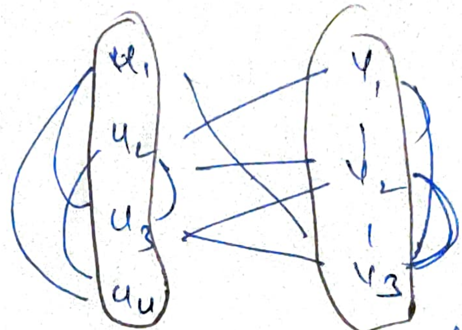
Consider the following simple graph.



here vertex set of G

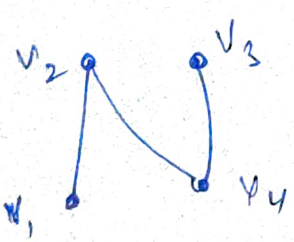
$\Rightarrow \{V\} \cup \{U\}$.

Clearly, G is a ~~b~~ bipartite graph, now complement of G will be :-

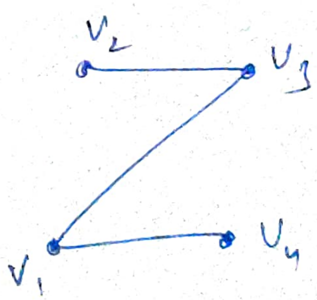


Here no partition of vertex set ~~is~~ is possible such that G' is bipartite. Hence, complement of bipartite Graph ~~need~~ need not ~~to~~ be a bipartite graph.

(15). Consider the following Graph G



G' will be



Now we, define a mapping f such that $f(v_1) = v_2$ $f(v_2) = v_3$, $f(v_3) = v_4$, $f(v_4) = v_1$

Now,

for edge $\{v_1, v_2\} \longrightarrow \{f(v_1), f(v_2)\} = \{v_2, v_3\}$ in G' exists.

$\{v_2, v_4\} \rightarrow \{f(v_2), f(v_4)\} = \{v_3, v_1\}$

$\{v_4, v_3\} \rightarrow \{f(v_4), f(v_3)\} = \{v_1, v_4\}$

Since, for each $\{v_i, v_j\}$ in G' $\{f(v_i), f(v_j)\}$ exist & we can show that for $\forall$ edge $\{v_i, v_j\}$ in G' $\{f^{-1}(v_i), f^{-1}(v_j)\}$ exists

$\therefore$ G & G' are isomorphic

Hence, G is G' are selff complementary graph.

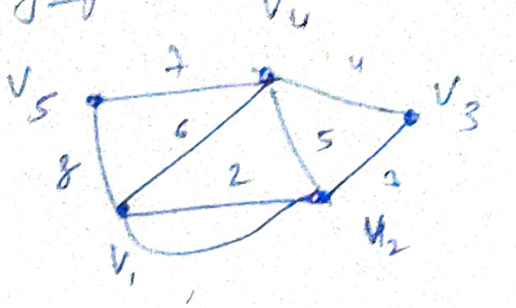
Qus: 16 Suppose G is a Euler Graph, then it contains a euler line, so while tracing this walk we observe that for every vertex v encountered in the walk we enter through one edge & exit through the other (even for terminal vertex) therefore every vertex must have even degree.

for sufficiency of the condition → let's suppose all vertices of G are of even degree, no let's start traversing from vertex v & since every vertex is of even degree we can exit from every vertex we enter, so the path will eventually end at v.

Now, if this is closed walk is h, say h is set also has even degree vertices.

edges and if this sets closed walks, as h & hence vertex h' also been seen vertices.

Also h' must touch h at point vertex a, since G is connected & so this walk can be combined with h', we repeat this process until we obtain a closed walk that traverse all the edges of G and hence G is a eular graph.

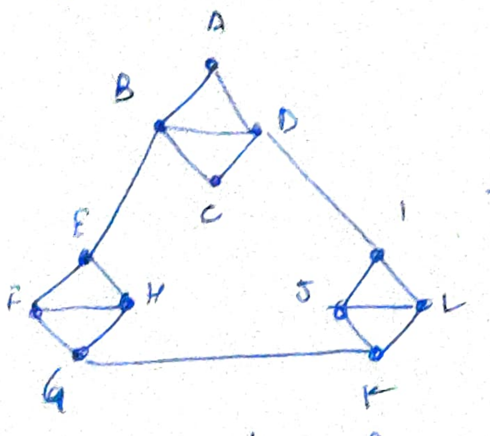


← G is eular

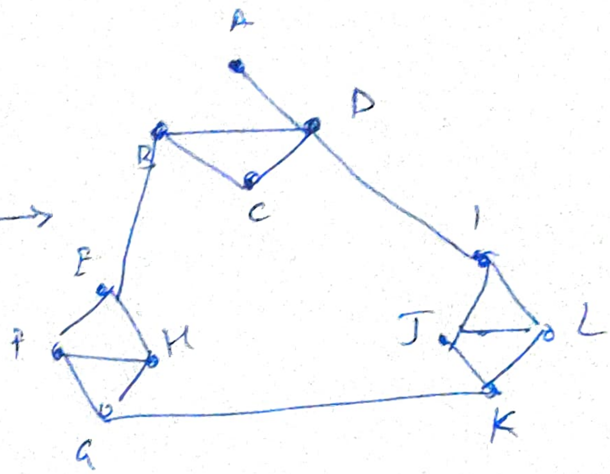
$|V| = 5$

$|E| = 8$

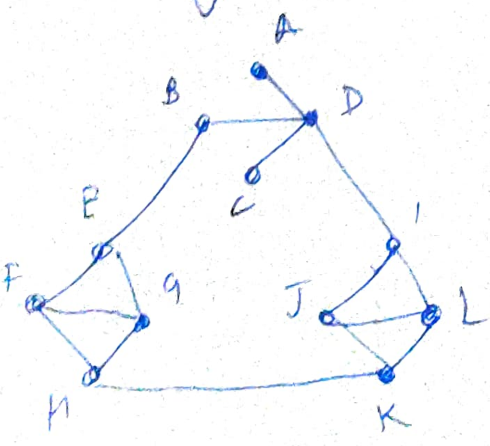
Qu: 17



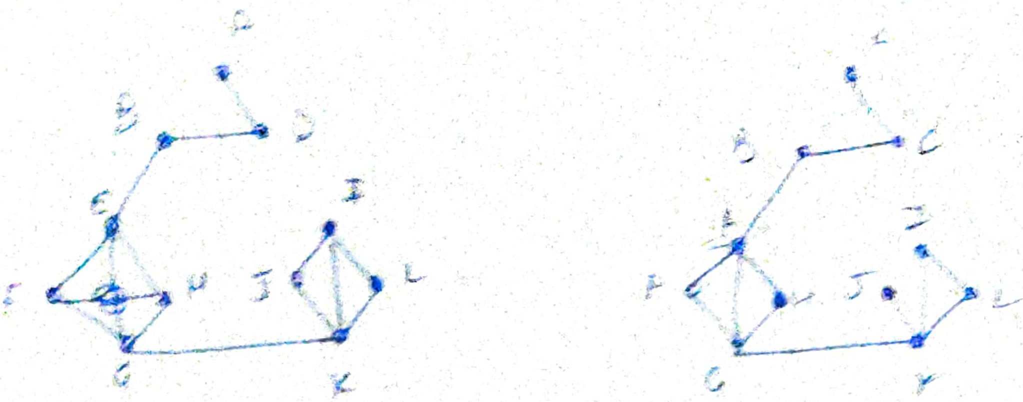
Starting from A, deleting AB



delete BC

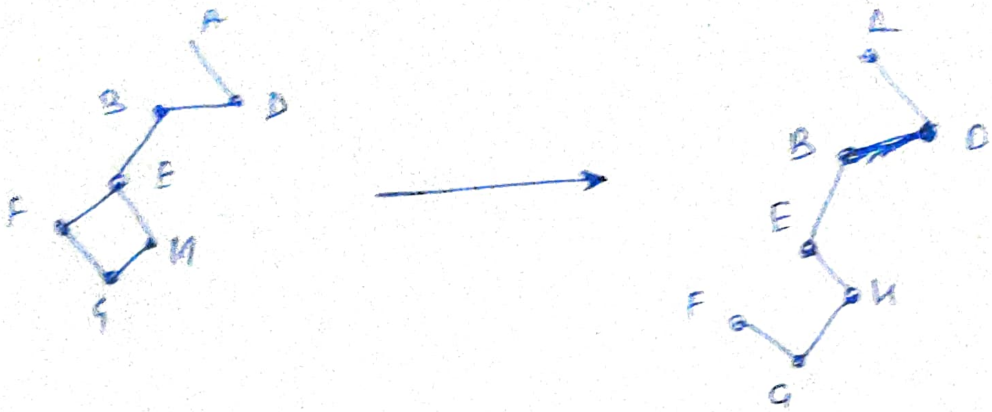


Delete DI edge



circuit so far → ABCDIJKILKG

and similarly

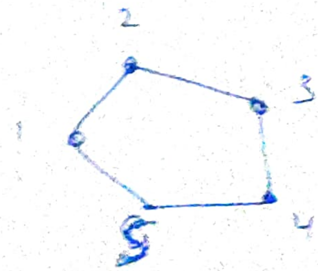


and the final circuit.

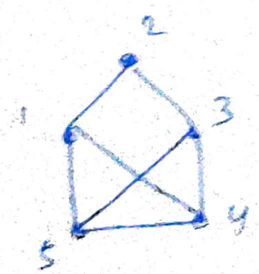
$\boxed{\text{ABCDIJKHLKGEFGHEBDA}}$

Qn 18 Hamilton and Eulerian

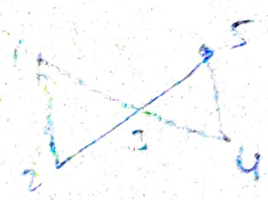
a)



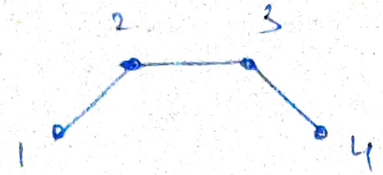
b)



c) Non-hamilton
& Eulerian

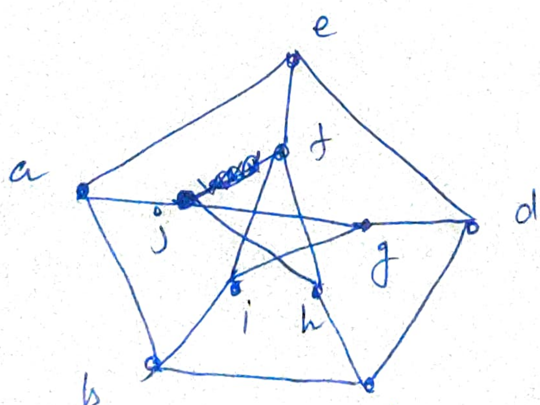


d) non-Hamilton & non-eulerian



Qu: 19 Petersen Graph :- The petersen graph is an indirected graph having 10 vertices & 15 edges, usually drawn as a pentagram within a pentagon corresponding vertices attached to each other.

The graph doesn't have a hamiltonian cycle, but it has a hamiltonian path, one such path is [baedehjgif]



Qu: 20

A simple graph with n vertices &

$m = \frac{1}{2}(n^2 - 3n + 6)$ edges. let (u, v) be two non-adjacent vertices of G. then consider the sub-graph H of G induced by the vertices $v_1, v_2, v_3 \dots v_{n-2}$. This is the sub-graph containing all edges of G with both endpoints from the set $\{v_1, v_2 \dots v_{n-2}\}$. Since the number of edges in H is at most ${}^{n-2}C_2 = \frac{1}{2}(n-2)(n-3)$

there's at least $m - {}^{n-2}C_2$ edges in G that have endpoints in u or v.

$$\deg(u) + \deg(v) \geq m - {}^{n-2}C_2$$

$$\geq \frac{n^2 - 3n + 6}{2} - \frac{(n-2)(n-3)}{2}$$

$$\geq \frac{n^2 - 3n + 6}{2} - \frac{n^2 - 5n + 6}{2}$$

$$\deg(u) + \deg(v) = \frac{2n}{2} = n$$

Qu: 21 A simple graph's is Hamiliton if $\deg(v) \geq (n/2)$ $\forall$ $v \in V(G)$ where n is the no. vertices in G & $n \geq 3$.

no; $\deg(u) \geq n/2$ and $\deg(v) \geq n/2$

$\deg(u) + \deg(v) \geq n$

Hence proved.

Q: 22

In order to show G be a hamiltonian if $\deg(u) + \deg(v) \geq n$ & v & u are non-adjacent, it is sufficient to show that every non-hamiltonian graph G does not obey the given conditions.

For G is a non-hamiltonian graph, let H be formed from G by adding edges one at a time that do not create a Hamiltonian cycle, until no more edges can be added.

let $x, y \in V(H)$ & non-adjacent,

$\Rightarrow$ adding xy would atleast one Hamiltonian cycle.

So, edges other than xy must form a Hamiltonian path say $v_1, v_2, v_3, \ldots, v_n$ with $x = v_1$, $y = v_n$

$\forall\ i \in [2, n]$, Consider 2 possible edges in H from v_1 to v_i & v_{i-1} to v_n at most one of these edges could be parent in H, otherwise the cycle $v_1 v_2, v_{i-1} v_n, v_{n-1}, v_i$ would be hamiltonian. Thus the edges incident to either v_1 or v_n be Hamiltonian cycle. Thus the edges incident to

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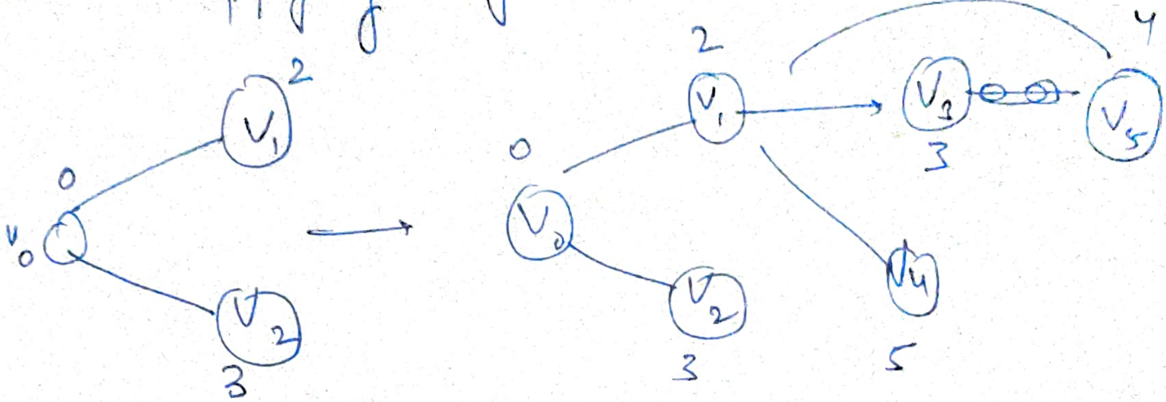
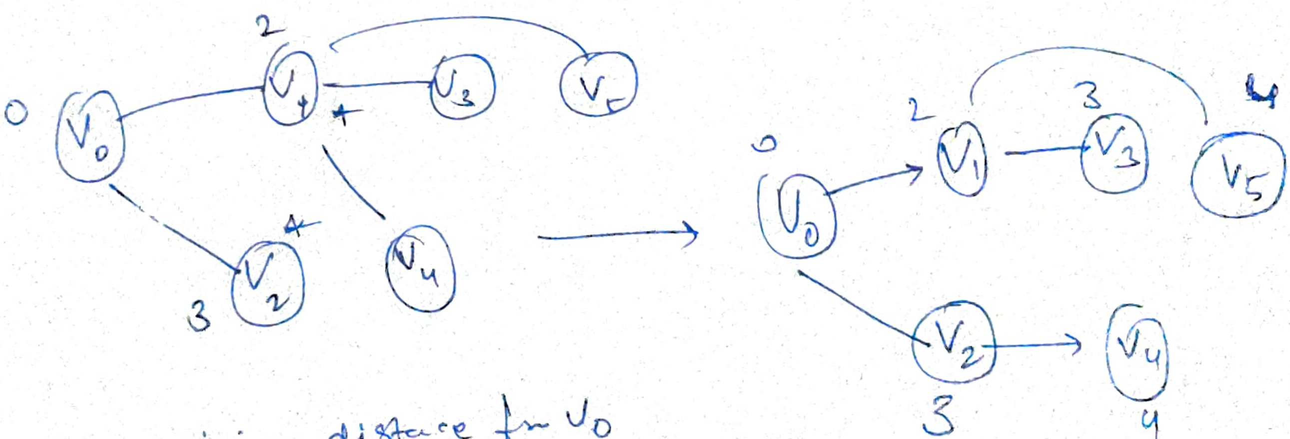
v_i or v_n is atmost equal to the number of choices of i; which is $n-1$.

∴ H doesn't obey the given condn (1), since vertex in G are atmost equal to H

→ G doesn't obey it

Qn 23 Applying dijkstra's algorithm.





final

Vertex	minimum distance from V_0
V_0	0
V_1	2
V_2	3
V_3	3
V_4	4
V_5	4