Point Visibility Graphs

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Integrated M.Sc in Mathematics and Computing

by

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CERTIFICATE

This is to certify that I have examined the thesis entitled **Point Visibility Graphs**, submitted by **Sriharsha Vangala**(Roll Number: *17MA20046*) a post graduate student of **Department of Mathematics** in partial fulfillment for the award of degree of Integrated M.Sc in Mathematics and Computing. I hereby accord my approval of it as a study carried out and presented in a manner required for its acceptance in partial fulfillment for the Post Graduate Degree for which it has been submitted.

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Chapter 1

Abstract

The Visibility Graph is a fundamental structure studied in the field of computational geometry. Visibility graphs are a way to encode the information that certain objects are visible from one another or not. They have wide range of applications in art gallery problem, robot motion planning, visibility analysis etc. Few other applications are finding euclidean shortest path in the presence of obstacles, object detection.

Understanding different computational complexities of Visibility graph is well researched area in literature. This report studies different computational complexities of Visibility Graph, introduce Visibility Grid Graph to find no.of Max-Cliques in Visibility Graphs.

Chapter 2

Introduction

Visibility Graph is extensively studied in literature, few applications of Visibility graph are Art Gallery Problem to estimate minimum no.of guards to cover specified area[1], finding euclidean shortest path in the presence of obstacles[8], robot motion planning .

Many properties like Vertex Cover, Independent Set and Maximum Clique, Feedback Vertex Set, Longest Induced Path, Colourability, Bisection, Cluster Vertex Deletion remain NP-Hard in PVG [4, 6, 7]. Ghosh and Roy[6] has extensively researched on paths of Visibility Graph and proved Hamiltonian property of Visibility Graph.

A *Point Visibility Graph (PVG)* is graph induced by set of points on plane, considering points as vertices in the graph, and connecting two vertices with an edge, if the points are visible from each other, i.e. line segment joining those points doesn't pass through other point. Consider P of n points $= \{p_1, p_2, \ldots, p_n\}$ in the plane. Let the corresponding PVG(point visibility graph) be $G(V, E)$, where p_i in P is associated with vertex v_i in V and edge v_iv_j in E if line joining p_i and p_j doesn't contain any other point p_k.

2.1 Organization of Report

In chapter 3, we introduce PVG and show computational complexities of few graph problems like Minimum Vertex Cover, Maximum Independent Set, Max-Clique, Colourability [4] etc. In Chapters 4 and 5, we study no.of Max-Cliques in graphs and PVG respectively, and introduce Visibility Grid Graph in section 5.1. In sections 5.2 and 5.3 we study different type of cliques in Visibility Grid Graph. In section 5.4 we simplify and summarize results.

Chapter 3

Point Visibility Graph

Definition 1. A graph $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$ is a *Point Visibility Graph* (PVG), if there exists a set of points $P = \{p_1, \ldots, p_n\}$ in the plane (each point p_i corresponds to vertex v_i) such that $\{v_i, v_j\} \in E$ if and only if there exists no other point in P on the line segment between p_i and p_j, that is, p_i and p_j are visible to each other. The point set P is also called the visibility embedding of G

Definition 2 (Transformation to G')**.** Given a graph $G = (V, E)$,not necessarily PVG, we add a vertex b_{uv} for every vertex pair $u \neq v \in V$ with $\{u, v\} \notin E$ and we connect b_{uv} to all vertices in V . We will call b_{uv} a blocker because it blocks the visibility between u and v in a corresponding visibility embedding. Finally, all blockers are connected to each other to obtain G'.

It can be observed that G is an induced subgraph of G' and G' is PVG[6]. This transformation takes polynomial time reductions in edges and vertices, so we use above transformation to prove NP-Hardness of few properties of PVG. Transformation is illustrated in Figure 3.1, b_0 and b_1 are blockers added.

3.1 Computational Graph Problems on *PVG*

In this section, we explain few standard computational graph problems like Minimum Vertex Cover, Maximum Independent set, Max-Clique problem.

3.1.1 Minimum Vertex Cover of *PVG*

Theorem 1. *Minimum Vertex Cover on PVG is NP-Hard*

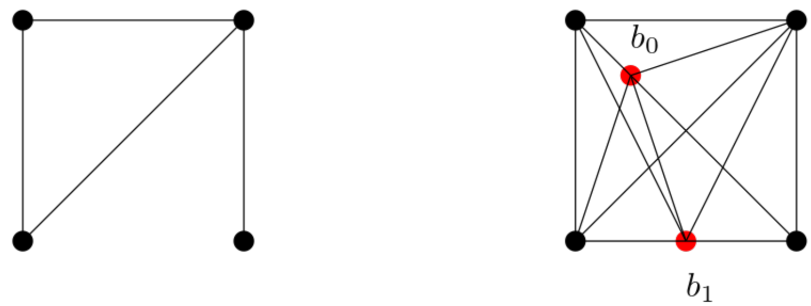

Figure 3.1: Illustration of Transformation G to G'

Proof. Consider a graph G, apply transformation mentioned above(Definition 2) on G to form G'. It can be observed that minimum vertex cover of G' is vertices in minimum vertex cover of G along with blockers added in transformation. As the transformation is polynomial time reducible, minimum vertex cover of G' is NP-Hard. Let the size of minimum vertex cover of G is k then minimum vertex cover of G' is $k + b$ where b is no.of blockers added in transformation. □

3.1.2 Maximum Independent Set of *PVG*

Theorem 2. *Maximum Independent Set on PVG is NP-Hard*

Proof. Consider a graph G, form G' using above transformation. Let Maximum Independent Set of G be D, As G is induced subgraph of G', D is independent set in G'. Blockers are connected to all the vertices of D, therefore blockers can not be included in independent set, vertices in $G - D$ are also connected to some vertex in D. Hence D is Maximum Independent Set of G'. As the transformation is polynomial time reducible , Maximum Independent set is NP-Hard □

3.1.3 Max-Clique of *PVG*

Theorem 3. *Max-Clique on PVG is NP-Hard*

Proof. Let G' be PVG formed by applying transformation on G. Let C be Max-Clique in G, as the blockers are connected to all the vertices of G', C along with blockers form Max-Clique in G'. Transformation is polynomial time reducible, hence Max-Clique in PVG is NP-Hard □

In [7], few other computational problems namely, Feedback Vertex Set, Longest Induced Path on PVG are shown to be NP-Hard.

Chapter 4

Max-Cliques in Graphs

A clique in a graph is a set of pairwise adjacent vertices. A maximal clique is a clique that cannot be extended by including one more vertex.

Moon and Moser have proved that Maximum No.of Max-Cliques in graph is of order $3^{\frac{n}{3}}$ in [9]. Let Maximum No.of Max-Cliques for graph G of order n be $f(n)$ then $f(n)$ is given by

$$f(n) = \begin{cases} 3^{\frac{n}{3}}, & \text{if } x \equiv 0 \pmod 3 \\ 4.3^{\frac{n-4}{3}}, & \text{if } x \equiv 1 \pmod 3 \\ 2.3^{\frac{n-2}{3}}, & \text{if } x \equiv 2 \pmod 3 \end{cases}$$

For completeness we describe the example by Moon and Moser [9] that proves that above values are possible. If $n \equiv 0 \pmod 3$ then let G be the complement of disjoint union of $n/3$ copies of K_3. If $n \equiv 1 \pmod 3$ then,G be the complement of disjoint union of K_4 and $(n-4)/3$ copies of K_3. If $n \equiv 2 \pmod 3$ then let G be the complement of disjoint union of K_2 and $(n-2)/3$ copies of K_3. Observe that no.of Max-Cliques is $f(n)$.

As the above mentioned graph is not PVG, lets try to explore upper bound of No.of Max-Cliques on PVG.

Chapter 5

Max-Cliques in PVG

A clique in a graph is a set of pairwise adjacent vertices. A maximal clique is a clique that cannot be extended by including one more vertex. Maximum no.of maximal cliques is found to be order $3^{\frac{n}{3}}$ [9] in graph with n vertices. In this Chapter, we try to find maximum no.of Max-Cliques for PVG class.

5.1 Visibility Grid Graph

In this section, we introduce special kind of PVG called *Visibility Grid Graph.* This is graph consists of blocks $\{B_1, B_2, \ldots, B_b\}$ similar to Grid Graph in [3] with an extra vertex p, where b is no.of blocks in the Graph. Let us call p as *Vertex of Convergence.*

Let us denote Visibility Grid Graph as $VG(k, l)$ where k is no.of vertices on each line converging at p and l denotes no.of adjacent lines where ith points on l lines are pairwise concurrent $\forall i \in 1 \ldots k$. For instance, $VG(3, 2)$ is shown in Figure 5.1, only few edges are shown in 5.1 others are omitted for readability(few are shown in dotted lines).

Lemma 1. At most two adjacent vertices on any line can be included in max-clique.

Proof. Let p_1, p_2, p_3 are concurrent. p_1 and p_2 is edge if p_1 and p_2 are adjacent. As p_1, p_2, p_3 are concurrent they can't be pairwise adjacent which implies they can not form clique. Hence, at most two adjacent points can be included in max-clique. $\square$

Max-Cliques in the $VG(k, l)$ can be studied by classifying them into two different types

- Max-Cliques which contain p, say P-Cliques

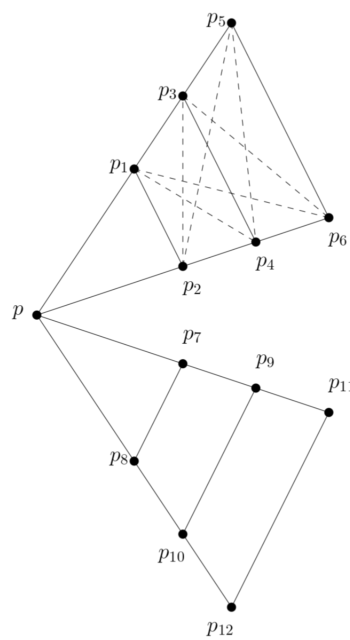

Figure 5.1: Example of $VG(3, 2)$

- Max-Cliques which does not contain p say B-Cliques

5.2 P-Cliques

Max-Cliques which include Vertex of Convergence are called P-Cliques.For instance, one of P-Cliques of $VG(3, 4)$ is shown in 5.2(Red Vertices form Max-Clique).

Lemma 2. No.of P-Clique in each block of $VG(k, l)$ is $l - 1$.

Proof. P-Cliques in each block contain vertex p and at most two adjacent vertices from $p_1, p_2, \ldots, p_l$(Lemma 1).Maximum no.of pairs of adjacent vertices in $\{p_1, p_2, \ldots, p_l\}$ is $l - 1$. Hence maximum no.of P-Cliques in each block is $l - 1$. □

Theorem 4. *No.of P-Cliques in* $VG(k, l)$ *is* $(l - 1)^b$

Proof. Vertices in P-Clique of block, say B_i, are connected to all the vertices of any P-Clique in blocks $B_j (j \neq i)$. Therefore P-Cliques of $VG(k, l)$ can be formed by combining P-Cliques in each block. As there are $l - 1$,from Lemma 2, distinct P-Cliques in each block, no.of P-Cliques in $VG(k, l)$ is $(l - 1)^b$. □

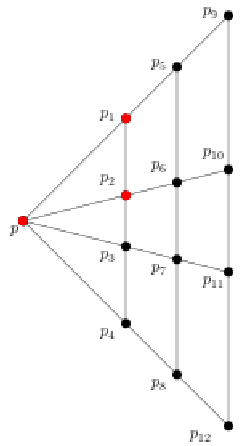

Figure 5.2: P-Clique in $VG(3,4)$

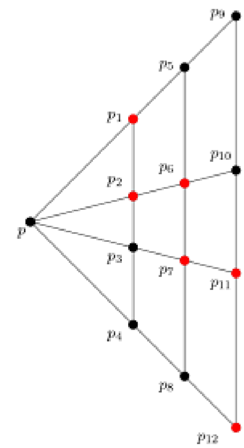

Figure 5.3: B-Clique in $VG(3,4)$

5.3 B-Cliques

Max-Clique which does not include Vertex of Convergence are called B-Cliques. For instance, one of B-Cliques of $VG(3,4)$ is shown in 5.3(Red Vertices form Max-Clique).

Let no.of B-Cliques in block is m, it is computed for different k and l and shown in table 5.1

k	l	m	k	l	m	k	l	m
2	2	1	4	2	9	6	2	25
2	3	4	4	3	18	6	3	80
2	4	9	4	4	42	6	4	232
2	5	16	4	5	98	6	5	710
2	6	25	4	6	232	6	6	2140
3	2	4	5	2	16	7	2	36
3	3	8	5	3	40	7	3	144
3	4	18	5	4	98	7	4	494
3	5	40	5	5	272	7	5	1700
3	6	80	5	6	710	7	6	5932

Table 5.1: No.of B-Cliques(m) for different k and l

Theorem 5. *No.of B-Cliques in $VG(k,l)$ is m^b*

Proof. Vertices in B-Clique of block, say B_i, are connected to all the vertices of any B-Clique in blocks $B_j (j \neq i)$. Therefore B-Cliques of $VG(k,l)$ can be formed by combining B-Cliques in each block. As there are m distinct B-Cliques in each block, no.of B-Cliques in $VG(k,l)$ is m^b. □

5.4 Max-Cliques in $VG(k, l)$

As Max-Cliques are classified into two disjoint classes P-Cliques and B-Cliques, no.of Max-Cliques in $VG(k, l)$ is sum of no.of P-Cliques and no.of B-Cliques i.e. $m^b + (l-1)^b$.

Theorem 6. *For large n, no.of Max-Cliques in $VG(k, l)$ is of order $m^{\frac{n}{l\times k}}$*

Proof. Consider no.of vertices in graph. $n = 1 + b \times l \times k$. Hence $b = \frac{n-1}{l\times k}$ which is $\frac{n}{l\times k}$ for large n. Hence no.of Cliques in $VG(k, l)$ is $m^{\frac{n}{l\times k}} + (l-1)^{\frac{n}{l\times k}}$,from Theorem 4 and Theorem 5, which is $max(m, l-1)^{\frac{n}{l\times k}}$ for large n. It can be empirically observed that $m > l - 1$. Therefore no.of Max-Cliques is of order $m^{\frac{n}{l\times k}}$ □

In [9], Moon et al. has proved that Maximum no.of Max-Cliques is of order $3^{\frac{n}{3}}$ for special kind of graphs, but for PVG ,specifically class of $VG(k, l)$ complexity is of order $(m^{\frac{1}{l\times k}})^n$. This varies for different values of l, k and empirically it has been observed that maximum of $m^{\frac{1}{l\times k}}$ is $4^{\frac{1}{5}}$, other values are tabulated in 5.2.This shows that no.of maximal cliques in PVG, specifically in $VG(k, l)$ is of order $4^{\frac{1}{5}}$.

k	l	$m^{\frac{1}{l\times k}}$	k	l	$m^{\frac{1}{l\times k}}$	k	l	$m^{\frac{1}{l\times k}}$
2	2	1	4	2	1.316074	6	2	1.307660
2	3	1.259921	4	3	1.272348	6	3	1.275637
2	4	1.316074	4	4	1.263145	6	4	1.254764
2	5	**1.319508**	4	5	1.257654	6	5	1.244635
2	6	1.307660	4	6	1.257654	6	6	1.237404
3	2	1.259921	5	2	1.319508	7	2	1.291708
3	3	1.259921	5	3	1.278804	7	3	1.267007
3	4	1.272348	5	4	1.257654	7	4	1.247971
3	5	1.278804	5	5	1.251361	7	5	1.236797
3	6	1.275637	5	6	1.244635	7	6	1.229810

Table 5.2: Value of $m^{\frac{1}{l*k}}$ different k and l

Chapter 6

Conclusion and Future Work

Point Visibility Graphs is growing area in literature because of its wide range of usage. This report analyzed few computational aspects of PVG and tried to find maximum no.of Max-Cliques in PVG using Visibility Grid Graphs.

Many of the standard computational problems are worked on PVGs, few areas like Dominating Set are not yet researched. It has been empirically shown that maximum no.of max-cliques in class of Visibility Grid Graphs is of order $4^{\frac{n}{5}}$, in Section 5.4, which is less than $3^{\frac{n}{3}}$, proved in [9]. This opens new area of research to theoretically estimate bound on Maximum No.of Max-Cliques in PVG.

As PVG are either path graphs or diameter 2 graphs, potential direction of research would be to extend properties of PVG to both the domains. For instance, as Vertex Cover is NP-Hard in PVGs(Section 3.1) implies Vertex Cover in diameter 2 graphs is also NP-Hard.

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