COMPUTER VISION

I. Symmetric matrices and Tridiagonalization

INTRODUCTION:

Symmetric matrices are matrices that are equal to their own transpose. In other words, if A is a square matrix, then it is symmetric if and only if $A = A^T$, where A^T is the transpose of A.

One important property of symmetric matrices is that they have real eigenvalues and orthogonal eigenvectors. This makes them very useful in many applications, particularly in linear algebra and physics.

Tridiagonalization is a process that transforms a given square matrix into a tridiagonal matrix, which is a matrix that has nonzero entries only on the main diagonal, the diagonal above the main diagonal, and the diagonal below the main diagonal. Tridiagonal matrices are easier to work with than general matrices, and many important algorithms for solving matrix problems (such as the QR algorithm for computing eigenvalues and eigenvectors) can be formulated in terms of tridiagonal matrices.

There are several methods for tridiagonalizing a given matrix, including Householder, Givens, and Jacobi methods. Each of these methods has its own advantages and disadvantages, and the choice of method depends on the specific problem at hand.

FORMULAS & PROCEDURE:

HOUSEHOLDER METHOD:

Let A be the given symmetric matrix and Q be the orthogonal matrix such that $Q^T A Q = T$, where T is tridiagonal. The Householder method involves a sequence of orthogonal Householder transformations of A to tridiagonal form T. The transformation matrix for each step is of the form:

$H = I - 2 vv^T$

where v is a vector, and I is the identity matrix. The transformation is applied to the submatrix of A below and to the right of the diagonal element being zeroed out, resulting in a new matrix A' with one more zero element on the diagonal.

The transformation vector v is chosen to be a multiple of the appropriate column of A, such that the resulting Householder transformation H annihilates all elements below the diagonal in the column.

The final tridiagonal matrix T can be obtained by applying a sequence of n-2 Householder transformations to A.

GIVENS METHOD:

Let A be the given symmetric matrix and Q be the orthogonal matrix such that Q^T A Q = T, where T is tridiagonal. The Givens method involves a sequence of orthogonal Givens rotations of A to tridiagonal form T. The transformation matrix for each step is of the form:

G = [cos(θ) sin(θ) -sin(θ) cos(θ)]

where theta is chosen such that the resulting Givens rotation G annihilates the appropriate off-diagonal element in A.

The final tridiagonal matrix T can be obtained by applying a sequence of

[(nxn)-[n+2(n-1)]]/2 Givens rotations to A.

JACOBIS METHOD:

Let A be the given symmetric matrix and Q be the orthogonal matrix such that Q^T A Q = T, where T is tridiagonal. The Jacobi method involves a sequence of orthogonal Jacobi rotations of A to tridiagonal form T. The transformation matrix for each step is of the form:

J = [cos(θ) -sin(θ) 0 0 ... 0 0 0

sin(θ) cos(θ) 0 0 ... 0 0 0

0 0 cos(φ) -sin(φ) ... 0 0

0 0 sin(φ) cos(φ) ... 0 0: :

: : ... : :

0 0 0 0 ... cos(φ) -sin(φ)

0 0 0 0 ... sin(φ) cos(φ)]

where theta and phi are chosen such that the resulting Jacobi rotation J annihilates the appropriate off-diagonal elements in A.

The final tridiagonal matrix T can be obtained by applying a sequence of O(n^2) Jacobi rotations to A.

<u>RESULTS, DISCUSSIONS & CONCLUSION:</u>

The **Householder, Givens, and Jacobi methods** are three widely used techniques for tridiagonalizing a symmetric matrix. Each method has its own strengths and weaknesses, and the choice of which method to use depends on the specific problem at hand.

The **Householder method** is known for its numerical stability and its ability to handle large matrices. However, it requires more computational effort than the Givens method, especially for larger matrices.

The Givens method is often preferred when the matrix is sparse, as it is computationally more efficient and has a lower memory requirement than the Householder method. However, it may not be as numerically stable as the Householder method.

The **Jacobi method** is an iterative method that can be used to tridiagonalize any symmetric matrix. It is not as efficient as the Householder or Givens methods, but it has the advantage of being relatively easy to implement and it can be used to compute all the eigenvalues and eigenvectors of a symmetric matrix. However, it can be slow to converge and is not as stable as the Householder method.

In general, the choice of tridiagonalization method depends on the specific problem and the characteristics of the matrix being tridiagonalized. For small and medium-sized matrices, the Householder method is often preferred due to its numerical stability. However, for large sparse matrices, the Givens method may be a better choice due to its lower computational cost and memory requirements. The Jacobi method can be useful when all eigenvalues and eigenvectors are needed, but may be slower and less stable than the other two methods.

The number of Iterations required for each method are stated below:

<u>Householder:</u> The final tridiagonal matrix T can be obtained by applying a sequence of n-2 Householder transformations to A.

<u>Givens:</u> The final tridiagonal matrix T can be obtained by applying a sequence of
[(nxn)-[n+2(n-1)]]/2 Givens rotations to A.

Jacobi: The final tridiagonal matrix T can be obtained by applying a sequence of O(n^2) Jacobi rotations to A.

Important Conclusions from the results:

- All the values in the diagonal of the final Jacobi matrix are indeed the eigen values of the covariance matrix.
- All the eigen values are similar for all the three methods.
- The eigenvectors of the respective tridiagonalized matrices are related to those of the covariance matrix by the relation $y_i=T^{-1}x_i$ for all the three methods.
- The total processing time for each method, and plotting the required processing time versus dimension **N** of the *N* x *N* matrix are seen in the figure below.