NEW SYLLABUS

SECOND YEAR B. Sc.
COMPUTER SCIENCE

SEMESTER-IV

COMPUTATIONAL GEOMETRY

MATHEMATICS (MTC 241) : Paper-I

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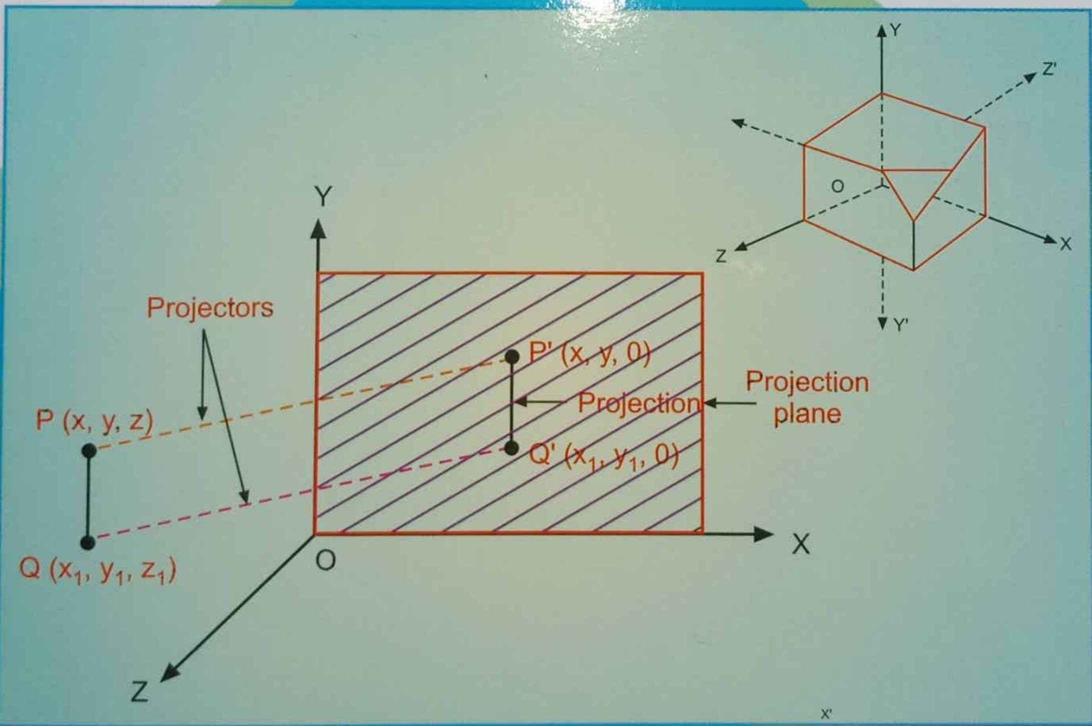

NIRALI PRAKASHAN
ADVANCEMENT OF KNOWLEDGE

SPPU New Syllabus

COMPUTATIONAL GEOMETRY

for

[MTC - 231]

Second Year B.Sc. (Computer Science)

Mathematics - Paper I, Semester - [redacted] IV

New Syllabus as per CBCS Pattern Credit - 2

June 2020

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Preface ...

We have great pleasure in presenting this text book on **COMPUTTIONAL GEOMETRY** to the students of S.Y.B.Sc. Computer Science Semester - IV, Mathematics Paper - I. This book is written strictly according to the new revised syllabus of Savitribai Phule Pune University to be implemented from June 2020.

We have taken utmost care to present the matter systematically and with proper flow of mathematical concepts. We begin the Chapter by Introduction and at the end the Summary of the Chapter is provided. We have added one significant feature: **"Think Over It"** in this new **edition**. Here, we have posed questions of simple, difficult and intuitive type in nature. It is expected that the students should think over it and try to find the answers. This will assess the understanding of the knowledge of the Chapter.

The book contains good number of solved problems and the number of graded problems in the exercises.

We are thankful to **Shri Dineshbhai Furia, Shri Jignesh Furia,** Mrs. Anagha Medhekar (Proof Reading and Co-ordination), Mr. Ilyas Shaikh, Mrs. Anjali Mule (Figure Drawing) and the staff of Nirali Prakashan for the great efforts that they have taken to publish the book in time.

We welcome the valuable suggestions from our colleagues' and readers for the improvement of the book.

PUNE **AUTHORS**

JANUARY 2021

Syllabus ...

1. Two Dimensional Transformations **(12 Lectures)**

1.1 Introduction.
1.2 Representation of points.
1.3 Transformations and matrices.
1.4 Transformation of points.
1.5 Transformation of straight lines
1.6 Mid-point Transformation
1.7 Transformation of parallel lines
1.8 Transformation of intersecting lines
1.5 Transformation: rotations, reflections, scaling, shearing.
1.6 Combined transformations.
1.7 Transformation of a unit square.
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2. Three Dimensional Transformations **(08 Lectures)**

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2.4 Rotation about – an axis parallel to co-ordinate axes, an arbitrary line
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3. Projection **(08 Lectures)**

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4. Plane and Space Curves **(08 Lectures)**

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4.5 Bezier Curves – Introduction, definition, properties (without proof), Curve fitting (up to n = 3), equation of the curve in matrix form (upto n = 3)

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Contents ...

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Chapter 1 ...

Two-Dimensional Transformations

Charle's Hermites

Charle's Hermites (1822-1901 A.D.) : At first he was Professor of mathematics in Šarbonne University. He left this place and joined in the same post in Ecole Polytechnic. He was very much influenced by Jacob's line. He wrote book on Theory of Functions and in 1873 A.D. he proved that $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} \dots$ is an incommensurable number.

Further he wrote on elliptic functions, Modular functions Theta functions, Theory of numbers and Invariants Theory, Hermitian Numbers and Hermitian Form which made him immortal in mathematics.

1.1 Introduction

Computational Geometry is a part of discrete mathematics that studies computational problems related to geometrical objects. The areas which use computational geometry are computer graphics, scientific calculations, computer games and robotics etc.

The representation and transformation of points and lines are the fundamentals of mathematics underlying the computer graphics. The objects are represented graphically by using the points and the lines joining term. The visual representation of an object needs scaling, translation, rotation, shearing etc. We shall discuss them one-by-one in this chapter.

In 2-dimensional geometry, a point is represented by its co-ordinates viz. abscissa and the ordinate. Thus, if P is a point with co-ordinates x and y, we write P(x, y).

We can represent the same point as 1×2 row matrix $[x \;\; y]$ or 2×1 column matrix $\begin{bmatrix} x \\ y \end{bmatrix}$. We shall be following the practice of representing the point as 1×2 row matrix $[x \;\; y]$ conveniently.

Likewise in three dimensional situation a point P whose co-ordinates are x, y, z; will be represented by 1×3 matrix $[x \;\; y \;\; z]$.

Once we accept to represent a point by a matrix, it is quite natural that any transformation of the point/line will be accomplished with the help of some other matrix say [T]. Such a matrix which accomplishes the transformation of a point/line is called a transformation matrix and it acts as a geometric operator.

Thus, if a point P is represented by the matrix [A] and [T] is the transformation matrix, then we write [A] [T] to mean that [T] operates on [A]. The multiplication [A] [T] produces another matrix [B] which is the transformed form P^* of the point P.

1.2 Transformation of Points

Let P be a point whose co-ordinates are x and y, so that the matrix formed by them is 1×2 matrix $[X] = [x \ y]$.

If $[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a general 2×2 transformation matrix, then multiplication of [X] by [T] leads to

$$[X]\,[T] = [x \ y] \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$= [ax + cy \quad bx + dy]$$
$$= [x^* \ y^*]$$

Thus, under the transformation T a point P with co-ordinates x and y is transformed to a point P^* with co-ordinates

$$x^* = ax + cy \text{ and}$$
$$y^* = bx + dy$$

Now depending upon the nature of values of the elements of [T], we have the following different cases.

First Case :

Suppose a = d = 1 and b = c = 0.

In this case, the matrix $[T] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is 2×2 identity matrix and we get

$$x^* = ax + cy = x$$
$$y^* = bx + dy = y$$

Hence, the change in the co-ordinates of the point P does not occur.

Second Case :

Suppose b = c = 0 and d = 1. Then

$$[X]\,[T] = [x \ y] \begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$$
$$\therefore \quad [X]\,[T] = [ax \ \ y] = [x^* \ y^*]$$

Here $x^* = ax$ suggests that a scale change is produced in the x-component of the position vector of a point P. The effect of this transformation is shown in Fig. 1.1 below.

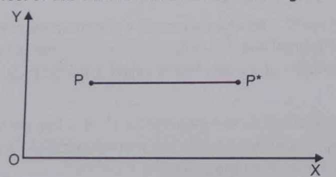


Fig. 1.1

Third Case :

Suppose b = c = 0. Then,

$$[X]\,[T] = [x \ y] \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$$
$$= [ax \ \ dy] = [x^* \ y^*]$$

Here, $x^* = ax$ and $y^* = dy$ means there is a scale change in the x-component as well as in y-component of the position vector of a point P. It is shown in Fig. 1.2.

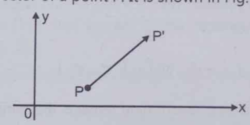


Fig. 1.2

We note that if $a \neq d$, then the scale changes in x and y components are different.

If $a = d > 1$, then equal enlargement occurs in both components while if $0 < a = d < 1$, then equal compression occurs in both components of a position vector.

Fourth case :

If b = c = 0, d = 1 and a = – 1, then

$$[\ x \ \ y\ ] \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = [\ -x \ \ y\ ] = [\ x^* \ \ y^*\ ]$$

This leads to the reflection through y-axis.

If b = c = 0, a = 1, d = – 1, then

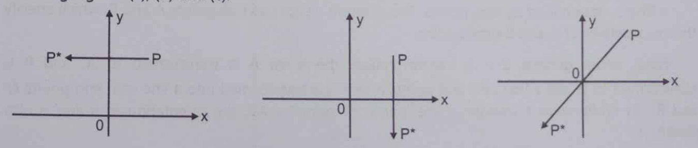
$$[\ x \ \ y\ ] \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = [\ x \ \ -y\ ] = [\ x^* \ \ y^*\ ]$$

This leads to the reflection in through x-axis.

If b = c = 0, a < 0, d < 0, taking a = d = – 1, we have

$$[\ x \ \ y\ ] \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = [\ -x \ \ -y\ ] = [\ x^* \ \ y^*\ ]$$

This means a reflection through origin has occurred. These three reflections are shown in the following Fig. 1.3 (a), (b) and (c).



(a) Reflection through y-axis **(b) Reflection through x-axis** **(c) Reflection through origin**

Fig. 1.3

Fifth case :

Suppose a = d = 1 and c = 0. In this case,

$$[\,x\ \ y\,]\begin{bmatrix}1 & b\\ 0 & 1\end{bmatrix} = [\,x\ \ \ bx+y\,] = [\,x^*\ \ y^*\,]$$

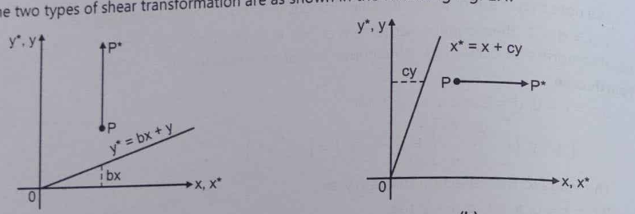
The effect is that x co-ordinate remains the same and y* is a linear combination of x and y. This effect is called shear. This transformation produces a shear proportional to the x co-ordinate.

Again, if a = d = 1, b = 0, then

$$[\,x\ \ y\,]\begin{bmatrix}1 & 0\\ c & 1\end{bmatrix} = [\,x+cy\ \ \ y\,] = [\,x^*\ \ y^*\,]$$

Under this shear transformation y co-ordinate remains the same $y^* = y$ and $x^* = x + cy$. It produces a shear proportional to the y co-ordinate.

The two types of shear transformation are as shown in the following Fig. 1.4.



(a) (b)

Fig. 1.4

Finally, we note that there is no any effect of 2 × 2 general transformation on origin.

$$[\,0\ \ 0\,]\begin{bmatrix}a & b\\ c & d\end{bmatrix} = [\,0\ \ 0\,]$$

This means origin is invarient under general transformation.

1.3 Transformation of Lines

A line is determined by two points. The position vectors of two points A and B, which specify the co-ordinates of A and B define a line.

Now, under general 2 × 2 transformation, the point A is transformed to A^* and B is transformed to B^*. So a line with end points A and B is transformed into a line with end points A* and B*. In addition to a change in the length of segment AB, the orientation of a line is also changed.

We illustrate this by an example.

Let A and B be the points with position vectors [1 2] and [2 0] respectively.

If $[T] = \begin{bmatrix}3 & 1\\ 1 & 2\end{bmatrix}$ is the transformation matrix, then

$$[1\ \ 2]\ [T] = [1\ \ 2]\begin{bmatrix}3 & 1\\ 1 & 2\end{bmatrix} = [5\ \ 5]$$

Also, $$[2\ \ 0]\ [T] = [2\ \ 0]\begin{bmatrix}3 & 1\\ 1 & 2\end{bmatrix} = [6\ \ 2]$$

Thus, under the transformation [T], the point A with position vector [1 2] is transformed to a point A* with position vector [5 5] and a point B with position vector [2 0] is transformed to a point B* with position vector [6 2].

The above two transformations are written in more compact form as below :

$$\begin{bmatrix}1 & 2\\ 2 & 0\end{bmatrix}\begin{bmatrix}3 & 1\\ 1 & 2\end{bmatrix} = \begin{bmatrix}5 & 5\\ 6 & 2\end{bmatrix}$$ This is shown in the following Fig. 1.5.

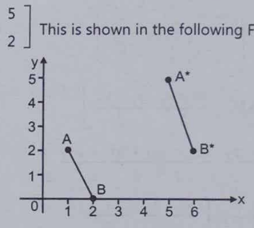


Fig. 1.5

We observe that the length as well as orientation of AB is changed.

1.4 Mid-point Transformation

When a line L joining the points A $[x_1\ y_1]$ and B $[x_2\ y_2]$ is transformed into a line L^* joining A^* $[x_1^*\ y_1^*]$ and B^* $[x_2^*\ y_2^*]$ under [T], it is clear that the points A and A^* correspond and also B and B^* correspond. In fact there is a one-one correspondence between the points of line L and those of L^*.

We shall now prove that the mid-point of AB corresponds to the mid-point of A^*B^*.

Fig. 1.6

As shown in the Fig. 1.6 the line AB is transformed into line A^*B^* under the transformation $[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$; where A $[x_1\ y_1]$, B $[x_2\ y_2]$, $A^*\ [x_1^*\ y_1^*]$ and $B^*\ [x_2^*\ y_2^*]$.

Also M and M^* are the mid-points of AB and A^*B^* respectively.

$$\begin{bmatrix} A \\ B \end{bmatrix} [\ T\ ] = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 + cy_1 & bx_1 + dy_1 \\ ax_2 + cy_2 & bx_2 + dy_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1^* & y_1^* \\ x_2^* & y_2^* \end{bmatrix} = \begin{bmatrix} A^* \\ B^* \end{bmatrix}$$

The mid-point M^* of A^*B^* is $M^* \left[\dfrac{x_1^* + x_2^*}{2} \quad \dfrac{y_1^* + y_2^*}{2} \right]$

$$\therefore \quad M^* \left[\frac{ax_1 + cy_1 + ax_2 + cy_2}{2} \quad \frac{bx_1 + dy_1 + bx_2 + dy_2}{2} \right] \qquad \text{... (1)}$$

Now, the mid-point M of AB is $M \left[\dfrac{x_1 + x_2}{2} \quad \dfrac{y_1 + y_2}{2} \right]$.

Therefore, under the transformation T it will be transformed to

$$[M]\ [T] = \left[\frac{x_1 + x_2}{2} \quad \frac{y_1 + y_2}{2} \right] \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \left[a\left(\frac{x_1 + x_2}{2}\right) + x\left(\frac{y_1 + y_2}{2}\right) \quad b_1\left(\frac{x_1 + x_2}{2}\right) + d\left(\frac{y_1 + y_2}{2}\right) \right]$$

$$= \left[\frac{ax_1 + ax_2 + cy_1 + cy_2}{2} \quad \frac{bx_1 + bx_2 + dy_1 + dy_2}{2} \right] \qquad \text{... (2)}$$

From (1) and (2) we see that $[M]\ [T] = [M^*]$.

Thus, mid-points of two lines correspond.

In general, if a point P divides AB in the ratio $\lambda : 1$, then under the transformation [T] it will be transformed into a point P^* that divides A^*B^* in the same ratio $\lambda : 1$. Hence, there is a one-one correspondence between the points on AB and the points on A^*B^*.

This is shown as below in Fig. 1.7.

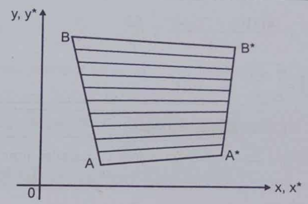


Fig. 1.7

Illustrative Example

Example 1.1 : Let A [0 1], B [2 2] and $T = \begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix}$ be the transformation matrix. Let P divides AB in the ratio 4 : 3.

(a) Find $[A^*] = [A]\ [T]$, $[B^*] = [B]\ [T]$.

(b) Find $[P^*] = [P]\ [T]$.

(c) Verify that P and P^* correspond.

Solution : (a) $\begin{bmatrix} A \\ B \end{bmatrix} [\ T\ ] = \begin{bmatrix} 0 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 4 \\ 14 & 10 \end{bmatrix}$

$$= \begin{bmatrix} A^* \\ B^* \end{bmatrix}$$

$\therefore$ A^* is the point $[\ 5 \quad 4\ ]$ and B^* is the point $[\ 14 \quad 10\ ]$.

(b) The point P [x y] divides AB in the ratio 4 : 3.

Therefore, by section formula we have,

$$x = \frac{(4)\ (2) + (3)\ (0)}{4 + 3} = \frac{8}{7} \quad \text{and}$$

$$y = \frac{(4)\ (2) + (3)\ (1)}{4 + 3} = \frac{11}{7}$$

Now, $$[P]\ [T] = \left[\frac{8}{7} \quad \frac{11}{7} \right] \begin{bmatrix} 2 & 1 \\ 5 & 4 \end{bmatrix} = \left[\frac{71}{7} \quad \frac{52}{7} \right]$$

So P^* is the point $\left[\dfrac{71}{7} \quad \dfrac{52}{7} \right]$.

(c) A point $[\ x^* \quad y^*\ ]$ which divides the join of A^*B^* in the ratio 4 : 3, has co-ordinates

$$x^* = \frac{(4)\ (14) + (3)\ (5)}{4 + 3} = \frac{71}{7}$$

$$y^* = \frac{(4)(10) + (3)(4)}{4+3} = \frac{52}{7}$$

Thus, the point $\begin{bmatrix} x^* & y^* \end{bmatrix} = \begin{bmatrix} \frac{71}{7} & \frac{52}{7} \end{bmatrix}$ is P^*.

1.5 Transformation of Parallel Lines

The effect of 2×2 transformation matrix on a pair of parallel lines is another pair of parallel lines.

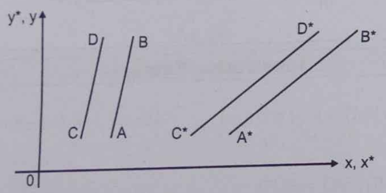


Fig. 1.8

As shown in the Fig. 1.8, AB, CD is a given pair of parallel lines. If $[A] = [x_1\ y_1]$ and $[B] = [x_2\ y_2]$, then AB and CD being parallel, they have the same slope $m = \frac{y_2 - y_1}{x_2 - x_1}$.

If $[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the transformation matrix, then

$$\begin{bmatrix} A \\ B \end{bmatrix}[T] = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 + cy_1 & bx_1 + dy_1 \\ ax_2 + cy_2 & bx_2 + dy_2 \end{bmatrix}$$

$$= \begin{bmatrix} x_1^* & y_1^* \\ x_2^* & y_2^* \end{bmatrix} = \begin{bmatrix} A^* \\ B^* \end{bmatrix}$$

The slope of line A^*B^* is

$$m^* = \frac{y_2^* - y_1^*}{x_2^* - x_1^*} = \frac{(bx_2 + dy_2) - (bx_1 + dy_1)}{(ax_2 + cy_2) - (ax_1 + cy_1)}$$

$$\therefore \quad m^* = \frac{b + d\left(\frac{y_2 - y_1}{x_2 - x_1}\right)}{a + c\left(\frac{y_2 - y_1}{x_2 - x_1}\right)}$$

$$\therefore \quad m^* = \frac{b + dm}{a + cm}$$

This shows that m^* is independent of x_1, y_1, x_2, y_2. Also m, a, b, c, d are the same for AB and CD. Hence, it immediately follows that m^* is the same for A^*B^* and C^*D^*.

Thus, parallel lines are transformed onto parallel lines. It is clear that under 2×2 transformation matrix, parallelogram gets transformed onto other parallelogram.

1.6 Transformation of intersecting lines

We shall prove that under the transformation matrix $[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ a pair of intersecting lines gets transformed onto a pair of intersecting lines. Further the point of intersection of original pair of lines is transformed onto a point of intersection of the transformed form of pair of lines.

First we obtain the transformed form of equation of the line $y = mx + h$; under the transformation matrix $[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Theorem 1 If the line $y = mx + h$ is transformed onto the line $y^* = m^*x^* + h^*$ under the transformation matrix $[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then (a) $m^* = \frac{b + dm}{a + cm}$ and (b) $h^* = h\left(\frac{ad - bc}{a + cm}\right)$.

Proof : (a) We have already proved in the preceding article that

$$m^* = \frac{b + dm}{a + cm}$$

(b) We find the value of h^* so that the equation $y = mx + h$ is transformed to the equation

$$y^* = m^*x^* + h^* \quad \text{... (1)}$$

But, $\quad [x\ y][T] = [x^*\ y^*]$ implies

$$[x\ y]\begin{bmatrix} a & b \\ c & d \end{bmatrix} = [x^*\ y^*]$$

$$\therefore \quad [ax + cy \quad bx + dy] = [x^*\ y^*]$$

$$\therefore \quad x^* = ax + cy \text{ and } y^* = bx + dy$$

After using these values of x^* and y^* in (1), we get

$$bx + dy = \left(\frac{b + dm}{a + cm}\right)(ax + cy) + h^*$$

The point (0, h) lies on the line $y = mx + h$.

$$\therefore \quad 0 + dh = \left(\frac{b + dm}{a + cm}\right)(0 + ch) + h^*$$

$$\therefore \quad h^* = dh - \left(\frac{b + dm}{a + cm}\right)(ch)$$

$$\therefore \quad h^* = h\left[d - \frac{c\,(b + dm)}{(a + cm)}\right]$$

$$\therefore \quad h^* = h\left(\frac{ad - bc}{a + cm}\right)$$ ■

Theorem 2 Let the intersecting lines $y = m_1x + h_1$ and $y = m_2 x + h_2$ be transformed under the transformation matrix $[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to

$$y^* = m_1^* x^* + h_1^*$$

and $$y^* = m_2^* x^* + h_2^* \text{ respectively}$$

Then,

(a) the transformed lines are intersecting.

(b) the point of intersection of original pair of lines is transformed to the point of intersection of new pair of lines, under the transformation matrix [T].

Proof : (a) The lines $y = m_1x + h_1$ ... (1)

and $y = m_2 x + h_2$ ... (2)

are intersecting. Therefore, their slopes are different; $m_1 \neq m_2$.

To find the point of intersection of (1) and (2) we write these equations in matrix form as below.

$$\left.\begin{aligned} -m_1x + y &= h_1 \\ -m_2 x + y &= h_2 \end{aligned}\right\}$$

$$\therefore \quad [x \;\; y]\begin{bmatrix} -m_1 & -m_2 \\ 1 & 1 \end{bmatrix} = [h_1 \;\; h_2]$$

$$\therefore \quad [x \;\; y] = [h_1 \;\; h_2]\begin{bmatrix} -m_1 & -m_2 \\ 1 & 1 \end{bmatrix}^{-1}$$

$$\therefore \quad [x \;\; y] = [h_1 \;\; h_2]\left(\frac{1}{m_2 - m_1}\right)\begin{bmatrix} 1 & m_2 \\ -1 & -m_1 \end{bmatrix}$$

$$\therefore \quad [x \;\; y] = \frac{1}{m_2 - m_1}[h_1 - h_2 \quad h_1m_2 - h_2 m_1] \qquad \because m_2 - m_1 \neq 0$$

$$\therefore \quad [x \;\; y] = \left[\frac{h_1 - h_2}{m_2 - m_1} \quad \frac{h_1m_2 - h_2 m_1}{m_2 - m_1}\right] \qquad \text{... (3)}$$

Now, the line $y = m_1x + h_1$ is transformed to $y^* = m_1^* x^* + h_1^*$ ... (4)

where, $$m_1^* = \frac{b + dm_1}{a + cm_1}$$

and $$h_1^* = h_1\left(\frac{ad - bc}{a + cm_1}\right)$$

Also the line $y = m_2 x + h_2$ is transformed to $y^* = m_2^* x^* + h_2^*$ ... (5)

where, $$m_2^* = \frac{b + dm_2}{a + cm_2}$$

and $$h_2^* = h_2\left(\frac{ad - bc}{a + cm_2}\right)$$

Now, we note that just like (3); if $\dfrac{h_1^* - h_2^*}{m_2^* - m_1^*}$ and $\dfrac{h_1^* m_2^* - h_2^* m_1^*}{m_2^* - m_1^*}$ exist then

$\left[\dfrac{h_1^* - h_2^*}{m_2^* - m_1^*} \quad \dfrac{h_1^* m_2^* - h_2^* m_1^*}{m_2^* - m_1^*}\right]$ will be the point of intersection of lines given by (4) and (5).

$$\frac{h_1^* - h_2^*}{m_2^* - m_1^*} = \frac{h_1\left(\frac{ad - bc}{a + cm_1}\right) - h_2\left(\frac{ad - bc}{a + cm_2}\right)}{\frac{b + dm_2}{a + cm_2} - \frac{b + dm_1}{a + cm_1}}$$

$$= \frac{a\,(h_1 - h_2) + c\,(h_1m_2 - h_2 m_1)}{m_2 - m_1}$$

It exists since $m_2 - m_1 \neq 0$.

Also $$\frac{h_1^* m_2^* - h_2^* m_1^*}{m_2^* - m_1^*} = \frac{b\,(h_1 - h_2) + d\,(h_1m_2 - h_2 m_1)}{m_2 - m_1}$$ exists since $m_2 - m_1 \neq 0$.

Thus, the lines given by (4) and (5) are intersecting and their point of intersection is

$$\left[\frac{a\,(h_1 - h_2) + c\,(h_1m_2 - h_2 m_1)}{m_2 - m_1} \quad \frac{b\,(h_1 - h_2) + d\,(h_1m_2 - h_2 m_1)}{m_2 - m_1}\right] \qquad \text{... (6)}$$

(b) Now, consider the transformation of the point of intersection of (1) and (2) under [T].

$$\left[\frac{h_1 - h_2}{m_2 - m_1} \quad \frac{h_1m_2 - h_2 m_1}{m_2 - m_1}\right]\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \left[\frac{a\,(h_1 - h_2) + c\,(h_1m_2 - h_2 m_1)}{m_2 - m_1} \quad \frac{b\,(h_1 - h_2) + d\,(h_1m_2 - h_2 m_1)}{m_2 - m_1}\right] \qquad \text{... (7)}$$

The results (6) and (7) imply that the point of intersection of lines (1) and (2) is transformed to the point of intersection of lines (4) and (5). ■

Illustrative Examples

Example 1.2 : A line segment joining A [4 9] and B [−2 1] is scaled uniformly by factor 2. What is the mid-point of the transformed line segment ? **(March 2006)**

Solution : The uniform scaling by a factor 2 is governed by the transformation matrix

$$[T] = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

We know that under the general transformation matrix, the mid-point of line segment AB is transformed onto the mid-point of the transformed line segment A^*B^*.

The mid-point M of segment AB; where A $[4 \ \ 9]$ and B $[-2 \ \ 1]$ is M $\begin{bmatrix} \frac{4-2}{2} & \frac{9+1}{2} \end{bmatrix}$ i.e. M $[1 \ \ 5]$.

$$\therefore \quad M^* = [M][T] = [1 \ \ 5]\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = [2 \ \ 10]$$

Example 1.3 : If a 2 × 2 transformation matrix $[T] = \begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix}$ is used to transform the line passing through the points A $\begin{bmatrix} 3 & -\frac{1}{2} \end{bmatrix}$ and B [0 1]., find the equation of the resulting line. **(April 2005)**

Solution : The equation of the line passing through the points A $\begin{bmatrix} 3 & -\frac{1}{2} \end{bmatrix}$ and B [0 1] is

$$\frac{y - y_1}{y_2 - y_1} = \frac{x - x_1}{x_2 - x_1}; \text{ (two point form)}$$

$$\therefore \quad \frac{y + \frac{1}{2}}{1 + \frac{1}{2}} = \frac{x - 3}{0 - 3}$$

$$\therefore \quad y = -\frac{1}{2}x + 1$$

Here $m = -\frac{1}{2}$ and $h = 1$.

The transformation matrix is

$$[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix}$$

For the transformed line $y^* = m^*x^* + h^*$, we have

$$m^* = \frac{b + dm}{a + cm} = \frac{3 + (2)\left(-\frac{1}{2}\right)}{1 + (-2)\left(-\frac{1}{2}\right)} = 1$$

$$h^* = h\left(\frac{ad - bc}{a + cm}\right) = 1 \times \frac{(1)(2) - (3)(-2)}{1 + (-2)\left(-\frac{1}{2}\right)} = 4$$

The equation of the transformed line is

$$y^* = x^* + 4$$

Example 1.4 : If the line y = 2x + 1 is transformed by the 2 × 2 transformation matrix $[T] = \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix}$, then find the equation of the transformed line.

Solution : We know that under the transformation matrix $[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the line $y = mx + h$ is transformed to the line $y^* = m^*x^* + h^*$;

where, $m^* = \dfrac{b + dm}{a + cm}$ and $h^* = h\left(\dfrac{ad - bc}{a + cm}\right)$.

Now $[T] = \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix}$ a = 4, b = 2, c = −1, d = 3.

Also $\quad y = 2x + 1 \Rightarrow m = 2, h = 1.$

$$\therefore \quad m^* = \frac{b + dm}{a + cm} = \frac{2 + (3)(2)}{4 + (-1)(2)} = \frac{8}{2} = 4$$

$$h^* = h\left(\frac{ad - bc}{a + cm}\right) = 1 \times \frac{(4)(3) - (2)(-1)}{4 + (-1)(2)} = 7$$

Therefore, equation of the transformed line is

$$y^* = m^*x^* + h^*$$

i.e. $\quad y^* = 4x^* + 7$

1.7 Rotation, Reflection, Scaling and Shearing Transformations

Rotation :

We obtain a transformation matrix corresponding to a transformation which rotates the position vector of a point; the rotation being about the origin.

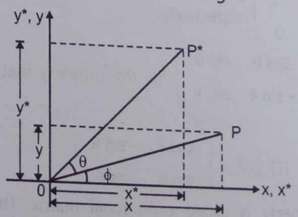


Fig. 1.9

shown in the Fig. 1.9, the position vector OP of the point P is such that OP = r and = ϕ. Then the point P [x y] is given by

$$x = r\cos\phi \text{ and } y = r\sin\phi \quad \text{... (1)}$$

et us rotate OP about the origin O through an angle θ so that the point P is transformed to

The point P^* [x^* y^*] is given by;

$$x^* = r\cos(\phi + \theta) \text{ and } y^* = r\sin(\phi + \theta) \quad \text{... (2)}$$

Now,

$$x^* = r\cos(\phi + \theta) = r(\cos\phi\cos\theta - \sin\phi\sin\theta)$$
$$= (r\cos\phi)\cos\theta - (r\sin\phi)\sin\theta$$
$$= x\cos\theta - y\sin\theta \quad \because \text{ by (1)} \quad \text{... (3)}$$

Also,

$$y^* = r\sin(\phi + \theta)$$
$$= r(\sin\phi\cos\theta + \cos\phi\sin\theta)$$
$$= (r\sin\phi)\cos\theta + (r\cos\phi)\sin\theta$$
$$= y\cos\theta + x\sin\theta \quad \because \text{ by (1)}$$
$$= x\sin\theta + y\cos\theta \quad \text{... (4)}$$

The equations (3) and (4) can be written in the matrix form as

$$[x^* \; y^*] = [x \; y]\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

The matrix $$[T] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

is the required transformation matrix for rotation of a position vector through an angle θ about the origin.

Some particular cases of the above result are as below.

If $\theta = 90^\circ$, then $$[T] = \begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

is the transformation matrix. Similarly, if $\theta = 180^\circ$ and 270°, then the transformation matrices are $[T] = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and $[T] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ respectively.

A rotation matrix $[T] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ has the property that its inverse is equal to its transpose.

$$[T]^{-1} = [T]' \; [T]^t = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

A matrix having this property is called orthogonal matrix. Thus the rotation matrix is orthogonal.

Reflection :

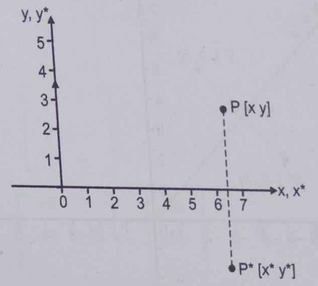


Fig. 1.10

As shown in the figure, the mirror image of the point P [x y] in x-axis is the point P^* [x^* y^*]. Thus P^* is reflection of P in x-axis. Here we see that the x co-ordinate of P and P^* is the same; while the y co-ordinate y of P^* is equal in magnitude but opposite in sign to the y co-ordinate of P.

The above transformation is achieved by the transformation matrix $[T] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

The reflection of a point P [x y] in y-axis, in the line y = x and in the line y = – x is shown below in Fig. 1.11.

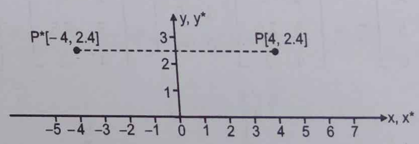


Fig. 1.11 : Reflection in y-axis ; $[T] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

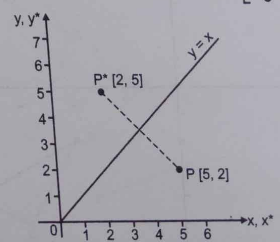


Fig. 1.12 : Reflection in the line y = x; $[T] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

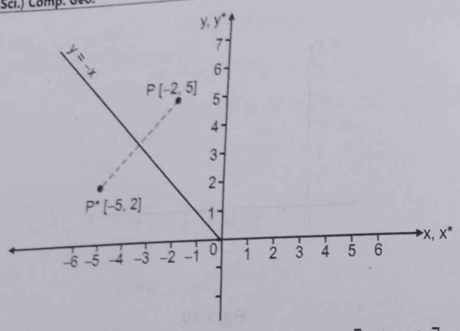


Fig. 1.13 : Reflection in the line y = – x; $[T] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

Illustration : Consider a triangle ABC with vertices A [2 1], B [3 2.5] and C [4 2]. We shall find reflection of this triangle in (a) x-axis, (b) y-axis, (c) the line y = x and (d) line y = – x.

(a) The transformation matrix for reflection in x-axis is

$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

We have $\begin{bmatrix} A \\ B \\ C \end{bmatrix} [T] = \begin{bmatrix} 2 & 1 \\ 3 & 2.5 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 3 & -2.5 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} A_1 \\ B_1 \\ C_1 \end{bmatrix}$

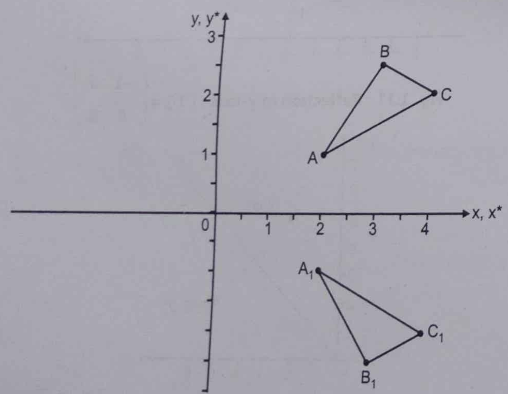


Fig. 1.14

(b) The transformation matrix for reflection in y-axis is

$$[T] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We have, $\begin{bmatrix} A \\ B \\ C \end{bmatrix} [T] = \begin{bmatrix} 2 & 1 \\ 3 & 2.5 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} -2 & 1 \\ -3 & 2.5 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} A_2 \\ B_2 \\ C_2 \end{bmatrix}$$

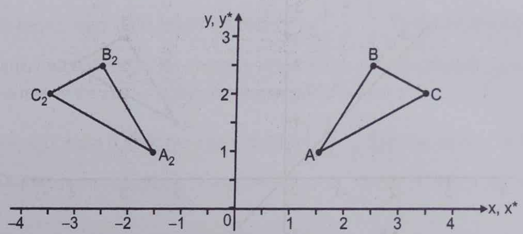


Fig. 1.15

(c) The transformation matrix for reflection in the line y = x is $[T] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

We have, $\begin{bmatrix} A \\ B \\ C \end{bmatrix} [T] = \begin{bmatrix} 2 & 1 \\ 3 & 2.5 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2.5 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} A_3 \\ B_3 \\ C_3 \end{bmatrix}$

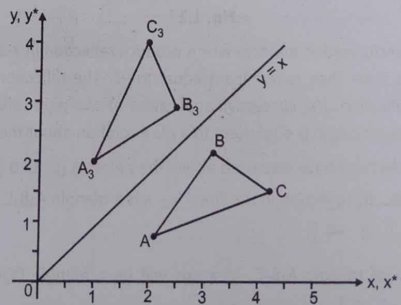


Fig. 1.16

(d) The transformation matrix for reflection in the line y = – x is $[T] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

We have, $\begin{bmatrix} A \\ B \\ C \end{bmatrix} [T] = \begin{bmatrix} 2 & 1 \\ 3 & 2.5 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$

$$= \begin{bmatrix} -1 & -2 \\ -2.5 & -3 \\ -2 & -4 \end{bmatrix} = \begin{bmatrix} A_4 \\ B_4 \\ C_4 \end{bmatrix}.$$

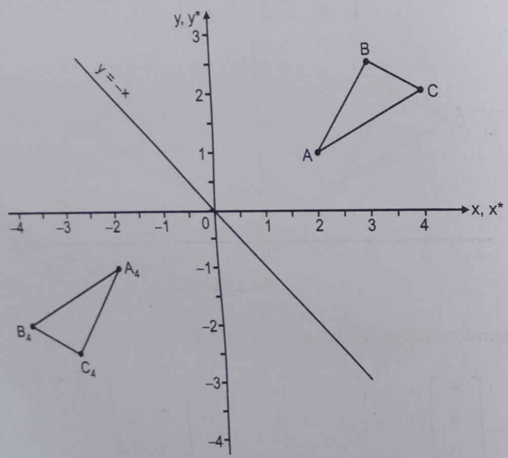


Fig. 1.17

Each of the four transformation matrices which produce reflection in x-axis, y-axis, the line y = x and the line y = – x, have their determinant equal to –1. The reflection produced by such a matrix is called pure reflection. The successive application of two pure reflection transformations about lines passing through origin is equivalent to a pure rotation about the origin.

In the illustration which we have discussed above the points A [2 1], B [3 2.5] and C [4 2] are vertices of a triangle ABC. Its reflection in the line y = – x is a triangle $A_4B_4C_4$ where A_4 [–1 –2], B_4 [–2.5 –3] and C_4 [–2 –4].

Now the reflection of triangle $A_4B_4C_4$ in x-axis will be a triangle PQR where, P [–1 2], Q [–2.5 3] and R [–2 4].

The same result is obtained by rotating the triangle ABC, about the point origin; through angle θ = 90°.

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} \begin{bmatrix} \cos 90^\circ & \sin 90^\circ \\ -\sin 90^\circ & \cos 90^\circ \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 2.5 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 \\ -2.5 & 3 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} P \\ Q \\ R \end{bmatrix}$$

Scaling and Shearing :

We have already seen in the beginning that if $[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the transformation matrix, then the scaling effect is governed by the entries a and d on the main diagonal of [T] while the shearing effect is governed by the entries b and c on the second diagonal of [T].

If a = d = 3, b = c = 0, then the effect of $[T] = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$ on the vertices of a triangle ABC is that 3 times enlargement i.e. uniform scaling occurs about the origin. This is shown in the Fig. 1.18 below.

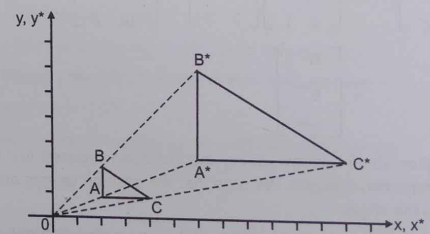


Fig. 1.18

We have A [2 1], B [2 2] and C [4 1].

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} [T] = \begin{bmatrix} 2 & 1 \\ 2 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 6 & 3 \\ 6 & 6 \\ 12 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} A^* \\ B^* \\ C^* \end{bmatrix}$$

On the other hand if $a = \frac{7}{2}$, $b = 2$, $c = d = 0$, then the effect of matrix $[T] = \begin{bmatrix} \frac{7}{2} & 0 \\ 0 & 2 \end{bmatrix}$ on the same triangle ABC is as shown below in Fig. 1.19, where there is a distortion due to non-uniform scale factors.

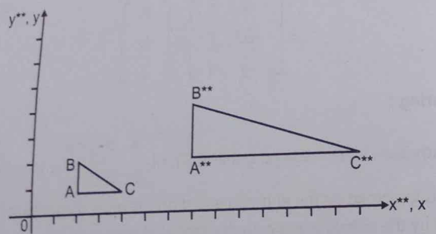


Fig. 1.19

As before here we have

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} [T] = \begin{bmatrix} 2 & 1 \\ 2 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} \frac{7}{2} & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 2 \\ 7 & 4 \\ 14 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} A^{**} \\ B^{**} \\ C^{**} \end{bmatrix}$$

The above discussion about uniform scaling and distortion suggests that there is an apparent translation of the transformed triangles. We note that the position vectors are scaled and not the points; with respect to the origin.

If the centroid of the figure is at the origin, then pure scaling without apparent translation takes place.

Consider a triangle PQR where P [1 2], Q [1 −1] and R [−2 −1]. The centroid of this angle is at the origin.

Now,
$$\begin{bmatrix} P \\ Q \\ R \end{bmatrix} [T] = \begin{bmatrix} 1 & 2 \\ 1 & -1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 3 & -3 \\ -6 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} P^{*} \\ Q^{*} \\ R^{*} \end{bmatrix}$$

This is represented in the Fig. 1.20.

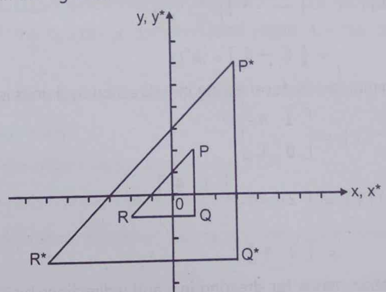


Fig. 1.20 : Pure scaling without apparent translation

Illustrative Example

Example 1.5 : Apply each of the following transformations on the point A [2 −4].

(a) Scaling in x co-ordinate by factor 3.

(b) Scaling in y co-ordinate by factor 2.

(c) Scaling in x and y co-ordinates by factors 3 and 2 respectively.

(d) Shearing in y-direction by 4 units.

(e) Shearing in x and y directions by 2 and 5 units respectively.

Solution : If [T] denotes the transformation matrix, then we have

$$[A][T] = [A^{*}]$$

(a) A transformation matrix for scaling in x co-ordinate by a factor 3 is

$$[T] = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \quad [A][T] = [2 \ \ -4] \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = [6 \ \ -4] = [A^{*}]$$

(b) A transformation matrix for scaling in y co-ordinate by a factor −2 is

$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\therefore \quad [A][T] = [2 \ \ -4] \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} = [2 \ \ 8] = [A^{*}]$$

(c) A transformation matrix for scaling in x and y co-ordinates by factors 3 and 2 respectively, is

$$[T] = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\therefore \quad [A][T] = [2 \ \ -4]\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= [6 \ \ -8] = [A^*]$$

(d) The transformation matrix for shearing in y-direction by 4 units is

$$[T] = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \quad [A][T] = [2 \ \ -4]\begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$$

$$= [2 \ \ 4]$$

(e) The transformation matrix for shearing in x and y directions by 2 and 5 units respectively is

$$[T] = \begin{bmatrix} 1 & 5 \\ 2 & 1 \end{bmatrix}$$

$$\therefore \quad [A][T] = [2 \ \ -4]\begin{bmatrix} 1 & 5 \\ 2 & 1 \end{bmatrix}$$

$$= [-6 \ \ 6]$$

1.8 Combined Transformations

The shape and the position of a curve in two dimensions or a surface in three dimensions is controlled by performing matrix operations on the position vectors of the points. In general, to obtain a desired orientation of original figure we need to perform two or more transformations.

Suppose, we perform successively three transformations on a point P [x y].

Let $[T_1]$, $[T_2]$ and $[T_3]$ be the respective transformations.

Then

$$[P][T_1] = [x \ \ y][T_1] = [x' \ \ y']$$

$$[x' \ \ y'][T_2] = [x'' \ \ y'']$$

$$[x'' \ \ y''][T_3] = [x^* \ \ y^*]$$

By using the laws of algebra of matrices, the same effect is obtained when the transformation matrix is [T]. Where,

$$[T] = [T_1][T_2][T_3]$$

$$[P][T] = [x \ \ y][T] = [x^* \ \ y^*]$$

The matrix [T] is the combined transformation matrix. It is called as concatenated transformation matrix. While finding the concatenated transformation matrix, the order in which the operations are performed, that order is to be maintained. The reason is that the matrix multiplication is not commutative.

Illustrative Example

Example 1.6 : Find the combined transformation matrix for the following sequence of transformations.

(i) Scaling in x and y co-ordinates by factors –1 and 2 respectively.

(ii) Reflection through x-axis.

(iii) Rotation about the origin through an angle $\theta = 270^\circ$.

Apply this combined transformation on the point P [2 –3].

Solution : The transformation matrix for scaling in x and y co-ordinates by factors –1 and 2 respectively is

$$[T_1] = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

The transformation matrix for reflection through x-axis is

$$[T_2] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The transformation matrix for rotation about the origin, through an angle $\theta = 270^\circ$ is

$$[T_3] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos 270^\circ & \sin 270^\circ \\ -\sin 270^\circ & \cos 270^\circ \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

If [T] denotes the combined transformation matrix, then

$$[T] = [T_1][T_2][T_3]$$

$$\therefore \quad [T] = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$\therefore \quad [T] = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$\therefore \quad [T] = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$$

If this transformation is applied on P [2 –3], then

$$[P][T] = [2 \ \ -3]\begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} = [6 \ \ 2]$$

$\therefore$ P^* is [6 2].

1.9 Transformation of the Unit Square

We discuss the transformation of unit square under the general transformation matrix $[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Also, we find the area of the transformed region in terms of the area of the original region.

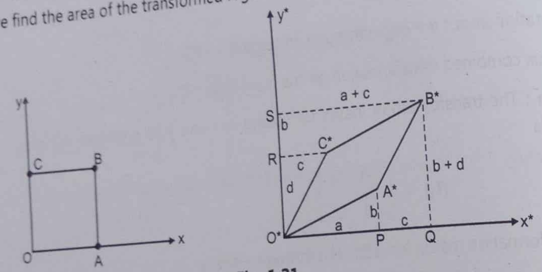


Fig. 1.21

As shown in the first figure, above, OABC is a unit square with O [0 0], A [1 0], B [1 1] and C [0 1].

Then under the transformation matrix $[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, the unit square OABC is transformed to parallelogram $O^*A^*B^*C^*$.

$$\begin{bmatrix} O \\ A \\ B \\ C \end{bmatrix} [T] = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ a & b \\ a+c & b+d \\ c & d \end{bmatrix} = \begin{bmatrix} O^* \\ A^* \\ B^* \\ C^* \end{bmatrix}$$

We note that the terms a and d on the main diagonal of [T] act as scale factors. Also the terms b and c on the second diagonal cause a shearing of the initial square in y and x directions respectively. Thus, the general 2 × 2 matrix produces a combination of shearing and scaling.

Now, we find the area of the parallelogram $O^*A^*B^*C^*$.

$$\begin{aligned} \text{Area}(\square\, O^*A^*B^*C^*) &= \text{Area}(\square\, O^*QB^*S) - \text{Area}(\Delta\, O^*PA^*) \\ &\quad - \text{Area}(\square\, PQB^*A^*) - \text{Area}(\Delta\, O^*C^*R) \\ &\quad - \text{Area}(\square\, RC^*B^*S) \end{aligned}$$

$$= (a + c)(b + d) - \frac{1}{2}ab - \frac{1}{2}c(b + b + d) - \frac{1}{2}cd - \frac{1}{2}b(c + a + c)$$

$$= ab + ad + cb + cd - \frac{1}{2}ab - bc - \frac{1}{2}cd - \frac{1}{2}cd - bc - \frac{1}{2}ba$$

$$= ad - bc$$

$$= \det [T]$$

From this we get,

Area of transformed region = (Area of initial region) (det [T])

Illustrative Examples

Example 1.7 : Apply each of the following transformations on the point P [3 – 2] :

(a) Scaling in x co-ordinate by 2 units and y co-ordinate by $\frac{1}{4}$ units.

(b) Shearing in x-direction – 2 units and y-direction by 5 units.

(c) Reflection through the line y = x.

(d) Rotation by 55° about origin.

(e) Reflection through the line y = – x.

Solution : (a) The transformation matrix for scaling in x co-ordinate by 2 units and y co-ordinate by $\frac{1}{4}$ units is $[T] = \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$.

$$[P][T] = [3 \; -2] \begin{bmatrix} 2 & 0 \\ 0 & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 6 & -\frac{1}{2} \end{bmatrix}$$

$$\therefore \qquad P^* \begin{bmatrix} 6 & -\frac{1}{2} \end{bmatrix}$$

(b) The transformation matrix for shearing in x-direction by – 2 units and y direction by units is

$$[T] = \begin{bmatrix} 1 & 5 \\ -2 & 1 \end{bmatrix}$$

$$[P][T] = [3 \; -2] \begin{bmatrix} 1 & 5 \\ -2 & 1 \end{bmatrix} = [7 \;\; 13]$$

$$\therefore \qquad P^* [7 \;\; 13]$$

(c) The transformation matrix for reflection through the line y = x is

$$[T] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$[P][T] = [3\ -2]\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = [-2\ 3]$$

$\therefore$ $P^*[-2\ 3]$

(d) The transformation matrix for rotation by 55° about the origin is

$$[T] = \begin{bmatrix} \cos 55° & \sin 55° \\ -\sin 55° & \cos 55° \end{bmatrix}$$

$$[P][T] = [3-2]\begin{bmatrix} 0.5735 & 0.8191 \\ -0.8191 & 0.5735 \end{bmatrix}$$

$$= [3.3587\ \ 1.3103]$$

$\therefore$ $P^*[3.3587\ \ 1.3103]$

(e) The transformation matrix for reflection through the line $y = -x$ is

$$[T] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$[P][T] = [3\ -2]\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = [2-3]$$

$\therefore$ $P^*[2-3]$

Example 1.8 : The position vectors of the line segment AB are A [2 3] and B [– 4 7]. It is transformed to the line A' B' by using the transformation matrix

$[T] = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Find the slope of A'B'.

Solution : The slope of the line AB where A [2 3] and B [– 4 7], is

$$m = \frac{y_1 - y_2}{x_1 - x_2} = \frac{3-7}{2+4} = -\frac{2}{3}$$

The transformation is

$$[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$a = 1,\ b = 2,\ c = 3,\ d = 4.$

The slope of the transformation line A'B' is $m^* = \frac{b + dm}{a + cm}$.

$$\therefore \quad m^* = \frac{2 + (4)\left(-\frac{2}{3}\right)}{1 + (3)\left(-\frac{2}{3}\right)} = \frac{2}{3}$$

Example 1.9 : A line $x + y = 3$ is transformed to another line by using 2×2 transformation matrix $[T] = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$. Obtain the equation of resulting line.

Solution : The line $x + y = 3$ has slope $m = -1$.

Let $y^* = m^* x^* + h^*$ be the transformed line under the transformation matrix

$$[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$$

Then, $$m^* = \frac{b + dm}{a + cm} = \frac{2 + (5)(-1)}{1 + (2)(-1)} = 3$$

$$h^* = h\left(\frac{ad - bc}{a + cm}\right) = 3\left[\frac{(1)(5) - (2)(2)}{1 + (2)(-1)}\right] = -3$$

The equation of the resulting line is

$$y^* = m^* x^* + h^*$$

$$\therefore \quad y^* = 3x^* - 3$$

Example 1.10 : A line L_1 is transformed to line $L_2 : 2.37\, x^* - 1.543\, y^* = 5.642$ under the transformation matrix $[T] = \begin{bmatrix} 3 & 2.5 \\ 2.1 & 4.3 \end{bmatrix}$. Find the equation of line L_1.

Solution : Let $y = mx + h$ be the equation of line L_1. The equation of line L_2 is

$$2.37\, x^* - 1.543\, y^* + 5.642 = 0$$

This can be written as

$$y^* = \left(\frac{2.37}{1.543}\right) x^* + \left(\frac{5.642}{1.543}\right)$$

i.e. $$y^* = 1.5359\, x^* + 3.6565$$

$$\therefore \quad m^* = 1.5359 \text{ and } h^* = 3.6565$$

The transformation matrix is

$$[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 3 & 2.5 \\ 2.1 & 4.3 \end{bmatrix}$$

$\therefore$ $a = 3,\ b = 2.5,\ c = 2.1,\ d = 4.3$

Now, $$m^* = \frac{b + dm}{a + cm}$$

$$\therefore \quad 1.5359 = \frac{2.5 + (4.3)(m)}{3 + (2.1)(m)}$$

This gives $m = 1.9611$.

$$h = h\left(\frac{ad - bc}{a + cm}\right)$$

Also,

$$\therefore \quad 3.6565 = h\left[\frac{(3)(4.3) - (2.5)(2.1)}{3 + 2.1\,m}\right]$$

$$\therefore \quad 3.6565 = h \times \left[\frac{12.9 - 5.25}{3 + (2.1)(1.9611)}\right]$$

$$\therefore \quad 3.6565 = h \times 1.0746$$

$$\therefore \quad h = 3.4026$$

Equation of original line L_1 is

$$y = 1.9611\,x + 3.4026$$

Example 1.11 : The line segment joining the points A [3 4], B [5 6] is transformed to the line segment A' B' by the transformation matrix

$$[T] = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}.$$ Find the mid-point of A' B'.

Solution : The mid-point of line segment AB where, A [3 4] and B [5 6] is

$$M\left[\frac{x_1 + x_2}{2} \quad \frac{y_1 + y_2}{2}\right] = [4 \;\; 5].$$

We know that under 2 × 2 transformation matrix, the mid-point of segment AB is transformed to the mid-point of segment A'B'.

$$[M^*] = [M]\,[T] = [4 \;\; 5]\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} = [13 \;\; 24]$$

$$\therefore \quad M^* = [13 \;\; 24]$$

Example 1.12 : Determine whether the transformation matrix $\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$ represents reflection. Justify.

(May 2006)

Solution : $[T] = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$, det. $[T]\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} = -1$

Also, $$[T]\,[T]^t = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

From the above two results, [T] represents a reflection.

Example 1.13 : Find the concatenated transformation matrix and apply it on a triangle with vertices O [0 0], A [1 2], B [3, – 1]. The sequence of transformations is given below :

(a) Rotation about origin through angle 50°.

(b) Shearing in y-direction by – 2.1 units.

(c) Uniform scaling by factor 2.

Solution : The transformation matrix for rotation through 50° about origin is,

$$[T_1] = \begin{bmatrix} \cos 50° & \sin 50° \\ -\sin 50° & \cos 50° \end{bmatrix}$$

$$= \begin{bmatrix} 0.6427 & 0.7660 \\ -0.7660 & 0.6427 \end{bmatrix}$$

The transformation matrix for shearing in y direction by – 2.1 units is

$$[T_2] = \begin{bmatrix} 1 & -2.1 \\ 0 & 1 \end{bmatrix}$$

The transformation matrix for uniform scaling by factor 2 is

$$[T_3] = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

The concatenated transformation matrix is

$$[T] = [T_1]\,[T_2]\,[T_3]$$

$$= \begin{bmatrix} 0.6427 & 0.7660 \\ -0.7660 & 0.6427 \end{bmatrix}\begin{bmatrix} 1 & -2.1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0.6427 & 0.7660 \\ -0.7660 & 0.6427 \end{bmatrix}\begin{bmatrix} 2 & -4.2 \\ 0 & 2 \end{bmatrix}$$

$$\therefore \quad [T] = \begin{bmatrix} 1.2854 & -1.1673 \\ -1.5320 & 4.5026 \end{bmatrix}$$

Now, the vertices of a triangle are O [0 0], A [1, 2] and B [3 – 1].

$$\begin{bmatrix} O^* \\ A^* \\ B^* \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}[T]$$

$$= \begin{bmatrix} 0 & 0 \\ 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1.2854 & -1.1673 \\ -1.5320 & 4.5026 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ -1.7786 & 7.8379 \\ 5.3882 & -8.0045 \end{bmatrix}$$

The vertices of a transformed triangle are

O^* [0 0], A^* [– 1.7786 7.8379] and B^* [5.3882 – 8.0045].

Example 1.14 : *Find the concatenated transformation matrix for the following sequence of transformations. First scaling of x co-ordinate by factor 3; followed by scaling of y co-ordinate by factor 5.*

If we apply this transformation on the circle with centre at origin and radius 4, then find the equation of the transformed figure.

Solution : The scaling of x co-ordinate by the transformation matrix

$$[T_1] = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

The transformation matrix for scaling of y co-ordinate by factor 5 is

$$[T_2] = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

The concatenated transformation matrix is

$$[T] = [T_1]\,[T_2] = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix}$$

Then, $\quad [x\ y]\,[T] = [x\ y] \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} = [3x\ \ 5y]$

The co-ordinates of the transformed point are : $x^* = 3x$, $y^* = 5y$.

Therefore, $x = \dfrac{x^*}{3}$, $y = \dfrac{y^*}{5}$.

Now, equation of the circle with centre at origin and radius 4 is $x^2 + y^2 = 16$.

$$\therefore \quad \left(\frac{x^*}{3}\right)^2 + \left(\frac{y^*}{5}\right)^2 = 16$$

$$\therefore \quad \frac{x^{*2}}{144} + \frac{y^{*2}}{400} = 1, \text{ which is ellipse.}$$

Example 1.15 : The triangle ABC with vertices A [3 –1], B [4 2] and C [2 1] is first rotated through 90° about the origin and then reflected through y-axis. Find the vertices of transformed triangle.

Solution : The transformation matrix for rotation through 90° about origin is

$$[T_1] = \begin{bmatrix} \cos 90° & \sin 90° \\ -\sin 90° & \cos 90° \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The transformation matrix for reflection through y-axis is

$$[T_2] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

The combined transformation matrix is

$$[T] = [T_1]\,[T_2] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Now, vertices of the triangle ABC are A [3 –1], B [4 2] and C [2 1].

$$\begin{bmatrix} A^* \\ B^* \\ C^* \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \end{bmatrix} [T] = \begin{bmatrix} 3 & -1 \\ 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} A^* \\ B^* \\ C^* \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 2 & 4 \\ 1 & 2 \end{bmatrix}$$

The vertices of the transformed triangle are A^* [–1 3], B^* [2 4] and C^* [1 2].

Example 1.16 : Find the concatenated transformation matrix and apply it on the triangle with vertices A [1 2], B [3 –1], C [2 1]. The sequence of transformation is given below :

(i) Rotation about origin through an angle θ = 50°.

(ii) Shearing in y-direction by – 2.1 units.

(iii) Uniform scaling by factor 2.

Solution : The transformation matrix for rotation about origin through an angle θ = 50° is

$$[T_1] = \begin{bmatrix} \cos 50° & \sin 50° \\ -\sin 50° & \cos 50° \end{bmatrix}$$

$$\therefore \quad [T_1] = \begin{bmatrix} 0.6427 & 0.7660 \\ -0.7660 & 0.6427 \end{bmatrix}$$

The transformation matrix for shearing in y-direction by – 2.1 units is;

$$[T_2] = \begin{bmatrix} 1 & -2.1 \\ 0 & 1 \end{bmatrix}$$

The transformation matrix for uniform scaling by factor 2 is

$$[T_3] = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

The concatenated transformation matrix is

$$[T] = [T_1]\,[T_2]\,[T_3]$$

$$[T] = \begin{bmatrix} 0.6427 & 0.7760 \\ -0.7660 & 0.6427 \end{bmatrix}\begin{bmatrix} 1 & -2.1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$\therefore \quad [T] = \begin{bmatrix} 0.6427 & 0.7660 \\ -0.7660 & 0.6427 \end{bmatrix}\begin{bmatrix} 2 & -4.2 \\ 0 & 2 \end{bmatrix}$$

$$\therefore \quad [T] = \begin{bmatrix} 1.2854 & -1.1673 \\ -1.5320 & 4.5026 \end{bmatrix}$$

The vertices of triangle are A [1 2], B [3 – 1], C [2 1].

If A^*, B^*, C^* are the vertices of a transformed triangle, under the transformation matrix [T], then

$$\begin{bmatrix} A^* \\ B^* \\ C^* \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}[T] = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1.2854 & -1.1673 \\ -1.5320 & 4.5026 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} A^* \\ B^* \\ C^* \end{bmatrix} = \begin{bmatrix} -1.7786 & 7.8379 \\ 5.3882 & -8.0045 \\ 1.0388 & 2.1680 \end{bmatrix}$$

$\therefore$ A^* [– 1.7786 7.8379], B^* [5.3882 – 8.0045] and C^* [1.0388 2.1680].

Example 1.17 : Show that under the transformation $[T] = \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix}$ the circle $(x - 2)^2 + (y + 1)^2 = 16$ is transformed to ellipse of area 48π square units. Verify the result by det. [T].

Solution : The transformation matrix is

$$[x^* \; y^*] = [x \; y]\,[T] = [x \; y]\begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} = [-3x \; -y]$$

$$\therefore \quad x^* = -3x,\; y^* = -y \quad x = -\frac{x^*}{3},\; y = -y^*.$$

The equation of the circle

$$\left(-\frac{x^*}{3} - 2\right)^2 + (-y^* + 1)^2 = 16$$

$$\therefore \quad \left(\frac{-x^* - 6}{3}\right)^2 + (-y^* + 1)^2 = 16$$

$$\therefore \quad \frac{(x^* + 6)^2}{144} + \frac{(y^* - 1)^2}{16} = 1$$

This represents ellipse having centre at [– 6 1], semimajor axis a = 12 and semiminor axis b = 4.

Area of ellipse = $\pi ab = 48\pi$ square units.

Radius of original circle is 4. Area of circle is 16π

$$\det [T] = \begin{vmatrix} -3 & 0 \\ 0 & -1 \end{vmatrix} = 3$$

$\therefore$ det [T] × Area of circle = $3 \times 16\pi = 48\pi$.

$\therefore$ Area of ellipse = det. [T] × Area of circle; verified.

Example 1.18 : If we apply transformation matrix $[T] = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$ on a square, then we get a parallelogram of area 64 cm². Find the length of each side of the original square.

Solution : The transformation matrix is $[T] = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$.

Area of transformed parallelogram

= det. [T] × Area of original square

$\therefore$ 64 = (6 – 2) × Area of original square

$\therefore$ 16 = Area of original square

The length of each side of the original square is $\sqrt{16} = 4$ cm.

Example 1.19 : A circle with area 25 cm² is transformed by using the transformation matrix $[T] = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}$. Find the area of the transformed figure. **(Oct. 2007)**

Solution : The transformation matrix is

$$[T] = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}$$

$$\det. [T] = \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} = 4$$

Area of transformed figure = det. [T] × Area of original circle

$= 4 \times 25$

$= 100$ cm²

Exercise 1.1

1. What is the effect of the transformation matrix :

   (a) $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ on a two dimensional object ?

   (b) $\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}$ on a two dimensional object ?

2. Apply each of the following transformation on the point P[3 7] :

   (a) Scaling in x co-ordinate by factor 2.

   (b) Scaling in y co-ordinate by factor $\frac{1}{3}$.

   (c) Uniform scaling by 3 units.

3. The points O[0 0], A[1 0], B[1 1] and C[0 1] are transformed by 2×2 transformation matrix [T] to the points O'[0 0], A'[2 3], B'[8 4] and C'[6 1] respectively. Find the matrix of transformation.

4. If a line segment joining the points A[1 1] and B[1 3] is transformed to the line segment $A^* B^*$ where A^*[8 4] and B^*[20 6], then find 2×2 transformation matrix.

5. The line $y = 2x + 1$ is transformed by 2×2 transformation matrix $T = \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix}$. Find the equation of the transformed line.

6. The two lines $L_1 : x + 2y = 2$ and $L_2 : x - y = 4$ are transformed by the transformation matrix $[T] = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$ to the lines L_1' and L_2' respectively. Find the point of intersection of L_1' and L_2'.

7. The lines $L_1 : 3x + 2y = 12$ and $L_2 : L\ 2x - 3y = 5$ are transformed to the line L_1' and L_2'. Find the point of intersection of L_1' and L_2'.

8. The lines AB and EF are transformed to the lines $A^* B^*$ and $E^* F^*$ respectively by the transformation matrix $T = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix}$. If A[−1 −1], $B\begin{bmatrix} 3 & \frac{5}{3} \end{bmatrix}$, $E\begin{bmatrix} -\frac{1}{2} & \frac{3}{2} \end{bmatrix}$ and F[3 −2] then find the point of intersection of $A^* B^*$ and $E^* F^*$.

9. What will be the effect, in combined transformations when the order of the transformations is changed ?

10. Show the combined two dimensional transformation first reflection about the x-axis and then about the line $y = -x$ is equivalent to the rotation about the origin by 270° :

11. (a) Find the angle of rotation required to rotate the line $y = 2x$ about the origin so that it is coincident with the x-axis.

    (b) Find the angle through which the line $y = -x$ is rotated so that it coincides with the x-axis.

12. Find the concatenated transformation matrix for the following sequence : First rotation about origin through 30° followed by shearing in x direction by −2 units.

13. Find the concatenated transformation matrix for the following transformations in order :

    (i) Rotation about origin through 45°.

    (ii) Shearing in x direction by −2 units.

    Apply it to the vector [1 2].

14. The line segment between the points A[−1 −3] and B[3 4] is transformed to the line segment A'B' by the transformation matrix $[T] = \begin{bmatrix} -1 & 4 \\ 4 & 2 \end{bmatrix}$. Find the mid-point of A'B'.

15. The line segment joining A[4 9] and B[−2 1] is scaled uniformly by factor 2. Find the mid-point of the transformed line segment.

16. Triangle $A^*B^*C^*$ is produced from triangle ABC by rotation about origin; in counter clockwise sense through angle $\frac{3\pi}{2}$. If $\begin{bmatrix} A^* \\ B^* \\ C^* \end{bmatrix} = \begin{bmatrix} 8 & 1 \\ 7 & 3 \\ 6 & 2 \end{bmatrix}$. Find the position vectors of the triangle ABC.

17. The rectangle $A^*B^*C^*D^*$ is produced by rotation about the origin in anticlockwise direction through an angle 135°. If $\begin{bmatrix} A^* \\ B^* \\ C^* \\ D^* \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0.71 & -0.71 \\ 0 & -1.42 \\ -0.71 & -0.71 \end{bmatrix}$, then find the position vectors of A, B, C, D.

18. If a square with sides 2 cm is reflected through y-axis, then find the area of the transformed figure ?

19. If the circle of circumference 14π is uniformly scaled by 2 units, find the area of the transformed figure.

20. If the circle of circumference 14π is uniformly scaled by 3 units, what is the area of the transformed figure ?

21. The triangle ABC with vertices [3 −1], B[4 2], C[2 1] is first rotated through 90° about the origin and then reflected through y-axis. Find the vertices of transformed triangle.

22. Consider the unit sequence at origin {[0 0], [1 0], [1 1], [0, 1]}. Apply the general 2×2 transformation $[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and show that the area of transformed parallelogram is equal to det. [T].

Answers 1.1

1. (a) Shearing in x co-ordinate by factor − 2.

   (b) Shearing in x co-ordinate by factor − 1.

2. (a) [6 7], (b) $\begin{bmatrix} 3 & \frac{7}{3} \end{bmatrix}$, (c) [9 21]

3. $[T] = \begin{bmatrix} 2 & 3 \\ 6 & 1 \end{bmatrix}$

4. $[T] = \begin{bmatrix} 2 & 3 \\ 6 & 1 \end{bmatrix}$

5. $y^* = 4x^* + 7$

6. $\begin{bmatrix} \frac{22}{3} & 2 \end{bmatrix}$

7. [1 5]

8. [1 1]

11. (a) $\tan^{-1}(-2)$, (b) 225°

12. $T\begin{bmatrix} -0.134 & 0.5 \\ -2.232 & 0.866 \end{bmatrix}$

13. $[T] = \begin{bmatrix} -0.7071 & 0.7071 \\ -2.1213 & 0.7071 \end{bmatrix}$, $P^*[-4.9497 \quad 2.1213]$

14. [1 5]

15. [2 10]

16. A[− 1 8], B[− 3 7], C[− 2 6]

17. A[0 0], B[− 1.0041 0], C[− 1.0042 1.0041], D[0 1.0041]

18. 4 cm^2

19. 196 π

20. 441 π

21. $A^*[-1 \quad 3]$, $B^*[2 \quad 4]$, $C^*[1 \quad 2]$

1.10 Solid Body Transformation

The transformation which do not change the shape and size of the object is known as solid ody transformation. Obviously, the rotation is a solid body transformation whereas the reflection not a solid body transformation.

We know that, under a general 2 × 2 transformation matrix the parallel lines are transformed parallel lines. This result however is not true in case of perpendicular lines.

Let us consider, two lines AB and CD where A[2 1], B[−3 2], C[4 −2] and D[5 3]. Their slopes $-\frac{1}{5}$ *and 5 respectively, so that the lines are perpendicular. If the transformation matrix is*

$$[T_1] = \begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix} \text{ then}$$

$$\begin{bmatrix} A \\ B \end{bmatrix}[T_1] = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

$$\begin{bmatrix} A \\ B \end{bmatrix}[T_1] = \begin{bmatrix} \frac{2}{5} & \frac{11}{5} \\ -\frac{17}{5} & -\frac{6}{5} \end{bmatrix} = \begin{bmatrix} A^* \\ B^* \end{bmatrix}$$

Also,
$$\begin{bmatrix} C \\ D \end{bmatrix}[T_1] = \begin{bmatrix} 4 & -2 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} \frac{3}{5} & \frac{4}{5} \\ -\frac{4}{5} & \frac{3}{5} \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} C \\ D \end{bmatrix}[T_1] = \begin{bmatrix} 4 & 2 \\ \frac{3}{5} & \frac{29}{5} \end{bmatrix} = \begin{bmatrix} C^* \\ D^* \end{bmatrix}$$

We observe that, the slope of the line A^*B^* is $\dfrac{\frac{11}{5}+\frac{6}{5}}{\frac{2}{5}+\frac{17}{5}} = \frac{17}{19}$.

and the slope of line C^* D^* is $\dfrac{2-\frac{29}{5}}{4-\frac{3}{5}} = -\frac{19}{17}$. Therefore, the lines A^*B^* and C^*D^* are perpendicular lines. In this case the perpendicular lines are transformed to perpendicular lines.

Now, consider the transformation matrix $[T_2] = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}$. We have

$$\begin{bmatrix} A \\ B \end{bmatrix}[T_2] = \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 5 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} A^{**} \\ B^{**} \end{bmatrix}$$

$$\begin{bmatrix} C \\ D \end{bmatrix}[T_2] = \begin{bmatrix} 4 & -2 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 19 & 13 \end{bmatrix} = \begin{bmatrix} C^{**} \\ D^{**} \end{bmatrix}$$

The slope of A^{**} B^{**} is $\frac{5+4}{7-0} = \frac{9}{7}$ and the slope of C^{**} D^{**} is $\frac{6-13}{2-19} = \frac{7}{17}$.

The lines A^{**} B^{**} and C^{**} D^{**} are not perpendicular.

Thus, under the transformation matrix $[T_2]$ the perpendicular lines are transformed to the lines which are not perpendicular.

We now obtain the condition under which the perpendicular lines are transformed to perpendicular lines. The required condition in its more general form is the condition under which the angle between two given lines is preserved under the transformation matrix [T].

Let $\bar{v}_1 = v_{1x}\,\bar{i} + v_{1y}\,\bar{j}$ and $\bar{v}_2 = v_{2x}\,\bar{i} + v_{2y}\,\bar{j}$ be the vectors represented by the line segments AB and CD.

Then, $\bar{v}_1 \cdot \bar{v}_2 = |\bar{v}_1|\,|\bar{v}_2| \cos\theta = v_{1x} v_{2x} + v_{1y} v_{2y}$ ... (1)

and $\bar{v}_1 \times \bar{v}_2 = |\bar{v}_1|\,|\bar{v}_2| \sin\theta\,\bar{k} = (v_{1x} v_{2y} - v_{1y} v_{2x})\,\bar{k}$ ... (2)

where, θ is the acute angle between the lines of action of $\bar{v}_1$ and $\bar{v}_2$.

If $[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the transformation matrix, then the transformed form of $\bar{v}_1$ and $\bar{v}_2$ is

$$\begin{bmatrix} \vec{v}_1^{\,*} \\ \vec{v}_2^{\,*} \end{bmatrix} = \begin{bmatrix} \bar{v}_1 \\ \bar{v}_2 \end{bmatrix} [T] = \begin{bmatrix} v_{1x} & v_{1y} \\ v_{2x} & v_{2y} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} a\,v_{1x} + c\,v_{1y} & b\,v_{1x} + d\,v_{1y} \\ a\,v_{2x} + c\,v_{2y} & b\,v_{2x} + d\,v_{2y} \end{bmatrix} \quad \text{... (3)}$$

From (3) we get

$$\begin{aligned} \vec{v}_1^{\,*} \cdot \vec{v}_2^{\,*} &= (a\,v_{1x} + c\,v_{1y})(a\,v_{2x} + c\,v_{2y}) + (b\,v_{1x} + d\,v_{1y})(b\,v_{2x} + d\,v_{2y}) \\ &= (a^2 + b^2)(v_{1x} v_{2x}) + (c^2 + d^2)(v_{1y} v_{2y}) \\ &\quad + (ac + bd)(v_{1x} v_{2y} + v_{1y} + v_{2x}) \end{aligned} \quad \text{... (4)}$$

Also $\bar{v}_1 \times \bar{v}_2 = [(a\,v_{1x} + c\,v_{1y})(b\,v_{2x} + d\,v_{2y}) - (b\,v_{1x} + d\,v_{1y})(a\,v_{2x} + c\,v_{2y})]\,\bar{k}$

$\therefore$ $\bar{v}_1 \times \bar{v}_2 = [(ad - bc)(v_{1x} v_{2y} - v_{1y} v_{2x})]\,\bar{k}$ ... (5)

Now, we assume that the angle between the vectors is preserved and also their magnitudes are preserved. This requires $\bar{v}_1 \cdot \bar{v}_2 = \vec{v}_1^{\,*} \cdot \vec{v}_2^{\,*}$.

and $\bar{v}_1 \times \bar{v}_2 = \vec{v}_1^{\,*} \times \vec{v}_2^{\,*}$

Then the comparison of (1) with (4) and the comparison of (2) with (5) gives

$$a^2 + b^2 = 1 \quad \text{... (6)}$$

$$c^2 + d^2 = 1 \quad \text{... (7)}$$

$$ac + bd = 0 \quad \text{... (8)}$$

$$ad - bc = 1 \quad \text{... (9)}$$

Now, $[T]\,[T]^t = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

$$= \begin{bmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \qquad \because \text{ by (6), (7) and (8)}$$

This means the matrix [T] is orthogonal.

Also $\det.[T] = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = 1 \qquad \because$ by (9)

Thus, if the transformation matrix [T] is orthogonal and det. [T] = 1, then the perpendicular lines are transformed to perpendicular lines and also the magnitudes are preserved.

Note : The transformation matrix of a reflective transformation is orthogonal with its determinant equal to – 1. In this case the magnitudes of vectors is preserved. However, the angle θ between original lines becomes $2\pi - \theta$ between transformed lines and hence the perpendicular lines are still transformed to perpendicular lines.

Note : The uniform scaling preserves the angle between the lines but in this case the magnitudes of vectors are not preserved.

1.11 Translations and Homogeneous Co-ordinates

We know that under the general 2×2 transformation matrix [T] every point P except the origin gets transformed to P^* i.e. $[P]\,[T] = [P^*]$ origin remains unchanged under [T]. Many times we require to change the origin also in the two dimensional plane. The modification of every point including origin, is achieved by translating the origin or any other point in the two dimensional plane. For this we write

$$x^* = ax + cy + m$$

$$y^* = bx + dy + n$$

Accordingly, there is a modification in the transformation matrix. For this we introduce homogeneous co-ordinates. If h is any real number, then the homogeneous co-ordinates of a non-homogeneous vector [x y] are [hx hy h]. In particular when h = 1, the homogeneous co-ordinates of [x y] are [x y 1]. Thus, [2 4 1], [6 12 3], [8 16 4] are different representations of the same point [2 4], in terms of homogeneous co-ordinates. Thus, the homogenous co-ordinate representation of a point is not unique.

Now we consider 3×3 transformation matrix

$$[T] = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ m & n & 1 \end{bmatrix}$$

The constants m and n appearing in this matrix are called as translation factors in x and y directions respectively.

The equations $x^* = ax + cy + m$

$y^* = bx + dy + n$

are written using homogeneous co-ordinates as

$$[x^*\ y^*\ 1] = [x\ y\ 1]\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ m & n & 1 \end{bmatrix}$$

We note that under this transformation, origin is also changed to the point [m n].

Illustrative Example

Example 1.20 : A point P [4 2 1] in a plane is transformed to the point P^* $[x^*\ y^*\ 1]$ under the homogeneous transformation matrix $[T] = \begin{bmatrix} 0 & -2 & 2 \\ -2 & 2 & -2 \\ 1 & 0 & 1 \end{bmatrix}$. Prove that $x^{*2} + y^{*2} = 1$.

(April 1994)

Solution : P [4 2 1] and $[T] = \begin{bmatrix} 0 & -2 & 2 \\ -2 & 2 & -2 \\ 1 & 0 & 1 \end{bmatrix}$ implies

$$[P]\,[T] = [4\ 2\ 1]\begin{bmatrix} 0 & -2 & 2 \\ -2 & 2 & -2 \\ 1 & 0 & 1 \end{bmatrix} = [-3\ -4\ 5]$$

Now, $[P^*] = [P]\,[T]$

$\therefore$ $P^* = [-3\ -4\ 5] = \left[-\frac{3}{5}\ -\frac{4}{5}\ 1\right]$

$\therefore$ $[x^*\ y^*\ 1] = \left[-\frac{3}{5}\ -\frac{4}{5}\ 1\right]$

$\therefore$ $x^* = -\frac{3}{5},\ y^* = -\frac{4}{5}$

Then $x^{*2} + y^{*2} = \left(-\frac{3}{5}\right)^2 + \left(-\frac{4}{5}\right)^2 = \frac{9}{25} + \frac{16}{25} = 1$

1.12 Rotation about an Arbitrary Point

We know that the rotation of a position vector about origin, through an angle θ is accomplished by the rotation matrix $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

The rotation about an arbitrary point [m n], through an angle θ requires three transformations in succession.

(i) Translation of [m n 1] to the origin :

$$\text{Translation matrix } [T_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -m & -n & 1 \end{bmatrix}.$$

(ii) Rotation about origin :

$$\text{Rotation matrix } [T_2] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(iii) Translation back to the point [m n 1] :

$$\text{Translation matrix } [T_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ m & n & 1 \end{bmatrix}.$$

The concatenated transformation matrix for rotation about an arbitrary point [m n 1] is

$$[T] = [T_1]\,[T_2]\,[T_3]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -m & -n & 1 \end{bmatrix}\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ m & n & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ m(1-\cos\theta) + n\sin\theta & n(1-\cos\theta) - m\sin\theta & 1 \end{bmatrix}$$

Illustrative Examples

Example 1.21 : Rotate the triangle ABC about its centroid through an angle 45°, where A [2 – 4], B [3 0] and C [–2 1]. **(April 2008)**

Solution : The vertices of a triangle ABC are A [2 – 4], B [3 0] and C [– 2 1].

At the centroid G [x y] we have

$$x = \frac{x_1 + x_2 + x_3}{3} = \frac{2 + 3 - 2}{3} = 1$$

$$y = \frac{y_1 + y_2 + y_3}{3} = \frac{-4 + 0 + 1}{3} = -1$$

$\therefore$ G [1 – 1]

The angle of rotation about G [1 – 1] is θ = 45°.

The required transformation matrix is

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} \cos 45° & \sin 45° & 0 \\ -\sin 45° & \cos 45° & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\therefore \quad [T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\therefore \quad [T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\sqrt{2}+1 & -1 & 1 \end{bmatrix}$$

Now, $$\begin{bmatrix} A \\ B \\ C \end{bmatrix} [T] = \begin{bmatrix} 2 & -4 & 1 \\ 3 & 0 & 1 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\sqrt{2}+1 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2\sqrt{2}+1 & -\sqrt{2}-1 & 1 \\ \frac{1}{\sqrt{2}}+1 & \frac{3}{\sqrt{2}}-1 & 1 \\ -\frac{5}{\sqrt{2}}+1 & -\frac{1}{\sqrt{2}}-1 & 1 \end{bmatrix}$$

The vertices of a transformed triangle $A^* B^* C^*$ are

$A^* [2\sqrt{2}+1 \; -\sqrt{2}-1]$,

$B^* \left[\frac{1}{\sqrt{2}}+1 \quad \frac{3}{\sqrt{2}}-1\right]$ and $C^* \left[-\frac{5}{\sqrt{2}}+1 \quad -\frac{1}{\sqrt{2}}-1\right]$

Example 1.22 : An object is rotated through angle 90° about the point [4 3]. Find the ansformation equations.

Solution : The rotation about the point [4 3] through an angle 90° is achieved by the ansformation matrix.

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & -3 & 1 \end{bmatrix} \begin{bmatrix} \cos 90° & \sin 90° & 0 \\ -\sin 90° & \cos 90° & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix}$$

$$\therefore \quad [T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & -3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix}$$

$$\therefore \quad [T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & -3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 4 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 7 & -1 & 1 \end{bmatrix}$$

Now, $[x^* \; y^* \; 1] = [x \; y \; 1] \; [T]$

$$\therefore \quad [x^* \; y^* \; 1] = [x \; y \; 1] \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 7 & -1 & 1 \end{bmatrix}$$

The transformation equations are

$$x^* = -y + 7$$

$$y^* = x - 1$$

1.13 Reflection Through an Arbitrary Line

As we have seen previously, the reflection in x-axis and y-axis governed by the reflective matrices $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ respectively.

If a line L does not pass through the origin, then the reflection of an object in this lines is accomplished as below.

(i) By choosing a suitable point [m n] on L, translate the line L and object in such a way that the line passes through the origin.

$$\text{Translation matrix } [T_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -m & -n & 1 \end{bmatrix}.$$

(ii) Rotate the resulting line about origin such that the resulting line coincides with x-axis (or y-axis).

Let T_2 be the rotation matrix.

(iii) In step (ii) if the resulting line coincides with x-axis, then take reflection of an object in x-axis.

$$\text{Reflection matrix } [T_3] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

In step (ii) if the resulting line coincides with y-axis, then take reflection of an object in y-axis.

$$\text{Reflection matrix } [T_3] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(iv) Rotate in backward direction by using the matrix $[T_4]$ which is inverse of $[T_2]$.

$$[T_4] = [T_2]^{-1}$$

(v) Translate back to original location by using the matrix $[T_5]$ which is inverse of $[T_1]$.

$$[T_5] = [T_1]^{-1}$$

Then the required transformation is accomplished by the combined transformation matrix.

$$[T] = [T_1]\,[T_2]\,[T_3]\,[T_4]\,[T_5]$$
$$= [T_1]\,[T_2]\,[T_3]\,[T_2]^{-1}\,[T_1]^{-1}$$

Illustrative Examples

Example 1.23 : Using concatenated matrix, reflect the triangle ABC through the line $y = 5$, where A [1 3], B [2 4] and C [3 5]. **(March 2006)**

Solution : The line $y = 5$ is parallel to x-axis and the point [0 5] is on the line.

We translate the line in such a way that the resulting line passes through the origin. By considering the point [0 5] on the line, the translation matrix is

$$[T_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}$$

Now, the resulting line coincides with x-axis.

Now, for reflection of Δ ABC in x-axis the reflection matrix is

$$[T_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For back translation the translation matrix is

$$[T_3] = [T_1]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}.$$

The concatenated transformation matrix is

$$[T] = [T_1]\,[T_2]\,[T_3]$$

$$\therefore \quad [T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 10 & 1 \end{bmatrix}$$

Now, the vertices of Δ ABC are A [1 3], B [2 4] and C [3 5].

$$\begin{bmatrix} A^* \\ B^* \\ C^* \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \end{bmatrix}[T] = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 4 & 1 \\ 3 & 5 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 10 & 1 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} A^* \\ B^* \\ C^* \end{bmatrix} = \begin{bmatrix} 1 & 7 & 1 \\ 2 & 6 & 1 \\ 3 & 5 & 1 \end{bmatrix}$$

The transformed triangle has vertices A^* [1 7], B^* [2 6] and C^* [3 5].

Example 1.24 : By using concatenated matrix, reflect the triangle ABC in the line $3x - 4y + 8 = 0$, where A [0 0], B [1 2] and C [4 4].

Solution : The line $3x - 4y + 8 = 0$ has slope $m = \tan\theta = \frac{3}{4}$.

Therefore, $\sin\theta = \frac{3}{5}$ and $\cos\theta = \frac{4}{5}$.

A point [0 2] is on the line. By considering this point we translate the line such that the resulting line passes through the origin.

The translation matrix is .

$$[T_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

Now, we rotate the line in such a way that the resulting line will coincide with x-axis. This requires the angle of rotation $-\theta$.

The rotation matrix is

$$[T_2] = \begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\therefore \quad [T_2] = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Next the reflection in x-axis is achieved by using the reflection matrix

$$[T_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Back rotation matrix is

$$[T_4] = [T_2]^{-1} = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} & 0 \\ -\frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Back translation matrix is

$$[T_5] = [T_1]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

The required transformation matrix is

$$[T] = [T_1]\,[T_2]\,[T_3]\,[T_4]\,[T_5]$$

$$\therefore \quad [T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{4}{5} & -\frac{3}{4} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{4}{5} & \frac{3}{5} & 0 \\ -\frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{7}{25} & \frac{24}{25} & 0 \\ \frac{24}{25} & -\frac{7}{25} & 0 \\ -\frac{48}{25} & \frac{64}{25} & 1 \end{bmatrix}$$

The vertices of a triangle ABC are A [0 0], B [1 2] and C [4 4]. If triangle $A^* B^* C^*$ is the transformed triangle, then

$$\begin{bmatrix} A^* \\ B^* \\ C^* \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \end{bmatrix} [T]$$

$$\begin{bmatrix} A^* \\ B^* \\ C^* \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 4 & 4 & 1 \end{bmatrix} \begin{bmatrix} \frac{7}{25} & \frac{24}{25} & 0 \\ \frac{24}{25} & -\frac{7}{25} & 0 \\ -\frac{48}{25} & \frac{64}{25} & 1 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} A^* \\ B^* \\ C^* \end{bmatrix} = \begin{bmatrix} -\frac{48}{25} & \frac{64}{25} & 1 \\ \frac{7}{25} & \frac{74}{25} & 1 \\ \frac{76}{25} & \frac{132}{25} & 1 \end{bmatrix}$$

The transformed triangle has vertices $A^* = \left[-\frac{48}{25} \;\; \frac{64}{25}\right]$, $B^*\left[\frac{7}{25} \;\; \frac{74}{25}\right]$ and $C^*\left[\frac{76}{25} \;\; \frac{132}{25}\right]$.

Example 1.25 : Find the concatenated transformation matrix for the following sequence of transformations : shearing in x-direction by – 2.5 units followed by translation in x and y-directions by – 1.2 and – 3.9 units respectively, followed by scaling in x and y co-ordinates by the factors $\frac{1}{2}$ and 2.1 units respectively.

Solution : The transformation matrix for shearing in x-direction by – 2.5 units is

$$[T_1] = \begin{bmatrix} 1 & 0 & 0 \\ -2.5 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

The transformation matrix for translation in x and y-directions by – 1.2 and – 3.9 units respectively is;

$$[T_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1.2 & -3.9 & 1 \end{bmatrix}$$

The transformation matrix for scaling in x and y co-ordinates by the factors $\frac{1}{2}$ and 2.1 respectively is

$$[T_3] = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 2.1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The concatenated transformation matrix is

$$[T] = [T_1]\,[T_2]\,[T_3]$$

$$\therefore \quad [T] = \begin{bmatrix} 1 & 0 & 0 \\ -2.5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1.2 & -3.9 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 2.1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad [T] = \begin{bmatrix} 1 & 0 & 0 \\ -2.5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 2.1 & 0 \\ -0.6 & -8.19 & 1 \end{bmatrix}$$

$$\therefore \quad [T] = \begin{bmatrix} 0.5 & 0 & 0 \\ -1.25 & 2.1 & 0 \\ -0.6 & -8.19 & 1 \end{bmatrix}$$

Example 1.26 : The position vector [2 5] is rotated about the point [4 3] by an angle $\theta = 90°$. By using homogeneous system obtain the position vector of the transformed point.

Solution : The concatenated transformation matrix for rotation about the point [4 3] by an angle $\theta = 90°$ is

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & -3 & 1 \end{bmatrix} \begin{bmatrix} \cos 90° & \sin 90° & 0 \\ -\sin 90° & \cos 90° & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix}$$

$$\therefore \quad [T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & -3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix}$$

$$\therefore \quad [T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & -3 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 4 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 7 & -1 & 1 \end{bmatrix}$$

Now, P [2 5 1]. The transformed point P^* is given by

$$[P^*] = [P]\,[T] = [2 \;\; 5 \;\; 1] \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 7 & -1 & 1 \end{bmatrix} = [2 \;\; 1 \;\; 1]$$

The position vector of P^* is [2 1].

Example 1.27 : Rotate the line segment AB; where A [2 1] and B [5 7], about it's point of trisection near A, by an angle of 35°.

Solution : A [2 1] and B [5 7]. By section formula the point of trisection of segment AB near A is [3 3].

The concatenated transformation matrix for rotation about the point [3 3] by an angle 35° is

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & -3 & 1 \end{bmatrix} \begin{bmatrix} \cos 35° & \sin 35° & 0 \\ -\sin 35° & \cos 35° & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix}$$

$$\therefore \quad [T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & -3 & 1 \end{bmatrix} \begin{bmatrix} 0.8191 & 0.5735 & 0 \\ -0.5735 & 0.8191 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 3 & 1 \end{bmatrix}$$

$$\therefore \quad [T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & -3 & 1 \end{bmatrix} \begin{bmatrix} 0.8191 & 0.5735 & 0 \\ -0.5735 & 0.8191 & 0 \\ 3 & 3 & 1 \end{bmatrix}$$

$$\therefore \quad [T] = \begin{bmatrix} 0.8191 & 0.5735 & 0 \\ -0.5735 & 0.8191 & 0 \\ 2.2632 & -1.1778 & 1 \end{bmatrix}$$

$$\begin{bmatrix} A^* \\ B^* \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix} [T] = \begin{bmatrix} 2 & 1 & 1 \\ 5 & 7 & 1 \end{bmatrix} \begin{bmatrix} 0.8191 & 0.5735 & 0 \\ -0.5735 & 0.8191 & 0 \\ 2.2632 & -1.1778 & 1 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} A^* \\ B^* \end{bmatrix} = \begin{bmatrix} 3.3279 & 0.7883 & 1 \\ 2.3442 & 7.4234 & 1 \end{bmatrix}$$

$\therefore$ A^* [3.3279 0.7883], B^* [2.3442 7.4234]

Example 1.28 : Rotate the triangle ABC with vertices A [1 2], B [3 6], C [3 1] through an angle 90° about the point [2 4]. **(October 2007)**

Solution : The concentrated transformation matrix for rotation about the point [2 4] through the angle 90° is

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -4 & 1 \end{bmatrix} \begin{bmatrix} \cos 90° & \sin 90° & 0 \\ -\sin 90° & \cos 90° & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix}$$

$$\therefore \quad [T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix}$$

$$\therefore \quad [T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & -4 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 6 & 2 & 1 \end{bmatrix}$$

The vertices of triangle ABC are A [1 2], B [3 6] and C [3 1].

If the triangle is A^* B^* C^*, the

$$\begin{bmatrix} A^* \\ B^* \\ C^* \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \end{bmatrix} [T] = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 6 & 2 & 1 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} A^* \\ B^* \\ C^* \end{bmatrix} = \begin{bmatrix} 4 & 3 & 1 \\ 0 & 5 & 1 \\ 5 & 5 & 1 \end{bmatrix}$$

$\therefore$ A^* [4 3], B^* [0 5], C^* [5 5]

Example 1.29 : Reflect the triangle ABC through the line $3x - y = 0$; where A $[-2\ -3]$, B $[-10\ -6]$, C $[-15\ -10]$.

Solution : The line $y = 3x$ passes through origin. Hence translation is not needed.

The line $y = 3x$ makes angle $\theta = \tan^{-1} 3$ with x-axis.

$$\therefore \quad \tan\theta = 3 \quad \sin\theta = \frac{3}{\sqrt{10}},\ \cos\theta = \frac{1}{\sqrt{10}}.$$

The transformation matrix for rotation about origin through an angle $-\theta$ is;

$$[T_1] = \begin{bmatrix} \cos(-\theta) & \sin(-\theta) & 0 \\ -\sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad [T_1] = \begin{bmatrix} \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} & 0 \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The transformation matrix for reflection is x-axis is

$$[T_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The transformation matrix for rotation about origin through an angle θ is $[T_3] = [T_1]^{-1} = [T_1]^t$.

$$\therefore \quad [T_3] = \begin{bmatrix} \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} & 0 \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The concatenated transformation matrix is $[T] = [T_1]\,[T_2]\,[T_3]$.

$$[T] = \begin{bmatrix} \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} & 0 \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} & 0 \\ -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad [T] = \begin{bmatrix} \frac{1}{\sqrt{10}} & -\frac{3}{\sqrt{10}} & 0 \\ \frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} & 0 \\ -\frac{3}{\sqrt{10}} & \frac{1}{\sqrt{10}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad [T] = \begin{bmatrix} -\frac{8}{10} & \frac{6}{10} & 0 \\ \frac{6}{10} & \frac{8}{10} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The vertices of triangle ABC have co-ordinates A $[-2, -3, 1]$, B$[-10, -6, 1]$, C$[-15, -10, 1]$.

If A^*, B^*, C^* are the vertices of transformed triangle, then

$$\begin{bmatrix} A^* \\ B^* \\ C^* \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \end{bmatrix} [T] = \begin{bmatrix} -2 & -3 & 1 \\ -10 & -6 & 1 \\ -15 & -10 & 1 \end{bmatrix} \begin{bmatrix} -\frac{4}{5} & \frac{3}{5} & 0 \\ \frac{3}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} A^* \\ B^* \\ C^* \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} & -\frac{18}{5} & 1 \\ \frac{22}{5} & -\frac{54}{5} & 1 \\ 6 & -17 & 1 \end{bmatrix}$$

$$\therefore \quad A^*\left[-\frac{1}{5}\ \ -\frac{18}{5}\ \ 1\right],\ B^*\left[\frac{22}{5}\ \ -\frac{54}{5}\ \ 1\right], C^*\,[6\ \ -17\ \ 1].$$

Example 1.30 : Find the concatenated transformation matrix for reflection through the line $y = -4$. Apply it to the position vector $[-1\ 2]$.

Solution : The point $[0\ -4]$ lies on the line $y = -4$.

The translation matrix is

$$[T_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

Now, the line coincides with x-axis. The reflection matrix for reflection through x-axis is

$$[T_2] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Back translation matrix is

$$[T_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}$$

The concatenated transformation matrix is $[T] = [T_1]\,[T_2]\,[T_3]$.

$$\therefore \quad [T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix}$$

$$\therefore \quad [T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -8 & 1 \end{bmatrix}$$

Now, P [–1 2 1]

$$[P^*] = [P]\,[T] = [-1\ 2\ 1]\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & -8 & 1 \end{bmatrix} = [-1\ -10\ 1]$$

The position vector of P^* is [–1 – 10].

Example 1.31 : Obtain the combined transformation matrix for the following sequence of transformations :

Reflection through the line y = – x, shearing in x and y directions by 3 and – 4, units receptively.

Translation in x and y direction by – 1 and 2 units respectively.

Apply it on the point P [3 – 8].

Solution : The transformation matrix for reflection through the line y = – x is

$$[T_1] = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The transformation matrix for shearing in x and y directions by 3 and – 4 units respectively.

$$[T_2] = \begin{bmatrix} 1 & -4 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

he transformation matrix for translation in x and y directions by –1 and 2 units respectively is

$$[T_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$$

The combined transformation matrix is $[T] = [T_1]\,[T_2]\,[T_3]$.

$$\therefore \quad [T] = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -4 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$$

$$\therefore \quad [T] = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -4 & 0 \\ 3 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$$

$$\therefore \quad [T] = \begin{bmatrix} -3 & -1 & 0 \\ -1 & 4 & 0 \\ -1 & 2 & 1 \end{bmatrix}$$

Now, we apply matrix [T] on the point P [3 – 8 1].

$$[P^*] = [P]\,[T] = [3\ -8\ 1]\begin{bmatrix} -3 & -1 & 0 \\ -1 & 4 & 0 \\ -1 & 2 & 1 \end{bmatrix}$$

$$\therefore \quad [P^*] = [-2\ \ -33\ \ 1].$$

Example 1.32 : Reflect the square with vertices O [0 0], A [2 0], B [2 2], C [0 2] through the line 2x – y + 4 = 0.

Solution : We take any point say [1 6] on the line 2x – y + 4 = 0.

The transformation matrix for translation is;

$$[T_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -6 & 1 \end{bmatrix}$$

The line 2x – y + 4 = 0 has slope tan θ = 2.

Therefore, $\sin\theta = \frac{2}{\sqrt{5}},\ \cos\theta = \frac{1}{\sqrt{5}},\ \sin(-\theta) = -\frac{2}{\sqrt{5}},\ \cos(-\theta) = \frac{1}{\sqrt{5}}$.

The transformation matrix for rotation by angle – θ is;

$$[T_2] = \begin{bmatrix} \cos(-\theta) & \sin(-\theta) & 0 \\ -\sin(-\theta) & \cos(-\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad [T_2] = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The translation matrix for reflection in x-axis is;

$$[T_3] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Back rotation matrix by an angle θ is

$$[T_4] = [T_2]^{-1} = [T_2]^t = \begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Back translation matrix is

$$[T_5] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 6 & 1 \end{bmatrix}$$

The concatenated matrix of translation is

$[T] = [T_1]\ [T_2]\ [T_3]\ [T_4]\ [T_5]$.

$$\therefore \quad [T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -6 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 6 & 1 \end{bmatrix}$$

The calculations give

$$[T] = \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ -\frac{16}{5} & \frac{8}{5} & 1 \end{bmatrix}$$

We apply this transformation matrix on the square with vertices O [0 0], A [2 0], B [2 2], C [0 2].

$$\begin{bmatrix} O^* \\ A^* \\ B^* \\ C^* \end{bmatrix} = \begin{bmatrix} O \\ A \\ B \\ C \end{bmatrix} [T]$$

$$\therefore \quad \begin{bmatrix} O^* \\ A^* \\ B^* \\ C^* \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 0 & 1 \\ 2 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} -\frac{3}{5} & \frac{4}{5} & 0 \\ \frac{4}{5} & \frac{3}{5} & 0 \\ -\frac{16}{5} & \frac{8}{5} & 1 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} O^* \\ A^* \\ B^* \\ C^* \end{bmatrix} = \begin{bmatrix} -\frac{16}{5} & \frac{8}{5} & 1 \\ -\frac{22}{5} & \frac{16}{5} & 1 \\ -\frac{14}{5} & \frac{22}{5} & 1 \\ -\frac{8}{5} & \frac{14}{5} & 1 \end{bmatrix}$$

The transformed figures has vertices with position vectors

$O^*\left[-\frac{16}{5}\ \frac{8}{5}\right]$, $A^*\left[-\frac{22}{5}\ \frac{16}{5}\right]$, $B^*\left[-\frac{14}{5}\ \frac{22}{5}\right]$, $C^*\left[-\frac{8}{5}\ \frac{14}{5}\right]$.

Think Over It

1. Prove that, the multiplication of transformation matrices for each of the following sequence of operations is commutative :
   (a) Two successive rotations. (b) Two successive translations.
   (c) Two successive scalings.
2. Prove that, a uniform scaling and a rotation form a commutative pair of operations but that, in general, scaling and rotation are not commutative operations.
3. Show that the order in which the transformations are performed is important by the transformation of a triangle A[1 0], B[0 1], C[1 1] by :
   (a) rotating 45° about the origin and then translating in the direction of vector ?
   (b) translating and then rotating.

4. A diamond shaped polygon has vertices $A[-1\ \ 0]$, $B[0\ \ -2]$, $C[1\ \ 0]$, $D[0\ \ 2]$. Reflect this polygon about :
   (a) the horizontal line $y = 2$.
   (b) the vertical line $x = 2$.
   (c) the line $y = x + 2$.

Points to Remember

1. The general 2×2 transformation matrix is $[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.
2. If $a = d > 1$ then $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$ produces pure enlargement of the co-ordinates of point P and if $a = d < 1$ then $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ produces compression of the co-ordinates of the point P.
3. If $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the transformation matrix, then the scaling effect is governed by the entries a and d on the main diagonal of [T]; while the shearing effect is governed by the entries b and c on the second diagonal of [T].
4. **Scaling :**
   (i) $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$ produces scaling in x-co-ordinate by factor a.
   (ii) $\begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix}$ produces scaling in y-co-ordinate by factor d.
   (iii) $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ produces scaling in x and y co-ordinate by units a and d respectively.
   (iv) $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ produces uniform scaling by factor a.
5. **Reflection :**
   (i) $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ represents reflection through x-axis.
   (ii) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ represents reflection through y-axis.
   (iii) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ represents reflection through the line $y = x$.
   (iv) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ represents reflection through the line $y = -x$.
   (v) $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ represents reflection through origin.

6. **Shearing :**
   (i) $\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}$ produces shearing along x-axis by factor c.
   (ii) $\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$ produces shearing y-axis by factor b.
   (iii) $\begin{bmatrix} 1 & b \\ c & 1 \end{bmatrix}$ produces shearing along x-axis and y-axis by factors b and c respectively.
7. **Mid-point transformation :**
   When line L joining the points $A[x_1\ \ y_1]$ and $B[x_2\ \ y_2]$ is transformed into a line L^* joining $A^*[x_1^*\ \ y_1^*]$ and $B^*\ [x_2^*\ \ y_2^*]$ under [T] then :
   (i) The mid-point of AB corresponds to the mid-point of $A^*\ B^*$.
   (ii) In general if a point P divides AB in the ratio $\lambda : 1$, then under the transformation [T] it will be transformed into a point P^* that divides $A^*\ B^*$ in the same ratio $\lambda : 1$. Hence, there is one-one correspondence between the points on AB and the points on $A^*\ B^*$.
8. Suppose a 2×2 transformation matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ transforms the line segment AB to the segment $A^*\ B^*$. If the slope of lie AB is m, then the slope of transformed line $A^*\ B^*$ is $m^* = \dfrac{b + dm}{a + cm}$.
9. The effect of 2×2 transformation matrix on a pair of parallel lines is another pair of parallel lines.
10. Under any 2×2 transformation matrix $[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ a pair of intersecting lines gets transformed onto a pair of intersecting lines. Further the point of intersection of original pair of lines to transformed onto a point of intersection of the transformed form of pair of lines.
11. The transformation matrix for rotation about origin through angle θ is $[T] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.
12. If the transformation matrix [T] is orthogonal and det. $[T'] = 1$, then perpendicular lines are transformed to perpendicular line.
13. If $[T] = -1$ then magnitudes of the vectors are preserved but; the angle θ between original lines becomes $2\pi - \theta$ between the transformed lines.
14. The uniform scaling preserves the angle between the lines but in this case the magnitudes of vectors are not preserved.
15. The equations $x^* = ax + cy + m$, $y^* = bx + dy + n$ are written in homogeneous co-ordinate system as
$$[x^*\ \ y^*\ \ 1] = [x\ \ y\ \ 1]\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ m & n & 1 \end{bmatrix}.$$
16. For reflection through an arbitrary line, we write
$$[T] = [T_1]\,[T_2]\,[T_3]\,[T_2]^{-1}\,[T_1]^{-1}$$
where, T_1 is the translation matrix.
T_2 is the rotation matrix about the origin. T_3 is the reflection matrix.

17. The concatenated transformation matrix for rotation about a point is

$$[T] = [T_1]\,[T_2]\,[T_3]$$

$$= \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ m(1-\cos\theta)+n\sin\theta & n(1-\cos\theta)-m\sin\theta & 1 \end{bmatrix}$$

Miscellaneous Exercise

(A) State whether the following statements are True or False :

1. The x'y' co-ordinate system results form of scaling of a units in the y-direction. Then the equation of the circle $(x')^2 + (y')^2 = 1$ in terms of xy co-ordinates is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
2. If two lines $2x - y = 7$ and $3x + y = 3$ are transformed under the transformation matrix $[T] = \begin{bmatrix} 3 & 4 \\ -1 & 2 \end{bmatrix}$ then the point of intersection of the transformed lines is [2 9].
3. The triangle ABC with position vectors [1 0], [0 1] and [−1 0] is transformed by $T = \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}$ to create a second triangle A'B'C'. Then the area of triangle A'B'C' is 8 units. State whether true or false.
4. Two parallel lines may intersect after any transformation. State whether it is true or false.
5. The effect of shearing in x-direction by 4 units and in y-direction by 3 units is [31 16].
6. A line with slope 2 is transformed using the matrix $[T] = \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix}$. Then the slope of transformed line is 3.
7. The line segment joining A[4 9] and B[−2 1] is scaled uniformly by factor 2. Then the mid-point of the transformed line segment is [2 10].
8. The effect of the transformation matrix $\begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$ on a two dimensional object is rotation about origin through angle 60° in clockwise sense.
9. If a 2 × 2 transformation matrix $[T] = \begin{bmatrix} 1 & 3 \\ -2 & 2 \end{bmatrix}$ is used to transform the line passing through two points $A\begin{bmatrix} 3 & -\frac{1}{2} \end{bmatrix}$ and B[0 1] then the equation of the resulting line is y = x + 4.
10. The transformation matrix $[T] = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ represents a solid body transformation, state whether it is true or false.
11. The general form of the matrix for rotation about a point P[h k] is
$$\begin{bmatrix} \cos\theta & -\sin\theta & h + k\sin\theta - h\cos\theta \\ \sin\theta & \cos\theta & k - h\sin\theta - k\cos\theta \\ 0 & 0 & 1 \end{bmatrix}$$

12. The general form of a scaling matrix with respect to a fixed point P[h k] is $\begin{bmatrix} a & 0 & -ah+h \\ 0 & b & -bx+k \\ 0 & 0 & 1 \end{bmatrix}$.
13. The 45° rotation of triangle A[0 0], B[1 1], C[5 2] about the origin leads to $A^*[0\ 0]$, $B^*[0\ 2]$, $C^*\begin{bmatrix} \frac{5}{\sqrt{2}} & \frac{7}{\sqrt{2}} \end{bmatrix}$.

ANSWERS

1. - True	2. - False	3. - True	4. - True	5. - True
6. - False	7. - True	8. - False	9. - True	10. - False
11. - True	12. - True	13. False		

(B) Multiple Choice Questions :

1. If we apply a shearing in x and y directions by 4 and − 3 units respectively onto the line $2x - y = 1$ and $x + y = 2$, then the co-ordinates of the point of intersection of resultant lines is ......
   (a) (− 5, 2) (b) (5, − 2)
   (c) (1, 2) (d) (3, − 5)
2. What is the determinant of the inverse of any pure rotation matrix ?
   (a) 1 (b) − 1
   (c) 0 (d) none of these
3. A unit square is transformed under 2 × 2 transformation matrix $[T] = \begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}$. Then the area of the transformed figure is ......
   (a) 4 (b) 6
   (c)) 8 (d) 1
4. If the transformation matrix $[T] = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$ is applied to a circle of radius 100 units then the area of resulting figure is ......
   (a) 3.14 sq. units (b) 31400 sq. units
   (c) 1000 sq. units (d) none of these
5. The circle of area 10 cm^2 is scaled uniformly by factor 2. Then the area of transformed figure is ......
   (a) 100 cm^2 (b) 60 cm^2
   (c) 40 cm^2 (d) none of these
6. A line with slope 2 is transformed using the matrix $T = \begin{bmatrix} 4 & 2 \\ -1 & 3 \end{bmatrix}$. Then the slope of the transformed line is ......
   (a) 1 (b) 3
   (c) 4 (d) 2

7. A line segment joining A[4 9] and B[−2 1] is scaled uniformly by factor 2. Then the mid-point of the transformed line segment is ......
   (a) [10 2] (b) [1 5]
   (c) [2 10] (d) [3 5]

8. The combined two dimensional transformation first reflection about x-axis and then about the line y = −x is equivalent to the rotation about the origin by ......
   (a) 136° (b) 270°
   (c) 45° (d) 120°

9. The effect of the transformation matrix $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ on a two dimensional object is ......
   (a) shearing in x co-ordinate by factor − 2.
   (b) Shearing in x co-ordinate by factor − 1.
   (c) rotation about origin through angle 60° in anticlockwise sense.
   (d) rotation about origin through angle 45° in anticlockwise sense.

10. The angle of rotation required to rotate the line, y = 2x about the origin, so that it coincides with the x-axis is ......
   (a) $\tan^{-1}\left(\frac{1}{2}\right)$ (b) $\tan^{-1}(-2)$
   (c) $\tan^{-1}(-3)$ (d) $\tan^{-1}(3)$

11. The concatenated transformation matrix for reflection through the line y = 4 is ......
   (a) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 10 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & 10 \end{bmatrix}$
   (c) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 10 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 10 & 1 \end{bmatrix}$

12. The slope of original line is $\frac{2}{5}$ and that of the transformed line is −1. Then [T] is ......
   (a) $\begin{bmatrix} -1 & 3 \\ 5 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -1 \\ 5 & 0 \end{bmatrix}$
   (c) $\begin{bmatrix} 1 & -3 \\ 5 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & 1 \\ 5 & 0 \end{bmatrix}$

13. The general form of a scaling matrix with respect to a fixed point P(h, k) is ......
   (a) $\begin{bmatrix} a & 0 & ah-h \\ 0 & b & bk-b \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} a & 0 & -ah+h \\ 0 & b & -bk+b \\ 0 & 0 & 1 \end{bmatrix}$
   (c) $\begin{bmatrix} a & 0 & ah \\ 0 & b & bk \\ 0 & 10 & 1 \end{bmatrix}$ (d) none of these

14. An object is rotated by 30° about the origin. The matrix which represents this rotation is ......
   (a) $\begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$ (b) $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$
   (c) $\begin{bmatrix} \frac{-\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{-1}{2} & \frac{-\sqrt{3}}{2} \end{bmatrix}$ (d) None of these

15. The concatenated transformation matrix for reflection through the line y = 4 is ......
   (a) $\begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 10 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 10 & 1 \end{bmatrix}$
   (c) $\begin{bmatrix} 0 & -1 & 0 \\ 0 & 10 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ (d) none of these

16. The line segment AB is rotated about the point [2 1] through 90°; where A[3 5] and B[4 1]. Then ......
   (a) A^*[−2 2 −1], B^*[2 31] (b) A^*[2 3 1], B^*[−2 2 1]
   (c) A^*[−2 −2 1], B^*[2 −3 1] (d) A^*[−2 2 1], B^*[2 3 1]

17. The 45° rotation of triangle A[0 0], B[1 1], C[5 2] about the point P[−1 −1] leads to ......
   (a) $A^*[-1\ \sqrt{2}]$, $B^*[1\ \sqrt{2}-1]$, $C^*[1\ \sqrt{2}]$
   (b) $A^*[-1\ \sqrt{2}-1]$, $B^*[-1\ \ 2\sqrt{2}-1]$, $C^*\left[3\sqrt{2}-1\ \ \frac{9}{\sqrt{2}}-\frac{9}{2}\right]$
   (c) $A^*[1\ -\sqrt{2}]$, $B^*[-1\ \ \sqrt{2}]$, $C^*[7\ \ -\sqrt{2}]$
   (d) $A^*[\sqrt{2}-1\ \ 2]$, $B^*[1\ \ 2\sqrt{2}]$, $C^*[2\sqrt{2}\ \ -1]$

Answers

1 - (b)	2 - (a)	3 - (c)	4 - (b)	5 - (c)	6 - (c)	7 - (c)
8 - (b)	9 - (a)	10 - (b)	11 - (c)	12 - (b)	13 - (b)	14 - (b)
15 - (b)	16 - (d)	17 - (d)				

(C) Theory Questions :

1. Explain different possible effects due to the entries of general 2 × 2 transformation matrix $[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. **(P.U. March 2008)**
2. Write the transformation matrix for reflection through the line y = x. **(P.U. Oct. 2012)**

3. Prove that, mid-point of the line segment AB is transformed to the mid-point of segment $A^* B^*$ under 2×2 transformation matrix [T]. **(P.U. Oct. 2012)**
4. Prove that under any 2×2 transformation matrix, the mid-point M of the line segment AB is transformed into the mid-point M^* of the transformed line segment $A^* B^*$.

(P.U. April 2005, Oct. 2007)

5. Suppose the 2×2 transformation matrix transforms the points P and Q to the points P^* and Q^* respectively. If R divides segment PQ internally in the ratio m : n then prove that its transformed point R^* divides segment $P^* Q^*$ internally in the ratio m : n. **(P.U. March 2009)**
6. Prove that, if the line y = mx + k is transformed to the line

$y^* = m^* x^* + k^*$ by 2×2 transformation matrix

$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $m^* = \dfrac{b + dm}{a + cm}$ and $K^* = K\left(\dfrac{ad - bc}{a + cm}\right)$ **(P.U. April 2005)**

7. If a line y = mx + k is transformed by using 2×2 transformation matrix $[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then obtain the equation of the transformed line. **(P.U. Oct. 2001)**
8. If a 2×2 transformation matrix is applied on a pair of parallel lines, then they are transformed to a pair of parallel lines. **(P.U. April 2007)**
9. If segments PQ and RS are transformed to $P^* Q^*$ and $R^* S^*$ respectively by a 2×2 transformation matrix, then PQ is parallel to RS, iff $P^* Q^*$ is parallel to $R^* S^*$. Prove or disprove.

(P.U. Oct. 2008)

10. If a 2×2 transformation matrix is applied on a pair of intersection lines then they are transformed to a pair of intersecting lines. Prove or disprove. **(P.U. April 2010)**
11. Derive the transformation matrix for rotation about origin through an angle θ. **(P.U. Oct. 2006)**
12. Derive the condition under which 2×2 transformation matrix [T] preserves magnitude and angle between two vectors. **(P. U. April 12)**
13. What is an apparent translation ? Explain how to obtain pure scaling without an apparent translation ?
14. Explain what is meant by a solid body transformation.
15. Define homogeneous co-ordinate in two dimensional structure.
16. Find the equation of the line y' = mx' + b in xy, co-ordinates, if the x'y' co-ordinate system results from a 90° rotation of the xy coordinate system.
17. What is the significance of h, when the point in two dimensions is represented as [x y h] ?
18. Derive the transformation that rotates an object point θ° about the origin. Write the matrix representation for this rotation.
19. Find the transformation that scales (with respect to the origin] by :
    (a) a units in x-direction.
    (b) b units in y-direction.
    (c) Simultaneously a units in x-direction and b units in y-direction.

20. Find the form of the matrix for reflection about a line L with slope m and y intercept (0 b).
21. Obtain the condition under which the perpendicular lines are transformed onto perpendicular lines; under the transformation matrix T.

(D) Numerical Problems :

1. An object is rotated about the point [5 3] through an angle $\frac{\pi}{2}$. Find the concatenated matrix of transformation.
2. Find the concatenated transformation matrix for the following sequence of transformations : shearing in x and y directions by −2 and 3 units respectively, followed by translation in x and y directions by − 3.2 and 1.6 units respectively, followed by scaling y co-ordinate by factor 4.
3. An object is rotated about the point [−1 2] through π^c. Find the concatenated matrix of transformation.
4. Find the matrix that represents rotation of an object by 30° about the origin. What are the new co-ordinates of the point [2 −4] after the rotation ?
5. Show that the parallel lines AB and CD are not transformed onto parallel lines under the transformation matrix $[T] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ where, A[1 2], B[2 4], C[2 6] and D[3 8].
6. Consider a triangle with vertices A[3 6], B[6 9], C[3 9]. Rotate the triangle about the point [−2 1] through an angle 35°. Write the position vectors of the transformed triangle.
7. By using concatenated matrix of transformation reflect the triangle ABC through the line y = 5; given that A[1 3], B[2 4] and C[3 5].
8. Rotate the line segment between the points A[1 2] and B[3 6] by $\frac{\pi}{2}$ about the mid-point of segment AB.
9. Obtain the concatenated transformation matrix and apply it on the line segment joining [1 2] and B[0 4]. The sequence of transformations is as below.
   Translation in y-direction by 2 units and then rotation about the origin by θ = − 40°.
10. Find the concatenated transformation matrix for the following sequence of transformations : Reflection through the line y = x, shearing in x and y directions by 1 and −2 units respectively.
    Apply it onto the point P[4 −2].
11. Reflect the triangle ABC through the line x − 2y + 4 = 0 where A[2 4], B[4 6] and C[2 6].
12. Find the concatenated transformation matrix for the following sequence of transformations : Shearing in y-direction by − 3.4 units, followed by translation in x and y directions by − 5.2 and − 1.3 units respectively, followed by scaling in x and y coordinates by factor 2.1 and 1.2 respectively.

Answers

1. $[T] = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 8 & -2 & 1 \end{bmatrix}$

2. $[T] = \begin{bmatrix} 1 & 12 & 0 \\ -2 & 4 & 0 \\ -3.2 & 6.4 & 1 \end{bmatrix}$

3. $[T] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ -2 & 4 & 1 \end{bmatrix}$

4. $\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}, \begin{bmatrix} \sqrt{3} + 2 \\ 1 - 2\sqrt{3} \end{bmatrix}$

6. A^* [0.228 7.963 1]

   B^* [− 0.0352 12.1408 1]

   C^* [− 2.4925 10.4203 1]

7. A^*[1 7], B^*[2 6], C^*[3 5]

8. A^*[4 3 1], B^* [0 5 1]

9. $[T] = \begin{bmatrix} 0.7660 & -0.6427 & 0 \\ 0.6427 & 0.7660 & 0 \\ -1.2854 & 0.4680 & 1 \end{bmatrix}$

   A^*[0.7660 1.3573 1], B^*[1.2854 3.5320 1]

10. $[T] = \begin{bmatrix} -3 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & -2 & 1 \end{bmatrix}$, P^*[−13 −2 1]

11. $A^*\begin{bmatrix} \frac{14}{5} & \frac{12}{5} \end{bmatrix}$, $B^*\begin{bmatrix} \frac{28}{5} & \frac{14}{5} \end{bmatrix}$, $C^*\begin{bmatrix} \frac{22}{5} & \frac{6}{5} \end{bmatrix}$

12. $[T] = \begin{bmatrix} 2.1 & -4.08 & 0 \\ 0 & 1.2 & 0 \\ -10.92 & -1.56 & 1 \end{bmatrix}$

✍✍✍

Chapter **2**...

Three-Dimensional Transformations

George David Birkhoff

George David Birkhoff (1884-1944) : He was an established mathematician of United States of America. He was born in Overisel of Mich. He had his first education from Lewis Institution III of Chicago. Then he joined Chicago University. After this he joined Harvard University where he obtained A.B. Degree in 1905 and A.M. Degree in 1906. At last he got Doctorate Degree from Chicago University in 1907. As a teacher he brought many improvements in the University of Wisconsin.

He made research on asymptotic problem, singular points and expansion problems and added many new facts. His first writing was based on Dynamics which was published in the year 1909.

He devoted his whole life in making research about differential calculus and Dynamics with addition of new ideas and style. He has created many new facts on Theory of stability, Problem of Three Bodies, geometrical Theorems of Poincare, dynamical systems and Ergodic Theory. He died suddenly at Cambiridge on 12, Nov., 1944 which created a great vacuum in research atmsophere.

2.1 Introduction

Construction of three-dimensional graphic images requires the use of three-dimensional geometrical and co-ordinate transformations. These transformations are formed by the composition of scaling, reflection, rotation and translation. All of the above transformations can be represented as a matrix transformation just as we have discussed in two-dimensional case.

Methods for geometric transformations in three dimensions are extended from two-dimensional methods by including considerations for the z co-ordinate in space. A point in three-dimensional space is represented by the position vector matrix [x y z].

We have introduced homogeneous co-ordinates in two dimensions. Now, we have to introduce same in three dimensional space.

A point [x y z] in three dimensional space is represented in the homogeneous co-ordinate system by a homogeneous co-ordinate vector matrix [x' y' z' h], where $x' = xh$, $y' = yh$, $z' = zh$, $h \in R$ (set of real numbers).

The transformation from homogeneous co-ordinates to three dimensional ordinary co-ordinates is given by;

$$\left[\frac{x'}{h} \quad \frac{y'}{h} \quad \frac{z'}{h} \; 1\right].$$

For three-dimensional homogeneous co-ordinates, the generalized 4×4 transformation matrix is given by;

$$[T] = \begin{bmatrix} a & b & c & p \\ d & e & f & q \\ g & i & j & r \\ l & m & n & s \end{bmatrix}$$

The above 4×4 transformation matrix can be divided into four parts as follows :

$$\begin{bmatrix} \underbrace{\begin{bmatrix} a & b & c \\ d & e & f \\ g & i & j \end{bmatrix}_{3 \times 3}}_{\text{Scaling, shearing, reflection, rotation transformation}} & \underbrace{\begin{bmatrix} p \\ q \\ r \end{bmatrix}_{3 \times 1}}_{\text{Perspective transformation}} \\ \underbrace{[l \ m \ n]_{1 \times 3}}_{\text{Translation transformation}} & \underbrace{[s]_{1 \times 1}}_{\text{Overall scaling transformation}} \end{bmatrix}$$

The upper left 3×3 sub-matrix gives transformations in the form of scaling, shearing, rotation and reflection. The lower left 1×3 sub-matrix gives translation. The upper right 3×1 sub-matrix gives perspective transformation and last 1×1 sub-matrix gives overall scaling.

2.2 Scaling

The diagonal terms of the general 4×4 transformation produce local and overall scaling. We consider here local shearing effect and overall shearing effect separately.

Local Shearing Effect : *Consider the transformation matrix,* $[T] = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & j & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Applying [T] on [X] = [x, y, z, 1] we get [X']

$\therefore$ $[X'] = [x', y', z', 1] = [X]\ [T]$

$$\Rightarrow \quad [x'\ y'\ z'\ 1] = [x\ y\ z\ 1] \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & j & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [ax\ ey\ jz\ 1]$$

$$\Rightarrow \quad x' = ax,\ y' = ey,\ z' = jz$$

Illustration : *Consider the rectangular parallelopiped where x = 2, y = 3 and z = 1.*

We can write homogeneous position vector matrix for the above rectangular parallelopiped as:

$$[X] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 2 & 3 & 1 & 1 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 2 & 3 & 0 & 1 \\ 0 & 3 & 0 & 1 \end{bmatrix} \text{(co-ordinate of the 8 vertices)} \quad \ldots (1)$$

We can convert rectangular parallelopiped into a unit cube by shearing factors $\frac{1}{2}$ unit along x-axis, $\frac{1}{3}$ unit along y-axis and 1 unit along z-axis. The local shearing transformation matrix is

$$[T] = \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \ldots (2)$$

Applying [T] on [X], we get

$$[X'] = [X]\ [T]$$

$$= \begin{bmatrix} 0 & 0 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 2 & 3 & 1 & 1 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 2 & 3 & 0 & 1 \\ 0 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad = [X'] \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

The matrix [X'] obtained above is the homogeneous position vector matrix of the unit cube.

Overall Shearing Effect : Overall scaling is obtained by using the fourth diagonal element in [T].

Therefore, consider the transformation matrix

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s \end{bmatrix}$$

Applying [T] on [X] = [x y z 1], we get [X']

∴ [X'] = [x' y' z' 1] = [X] [T]

$$\Rightarrow \quad [x'\ y'\ z'\ 1] = [x\ y\ z\ 1]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s \end{bmatrix} = [x\ y\ z\ 1]$$

The homogeneous co-ordinates [x y z s] can be written as $\left[\frac{x}{s}\ \frac{y}{s}\ \frac{z}{s}\ 1\right]$.

Hence, we have $x' = \frac{x}{s}$, $y' = \frac{y}{s}$, $z' = \frac{z}{s}$.

The same overall scaling effect is obtained by using the transformation matrix

$$[T] = \begin{bmatrix} \frac{1}{s} & 0 & 0 & 0 \\ 0 & \frac{1}{s} & 0 & 0 \\ 0 & 0 & \frac{1}{s} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

i.e. $$[x'\ y'\ z'\ 1] = [x\ y\ z\ 1]\begin{bmatrix} \frac{1}{s} & 0 & 0 & 0 \\ 0 & \frac{1}{s} & 0 & 0 \\ 0 & 0 & \frac{1}{s} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \left[\frac{x}{s}\ \frac{y}{s}\ \frac{z}{s}\ 1\right]$$

$$\Rightarrow \quad x' = \frac{x}{s},\ y' = \frac{y}{s},\ z' = \frac{z}{s}$$

Illustration : The homogeneous position vector matrix for the unit cube is

$$[X] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

If we want the size of the cube four times the unit cube, the transformation matrix for it will be

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

Applying [T] on [X] = [x y z 1], we get [X']

∴ [X'] = [x' y' z' 1] = [X] [T]

$$\Rightarrow \quad [x'\ y'\ z'\ 1] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & \frac{1}{4} \\ 1 & 0 & 1 & \frac{1}{4} \\ 1 & 1 & 1 & \frac{1}{4} \\ 0 & 1 & 1 & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{4} \\ 1 & 0 & 0 & \frac{1}{4} \\ 1 & 1 & 0 & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{4} \end{bmatrix}$$

The homogeneous co-ordinate factor for each of the transformed position vectors is $h = \frac{1}{4}$.

To obtain the ordinary co-ordinates in three dimensional space, each position vector should be multiplied by 4.

Thus, $$[X'] = [x'\ y'\ z'\ 1] = \begin{bmatrix} 0 & 0 & 4 & 1 \\ 4 & 0 & 4 & 1 \\ 4 & 4 & 4 & 1 \\ 0 & 4 & 4 & 1 \\ 0 & 0 & 0 & 1 \\ 4 & 0 & 0 & 1 \\ 4 & 4 & 0 & 1 \\ 0 & 4 & 0 & 1 \end{bmatrix}$$

It should be noted here that, in transformation matrix

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s \end{bmatrix}$$

If $s < 1$, we get uniform expansion of the position vectors, and if $s > 1$, we get uniform compression of the position vectors.

2.3 Shearing

We have generalized 4×4 transformation matrix as

$$\begin{bmatrix} a & b & c & p \\ d & e & f & q \\ g & i & j & r \\ l & m & n & s \end{bmatrix}$$

Putting $a = e = j = s = 1$, and $l = m = n = p = q = r = 0$, the above transformation can be written as

$$[T] = \begin{bmatrix} 1 & b & c & 0 \\ d & 1 & f & 0 \\ g & i & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, applying [T] on $[X] = [x\ y\ z\ 1]$, we get

$$[X'] = [x'\ y'\ z'\ 1] = [X]\,[T]$$

$$\Rightarrow \quad [X'] = [x\ y\ z\ z] \begin{bmatrix} 1 & b & c & 0 \\ d & 1 & f & 0 \\ g & i & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad [x'] = [x + dy + gz,\ y + bx + iz,\ z + cx + fy]$$

$$\Rightarrow \quad x' = x + dy + gz,\ y' = y + bx + iz,\ z' = z + cx + fy$$

Thus, the non-diagonal terms in the upper left 3×3 sub-matrix of the generalized 4×4 transformation matrix produce *shearing effect* in three dimensional space.

Illustrative Examples

Example 2.1 : Write the transformation matrix for each of the following transformations and apply it on unit cube.

(i) Shear in y-co-ordinate proportional to z-co-ordinate by factor 4.

(ii) Shear in z-co-ordinate proportional to x-co-ordinate and y-co-ordinate by factors 3 and 2 respectively.

Solution : (i) The transformation matrix [T] for shearing in y-co-ordinate proportional to z-co-ordinate by factor 4 is

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying [T] on unit cube, we get,

$$[X'] = [X]\,[T] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 1 & 1 \\ 1 & 4 & 1 & 1 \\ 1 & 5 & 1 & 1 \\ 0 & 5 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

(ii) The transformation matrix [T] for shearing in z-co-ordinate proportional to x-co-ordinate and y-co-ordinate by factors 3 and 2 respectively is;

$$[T] = \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying [T] on unit cube, we get

$$[X'] = [X]\,[T] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 4 & 1 \\ 1 & 1 & 6 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 3 & 1 \\ 1 & 1 & 5 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

2.4 Rotation

Rotation in three dimensions is more complex than rotation in two dimensions. It requires the prescription of an angle of rotation and axis of rotation.

The positive rotation angles produce counter clockwise rotations about a co-ordinate axis, if we are looking along the positive half of the axis towards the origin.

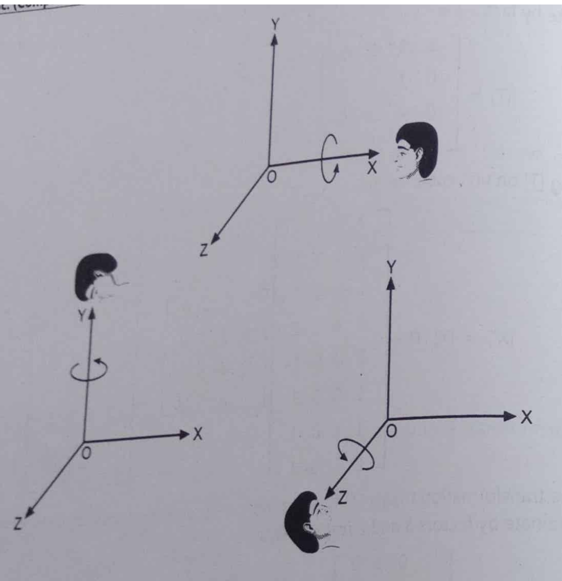


Fig. 2.1

In three dimensions, it is helpful to devise transformations for rotation about each of the three co-ordinate axes.

Rotation about x-axis through an angle θ :

$$x' = x$$
$$y' = y\cos\theta - z\sin\theta$$
$$z' = y\sin\theta + z\cos\theta$$

which can be written in homogeneous co-ordinate form as

$$[x'\ y'\ z'\ 1] = [x\ y\ z\ 1]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{... (1)}$$

Rotation about y-axis through an angle θ :

$$z' = z\cos\theta - x\sin\theta$$
$$x' = z\sin\theta + x\cos\theta$$
$$y' = y$$

ich can be written in homogeneous co-ordinate form as

$$[x'\ y'\ z'\ 1] = [x\ y\ z\ 1]\begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{... (2)}$$

Rotation about z-axis through an angle θ :

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$
$$z' = z$$

which can be written in homogeneous co-ordinate form as

$$[x'\ y'\ z'\ 1] = [x\ y\ z\ 1]\begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{... (3)}$$

It should be noted that the transformation equations for rotations about the co-ordinate axes are obtained with a cyclic permutation of the co-ordinate parameters x, y and z i.e. we use the replacements.

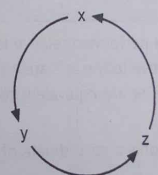


Fig. 2.2

It should be further noted that the three-dimensional rotations are obtained by using matrix multiplication and we know that the matrix multiplication is not commutative. Therefore, the order of multiplication affects the final result.

Illustrative Example

Example 2.2 : Write the transformation matrix for each of the following :

(i) Rotation about x-axis through an angle 50°

(ii) Rotation about z-axis through an angle – 25°.

Solution : (i) Rotation about x-axis through an angle $\theta = 50°$:

From equation (1), we have

$$T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 50° & \sin 50° & 0 \\ 0 & -\sin 50° & \cos 50° & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad [T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.6428 & 0.7660 & 0 \\ 0 & -0.7660 & 0.6428 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(ii) Rotation about z-axis through an angle $\theta = -25°$:

From equation (3), we have

$$[T] = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad [T] = \begin{bmatrix} 0.9063 & -0.4663 & 0 & 0 \\ 0.4663 & 0.9063 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.5 Reflection

Reflection in three dimension can be performed relative to a selected axis or with respect to a selected reflection plane. Reflections relative to the axis are equivalent to 180° rotations about that axis and reflections with respect to a plane are equivalent to 180° rotations in four-dimensional space.

In a reflection through the XY-plane, only z-co-ordinate of the position vector changes in sign. Thus, the transformation matrix for a reflection through the XY-plane is

$$[R_f] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformation matrix for a reflection through the YZ-plane is

$$[R_f] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformation matrix for a reflection through the ZX-plane is

$$[R_f] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 2.3 : Reflect the following object [X] through the YZ-plane.

$$[X] = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 \\ 2 & 0 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 \end{bmatrix}$$

Solution : Here the transformation matrix for reflection through YZ-plane is

$$[R_f] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ Applying [T] on [X], we get

$$[X'] = [X]\,[T] = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 2 & 1 \\ 2 & 0 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 1 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & 1 \\ -2 & 0 & 1 & 1 \\ -2 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & 0 & 2 & 1 \\ -2 & 0 & 2 & 1 \\ -2 & 1 & 2 & 1 \\ -1 & 1 & 2 & 1 \end{bmatrix}$$

2.6 Translation

We know that in the generalized 4×4 transformation matrix, the lower left 1×3 sub-matrix produces *translation*. Let the three-dimensional translation matrix be

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ l & m & n & 1 \end{bmatrix}$$

Applying [T] on [X] = [x, y, z, 1] we get

$$[X'] = [x'\ y'\ z'\ 1] = [X]\,[T]$$

$$\Rightarrow \quad [x'\ y'\ z'\ 1] = [x\ y\ z\ 1] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ l & m & n & 1 \end{bmatrix}$$

$\Rightarrow \quad [x'\ y'\ z'\ 1] = [x + l \quad y + m \quad z + n \quad 1]$

$$\Rightarrow \quad x' = x + l$$
$$y' = y + m$$
$$z' = z + n$$

Thus, the elements l, m, n produce translation in x, y, z co-ordinates respectively.

Illustrative Example

Example 2.3 : Obtain the transformation matrix for translation in x-co-ordinate by 2 units, in y-co-ordinate by – 7 units and in z-co-ordinate by 11 units. Also obtain the translated homogeneous co-ordinates.

Solution : Here,

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & -7 & 11 & 1 \end{bmatrix}$$

Applying [T] on [X], we get

$$[X'] = [X]\,[T]$$

$$\Rightarrow \quad [x'\ y'\ z'\ 1] = [x\ y\ z\ 1]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & -7 & 11 & 1 \end{bmatrix}$$

$$\Rightarrow \quad [x'\ y'\ z'\ 1] = [x + 2 \quad y - 7 \quad z + 11 \quad 1]$$

$$\Rightarrow \quad x' = x + 2,\ y' = y - 7,\ z' = z + 11$$

which are the translated homogeneous co-ordinates.

2.7 Multiple Transformations

If we apply the transformation matrices $[T_1]$, $[T_2]$, $[T_3]$,... on [X] successively, we get

$$[X'] = [X]\,[T] = [X]\,[T_1]\,[T_2]\,[T_3] \ldots$$

Alternatively, successive transformation matrices can be combined or concatenated in a single 4×4 transformation matrix. i.e. $[T] = [T_1]\,[T_2]\,[T_3] \ldots$ which can be applied on [X] to produce [X']. It should be noted that the order in the matrix multiplication should be same as the order in the application of successive transformation matrices.

Illustrative Examples

Example 2.4 : Find the concatenated transformation matrix for successive transformations.

(i) Reflection in XY-plane.

(ii) Translate in x, y, z direction by 4, 2, 1 units respectively.

(iii) Shearing in x co-ordinate by a factor – 4 units proportional to z-co-ordinate.

Solution : (i) The transformation matrix for reflection in XY-plane is

$$[R_f] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(ii) The transformation matrix for translation in x, y, z directions by 4, 2, 1 units respectively is

$$[T_2] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4 & 2 & 1 & 1 \end{bmatrix}$$

(iii) The transformation matrix for shearing in x-co-ordinate by a factor – 4 units proportional to z-co-ordinate is

$$[T_3] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The concatenated transformation matrix is

$$[T] = [R_f]\,[T_2]\,[T_3]$$

$$\Rightarrow \quad [T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 4 & 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad [T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 4 & 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -4 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad [T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 2 & 1 & 1 \end{bmatrix}$$

Example 2.5 : Find the concatenated transformation matrix for the rotation about x-axis through 90° followed by the rotation about z-axis through 180°. Apply the resulting concatenated transformation matrix on the position vector [2 2 1].

Solution : (i) Rotation about x-axis through θ = 90° :

$$[T_1] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(ii) Rotation about z-axis through θ = 180° :

$$[T_2] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore, the concatenated transformation matrix is

$$[T] = [T_1]\,[T_2]$$

$$\Rightarrow \quad [T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad [T] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \ldots (1)$$

Applying [T] on the homogeneous co-ordinate position vector [X] = [2 2 1 1], we get

$$[X'] = [X]\,[T]$$

$$= [2\ 2\ 1\ 1]\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = [-2\ \ 1\ \ 2\ \ 1]$$

Note : Here.

$$[T_2]\,[T_1] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \ldots (2)$$

∴ (1) and (2) ⇒ $[T_1]\,[T_2] \neq [T_2]\,[T_1]$

Example 2.6 : Obtain the concatenated matrix representation of the following transformations in order in a three space.

(i) Translation in x, y and z directions by – 1, 2, 1 units respectively.

(ii) Rotation about z-axis by 90°.

(iii) Reflection in XY plane.

Hence find the transformed position vector of the point A [3 2 1].

Solution : The concatenated transformation matrix is

$$[T] = [T_r]\,[R_z]\,[R_f]$$

where, T_r = Translation matrix

R_z = Rotation matrix

R_f = Reflection matrix

$$\therefore \quad [T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad [T] = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -2 & -1 & -1 & 1 \end{bmatrix}$$

Applying [T] on A [3 2 1], we get

$$[A'] = [A]\,[T] = [3\ \ 2\ 1\ \ 1]\begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -2 & -1 & -1 & 1 \end{bmatrix}$$

$$\Rightarrow \quad [A'] = [-4\ \ 2\ -2\ \ 1]$$

Example 2.7 : Obtain the concatenated transformation matrix for the following sequence of transformation :

(i) Scaling in x and z co-ordinates by factors 2 and 3 respectively.

(ii) Reflection through YZ-plane.

Solution : Here, the concatenated transformation matrix [T] is given by

$$[T] = [S]\,[R_f]$$

where, S = Scaling matrix, R_f = Reflection matrix

$$\therefore \quad [T] = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad [T] = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 2.8 : Find the concatenated transformation matrix for rotation about x-axis through an angle 60° followed by a rotation about y-axis through an angle 90°. Apply this on [X], where,

$$[X] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 2 & 3 & 1 & 1 \end{bmatrix}$$

Solution : Concatenated transformation matrix [T] is

$$[T] = [R_1]\,[R_2]$$

where, R_1 = Rotation matrix about x-axis

R_2 = Rotation matrix about y-axis

$$\therefore \quad [T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 60° & \sin 60° & 0 \\ 0 & -\sin 60° & \cos 60° & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 90° & 0 & -\sin 90° & 0 \\ 0 & 1 & 0 & 0 \\ \sin 90° & 0 & \cos 90° & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad [T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.5 & 8.66 & 0 \\ 0 & -0.866 & 0.5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad [T] = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0.866 & 0.5 & 0 & 0 \\ 0.5 & -0.866 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying [T] on [X], we get transformed position vectors [X']

i.e. $[X'] = [X]\,[T]$

$$\Rightarrow \quad [X'] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 2 & 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0.866 & 0.5 & 0 & 0 \\ 0.5 & -0.866 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad [X'] = \begin{bmatrix} 0.5 & -0.866 & 0 & 1 \\ 0.5 & -0.866 & -2 & 1 \\ 3.098 & 0.634 & -2 & 1 \end{bmatrix}$$

2.8 Rotation about an Axis Parallel to Co-ordinate Axis

In the article 2.4 we have discussed the rotation about the x, y and z axes and obtained the transformation matrices about these axes. Now, we have to discuss the rotation of an object about a line parallel to co-ordinate axis.

Consider the original frame of reference in three dimensional space. We call it a fixed global axis system. Now considering any point in this space, draw lines parallel to – x, – y, – z axes. These three lines constitute the *local axis system.*

When an object is to be rotated about an axis that is parallel to one of the co-ordinate axes, we can attain the rotation with the following transformation sequence :

1. Translate the object so that the rotation axis (local axis) coincides with the parallel co-ordinate axis (fixed global axis) (matrix T_r).
2. Perform the specified rotation about that axis (i.e. rotation about the respective global axis through an angle θ) (matrix R).
3. Apply inverse translation (i.e. translate the object so that the rotation axis is moved back to its original position.

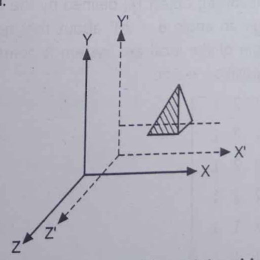


Fig. 2.3 : Original position of the object

Thus, the concatenated transformation matrix will be

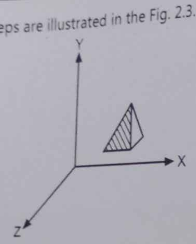
$$[T] = [T_r]\,[R]\,[T_r]^{-1}$$

where, $[T_r]$ ≡ Translation matrix

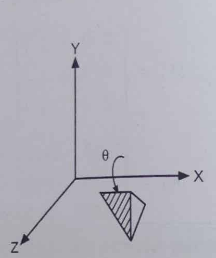
$[R]$ ≡ Rotation matrix

$[T_r]^{-1}$ ≡ Inverse of the translation matrix.

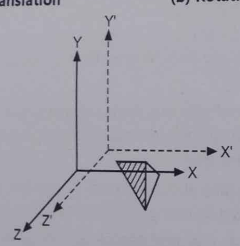
These steps are illustrated in the Fig. 2.3.



(a) Translation



(b) Rotation through (θ)



(c) Translation to original position

Fig. 2.4

Illustrative Examples

Example 2.9 : Rotate the following object [X] defined by the position vectors (relative to the global xyz-axis system) through an angle θ = 30° about the local x-axis passing through the centroid of the block. The origin of the local axis system is assumed to be the centroid of the block. Obtain the transformed position vector.

$$[X] = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix}$$

Solution : The centroid of the given object [X] is

$$(x_c, y_c, z_c) = \left(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}\right)$$

Here, the concatenated transformation matrix is

$$[T] = [T_r]\,[R]\,[T_r]^{-1} \quad \text{... (1)}$$

where,

$$[T_r] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -y_c & -z_c & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{3}{2} & -\frac{3}{2} & 1 \end{bmatrix}$$

$$[R] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.866 & -0.5 & 0 \\ 0 & 0.5 & 0.866 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[T_r]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & y_c & z_c & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{3}{2} & \frac{3}{2} & 1 \end{bmatrix}$$

$$\therefore (1) \Rightarrow [T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{3}{2} & -\frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.866 & 0.5 & 0 \\ 0 & -0.5 & 0.866 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{3}{2} & \frac{3}{2} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.866 & 0.5 & 0 \\ 0 & -0.5 & 0.866 & 0 \\ 0 & 0.951 & -0.549 & 1 \end{bmatrix}$$

Applying [T] on [X], we get

$$[X'] = [X]\,[T]$$

$$\Rightarrow \quad [X'] = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 2 & 2 & 1 & 1 \\ 1 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.866 & 0.5 & 0 \\ 0 & -0.5 & 0.866 & 0 \\ 0 & 0.951 & -0.549 & 1 \end{bmatrix}$$

$$\Rightarrow \quad [X'] = \begin{bmatrix} 1 & 0.817 & 1.683 & 1 \\ 2 & 0.817 & 1.683 & 1 \\ 2 & 1.683 & 2.183 & 1 \\ 1 & 1.683 & 2.183 & 1 \\ 1 & 1.317 & 0.817 & 1 \\ 2 & 1.317 & 0.817 & 1 \\ 2 & 2.183 & 1.317 & 1 \\ 1 & 2.183 & 1.317 & 1 \end{bmatrix}$$

Example 2.10 : Obtain transformed position vectors of the vertices of Δ ABC, when Δ ABC is rotated through an angle 90° about the local x-axis passing through A [– 1 2 2 1], B [2 1 2 1], C [2 3 2 1].

Solution : Position vector of centroid of a Δ ABC in homogeneous co-ordinate system is

$$[x_C \; y_C \; z_C \; 1] = \left[\frac{-1+2+2}{3} \; \frac{2+1+3}{3} \; \frac{2+2+2}{3} \; 1\right]$$

$$= [1 \; 2 \; 2 \; 1]$$

The concatenated transformation matrix [T] is

$$[T] = [T_r]\,[R\,]\,[T_r]^{-1} \qquad \text{... (1)}$$

where, $[T_r]$ = Translation matrix

$[R]$ = Rotation matrix

$[T_r]^{-1}$ = Inverse of translation matrix

$$(1) \Rightarrow \quad [T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 \end{bmatrix}$$

$$\Rightarrow \quad [T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 \end{bmatrix}$$

$$\Rightarrow \quad [T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 4 & 0 & 1 \end{bmatrix}$$

Applying [T] on [X], we get

$$[X'] = \begin{bmatrix} A' \\ B' \\ C' \end{bmatrix} = [X]\,[T]$$

$$\Rightarrow \quad [X'] = \begin{bmatrix} -1 & 2 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 2 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 4 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} A' \\ B' \\ C' \end{bmatrix} = \begin{bmatrix} -1 & 2 & 2 & 1 \\ 2 & 2 & 1 & 1 \\ 2 & 2 & 3 & 1 \end{bmatrix}$$

which are the transformed position vectors of Δ ABC.

2.9 Rotation about an Arbitrary Axis in Space

When an object is to be rotated about an axis which is not parallel to x-axis, y-axis and z-axis, we have to apply some additional transformations. In this case, we also need rotations to align the axis with a selected co-ordinate axis and to bring the axis back to its original orientation. Given the specifications for the rotation of arbitrary axis and the rotation angle, we can accomplish the required rotation about an arbitrary axis in the following five steps :

(We can transform the rotation axis on to any of the three co-ordinate axes. Here the choice of z-axis is arbitrary.)

Step 1 : Translate the object so that rotation axis passes through the origin of the co-ordinate system.

Step 2 : Rotate the object so that the axis of rotation coincides with one of the co-ordinate axes (say z-axis).

Step 3 : Perform the specified rotation (say through angle δ) about z-axis.

Step 4 : Apply inverse rotations to bring the rotation axis back to its original orientation.

Step 5 : Apply the inverse translation to bring the rotation axis back to its original positions.

Above five transformations for obtaining a combined matrix for rotation about an arbitrary axis, with the rotation axis projected on to the z-axis are shown in the following figures.

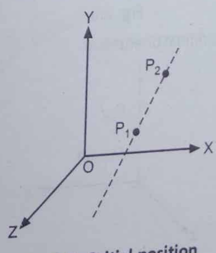


Fig. 2.5 : Initial position

Step 1 : *Translate P_1 to the origin*

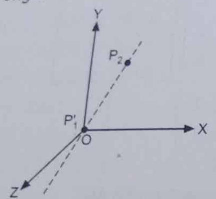


Fig. 2.6

Step 2 : *Rotate P'_2 for the z-axis*

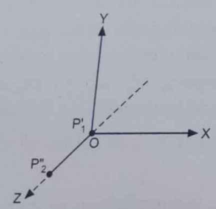


Fig. 2.7

Step 3 : Rotate the object around the z-axis

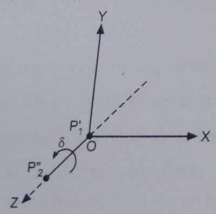


Fig. 2.8

Step 4 : Rotate the axis to the original orientation.

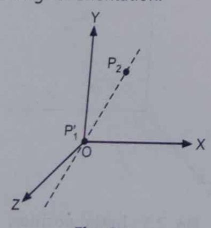


Fig. 2.9

Step 5 : Translate the rotation axis to the original position.

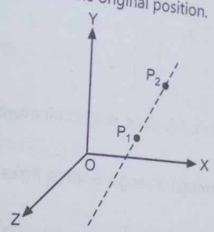


Fig. 2.10

For making an arbitrary axis passing through the origin coincides with one of the co-ordinate axes requires two rotations about the remaining two co-ordinate axes.

Procedure : Make the arbitrary rotation axis coincide with the z-axis, first rotate about x-axis and then about y-axis. For the determination of the rotation angle α about x-axis, we have to place the arbitrary axis in XZ-plane, project the unit vector along the axis on YZ-plane.

Let c_x, c_y, c_z be the direction cosines of the line OP passing through the origin. The position vector of the point P will be $[c_x, c_y, c_z]$.

Also $\quad |OP| = \sqrt{c_x^2 + c_y^2 + c_z^2} = 1 \qquad (\because l^2 + m^2 + n^2 = 1)$

$|OM| = c_y, |LM| = c_x, |MQ| = c_z$

Let $\quad |OQ| = d$

In right angled Δ OQR,

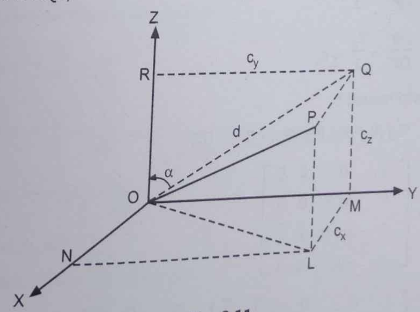


Fig. 2.11

$$OQ^2 = OR^2 + RQ^2 = OM^2 + RQ^2$$

$$\Rightarrow \quad d^2 = c_y^2 + c_z^2$$

$$\Rightarrow \quad d = \sqrt{c_y^2 + c_z^2}$$

Again $\cos \alpha = \frac{OR}{OQ} = \frac{OM}{OQ} = \frac{c_z}{d}$

and $\sin \alpha = \frac{RQ}{OQ} = \frac{c_y}{d}$

After rotation about the x-axis into XZ-plane, the z component of the unit vector is d and the x component is c_x.

Now, we rotate about y-axis through an angle β which make the arbitrary axis coincide with z-axis.

In right angled Δ ONP',

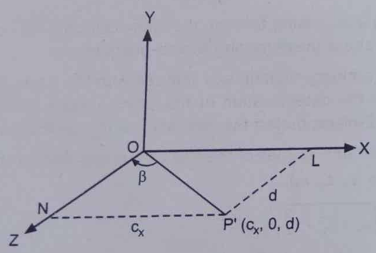


Fig. 2.12

$$\cos \beta = \frac{ON}{OP'} = \frac{LP'}{1} = d \qquad (\because |OP'| = 1)$$

and $$\sin \beta = \frac{NP'}{OP'} = \frac{c_x}{1} = c_x$$

The combined transformation is

$$[M] = [T_r]\,[R_x]\,[R_y]\,[R_\delta]\,[R_y]^{-1}\,[R_x]^{-1}\,[T_r]^{-1}$$

where, $$[T_r] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & 0 & 0 \\ 0 & -\sin\alpha & 1 & 0 \\ -x_0 & -y_0 & -z_0 & 1 \end{bmatrix}$$

$$[R_x] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha & 0 \\ 0 & -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c_z/d & c_y/d & 0 \\ 0 & -c_y/d & c_z/d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[R_y] = \begin{bmatrix} \cos(-\beta) & 0 & -\sin(-\beta) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(-\beta) & 0 & \cos(-\beta) & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} d & 0 & c_x & 0 \\ 0 & 1 & 0 & 0 \\ -c_x & 0 & d & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[R_\delta] = \begin{bmatrix} \cos\delta & \sin\delta & 0 & 0 \\ -\sin\delta & \cos\delta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Illustrative Examples

Example 2.11 : Consider the line with direction ratios 1, 1, 1 and passing through the origin. Determine angles through which the line should be rotated about x-axis and then about y-axis; so that it coincides with z-axis.

Solution : Direction cosines of the line are

$$(c_x, c_y, c_z) = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$d^2 = c_y^2 + c_z^2$$

$$\Rightarrow \quad d = \sqrt{c_y^2 + c_z^2}$$

$$\Rightarrow \quad d = \sqrt{\frac{1}{3} + \frac{1}{3}} = \sqrt{\frac{2}{3}}$$

Let the line makes an angle α with x-axis.

$$\Rightarrow \quad \cos\alpha = \frac{c_z}{d} = \frac{1}{\sqrt{2}}$$

and $$\sin\alpha = \frac{c_y}{d} = \frac{1}{\sqrt{2}}$$

since $$\cos\alpha = \sin\alpha = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \alpha = 45°$$

Let the line makes an angle β with y-axis.

$$\Rightarrow \quad \cos\beta = d = \sqrt{\frac{2}{3}}$$

$$\beta = \cos^{-1}\sqrt{\frac{2}{3}} = 35.26°$$

Thus, α = 45° and β = 35.26°

Example 2.12 : Consider the line with direction ratios 1, – 2, 2 and passing through the origin. Determine the angles through which the line should be rotated about x-axis and then about y-axis, so that it coincides with z-axis.

Solution : Direction cosines of the line are

$$(c_x, c_y, c_z) = \left(\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)$$

$$d = \sqrt{c_y^2 + c_z^2} = \sqrt{\frac{4}{9} + \frac{4}{9}} = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$$

Let the line makes an angle α with x-axis.

$$\Rightarrow \quad \cos\alpha = \frac{c_z}{d} = \frac{2}{3} \cdot \frac{3}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\sin\alpha = \frac{c_y}{d} = \left(-\frac{2}{3}\right) \cdot \frac{3}{2\sqrt{2}} = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \alpha = -45°$$

Again, let the line makes an angle β with y-axis.

$$\Rightarrow \quad \cos\beta = d = \frac{2\sqrt{2}}{3}$$

$$\Rightarrow \quad \beta = \cos^{-1}\left(\frac{2\sqrt{2}}{3}\right)$$

$$\Rightarrow \quad \beta = 19.47°$$

Thus, $\alpha = -45°$ and $\beta = 19.47°$

2.10 Reflection through an Arbitrary Plane

In article 2.5 we have discussed the reflection of an object through the co-ordinate planes. But it may be required to reflect an object through the plane other than the co-ordinate plane.

The procedure for getting the concatenated transformation for the reflection through an arbitrary plane is given below.

1. Translate the point (lies in the reflection plane) to the origin of the co-ordinate system.
2. Rotate through an angle α about x-axis, where $\cos\alpha = \frac{c_z}{d}$, $\sin\alpha = \frac{c_y}{d}$.
3. Rotate through an angle $-\beta$ about y-axis, where $\cos\beta = d$, $\sin\beta = c_x$.
4. Reflect the object through the plane z = 0
5. Perform inverse rotation about y-axis.
6. Perform inverse rotation about x-axis.
7. Perform inverse translation.

Using the above steps successively, the concatenated transformation can be written as

$$[M] = [T_r]\,[R_x]\,[R_y]\,[R_f]\,[R_y]^{-1}\,[R_x]^{-1}\,[T_r]^{-1}$$

where the matrices $[T_r]$, $[R_x]$ $[R_y]$ are given in the previous article 2.9.

Illustrative Examples

Example 2.13 : It is required to reflect in the face ABC of the pyramid OABC given by O [0 0 0], A [3 1.5 2], B [2.5 2 2], C [3 2 1.5]. Determine the transformation matrix for the translation and rotation so that the required face coincides with the XY-plane.

Solution : For the translation choose the point A to the origin which gives us translation matrix

$$[T_r] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & -\frac{3}{2} & -2 & 1 \end{bmatrix}$$

The normal $\bar{n}$ to the reflection plane containing A, B, C is

$$\bar{n} = \overline{AC} \times \overline{AB} = (\bar{C} - \bar{A}) \times (\bar{B} - \bar{A})$$

$$= \begin{bmatrix} i & j & k \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

$$= \frac{1}{4}\,i + \frac{1}{4}\,j + \frac{1}{4}\,k$$

$$= \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

Normalizing gives direction cosines $c_x\ c_y\ c_z$ as

$$[c_x, c_y, c_z] = \left[\frac{\frac{1}{4}}{\sqrt{\frac{1}{16} + \frac{1}{16} + \frac{1}{16}}} \quad \frac{\frac{1}{4}}{\sqrt{\frac{1}{16} + \frac{1}{16} + \frac{1}{16}}} \quad \frac{\frac{1}{4}}{\sqrt{\frac{1}{16} + \frac{1}{16} + \frac{1}{16}}}\right]$$

$$= \left[\frac{1}{\sqrt{3}} \; \frac{1}{\sqrt{3}} \; \frac{1}{\sqrt{3}}\right]$$

$$d = \sqrt{c_y^2 + c_z^2} = \sqrt{\frac{2}{3}}$$

$$\cos\alpha = \frac{c_z}{d} = \frac{\frac{1}{\sqrt{3}}}{\sqrt{\frac{2}{3}}} = \frac{1}{\sqrt{2}}$$

$$\sin\alpha = \frac{c_y}{d} = \frac{1}{\sqrt{2}}$$

$$\cos\beta = d = \sqrt{\frac{2}{3}}$$

$$\sin\beta = c_x = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \quad \alpha = 45°, \ \beta = 35.26°$$

From the previous article 2.9, we have

$$[R_x] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[R_y] = \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} & 0 \\ 0 & 1 & 0 & 0 \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad [T_r]\,[R_x]\,[R_y] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & -3/2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} & 0 \\ 0 & 1 & 0 & 0 \\ -1/\sqrt{3} & 0 & 2/\sqrt{6} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} & 0 \\ -5/2\sqrt{6} & 1/2\sqrt{2} & -13/2\sqrt{3} & 1 \end{bmatrix} \quad \ldots (1)$$

Reflection through the arbitrary plane is now given by reflection through the z = 0 plane (see article 2.5).

$$\therefore \quad [R_f] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \ldots (2)$$

$$[R_y]^{-1}\,[R_x]^{-1}\,[T_r]^{-1} = \begin{bmatrix} 2/\sqrt{6} & -1/\sqrt{6} & -1/\sqrt{6} & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} & 0 \\ 3 & 3/2 & 2 & 1 \end{bmatrix} \quad \ldots (3)$$

$$\therefore \quad [M] = [T_r]\,[R_x]\,[R_y]\,[R_f]\,[R_y]^{-1}\,[R_x]^{-1}\,[T_r]^{-1}$$

$$= \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} & 0 \\ -5/2\sqrt{6} & 1/2\sqrt{2} & -13/2\sqrt{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} & 0 \\ 3 & 3/2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & -2/3 & -2/3 & 0 \\ -2/3 & 1/3 & -2/3 & 0 \\ -2/3 & -2/3 & 1/3 & 0 \\ 13/3 & 13/3 & 13/3 & 1 \end{bmatrix}$$

Example 2.14 : Rotate the line segment joining A [1 2 3], B [4 5 6] about y-axis through 30°.

Solution : $\theta = 30°$, $\cos 30° = 0.8660$, $\sin 30° = 0.5$

$$[R_y] = \begin{bmatrix} 0.8660 & 0 & -0.5 & 0 \\ 0 & 1 & 0 & 0 \\ 0.5 & 0 & 0.8660 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[X] = \begin{bmatrix} A \\ B \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 1 \end{bmatrix}$$

$$[X'] = [X]\,[R_y] = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 1 \end{bmatrix} \begin{bmatrix} 0.8660 & 0 & -0.5 & 0 \\ 0 & 1 & 0 & 0 \\ 0.5 & 0 & 0.8660 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2.368 & 2 & 2.098 & 1 \\ 6.464 & 5 & 3.196 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} A' \\ B' \end{bmatrix} = \begin{bmatrix} 2.368 & 2 & 2.098 \\ 6.464 & 5 & 3.196 \end{bmatrix}$$

Example 2.15 : Rotate the line segment AB, through where A [1 2 4], B [2 2 1] about the local x-axis passes through the point P [2 3 4] through an angle 65°.

Solution : Here

$$[X] = \begin{bmatrix} 1 & 2 & 4 & 1 \\ 2 & 2 & 1 & 1 \end{bmatrix}$$

$$T_r = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & -4 & 1 \end{bmatrix}$$

$\cos 65° = 0.4226$, $\sin 65° = 0.9063$

$$[R_x] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.4226 & 0.9063 & 0 \\ 0 & -0.9063 & 0.4226 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[T] = [T_r]\,[R_x]\,[T_r]^{-1}$$

$$\Rightarrow \quad [T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.4226 & 0.9063 & 0 \\ 0 & -0.9063 & 0.4226 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 3 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.4226 & 0.9063 & 0 \\ 0 & -0.9063 & 0.4226 & 0 \\ 0 & 2.3574 & -4.4093 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 3 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.4226 & 0.9063 & 0 \\ 0 & -0.9063 & 0.4226 & 0 \\ 0 & 2.3574 & -4.4093 & 1 \end{bmatrix}$$

Now, $[X'] = [X]\,[T]$

$$= \begin{bmatrix} 1 & 2 & 4 & 1 \\ 2 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.4226 & 0.9063 & 0 \\ 0 & -0.9063 & 0.4226 & 0 \\ 0 & 2.3574 & -4.4093 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2.5779 & 3.0937 & 1 \\ 2 & 5.2968 & 1.8259 & 1 \end{bmatrix}$$

$$\Rightarrow \quad A' = [\,1 \;\; 2.5779 \;\; 3.0937\,]$$

$$B' = [\,2 \;\; 5.2968 \;\; 1.8259\,]$$

Example 2.16 : Reflect the pyramid OABC with O [0 0 0], A [1 1 0], B [0 1 0], C [0 0 1] in the plane x = 3.

Solution : For reflection, combined transformation matrix [T] is

$$[T] = [T_r]\,[R_f]\,[T_r]^{-1}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 6 & 0 & 0 & 1 \end{bmatrix}$$

Example 2.17 : Apply shearing transformation

$$[T] = \begin{bmatrix} 1 & -0.85 & 0.25 & 0 \\ -0.75 & 1 & 0.7 & 0 \\ 0.5 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ on the unit cube.}$$

Solution : Applying [T] on unit cube, we get

$$[X'] = [X]\,[T]$$

$$= \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -0.85 & 0.25 & 0 \\ -0.75 & 1 & 0.7 & 0 \\ 0.5 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 1 & 1 & 1 \\ 1.5 & 0.15 & 1.25 & 1 \\ 0.75 & 1.15 & 1.95 & 1 \\ -0.25 & 2 & 1.7 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & -0.85 & 0.25 & 1 \\ 0.25 & 0.15 & 0.95 & 1 \\ -0.75 & 1 & 0.7 & 1 \end{bmatrix}$$

Example 2.18 : Write the transformation matrix for each of the following :

(i) Rotation about x-axis through $\theta = -90°$.

(ii) Rotation about y-axis through $\theta = 90°$.

Apply each transformation on the rectangular parallelopiped whose position vector matrix is

$$[X] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 3 & 0 & 1 & 1 \\ 3 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 \\ 3 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

Solution : (i) Transformation matrix for rotation about x-axis through $\theta = -90°$ is

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying [T] on [X], we get

$$[X'] = [X]\,[T] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 3 & 0 & 1 & 1 \\ 3 & 2 & 1 & 1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 \\ 3 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 3 & 1 & 0 & 1 \\ 3 & 1 & -2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 \\ 3 & 0 & -2 & 1 \\ 0 & 0 & -2 & 1 \end{bmatrix}$$

(ii) Transformation matrix for rotation about y-axis through $\theta = 90°$ is

$$[T] = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying [T] on [X], we get

$$[X'] = [X]\,[T] = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 3 & 1 & 0 & 1 \\ 3 & 1 & -2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 \\ 3 & 0 & -2 & 1 \\ 0 & 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & -3 & 1 \\ 1 & 2 & -3 & 1 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -3 & 1 \\ 0 & 2 & -3 & 1 \\ 0 & 2 & -3 & 1 \end{bmatrix}$$

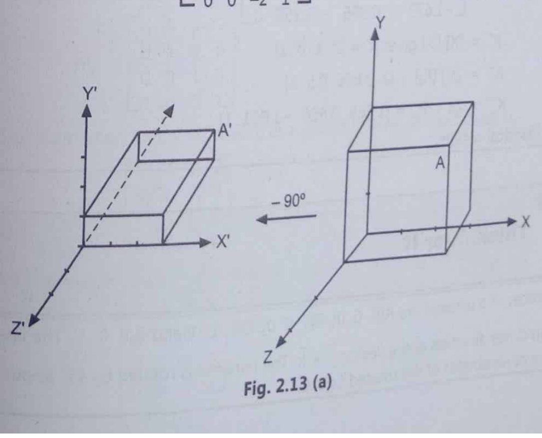


Fig. 2.13 (a)

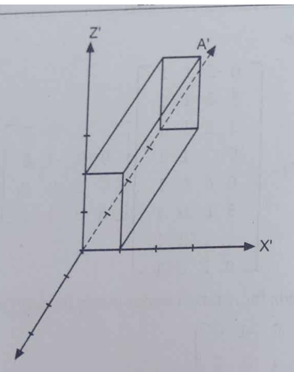


Fig. 2.13 (b)

Example 2.19 : Obtain the concatenated transformation matrix for the translation in x, y, z directions by −1, −1, −1 respectively followed by 30° rotation about x-axis and 45° rotation about y-axis. Apply it [3 2 1 1]. Also show that the concatenated matrix yield the same result as individually applied matrix.

Solution : The concatenated transformation matrix [T] is $[T] = [T_r]\ [T_x]\ [T_y]$

$$\therefore \quad [T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.866 & 0.5 & 0 \\ 0 & -0.5 & 0.66 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.707 & 0 & -0.707 & 0 \\ 0 & 1 & 0 & 0 \\ 0.707 & 0 & 0.707 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.707 & 0 & -0.707 & 0 \\ 0.354 & 0.866 & 0.354 & 0 \\ 0.612 & -0.5 & 0.612 & 0 \\ -1.673 & -0.366 & -0.259 & 1 \end{bmatrix}$$

$X' = [X]\ [T_r]$ gives $X' = [2\ \ 1\ \ 0\ \ 1]$

$X'' = [X']\ [R_x] = [2\ \ 0.866\ \ 0.5\ \ 1]$

$X''' = [X'']\ [R_y] = [1.768\ \ 0.866\ \ -1.061\ \ 1]$

We get identical results.

Think Over It

1. The vertices of a pyramid are A[0 0 0], B[1 0 0], C[0 1 0] and D[0 0 1]. The line L passing through C has direction of the vector $\bar{j} + \bar{k}$. The pyramid is rotated by 45° about the line L. Find the co-ordinates of the rotated figure.

Points to Remember

1. For three dimensional homogeneous co-ordinates the generalized 4 × 4 transformation matrix is given by,

$$[T] = \begin{bmatrix} a & b & c & p \\ d & e & f & q \\ g & i & j & r \\ l & m & n & s \end{bmatrix}$$

The above 4 × 4 transformation matrix can be divided into four parts as : scaling, shearing, reflection, rotation transformation.

$$\text{Transformation}\left[\begin{array}{c|c} \begin{bmatrix} a & b & c \\ d & e & f \\ g & i & j \end{bmatrix}_{3\times 3} & \begin{bmatrix} p \\ q \\ r \end{bmatrix}_{3\times 1} \\ \hline [l\ m\ n]_{1\times 3} & [s] \end{array}\right] \begin{array}{l} \text{Perspective} \\ \\ \text{Overall scaling} \end{array}$$

The upper left, 3 × 3 submatrix gives transformation in the form of scaling, shearing rotation, reflection.

The lower left 1 × 3 submatrix gives translation.

The upper right 3 × 1 submatrix gives pressure transformation.

The lower right 1 × 1 submatrix corresponds to overall scaling.

2. Three dimensional scaling :

(i) Transformation matrix for scaling in y-coordinate by 'e' units is given by

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(ii) Transformation matrix for scaling in x co-ordinate by a units is given by

$$[T] = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(iii) Transformation matrix for scaling in z co-ordinate by j units is given by

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & j & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(iv) Transformation matrix for uniform scaling by 'a' units is given by

$$[T] = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix}$$

It should be noted that, if $0 < a < 1$, then it gives uniform compression of the position vectors and if $a > 1$ then it gives uniform expansion of the position vectors.

(v) Transformation matrix for overall scaling by factor is given by,

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s \end{bmatrix}$$

It should be noted that, if $0 < s < 1$, then it gives uniform expansion of the position vectors and if $s > 1$ then uniform compression of the position vectors.

(vi) Transformation matrix for scaling in x co-ordinate y co-ordinate and z-co-ordinates by factors a, e and j respectively is

$$[T] = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & j & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3. **Three dimensional shearing :**

(i) Transformation matrix for shearing in x-co-ordinate proportional to y and z axes by factor 'd' and 'g' respectively is,

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ d & 1 & 0 & 0 \\ g & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(ii) Transformation matrix for shearing in y-co-ordinate proportional to x and z axes by factor b and i respectively is,

$$[T] = \begin{bmatrix} 1 & b & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & i & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(iii) Transformation matrix for shearing in z-co-ordinate proportional to x and y axes by factors 'c' and 'f' respectively is,

$$[T] = \begin{bmatrix} 1 & 0 & c & 0 \\ 0 & 0 & f & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4. **Three dimensional rotations :**

(i) The transformation matrix for rotation about X-axis through an angle θ is,

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(ii) The transformation matrix for rotation about y-axis through an angle θ is,

$$[T] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ \sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(iii) The transformation matrix for rotation about z-axis through an angle θ is,

$$[T] = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

5. **Three dimensional reflection :**

(i) The transformation matrix for reflection through XOY plane (i.e. z = 0 plane) is,

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(ii) The transformation matrix for reflection through YOZ plane (i.e. x = 0 plane) is,

$$[T] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(iii) The transformation matrix for XOZ plane (i.e. y = 0 plane) is,

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

6. **Three dimensional translation :** The transformation matrix for translation in X, Y, Z direction by units l, m, n respectively is,

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ l & m & n & 1 \end{bmatrix}$$

Miscellaneous Exercise

(A) State whether the following statement is True or False. Justify your answer.

1. The translation matrix for scaling in X, Y, Z co-ordinates by factor 2, −3, 4 units respectively is,

$$[T] = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. The transformation produced by the following transformation matrix,

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

is a scaling in x co-ordinate.

3. The transformation matrix to translate the point P[2 1 −3] to origin is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -2 & 1 & -3 & 1 \end{bmatrix}$.

4. The transformation matrix for scaling x and y co-ordinate by factor $\frac{1}{3}$ and 2 units respectively is,

$$[T] = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

ANSWERS

1. - True	2. - False	3. - True	4. - False

(B) Multiple Choice Questions : Choose the correct alternative from given alternatives.

1. The effect of the 4 × 4 transformation matrix, $[T] = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ on any object in space is ……
   (a) Shearing in y co-ordinate proportional to X and Z axis by factors −1 and 3.
   (b) Shearing in Z co-ordinate proportional to X and Y axis by factor 2 and −1.
   (c) Shearing in y-co-ordinate proportional to x and z axis by factor −1 and 3. Also shearing in z co-ordinate proportional to X and y axis by factor 2 and −1.
   (d) None of the above.

2. The transformation matrix for rotation about x-axis through an angle θ = 45° is ……

(a) $[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(b) $[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(c) $[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(d) None of these

3. The transformation matrix for reflection through the plane z = −3 is ……

(a) $[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(b) $[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -6 & 1 \end{bmatrix}$

(c) $[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -6 & 1 \end{bmatrix}$

(d) none of these

4. The matrix, $[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -5 & 0 & 0 & 1 \end{bmatrix}$ is ……
   (a) the transformation matrix for translation in y direction by −5 units.
   (b) the transformation matrix for translation in z direction by −5 units.
   (c) the transformation matrix for translation in x direction by −5 units.
   (d) none of above.

5. The three dimensional translation matrix [T] is ……

(a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ l & m & n & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 0 & 0 & 0 & l \\ 0 & 1 & 0 & m \\ 0 & 0 & 1 & n \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} l & m & n & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} 1 & 0 & 0 & l \\ 0 & 1 & 0 & m \\ 0 & 0 & 1 & n \\ l & m & n & 1 \end{bmatrix}$

6. The transformation matrix for a reflection through xy plane is ......

(a) $\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

7. The transformation matrix, $[T] = \begin{bmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ corresponds to rotation by an angle ϕ about ......

(a) x-axis (b) y-axis

(c) z-axis (d) none of these

8. For a pure reflection the determinant of the reflection matrix is ......

(a) 1 (b) −1

(c) 0 (d) none of these

Answers

1 - (c)	2 - (a)	3 - (b)	4 - (c)	5 - (a)	6 - (c)
7 - (b)	8 - (b)				

(C) Theory Questions :

1. Obtain three dimensional transformation matrix for :
   (i) Translation (ii) Scaling
   (iii) Shearing (iv) Rotation about an arbitrary axis
   (v) Reflection
2. Explain the rotation about :
   (i) an axis parallel to co-ordinate axis.
   (ii) an arbitrary axis in space.
3. State the various steps for getting the concatenated transformation for the reflection through an arbitrary plane.

(D) Numerical Problems :

1. Find the combined transformation matrix for the following sequence of transformations : Rotation about X-axis by an angle 30°, followed by rotation about Y-axis by an angle 30°, followed by scaling in X and Z co-ordinates by factor of 3 and 2 respectively. Apply it on the point [2 1 3].

2. Obtain the concatenated transformation matrix for the following sequence of transformation : Shearing in x co-ordinate by a factor of 2 units proportional to Z co-ordinate followed by reflection YZ plane.
3. Find the concatenated transformation matrix for rotation about X-axis through an angle 60°, followed by a rotation about y-axis through an angle 90°. Apply it on $[X] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 2 & 0 & 1 & 1 \\ 2 & 3 & 1 & 1 \end{bmatrix}$.
4. Find the concatenated transformation matrix for the following transformations in order : Translate in X, Y, Z directions by −2, −2, −2 units respectively. Rotation about X-axis by angle 45°. Reduce to half its size.
5. What is the effect of 4 × 4 transformation matrix

$$[T] = \begin{bmatrix} 1 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ on any object in space ?

6. The plane x + 2y + 2z = 0 is to be rotated, so that it coincides with the z = 0 plane. Determine the required angles of rotation about the X-axis and Y-axis.
7. Consider the line having direction ratios 1, 1, 1. Determine the angle of rotation about X-axis and then about Y-axis, so that the line coincides with the z-axis.
8. Find the concatenated transformation matrix for the following sequence of transformations : Translate in X, Y and Z directions by −2, −2, −2 units respectively; followed by rotation about X-axis by an angle 45° followed by uniform scaling by $\frac{1}{2}$ units.
9. Obtain the transformed position vectors of the vertices of ΔABC, when ΔABC is rotated through an angle 90° about the local X-axis passing through the centroid of ΔABC; where A[−1 2 2 1], B[2 1 2 1], C[2 3 2 1].
10. A line L has direction ratios 1, −2, 2 and it passes through origin. Determine the angles through which the line should be rotated about Y-axis and then about Z-axis.
11. Find the concatenated transformation matrix for the following sequence of transformations : First translate in X and Z direction by 2 and 1 units respectively, followed by shearing in Y-direction proportional to X and Z coordinate with $\frac{1}{2}$ and 3 units respectively, followed by reflection through XZ plane. Apply it on origin.

Answers

1. $[T] = \begin{bmatrix} 2.598 & 0 & -1 & 0 \\ 0.75 & 0.8660 & 0.8660 & 0 \\ 1.299 & -0.5 & 1.4998 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$P^* = [9.843 \; -0.634 \; 3.3654]$

2. $[T] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

3. $[T] = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0.8660 & 0.5 & 0 & 0 \\ 0.5 & -0.8660 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$[X^*] = \begin{bmatrix} 0.5 & -0.8660 & 0 & 1 \\ 0.5 & -0.8660 & -2 & 1 \\ 3.098 & 0.6340 & -2 & 1 \end{bmatrix}$

4. $[T] = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.3535 & 0.3535 & 0 \\ 0 & -0.3535 & 0.3535 & 0 \\ -1 & 0 & -1.4140 & 1 \end{bmatrix}$

5. Shearing in y co-ordinate proportional to x-co-ordinate by factor −2 and proportional to z co-ordinate by factor 3.

6. 45°, 35.263°

7. $\alpha = 45°$, $\beta = 45°$

8. $\alpha = 45°$, $\beta = \tan^{-1}\left(\frac{1}{2\sqrt{2}}\right)$

9. $[T] = \begin{bmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0.3535 & 0.3535 & 0 \\ 0 & -0.3535 & 0.3535 & 0 \\ -1 & 0 & -1.4140 & 1 \end{bmatrix}$

$A^* = [-1\ \ 2\ \ 2]$, $B^*[2\ \ 2\ \ 1]$, $C[2\ \ 2\ \ 3]$.

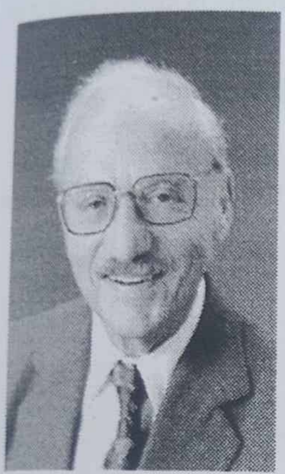
10. $-\frac{\pi}{4}$ and $\tan^{-1}\left(\frac{1}{2\sqrt{2}}\right)$

11. $[T] = \begin{bmatrix} 1 & -1/2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 2 & -4 & 1 & 1 \end{bmatrix}$

✍✍✍

Chapter **3...**

Projection



George David Birkhoff

George Bernard Dantzig, professor emeritus of operations research and of computer science who devised the "simplex method" and invented linear programming (which is not related to computer programming), died May 13 at his Stanford home of complications from diabetes and cardiovascular disease. He was 90 years old. A funeral service has been the mathematicians and held. "George B. Dantzig is regarded by most experts as having been the initiator of and leading figure in the revolutionary scientific development of mathematical programming as a powerful method for optimally managing resources in literally thousands of applications in industry and government in the last three decades," said Arthur F. Veinott Jr., professor of management science and engineering.

"So pervasive is the influence of Dantzig's simplex method that experts have estimated that from 10 percent to 25 percent of all scientific computation is devoted to it. Indeed, that method is probably the single most widely used algorithm originated in the last six decades."

3.1 Introduction

Geometric theorems are developed for affine and perspective geometry. The theorems of affine geometry are identical to the theorems of Euclidean geometry. In Euclidean geometry and affine geometry; parallelism of lines is important. In perspective geometry; the lines are generally non-parallel. An affine transformation is a combination of linear transformations; for example, rotation followed by translation etc. The affine and perspective transformations are both three dimensional transformations. their view in 2-d surface needs a projection from three space to two space.

3.2 Affine and Perspective Geometry

In affine geometry, lines are parallel (i.e. parallelism concept is important) whereas in perspective geometry, lines are not generally parallel. Affine and perspective transformations are three-dimensional transformation i.e. these are transformations from one three-dimensional space to another three-dimensional space.

Affine transformations have the general properties that the parallel lines are transformed into parallel lines and finite points transformed to finite points. Translation, rotation, scaling, shearing and reflection are examples of two-dimensional affine transformations.

Projection : A geometrical transformation in which one line, shape etc. is converted into another according to certain geometrical rule is called *projection*.

Plane Geometric Projection : The result of the projection of an object from three-dimensional space to two-dimensional plane is called the *plane geometric projection*.

The projection matrix obtained from three-dimensional space to two-dimensional plane always contains a zero column.

The plane geometric projections of an object are obtained by the intersection of lines (called *projectors*) with the plane (called the *projection plane*).

Centre of Projection : An arbitrary point from which the lines are projected is called centre of *projection*.

Projector : A ray drawn from an object point to the centre of projection is called a projector.

Two Basic Methods of Projection :

Parallel projection : If the centre of projection is located at infinity then all the projectors are parallel lines. The resulting projection is called the *parallel projection.*

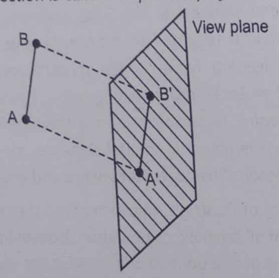


Fig. 3.1 : Parallel projection

Perspective Projection : If the centre of projection is located at finite point in three-dimensional space, the resulting projection is *perspective projection.*

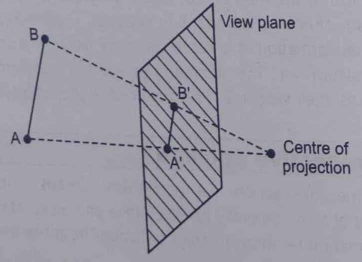
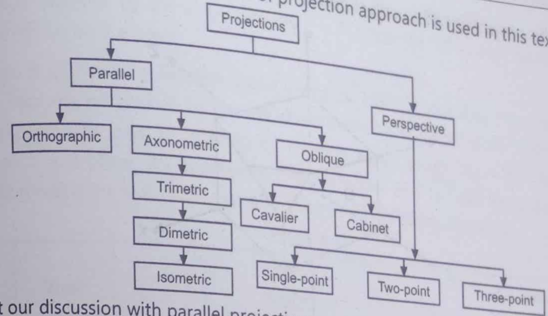


Fig. 3.2 : Perspective projection

Parallel projection methods are used by drafters and engineers to create drawings of an object which preserve its relative proportion. Accurate views of the various sides of an object are obtained by this parallel projection method but this method does not give a realistic representation of the appearance of the three-dimensional object. A perspective projection method produces realistic views but does not preserve relative proportions. The techniques of perspective projection are generalizations of the principles used by artists in the preparation of perspective drawing of an object in three-dimensional space.

The movable object and the fixed centre of projection approach is used in this text.



We start our discussion with parallel projection.

3.3 Orthographic Projections

The projection formed by parallel projectors such that they are perpendicular to the projection plane, is called *orthographic projection.*

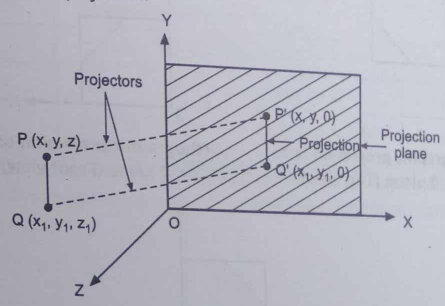


Fig. 3.3

The orthographic projections are projections on one of the co-ordinate planes x = 0 (yz-plane), y = 0 (xz-plane), z = 0 (xy-plane) respectively. If the point is projected onto z = 0 plane, the matrix for the projection onto z = 0 plane is

$$[P_z] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The matrices for the orthographic projections onto x = 0 and y = 0 planes are :

$$[P_x] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } [P_y] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Orthographic projections of an object onto x = 0, y ... planes are shown below.

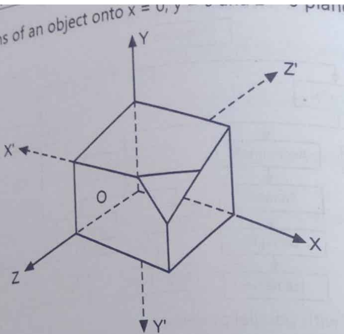
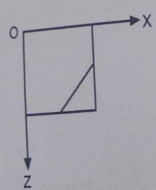
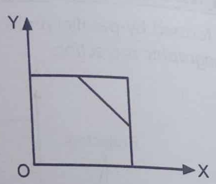


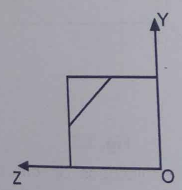
Fig. 3.4 : Object



Orthographic projection onto y = 0 plane (Top view)



Orthographic projection onto z = 0 plane (Front view)



Orthographic projection onto x = 0 plane (Right side view)

Fig. 3.5

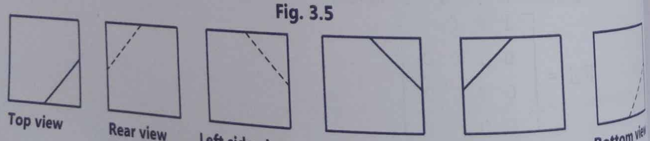


Fig. 3.6

The front view, side view and top view are sometimes called front elevation, side elevation and the plane view. Engineering and architectural drawings generally use orthographic projection because it accurately shows the true size and shape of the single plane face of the object.

It should be noted that all six views are obtained by the combinations of reflection, rotation and translation followed by the projection on z = 0 plane (i.e. XY plane) from the centre of projection at infinity on the z-axis.

Rear view is obtained by rotation about y-axis by 180°, followed by the projection on the plane z = 0. Top view is obtained by rotation about x-axis by + 90°, followed by the projection on z = 0 plane. Left side view is obtained by rotation about y-axis by + 90°, followed by the projection on z = 0 plane. We summarize this procedure and we can write the concatenated transformation matrix for each of the views as given in the following Table 3.1.

Table 3.1

No.	View	Rotation	Concatenated Transformation matrix
1.	Front	–	$T = [P_z]$
2.	Top	Rotation about x-axis through θ = 90°	$T = [R_x]\ [P_z]$
3.	Bottom	Rotation about x-axis through θ = – 90°	$T = [R_x]\ [P_z]$
4.	Left	Rotation about y-axis through θ = 90°	$T = [R_y]\ [P_z]$
5.	Right	Rotation about y-axis through θ = – 90°	$T = [R_y]\ [P_z]$
6.	Rear	Rotation about y-axis through θ = 180°	$T = [R_y]\ [P_z]$

Illustrative Examples

Example 3.1 : Develope the top view for the object whose position vectors are given below :

$$[X] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0.5 & 1 & 1 \\ 0.5 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0.5 & 1 \end{bmatrix}$$

Solution : Here, for top-view, the concatenated transformation matrix is;

$$[T] = [R_x]\ [P_z] \quad (\theta = 90°)$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Applying [T] on [X], we get

$$[X'] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0.5 & 1 & 1 \\ 0.5 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & 1 \\ 0.5 & -1 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & -0.5 & 0 & 1 \end{bmatrix}$$

3.4 Axonometric Projections

Axonometric projections are orthographic projections in which the direction of projection is not parallel to any of the three co-ordinate axes.

It is constructed by manipulating the object :

(i) Using rotations and translations such that atleast three adjacent faces are shown.

(ii) The result is then projected from the centre of projection at infinity, usually onto the z = 0 plane. It should be noted that the axonometric projection does not accurately show its correct size and shape, unless a face is parallel to the plane of the projection.

In an axonometric projection, we first apply rotation or translation which preserve the length of the vectors. Then we apply orthographic projection where all the projectors are perpendicular to the projection plane, hence the lengths of the vectors are shortened. However, in an axonometric projection, the relative lengths of parallel lines remain constant. (i.e. parallel lines are equally foreshortened). First we define the foreshortening factor.

Foreshortening factor : It is the ratio of the projected length of line to its real length. i.e. if P'Q' is the projected length of the real line PQ then foreshortening factor is given by $\frac{|P'Q'|}{|PQ|}$

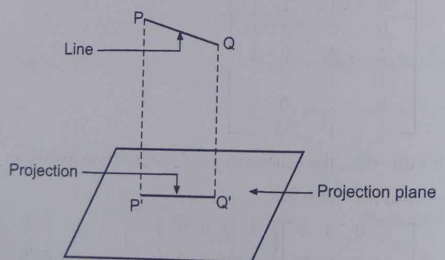


Fig. 3.7

The foreshortening factors along the projected principal axes are denoted by f_x, f_y, f_z.

Principal Foreshortening Factors :

Let OA, OB, OC be unit vectors along x, y, z axes respectively.

$$\therefore \quad [U] = \begin{bmatrix} A \\ B \\ C \\ O \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Let [T] be the transformation, for an axonometric projection.

$$\text{Then} \quad [U^*] = [U]\,[T] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} [T]$$

$$= \begin{bmatrix} x_x^* & y_x^* & 0 & 0 \\ x_y^* & y_y^* & 0 & 0 \\ x_z^* & y_z^* & 0 & 0 \\ x_0^* & y_0^* & 0 & 0 \end{bmatrix}$$

$$f_x = \frac{|O^* A^*|}{|OA|} = \frac{\sqrt{(x_x^* - x_0^*)^2 + (y_x^* - y_0^*)^2}}{1}$$

$$f_y = \sqrt{(x_y^* - x_0^*)^2 + (y_y^* - y_0^*)^2}, \quad f_z = \sqrt{(x_z^* - x_0^*)^2 + (y_z^* - y_0^*)^2}$$

If the translations are not applied, then $x_0^* = y_0^* = 0$.

$$\therefore \quad f_x = \sqrt{x_x^{*2} + y_x^{*2}}, \quad f_y = \sqrt{x_y^{*2} + y_y^{*2}}, \quad f_z = \sqrt{x_z^{*2} + y_z^{*2}}$$

Further,

$$\text{if} \quad [T] = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & 0 & 0 \\ 1 & m & 0 & 1 \end{bmatrix}$$

then $[U^*] = [U]\,[T]$

$$\begin{bmatrix} A^* \\ B^* \\ C^* \\ O^* \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & a_{32} & 0 & 0 \\ 1 & m & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}+1 & a_{12}+m & 0 & 1 \\ a_{21}+1 & a_{22}+m & 0 & 1 \\ a_{31}+1 & a_{32}+m & 0 & 1 \\ 1 & m & 0 & 1 \end{bmatrix}$$

Here $f_x = \dfrac{|O^*A^*|}{|OA|} = \sqrt{a_{11}^2 + a_{12}^2}$

Similarly, $f_y = \sqrt{a_{21}^2 + a_{22}^2}$

$f_z = \sqrt{a_{31}^2 + a_{32}^2}$

There are three types of axonometric projections :

(i) Trimetric projection (ii) Dimetric projection (iii) Isometric projection

(I) Trimetric Projection :

A trimetric projection is obtained by arbitrary rotations, in arbitrary order about any or all co-ordinate axes, followed by parallel (orthographic) projection on z = 0 plane. Trimetric projection is formed by first rotation about y-axis through an angle ϕ and then rotation about x-axis through an angle θ, followed by parallel (orthographic) projection on the z = 0 plane.

Thus, the concatenated trimetric projection is given by;

$$[T] = [R_y]\,[R_x]\,[P_z]$$

$$= \begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi & \sin\phi\sin\theta & 0 & 0 \\ 0 & \cos\theta & 0 & 0 \\ \sin\phi & -\cos\phi\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \dots (1)$$

Therefore foreshortening factor along

x-axis $= f_x = \sqrt{\cos^2\phi + \sin^2\phi\sin^2\theta}$

Therefore foreshortening factor along y-axis $= f_y = \cos\theta$

Therefore foreshortening factor along

z-axis $= f_z = \sqrt{\sin^2\phi + \cos^2\phi\sin^2\theta}$

Illustrative Examples

Example 3.2 : Determine the foreshortening factors f_x, f_y, f_z if the transformation matrix for axonometric projection is given by;

$$[T] = \begin{bmatrix} 0.5 & 0.43 & 0 & 0 \\ 0 & 0.86 & 0 & 0 \\ 0.86 & 0.25 & 0 & 0 \\ 3.58 & 0.75 & 0 & 1 \end{bmatrix}$$

Solution : The foreshortening factors f_x, f_y, f_z are given by the following formula,

$$f_x = \sqrt{a_{11}^2 + a_{12}^2} = \sqrt{(0.5)^2 + (0.43)^2} = 0.6594$$

$$f_y = \sqrt{a_{21}^2 + a_{22}^2} = \sqrt{(0)^2 + (0.86)^2} = 0.860$$

$$f_z = \sqrt{a_{31}^2 + a_{32}^2} = \sqrt{(0.86)^2 + (0.25)^2} = 0.8956$$

Example 3.3 : Obtain the transformation matrix for the trimetric projection formed by rotation about the y-axis through an angle $\phi = 30°$, followed by rotation about the x-axis through an angle $\theta = 45°$, and then orthographic projection on the z = 0 plane. Also determine the principal foreshortening factor.

Solution : The concatenated trimetric projection is given by;

$$[T] = [R_y]\,[R_x]\,[P_z]$$

$$= \begin{bmatrix} \cos\phi & \sin\phi\sin\theta & 0 & 0 \\ 0 & \cos\theta & 0 & 0 \\ \sin\phi & -\cos\phi\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{[Using equation (1)]}$$

$$= \begin{bmatrix} \cos 30° & \sin 30°\sin 45° & 0 & 0 \\ 0 & \cos 45° & 0 & 0 \\ \sin 30° & -\cos 30°\sin 45° & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (\because \phi = 30°, \theta = 45°)$$

$$= \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{\sqrt{2}}{4} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & 0 \\ \frac{1}{2} & \frac{-\sqrt{6}}{4} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The principal foreshortening factors are :

$$f_x = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{\sqrt{2}}{4}\right)^2} = \sqrt{\frac{3}{4} + \frac{2}{16}} = 0.935$$

$$f_y = \sqrt{0^2 + \left(\frac{\sqrt{2}}{2}\right)^2}$$

$$= \sqrt{\frac{2}{4}} = 0.707$$

$$f_z = \sqrt{\left(\frac{\sqrt{1}}{2}\right)^2 + \left(\frac{-\sqrt{6}}{4}\right)^2}$$

$$= \sqrt{\frac{1}{4} + \frac{6}{16}} = 0.791$$

Example 3.4 : Develop the transformation matrix for trimetric projection obtained by rotation about y-axis through an angle $\phi = 30°$, followed by rotation about x-axis through an angle $\theta = 45°$, and then orthographic projection on z = 0 plane. Apply it on the cube whose position vectors are given below :

$$[X] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0.5 & 1 & 1 \\ 0.5 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0.5 & 1 \end{bmatrix}$$

Also obtain foreshortening factors f_x, f_y, f_z.

Solution : From the previous example (1), the concatenated transformation matrix [T] is

$$[T] = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{\sqrt{2}}{4} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & 0 \\ \frac{1}{2} & \frac{-\sqrt{6}}{4} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformed position vectors [X'] are obtained by applying [T] on [X], we get

$$[X'] = [X]\ [T] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0.5 & 1 & 1 \\ 0.5 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{\sqrt{2}}{4} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & 0 \\ \frac{1}{2} & \frac{-\sqrt{6}}{4} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & 0.612 & 0 & 1 \\ 1.366 & -0.259 & 0 & 1 \\ 1.366 & 0.095 & 0 & 1 \\ 0.933 & 0.272 & 0 & 1 \\ 0.5 & 0.095 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0.866 & 0.354 & 0 & 1 \\ 0.866 & 1.061 & 0 & 1 \\ 0 & 0.707 & 0 & 1 \\ 1.116 & 0.754 & 0 & 1 \end{bmatrix}$$

The foreshortening ratios are obtained by applying [T] on unit vectors along the principal axes.

$$\therefore \quad [U]\ [T] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{\sqrt{2}}{4} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 0 & 0 \\ \frac{1}{2} & \frac{-\sqrt{6}}{4} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{\sqrt{2}}{4} & 0 & 1 \\ 0 & \frac{\sqrt{2}}{2} & 0 & 1 \\ \frac{1}{2} & \frac{-\sqrt{6}}{4} & 0 & 1 \end{bmatrix}$$

Foreshortening factors are :

$$f_x = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{\sqrt{2}}{4}\right)^2} = 0.935$$

$$f_y = \frac{\sqrt{2}}{2} = 0.707$$

$$f_z = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{-\sqrt{6}}{4}\right)^2} = 0.791$$

Example 3.5 : Develop the transformation matrix for trimetric projection obtained by rotation about y-axis through an angle $\phi = 45°$, followed by rotation about x-axis through an angle $\theta = 60°$ and then orthographic projection on z = 0 plane. Apply it on the cube whose position vectors are given below :

$$[X] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0.5 & 1 & 1 \\ 0.5 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0.5 & 1 \end{bmatrix}$$

Also obtain the foreshortening factors f_x, f_y, f_z.

Solution : The concatenated transformation matrix for trimetric projection is given by;

$$[T] = [R_y]\,[R_x]\,[P_z]$$

$$= \begin{bmatrix} \cos\phi & \sin\phi\sin\theta & 0 & 0 \\ 0 & \cos\theta & 0 & 0 \\ \sin\phi & -\cos\phi\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos 45° & \sin 45°\sin 60° & 0 & 0 \\ 0 & \cos 60° & 0 & 0 \\ \sin 45° & -\cos 45°\sin 60° & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (\because \phi = 45°, \theta = 60°)$$

$$\Rightarrow \quad [T] = \begin{bmatrix} 0.7071 & 0.6124 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0.7071 & -0.6124 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformed position vectors [X'] are obtained by applying [T] on [X], we get;

$$[X'] = [X]\,[T] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0.5 & 1 & 1 \\ 0.5 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 0.7071 & 0.6124 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0.7071 & -0.6124 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.7071 & -0.6123 & 0 & 1 \\ 1.4142 & 0 & 0 & 1 \\ 1.4142 & 0.25 & 0 & 1 \\ 1.0607 & 0.1939 & 0 & 1 \\ 0.7071 & -0.1123 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0.7071 & 0.6124 & 0 & 1 \\ 0.7071 & 1.1124 & 0 & 1 \\ 0 & 0.5 & 0 & 1 \\ 1.0607 & 0.8062 & 0 & 1 \end{bmatrix}$$

The foreshortening ratios are obtained by applying [T] on unit vectors along the principal axes.

$$\therefore \quad [U]\,[T] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.7071 & 0.6124 & 0 & 0 \\ 0 & 0.5 & 0 & 0 \\ 0.7071 & -0.6124 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.7071 & 0.6124 & 0 & 1 \\ 0 & 0.5 & 0 & 1 \\ 0.7071 & -0.6124 & 0 & 1 \end{bmatrix}$$

Foreshortening factors are :

$$f_x = \sqrt{(0.7071)^2 + (0.6124)^2} = 0.8749$$

$$f_y = \sqrt{(0.5)^2} = 0.50$$

$$f_z = \sqrt{(0.7071)^2 + (-0.6124)^2} = 0.8749$$

(II) Dimetric Projection :

A dimetric projection is a trimetric projection with any two of the three foreshortening factors are equal and the third is arbitrary.

A dimetric projection is formed by the rotation about y-axis through an angle ϕ, followed by the rotation about x-axis through an angle θ and then orthographic projection on z = 0 plane.

The resulting transformation matrix is given by

$$[T] = [R_y]\,[R_x]\,[P_z]$$

$$= \begin{bmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad [T] = \begin{bmatrix} \cos\phi & \sin\phi\sin\theta & 0 & 0 \\ 0 & \cos\theta & 0 & 0 \\ \sin\phi & -\cos\phi\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The unit vectors on x, y, z principal axes transformed to

$$[U'] = [U]\,[T] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi\sin\theta & 0 & 0 \\ 0 & \cos\theta & 0 & 0 \\ \sin\phi & -\cos\phi\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad [U'] = \begin{bmatrix} \cos\phi & \sin\phi\sin\theta & 0 & 0 \\ 0 & \cos\theta & 0 & 0 \\ \sin\phi & -\cos\phi\sin\theta & 0 & 0 \end{bmatrix}$$

Here, foreshortening factors are

$$f_x^2 = \cos^2\phi + \sin^2\phi\sin^2\theta \qquad \ldots (1)$$

$$f_y^2 = \cos^2\theta \qquad \ldots (2)$$

$$f_z^2 = \sin^2\phi + \cos^2\phi\sin^2\theta \qquad \ldots (3)$$

$$f_x = f_y \qquad \ldots (4)$$

Choose, equations (1) and (2) $\Rightarrow$

$$\cos^2\phi + \sin^2\phi\sin^2\theta = \cos^2\theta$$

$$\Rightarrow \quad 1 - \sin^2\phi + \sin^2\phi\sin^2\theta = 1 - \sin^2\theta$$

$$\Rightarrow \quad \sin^2\phi(\sin^2\theta - 1) = -\sin^2\theta$$

$$\Rightarrow \quad \sin^2\phi = \frac{\sin^2\theta}{1 - \sin^2\theta} \qquad \ldots (5)$$

Equation (3) $\Rightarrow$

$$\begin{aligned} f_z^2 &= \sin^2\phi + \cos^2\phi\sin^2\theta \\ &= \sin^2\phi + (1 - \sin^2\phi)\sin^2\theta \\ &= \sin^2\phi + \sin^2\theta - \sin^2\phi\sin^2\theta \\ &= \sin^2\phi(1 - \sin^2\theta) + \sin^2\theta \\ &= \left(\frac{\sin^2\theta}{1 - \sin^2\theta}\right)(1 - \sin^2\theta) + \sin^2\theta \qquad [\because \text{using (5)}] \\ &= \sin^2\theta + \sin^2\theta \\ &= 2\sin^2\theta \end{aligned}$$

$$\Rightarrow \quad \sin^2\theta = \frac{f_z^2}{2} \qquad \ldots (6)$$

Equation (5) $\Rightarrow$

$$\sin^2\phi = \frac{\frac{f_z^2}{2}}{1 - \frac{f_z^2}{2}} = \frac{f_z^2}{2 - f_z^2} \qquad \ldots (7)$$

Thus, equations (6) and (7) $\Rightarrow$ $\theta = \sin^{-1}\left(\frac{\pm f_z}{\sqrt{2}}\right)$

and $\phi = \sin^{-1}\left(\frac{\pm f_z}{\sqrt{2 - f_z^2}}\right)$

This shows that the range of foreshortening factor is $0 \le f_z \le 1$. Foreshortening factor $f_{(z)}$ gives four possible dimetric projections for the following angles :

(i) $\theta > 0, \phi > 0$

(ii) $\theta > 0, \phi < 0$

(iii) $\theta < 0, \phi > 0$

(iv) $\theta < 0, \phi < 0$

Example 3.6 : Determine the dimetric projection for the for shortening factor along z-axis of $\frac{1}{2}$, for the cube with the corner cut-off whose position vector is

$$[X] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0.5 & 1 & 1 \\ 0.5 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0.5 & 1 \end{bmatrix}$$

Solution : Here $\theta = \sin^{-1}\left(\frac{\pm f_z}{\sqrt{2}}\right) = \sin^{-1}\left(\frac{\pm 1}{2\sqrt{2}}\right) = \sin^{-1}(\pm 0.35\ 355) = \pm 20.705°$

and $\phi = \sin^{-1}\left(\frac{\pm f_z}{\sqrt{2 - f_z^2}}\right)$

$$= \sin^{-1}\left(\frac{\pm 1}{2\sqrt{2 - \frac{1}{4}}}\right)$$

$$= \sin^{-1}(\pm 0.378)$$

$$= \pm 22.208°$$

Choosing $\theta > 0, \phi > 0$ i.e. $\theta = 20.705°, \phi = 22.208°$

$$[T] = \begin{bmatrix} \cos\phi & \sin\phi\sin\theta & 0 & 0 \\ 0 & \cos\phi & 0 & 0 \\ \sin\phi & -\cos\phi\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(22.208°) & \sin(22.208°)\sin(22.705°) & 0 & 0 \\ 0 & \cos(22.208°) & 0 & 0 \\ \sin(22.208°) & -\cos(22.208°)\sin(22.208°) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 0.926 & 0.134 & 0 & 0 \\ 0 & 0.935 & 0 & 0 \\ 0.378 & -0.327 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformed position vectors [X'] are obtained by applying [T] on [X],

$$[X'] = [X]\,[T] = \begin{bmatrix} 0.378 & -0.327 & 0 & 1 \\ 1.304 & -0.194 & 0 & 1 \\ 1.304 & 0.274 & 0 & 1 \\ 0.841 & 0.675 & 0 & 1 \\ 0.378 & 0.608 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0.926 & 0.134 & 0 & 1 \\ 0.926 & 1.069 & 0 & 1 \\ 0 & 0.935 & 0 & 1 \\ 1.115 & 0.905 & 0 & 1 \end{bmatrix}$$

(III) Isometric Projection :

In an isometric projection, all the three principal axes are foreshortened equally so that relative proportions are maintained. This creates no confusion and no error in manual construction. Obviously, this is not the case in previous projections where scaling factors are different for the three principal axes.

Thus, isometric projection is the projection where all the three foreshortening factors are equal. i.e. in isometric projection, $f_x = f_y = f_z$

To determine the angles θ and ϕ :

In dimetric projection, equations (6) and (2) are **(April 2013)**

$$\sin^2\theta = \frac{f_z^2}{2} \Rightarrow f_z^2 = 2\sin^2\theta$$

$$f_z^2 = \cos^2\theta$$

Now, in isometric projection, $f_y = f_z = f_x$

$$2\sin^2\theta = \cos^2\theta$$

$$\Rightarrow \quad 2\sin^2\theta = 1 - 2\sin^2\theta$$

$$\Rightarrow \quad 3\sin^2\theta = 1$$

$$\Rightarrow \quad \sin\theta = \pm\sqrt{\frac{1}{3}}$$

$$\Rightarrow \quad \theta = \sin^{-1}\left(\pm\sqrt{\frac{1}{3}}\right) = \pm 35.26°$$

Again, in dimetric projection, equation (5) is

$$\sin^2\phi = \frac{\sin^2\theta}{1-\sin^2\theta}$$

$$\Rightarrow \quad \sin^2\phi = \frac{\frac{1}{3}}{1-\frac{1}{3}}$$

$$\Rightarrow \quad \phi = \sin^{-1}\left(\pm\frac{1}{\sqrt{2}}\right) = \pm 45°$$

Note that there are four possible isometric projections.

The foreshortening factors for an isometric projection are

$$f_x = f_y = f_z$$

$$\Rightarrow \quad f_x^2 = f_z^2 = f_y^2$$

$$\Rightarrow \quad f_x^2 = f_z^2 = f_y^2 = \cos^2\theta = 1 - \sin^2\theta = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\Rightarrow \quad f_x = f_y = f_z = \sqrt{\frac{2}{3}} = 0.8165$$

Illustrative Examples

Example 3.7 : Obtain isometric projection of the line segment between the points A [1 –2 1] and B [3 1 –6].

Solution : For isometric projection, $f_x = f_y = f_z = 0.8165$

Let,

ϕ = Rotation about y-axis = ± 45°

θ = Rotation about x-axis = ± 35.26°

Choosing $\theta > 0, \phi > 0$

$\therefore$ The transformation matrix [T] is

$$[T] = \begin{bmatrix} \cos\phi & \sin\phi\sin\theta & 0 & 0 \\ 0 & \cos\theta & 0 & 0 \\ \sin\phi & -\cos\phi\sin\phi & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0.7071 & 0.4082 & 0 & 0 \\ 0 & 0.8165 & 0 & 0 \\ 0.7071 & -0.4082 & 0 & 0 \end{bmatrix}$$

The projected terminal vertices A' and B' are obtained by applying [T] on $\begin{bmatrix} A \\ B \end{bmatrix}$.

Thus $\begin{bmatrix} A' \\ B' \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix} [T]$

$$\Rightarrow \begin{bmatrix} A' \\ B' \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 & 1 \\ 3 & 1 & -6 & 1 \end{bmatrix} \begin{bmatrix} 0.7071 & 0.4082 & 0 & 0 \\ 0 & 0.8165 & 0 & 0 \\ 0.7071 & -0.4082 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} A' \\ B' \end{bmatrix} = \begin{bmatrix} 1.4142 & -1.633 & 0 & 1 \\ -2.1214 & 4.4903 & 0 & 1 \end{bmatrix}$$

Example 3.8 : Determine the isometric projection transformation for $\phi = -45°$ and $\theta = +35.26439°$. Apply it on the object [X], where,

$$[X] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0.5 & 1 & 1 \\ 0.5 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0.5 & 1 \end{bmatrix}$$

Solution : Here $\phi = -45$, $\theta = 35.26439$

$\therefore$ The transformation matrix [T] is

$$[T] = \begin{bmatrix} \cos\phi & \sin\phi\sin\theta & 0 & 0 \\ 0 & \cos\theta & 0 & 0 \\ \sin\phi & -\cos\phi\sin\phi & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(-45°) & \sin(-45°)\sin(35.26439°) & 0 & 0 \\ 0 & \cos(35.26439°) & 0 & 0 \\ \sin(-45°) & -\cos(-45°)\sin(35.26439°) & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.707 & -0.408 & 0 & 0 \\ 0 & 0.816 & 0 & 0 \\ -0.707 & -0.408 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformed position vectors [X'] are obtained by applying [T] on [X].

$$\therefore \quad [X'] = [X]\,[T] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0.5 & 1 & 1 \\ 0.5 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0.5 & 1 \end{bmatrix} \begin{bmatrix} 0.707 & -0.408 & 0 & 0 \\ 0 & 0.816 & 0 & 0 \\ -0.707 & -0.408 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.707 & -0.408 & 0 & 1 \\ 0 & -0.816 & 0 & 1 \\ 0 & -0.406 & 0 & 1 \\ -0.354 & 0.204 & 0 & 1 \\ -0.707 & 0.408 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0.707 & -0.408 & 0 & 1 \\ 0.707 & 0.408 & 0 & 1 \\ 0 & 0.816 & 0 & 1 \\ 0.354 & 0.204 & 0 & 1 \end{bmatrix}$$

For $\phi < 0$, $\theta > 0$, the result is shown in Fig. 3.21 below.

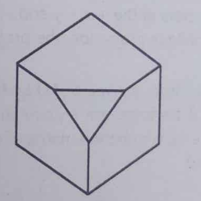


Fig. 3.8

3.5 Oblique Projections

An oblique projection is formed by parallel projectors from the centre of projection at infinity that intersect the plane of projection at an oblique angle. It illustrates the three-dimensional shape of the object. There may be shortening or enlarging the lengths of the vector (see figures below). If the vector is parallel to the projection plane, then the length of the vector remains the same i.e. in this case, it preserves the length. Thus, if the face of the object is parallel to the projection plane, then we get the correct size and shape of the projected figure.

Here, we are discussing only two oblique projections, namely cavalier and cabinet projections.

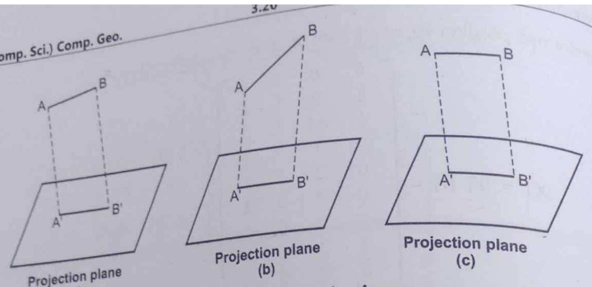


Fig. 3.9 : Oblique projection

Cavalier Projection :

In an oblique projection, when the angle between the oblique projectors and the plane of the projection is 45°, then the projection is called the cavalier projection. In this projection, the foreshortening factors $f_x = f_y = f_z$ for all three principal directions are equal i.e. $f_x = f_y = f_z$.

Cabinet Projection :

An oblique projection for which the foreshortening factor for edges perpendicular to the plane of the projection is $\frac{1}{2}$, is called the *cabinet projection*. In this projection, the angle between the projectors and the plane of the projection is of the $\cot^{-1}\left(\frac{1}{2}\right) = 63.435°$

To determine the transformation matrix [T] for an oblique projection : We know that [1 0 0], [0 1 0], [0 0 1] are the unit vectors at the + x, – y and – z axes respectively. Let A [0 0 1] be the unit vector along z-axis. For an oblique projection, the projectors AC and BO make an angle β with the plane z = 0.

The projector BO can be obtained from the vector AO by translating the point A [0 0 1] to the point B [–a –b 1]. It is obvious that if we transform a point that is represented by homogeneous co-ordinates, then we cannot use the transformation matrix of order 2 × 2. In this case, the general transformation matrix is of order 3 × 3.

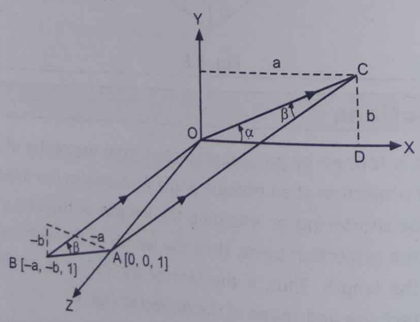


Fig. 3.10

Thus 3 × 3 transformation matrix is

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -a & -b & 1 \end{bmatrix}$$

In three-dimensional space, translation in two-dimensions is nothing but shearing in x and y directions by the factors proportional to z-co-ordinate in three-dimensional space. The resulting transformation matrix is

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -a & -b & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Projection on z = 0 plane gives

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -a & -b & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{... (1)}$$

In Δ OCD, $\sin\alpha = \frac{b}{f} \Rightarrow b = f\sin\alpha$

$$\cos\alpha = \frac{a}{f} \Rightarrow a = f\cos\alpha$$

Thus, the transformation matrix for an oblique projection is

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -f\cos\alpha & -f\sin\alpha & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{... (2)}$$

Note : (1) For orthographic projection :

$$f = 0, \beta = 90°$$

(2) For cavalier projection :

$$f = 1, \beta = 45°$$

(3) For cabinet projection :

$$f = \frac{1}{2}, \beta = 63.435°$$

Illustrative Examples

Example 3.9 : Let

$$[X] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 5 & 5 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 5 & 5 & 0 & 1 \end{bmatrix}$$

Obtain the cabinet projection of [X] with $\alpha = 30°$.

Solution : The transformation matrix for an oblique projection is

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -f\cos\phi & -f\sin\alpha & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For cabinet projection : $f = \frac{1}{2}$, $\beta = 30°$ (given)

$$\therefore \quad [T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{\sqrt{3}}{4} & -\frac{1}{4} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -0.433 & -0.25 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Cabinet projection [X'] is obtained by applying [T] on [X].

$$[X'] = [X]\,[T]$$

$$= \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 5 & 5 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 5 & 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -0.433 & -0.25 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.433 & -0.25 & 0 & 1 \\ 0.567 & -0.25 & 0 & 1 \\ 4.567 & 4.75 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 5 & 5 & 0 & 1 \end{bmatrix}$$

Example 3.10 : Obtain the transformation matrix for a cavalier projection for $\alpha = 45°$.

Solution : The transformation matrix for an oblique projection is

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -f\cos\alpha & -f\sin\alpha & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For cavalier projection :

$f = 1$, here $\alpha = 45°$

$$\therefore \quad [T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\cos 45° & -\sin 45° & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -0.707 & -0.707 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 3.11 : Find the cabinet projection of the object represented by the following position vector matrix [X] with a horizontal inclination $\alpha = 25°$.

$$[X] = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & -1 \\ -1 & -2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

Solution : The transformation matrix for an oblique projection is

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -f\cos\alpha & -f\sin\alpha & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For cabinet projection :

$f = \frac{1}{2}$, here $\alpha = 25°$

$$\therefore \quad [T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{\cos 25°}{2} & -\frac{\sin 25°}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -0.4532 & -0.2114 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformed position vectors for the cabinet projection are

$$[X'] = [X]\,[T] = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 4 & -1 & 1 \\ -1 & -2 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -0.4531 & -0.2113 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5469 & 1.7887 & 0 & 1 \\ 3.4531 & 4.2113 & 0 & 1 \\ -1.4531 & -2.2113 & 0 & 1 \\ -1.5468 & 0.7887 & 0 & 1 \end{bmatrix}$$

Example 3.12 : Find the cavalier and cabinet projections of the unit cube determined by the vertices A [0 0 0], B [1 0 0], C [1 1 0], D [0 1 0], E [0 1 1], F [0 0 0], G [1 0 1], H [1 1 1]. (Take the horizontal inclination angle $\alpha = 30°$)

Solution : The transformation matrix for an oblique projection is

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -f\cos\alpha & -f\sin\alpha & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For cavalier projection : $f = 1$, here $\alpha = 30°$

$$\therefore \quad [T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\cos 30° & -\sin 30° & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -0.8660 & -0.5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformed position vectors [X'] for cavalier projection are

$$[X'] = [X]\,[T]$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -0.8660 & -0.5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ -0.8660 & 0.5 & 0 & 1 \\ -0.8660 & -0.5 & 0 & 1 \\ 0.1340 & -0.5 & 0 & 1 \\ 0.1340 & 0.5 & 0 & 1 \end{bmatrix}$$

For cabinet projection : $f = \frac{1}{2}$, here $\alpha = 30°$

$$\therefore \quad [T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{\cos 30}{2} & -\frac{\sin 30}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -0.433 & -0.25 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformed position vectors [X'] for cabinet projection are

$$[X'] = [X]\,[T] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -0.433 & -0.25 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ -0.433 & 0.75 & 0 & 1 \\ -0.433 & -0.25 & 0 & 1 \\ 0.567 & -0.25 & 0 & 1 \\ 0.567 & 0.75 & 0 & 1 \end{bmatrix}$$

3.6 Perspective Transformations

We know that when the general 4×4 transformation matrix partitioned into four separate sections (sub-matrices), the upper right 3×1 sub-matrix produces a perspective transformation. We have also discussed in the previous article 3.2 that the perspective transformation is the transformation from three-dimensional space to three-dimensional space. In perspective transformation, the parallel lines seem to converge. The objects and lengths appear smaller as their distance from the centre of the projection increases.

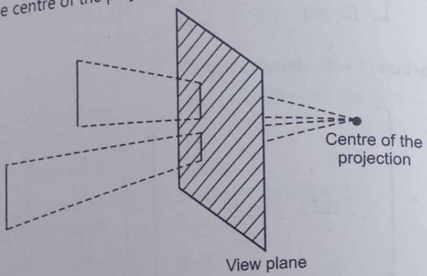


Fig. 3.11

Perspective Projection of a Point :

A single-point perspective transformation is given by

$$[x\ y\ z\ 1]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{bmatrix} = [x\ y\ z\ rz + 1] \quad \text{...(1)}$$

where, $rz + 1 \neq 1$.

The ordinary co-ordinates are obtained by dividing throughout by $rz + 1$, we get

$$[x'\ y'\ z'\ 1] = \left[\frac{x}{rz+1}\ \frac{y}{rz+1}\ \frac{z}{rz+1}\ 1\right] \quad \text{...(2)}$$

The perspective projection on $z = 0$ plane is given by

$$[T] = [P_r]\,[P_z]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & r \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{... (3)}$$

Applying [T] on [x y z 1], we get

$$[x\ y\ z\ 1]\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & r \\ 0 & 0 & 0 & 1 \end{bmatrix} = [x\ y\ 0\ rz + 1] \quad \text{... (4)}$$

where, $rz + 1 \neq 1$.

The ordinary co-ordinates are obtained by dividing $rz + 1$, we get

$$[x'\ y'\ z'\ 1] = \left[\frac{x}{rz+1}\ \frac{y}{rz+1}\ 0\ 1\right] \quad \text{... (5)}$$

We have to show that $[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & r \\ 0 & 0 & 0 & 1 \end{bmatrix}$ produces the perspective projection on $z = 0$ plane.

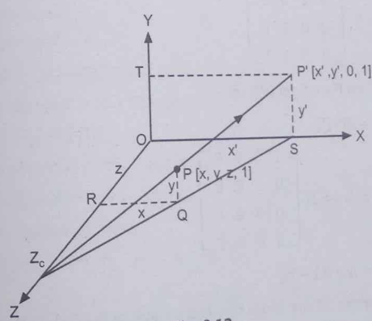


Fig. 3.12

Consider the point P [x z 1] in the space from the centre of projection at z_C. The perspective projection of the P (x y z 1) z = z' = 0 plane at point [x' y' z' 0 1].

The co-ordinates of the point P' are obtained in terms of Z_C as follows,

$$\Delta Z_CRQ \equiv \Delta Z_C OS$$

$$\Rightarrow \quad \frac{|OS|}{|RQ|} = \frac{|OZ_C|}{|RZ_C|} = \frac{|Z_CS|}{|Z_CQ|}$$

$$\Rightarrow \quad \frac{x'}{x} = \frac{z_c}{z_c - z}$$

$$\Rightarrow \quad x' = \frac{x - z_c}{z_c - z}$$

$$\Rightarrow \quad x' = \frac{x}{1 - \frac{z}{z_c}}$$

Similarly considering $\Delta P'SZ_c \equiv \Delta PQR$, we get

$$y' = \frac{y}{1 - \frac{z}{z_c}}$$

Choose $r = -\frac{1}{z_c}$. The co-ordinates to the point P' can be written as

$$P'\left[\frac{x}{1+rz} \; \frac{y}{1+rz} \; 0 \; 1\right]$$

Thus if
$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & r \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

then, applying [T] on [P], we get [P'], i.e.

$$[P'] = [P]\,[T]$$

$$= [x\; y\; z\; 1] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & r \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= [x\; y\; 0\; 1 + rz]$$

Note : (i) If the centre of the projection is on x-axis at $[x_c \; 0 \; 0 \; 1]$ then the corresponding transformation matrix is

$$[T] = \begin{bmatrix} 0 & 0 & 0 & -\frac{1}{x_c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(ii) If the centre of the projection is on y-axis at $[0 \; y_c \; 0 \; 1]$ then the corresponding transformation matrix is

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{y_c} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(iii) If the centre of the projection is on z-axis at $[0 \; 0 \; z_c \; 1]$ then the corresponding transformation matrix is

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{z_c} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Illustrative Examples

Example 3.13 : Obtain the concatenated matrix representation of the following transformation in order :

(α) Rotation about y-axis by $\phi = -30°$

(β) Rotation about x-axis by $\theta = 45°$

(γ) Projected onto z = 0 plane from a centre of projection $z = z_c = 2.5$.

Solution : The concatenated matrix [T] is given by

$$[T] = [R_y]\,[R_x]\,[T_p]$$

where, $[R_y]$ = Rotation matrix about y-axis

$[R_x]$ = Rotation matrix about x-axis

$[T_p]$ = Projection matrix

$$\therefore \quad [T] = \begin{bmatrix} \cos(-30°) & 0 & -\sin(-30°) & 0 \\ 0 & 1 & 0 & 0 \\ \sin(-30°) & 0 & \cos(-30°) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 45° & \sin 45° & 0 \\ 0 & -\sin 45° & \cos 45° & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2.5} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.866 & 0 & 0.5 & 0 \\ 0 & 1 & 0 & 0 \\ -0.5 & 0 & 0.866 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.707 & 0.707 & 0 \\ 0 & -0.707 & 0.707 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.866 & -0.3535 & 0 & -0.1414 \\ 0 & 0.707 & 0 & -0.2828 \\ -0.5 & -0.6123 & 0 & -0.2449 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 3.14 : An object is translated by 2, 3, 4 units along x, y and z axes respectively and then a single point perspective projection on the z = 0 plane from the centre of projection on z-axis at $z_c = 50$ is performed. Obtain concatenated matrix of transformation.

Solution : The concatenated transformation matrix [T] is

$$[T] = [T_r]\,[T_p]$$

where, $[T_r]$ = Translation matrix

$[T_p]$ = Projection matrix on z = 0 plane from $z_C = 50$

$$\therefore \quad [T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.02 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.02 \\ 2 & 3 & 0 & 0.92 \end{bmatrix}$$

Example 3.15 : Obtain perspective projection onto the y = 0 plane from the centre of projection C' [0 5 0] of the Δ ABC, where,

$$\begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 15 & 17 & -20 \\ 25 & 10 & 40 \\ 30 & 15 & 40 \end{bmatrix}$$

Solution : The centre of projection is on y-axis at C' [0 5 0 1], the corresponding transformation matrix is

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{5} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

[see note (ii)]

The perspective projection is given by

$$\begin{bmatrix} A' \\ B' \\ C' \end{bmatrix} = \begin{bmatrix} A \\ B \\ C \end{bmatrix} [T]$$

$$= \begin{bmatrix} 15 & 17 & -20 & 1 \\ 25 & 10 & 40 & 1 \\ 30 & 15 & 40 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{5} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 0 & -20 & -2.4 \\ 25 & 0 & 40 & -1 \\ 30 & 0 & 40 & -2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} A' \\ B' \\ C' \end{bmatrix} = \begin{bmatrix} -6.25 & 0 & 8.33 & 1 \\ -25 & 0 & -40 & 1 \\ -15 & 0 & -20 & 1 \end{bmatrix}$$

3.7 The Effect of Perspective Transformations and Vanishing Point

The Fig. 3.26 shows the perspective projection on the line AB parallel to z-axis on the z = 0 plane, into the line A* B* in the z = 0 plane, from the centre of projection at $-\frac{1}{r}$ on z-axis. Obviously this transformation contains two steps: the perspective transformation and orthographic projection. i.e. $[T] = [P_r]\,[P_z]$. The perspective transformation of the line AB yields the line A' B' in three-dimensional space and the orthographic projection of the line A' B' in three-dimensional space from the centre of projection at infinity of z-axis yields the line A* B*.

The line A'B' intersects the z = 0 plane (the line AB and the line A'B' intersect in z = 0 plane at the same point). The line A'B' also intersects z-axis at point $z = \frac{1}{r}$. The perspective transformation has transformed the point of intersections at infinity of the line AB which is parallel to z-axis and z-axis itself into finite point at $z = \frac{1}{r}$, on z-axis. The point $\begin{bmatrix} 0 & 0 & \frac{1}{r} & 1 \end{bmatrix}$ is called the **'vanishing point'** on z-axis. The vanishing point lies at equidistance on the opposite side of the plane of projection from the centre of the projection.

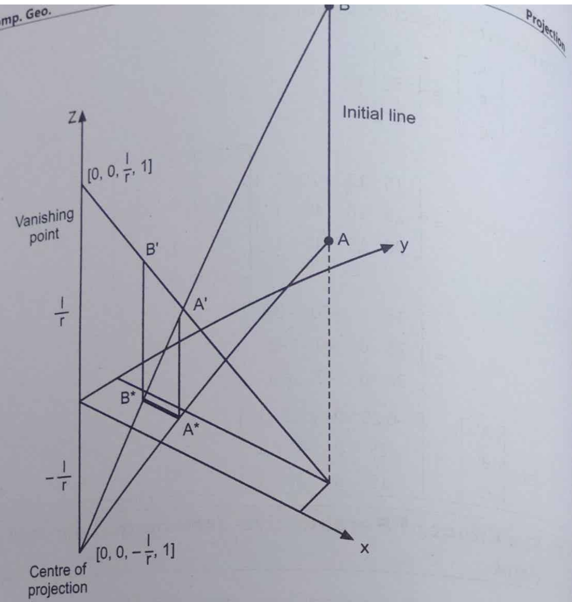


Fig. 3.13

The perspective transformation of the point at infinity on positive z-axis is

$$[0\ 0\ 1\ 0]\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & r\\0 & 0 & 0 & 1\end{bmatrix} = [0\ 0\ 1\ r]$$

The point $[x^*\ y^*\ z^*\ 1] = \left[0\ 0\ \frac{1}{r}\ 1\right]$ which is the transformed point at infinity is now a finite point on positive z-axis.

From the previous article 2.15 and from the above discussion, we have

For positive x-axis :

(i) A single-point perspective transformation is given by;

$$[x\ y\ z\ 1]\begin{bmatrix}1 & 0 & 0 & p\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix} = [x\ y\ z\ (px + 1)]$$

(ii) Ordinary co-ordinates are

$$[x^*\ y^*\ z^*\ 1] = \left[\frac{x}{px+1}\ \frac{y}{px+1}\ \frac{z}{px+1}\ 1\right]$$

(iii) Centre of projection is at $\left[-\frac{1}{p}\ 0\ 0\ 1\right]$.

(iv) Vanishing point is at $\left[\frac{1}{p}\ 0\ 0\ 1\right]$.

For positive y-axis :

(i) A single point perspective transformation is given by

$$[x\ y\ z]\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & q\\0 & 0 & 1 & 0\\0 & 0 & 0 & 1\end{bmatrix} = [x\ y\ z\ (qy + 1)]$$

(ii) Ordinary co-ordinates are

$$[x^*\ y^*\ z^*\ 1] = \left[\frac{x}{qy+1}\ \frac{y}{qy+1}\ \frac{z}{qy+1}\ 1\right]$$

(iii) Centre of projection is at $\left[0\ -\frac{1}{q}\ 0\ 1\right]$

(iv) Vanishing point is at $\left[0\ \frac{1}{q}\ 0\ 1\right]$.

For positive z-axis :

(i) A single-point perspective transformation is given by

$$[x\ y\ z\ 1]\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & r\\0 & 0 & 0 & 1\end{bmatrix} = [x\ y\ z\ rz + 1]$$

(ii) Ordinary co-ordinates are

$$[x^*\ y^*\ z^*\ 1] = \left[\frac{x}{px+1}\ \frac{y}{px+1}\ \frac{z}{px+1}\ 1\right]$$

(iii) Centre of projection is at $\left[0\ 0\ -\frac{1}{r}\ 1\right]$.

(iv) Vanishing point is at $\left[0\ 0\ \frac{1}{q}\ 0\ 1\right]$

Illustrative Examples

Example 3.16 : Perform the perspective projection onto the z = 0 plane of the unit cube (whose position vectors are given below) from the centre of projection at $z_c = 10$ on z-axis.

$$[X] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Solution : A perspective projection onto the z = 0 plane is given by

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & r \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \left(\because r = \frac{1}{z_c} = -\frac{1}{10} = -0.1\right)$$

Applying [T] on [X], we get

$$[X'] = [X]\,[T] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0.9 \\ 1 & 0 & 0 & 0.9 \\ 1 & 1 & 0 & 0.9 \\ 0 & 1 & 0 & 0.9 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1.11 & 0 & 0 & 1 \\ 1.11 & 1.11 & 0 & 1 \\ 0 & 1.11 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

The result is shown in the following Fig. 3.14.

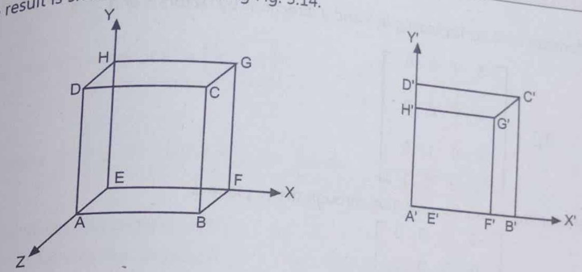


Fig. 3.14

Example 3.17 : Obtain the principal vanishing points for the perspective transformations given by

$$[T] = \begin{bmatrix} 0.866 & -0.354 & 0 & -0.141 \\ 0 & 0.707 & 0 & -0.283 \\ -0.5 & -0.612 & 0 & -0.245 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution : The points at infinity on x, y and z axes are [1 0 0 0], [0 1 0 0], [0 0 1 0]. Transforming the points at infinity on x, y and z axes yields

$$[VP]\,[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0.866 & -0.354 & 0 & -0.141 \\ 0 & 0.707 & 0 & -0.283 \\ -0.5 & -0.612 & 0 & -0.245 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{array}{c} \begin{bmatrix} -6.142 & 2.5 & 0 & 1 \\ 0 & -2.5 & 0 & 1 \\ 2.04 & 2.5 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{l} VP_x \\ VP_y \\ VP_z \end{array}$$

Example 3.18 : Find the combined transformation matrix for the following sequence of transformations : Translation in x, y and z directions by –1, 2 and 1 units respectively, followed by scaling in x and y directions by factors 3 and $\frac{1}{2}$ respectively, followed by reflection through the xy-plane. Apply it on the point [1 3 2]. **(April 2008)**

Solution : The transformation matrix for translation in x, y, z directions by –1, 2, 1 units is

$$[T_1] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 2 & 1 & 1 \end{bmatrix}$$

The transformation matrix for scaling in x and y directions by factors 3 and $\frac{1}{2}$ is

$$[T_2] = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformation matrix for reflection through the yz plane is

$$[T_3] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The combined transformation matrix is

$$[T] = [T_1]\,[T_2]\,[T_3]$$

$$\therefore \quad [T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad [T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad [T] = \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 1 & 1 & 1 \end{bmatrix}$$

The point P [1 3 2].

$$[P^*] = [P]\,[T] = [1 \;\; 3 \; 2 \; 1] \begin{bmatrix} -3 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 1 & 1 & 1 \end{bmatrix}$$

$$\therefore \quad [P^*] = \left[0 \;\; \frac{5}{2} \; 3 \;\; 1\right]$$

The position vector of P^* is $\left[0 \;\; \frac{5}{2} \;\; 3\right]$.

Example 3.19 : By considering the position vector matrix $[X] = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 4 & -1 \\ -1 & -2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$ with horizontal inclination angle $\alpha = 25°$.

(a) Find the caviler projection. (b) Find the cabinet projection.

Solution : $\alpha = 25°$, $\cos \alpha = 0.9063$, $\sin 25° = 0.4226$, $f = 1$.

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -f\cos\alpha & -f\sin\alpha & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad [T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -0.9063 & -0.4226 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[X]\,[T] = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 4 & -1 & 1 \\ -1 & -2 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -0.9063 & -0.4226 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad [X^*] = \begin{bmatrix} 0.9937 & 1.5774 & 0 & 1 \\ 3.9063 & 4.4226 & 0 & 1 \\ -1.9063 & -2.4226 & 0 & 1 \\ 1.0937 & 0.5774 & 0 & 1 \end{bmatrix}$$

(b) $\alpha = 25°$, $\cos \alpha = 0.9063$, $\sin \alpha = 0.4226$, $f = \frac{1}{2}$.

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -f\cos\theta & -f\sin\alpha & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad [T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -0.4531 & -0.2113 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[X][T] = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 4 & -1 & 1 \\ -1 & -2 & 1 & 1 \\ 2 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -0.4531 & -0.2113 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad [X^*] = \begin{bmatrix} 0.5469 & 1.7887 & 0 & 1 \\ 3.4531 & 4.2113 & 0 & 1 \\ -1.4531 & -2.2113 & 0 & 1 \\ 1.5469 & 0.7887 & 0 & 1 \end{bmatrix}$$

Example 3.20 : Find the concatenated transformation matrix for the following sequence of transformation :

First a rotation about x-axis by 75°, followed by a translation in x, y and z-directions by 1, 2 and 3 units respectively, followed by a perspective projection onto y = 0 plane from the centre of projection at the point [0 5 0]. **(March 2006)**

Solution : The transformation matrix for notation about x-axis by an angle 75° is;

$$[T_1] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 75° & \sin 75° & 0 \\ 0 & -\sin 75° & \cos 75° & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad [T_1] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.2588 & 0.9659 & 0 \\ 0 & -0.9659 & 0.2588 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The transformation matrix for translation in x, y and z directions by 1, 2 and 3 units is

$$[T_2] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

The transformation matrix for perspective projection onto y = 0 plane from the centre of projection at the point [0 5 0] is

$$[T_3] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{5} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The concatenated transformation matrix is

$$[T] = [T_1]\,[T_2]\,[T_3]$$

$$\therefore \quad [T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.2588 & 0.9659 & 0 \\ 0 & -0.9659 & 0.2588 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{5} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad [T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.2588 & 0.9659 & 0 \\ 0 & -0.9659 & 0.2588 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.2 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 3 & 0.6 \end{bmatrix}$$

$$\therefore \quad [T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0.9659 & -0.0517 \\ 0 & 0 & 0.2588 & 0.1931 \\ 1 & 0 & 3 & 0.6 \end{bmatrix}$$

Example 3.21 : Consider a triangle with vertices A [2 2 1], B [1 – 1 4] and C [3 – 1 2]. Find the perspective projection of Δ ABC onto Z = 0 plane with centre of projection at $Z_C = 20$ on the Z-axis. **(Oct. 2008)**

Solution : $Z_C = 20 \Rightarrow -\frac{1}{Z_C} \quad -\frac{1}{20} = -0.05.$

$$[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.05 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We apply [T] on triangle ABC.

$$\begin{bmatrix} A^* \\ B^* \\ C^* \end{bmatrix} = \begin{bmatrix} 2 & 2 & -1 & 1 \\ 1 & -1 & 4 & 1 \\ 3 & -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.05 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} A^* \\ B^* \\ C^* \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 & 1.05 \\ 1 & -1 & 0 & 0.80 \\ 3 & -1 & 0 & 0.90 \end{bmatrix}$$

For normalization we divide in the first row by 1.05, in the second row by 0.80 and in the third row by 0.90.

$$\begin{bmatrix} A^* \\ B^* \\ C^* \end{bmatrix} = \begin{bmatrix} 1.9047 & 1.9047 & 0 & 1 \\ 1.125 & -1.125 & 0 & 1 \\ 3.3333 & -1.1111 & 0 & 1 \end{bmatrix}$$

Example 3.22 : Find the transformation matrix for the trimetric projection formed by a 40° rotation about y-axis, followed by 75° rotation about x-axis and then parallel projection onto z = 0 plane. Also find all foreshortening factors. **(Oct. 2007)**

Solution : $\phi = 40°$, $\theta = 75°$, Given :

$$[T] = [R_y]\,[R_x]\,[P_z]$$

$$[T] = \begin{bmatrix} \cos\phi & 0 & -\sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ \sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} \cos 40° & 0 & -\sin 40° & 0 \\ 0 & 1 & 0 & 0 \\ \sin 40° & 0 & \cos 40° & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 75° & \sin 75° & 0 \\ 0 & -\sin 75° & \cos 75° & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 0.7660 & 0 & -0.6427 & 0 \\ 0 & 1 & 0 & 0 \\ 0.6427 & 0 & 0.7660 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.2588 & 0.9659 & 0 \\ 0 & -0.9659 & 0.2588 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 0.7660 & 0 & -0.6427 & 0 \\ 0 & 1 & 0 & 0 \\ 0.6427 & 0 & 0.7660 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.2588 & 0.9659 & 0 \\ 0 & -0.9659 & 0.2588 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad [T] = \begin{bmatrix} 0.7660 & 0.6207 & 0 & 0 \\ 0 & 0.2588 & 0 & 0 \\ 0.6427 & -0.7398 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We apply [T] on the unit matrix [U].

$$[U]\,[T] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0.7660 & 0.6207 & 0 & 0 \\ 0 & 0.2588 & 0 & 0 \\ 0.6427 & -0.7398 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.7660 & 0.6207 & 0 & 1 \\ 0 & 0.2588 & 0 & 1 \\ 0.6427 & -0.7398 & 0 & 1 \end{bmatrix}$$

$$f_x = \sqrt{(0.7660)^2 + (0.6207)^2} = \sqrt{0.5867 + 0.3852}$$

$$\therefore \quad f_x = \sqrt{0.9719} = 0.9858$$

$$f_y = \sqrt{(0)^2 + (0.2588)^2} = 0.2588$$

$$f_z = \sqrt{(0.6427)^2 + (-0.7398)^2}$$

$$= \sqrt{0.4130 + 0.5473} = \sqrt{0.9603}$$

$$= 0.9799$$

f_x, f_y, f_z are foreshortening ratios.

Example 3.23: Obtain the concatenated matrix for the following sequence of transformations:

Translate by 2, 3 and 4 units along x, y and z-axis respectively. Rotate about x-axis through π^c. Single point perspective projection from the centre on z-axis at point [0 0 5]. Apply it on the point [1 2 3]. **(Oct. 2007)**

Solution : The translation matrix $[T_1]$ for translation by 2, 3, 4 units on x, y, z axes is

$$[T_1] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 3 & 4 & 1 \end{bmatrix}$$

The rotation matrix for rotation about x-axis through π^c is

$$[T_2] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\pi & \sin\pi & 0 \\ 0 & -\sin\pi & \cos\pi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. An arbitrary point from which the lines are projected is called centre of projection.
3. If the centre of projection is located at infinity, then all the projections are parallel lines and the resulting projection is called parallel projection.
4. If the centre of projection is located at finite point in three dimensional space, the resulting projection is perspective projection.
5. The projection formed by parallel projectors, such that they are perpendicular to the projection plane, is called orthographic projection.
6. The orthographic projection in which the direction of projection is not parallel to any of the three co-ordinate axes, is called axonometric projection.
7. Principal forecasting factors.
8. Trimetric projection.
9. Oblique projection.
10. Cabinet projection.

Miscellaneous Exercise

(A) State whether the following statement is True or False :

1. The transformation matrix for reflection through YOZ plane is

$$[T] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. The transformation matrix for rotation about X-axis through an angle $\frac{\pi}{4}$ is

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

3. The transformation given in $[T] = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ causes reflection through x = 0.

4. The matrix for projection onto Z = 0 plane is $[P_z] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

ANSWERS

1. - True	2. - True	3. - False	4. - True

(B) Multiple Choice Questions : Choose the correct alternative from given alternatives.

1. Axonometric projection is orthographic projection in which the direction of projection is not parallel to ……
   (a) x-axis (b) y-axis
   (c) z-axis (d) all three axes
2. The transformation matrix for a cavalier projection for α = 30° is ……

(a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \sqrt{3} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

3. The transformation matrix for translation in x, y, z directions by −1, 2, 1 units is ……

(a) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 2 & 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 2 & 1 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 \end{bmatrix}$ (d) none of these

Answers

1 - (d)	2 - (b)	3 - (a)

(C) Theory Questions :

1. Give the classification of perspective parallel projection.
2. Explain parallel projection and transformation matrix.
3. What is homogeneous co-ordinate system ? Explain the need of homogeneous co-ordinates.
4. Explain the concept of vanishing point in perspective projection.
5. Explain the classification of parallel projection. Discuss the applications of parallel projections.
6. What are parallel perspective projections ? Give classification of both.
7. Derive the transformation matrix for perspective projection.

8. Write definitions of :
   (i) Parallel projection
   (ii) Perspective projection
   (iii) orthographic projection
   (iv) Axonometric projection
   (v) Isometric projection
   (vi) Diametric projection.
9. Derive the transformation matrix for oblique projection.
10. Write an algorithm for reflection of an object through an arbitrary plane in space.
11. State various types of axonometric projection.
12. What is the difference between cavalier and cabinet projection ? Derive the transformation matrix for an oblique projection and hence find the conditions for cavalier and cabinet projection.
13. Explain : (i) centre of projection, (ii) parallel projection.
14. Define orthographic projection. Write a matrix of orthographic projection.
15. With usual notations, derive the expression for

    θ = angle of rotation about x-axis

    ϕ = angle of rotation about y-axis for diametric projection

(D) Numerical Problems :

1. Determine the principle for shortening factors, if the matrix for axonometric projection is given by;

$$[T] = \begin{bmatrix} 0.87 & 0 & 0 & 0 \\ -0.05 & -0.69 & 0 & 0 \\ 0.08 & -0.74 & 0 & 0 \\ 3.1 & 2.7 & 0 & 1 \end{bmatrix}$$

2. In trimetric projection the rotation about y-axis through an angle 30° is followed by rotation about x-axis through an angle 45° and then parallel projection onto the Z = 0 plane. Find the matrix of transformation. Also find the foreshortening ratios.
3. Find the cavalier and cabinet projections of the unit cube determined by the vectors, A[0 0 0], B[1 0 0], C[1 1 0], D[0 1 0], E[0 1 1], F[0 0 1], G[1 0 1] and H[1 1 1]. Take the horizontal inclination angle $\alpha = 30°$.
4. Determine the foreshortening factors f_x, f_y, f_z if the transformation matrix for axonometric projection is given by $\begin{bmatrix} 0.5 & 0.43 & 0 & 0 \\ 0 & 0.86 & 0 & 0 \\ 0.86 & 0.25 & 0 & 0 \\ 3.58 & 0.75 & 0 & 1 \end{bmatrix}$.
5. Obtain isometric projection of the line segment between the points A[1 −2 1] and B[3 1 −6].
6. Find the cavalier projection with $\alpha = 30°$ and cabinet projection with $\alpha = 25°$ of the object represented by the matrix $[X] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$.

7. Determine the diametric projection if the foreshortening factor along z-axis $\frac{1}{5}$ with $\theta > 0$, $\phi > 0$.
8. Determine the isometric projection for $\phi = -45°$, $\theta = -35.26°$. Apply it on P[1 2 1].
9. Write the transformation matrix for perspective projection onto the z = 0 plane from the centre of projection at the point [0 0 5].
10. Write the transformation matrix for diametric projection with $f_x = \frac{3}{8}$, $\phi > 0$, $\theta > 0$.
11. Obtain single point perspective projection onto z = 0 plane of the object represented by X from the centre of projection $Z_c = 4$ on z-axis. $X = \begin{bmatrix} 0 & 3 & 1 \\ 1 & 1 & 0 \\ 1.5 & 0 & -2 \end{bmatrix}$.
12. Give an example of axonometric projection.
13. Develop the cavalier and cabinet projection for $\alpha = 120°$ of the object $X = \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$.
14. Write the transformation matrix for orthographic projection to create bottom view of the object.
15. Write the transformation matrix for the perspective projection from the centre of projection Z_b (0 50 0) onto the y = 0 plane.
16. Write the transformation matrix which is required to transform the plane z = 0 to the plane x = 5.

Answers

1. $f_x = 4.8011$, $f_y = 3.6527$, $f_z = 3.7355$
2. $[T] = \begin{bmatrix} 0.8660 & 0.3535 & 0 & 0 \\ 0 & 0.7071 & 0 & 0 \\ 0.5 & -0.6123 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
3. Cavalier projection, Cabinet projection

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ -0.866 & 0.5 & 0 & 1 \\ -0.866 & -0.5 & 0 & 1 \\ 0.134 & -0.5 & 0 & 1 \\ 0.134 & 0.5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ -0.866 & 0.5 & 0 & 1 \\ -0.866 & -0.5 & 0 & 1 \\ 0.134 & -0.5 & 0 & 1 \\ 0.134 & 0.5 & 0 & 1 \end{bmatrix}$$

4. $f_x = 4.2472$, $f_y = 23.9253$, $f_z = 4.5512$
5. $\begin{bmatrix} A^* \\ B^* \end{bmatrix} = \begin{bmatrix} 1.4142 & -1.633 & 0 & 1 \\ -2.1214 & 4.4903 & 0 & 1 \end{bmatrix}$
6. $\begin{bmatrix} -0.866 & -0.5 & 0 & 1 \\ 0.134 & 0.5 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} -0.4531 & -0.2113 & 0 & 1 \\ 0.5469 & -0.2113 & 0 & 1 \end{bmatrix}$

7. $[T] = \begin{bmatrix} 0.9897 & 0.0201 & 0 & 0 \\ 0 & 0.9899 & 0 & 0 \\ 0.1428 & -0.1399 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

8. $P^* = [0 \quad 0.8168 \quad 0]$

9. $[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -0.2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

10. $[T] = \begin{bmatrix} 0.9614 & -0.07330 & 0 & 0 \\ 0 & 0.9642 & 0 & 0 \\ 0.2750 & -0.2550 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

11. $X^* = \begin{bmatrix} 0 & -2.4 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

12. Following are some of the examples of axonometric projection :
    (i) Shadow created by sun.
    (ii) Image of plane occurred by telecasting.

13. The transformation matrix by Cavalier projection is, $\begin{bmatrix} 2 & 0.268 & 0 & 1 \\ 1 & 0.732 & 0 & 1 \end{bmatrix}$

    The transformation matrix is by cabinet projection is, $\begin{bmatrix} 1.5 & 1.134 & 0 & 1 \\ 0.5 & 0.134 & 0 & 1 \end{bmatrix}$

14. $[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

15. $[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{50} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

16. $[T] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & 1 \end{bmatrix}$

Chapter 4...

Plane and Space Curves

(A) Plane Curves

T.J. Sleiltjes
(1856-1894 A.D.)

He was a personal friend of Charles Hermite due to whose efforts he obtained the post of professor of mathematics at Tavlouse. He made research on Integration of Calculus. He discussed many mathematical problems with Hermite by means of personal letters. They were so deep rooted to mathematics that the contents of the letters were published as "Correspondence" in 1905 after his death in which he made vivid and elaborate discussions about complex variables. The integrals which he had indicated therein are now-a-days called "Steiltjes Integral".

4.1 Introduction

We know that variety of techniques is available for drawing and designing curves. Here, we are going to discuss two-dimensional curve generation techniques. A curve is two-dimensional if it lies completely in one plane. In this chapter, we discuss only the conic sections.

4.2 Curve Representation

A curve may be represented as a collection of points, provided that the points are properly spaced and are connected by short straight line segment. Even though the curve may be adequately represented as a collection of points, the analytical representation has various advantages. By an analytical representation, properties such as radius of curvature and scope can be easily determined. When the curve is represented by a collection of points, the intermediate points are obtained by using interpolation, whereas any point on an analytically represented curve can be precisely determined.

There are various techniques for analytical representation of two-dimensional curves. Analytical definition of curve from the known set of data points is the interpolation problem. A curve that passes through all given set of data points is said to fit the given data.

If the curve does not pass through the given set of data points, the curve is said to fail the data. The technique for it is the method of least squares approximation. This method gives curve of the form $y = f(x)$. For example, ($y = ax^b$, $y = ae^{bx}$) etc. where a, b are constant.

4.3 Non-parametric Curves

For a plane curve, the non-parametric form is given by,

$$y = f(x) \quad \text{... (1)}$$

For example, equation of straight line $y = mx + c$ is in the non-parametric form.

A general second degree equation

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$$

represents a conic, which is two-dimensional curve.

(1) represents an ellipse if $ab - h^2 > 0$; a parabola, if $ab - h^2 = 0$; and a hyperbola if $ab - h^2 < 0$. If coefficient of x^2 and y^2 are same and there is no term in xy then we get a circle.

A straight line is obtained by putting $a = b = h = 0$.

The equation (1) can then becomes

$$2gx + 2fy + c = 0, \text{ which represents a line.}$$

4.4 Parametric Curves

In parametric form, each point on the curve is represented by a function of single parameter t. For two-dimensional curve, the cartesian co-ordinates of a point P on the curve with t as parameter are given by,

$$x = x(t), y = y(t)$$

and the position vector of a point P on it is given by,

$$P(t) = [x(t) \;\; y(t)]$$

The slope of the curve is,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}$$

where dash (´) represents differentiation with respect to t.

The curve end points and the length are fixed by the range of parameter. It is convenient to take t between 0 and 1.

If P_1 and P_2 are two position vectors of points on the straight line, the parametric representation of the straight line segment between two position vectors P_1 and P_2 is,

$$P(t) = P_1 + (P_2 - P_1)\,t \qquad 0 \le t \le 1$$

or

$$\left.\begin{aligned} x(t) &= x_1 + (x_2 - x_1)\,t \\ y(t) &= y_1 + (y_2 - y_1)\,t \end{aligned}\right\} \quad 0 \le t \le 1$$

Example : For the position vectors P_1 [3 4] and P_2 [5 2], determine the parametric representation of the line segment between them. Also determine the slope and tangent vector of the line segment. The parametric representation of the straight line is,

$$P(t) = P_1 + (P_2 - P_1)\,t$$

$$\Rightarrow \quad P(t) = [3\;4] + [2\;-2]\,t \;; \quad 0 \le t \le 1$$

Parametric representation of x and y components are,

$$x(t) = x_1 + (x_2 - x_1)\,t$$

$$\Rightarrow \quad x(t) = 3 + 2t$$

$$y(t) = y_1 + (y_2 - y_1)\,t$$

$$\Rightarrow \quad y(t) = 4 + (-2)\,t$$

Tangent vector $\quad P'(t) = [x'(t) \;\; y'(t)]$

$$= [2 \; -2]$$

Slope of the line segment is,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$= \frac{-2}{2} = -1$$

Remark : A comparison of non-parametric and parametric representations for a circle in the first quadrant :

The non-parametric representation of the unit circle in the first quadrant is,

$$y = \sqrt{1 - x^2}\,, \quad (0 \le x \le 1) \qquad \text{... (1)}$$

and its parametric representation is

$$\left.\begin{aligned} x &= \cos\theta \\ y &= \sin\theta \end{aligned}\right\} \quad (0 \le \theta \le 2\pi) \qquad \text{... (2)}$$

$$\Rightarrow \quad P(\theta) = [x \;\; y]$$

$$= [\cos\theta \;\; \sin\theta] \; (0 \le \theta \le 2\pi) \qquad \text{... (3)}$$

This representation may not be unique.

For example,

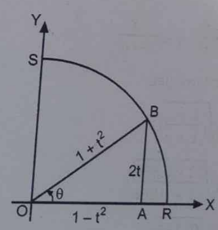


Fig. 4.1

$$P(t) = \left[\frac{1 - t^2}{1 + t^2} \;\; \frac{2t}{1 + t^2}\right] \qquad (0 \le t \le 1) \qquad \text{... (4)}$$

In a right angled Δ OAB,

$$x = \cos\theta = \frac{OA}{OB} = \frac{1 - t^2}{1 + t^2}, \quad 0 \le \theta \le \frac{\pi}{2}$$

$$y = \sin\theta = \frac{AB}{OB} = \frac{2t}{1 + t^2}, \quad 0 \le t \le 1$$

which also represents part of unit circle in first quadrant.

Illustrative Examples

Example 4.1 : Determine the value of y on the unit circle, given that $x = 0.866$.

Solution : The non-parametric representation of the unit circle in the first quadrant is given by,

$$y = \sqrt{1-x^2}\ ,\ 0 \le x \le 1$$

$$\Rightarrow \quad y = \sqrt{1-(0.866)^2}$$

$$\Rightarrow \quad y = \sqrt{1-0.7499}$$

$$\Rightarrow \quad y = 0.5$$

First, we have to find parameter t in terms of x and then in terms of y.

The parametric representation of unit circle is,

$$x = \cos\theta,\ y = \sin\theta$$

$$\Rightarrow \quad \theta = \cos^{-1}(x)$$

$$\Rightarrow \quad \theta = \cos^{-1}(0.866) = 30°$$

$$y = \sin 30° = 0.5$$

Thus, $\quad y = 0.5$

Alternatively, we have

$$x = \frac{1-t^2}{1+t^2},\ y = \frac{2t}{1+t^2}$$

Solving these two equations for t, we get,

$$t = \sqrt{\left(\frac{1-x}{1+x}\right)}$$

$$\Rightarrow \quad t = \sqrt{\left(\frac{1-0.866}{1+0.866}\right)}$$

$$= \sqrt{\left(\frac{0.134}{1.866}\right)}$$

$$\Rightarrow \quad t = 0.2679$$

Thus, $$y = \frac{2t}{1+t^2}$$

$$= \frac{2\,(0.2679)}{1+(0.2679)^2} = 0.5$$

$$\Rightarrow \quad y = 0.5$$

For finding the unknown value of an variable, an iterative technique is used for parametric representations. Both representations (parametric and non-parametric) find useful applications in computer graphics.

4.5 Parametric Representation of a Circle

We know that,

(i) The equation of a circle with centre at the origin (0, 0) and radius r is,

$$x^2 + y^2 = r^2$$

(ii) The equation of the circle with centre at (h, k) and radius r is,

$$(x-h)^2 + (y-k)^2 = r^2$$

The circle $x^2 + y^2 = r^2$ has the following two sets of parametric equations :

$$\left.\begin{array}{ll}\text{(i)} & x = r\cos\theta,\ y = r\sin\theta \\ \text{(ii)} & x = r\sin\theta,\ y = r\cos\theta\end{array}\right\}\ 0 \le \theta \le 2\pi$$

The circle $(x-h)^2 + (y-k)^2 = r^2$ has the following two sets of parametric equations :

$$\left.\begin{array}{ll}\text{(i)} & x = h + r\cos\theta,\ y = k + r\sin\theta \\ \text{(ii)} & x = h + r\sin\theta,\ y = k + r\cos\theta\end{array}\right\}\ 0 \le \theta \le 2\pi$$

Iteration process :

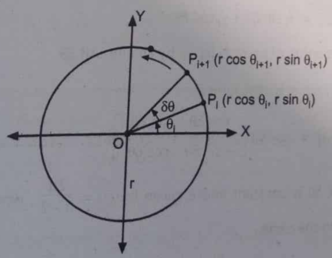


Fig. 4.2

Let the point P_i (where i = 1, 2, 3, ..., n) lies evenly on the origin centered circle having radius r. Since the points on the circle are uniformly spaced, there is an equal increment in the angle θ. Let this increment be θ.

Now, at point P_i (x_i, y_i), we have,

$$x_i = r\cos\theta_i$$

$$y_i = r\sin\theta_i \qquad \text{... (1)}$$

∴ At point P_{i+1} (x_{i+1}, y_{i+1}), we have

$$x_{i+1} = r\cos\theta_{i+1} \qquad \text{... (2)}$$

$$y_{i+1} = r\sin\theta_{i+1}$$

But $\theta_{i+1} = \theta_i + \delta\theta$

$\therefore$ (1) $\Rightarrow$ $x_{i+1} = r\cos(\theta_i + \delta\theta)$

$= r[\cos\theta_i \cos\delta\theta - \sin\theta_i \sin\delta\theta]$

$= (r\cos\theta_i)\cos\delta\theta - (r\sin\theta_i)\sin\delta\theta$

$= x_i\cos\delta\theta - y_i\sin\delta\theta$

Similarly (2) $\Rightarrow$ $y_{i+1} = r\sin(\theta_i + \delta\theta)$

$= r[\sin\theta_i\cos\delta\theta + \cos\theta_i\sin\delta\theta]$

$= (r\sin\theta_i)\cos\delta\theta + (r\cos\theta_i)\sin\delta\theta$

$= y_i\cos\delta\theta + x_i\sin\delta\theta$

$= x_i\sin\delta\theta + y_i\cos\delta\theta$

Thus, we get the recursion equations

$$x_{i+1} = x_i\cos\delta\theta - y_i\sin\delta\theta$$
$$y_{i+1} = x_i\sin\delta\theta + y_i\cos\delta\theta \quad \text{... (3)}$$

which represents the rotation of point $P_i(x_i, y_i)$ by an amount $\delta\theta$.

Equation (3) can be written in matrix form as,

$$[x_{i+1} \;\; y_{i+1}] = [x_i \;\; y_i]\begin{bmatrix} \cos\delta\theta & \sin\delta\theta \\ -\sin\delta\theta & \cos\delta\theta \end{bmatrix}$$

Note : The increment $\delta\theta$ is constant and is given by $\delta\theta = \frac{2\pi}{n-1}$, where n is the number of uniformly spaced points on the circle.

Since a circle is a closed curve, the first point ($\theta = 0$) and the last point ($\theta = 2\pi$) coincide. Thus, for getting n unique points on the circle, it is necessary to calculate (n + 1) points.

Thus, $\delta\theta = \frac{2\pi}{(n+1-1)}$

$= \frac{2\pi}{n}$

Hence, for open curve, $\delta\theta = \frac{2\pi}{n-1}$

and for closed curve, $\delta\theta = \frac{2\pi}{n}$

Parametric Generation of Circle :

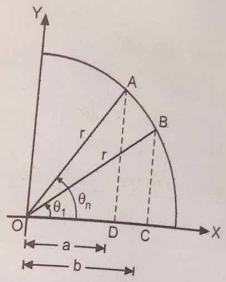


Fig. 4.3

Draw the circular arc of the origin centered circle of radius r in the first quadrant. Consider uniformly spaced n points on the arc of the circle.

If arc lies in range $a \le x \le b$.

At point B, in a right angled Δ OBC,

$$\cos\theta_1 = \frac{OC}{OB}$$

$$\Rightarrow \quad \cos\theta_1 = \frac{b}{r}$$

$$\Rightarrow \quad \theta_1 = \cos^{-1}\left(\frac{b}{r}\right)$$

Therefore at point A, in a right angled Δ OAD

$$\theta_n = \cos^{-1}\left(\frac{a}{r}\right)$$

Since n points are to be generated on this open arc, the range is uniformly divided into (n – 1) parts.

$$\therefore \quad \delta\theta = \frac{\theta_n - \theta_1}{n-1}$$

Theorem 1 Obtain an algorithm to generate uniformly spaced n points on the circle $x^2 + y^2 = r^2$.

The given circle is $x^2 + y^2 = r^2$

Algorithm :

Step 1 : First, calculate : $\delta\theta = \frac{2\pi}{n}$

Step 2 : Obtain the values of $\cos\delta\theta$ and $\sin\delta\theta$.

Step 3 : Here, initial points are

$$\left.\begin{aligned} x_1 &= r \\ y_1 &= 0 \end{aligned}\right\} \Rightarrow [x_1 \ y_1] = [r, 0]$$

Step 4 : Next obtain $[x_2 \ y_2]$, $[x_3 \ y_3]$, ... etc.

Using $$[x_{i+1} \ y_{i+1}] = [x_i \ y_i]\begin{bmatrix} \cos\delta\theta & \sin\delta\theta \\ -\sin\delta\theta & \cos\delta\theta \end{bmatrix}$$

Step 5 : Represent the points generated by the position vector matrix [X].

Theorem 2 Obtain an algorithm to generate uniformly spaced n points on the circle $(x-h)^2 + (y-k)^2 = r^2$.

Algorithm : The given circle is $(x-h)^2 + (y-k)^2 = r^2$

Step 1 : First, calculate $\delta\theta = \frac{2\pi}{n}$

Step 2 : Calculate the values of $\cos\delta\theta$ and $\sin\delta\theta$.

Step 3 : Generate n points on the circle $x^2 + y^2 = 1$. i.e. on the circle with centre (0, 0) and radius 1.

Step 4 : Let $[X]_{n\times 3}$ denote the n points on the circle $x^2 + y^2 = 1$.

Step 5 : Obtain the scaling matrix [S], for scaling x and y directions by a factor r.

Step 6 : Next, obtain the translation matrix $[T_r]$, by translating origin (0, 0) to new origin (h, k).

Step 7 : From steps 5 and 6, the concatenated transformation matrix is,

$$[T] = [S]\,[T_r]$$

Step 8 : The points on the circle $(x-h)^2 + (y-k)^2 = r^2$ are given by

$$[X'] = [X]\,[T]$$

Step 9 : Write [X'] with two-dimensional co-ordinates.

Illustrative Examples

Example 4.2 : Obtain 8 uniformly spaced points on the circle $x^2 + y^2 = 1$.

Solution : Here n = 8.

$$\therefore \quad \delta\theta = \frac{2\pi}{n} = \frac{2\pi}{8} = \frac{\pi}{4}$$

Starting with $\theta_1 = 0$, the initial values of x and y are x_1 and y_1.

$$\therefore \quad x_1 = r\cos\theta_1 = (1)\cos(0) = 1 \qquad (\because r = 1)$$

$$y_1 = r\sin\theta_1 = (1)\sin(0) = 0$$

The remaining seven unique points can be obtained.

First, $$\sin\delta\theta = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$\cos\delta\theta = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$\therefore \quad x_2 = x_1\cos\delta\theta - y_1\sin\delta\theta = (1)\left(\frac{1}{\sqrt{2}}\right) - 0\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$$

$$y_2 = x_1\sin\delta\theta + y_1\cos\delta\theta = (1)\left(\frac{1}{\sqrt{2}}\right) + 0\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$$

The remaining values of x and y are shown in the table below :

i	x_i	y_i
1	1	0
2	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
3	0	1
4	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$
5	−1	0
6	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
7	0	−1
8	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$
9	1	0

Example 4.3 : Let [X] represent n points on the circle $x^2 + y^2 = 1$, obtain matrix representing the circle $(x+3)^2 + (y-2)^2 = 16$.

Solution : Let $[X]_{n\times 3}$ + n points on the circle $x^2 + y^2 = 1$ and $[X']_{n\times 3}$ + n points on the circle $(x+3)^2 + (y-2)^2 = 1$.

The combined 3 × 3 transformation is obtained, first scaling by the factor 4 and then translating the origin (0, 0) to new origin (− 3, 2).

$$\therefore \quad [T] = [S]\,[T_r]$$

$$\Rightarrow \quad [T] = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ -3 & 2 & 1 \end{bmatrix}$$

Applying [T] on [X], we get,

$$[X'] = [X]\,[T]$$

$$\Rightarrow \quad [X'] = [X]\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ -3 & 2 & 1 \end{bmatrix}$$

Example 4.4 : Let [X] represents n points of the origin centered circle of radius 2. Obtain matrix representing circle of radius 2 with centre located at (2, 2).

Solution : Let $[X]_{n \times 3}$ + n points on the circle $x^2 + y^2 = 2$ and $[X']_{n \times 3}$ + n points on the circle $(x-2)^2 + (y-2)^2 = 2$.

The combined 3 × 3 transformation is obtained, first scaling by the factor 2 and then translating the centre (0, 0) of the circle to the point (2, 2).

$$\therefore \quad [T] = [S]\,[T_r]$$

$$\Rightarrow \quad [T] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

$$\Rightarrow \quad [X'] = [X]\,[T]$$

$$\Rightarrow \quad [X'] = [X] \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

Example 4.5 : Obtain 4 uniformly spaced points in the first quadrant of the unit circle with centre at origin.

Solution : Since the arc of the circle lies in the first quadrant, here $0 \le \theta \le \frac{\pi}{2}$.

$$\therefore \quad \delta\theta = \frac{\frac{\pi}{2}}{n} = \frac{\frac{\pi}{2}}{4} = \frac{\pi}{8} \qquad (\because n = 4)$$

For $\theta = 0$

$$x_1 = r\cos\theta_1 = (1)\cos(0) = 1$$

$$y_1 = r\sin\theta_1 = (1)\sin(0) = 0$$

For getting the remaining points, we have

$$\sin\delta\theta = \sin\frac{\pi}{8} = 0.3827$$

$$\cos\delta\theta = \cos\frac{\pi}{8} = 0.9239$$

$$x_2 = x_1\cos\delta\theta - y_1\sin\delta\theta = (1)(0.9239) - (0)(0.3827) = 0.9239.$$

$$y_2 = x_1\sin\delta\theta + y_1\cos\delta\theta = (1)(0.3827) + (0)(0.9239) = 0.3827$$

The other uniformly spaced points are shown in the table below.

i	x_i	y_i
1	1	0
2	0.9239	0.3827
3	0.7070	0.7070
4	0.3827	0.9239
5	0	1

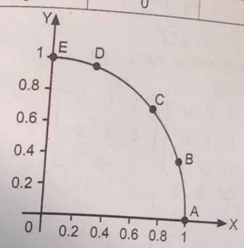


Fig. 4.4

Example 4.6 : Write an algorithm to generate 36 points on the circle $(x-2)^2 + (y+2)^2 = 25$. **(April 1995)**

Solution : The given circle is, $(x-2)^2 + (y+2)^2 = 25$

Here,

(i) Number of points on the circle = n = 36.

(ii) Radius of given circle = r = 5.

(iii) Centre of the given circle = (h, k) = (2, − 2).

Step 1 : Compute, $\delta\theta = \frac{2\pi}{n} = \frac{2\pi}{36} = \frac{\pi}{18}$

Step 2 : $\cos\delta\theta = \cos\frac{\pi}{18} = 0.9848$

$$\sin\delta\theta = \sin\frac{\pi}{18} = 0.1736$$

Step 3 : Generate 36 points on the circle $x^2 + y^2 = 1$.

Step 4 : Let $[X]_{36 \times 3}$ denote the 36 points on the circle $x^2 + y^2 = 1$, using

$$[x_{i+1} \;\; y_{i+1}] = [x_i \;\; y_i] = \begin{bmatrix} \cos\delta\theta & \sin\delta\theta \\ -\sin\delta\theta & \cos\delta\theta \end{bmatrix}$$

⇒ where i = 1, 2, 3 ... 36 and initial points are,

$$x_1 = 1, \;\; y_1 = 0$$

i.e. $[x_1 \;\; y_1] = [1, 0]$.

Step 5 : Scaling matrix [S], for scaling x and y directions by a factor 5 is,

$$[S] = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 6 : Translation matrix $[T_r]$ by translating (0, 0) to (2, – 2) is,

$$[T_r] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}$$

Step 7 : The concatenated transformation matrix [T] is,

$$[T] = [S]\,[T_r]$$

$$\Rightarrow \quad [T] = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -2 & 1 \end{bmatrix}$$

$$\Rightarrow \quad [T] = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 2 & -2 & 1 \end{bmatrix}$$

Step 8 : The 36 points on the circle $(x-2)^2 + (y+2)^2 = 25$ are given by,

$$[X']_{36\times 3} = [X']_{36\times 3}\,[T]$$

Step 9 : Write [X'] with two-dimensional co-ordinates.

Example 4.7 : Generate 8 points on the circle, $(x-2)^2 + (y-4)^2 = 25$

Solution : The given circle is,

$$(x-2)^2 + (y-4)^2 = 25$$

Here, number of points = n = 8

radius = r = 5

centre = (h, k) = (2, 4)

$$\delta\theta = \frac{2\pi}{8} = \frac{\pi}{4}$$

8 points on the circle $x^2 + y^2 = 1$ are given below :

$$[X] = \begin{bmatrix} 1 & 0 & 1 \\ 0.7071 & 0.7071 & 1 \\ 0 & 1 & 1 \\ -0.7071 & 0.7071 & 1 \\ -1 & 0 & 1 \\ -0.7071 & -0.7071 & 1 \\ 0 & -1 & 1 \\ 0.7071 & -0.7071 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

Scaling matrix [S], for scaling x and y directions by a factor 5 is

$$S = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Translating matrix $[T_r]$ by translating (0, 0) to (2, 8/4)

$$[T_r] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix}$$

The combined transformation matrix [T] is,

$$[T] = [S]\,[T_r]$$

$$\Rightarrow \quad [T] = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 4 & 1 \end{bmatrix}$$

$$\Rightarrow \quad [T] = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 2 & 4 & 1 \end{bmatrix}$$

The points [X'] on the circle $(x-2)^2 + (y-4)^2 = 25$ are obtained by applying [T] on [X].

$$[X'] = \begin{bmatrix} 1 & 0 & 1 \\ 0.7071 & 0.7071 & 1 \\ 0 & 1 & 1 \\ -0.7071 & 0.7071 & 1 \\ -1 & 0 & 1 \\ -0.7071 & -0.7071 & 1 \\ 0 & -1 & 1 \\ 0.7071 & -0.7071 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 2 & 4 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 4 & 1 \\ 5.5355 & 7.5355 & 1 \\ 2 & 9 & 1 \\ -1.5355 & 7.5355 & 1 \\ -3 & 4 & 1 \\ -1.5355 & 0.4645 & 1 \\ 2 & -1 & 1 \\ 5.5355 & 0.4645 & 1 \\ 7 & 4 & 1 \end{bmatrix}$$

$\therefore$ Eight points are :

i	x_i	y_i
1	7	4
2	5.5355	7.5355
3	2	9
4	– 1.5355	7.5355
5	– 3	4
6	– 1.5355	0.4645
7	2	– 1
8	5.5355	0.4645
9	7	4

Example 4.8 : Generate 5 uniformly spaced points on the arc of a circle $x^2 + y^2 = 49$, in the first quadrant. **(March 2006)**

Solution : Given circle $x^2 + y^2$ and n = 5.

Therefore, $\delta\theta = \frac{\pi/2}{n-1} = \frac{\pi/2}{4} = \frac{\pi}{8}$

Then $\cos\delta\theta = \cos\frac{\pi}{8} = 0.9238$

and $\sin\delta\theta = 0.3826$

The initial point is,

$$P_1[x_1\ y_1] = [7\ \ 0]$$

Now, $[x_{i+1}\ y_{i+1}] = [x_i\ y_i]\begin{bmatrix}\cos\delta\theta & \sin\delta\theta\\ -\sin\delta\theta & \cos\delta\theta\end{bmatrix}$

$\therefore$ $[x_{i+1}\ y_{i+1}] = [x_i\ y_i]\begin{bmatrix}0.9238 & 0.3826\\ -0.3826 & 0.9238\end{bmatrix}$

$i = 1 \Rightarrow [x_2\ y_2] = [x_1\ y_1]\begin{bmatrix}0.9238 & 0.3826\\ -0.3826 & 0.9238\end{bmatrix}$

$[x_2\ y_2] = [7\ \ 0]\begin{bmatrix}0.9238 & 0.3826\\ -0.3826 & 0.9238\end{bmatrix}$

$\therefore$ $[x_2\ y_2] = [6.4666\ \ 2.6782]$

i = 2 implies, $[x_3\ y_3] = [x_2\ y_2]\begin{bmatrix}0.9238 & 0.3826\\ -0.3826 & 0.9238\end{bmatrix}$

$\therefore$ $[x_3\ y_3] = [4.9492\ \ 4.9482]$

i = 3 implies, $[x_4\ y_4] = [x_3\ y_3]\begin{bmatrix}0.9238 & 0.3826\\ -0.3826 & 0.9238\end{bmatrix}$

$\therefore$ $[x_4\ y_4] = [4.9492\ \ 4.9482]\begin{bmatrix}0.9238 & 0.3826\\ -0.3826 & 0.9238\end{bmatrix}$

$\therefore$ $[x_4\ y_4] = [2.6789\ \ 6.4646]$

The point, $[x_5\ y_5] = [0\ \ 7]$

The five uniformly spaced points are shown in the following table.

Value of i	Point $[x_i\ \ y_i]$
1	[7 0]
2	[6.4666 2.6782]
3	[4.9482 4.9482]
4	[2.6789 6.4646]
5	[0 7]

Example 4.9 : Generate 5 uniformly spaced points on the arc of a circle $x^2 + y^2 = 49$, in the first quadrant.

Solution : The equation of circle is $x^2 + y^2 = 49$ and n = 5.

Therefore, $\delta\theta = \frac{\pi/2}{n-1} = \frac{\pi/2}{4}$

$\therefore$ $\delta\theta = \frac{\pi}{8}$

$\cos\delta\theta = \cos\frac{\pi}{8} = 0.9238$

$\sin\delta\theta = \sin\frac{\pi}{8} = 0.3826$

The initial point is $P_1[x_1\ y_1] = [7\ \ 0]$

Now, $[x_{i+1}\ y_{i+1}] = [x_i\ y_i]\begin{bmatrix}\cos\delta\theta & \sin\delta\theta\\ -\sin\delta\theta & \cos\delta\theta\end{bmatrix}$

$= [x_i\ y_i]\begin{bmatrix}0.9238 & 0.3826\\ -0.3826 & 0.9238\end{bmatrix}$

i = 1 gives, $[x_2\ y_2] = [x_1\ y_1]\begin{bmatrix}0.9238 & 0.3826\\ -0.3826 & 0.9238\end{bmatrix}$

$\therefore$ $[x_2\ y_2] = [7\ \ 0]\begin{bmatrix}0.9238 & 0.3826\\ -0.3826 & 0.9238\end{bmatrix}$

$\therefore$ $[x_2\ y_2] = [6.4666\ \ 2.6782]$

i = 2 gives, $[x_3\ y_3] = [x_2\ y_2]\begin{bmatrix}0.9238 & 0.3826\\ -0.3826 & 0.9238\end{bmatrix}$

$\therefore$ $[x_3\ y_3] = [14.9492\ \ 4.9482]$

As above, $[x_4 \; y_4] = [2.6789 \;\; 6.4646]$

and $[x_5 \; y_5] = [0 \;\; 7]$

The 5 uniformly spaced points are shown in the following table.

i	$[x_i \; y_i]$
1	[7 0]
2	[16.4666 2.6782]
3	[14.9492 4.9482]
4	[12.6789 6.4646]
5	[0 7]

Think Over It

For origin centered circle of radius r; obtain in usual notation

$$x_{i+1} = x_i \cos \delta\theta - y_i \sin \delta\theta$$
$$y_{i+1} = x_i \sin \delta\theta + y_i \cos \delta\theta$$

Points to Remember

1. A geometrical transformation in which one line, shape etc. is converted into another according to certain geometrical rule is called a projection.

Miscellaneous Exercises

(A) State whether the following statement are True or False :

1. In a parametric curve the curve end points and the length are fixed by the range of the parameter.
2. The non-parametric representation of the unit circle in the first quadrant is
$$y = \sqrt{1 - x^2}; \; -1 \le x \le 1$$
3. The parametric representation of unit circle in the first quadrant is
$$x = \cos \theta, \; y = \sin \theta; \; 0 \le \theta \le \frac{\pi}{2}.$$

ANSWERS

1. - True	2. - False	3. True

(B) Multiple Choice Questions

1. If P_1 and P_2 are two position vectors of points on the straight line, then parametric representation of the straight line segment between two position vectors P_1 and P_2 is ......
   (a) $P(t) = P_1 + (P_1 - P_2)\, t; \; 0 \le t \le 1$
   (b) $P(t) = P_1 + (P_2 - P_1)\, t; \; 0 \le t \le 1$
   (c) $P(t) = P_1 + (P_2 + P_1)\, t$
   (d) none of these
2. In usual notations $[x_{i+1} \; y_{i+1}]$ is given by ......
   (a) $[x_i \; y_i]\begin{bmatrix} \cos \delta\theta & -\sin \delta\theta \\ \sin \delta\theta & \cos \delta\theta \end{bmatrix}$
   (b) $[x_i \; y_i]\begin{bmatrix} \cos \delta\theta & \sin \delta\theta \\ -\sin \delta\theta & -\cos \delta\theta \end{bmatrix}$
   (c) $[x_i \; y_i]\begin{bmatrix} \cos \delta\theta & \sin \delta\theta \\ -\sin \delta\theta & \cos \delta\theta \end{bmatrix}$
   (d) none of these

Answers

1 - (b)	2 - (c)

(C) Theory Questions :

1. Derive the iterative matrix for generation of uniformly spaced n points on the circumference of the circle $x^2 + y^2 = r^2$.
2. State two sets of parametric equations for the circle $x^2 + y^2 = r^2$.
3. State two sets of parametric equations for the circle $(x - h)^2 + (y - k)^2 = r$.
4. Obtain an algorithm to generate uniformly spaced n points on the circle $x^2 + y^2 = r^2$.
5. Prove with usual notations $\delta\theta = \dfrac{\theta_n - \theta_1}{n - 1}$.
6. Write an algorithm to generate n equispaced points on the circumference of the circle $(x - h)^2 + (y - k)^2 = r^2$.

(D) Numerical Problems :

1. Generate 5 uniformly spaced points on the arc of the circle $x^2 + y^2 = 49$, in the second quadrant.
2. Generate 8 uniformly spaced points on the circle $(x - 3)^2 + (y + 1)^2 = 16$.
3. The position vectors of two points are $P_1[1 \;\; 2]$ and $P_2[4 \;\; 3]$. Determine the representation of the line segment between them and find the slope and tangent vector of the line segment.
4. The circle C is unit circle centered at origin. For a point P on it; x co-ordinate is 0.5. Determine the y-co-ordinate of P in two ways viz. without using parameter and by using parameter.
5. Obtain equispaced 6 points on the circle $(x - 1)^2 + (y + 2)^2 = 9$.

Answers

1. [10 7], [−2.6782 6.4666], [−4.9482 4.9492],

   [−6.4646 2.6789], [−7 0]

2. [7 −1], [5.8284 1.8284], [3 3], [0.1716 1.8284],

   [−1 −1], [10.1716 −3.8284], [3 −5] [5.8284 −3.8284]

3. Parametric representation is P(t) = [1 2] + [3 1] t; $0 \le t \le 1$ parametric representation of x and y co-ordinates are

   $x(t) = i + 3t,\ y(t) = 2 + t$, where $0 \le t \le 1$.

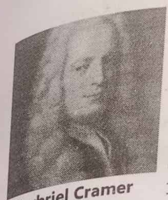
   The slope is $\frac{1}{3}$. The tangent vector is $\bar{v} = 3\bar{i} + \bar{j}$

4. y = 0.866

5. [4 −2], $\left[\frac{5}{2}\ \ \frac{3\sqrt{3}-4}{2}\right]$, $\left[-\frac{1}{2}\ \ \frac{3\sqrt{3}-4}{2}\right]$, [2 −2],

   $\left[-\frac{1}{2}\ \ \frac{3\sqrt{3}-4}{2}\right]$, $\left[\frac{5}{2}\ \ -\frac{3\sqrt{3}+4}{2}\right]$

(B) Space Curves



Garbriel Cramer (1703-1758 A.D.)

He was born in Hallstin family in Geneva. He passed some years at Strassburg and returned back to Geneva. When he was only 20 years old he joined as Professor of mathematics in Geneva University in the year 1824 A.D.

His book on Geometry was published in 1732 A.D. The demand for his History of mathematics was so much that it was published in 1739, 1741, 1748, 1750 A.D. successively.

Further his book on Algebraic curves was also published in 1750. All his books were published in Geneva. He published many research papers on Physics with mathematical background.

He died at Geneva when he was only 55 years old leaving friends and admirers by throwing all in the ocean of grief.

4.6 Introduction

Space curves play an important role in engineering, the design of automobile bodies, aircraft and space-craft surfaces, ship hulls. They also play an important role in medical science and geology.

Space curves and surfaces can be generated from the given set of user-specific data points. More precisely, the points that control the shape of the curve in a predictable ways, these points are called *control points*. (See Fig. 4.5)

Fig. 4.5

(Control points indicated by dots govern the shape of a curve.)

A polynomial function of degree n in x given by,

$$\sum_{i=0}^{n} a_i x^i.$$

When n = 1, we get first degree equation; when n = 2, we get a quadratic; when n = 3, we get a cubic polynomial and so on. Polynomials are useful in graphics, including the design of object shapes. Designing object shapes is done by specifying a few points to define a general curve contour, and then fitting the selected points with a polynomial.

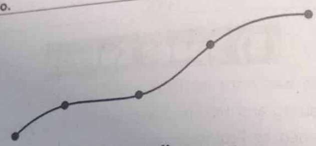


Fig. 4.6 : Spline curve

One way to accomplish the curve fitting is to construct the cubic polynomial sections between specified points. Continuous curves that are formed with pieces of polynomial function curves are called *spline curves*.

In drafting, a spline is defined as a flexible strip used to produce a smooth curve through specified data points. Spline curves are shaped by distributing several small weights along the length of the strip. By changing the number and position of the weights, the spline curve is made to pass through the designated set of points so that the resulting curve appears smooth.

We specify a spline curve by giving data points (control points), which indicates the shape of the curve. These control points are then fitted with piecewise continuous parametric polynomial functions in two ways.

When polynomial sections are fitted such that the curve passes through each control point, the resulting curve is said to *interpolate* the set of control points. It is shown in Fig. 4.7 below.

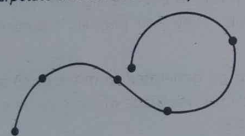


Fig. 4.7

When the polynomial sections are fitted to the general control point path without necessarily passing through any control points, the resulting curve is said to approximate the set of control points. See Fig. 4.20 below.

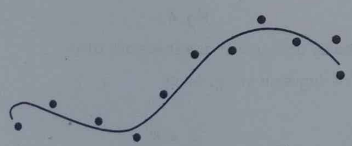


Fig. 4.8

A spline curve is defined, modified and manipulated with operations on the control points. A spline curve can be translated, scaled or rotated with transformation applied to the control points.

A convex polygon boundary that encloses a set of control points is called a *convex hull*.

A convex hull shapes for two sets of control points are shown by dotted lines in the Fig. 4.9 (a) and (b).

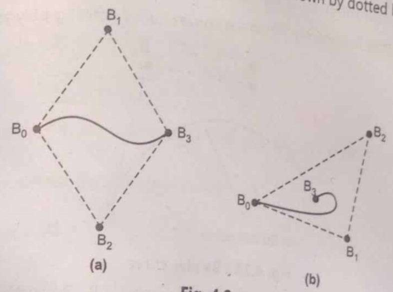


Fig. 4.9

There are three methods for representation of spline curve :

(i) Using the set of boundary conditions imposed on the spline curve.

(ii) Using the matrix which characterizes the spline curve.

(iii) Using the set of blending functions (or Bernstein basis functions) which determine how specified geometric constrains on the curve are combined for calculating positions along the curve path.

The mathematical description of a space curve is obtained without previous knowledge regarding the shape of the curve.

In this chapter, we discuss two techniques : Be'zier curves and their generalization to B-spline curves. These techniques are characterized by the fact that any points on the curve pass through the control points used to define the curve.

4.7 Be'zier Curves

First, we discuss several properties of Be'zier curves.

Properties :

(i) It always passes through the first and last control points. See Fig. 4.10.

(ii) Be'zier curves has variation-diminishing property. This means it never oscillates widely from its defining control points.

(iii) Bernstein basis functions are real.

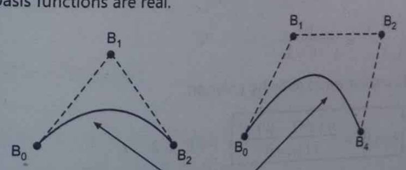


Fig. 4.10 : Be'zier curves

(iv) The curve follows the shape of the defining polygon.

(v) The curve is invariant under affine transformation.

(vi) The Be'zier curve lies entirely within the convex hull of defining polygon.

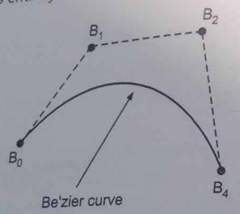


Fig. 4.11 : Be'zier curve

(vii) The direction of the tangent vector at the end points B_0, B_n is the same as that of the vector determined by the first and last segments $\overline{B_0 B_1}$, $\overline{B_{n-1} B_n}$ of the defining polygon.

(viii) Be'zier curve is a polynomial of degree one less than the number of control points used.

Be'zier curves are generally found in painting, drawing, packages and CAD systems, as they are reasonably powerful in curve design. Methods for determining position of co-ordinate along a Be'zier curve can be set-up by using recurrence relations.

A parametric equation of the Be'zier curve is,

$$[P(t)] = (x(t), y(t))$$

$$P(t) = \sum_{i=0}^{n} B_i J_{n,i}(t), \quad 0 \le t \le 1 \qquad \text{... (1)}$$

i.e. $P(t) = B_0 J_{n,0}(t) + B_1 J_{n,1}(t) + B_2 J_{n,2}(t) \ldots + B_n J_{n,n}(t)$

where $J_{n,i}(t)$ is the Bernstein function or blending function, which is defined as,

$$J_{n,i}(t) = \binom{n}{i} t^i (1-t)^{n-i}, \; i = 0, 1, 2, \ldots\ldots n \qquad \text{... (2)}$$

with $\quad \binom{n}{i} = \dfrac{n!}{i!(n-1)!}$

and $B_0, B_1, B_2, \ldots, B_n$ are known as defining polygon.

(2) $\quad \boxed{J_{n,i}(t) = \dfrac{n!\, t^i (1-t)^{n-i}}{i!(n-i)!}}, \; i = 0, 1, 2, \ldots n \qquad \text{... (3)}$

For the first point on the Be'zier curve i.e. at t = 0;

$$J_{n,0}(0) = \frac{n!(1)^0(1-0)^{n-0}}{(1!)(n!)} = 1, \text{ for } i = 0$$

and $\quad J_{n,i}(0) = \dfrac{n!\,0^i(1-0)^{n-i}}{i!(n-i)!} = 0, \text{ for } i \neq 0$

$\therefore \quad \boxed{P(0) = B_0 J_{n,0}(0) = B_0}$

Thus, the first point on the Be'zier curve and the first point on its defining polygon are coincident.

For the last point on the Be'zier curve i.e. at t = 1 :

(3) $\quad J_{n,n}(1) = \dfrac{n!(1)^n(0)^{n-n}}{n!(1)} = 1, \text{ for } i = n$

and $\quad J_{n,i}(1) = \dfrac{n!\,t^i(1-1)^{n-i}}{n!(1)} = 0, \text{ for } i \neq n$

$\therefore \quad \boxed{P(1) = B_n J_{n,n}(1) = (1) = B_n}$

Thus, the last point on the Be'zier curve and the last point on its defining polygon are coincident.

Note : $\quad \sum_{i=0}^{n} J_{n,i}(t) = 1$

Here we consider some cases (for n = 2, 3, 4 etc.) which will be helpful for solving some examples.

When n = 2 :

(1)
$$\begin{aligned} P(t) &= B_0 J_{2,0}(t) + B_1 J_{2,1}(t) + B_2 J_{2,2}(t). \\ &= B_0 (1-t)^2 + B_1\, 2t(1-t) + B_2 t^2 \\ &= (t^2 - 2t + 1) B_0 + (-2t^2 + 2t) B_1 + t^2 B_2 \qquad \text{... (4)} \end{aligned}$$

This can be written in matrix form as,

$$P(t) = [\,t^2 \;\; t \;\; 1\,] \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \\ B_2 \end{bmatrix}$$

$$P(t) = [T]\,[N]\,[G]$$

When n = 3 :

(1) $P(t) = J_{3,0}(t) B_0 + J_{3,1}(t) B_1 + J_{3,2}(t) B_2 + J_{3,3}(t) B_3$

$= (1-t)^3 B_0 + 3t(1-t)^2 B_1 + 3t^2(1-t) B_2 + t^3 B_3$ [using equation (3)]

$= (-t^3 + 3t^2 - 3t + 1) B_0 + (3t^3 - 6t^2 + 3t) B_1 + (-3t^3 - 3t^2) B_2 + t^3 B_3$ ... (5)

This can be written in matrix form as,

$$P(t) = [t^3\ t^2\ t\ 1]\begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

$$\therefore \quad P(t) = [T]\,[N]\,[G]$$

When n = 4 :

$$= J_{4,0}(t)\,B_0 + I_{4,1}(t)\,B_1 + J_{4,2}(t)\,B_2 + J_{4,3}(t)\,B_3 + J_{4,4}(t)\,B_4$$

On expansion, we get

$$P(t) = (-t^4 - 4t^3 + 6t^2 - 4t + 1)\,B_0 + (-4t^4 + 12t^3 - 12t^2 + 4t)\,B_1 + (6t^4 - 12t^3 + 6t^2)\,B_2 + (-4t^4 + 4t^3)\,B_3 + t^4 B_4 \quad \ldots (6)$$

This can be written in matrix form as,

$$P(t) = [t^4\ t^3\ t^2\ t\ 1]\begin{bmatrix} 1 & -4 & 6 & -4 & 1 \\ -4 & 12 & -12 & 4 & 0 \\ 6 & -12 & 6 & 0 & 0 \\ -4 & 4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \\ B_4 \end{bmatrix}$$

$$\therefore \quad P(t) = [T]\,[N]\,[G]$$

Illustrative Examples

Example 4.10 : If B_0 [1 1], B_1 [2 3], B_2 [4 3] and B_3 [3 1] are the vertices of a Be'zier polygon, then determine the point $P\left(\frac{1}{2}\right)$ of the Be'zier curve.

Solution : The parametric equation of the Be'zier curve is,

$$P(t) = J_{3,0}(t)\,B_0 + J_{3,1}(t)\,B_1 + J_{3,2}(t)\,B_2 + J_{3,3}(t)\,B_3,\quad 0 \le t \le 1.$$

$$= (1-t)^3 B_0 + 3t(1-t)^2 B_1 + 3t^2(1-t)\,B_2 + t^3 B_3$$

$$= (1-t)^3 [1\ 1] + 3t(1-t)^2 [2\ 3] + 3t^2(1-t)\,[4\ 3] + 1^3 [3\ 1]$$

$$\therefore \quad P\left(\frac{1}{2}\right) = (0.5)^3 [1\ 1] + 3(0.5)(0.5)^2 [2\ 3] + 3(0.5)^2(0.5)\,[4\ 3] + (0.5)^3 [3\ 1]$$

$$= 0.125\,[1\ 1] + 0.375\,[2\ 3] + 0.375\,[4\ 3] + 0.125\,[3\ 1]$$

$$= [2.75\ 2.5]$$

Example 4.11 : Find the parametric equation of a Be'zier curve determined by control points B_0 [2 1], B_1 [4 3], B_2 [6 0.5] and hence find the position vector of the point corresponding to parameter value t = 0.43.

Solution : The parametric equation of the Be'zier curve is,

$$P(t) = J_{2,0}(t)\,B_0 + J_{2,1}(t)\,B_1 + J_{2,2}(t)\,B_2$$

$$= (1-t)^2 B_0 + 2t(1-t)\,B_1 + t^2 B_2$$

$$= (1-t)^2 [2\ 1] + 2t(1-t)\,[4\ 3] + t^2 [6\ 0.5]$$

$$\therefore \quad P(0.43) = (1-0.43)^2 [2\ 1] + 2(0.43)(1-0.43)\,[4\ 3] + (0.43)^2 [6\ 0.5]$$

$$= 0.3249\,[2\ 1] + 0.4902\,[4\ 3] + 0.1849\,[6\ 0.5]$$

$$= [28.4694\ \ 1.3151]$$

$$P(0.25) = (1-0.25)^2 [2\ 1] + 2(0.25)(1-0.25)\,[4\ 3] + (0.25)^2 [6\ 0.5]$$

$$= 0.5625\,[2\ 1] + 0.375\,[4\ 3] + 0.0625\,[6\ 0.5]$$

$$= [36.375\ 1.4375]$$

Example 4.12 : Find the parametric equation of a Be'zier curve determined by control points B_0 [1 0], B_1 [2 3] and B_2 [4 1] and hence find the position vector of the point corresponding to parameter values t = 0.1, 0.2, 0.3, ... 0.9.

Solution : The parametric equation of the Be'zier curve is,

$$P(t) = J_{2,0}(t)\,B_0 + J_{2,2}(t)\,B_1 + J_{2,2}(t)\,B_2$$

$$= (1-t)^2 B_0 + 2t(1-t)\,B_1 + t^2 B_2$$

$$= (1-t)^2 [1\ 0] + 2t(1-t)\,[2\ 3] + t^2 [4\ 1]$$

$$\therefore \quad P(0.1) = (1-0.1)^2 [1\ 0] + 2(0.1)(1-0.1)\,[2\ 3] + (0.1)^2 [4\ 1]$$

$$= 0.81\,[1\ 0] + 0.18\,[2\ 3] + 0.01\,[4\ 1] = [1.21\ 0.55]$$

$$P(0.2) = 0.64\,[1\ 0] + 0.32\,[2\ 3] + 0.04\,[4\ 1] = [1.44\ 1]$$

$$P(0.3) = 0.49\,[1\ 0] + 0.42\,[2\ 3] + 0.09\,[4\ 1] = [1.69\ 1.35]$$

$$P(0.4) = 0.36\,[1\ 0] + 0.48\,[2\ 3] + 0.16\,[4\ 1] = [1.96\ 1.60]$$

$$P(0.5) = 0.25\,[1\ 0] + 0.45\,[2\ 3] + 0.25\,[4\ 1] = [2.15\ 1.60]$$

$$P(0.6) = 0.16\,[1\ 0] + 0.48\,[2\ 3] + 0.36\,[4\ 1] = [2.56\ 1.80]$$

$$P(0.7) = 0.09\,[1\ 0] + 0.42\,[2\ 3] + 0.49\,[4\ 1] = [2.89\ 1.75]$$

$$P(0.8) = 0.04\,[1\ 0] + 0.32\,[2\ 3] + 0.64\,[4\ 1] = [3.24\ 1.60]$$

$$P(0.9) = 0.01\,[1\ 0] + 0.18\,[2\ 3] + 0.81\,[4\ 1] = [3.61\ 1.35]$$

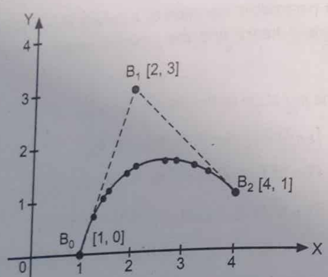


Fig. 4.12 : Be'zier curve and Be'zier polygon

Example 4.13 : Find the parametric equation of the Be'zier curve, where B_0 [0 2], B_1 [2 3], B_2 [3 2] and B_3 [2 0] are the vertices of the Be'zier polygon. Also find the position vectors of the points on the curve corresponding to parameter values 0.1, 0.2, 0.3, ..., 0.9.

Solution : The parametric equation of the Be'zier curve is,

$$P(t) = J_{3,0}(t)\,B_0 + J_{3,2}(t)\,B_1 + J_{3,2}(t)\,B_2 + J_{3,3}(t)\,B_3$$

$$= (1-t)^3 B_0 + 3t(1-t)^2 B_1 + 3t^2(1-t) B_2 + t^3 B_3.$$

$$= (1-t)^3 [0\ 2] + 3t(1-t)^2 [2\ 3] + 3t^2(1-t)[3\ 2] + t^3 [2\ 0].$$

$$P(0.1) = 0.729\,[0\ 2] + 0.243\,[2\ 3] + 0.027\,[3\ 2] + 0.001\,[2\ 0]$$
$$= [0.569\ 2.41].$$

$$P(0.2) = 0.512\,[0\ 2] + 0.384\,[2\ 3] + 0.096\,[3\ 2] + 0.008\,[2\ 0]$$
$$= [1.072\ 2.368].$$

$$P(0.3) = 0.343\,[0\ 2] + 0.441\,[2\ 3] + 0.189\,[3\ 2] + 0.027\,[2\ 0]$$
$$= [1.503\ 2.387].$$

$$P(0.4) = 0.216\,[0\ 2] + 0.432\,[2\ 3] + 0.288\,[3\ 2] + 0.064\,[2\ 0]$$
$$= [1.856\ 2.304].$$

$$P(0.5) = 0.125\,[0\ 2] + 0.375\,[2\ 3] + 0.375\,[3\ 2] + 0.125\,[2\ 0]$$
$$= [2.125\ 2.125].$$

$$P(0.6) = 0.064\,[0\ 2] + 0.288\,[2\ 3] + 0.432\,[3\ 2] + 0.216\,[2\ 0]$$
$$= [2.304\ 1.856].$$

$$P(0.7) = 0.027\,[0\ 2] + 0.189\,[2\ 3] + 0.441\,[3\ 2] + 0.343\,[2\ 0]$$
$$= [2.387\ 1.503].$$

$$P(0.8) = 0.008\,[0\ 2] + 0.096\,[2\ 3] + 0.384\,[3\ 2] + 0.512\,[2\ 0]$$
$$= [2.368\ 1.072].$$

$$P(0.9) = 0.001\,[0\ 2] + 0.027\,[2\ 3] + 0.243\,[3\ 2] + 0.729\,[2\ 0]$$
$$= [2.241\ 0.569].$$

See Fig. 4.9.

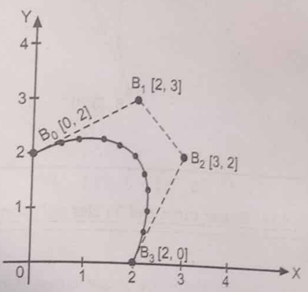


Fig. 4.13 : Be'zier curve and Be'zier polygon

Example 4.14 : Given B_0 [1 1], B_1 [2 3], B_2 [4 3] and B_3 [3 1] the vertices of the Be'zier curve at t = 0, 0.15, 0.35, 0.5, 0.65, 0.85, 1.

Solution : The parametric equation of the Be'zier curve is,

$$P(t) = J_{3,0}(t)\,B_0 + J_{3,1}(t)\,B_1 + J_{3,2}(t)\,B_2 + J_{3,3}(t)\,B_3$$

$$= (1-t)^3 B_0 + 3t(1-t)^2 B_1 + 3t^2(1-t) B_2 + t^3 B_3.$$

$$= (1-t)^3 [1\ 1] + 3t(1-t)^2 [2\ 3] + 3t^2(1-t)[4\ 3] + t^3 [3\ 1].$$

$$P(0) = [1,\ 1] \qquad (\because P(0) = B_0)$$

$$P(0.15) = 0.674\,[1\ 1] + 0.325\,[2\ 3] + 0.058\,[4\ 3] + 0.003\,[4\ 3]$$
$$= [1.5\ 1.765]$$

$$P(0.35) = 0.275\,[1\ 1] + 0.444\,[2\ 3] + 0.239\,[4\ 3] + 0.042\,[4\ 3]$$
$$= [2.248\ 2.367]$$

$$P(0.5) = 0.125\,[1\ 1] + 0.375\,[2\ 3] + 0.375\,[4\ 3] + 0.125\,[4\ 3]$$
$$= [2.75\ 2.5]$$

$$P(0.65) = 0.042\,[1\ 1] + 0.239\,[2\ 3] + 0.444\,[4\ 3] + 0.275\,[4\ 3]$$
$$= [3.122\ 2.367]$$

$$P(0.85) = 0.003\,[1\ 1] + 0.058\,[2\ 3] + 0.325\,[4\ 3] + 0.614\,[4\ 3]$$
$$= [3.248\ 1.765]$$

$$P(1) = [3,\ 1] \qquad (\because P(1) = B_3)$$

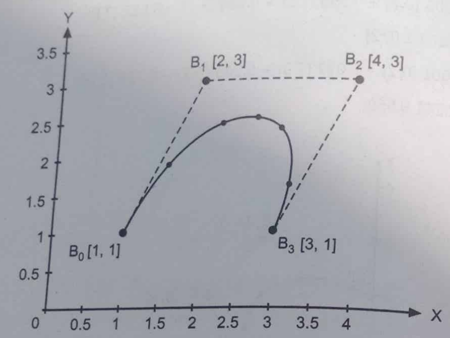


Fig. 4.14 : Be'zier curve and Be'zier polygon

4.8 B-spline Curves

In the article 4.2, we have discussed about the Be'zier curve. It has many advantages and applications in various fields. But it has some drawbacks which have been overcome in B-spline curve. First, the degree of the polynomial from which the curve is defined depends on the number of control points (vertices of the defining polygon). That is if the number of control points is 4, we get a cubic curve, if the number of control points is 5, we get a fourth degree curve. Thus, if the number of control points is (n + 1), we get n^{th}-degree curve. Secondly, the Bernstein function used in the parametric representation of the Be'zier have global nature which means that the value of the Bernstein function $J_{n,\,i}(t)$ given by,

$$J_{n,\,i}(t) = \binom{n}{i} t^i (1-t)^{n-i};\ 0 \le t \le 1.$$

is non-zero over the entire range of parameter. Thus, Be'zier curves do not allow for local control of the curve shape. If we decide to change the position of the control points, the entire curve will be affected.

The theory of B-splines was first introduced by Schoenberg. Let P (t) be the position vector of any point on the curve, where t is a parameter. The parametric equation of B-spline curve is given by,

$$P(t) = \sum_{i=0}^{n} N_i\,k(t)\,B_i\ ;\ t_{min} \le t \le t_{max} \qquad \text{... (1)}$$

$$\text{i.e. } 0 \le t \le (n - K' + 2)$$

where, B_i (i = 0, 2, ... n) be the control points and N_i, k are the normalized B-spline basis function (or blending function).

The normalized B-spline functions of order k (degree k − 1) are defined by recursion formula as follows.

$$N_{i,\,1}(t) = 1, \text{ if } x_i \le t, \le x_{i+1}$$

$$N_{i,\,1}(t) = 0, \text{ otherwise}$$

and $$N_{i,\,k}(t) = \frac{(t - x_i)\,N_{i,\,k-1}(t)}{x_{i+k-1} - x_i} + \frac{(x_{i+k}\,t)\,N_{i+1,\,k-1}(t)}{x_{i+k} - x_{i+1}} \qquad \text{... (2)}$$

While calculating $N_{i,\,k}(t)$, the denominator may turn out to be 0. So it should be noted that ... (3)

the term $\frac{0}{0}$ is not meaningless, but it is assigned the value 0. Thus, here the convention $\frac{0}{0} = 0$ is permissible.

The parameter range [0 n − k + 2] is subdivided into (n + k) sub-intervals by the points $x_0, x_1, x_2 \ldots x_n + k$, satisfying the relation $x_i \le x_i + 1$. These points are represented in the form of vector $[x_0, x_1, x_2 \ldots x_n + k]$ called as knot vector.

Knot vector is given by,

$$x_i = 0 \quad , \text{if } i < k$$

$$x_i = i - k + 1 \quad , \text{if } k \le i \le n$$

$$x_i = n - k + 2 \quad , \text{if } i > n \qquad \text{... (4)}$$

where, $i \le n \le n + k$.

From the B-spline basis function formulas (2) and (3), it follows that the basis function $N_{i,\,k}$ of given order k depends on the basis functions $N_{i,\,k-1}$, $N_{i-1,\,k-1}$ of order k − 1 for a given basis function $N_{i,\,k}$. This dependence is shown in triangular term as under :

$N_{i,\,k}$

$N_{i,\,k-1}$ $\quad N_{i+1,\,k-1}$

$N_{i,\,k-2}$ $\quad N_{i+1,\,k-2}$ $\quad N_{i+2,\,k-2}$

$\vdots \quad \vdots \quad \vdots$

$N_{i,\,1}$ $\quad N_{i+1,\,1}$ $\quad N_{i+2,\,1}$ .... $N_{i+3,\,1}$ .... $N_{i+k-1,\,1}$

Fig. 4.15

For illustration, we prepare here, the parametric equation of the B-spline curve, where k = 2, n = 5.

n = 5 ⇒ there are 6 control points, let it be $B_0, B_1, B_2, B_3, B_4, B_5$.

From (1), the parametric equation of the B-spline curve is,

$$P(t) = \sum_{i=0}^{5} N_{i,k}(t) B_i, \; 0 \le t \le 5 - 2 + 2 \text{ i.e. } 0 \le t \le 5$$

$$= N_{0,2}(t) B_0 + N_{1,2}(t) B_1 + N_{2,2}(t) B_2 + N_{3,2}(t) B_3 + N_{4,2}(t) B_4 + N_{5,2}(t) B_5 \quad \text{... (5)}$$

We can subdivide the range [0, 5] by the knot vectors $x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7$ using equation (4) as follows :

$$x_0 = 0, x_1 = 0, x_2 = 1, x_3 = 1, x_4 = 3, x_5 = 4, x_6 = 5, x_7 = 5$$

i.e. $[x_0 \; x_1 \; x_2 \; x_3 \; x_4 \; x_5 \; x_6 \; x_7] = [0\;0\;1\;2\;3\;4\;5\;5]$

Here, we require basis function of order two, hence from equation (3), we have

$$N_{0,2}(t) = (1-t) N_{1,1}(t)$$

$$N_{1,2}(t) = t N_{1,1}(t) + (2-t) N_{2,1}(t)$$

$$N_{2,2}(t) = (t-1) N_{2,1}(t) + (3-t) N_{3,1}(t)$$

$$N_{3,2}(t) = (t-2) N_{3,1}(t) + (4-t) N_{4,1}(t)$$

$$N_{4,2}(t) = (t-3) N_{4,1}(t) + (5-t) N_{5,1}(t)$$

$$N_{5,2}(t) = (t-4) N_{5,1}(t)$$

Substituting all these values in equation (5), we get the parametric equation of the B-spline curve as follows :

$$P(t) = (1-t) N_{1,1}(t) B_0 + [(t-0) N_{1,1}(t) + (2-t) N_{2,1}(t)] B_1$$

$$+ [(t-1) N_{2,1}(t) + (3-t) N_{3,1}(t)] B_2$$

$$+ [(t-2) N_{3,1}(t) + (4-t) N_{4,1}(t)] B_3$$

$$+ [(t-3) N_{4,1}(t) + (5-t) N_{5,1}(t)] B_4$$

$$+ (t-4) N_{5,1}(t) B_5 \quad \text{... (6)}$$

Now, using equation (2) i.e.

$$N_{i,1}(t) = 1 \; ; \text{if } 0 \le t < 5$$

$$N_{i,1}(t) = 0 \; ; \text{otherwise}$$

From equation (6), we have,

$$P(t) = (1-t) B_0 + t B_1, \quad \text{if } 0 \le t < 1$$

$$P(t) = (2-t) B_1 + (t-1) B_2, \quad \text{if } 1 \le t < 2$$

$$P(t) = (3-t) B_2 + (t-2) B_3, \quad \text{if } 2 \le t < 3$$

$$P(t) = (4-t) B_3 + (t-3) B_4, \quad \text{if } 3 \le t < 4$$

and $$P(t) = (5-t) B_4 + (t-4) B_5, \quad \text{if } 4 \le 1 < 5 \quad \text{... (7)}$$

When, $t = 0$, (7) $P(0) = B_1$

When, $t = 5$, (7) $P(5) = B_5$

This means that the B-spline curve always passes through the first and last control points. The equation (7) is the parametric equation of B-spline curve of order 2, with 6 control points. B-spline curves have the following properties.

Properties :

(1) The sum of the B-spline basis functions, for any parameter value t is equal to 1.

i.e. $$\sum_{i=0}^{n} N_{i,k}(t) = 1$$

(2) Each B-spline basis function lies entirely within 0 and 1. i.e. $0 \le N_{i,k}(t) \le 1$, for any parameter t.

(3) For (n + 1) control points, the curve is described with (n + 1) basis functions.

(4) The B-spline curve generally follows the shape of the defining polygon.

(5) The B-spline curve lies completely within the convex hull of the defining polygon.

(6) Any affine transformation can be applied to the B-spline curve by applying it to the defining polygon vertices (control points).

(7) The B-spline curve exhibits variation-diminishing property.

(8) Any control point can affect the shape of at most k curve sections.

(9) The parameter range [0 n – k + 2] is divided into (n + k) sub-intervals with the break-up points $x_0, x_1, x_2 \ldots x_{n+k}$, satisfying the relation $x_i \le x_{i+1}$. It is represented in the form of vector $[x_0, x_1 \ldots x_{n+k}]$ called as knot vector.

Illustrative Examples

Example 4.15 : Find the parametric equation of a Be'zier curve determined by control points B_1 [1 0], B_1 [2 3], B2 [4 1] and find the position vectors of the points on the curve corresponding to the parameter values t = 0.3, 0.6, 0.9. **(Oct. 2007)**

Solution : The three control points B_0 [1 0], B_1 [2 3] and B_2 [4 1] are given.

Therefore, $n = 2$

$$J_{2,0}(t) = \binom{2}{0} t^0 (1-t)^2 = (1)(1)(1-t^2) = (1-t)^2$$

$$J_{2,1}(t) = \binom{2}{1} t^1 (1-t)^1 = (2)\, t(1-t) = 2t(1-t)$$

$$J_{2,2}(t) = \binom{2}{2} t^2 (1-t)^0 = (1)\, t^2 (1) = t^2.$$

The parametric equation of Be'zier curve is

$$[P(t)] = J_{2,0}(t)[B_0] + J_{2,1}(t)[B_1] + J_{2,2}(t)[B_2]$$

$\therefore \quad [P(t)] = (1-t)^2 [1\ 0] + 2t(1-t)[2\ 3] + t^2 [4\ 1]$

t = 0.3 gives

$$[P(0.3)] = (1-0.3)^2 [1\ 0] + (2)(3)(1-0.3)[2\ 3] + (0.3)^2[4\ 1]$$

$$\therefore \quad [P(0.3)] = 0.49[1\ 0] + 0.42[2\ 3] + 0.09[4\ 1]$$

$$= [0.49\ 0] + [0.84\ \ 1.26] + [0.36\ 0.09]$$

$$= [1.69\ 1.35]$$

$$[P(0.6)] = (1-0.6)^2[1\ 0] + (2)(0.6)(1-0.6)[2\ 3] + (0.6)^2[4\ 1]$$

$$\therefore \quad [P(0.6)] = 0.16[1\ 0] + 0.48[2\ 3] + 0.36[4\ 1]$$

$$\therefore \quad [P(0.6)] = [0,.16\ 0] + [0.96\ 1.44] + [1.44\ \ 0.36]$$

$$\therefore \quad [P(0.6)] = [2.56\ \ 1.80]$$

$$[P(0.9)] = (1-0.9)^2[1\ \ 0] + (2)(0.9)(1-0.9)[2\ 3] + (0.9)^2[4\ 1]$$

$$\therefore \quad [P(0.9)] = 0.01[1\ 0] + 0.18[2\ 3] + 0.81[4\ 1]$$

$$\therefore \quad [P(0.9)] = [0.01\ \ 0] + [0.36\ 0.54] + [3.24\ 0.81]$$

$$\therefore \quad [P(0.9)] = [3.61\ 1.35]$$

t	Position vector
0.3	[1.69 1.35]
0.6	[2.56 1.80]
0.9	[3.61 1.35]

Example 4.16 : Find the parametric equation of the Be'zier curve with control points $B_0[-2\ 1]$, $B_1[1\ 3]$, $B_2[6\ -1]$. Find the point corresponding to the parameter value t = 0.357. **(Oct. 2008)**

Solution : The three control points $B_0[-2\ 1]$, $B_1[1\ 3]$, $B_2[6-1]$ are given.

Therefore, $n = 2$

$$J_{2,0}(t) = \binom{2}{0} t^0 (1-t)^2 = (1-t)^2$$

$$J_{2,1}(t) = \binom{2}{1} t^1 (1-t)^1 = 2t(1-t)$$

$$J_{2,2}(t) = \binom{2}{2} t^2 (1-t)^0 = t^2$$

The parametric equation of Be'zier curve is

$$[P(t)] = J_{2,0}(t)[B_0] + J_{2,1}(t)[B_1] + J_{2,2}(t)[B_2]$$

$$\therefore \quad [P(t)] = (1-t)^2[-2\ 1] + 2t(1-t)[1\ \ 3] + t^2[6\ \ -1]$$

Now, $t = 0.357$ gives

$$[P(0.357)] = (1-0.357)^2[-2\ 1] + (2)(0.357)(1-0.357)[1\ 3] + (0.357)^2[6-1]$$

$$\therefore \quad [P(0.357)] = 0.4134[-2\ 1] + 0.4591[1\ 3] + 0.1274[6\ -1]$$

$$\therefore \quad [P(0.357)] = [-0.8264\ \ 0.4134] + [0.4591\ 1.3773] + [0.7644 - 0.1274]$$

$$\therefore \quad [P(0.357)] = [0.3971\ \ 1.6633]$$

Example 4.17 : If $B_0[2\ 1]$, $B_1[4\ 4]$, $B_2[5\ 3]$, $B_3[5\ 1]$ are the vertices of a Be'zier polygon, then determine the points at t = 0.2, 0.4, 0.6, 0.8 of the Be'zier curve. Also trace the Be'zier curve.

Solution : The 4 control points are $B_0[2\ 1]$, $B_1[4\ 4]$, $B_2[5\ 3]$ and $B_3[5\ 1]$.

Therefore, $n = 3$

$$J_{3,0}(t) = \binom{3}{0} t^0 (1-t)^3 = (1-t)^3$$

$$J_{3,1}(t) = \binom{3}{1} t^1 (1-t)^2 = 3t(1-t)^2$$

$$J_{3,2}(t) = \binom{3}{2} t^2 (1-t)^1 = 3t^2(1-t)$$

$$J_{3,3}(t) = \binom{3}{3} t^3 (1-t)^0 = t^3$$

The parametric equation of Be'zier curve is

$$[P(t)] = J_{3,0}(t)[B_0] + J_{3,1}(t)[B_1] + J_{3,2}(t)[B_2] + J_{3,3}(t)[B_3]$$

$$\therefore \quad [P(t)] = (1-t)^3[2\ 1] + 3t(1-t)^2[4\ 4] + 3t^2(1-t)[5\ 3] + t^3[5\ 1]$$

Now, $t = 0.2$ gives

$$[P(0.2)] = (1-0.2)^3[2\ 1] + (3)(0.2)(1-0.2)^2[4\ 4] + (3)(0.2)^2(1-0.2)[5\ 3] + (0.2)^3[5\ 1]$$

$$\therefore \quad [P(0.2)] = 0.512[2\ 1] + 0.384[4\ 4] + 0.096[5\ 3] + 0.008[5\ 1]$$

$$\therefore \quad [P(0.2)] = [1.024\ 0.512] + [1.536\ 1.536] + [0.480\ 0.288] + [0.040\ 0.008]$$

$$\therefore \quad [P(0.2)] = [3.080\ \ 2.344]$$

In the same way we get

$$[P(0.4)] = [3.920\ \ 2.872]$$

$$[P(0.6)] = [4.520\ 2.728]$$

$$[P(0.8)] = [4.880\ \ 2.056]$$

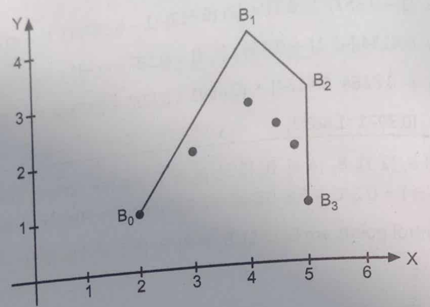


Fig. 4.12

The encircled point give the shape of Be'zier curve.

Think Over It

- The mathematical description of a space curve is generated without any proper knowledge of the shape of the curve.
- The Be'zier curves and their powerful generalization to B-spline curves are two techniques, which are characterized by the fact that some points on the curve pass through the control points used to define the curve.

Points to Remember

1. Polynomial of Be'zier curve for various number of control points (n)

Control points	**Be'zeir curve**
B_0	$P(t) = B_0$ point itself
B_0, B_1	$P(t) = (1 - t) B_0 + B_1$ linear polynomial
B_0, B_1, B_2	$P(t) = (1 - t)^2 B_0 + 2(1 - t) (t) B_1 + t^2 B_2$ Quadratic polynomial
B_0, B_1, B_2, B_3	$P(t) = (1 - t)^3 B_0 + 3(1 - t)^2 t B_1 + 3(1 - t) t^2 B_2 + t^3 B_3$

Matrix representation of Be'zeir polynomial P(t)

Control points	**P(t)**
B_0	$P(t) = [1]\, B_0$
B_0, B_1	$P(t) = [t \;\; 1] \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \end{bmatrix}$
B_0, B_2, B_3	$P(t) = [t^2 \;\; t \;\; 1] \begin{bmatrix} 1 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \\ B_2 \end{bmatrix}$
B_0, B_1, B_2, B_3	$P(t) = [t^3 \;\; t^2 \;\; t \;\; 1] \begin{bmatrix} 1 & 3 & -3 & 1 \\ -3 & -6 & 3 & 0 \\ 3 & 3 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{bmatrix}$

Miscellaneous Exercise

(A) State whether the following statement are True or False :

1. Be'zeir curve is a polynomial of degree one less than the number of control points.
2. Any Be'zeir curve lies within the convex hull of the control points.
3. In a Be'zeir curve, the degree of the polynomial defining the curve segment is one more than the number of defining polygon points.

ANSWERS

1. - True	2. True	3. False

(B) Multiple Choice Questions

1. The control points of a Be'zeir curve are $B_0[2 \;\; 1]$, $B_1[4 \;\; 3]$ and $B_2[6 \;\; 0.5]$. The vector of the point corresponding to parameter t = 0.5 is ......
   (a) [3.8 1.85] (b) [1.66 0.8]
   (c) [4 1.875] (d) [3 2.45]
2. For a cubic n = 3 the maximum value of $J_{3,1}$ is ......
   (a) $\frac{9}{4}$ (b) $\frac{4}{9}$
   (c) $\frac{3}{5}$ (d) $\frac{5}{3}$

Answers

1 - (c)	2 - (b)

(C) Theory Questions :

1. State the properties of Be'zeir curve.
2. Explain the term : Convex hull.
3. Explain : Interpolation and approximation of the set of control points.

4. State general parametric equation of the Be'zeir curve and obtain the matrix representation of the Be'zeir cubic curve.
5. Write various properties of the Be'zeir curve.
6. Write the applications of Be'zeir curve.

(D) Numerical Problems :

1. Find the parametric equation of a Be'zeir curve, determine by the control points $B_0[2\ \ 1]$, $B_1[3\ \ 5]$, $B_2[4\ \ 3]$ and find the position vectors of the points on the curve, corresponding to the parameter values t = 0.24, 0.85.
2. Find the parametric equation of the Be'zeir curve determined by the control points $B_0[1\ \ 0]$, $B_1[2\ \ 5]$, $B_2[4\ \ 6]$ and $B_3[6\ \ 2]$. Also determine the position vector of a point on the curve, for which the value of parameter is t = 0.4.
3. Find the parametric equation of the Be'zeir curve, determined by the control points $B_1[-1\ \ 2]$, $B_1[2\ \ 4]$ and $B_2[4\ \ 1]$. Also find the point on the curve corresponding to the parameter value t = 0.75.
4. Find the parametric equation of a Be'zeir curve, determined by a Be'zeir polygon $B_0[-1\ \ -1]$, $B_1[2\ \ 3]$, $B_2[3\ \ 3]$, $B_3[5\ \ 2]$. Also find P(0.6).
5. Find the parametric equation of the Be'zeir curve with control points $B_0[-2\ \ 1]$, $B_1[1\ \ 3]$, $B_2[6\ \ -1]$. Also find the point corresponding to t = 0.357.
6. Find the parametric equation of a Be'zeir equation of a Be'zeir curve determined by control points $B_0[0\ \ 2]$, $B_1[2\ \ 3]$ and $B_2[2\ \ 0]$. Also find position vectors of points on the curve corresponding to parameter value t = 0.2, 0.4, 0.6.

Answers

1. $P(t) = (1 - t)^2 [2\ \ 1] + 2t(1 - t) [3\ \ 5] + t^2[4\ \ 3]$
   $[P(0.24)] = [2.48\ \ 2.5744]$
   $[P(0.85)] = [3.07\ \ 2.415]$
2. $P(t) = (1 - t)^3 [1\ \ 0] + 3t (1 - t)^2 [2\ \ 5] + 3t^2 (1 - t) [4\ \ 6] + t^3 [6\ \ 2]$
   $[P(0.4)] = [2.616\ \ 4.016]$
3. $P(t) = (1 - t)^2 B_0 + 2(1 - t)\, t\, B_1 + t^2 B_2$
   $P(0.75) = [2.9375\ \ 2.1875]$
4. $P(t) = (1 - t)^3 B_0 + 3(1 - t)^2 tB_1 + 3(1 - t)\, t^2 B_2 + t^3 B_3,$
   $P(0.6) = [2.3177\ \ 1.6726]$
5. $P(t) = (1 - t)^2 B_0 + 2(1 - t)(t) B_1 + t^2 B_2,$
   $P(0.357) = [0.3968\ \ 1.6633]$
6. $P(t) = (1 - t)^3 B_0 + 3(1 - t)^2 t B_1 + 3(1 - t)\, t^2 B_2 + t^3 B_3,$
   $P(0.2) = [1.072\ \ 2.368]$, $P(0.4) = [1.856\ \ 2.305]$
   $P(0.6) = [2.305\ \ 1.856]$