09 09 20

18

11/09/20

Unit 3 : TREES

Tree - A connected graph without any circuits.

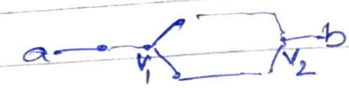
Vertices — Leaf — degree = 1
— Internal — otherwise

Tree is a simple graph {No self loops & parallel edges}

Th There is one & only one path between every pair of vertices in a Tree

Proof Let T be a tree
By defn (connected graph) ∃ at least one path between every pair of vertices.

let ∃ vertices a & b in the tree T such that ∃ 2 paths between a & b

case II 

case I 

In both cases ∃ exists a circuit in path1 ∪ path2 between a & b.
As it contradicts the definition of Tree having no circuits, two vertices must always have exactly one path between them
Hence Proved

Th If ∃ exactly one path b/w any pair of vertices, then the given graph is a tree.

∃ a path → hypothesis
∃ No circuit → only one path between any pair of vertices
— hypothesis
⇒ Tree

11 09 20
19

Th let T be a tree with n vertices, then T has n-1 edges

Proof We prove the theorem by induction

base step. consider $n=1$
$|E| = 0$ as there can be no self loops
consider $n=2$
$|E| = 1 = 2-1$
consider $n=3$
$|E| = 2 = 3-1$
as $|E| \geq 3$ result in a circuit and $|E| \leq 1$ result in a disconnected graph

hypothesis. we assume the result holds for trees with vertices $(n) < k$

induction we prove the result for $n=k$
we remove any one edge from T with k vertices $\Rightarrow$ we get two disconnected components with $|V| = m_1$ & $|V| = m_2$ with $m_1 + m_2 = k$
as $\exists$ only one path b/w any pair of vertices (defn)

for component one $|V| = m_1 \Rightarrow |E| = m_1 - 1$
for component two $|V| = m_2 \Rightarrow |E| = m_2 - 1$ } hypothesis.

for the tree, $|E| = |E|_1 + |E|_2 + 1 = m_1 - 1 + m_2 - 1 + 1$
$= k-1$

$\Rightarrow$ By induction, ~~we prov~~ the theorem is proved

also used as defn.

$\Rightarrow$ A connected graph of $|V|-1$ edges is a tree (circuitless)

leaves / pendant vertices

A tree ($|V| \geq 2$) has at least 2 leaves

say $|V| = n \Rightarrow |E| = |V| - 1 = n-1$

Using Handshaking lemma

$$\sum_{v \in V} d(v) = 2|E| = 2(n-1)$$

assuming p leaves

$$p + \sum_{\substack{v \in V \\ d(v) \geq 2}} d(v) = 2n-2$$

$$p + 2(n-p) \leq 2n-2 \quad \{ d(v) \geq 2 \}$$

$$p + 2n - 2p \leq 2n - 2$$

$$-p \leq -2$$

$$p \geq 2$$

$\Rightarrow$ the tree has at least 2 leaves.

Distance : In a connected graph G, the dist $d(v_i, v_j)$ b/w v_i & $v_j \in V$ is the length of the shortest path b/w v_i & v_j.

for a tree – the length of path b/w the 2 vertices

Distance is A metric space

i.e. $d(a,a) = 0$

$d(a,b) = d(b,a)$

$d(a,c) \leq d(a,b) + d(b,c)$

Eccentricity (Associated No.)

In a graph G, eccentricity of a vertex v, $E(v)$ is defined as the distance (as above) of v from the vertex farthest from v.

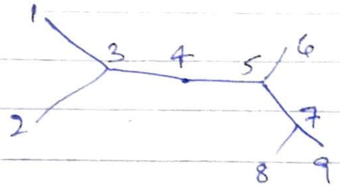
i.e. $E(v) = \max \{ dist(v,u) \mid u \in V(G) \}$

11 09 20
21

Center of a tree

The vertex in a tree with minimum eccentricity

eg



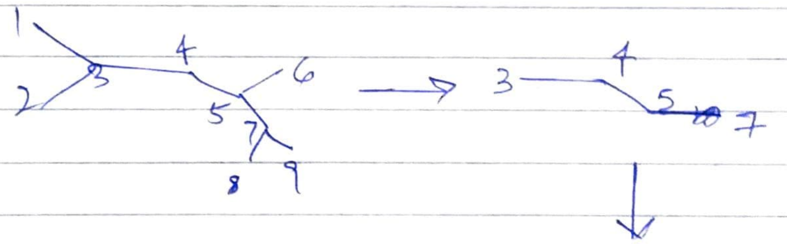
E(1) = E(2) = 5
E(3) = 4
$\boxed{E(4) = E(5) = 3}$
E(6) = 4 = E(7)
E(8) = E(9) = 5

4, & 5 → center

|center| < one / two (bi center)

{ while (|V| > 2)
  remove pendant vertices

→ the result with |V| = 1 or |V| = 2 are center(s).



Bi center ⟸ 4 — 5

14/09/20

Radius of a tree → Eccentricity of the center

Diameter of a tree → Length of longest path

diameter ≠ 2× radius

dist → length of path
eccentricity → max dist
Center → min eccentricity vertex
radius → eccentricity of center (min eccentri
diameter → length of longest path

Rooted tree

A tree in which exactly one vertex is differentiated from all other vertices

e.g for binary tree ∃ exactly one vertex w/ deg = 2
else deg = 1 or 3.

The differentiated vertex is called Root.

Binary Tree

* only one vertex with degree 2, rest deg = 1 or 3.

* no. of vertices $= 2^{m} - 1 = n$ (levels) | 1 of deg 2 + (n-1) even
* no. of pendant vertices $= \frac{n+1}{2}$

* no. of edges $= n - 1$

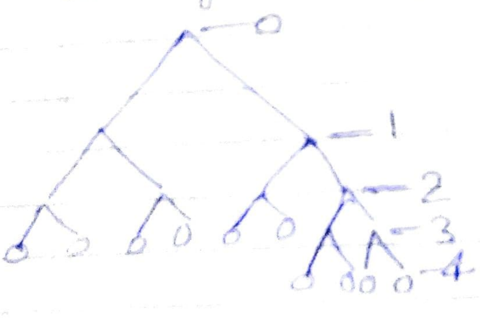
$$p + 2 + (n-p-1)3 = 2n - 2 \quad \{\text{Handshaking Lemma}\}$$
$$p + 2 + 3n - 3p - 3 = 2n - 2$$
$$n + 1 = 2p$$
$$\frac{n+1}{2} = p$$

~~Non pendant a.k.a intermediate~~

* Non pendant a.k.a internal vertices
$$= n - \frac{n+1}{2} = \frac{n-1}{2}$$

* Path Length of a Binary tree

Sum of levels of the pendant vertices.



Path length $= 6(3) + 4(4)$
$= 34$

18 09 20
25

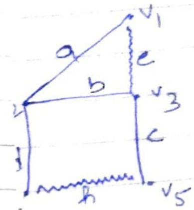
23

Spanning Tree

Given a graph G, a tree T is called a spanning tree of G if.

- T is a subgraph of G
- T contains all vertices of G.

ndamental circui



T_1

(d, b, c, h)
||| fundamental ck
(a, b, e)

if G is connected, always ≥ 1 spanning trees.
(= 1 if G is a tree)

if G is disconnected
$\exists$ spanning tree(s) for each component
spanning forest i.e. union spanning tree of individual components.

given a graph (connected)
- w/o circuits → itself a tree
- w/ circuits → remove edges so that $\exists$ no circuits ity

or

→ add edges 1 by 1 (n-1) to form tree.

= #chord
= e-n+k
[|E|-|V|

* Branches - edges of the spanning tree T

* Chords - edges of the graph G which are not in Tree T.

	connected	disconnected (k comp)
Branches	n-1 |V|	n-k
Chords	|E| - n + 1	|E| - (n-k) || e

1 ckt (-1 branc

Thec form
(f^m circui

Rank of graph → n-k

Nullity of graph → e - n + k

Rank + Nullity = e = |E|

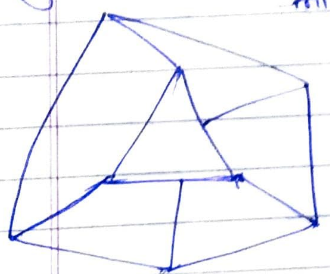
Rank - Nullity Theorem

Rank + Nullity = No. of edges

Rank = No. of branches = no. of edges in tree

Nullity = No. of chords = no. of edges (4) NOT in tree

e.g. the diag given below is a field, how many minimum edges we need to remove to drain all water.



$|V| = 10$

$|E| = \cancel{13}\ 15$

for spanning tree $|E| = 9$

$\Rightarrow$ remove $\underline{6}$ edges

$\hookrightarrow$ need a circuitless graph (connected for min^m change)

$\Rightarrow$ a spanning tree

no. of chords = $e - (n-1) = 15 - 10 + 1$

$= 6$ edges to remove

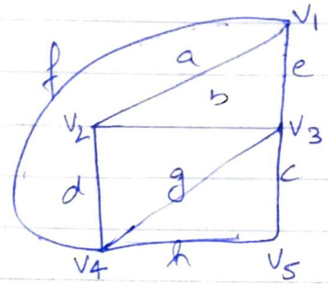
Fundamental ~~theorem~~ circuits

$G \longrightarrow T_1$ (spanning tree)

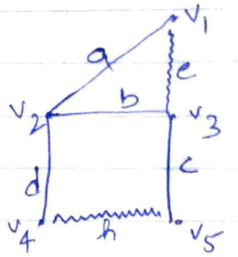
$\underbrace{T_1 + (u,v)}$ → a chord

only one circuit formed → Fundamental circuit wrt. T_1.

eg



→

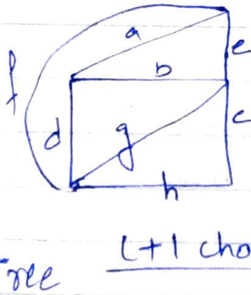


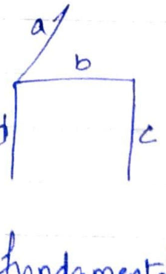
T_1

Various fundamental ckt. ⟹ (d, b, c, h)
||| fundamental ckt
(a, b, e)

★ No. of fundamental circuits = Nullity = # chords
= e − n + k
[|E| − |V| + |compl|]

Generation of spanning tree



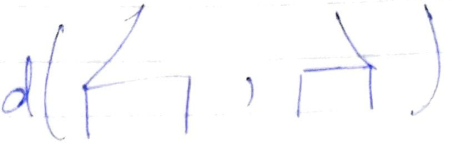


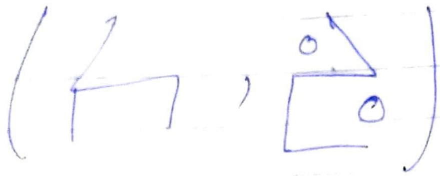
sp tree $\xrightarrow{(+1 \text{ chord})}$ fundamental ckt $\xrightarrow{(-1 \text{ branch}) \text{ from ckt.}}$ sp tree

★ Cyclic Interchange / Elementary tree formation
sp tree + chord − branch (f^{m} circuit) = new sp tree

Distance b/w two spanning trees

dist(T_1, T_2) = size of $T_1 \setminus T_2$
= # edges in T_1 & not in T_2.

eg d() = 1

d() = 2

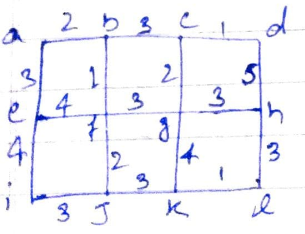
Minimal Spanning Tree (MST)

weighted graph

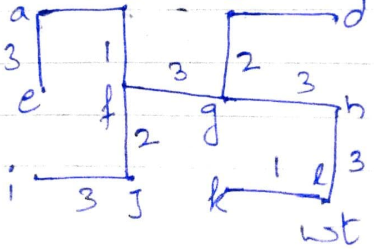
MST → sp. Tree with min^m cost (weight)

Prim's Algorithm
An edge of min^m wt, not forming a ckt & incident on vertex in the tree

Kruskal's Algorithm
An edge is added which is of min^m weight & not forming a circuit.



MST $|V| = 12 = n$
branches = $n-1$ = 11



wt = 24.