NEW SYLLABUS

SECOND YEAR B. Sc.
COMPUTER SCIENCE

SEMESTER-IV

OPERATIONS RESEARCH

MATHEMATICS (MTC 242) : Paper-II

M. D. BHAGAT
N. M. PHATANGARE
R. S. BHAMARE
Dr. A. S. KHAIRNAR
Dr. S. G. PURANE

CBCS
2 CREDITS

NIRALI PRAKASHAN
ADVANCEMENT OF KNOWLEDGE

SPPU New Syllabus

OPERATIONS RESEARCH

For Second Year B. Sc. (Computer Science)
MTC : 242 : Mathematics – Paper-II : Semester-IV
As Per Savitribai Phule Pune University
Revised Syllabus Effective from June 2020

M. D. Bhagat
Ex-Head, Mathematics Dept.
Tuljaram Chaturchand College,
Baramati (Dist. Pune)

R. S. Bhamare
Ex-Head, P.G. Dept. of Mathematics
New Arts, Science & Commerce College
Ahmednagar.

N. M. Phatangare
Assistant Prof., Department of Mathematics
Fergusson College,
Pune 411 004

Dr. A. S. Khairnar
Assistant Professor & HOD of Mathematics
MES's Abasaheb Garware College,
Pune - 411 004

Dr. S. G. Purane
Associate Prof. & HOD of Mathematics,
PES's Jamkhed Mahavidyalaya,
Jamkhed, Ahmednagar 413 201

Price ₹ 290.00

NIRALI PRAKASHAN
ADVANCEMENT OF KNOWLEDGE

N5544

S.Y.B.Sc. (Comp. Sci.) OPERATIONS RESEARCH **ISBN : 978-93-90506-72-9**

First Edition : January 2021

Published By :
NIRALI PRAKASHAN
Abhyudaya Pragati, 1312, Shivaji Nagar
Off J.M. Road, Pune – 411005
Tel - (020) 25512336/37/39, Fax - (020) 25511379
Email : niralipune@pragationline.com

POD

Printed By :
YOGIRAJ PRINTERS AND BINDERS
Survey No. 10/1A, Ghule Industrial Estate,
Nanded Gaon Road,
Nanded, Pune - 411041
Mobile No. 9404233041/9850046517

DISTRIBUTION CENTRES

PUNE

Nirali Prakashan (For orders within Pune) : 119, Budhwar Peth, Jogeshwari Mandir Lane, Pune 411002, Maharashtra
Tel : (020) 2445 2044, 66022708, Fax : (020) 2445 1538; Mobile : 9657703145
Email : niralilocal@pragationline.com

Nirali Prakashan (For orders outside Pune) : S. No. 28/27, Dhayari, Near Asian College Pune 411041
Tel : (020) 24690204 Fax : (020) 24690316; Mobile : 9657703143
Email : bookorder@pragationline.com

MUMBAI

Nirali Prakashan : 385, S.V.P. Road, Rasdhara Co-op. Hsg. Society Ltd.,
Girgaum, Mumbai 400004, Maharashtra; Mobile : 9320129587
Tel : (022) 2385 6339 / 2386 9976, Fax : (022) 2386 9976
Email : niralimumbai@pragationline.com

DISTRIBUTION BRANCHES

JALGAON

Nirali Prakashan : 34, V. V. Golani Market, Navi Peth, Jalgaon 425001, Maharashtra,
Tel : (0257) 222 0395, Mob : 94234 91860; Email : niralijalgaon@pragationline.com

KOLHAPUR

Nirali Prakashan : New Mahadvar Road, Kedar Plaza, 1st Floor Opp. IDBI Bank, Kolhapur 416 012
Maharashtra. Mob : 9850046155; Email : niralikolhapur@pragationline.com

NAGPUR

Pratibha Book Distributors : Above Maratha Mandir, Shop No. 3, First Floor,
Rani Jhanshi Square, Sitabuldi, Nagpur 440012, Maharashtra
Tel : (0712) 254 7129; Email : pratibhabookdistributors@gmail.com

DELHI

Nirali Prakashan : Omkar House, Ground Floor, Room No. 2, 4575/15, Agarwal Lane, Ansari Road,
Daryaganj, New Delhi 110002 Mob : +91 9555778814/+91 9818561840
Email : niralidelhi@pragationline.com

BENGALURU

Nirali Prakashan : Maitri Ground Floor, Jaya Apartments, No. 99, 6th Cross, 6th Main,
Malleswaram, Bengaluru 560003, Karnataka; Mob : 9449043034
Email: niralibangalore@pragationline.com

Other Branches : Hyderabad, Chennai

niralipune@pragationline.com | www.pragationline.com

Also find us on www.facebook.com/niralibooks

Preface ...

We have great pleasure in presenting this text book on **OPERATION RESEARCH** to the students of S.Y.B.Sc. Computer Science Semester - IV, Mathematics Paper - II. This book is written strictly according to the new revised syllabus of Savitribai Phule Pune University to be implemented from June 2020.

We have taken utmost care to present the matter systematically and with proper flow of mathematical concepts. We begin the Chapter by Introduction and at the end the Summary of the Chapter is provided. We have added one significant feature: **"Think Over It"** in this new **edition**. Here, we have posed questions of simple, difficult and intuitive type in nature. It is expected that the students should think over it and try to find the answers. This will assess the understanding of the knowledge of the Chapter.

The book contains good number of solved problems and the number of graded problems in the exercises.

We are thankful to **Shri Dineshbhai Furia, Shri Jignesh Furia,** Mrs. Anagha Medhekar (Proof Reading and Co-ordination), Mr. Rahul Thorat, Mrs. Yojana Deshpande (Figure Drawing) and the staff of Nirali Prakashan for the great efforts that they have taken to publish the book in time.

We welcome the valuable suggestions from our colleagues' and readers for the improvement of the book.

PUNE **AUTHORS**

JANUARY 2021

Syllabus ...

1. Linear Programming Problem I **(12 L)**

1.1 Introduction Definition and Examples

1.2 Problem solving using Graphical method

1.3 Theory of Linear Programming, Slack and surplus variables, Standard form of LPP, Some important definitions, Assumptions in LPP, Limitations of Linear programming, Applications of Linear programming, Advantages of Linear programming Techniques

1.4 Simplex method, Big- M-method

2. Linear Programming Problem II **(08 L)**

2.1 Special cases of LPP : Alternative solution, Unbounded solution, Infeasible solution

2.2 Duality in Linear Programming, Primal to dual conversion, Examples

3. Assignment Models **(06 L)**

3.1 Assignmment Model -Introduction

3.2 Hungerian method for Assignment problem

4. Transportation Models **(10 L)**

4.1 Introduction, Tabular representation

4.2 Methods of IBFS (North-West rule, Matrix-minima, Vogel's Approximation), Algorithms

4.3 The Optimality Test of Transportation Model (MODI method only)

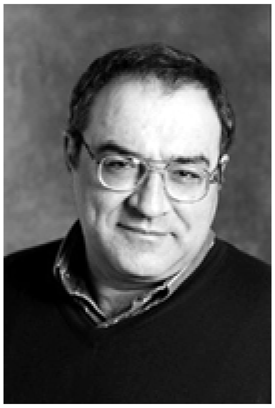
❑❑❑

Contents ...

☆☆☆

Chapter 1...

Linear Programming Problem I



Leonid Khachiyan

Leonid Genrikhovich Khachiyan (Armenian: Լեոնիդ Գենրիխովիչ Խաչիյան; Russian: Леонид Генрихович Хачиян; May 3, 1952 – April 29, 2005) was a Soviet mathematician of Armenian descent who taught Computer Science at Rutgers University. He was most famous for his Ellipsoid algorithm (1979) for linear programming, which was the first such algorithm known to have a polynomial running time. Even though this algorithm was shown to be impractical due to the high degree of the polynomial in its running time, it has inspired other randomized algorithms for convex programming and is considered a significant theoretical breakthrough.

1.1 Linear Programming Problem (L.P.P.)

Linear programming uses mathematics to describe the problem of concern. The term linear means that all the mathematical functions in this model are linear and the word "programming" is synonym with planning. Thus, linear programming deals with that section of programming problems for which all relations among the variables are linear. The general linear programming problem optimizes a linear function of several variables subject to a system of equalities and/or inequalities. The optimization of a linear function includes either a maximization or a minimization of this function.

For example, in an industrial plant the objective may be viewed as maximizing gains or productive time or minimizing cost or idle time. The optimization suggests the common goal of achieving the best solution to the linear programming model.

1.1.1 Some Definitions

Objective Function : The linear function to be optimized is called "*objective function*" of the L.P.P.

Constraints : The linear conditions in terms of equalities and/or inequalities imposed upon the variables to be determined are called *constraints* or *restrictions of L.P.P.*

Decision Variables : The unknown variables in terms of which "objective function" is expressed are called *'decision variables'* of the L.P.P.

1.1.2 Applications of Linear Programming

There are several fields of applications of operations research like military, industries, insurance companies, banks, agriculture, planning etc.

In industries all mega companies have several departments within the company itself, like Production department to minimize the cost of production, Marketing department to maximize the amount of selling product and to minimize the cost of selling, Finance department to minimize the capital required to maintain any level of business. There are other departments like research and development. In all these departments, co-ordination is maintained and linear programming helps each department to optimize their requirements and make overall work of the company effective and the best output.

Linear programming is used in the fields like, Diet Problem, Blending problem, Transportation problems. In diet problem, linear programming provides optimal food mix for meeting the nutritional needs of human beings, animals or a broiler at the least cost. Linear programming is also used in allocation problems, a problem which involves allocation (allotment) of available resources to the jobs (or activities) that are to be done. Allocation problems can be further subdivided into the following three types of problem :

(i) Linear Programming Problems (L.P.P.)
(ii) Transportation Problems (T.P.)
(iii) Assignment Problems (A.P.)

Let us illustrate the definition of linear programming problem by an example.

(i) Suppose a factory decides to manufacture two kinds of products say A and B. Let the profit per unit of product A and product B be ₹ 5 and ₹ 7 respectively. Each unit of product A requires 6 machine hours and that of product B requires 5 machine hours. Each unit of product A requires 10 units of raw material, whereas each unit of product B requires 6 units of raw material. The maximum available machine hours and material units are 220 and 320, respectively. A maximum of 100 units are required of product B. Determine the number of units to be manufactured of products A and B. This problem can be represented in mathematical linear equalities and/or inequalities.

For this, suppose x_1 and x_2 denote the number of units to be produced of products A and B respectively. The above information can be summarized in tabular form :

	Product A	Product B
Number of units produced (decision variables)	x_1	x_2
Machine hours required	$6x_1$	$5x_2$
Raw material required in units	$10x_1$	$6x_2$
Maximum requirements	No unit	100
Profit obtained	$5x_1$	$7x_2$

From above information we see that the objective function of this problem is to maximize the net profit $5x_1 + 7x_2$ for the two products. This maximum profit is subject to the conditions :

(i) The machine hours constraint, i.e.
$6x_1 + 5x_2 \leq 220$, since the total number of machine hours available is 220.

(ii) The raw material constraint, viz
$10x_1 + 6x_2 \leq 320$, as the total number units of raw material available is 320.

(iii) The maximum requirement constraints, viz.
$x_1 \geq 0$, since negative product has no physical meaning and $x_2 \leq 100$, since the maximum number of units required that of product B is 100.

This problem is usually written in the following form;

maximise : $z = 5x_1 + 7x_2$

Subject to

$$\begin{aligned} 6x_1 + 5x_2 &\leq 220 \\ 10x_1 + 6x_2 &\leq 320 \\ x_2 &\leq 100 \\ x_1 &\geq 0, \quad x_2 \geq 0 \end{aligned}$$

where, z is the value of the objective function.

The above illustration also explains how to formulate (or formulation) a given linear programming problem.

1.2 Linear Programming Formulation

Let us see some more examples, where we formulate the given problem in mathematical model, at the same time we see the various areas of the applications of L.P.P.

Illustrative Examples

Example 1.1 : A company sells two different products A and B. The company makes a profit of ₹ 30 and ₹ 40 per unit on the products A and B respectively. The two products are produced in a common production process and are sold in two different markets. The production process has a capacity of 35,000 man-hours. It takes two hours to produce one unit of A and 3 hours to produce one unit of B. The market has been surveyed and company feels that the maximum number of units of A that can be sold is 12000 units and the maximum of B is 8000 units. Assuming that the products can be sold in any circumstances, formulate the L.P.P.

Solution : Suppose company produces x_1 units of product A and x_2 units that of B. Then the profit function will be $30x_1 + 40x_2$. Subject to the conditions that $2x_1 + 3x_2 \leq 35,000$ and $x_1 \leq 12000$, $x_2 \leq 8000$, with $x_1 \geq 0$, $x_2 \geq 0$. Thus LPP formulation is;

Maximize : $Z = 30x_1 + 40x_2$

Subject to

$$\begin{aligned} 2x_1 + 3x_2 &\leq 35000 \\ x_1 &\leq 12000 \\ x_2 &\leq 8000 \\ \text{and} \quad x_1 \geq 0,\ x_2 &\geq 0. \end{aligned}$$

Example 1.2 : Suppose in a hospital, it is decided that each patient should be given at least 3, 4, 5 units of nutrients say A, B, C respectively. Suppose there are 4 foods say f_1, f_2, f_3 and f_4 available. Let the following table shows the nutrients A, B, C present per unit in the foods f_1, f_2, f_3 and f_4.

Nutrients	Foods				Requirement of nutrients in units
	f_1	f_2	f_3	f_4	
A	0.5	1	3	1.5	3
B	1	2	0	2.5	4
C	2	1.5	0.5	0	5

Suppose the cost per unit of the foods f_1, f_2, f_3 and f_4 is ₹ 1, ₹ 2, ₹ 3, ₹ 0.5 respectively. The problem is to find the best diet (the food combination) that can be supplied at minimum cost, satisfying the daily requirements of the patient.

Solution : Suppose x_1, x_2, x_3 and x_4 be the quantities of the foods f_1, f_2, f_3 and f_4 respectively, that constitute the best daily diet. Then problem is

Minimize $Z = x_1 + 2x_2 + 3x_3 + 0.5x_4$

Subject to

$$\begin{aligned} 0.5x_1 + x_2 + 3x_3 + 1.5x_4 &\geq 3 \\ x_1 + 2x_2 + 2.5x_4 &\geq 4 \\ 2x_1 + 1.5x_2 + 0.5x_3 &\geq 5 \\ \text{with} \quad x_1,\ x_2,\ x_3,\ x_4 &\geq 0. \end{aligned}$$

Example 1.3 : A firm can produce three types of clothes say A, B and C. The clothes are made of three colours of wools say, red, green and blue. One unit of cloth A needs 2 meters of red wool and 3 meters of blue wool; one unit of cloth B requires 3 meters of red wool, 2 meters of green and 2 meters of blue wool and one unit of cloth C requires 5 meters of green wool and 4 meters of blue wool. The firm has only a stock of 800 meters of red wool, 1000 meters of green wool and 1500 meters of blue wool. Suppose that the profit per unit of clothes A, B and C is ₹ 3, ₹ 4 and ₹ 5 respectively. Determine how the firm should use the available material, so as to maximize the income from the finished clothes.

Solution : Suppose the firm produces x_1, x_2 and x_3 units of clothes A, B and C respectively. Thus red wool required will be $2x_1 + 3x_2$ meters, $3x_1 + 2x_2 + 4x_3$ meters of blue wool and $2x_2 + 5x_3$ meters of green wool. Also the profit of the company will be

$$3x_1 + 4x_2 + 5x_3.$$

Thus the problem is

Maximize : $Z = 3x_1 + 4x_2 + 5x_3$

Subject to

$$\begin{aligned} 2x_1 + 3x_2 &\leq 800 \\ 2x_2 + 5x_3 &\leq 1000 \\ 3x_1 + 2x_2 + 4x_3 &\leq 1500 \\ \text{and} \quad x_1, x_2, x_3 &\geq 0. \end{aligned}$$

Example 1.4 : A factory manufactures three products. These products are processed through three distinct stages. The time required to manufacture a unit of each of three products and the daily capacity of the stages are given by the following table :

Stage	Time per unit (minutes)			Stage Capacity min/day
	Product 1	Product 2	Product 3	
1	2	3	4	520
2	4	6	–	460
3	–	5	2	490

It is required to determine the daily number of units to be produced of each product, given that the profits per unit of products 1, 2 and 3 are 3, 5 and 6 respectively. Suppose that all the amounts produced are absorbed by the market.

Solution : Let x_1, x_2, x_3 be the number of units to be manufactured of products 1, 2, 3 respectively. As all the units manufactured are absorbed by the market, the net profit becomes $3x_1 + 5x_2 + 6x_3$. For each of the three stages the total time consumed by all three products should not exceed the capacity of the stage. So we have

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &\leq 520 \\ 4x_1 + 6x_2 &\leq 460 \\ 5x_2 + 2x_3 &\leq 490 \end{aligned}$$

Therefore the problem is to

Maximize : $Z = 3x_1 + 5x_2 + 6x_3$

$$\begin{aligned} \text{Subject to} \quad 2x_1 + 3x_2 + 4x_3 &\leq 520 \\ 4x_1 + 6x_2 &\leq 460 \\ 5x_2 + 2x_3 &\leq 490 \\ \text{and} \quad x_1, x_2, x_3 &\geq 0. \end{aligned}$$

Example 1.5 : A paper mill received three orders for paper rolls with the widths and length indicated in the following table :

Order No.	Width (meters)	Length (meters)
1	5	10,000
2	7	30,000
3	9	20,000

Rolls are produced in the mill in two standard widths 10 and 20 meters which are slit to the sizes specified by the orders. There is no limit on the lengths of the standard rolls. The objective is to determine the production schedule that minimizes the firm losses while satisfying the given demand.

Solution : Let x_{ij} be the length of the i^{th} roll (i = 1 for 10 meter and i = 2 for 20 meter roll) which is cut according to the j^{th} pattern. The table below shows the possible patterns for both standard rolls.

Width	i = 1 (10 mt)			i = 2 (20 mt)					
	x_{11}	x_{12}	x_{13}	x_{21}	x_{22}	x_{23}	x_{24}	x_{25}	x_{26}
5 mt.	2	0	0	4	2	2	1	0	0
7 mt.	0	1	0	0	1	0	2	1	0
9 mt.	0	0	1	0	0	1	0	1	2
Time loss in mts	0	3	1	0	3	1	1	4	2

Let s_1, s_2 and s_3 be the surplus lengths produced of the rolls with widths 5, 7 and 9 respectively. Then

Minimize : $Z = 3x_{12} + x_{13} + 3x_{22} + x_{23} + x_{24} + 4x_{25} + 2x_{26} + 5s_1 + 7s_2 + 9s_3$

Subject to

$$2x_{11} + 4x_{21} + 2x_{22} + 2x_{23} + x_{24} - s_1 = 10{,}000$$

$$x_{12} + x_{22} + 2x_{24} + x_{25} - s_2 = 30{,}000$$

$$x_{13} + x_{23} + x_{25} + 2x_{26} - s_3 = 20{,}000$$

$$x_{ij} \geq 0, \quad s_r \geq 0 \text{ for all i, j and r.}$$

Exercise (1.1)

1. A firm produces three products. These products are processed on three different machines. The time required to manufacture one unit of each of these three products and daily capacity of the three machines is given in the table below :

Machine	Time per unit (minutes) Product-1	Product-2	Product-3	Machine Capacity (minutes/day)
M_1	2	3	2	440
M_2	4	–	3	470
M_3	2	5	–	430

Determine the daily number of units to be produced for each product to maximize the profit, if the profit per unit for product 1, 2 and 3 is ₹ 4, ₹ 3 and ₹ 6 respectively. It is assumed that all products produced are consumed in the market. Formulate the problem.

2. A chemical company produces two products X and Y. Each unit of product X requires 3 hours on operation - I and 4 hours on operation - II, while each unit of product Y requires 4 hours on operation – I and 5 hours on operation - II. The total available time for operation - I

and II is 20 hours and 26 hours respectively. The production of each unit of product Y also results in two units of a by-product Z at no extra cost.

Product X sells at profit of ₹ 10 per unit, while Y sells at profit of ₹ 20 per unit. By-product Z brings a unit profit of ₹ 6, if sold; in case it cannot be sold, the destruction cost is ₹ 4 per unit. Market survey indicates that more than five units of Z can be sold. Determine the quantities of X and Y to be produced, keeping Z in mind, so that the profit earned is maximum. Formulate the L.P.P.

3. A person wants to decide the constraints of a diet which will fulfil his daily requirements of proteins, fats and carbohydrates at the minimum cost. The choice is to be made from four different types of foods. The yields per unit of these foods are given below :

Food type	**Yield per unit**			**Cost per unit**
	Proteins	**Fats**	**Carbohydrates**	
1	p_1	f_1	c_1	d_1
2	p_2	f_2	c_2	d_2
3	p_3	f_3	c_3	d_3
4	p_4	f_4	c_4	d_4
Maximum daily requirements	P	F	C	

Formulate the L.P.P.

4. Three products are processed through three different operations. The time in minutes required to process per unit of each product, the daily capacity of operations (in minutes per day) and profit per unit of each product are given :

Operation	**Time per unit in minutes**			**Operation capacity minutes per day**
	Product-1	**Product-2**	**Product-3**	
1	2	3	2	450
2	4	0	3	480
3	1	5	0	440
Profit per unit	4	3	6	–

Formulate the L.P.P.

5. A company can produce three types of cloth say A, B and C. Three kinds of wool are required for it, say R, G and V. One unit length of type A cloth needs 2 meters wool of type R, 3 meters of type V; one unit of length of type B cloth needs 3 meters wool of type R, 2 meters of type G and 2 meters of type V; and one unit of type C cloth needs 5 meters of wool of type G and 4 meters of type V. A company has only a stock of 80 meters of R, 100 meters of G and 150 meters of V wool. The profit per unit of type A, B and C is ₹ 3, ₹ 5 and ₹ 4 respectively. Determine how the firm should use the available material so as to maximize the income from the finished cloth.

6. A company is contracted to receive 60,000 kg of ripe tomatoes at ₹ 7 per kg from which it produces both canned tomato juice and tomato paste. The canned products are packaged in cases of 24 cans each. A single can of juice requires 1 kg of fresh tomatoes, whereas that of

paste requires $\frac{1}{3}$ kg only. The company's share of the market is limited to 2000 cases of juice and 6000 cases of paste. The wholesale prices per case of juice and paste stand at ₹ 300 and ₹ 200 respectively. Formulate the problem.

7. A firm manufactures two products A and B on which the profit earned per unit are ₹ 3 and ₹ 4 respectively. Each product is processed on two machines M_1 and M_2. Product A requires one minute of processing time on M_1 and *two* minutes on M_2, while B requires one minute on M_1 and one minute on M_2. Machine M_1 is available for not more than 450 minutes, while machine M_2 is available for 600 minutes during any working day. Formulate the L.P.P.

8. A company has two bottling plants, say S and M. Each plant produces three drinks, say A, B and C. The number of bottles produced per day are as follows :

Drink	**Plant-S**	**Plant-M**
A	1500	1500
B	3000	1000
C	2000	5000

A market survey indicates that during a month of April there will be a demand of 20,000 bottles of drink A, 40,000 bottles of B and 44,000 bottles of C. The operating costs per day for plants S and M are ₹ 600 and ₹ 400. Formulate the L.P.P., so as to minimize the production cost, while still meeting the market demand.

9. A firm uses lathes, milling machines and grinding machines to produce two machine parts. The following table represents machining times required for each part, the machining times available on different machines and profit on each machine part.

Time of machine	**Time required for (in minutes)**		**Time available in minutes**
	Part – I	**Part – II**	
Lathes	12	6	3000
Milling machines	4	10	2000
Grinding	2	3	900
Profit per unit	₹ 40	₹ 100	

Formulate L.P.P.

Answers (1.1)

1. Maximize $Z = 4x_1 + 3x_2 + 6x_3$

Subject to

$$\begin{aligned} 2x_1 + 3x_2 + 2x_3 &\leq 440 \\ 4x_1 + 3x_3 &\leq 470 \\ 2x_1 + 5x_2 &\leq 430 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

2. Maximize $Z = 10x_1 + 20x_2 + 6x_3 - 4x_4$

Subject to

$$\begin{aligned} 3x_1 + 4x_2 &\leq 20 \\ 4x_1 + 5x_2 &\leq 26 \\ -2x_2 + x_3 + x_4 &= 0 \\ x_3 &\leq 5 \end{aligned}$$

where $x_1, x_2, x_3, x_4 \geq 0.$

3. Minimize $Z = x_1 d_1 + x_2 d_2 + x_3 d_3 + x_4 d_4$

Subject to

$$\begin{aligned} x_1 p_1 + x_2 p_2 + x_3 p_3 + x_4 p_4 &\geq P \\ x_1 f_1 + x_2 f_2 + x_3 f_3 + x_4 f_4 &\geq F \\ x_1 c_1 + x_2 c_2 + x_3 c_3 + x_4 c_4 &\geq C. \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

4. Maximize $Z = 4x_1 + 3x_2 + 6x_3$

Subject to

$$\begin{aligned} 2x_1 + 3x_2 + 2x_3 &\leq 450 \\ 4x_1 + 3x_3 &\leq 480 \\ x_1 + 5x_2 &\leq 440 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

5. Maximize $Z = 3x_1 + 5x_2 + 4x_3$

Subject to

$$\begin{aligned} 2x_1 + 3x_2 &\leq 80 \\ 2x_2 + 5x_3 &\leq 100 \\ 3x_1 + 2x_2 + 4x_3 &\leq 150 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

6. Maximize $Z = 300x_1 + 200x_2$

Subject to

$$\begin{aligned} 24x_1 + 8x_2 &= 60{,}000 \\ x_1 &\leq 2000 \\ x_2 &\leq 6000 \end{aligned}$$

x_1, x_2 are number of cannes of juice and paste respectively and $x_1, x_2 \geq 0$.

7. Maximize $Z = 3x_1 + 4x_2$

Subject to

$$\begin{aligned} x_1 + x_2 &\leq 450 \\ 2x_1 + x_2 &\leq 600 \\ x_1, x_2 &\geq 0. \end{aligned}$$

8. Minimize $Z = 600x_1 + 400x_2$

Subject to

$$\begin{aligned} 1500x_1 + 1500x_2 &\geq 20{,}000 \\ 3000x_1 + 1000x_2 &\geq 40{,}000 \\ 2000x_1 + 5000x_2 &\geq 44{,}000. \end{aligned}$$

$x_1, x_2 \geq 0.$

9. Maximize $Z = 40x_1 + 100x_2$

Subject to

$$\begin{aligned} 12x_1 + 6x_2 &\leq 3000 \\ 4x_1 + 10x_2 &\leq 2000 \\ 2x_1 + 3x_2 &\leq 900 \\ x_1, x_2 &\geq 0. \end{aligned}$$

1.3 Graphical Solution to Linear Programming Problem (L.P.P.)

Example 1.6 : Formulate mathematical model of the following L.P.P., and solve it by graphically.

A company produces both interior and exterior house points for wholesale distribution. The two basic raw materials, A and B are used to manufacture the paints. The maximum availability of A is 6 tons a day; that of B is 8 tons a day. The daily requirements of the raw materials per ton of interior and exterior paints are summarized in the following table :

	Tons of raw material per ton of paint		Maximum availability
	Exterior	**Interior**	
Raw material A	1	2	6
Raw material B	2	1	8

A market survey has established that the daily demand for interior paint cannot exceed that of exterior paint by more than 1 ton. The survey also shows that the maximum demand for interior paint is limited to 2 tons daily.

The wholesale price per tone is ₹ 12,000 for exterior paint and ₹ 8000 for interior paint.

How much interior and exterior paints should the company produce daily to maximize gross income ?

Solution : Formulation in the form of mathematical model.

Let us suppose that company produces x_1 tons of exterior paint and x_2 tons of interior paints. Since each ton of exterior paint sells for ₹ 12,000, the gross income from selling x_1 tons is $12x_1$ thousand rupees. Similarly, the gross income from x_2 tons of interior paint is $8x_2$ thousand rupees. Thus the total gross income becomes the sum of the two. So if Z represents the total gross income (in thousand rupees), the objective function may be written mathematically as
$Z = 12x_1 + 8x_2$.

Now $\begin{pmatrix} \text{use of raw material} \\ \text{by both paints} \end{pmatrix} \leq \begin{pmatrix} \text{maximum raw material} \\ \text{available} \end{pmatrix}$

This gives rise to two constraints

$$x_1 + 2x_2 \leq 6 \qquad \text{(when raw material A used)}$$
$$2x_1 + x_2 \leq 8 \qquad \text{(when raw material B used)}$$

Now it is given that $\begin{pmatrix} \text{excess amount of interior} \\ \text{over exterior paint} \end{pmatrix} \leq 1$ ton per day.

(demand for interior paint) ≤ 2 tons per day.

Mathematically, these are expressed, as

$$x_2 - x_1 \leq 1 \quad \text{and} \quad x_2 \leq 2$$

Thus, determine the tons of interior and exterior paints, x_2 and x_1 to be produced to

$$\text{Maximize } Z = 12x_1 + 8x_2$$
$$\text{Subject to} \quad \begin{aligned} x_1 + 2x_2 &\leq 6 \\ 2x_1 + x_2 &\leq 8 \\ -x_1 + x_2 &\leq 1 \\ x_2 &\leq 2 \end{aligned}$$

$x_1 \geq 0, \quad x_2 \geq 0.$

Now we proceed to solve the L.P.P. mathematical model by graphical method.

The first step in the graphical method is to plot the feasible solutions, or solution space, which satisfies all the constraints simultaneously. The non-negativity restrictions $x_1 \geq 0$, $x_2 \geq 0$ confine that all the feasible values to the first quadrant. First the constraint $x_1 + 2x_2 \leq 6$ should be written in equality form viz, $x_1 + 2x_2 = 6$, which is an equation of a straight line, so we can plot this straight line in the first quadrant. Similarly, converting remaining constraints in the form of equalities, we plot all the straight lines. The region in which each constraint holds when the inequality is activated is indicated by the direction of the arrow on the associated line. The resulting solution space is shown in the Fig. 1.1 by the area ABCDE.

$$
\begin{aligned}
x_1 + 2x_2 &\leq 6 && \dots (1)\\
2x_1 + x_2 &\leq 8 && \dots (2)\\
-x_1 + x_2 &\leq 1 && \dots (3)\\
x_2 &\leq 2 && \dots (4)\\
x_1 &\geq 0 && \dots (5)\\
x_2 &\geq 0 && \dots (6)
\end{aligned}
$$

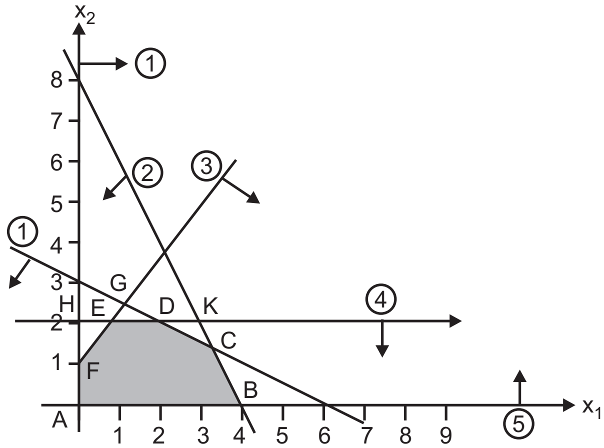


Fig. 1.1

Each point within and on the boundary of the solution space ABCDE satisfies all the constraints and hence represents a feasible point. Although there is an infinity of feasible points in the solution space, the *optimum solution* can be determined by observing the direction in which the objective function $12x_1 + 8x_2$ increases. The parallel lines representing the objective function are plotted by assigning (arbitrary) increasing values to $Z = 12x_1 + 8x_2$ to determine both the slope and direction in which total objective function increases. This is depicted in the Fig. 1.1. Here we used Z = 12, and Z = 24.

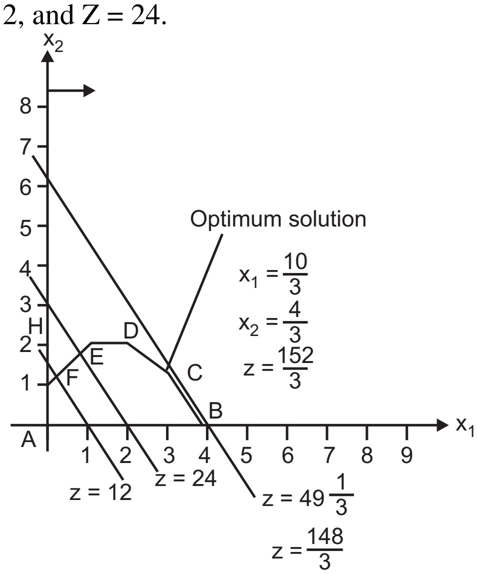


Fig. 1.2

To find the optimum (maximum) solution we move the objective function (profit line or revenue-line) line upto the point where any further increase in profit line would render an infeasible solution. The Fig. 1.2 shows that the optimum solution occurs at point C. Since x_1 and x_2 are determined by solving the intersection of lines (1) and (2), we find values of x_1 and x_2 by solving the equations $x_1 + 2x_2 = 6$ and $2x_1 + x_2 = 8$ simultaneously. The two equations yield $x_1 = \frac{10}{3}$ and $x_2 = \frac{4}{3}$. Thus the solution says that the daily production should be $\frac{10}{3}$ tons of exterior paint and $\frac{4}{3}$ tons of interior paint. The associated total gross income would be $Z = 12 \times \frac{10}{3} + 8 \times \frac{4}{3} = \frac{152}{3}$ thousand rupees.

Remark : The graphical method of solving L.P.P. can be used when the objective function contains only two variables. However, it is impossible to solve L.P.P. whose objective function contains three or more variables.

1.3.1 Summary

To get the region of solution or solution space treat all constraints including non-negativity restrictions as equations. In this, when non-negativity restrictions are converted to equality, they become equations of axes. These non-negativity constraints restrict the solution space to first quadrant only.

Draw the lines corresponding to equalities which outline the solution space.

It can be shown that the optimum solution always exists and exists at, at least one of the vertices, i.e. extreme point of the solution space.

To locate the optimum solution, two methods are used :

(i) Compute the value of objective function at all the vertices (extreme points) and select the one which gives optimum value of objective function as solution point.

(ii) Draw a line of profit in feasible region. Move it parallel to itself away from the origin, if the objective function is maximization type till it passes through farthest vertex of the solution space. The co-ordinates of this vertex give optimum solution of the problem.

If the problem is of minimization type then profit line is to be moved parallel to itself closer to the origin till it passes through nearest extreme point of solution space to origin. The corresponding co-ordinates give optimum solution.

If the optimum solution exist at only one extreme point, it is an unique solution. If the profit line is parallel to a side of a convex set which bounds solution space and if the corresponding two extreme points give same optimum solution, then the problem has alternative optimum solution, then in such case, the problem has optimum solution rather at each and every point on the line segment joining these extreme points is an optimum solution of the problem. Hence in this case there are infinitely many optimum solutions.

For example, consider the following L.P.P.

Illustrative Examples

Example 1.7 : Solve the following L.P.P. by graphical method :

$$\begin{aligned} \text{Max : } Z &= x_1 + 2x_2 \\ \text{Subject to } x_1 + 2x_2 &\le 20 \\ x_1 + x_2 &\le 12 \\ x_1 \ge 0, \quad x_2 &\ge 0. \end{aligned}$$

Solution : The feasible region is given by ABCD as shown in Fig. 1.3.

Extreme points are A (0, 0), B (0, 6), C (2, 4), D (5, 0). The value of Z at extreme points :

$$\begin{aligned} Z_A &= 0 \\ Z_B &= 1 \times 12 + 2 \times 0 = 12 \\ Z_C &= 1 \times 4 + 2 \times 8 = 20 \\ Z_D &= 1 \times 0 + 2 \times 10 = 20 \end{aligned}$$

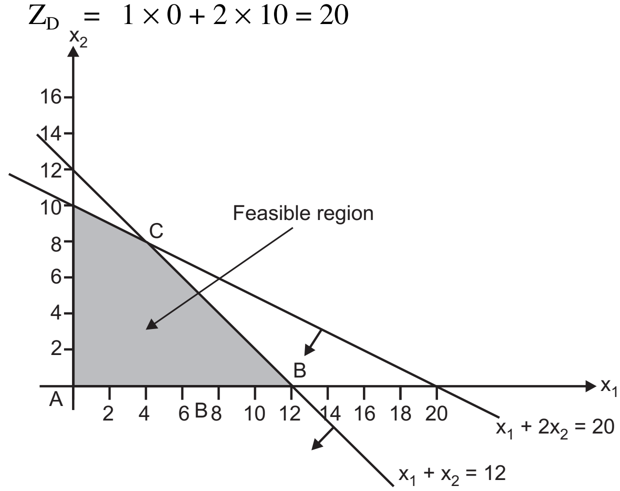


Fig. 1.3

Thus, the maximum of Z = 20 occurs at C and D. There are infinitely many points on line segment CD and each of them provide optimum solution.

In some cases, the problem does not possess an optimal solution but it is possible to find better feasible solution by continuously improving the objective function value. If the improvement continues indefinitely, it possesses *unbounded solution.*

Example 1.8 : Solve the following L.P.P. by graphical method.

Maximum : $Z = 6x_1 + 11x_2$

Subject to
$$\begin{aligned} 2x_1 + x_2 &\leq 104 \\ x_1 + 2x_2 &\leq 76 \\ x_1 &\geq 0, \quad x_2 \geq 0. \end{aligned}$$

Solution : First consider the constraints as equalities

$$\begin{aligned} 2x_1 + x_2 &= 104 \\ x_1 + 2x_2 &= 76, \end{aligned}$$

which are equations of straight lines in two dimensional plane. Draw these lines in plane.

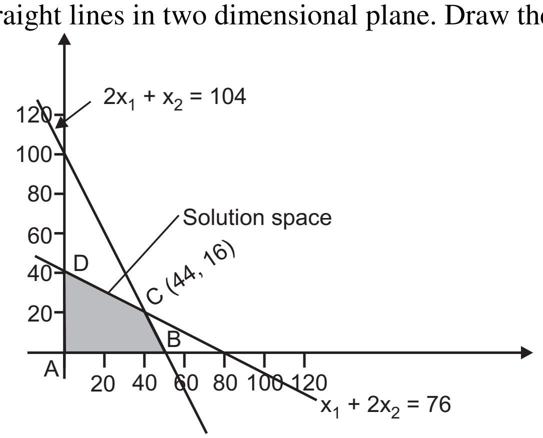


Fig. 1.4

We see that maximum of Z occurs at C (44, 16), which is a point of intersection of the lines $2x_1 + x_2 = 104$ and $x_1 + 2x_2 = 76$.

Thus, $Z_{max} = 6 \times 44 + 11 \times 16 = 440$,

with $x_1 = 44$ and $x_2 = 16$.

Example 1.9 : Solve the following by graphical method :

Maximum : $Z = 5x_1 + 7x_2$

Subject to
$$\begin{aligned} x_1 + x_2 &\leq 4 \\ 3x_1 + 8x_2 &\leq 24 \\ 10x_1 + 7x_2 &\leq 35 \\ x_1, x_2 &\geq 0. \end{aligned}$$

Solution : Considering inequalities as equalities :

$x_1 + x_2 = 4$, $3x_1 + 8x_2 = 24$, $10x_1 + 7x_2 = 35$.

Now draw these lines in xy-plane.

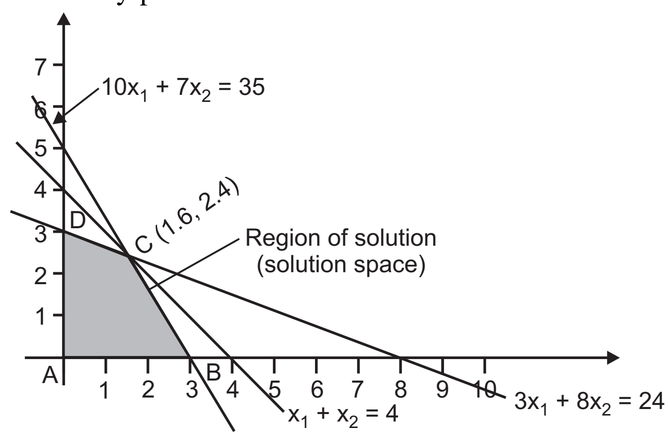


Fig. 1.5

The extreme vertices are A (0, 0), B (4, 0), C (1.6, 2.4), D (0, 3). We see that maximum of Z occurs at C (1.6, 2.4). Hence

$Z_{max} = 5 \times 1.6 + 7 \times 2.4 = 24.8$, with $x_1 = 1.6$, $x_2 = 2.4$.

Example 1.10 : Formulate the following problem and solve it by graphical method.

A toy company manufactures two types of dolls; ordinary doll A and a deluxe doll B. Each doll of type B takes twice as long to produce one of type A, and the company would have time to make a maximum of 2000 per day if it produced only the ordinary dolls. The supply of plastic is sufficient to produce 1500 dolls per day (both A and B combined). The deluxe version requires a fancy dress of which there are only 600 per day available. If the company makes a profit of ₹ 3 and ₹ 5 per doll respectively on dolls A and B, how many of each be produced per day in order to maximize the profit ?

Solution : Formulation : Suppose x_1 dolls of type A and x_2 dolls of type B are produced per day.

$\therefore$ Total profit $Z = 3x_1 + 5x_2$.

Total time consumed to produce x_1 and x_2 dolls of type A and B respectively will be $x_1 + 2x_2$, which should be less than 2000.

$\therefore \quad x_1 + 2x_2 \leq 2000$

Since plastic available is just sufficient to produce 1500 dolls,

$\therefore \qquad x_1 + x_2 \ \leq \ 1500.$

Again, fancy dress is available for 600 dolls per day only.

$\therefore \qquad x_2 \ \leq \ 600.$

Thus the L.P.P. is as follows :

Maximize : $Z = 3x_1 + 5x_2$

Subject to

$$\begin{aligned} x_1 + 2x_2 &\leq 2000 \\ x_1 + x_2 &\leq 1500 \\ x_2 &\leq 600; \\ x_1 &\geq 0, \quad x_2 \geq 0. \end{aligned}$$

Graphical solution : Consider inequalities as equalities as before and draw the lines in XY-plane.

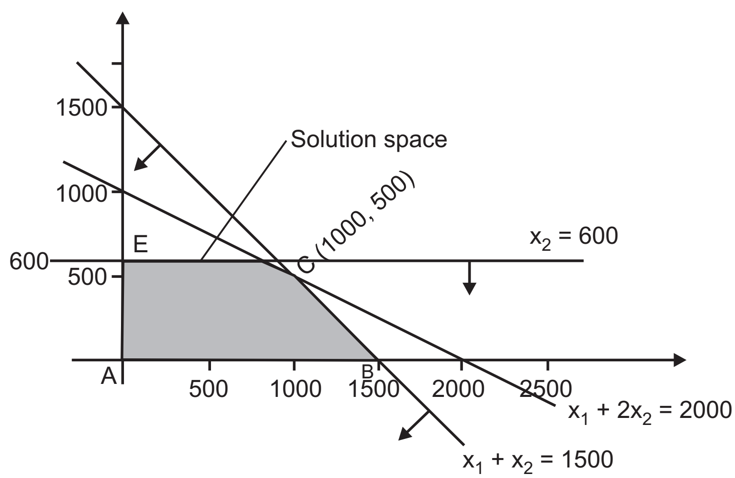


Fig. 1.6

The extreme points are A (0, 0), B (1500, 0), C (1000, 500), D (800, 600), E (0, 600).

The maximum of Z occurs at C (1000, 500). Therefore

$$Z_{max} = 3 \times 1000 + 5 \times 500 = 5500$$

with $x_1 = 1000$ and $x_2 = 500$.

Example 1.11 : Solve the following by graphical method.

Minimize : $Z = 3x_1 + 5x_2$

Subject to
$$\begin{aligned} -3x_1 + 4x_2 &\leq 12 \\ 2x_1 - x_2 &\geq -2 \\ 2x_1 + 3x_2 &\geq 12 \\ x_1 \leq 4, \ x_2 &\geq 2 \ \text{ and} \\ x_1 \geq 0, \quad x_2 &\geq 0. \end{aligned}$$

Solution : Considering the constraints as equalities,

$$\begin{aligned} -3x_1 + 4x_2 &= 12 \\ 2x_1 - x_2 &= -2 \end{aligned}$$

$$2x_1 + 3x_2 = 12$$
$$x_1 = 4$$
$$x_2 = 2$$

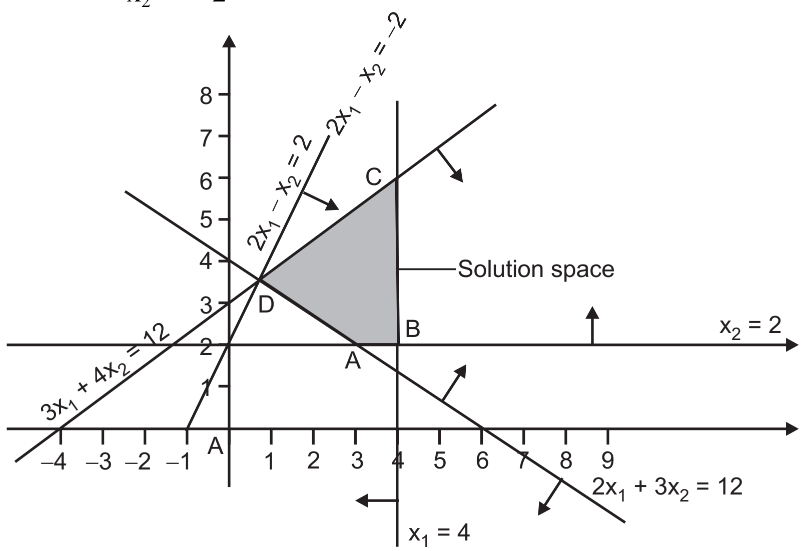


Fig. 1.7

The arrows on each line show the direction of inequalities. The extreme points are A (3, 2), B (2, 4), C (4, 6), and the point D is not single point, though it seems to be intersection of the three lines, it is not like that. Drawing profit line $3x_1 + 5x_2$ = constant closer to the origin, we see that it passes through the extreme point A (3, 2), which is nearest to the origin. Therefore minimum of

$$Z = 3 \times 3 + 5 \times 2 = 19$$

with $x_1 = 3$ and $x_2 = 2$.

Example 1.12 : Tellitell Television company operates two assembly lines Line I and Line II. Each line is used to assembly the components of three types of televisions : colour, standard and economy. The expected daily production on each line is as follows :

TV model	Line I	Line II
Colour	3	1
Standard	1	1
Economy	2	6

The daily running costs for two lines average ₹ 6000 for line I and ₹ 4000 for Line II. It is given that the company must produce at least 24 colour, 16 standard and 48 economy TV sets for which an order is pending.

Formulate the above problem as a L.P.P. to minimize the total cost and solve it by graphical method. **(Oct. 2007)**

Solution : Suppose Line I operates for x days and Line II for y days. Then the total cost of running is C = 6000 x + 4000 y ₹ It is to be minimized.

Problem constraints :

Colour TV sets : $3x + y \geq 24$

Standard TV sets : $x + y \geq 16$

Economy TV sets : $2x + 6y \geq 48$.

Non-negativity constraints : $x \geq 0,\ y \geq 0$.

L.P.P. is

Minimize $C = 6000x + 4000y$

Subject to
$$3x + y \geq 24$$
$$x + y \geq 16$$
$$2x + 6y \geq 48$$
$$x \geq 0, \quad y \geq 0$$

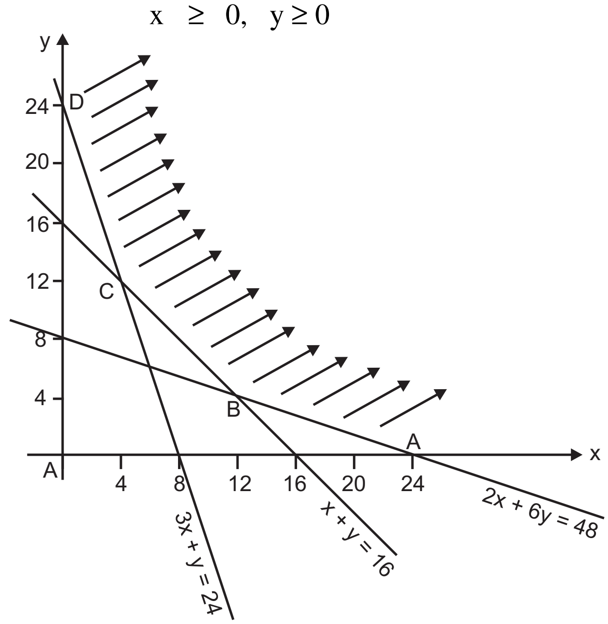


Fig. 1.8

The feasible region is shown by arrows.

The vertices of feasible region are A (24, 0), B (12, 4), C (4, 12), D (0, 24).

The following table gives the cost of production at each vertex.

Table

Vertex	C = 6000 x + 4000 y
A (24, 0)	₹ 144000
B (12, 4)	₹ 88000
C (4, 12)	₹ 72000
D (0, 24)	₹ 96000

The minimum cost of production is ₹ 72000 if Line I operates for 4 days and Line II operates for 12 days.

Example 1.13 : A XYZ company owns a small paint factory that produces both interior and exterior house points for wholesale distribution. Two basic raw materials A and B are used to manufacture the points. The maximum availability of A is 6 tons a day; that of B is 8 tons a day. The daily requirements of the raw materials per ton of interior and exterior are summarized in the following table.

Tons of raw material per ton of paints

		Exterior	Interior
Raw	A	1	2
Material	B	2	1

A market survey established that the daily demand for interior paint cannot exceed that of exterior paint by more than 1 ton. The survey also shows that the maximum demand for interior paint is limited to 2 tons daily.

The wholesale price per ton is ₹ 3000 for exterior and ₹ 2000 for interior paint.

Formulate the linear programming problem to maximum gross income and solve it graphically. **(April 2008)**

Solution : Suppose x tons of exterior and y tons of interior paint are produced daily. Then the gross daily income is $Z = 3000x + 2000y$ ₹ It is to be maximized.

The production requires $x + 2y$ tons of raw material A and $2x + y$ tons of raw material B.

$\therefore \quad x + 2y \leq 6$ and $2x + y \leq 8$.

Due to market conditions,

$$y \leq 2 \quad \text{and} \quad y \leq x + 1$$

L.P.P. is

Maximize $Z = 3000x + 2000y$

Subject to

$$\begin{aligned} x + 2y &\leq 6 \\ 2x + y &\leq 8 \\ y &\leq 2 \\ x - y &\geq -1 \\ x &\geq 0, \quad y \geq 0 \end{aligned}$$

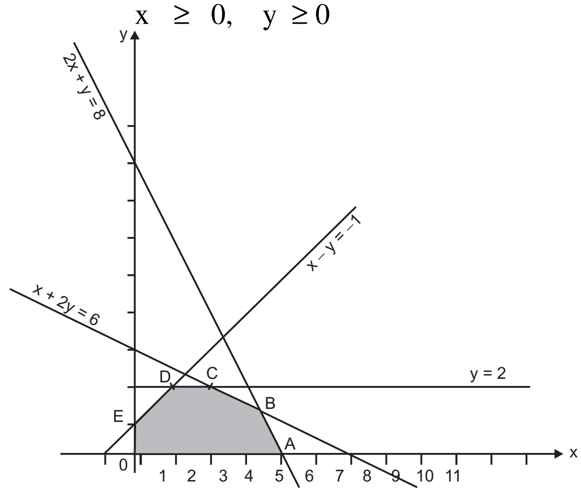


Fig. 1.9

The feasible region is shown dotted.

The vertices of feasible region are O (0, 0), A (4, 0), B $\left(\frac{10}{3}, \frac{4}{3}\right)$, C (2, 2), D (1, 2), E (0, 1).

The value of Z at each vertex is given in the following table.

Table 1.1

Vertex	**Z = 3000 x + 2000 y**
O (0, 0)	0
A (4, 0)	12000
B $\left(\frac{10}{3}, \frac{4}{3}\right)$	12666.66
C (2, 2)	10000
D (1, 2)	7000
E (0, 1)	2000

The value of Z is maximum when $x = \frac{10}{3}$ and $y = \frac{4}{3}$.

$$Z_{max} = ₹\ 12666.66\ .$$

Example 1.14 : An electric company manufactures two models at two different plants. The daily capacity of the first plant is 60 radios and that of second is 75 radios. Each unit of the first model requires 10 pieces of a certain electric component, whereas each unit of the second model requires 8 pieces of the same component. The maximum daily availability of the special component is 800 pieces. The profit per unit of the first model is ₹ 300 and that of the second model is ₹ 200. Formulate the L.P.P. and solve by graphical method. **(Oct. 2008)**

Solution : Suppose first plant produces x units of model one and second plant produces y units of model two.

Then the total profit is P = ₹ 300 x + 200 y

It is to be maximized.

Problem constraints : The capacity of the first plant is 60. Therefore, $x \le 60$. The capacity of the second plant is 75. $\therefore$ $y \le 75$.

The total number of electric components required is $10x + 8y$ and its maximum availability is 800. $\therefore$ $10x + 8y \le 800$.

Non-negativity constraints : $x \ge 0$, $y \ge 0$.

L.P.P. is

$$\begin{aligned} \text{Maximize } P &= 300x + 200y \\ \text{Subject to} \quad x &\le 60 \\ y &\le 75 \\ 10x + 8y &\le 800 \\ x \ge 0, \quad y &\ge 0 \end{aligned}$$

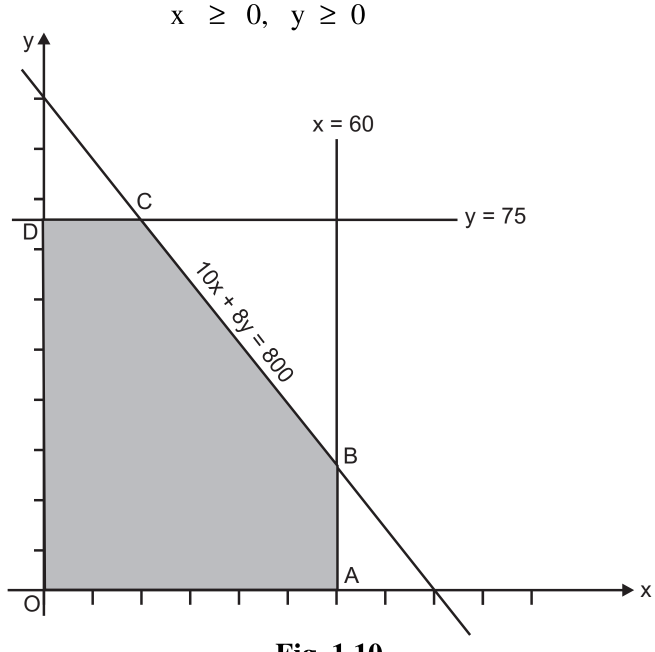


Fig. 1.10

The feasible region is shown dotted.

The vertices of the feasible region are O (0, 0), A (60, 0), B (60, 25), C (20, 75), D (0, 75).

The value of the profit P at the vertices of feasible region is given in the following table.

Table

Vertex	P = 300x + 200y
O (0, 0)	0
A (60, 0)	18000
B (60, 25)	23000
C (20, 75)	21000
D (0, 75)	15000

The maximum profit is ₹ 23000, when first plant produces 60 units and the second plant produces 25 units.

Exercise (1.2)

Solve the following problems by graphical method.

1. Maximize $Z = 12x_1 + 8x_2$, subject to

$$\begin{aligned} x_1 + 2x_2 &\leq 6 \\ 2x_1 + x_2 &\leq 8 \\ -x_1 + x_2 &\leq 1 \\ x_2 &\leq 2 \\ x_1,\ x_2 &\geq 0. \end{aligned}$$

2. Maximize $Z = x_1 + 2x_2$, subject to

$$\begin{aligned} x_1 + 2x_2 &\leq 20 \\ x_1 + x_2 &\leq 12 \\ x_1,\ x_2 &\geq 0. \end{aligned}$$

3. Maximize $Z = 6x_1 + 11x_2$, subject to

$$\begin{aligned} 2x_1 + x_2 &\leq 104 \\ x_1 + 2x_2 &\leq 76 \\ x_1, x_2 &\geq 0. \end{aligned}$$

4. Maximize $Z = 5x_1 + 7x_2$, subject to

$$\begin{aligned} x_1 + x_2 &\leq 4 \\ 3x_1 + 8x_2 &\leq 24 \\ 10x_1 + 7x_2 &\leq 35 \\ x_1, x_2 &\geq 0. \end{aligned}$$

5. Maximize $Z = 3x_1 + 5x_2$, subject to

$$\begin{aligned} x_1 + 2x_2 &\leq 2000 \\ x_1 + x_2 &\leq 1500 \\ x_2 &\leq 600 \\ x_1, x_2 &\geq 0. \end{aligned}$$

6. Minimize $Z = 3x_1 + 5x_2$, subject to

$$\begin{aligned} -3x_1 + 4x_2 &\leq 12 \\ 2x_1 - x_2 &\geq -2 \\ 2x_1 + 3x_2 &\geq 12 \\ x_1 \leq 4, \quad x_2 \geq 2 \text{ and } \quad x_1,\ x_2 &\geq 0. \end{aligned}$$

7. Maximize $Z = 5x_1 + 3x_2$, subject to

$$\begin{aligned} 3x_1 + 5x_2 &\le 15 \\ 6x_1 + 2x_2 &\le 24 \\ x_1,\ x_2 &\ge 0. \end{aligned}$$

8. Minimize $Z = 4x_1 + x_2$, subject to

$$\begin{aligned} 3x_1 + x_2 &= 3 \\ 4x_1 + 3x_2 &\ge 6 \\ x_1 + 2x_2 &\le 4 \\ x_1,\ x_2 &\ge 0. \end{aligned}$$

9. Maximize $Z = 3x_1 + 2x_2$, subject to

$$\begin{aligned} 2x_1 + x_2 &\le 2 \\ 3x_1 + 4x_2 &\ge 12 \\ x_1,\ x_2 &\ge 0. \end{aligned}$$

10. Maximize $Z = 40x_1 + 35x_2$, subject to

$$\begin{aligned} 2x_1 + 3x_2 &\le 60 \\ 4x_1 + 3x_2 &\le 96 \\ x_1,\ x_2 &\ge 0. \end{aligned}$$

11. Maximize $Z = 6x_1 + 10x_2$, subject to

$$\begin{aligned} 2x_1 + x_2 &\le 104 \\ x_1 + 2x_2 &\le 76 \\ x_1,\ x_2 &\ge 0. \end{aligned}$$

12. Maximize $Z = 40x_1 + 100x_2$, subject to

$$\begin{aligned} 12x_1 + 6x_2 &\le 3000 \\ 4x_1 + 10x_2 &\le 2000 \\ 2x_1 + 3x_2 &\le 900 \\ x_1,\ x_2 &\ge 0. \end{aligned}$$

13. Maximize $Z = 5x_1 + 7x_2$, subject to

$$\begin{aligned} x_1 + x_2 &\le 4 \\ 3x_1 + 8x_2 &\le 24 \\ 10x_1 + 7x_2 &\le 35 \\ x_1,\ x_2 &\ge 0. \end{aligned}$$

14. Minimize $Z = 3x_1 + 5x_2$, subject to

$$\begin{aligned} -3x_1 + 4x_2 &\le 12 \\ 2x_1 - x_2 &\ge -2 \\ 2x_1 + 3x_2 &\ge 12 \\ x_1 &\le 4 \\ x_2 &\ge 2 \end{aligned}$$

with $x_1,\ x_2 \ge 0.$

15. Maximize $Z = 300x_1 + 200x_2$, subject to

$$\begin{aligned} 24x_1 + 8x_2 &\leq 60{,}000 \\ x_1 &\leq 2000 \\ x_2 &\leq 6000 \\ x_1, x_2 &\geq 0. \end{aligned}$$

16. Maximize $Z = 3x_1 + 4x_2$, subject to

$$\begin{aligned} x_1 + x_2 &\leq 450 \\ 2x_1 + x_2 &\leq 600 \\ x_1, x_2 &\geq 0. \end{aligned}$$

17. Maximize $Z = 40x_1 + 100x_2$, subject to

$$\begin{aligned} 12x_1 + 6x_2 &\leq 3000 \\ 4x_1 + 10x_2 &\leq 2000 \\ 2x_1 + 3x_2 &\leq 900 \\ x_1, x_2 &\geq 0. \end{aligned}$$

18. Minimize $Z = 600x_1 + 400x_2$, subject to

$$\begin{aligned} 1500x_1 + 1500x_2 &\geq 20{,}000 \\ 3000x_1 + 1000x_2 &\geq 40{,}000 \\ 2000x_1 + 5000x_2 &\geq 44{,}000 \\ x_1, x_2 &\geq 0. \end{aligned}$$

Answers (1.2)

1. $x_1 = \frac{10}{3}$, $x_2 = \frac{4}{3}$, $Z_{max.} = \frac{152}{3}$.
2. $x_1 = 4$, $x_2 = 8$ (or $x_1 = 0$, $x_2 = 10$) with $Z_{max} = 20$. It has infinitely many solutions.
3. $x_1 = 44$, $x_2 = 16$ with $Z_{max.} = 440$.
4. $x_1 = 1.6$, $x_2 = 2.4$ with $Z_{max.} = 24.8$
5. $x_1 = 1000$, $x_2 = 500$ with $Z_{max.} = 5500$.
6. $x_1 = 3$, $x_2 = 2$ with $Z_{min.} = 19$.
7. $x_1 = \frac{15}{4}$, $x_2 = \frac{3}{4}$, with $Z_{max.} = 21$.
8. $x_1 = \frac{2}{5}$, $x_2 = \frac{9}{5}$ with $Z_{min.} = \frac{17}{5}$.
9. No feasible solution.
10. $x_1 = 18$, $x_2 = 8$, with $Z_{max.} = 1000$.
11. $x_1 = 44$, $x_2 = 16$, $Z_{max} = 424$.
12. $x_1 = 187.5$, $x_2 = 125$ (or, $x_1 = 0$, $x_2 = 200$) $Z_{max} = 20{,}000$.
13. $x_1 = \frac{8}{5}$, $x_2 = \frac{12}{5}$ with $Z_{max.} = \frac{124}{5}$.
14. $x_1 = 3$, $x_2 = 2$, with $Z_{min.} = 19$.
15. $x_1 = 500$, $x_2 = 6000$, with $Z_{max.} = 13{,}5000$
16. $x_1 = 0$, $x_2 = 450$, with $Z_{max.} = 1800$.
17. $x_1 = 0$, $x_2 = 300$ with $Z = 30{,}000$.
18. $x_1 = 1200$, $x_2 = 300$, $Z = 88000$.

1.4 General Linear Programming Problems

1.4.1 General Linear Programming Problem

A programming problem in which all the functions, (objective as well as constraints) are linear is known as linear programming problem. The general L.P.P. can be described as follows.

Given a set of 'm' linear inequalities or equalities in n-variables (decision variables) and the objective function is to be optimized (minimize or maximize). The solution to the problem is to find non-negative values of these n-variables which satisfy all the constraints simultaneously and optimizes the objective function. The mathematical model is given in the following form :

Optimize : $Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$ … (1.1)

Subject to

$$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n \ (\leq, =, \geq) \ b_1$$
$$a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n \ (\leq, =, \geq) \ b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad \ldots (1.2)$$
$$a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n \ (\leq, =, \geq) \ b_m$$

where, for each constraint one and only one of the signs $\leq$, $=$ or $\geq$ holds, but the sign may vary from one constraint to other. The non-negativity constraints are

$$x_j \geq 0, \quad j = 1, 2, \ldots, n. \qquad \ldots (1.3)$$

Here optimize means either maximize or minimize.

The above L.P.P. in n-variables with m inequalities can be expressed in matrix form as :

Optimize : $Z = C^t X$, subject to

$$AX = B \text{ and}$$

$x_j \geq 0$, for $j = 1, 2, \ldots, n$:

where, $$C = \begin{bmatrix} c_1 \\ c_2 \\ : \\ c_n \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ : \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ : \\ b_m \end{bmatrix}$$

and $$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ : & : & \ldots & : \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

Here C^t means the transpose of the column matrix C.

1.4.2 The Canonical Form

The L.P.P. is said to be in canonical form if (i) the objective function is of the maximization type, (ii) all the constraints are '$\leq$' type and (iii) all the basic (decision) variables are non-negative.

The mathematical form of the L.P.P. in canonical form is given by :

$$\text{Maximize : } Z = \sum_{j=1}^{n} c_j x_j$$

Subject to $\sum_{j=1}^{n} a_{ij} x_j \le b_i, \; i = 1, 2, \ldots, m$ and $x_j \ge 0, \; j = 1, 2, \ldots, n$.

Remark : (1) If the objective function is of minimization type, i.e.

Minimize : $Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$ … (i)

we can convert this to maximization type as the equation (i) is equivalent to

Maximize : $Z_0 = -c_1 x_1 - c_2 x_2 - \ldots - c_n x_n = -Z$.

(2) If any of the constraint is '≥' type, multiplying by (–1) on both sides we can convert it to '≤' type.

For example, if

$$-3x_1 + 4x_2 + 5x_3 \ge -5, \text{ which is equivalent to}$$

$$3x_1 - 4x_2 - 5x_3 \le 5.$$

(3) If a constraint is '=' type, then it can be resolved to '≤' type.

For example, if

$$2x_1 - 4x_2 + 3x_3 = -2 \qquad \ldots \text{(ii)}$$

then this is equivalent to two inequalities :

$$2x_1 - 4x_2 + 3x_3 \le -2$$

and $$2x_1 - 4x_2 + 3x_3 \ge -2,$$

Next these two inequalities are equivalent to

$$2x_1 - 4x_2 + 3x_3 \le -2$$

and $$-2x_1 + 4x_2 - 3x_3 \le 2 \qquad \ldots \text{(iii)}$$

Thus, the constraint '=' type can be equivalently converted to '≤' type as in equation (iii).

Thus, we see from above remarks, that any L.P.P. (may be minimization type) can be converted to L.P.P. in *canonical form.*

1.4.3 Linear Programming Problem in Standard Form

The L.P.P. is said to be in *standard form* if

(i) The objective function is of either maximization or minimization type,

(ii) All constraints are equations except the non-negativity constraints, and

(iii) The right hand side constants of the constraints must be non-negative.

Standard L.P.P. in mathematical form can be written as :

Optimize (i.e. minimize or maximize)

$$Z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n$$

Subject to

$$a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n = b_1$$

$$a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n = b_2$$

$$a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n = b_m$$

and $x_j \geq 0$, $j = 1, 2, \ldots, n$

$b_i \geq 0$, $i = 1, 2, \ldots, m$.

The L.P.P. in standard form can be written in matrix form as :

Optimize $Z = C^t X$

Subject to $AX = B$, with

$x_j \geq 0$, $j = 1, 2, \ldots, n$ and

$b_i \geq 0$, $i = 1, 2, \ldots, m$.

Here A, X, B and C are matrices as considered in article 1.4.1.

1.5 Some Important Definitions

1. Solution to L.P.P. : A set X of n real numbers $x_1, x_2, \ldots, x_n$ which satisfies all the constraints of the general L.P.P. (specified as in (1.2)) is called the *solution* to the general L.P.P.

2. Feasible region : The *feasible region* is the set S of all solutions of general L.P.P. OR, the set of points in solution space is called *feasible region.*

3. Feasible solution : A solution X to the general L.P.P. is called *feasible solution* if it satisfies non-negativity constraints.

Remark : Since in general L.P.P. we consider the number of unknowns more than the number of constraints, hence there always exists a non-trivial solution for the general L.P.P.

As the standard form of general L.P.P. requires all constraints to be equalities and the mathematical procedure to solve any L.P.P. needed to be in standard form, the inequalities in L.P.P. needed to be converted into equalities by adding or subtracting suitable non-negative constants, which are called *slack* or *surplus* variables.

4. Slack variables : If a constraint in given L.P.P. is '≤' type, then in order to make it an equation, it requires addition of some non-negative variable, on the left hand side of such inequality. Such variables are called *slack variables.*

5. Surplus variables : If a constraint is of the '≥' type, then in order to convert it to equality the *subtraction* of some non-negative variable is required. Such a variable is called *surplus variable.*

6. Basic solution : A solution X of the general L.P.P., obtained by setting $(n - m)$ of its variables equal to zero, is called a *basic solution.*

7. Basic variables : A variable is said to be *basic* in an equation if it occurs with unit coefficient in it and with zero coefficient in other equations.

The variables which are not basic variables in L.P.P. are called *non-basic* variables.

8. Basic feasible solution : A feasible solutions is called *basic feasible solution* (b.f.s.) if it is also a basic solution.

9. Non-degenerate B.F.S. : If each of the m variables of basic feasible solution is positive, the basic feasible solution is called *non-degenerate* b.f.s.

10. Degenerate B.F.S. : In other words, a b.f.s. is called degenerate if one or more of the basic variables are at zero level. **(Mach 2010)**

11. Optimal solution : A solution X_0 is said to be an *optimal solution* if it is feasible, the value of the objective function $C^t X_0$ is optimum to any other feasible solution.

12. Alternate solution : When a L.P.P. has more than one solution giving the same optimum value of the problem, it is said to have *alternate* solution.

To find possible basic solutions :

To illustrate this we consider the following system of linear equations :

$$\begin{aligned} 4x_1 + 3x_2 + x_3 &= 6 \\ x_1 + 2x_2 + 5x_3 &= 4 \end{aligned} \qquad \ldots \text{(i)}$$

The given system can be written in matrix form as :

AX = B, where,

$$A = \begin{bmatrix} 4 & 3 & 1 \\ 1 & 2 & 5 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad B = \begin{bmatrix} 6 \\ 4 \end{bmatrix}.$$

Since there are 3-variables and two equations, to obtain basic solution, we need to set one of the three variables equal to zero. This can be done in three ways : (i) By setting $x_3 = 0$, (ii) $x_2 = 0$, (iii) $x_1 = 0$.

Case (i) when $x_3 = 0$, we can solve the two equations in two unknowns (x_1 and x_2), which can be manipulated as : $x_1 = 0$, $x_2 = 2$ (basic). Thus in this case, the basic solution is $x_1 = 6$, $x_2 = 4$, $x_3 = 0$.

Case (ii) when $x_2 = 0$, we obtain

$$x_1 = \frac{26}{19}, \quad x_3 = \frac{10}{19} \text{ (basic variables)}$$

So the basic solution is $x_1 = \frac{26}{19}$, $x_3 = \frac{10}{19}$, $x_2 = 0$.

Case (iii) when $x_1 = 0$, we obtain $x_2 = 2$, $x_3 = 0$ as basic variables.

Thus, the basic solution is $x_2 = 2$, $x_3 = 0$ and $x_1 = 0$.

Note : (i) Since solutions (i), (ii) and (iii) satisfy non-negativity constraints, so all three solutions are *basic feasible solutions.*

(ii) The solution (ii) is *non-degenerate*, whereas the solutions in (i) and (iii) are *degenerate* basic feasible solutions.

Pivot operation : A pivot operation is a sequence of elementary operations which reduce the given system to an equivalent system in which a specified variable has an unit coefficient in one equation and zero coefficient in the remaining equations.

We state the following theorem with proof as it is beyond the scope of the book.

Theorem 1 If S is bounded feasible region then the optimal solution corresponds to an extreme point of the set S. Further, if it corresponds to more than one extreme point of the set S, there are infinitely many optimal solutions to the given L.P.P.

1.6 Assumptions in Linear Programming Problem

Following are the assumptions in L.P.P. that limit it applicability.

(i) Proportionality : A primary requirement of L.P.P. is that the objective function and every constraint function must be linear.

(ii) Additivity : As discussed in Example 1.4 additivity means if it takes t_1 minutes on stage 1 to processed product 1 and t_2 minutes to processed product 2, t_3 minutes to processed product 3, then time on stage 1 devoted to processes all product it $t_1 + t_2 + t_3$ for one unit, provided the time required to change the stage from product 1 to 2 is negligible.

(iii) Multiplicity : It requires :

(a) If it takes one minute to processes a single product on given stage, it will take 10 minute to processes 10 such product and

(b) The total profit from selling a given number of units is the unit profit times the number of units sold.

(iv) Divisibility : It means that the functional levels of variables must be permissible besides integral values.

(v) Deterministic : All the parameters in the LP models are assumed to be known exactly. While in actual practice, production may depends upon chance also, such type of problems, where some of coefficients are not known, are discussed in the extension of sensitivity analysis know as parametric programming.

Significance of Assumptions : A practical problem which completely statistics all the above assumptions for LP is very rare indeed. Therefore the user should be fully aware of the assumptions and approximations involved and should satisfy himself that they are justified before proceeding to apply LP approach.

1.7 Limitations of Linear Programming

Following are some limitation associated with LP techniques :

(i) In some problems objective functions and constraints are not linear. Generally, in real life situations concerning business and industrial problems constraints are not linearly treated to variables.

(ii) There is no guarantee to getting integer valued solutions, for example, in finding out how many men and machines would be required to perform a particular job, rounding-off the solution to the nearest integer will not give an optimal solution. Integer programming deals with such problem.

(iii) LP models does not take into consideration the effect of time and uncertainty. Thus the model should be defined in such a way that any change due to internal as well as external factors can be incorporated.

(iv) Sometimes large-scale problems cannot be solved with LP techniques even when computer facility is available. Such difficulty may be removed by decomposing the main problem into several small problems and then solving them separately.

(v) Parameters appearing in the model are assumed to be constant. But, in real life situations they are neither constant nor deterministic.

(vi) LP deals with only single objective, whereas in real life situations problems come across with multi-objective Goal programming and multi-objective programming deal with such problems.

1.8 Applications of Linear Programming

In this course we are going to study some applications of LP, like Personnel Assignment Problem, Transportation Problem. Following are some applications of LP reader can google for details of it :

1. Agricultural applications
2. Military applications

3. Production Management
4. Marketing Management
5. Manpower Management
6. Physical Distribution
7. Efficiencing on operation of system of Dams
8. Optimum Estimation of Executive Compensation
9. Administration Management
10. Education Management
11. Inventory Control Management
12. Capital Budgeting Management etc.

1.9 Advantages of Linear Programming Techniques

Following are some advantages of LP techniques :

1. LP techniques helps us in making the optimum utilization of productive resources. It also indicates how a decision-maker can employ this productive factors most effectively by choosing and allocating these resources.

2. The quality of decisions may also be improved by LP techniques. The use of this technique becomes more objective and less subjective.

3. LP techniques provides practically applicable solutions since there might be other constraints operating outside the problem which must also be taken into consideration just become, so many units must be produced does not mean that all those can be sold. So the necessary modification of its mathematical solution is required for the sake of convenience to the decision maker.

4. In production processes, high lighting of bottlenecks is most significant advantage of this technique. For example, when bottlenecks occur, some machines cannot meet the demand while other remain idle for some time.

1.10 Simplex Method

The simplex method for solving a L.P.P. which is developed by G. Dantzig is an iterative procedure. The method is applicable when an L.P.P. is expressed in standard form. The general nature of the simplex method is as follows :

(i) Start with an initial basic feasible solution (I.B.F.S.) from the equivalent system obtained by pivot operations. (ii) Improve this I.B.F.S. with a better objective function value. At this step, the simplex method ignores the previous I.B.F.S. The objective function behaves like an indicator of improvement. (iii) The improvement in objective function value is made possible until it becomes optimal. This is eased by simplex method without checking for all extreme points. As the basic feasible solution becomes optimal, the simplex procedure terminates.

1.10.1 The Simplex Method in Table Form

The various steps of the simplex method can be carried out in a more compact manner by using a table form to represent the constraints and the objective function. The various calculations can be made easy and mechanical, which are simple to remember. The use of simplex method in table form made the simplex method more efficient and convenient for computer programming.

We illustrate the simplex procedure in table form by following the particular example.

Illustrative Examples

Example 1.15 : Maximize $Z = 5x_1 + 3x_2$

Subject to
$$3x_1 + 5x_2 \le 15$$
$$6x_1 + 2x_2 \le 24$$
$$x_1,\ x_2 \ge 0.$$

Converting this problem to standard form

Maximize $Z = 5x_1 + 3x_2 + 0.s_1 + 0.s_2$

Subject to
$$3x_1 + 5x_2 + s_1 = 15$$
$$6x_1 + 2x_2 + s_2 = 24$$

$x_1, x_2, s_1, s_2 \ge 0$, where s_1, s_2 are slack variables.

Since, it is easy to see that s_1 and s_2 are basic variables and x_1 and x_2 are non-basic, the I.B.F.S. is given by $s_1 = 15$, $s_2 = 24$ (basic variables) and $x_1 = 0 = x_2$ (non-basic variables) with objective function value $Z = 0$. We write down the problem in table form in the following table 1.2.

Table 1.2

C_B	C_i / X_B Basis	5 x_1	3 x_2	0 s_1	0 s_2	Constants	Ratio
0	s_1	3	5	1	0	15	15/3 = 5
0	s_2	6	2	0	1	24	24/6 = 4
$Z_j - C_j$		**–5**	**–3**	**0**	**0**	**Z = 0**	

Here C_i row represents the coefficients of the respective variables in objective function. X_B is the column of basis elements or basic variables, and C_B represents the corresponding coefficients of basic variables in X_B that in objective function. In the body of the table there are coefficients of the respective variables in the two constraints of the problem. The first right hand side column represents the right hand side constants of the constraints. The value of the objective function is calculated by;

$$Z = C_B^t \times B. \text{ In present case}$$
$$C_B^t = (0, 0) \text{ and } X_B = \begin{pmatrix} 15 \\ 24 \end{pmatrix}.$$

Therefore, $$Z = (0, 0)\begin{pmatrix} 15 \\ 24 \end{pmatrix} = 0 \times 15 + 0 \times 24 = 0.$$

In order to check the above I.B.F.S. to be optimal, we require to calculate relative profits of all non-basic variables. The relative profit coefficient of the j^{th} variable, denoted by $Z_j - C_j$, is given by

$Z_j - C_j = C_B^t \times$ (The column of the coeffs. of the j^{th} variable in the constraints) $- C_j$

These relative profits of the corresponding variables are written as row at the bottom of the table 1.2. In present case, we calculate $Z_j - C_j$ for each j :

$$Z_{x_1} - C_{x_1} = Z_1 - C_1 = (0, 0)\begin{pmatrix} 3 \\ 6 \end{pmatrix} - 5 = 0 - 5 = -5$$

$$Z_{x_2} - C_{x_2} = Z_2 - C_2 = (0, 0)\begin{pmatrix}5\\2\end{pmatrix} - 3 = 0 - 3 = -3.$$

$$Z_{s_1} - C_{s_1} = Z_3 - C_3 = (0, 0)\begin{pmatrix}1\\0\end{pmatrix} - 0 = 0$$

$$Z_{s_2} - C_{s_2} = Z_4 - C_4 = (0, 0)\begin{pmatrix}0\\1\end{pmatrix} - 0 = 0.$$

There are some negative values in $Z_j - C_j$ row, showing that the current basic feasible solution is not optimal. Observe that the relative profit coefficients of basic variables are zero. The non-basic variables give negative relative profit coefficient.

For better improvement we select a non-basic variable largest in magnitude with negative sign to enter the basis. In the present case it is x_1, so we say that x_1 is entering variable, the corresponding column to x_1 is called **pivot column.** Now, one of the basic variables must leave the basis. To decide this, we apply minimum ratio rule by calculating limits for constraints as follows.

Row No.	Basic variable	Upper limit for x_1 ratio
1	s_1	15/3 = 5
2	s_2	24/6 = 4 minimum

The minimum of the ratios, i.e., min (5, 4) = 4, which corresponds to second row. Thus, x_1 enters the basis through second equation and the corresponding row (here second row) is called pivot row or key row and s_2 is leaving variable. Thus s_2 is to be reduced to zero and x_1 increases to 4. The element at the intersection of pivot column and pivot row is called *pivot* element, it is encircled in table 2.

The new basis will contain s_1 and x_1. The new equivalent system is obtained by performing a pivot operations as follows :

(i) Divide the pivot row by pivot element, here it is 5, to have the coefficient of x_1 unity.

(ii) Multiply the row obtained in (i) by (–3) and add it to the first row to have coefficient of x_1 zero in it.

The new table is obtained.

Table 1.3

C_B	C_j / X_B Basis	5 x_1	3 x_2	0 s_1	0 s_2	Constants	Ratio
0	s_1	0	4	1	–1/2	3	3/4
5	x_1	1	3	0	1/6	4	$\frac{4}{1/3} = 12$
$Z_j - C_j$		0	$-\frac{4}{3}$	0	$\frac{5}{6}$	Z = 20	

We see that the new basic feasible solution is given by $s_1 = 3$, $x_1 = 4$ (basic variables) and $x_2 = 0 = s_2$ (non-basic variables),

$$\text{with } Z = 5 \times 4 + 0 \times 3 = 20.$$

The new relative profit coefficients and ratios are calculated as before.

We observe that $Z_2 - C_2 = -\frac{4}{3} < 0$, showing that the basic feasible solution obtained in second state is *not* optimal. Arguing as before, we see that second column corresponding to $-\frac{4}{3}$ is pivot column and minimum ratio rule shows that first row is pivot row and '4' is pivot element. This shows that s_1 leaves the basis and x_2 enters the basis. The new table is obtained by pivot operations as before.

Table 1.4

C_B	C_i / X_B Basis	5 x_1	3 x_2	0 s_1	0 s_2	Constants	Ratio
3	x_2	0	1	1/4	–1/8	3/4	
5	x_1	1	0	–1/12	5/24	15/4	
$Z_j - C_j$		0	0	$\frac{3}{4} - \frac{5}{12} = \frac{1}{3}$	$-\frac{3}{8} - \frac{25}{24} = \frac{2}{3}$	Z = 21	

We observe that all the relative profit coefficients are positive (non-negative). The present solution is $x_1 = \frac{15}{4}$, $x_2 = \frac{3}{4}$ (basic variables) ; $s_1 = 0 = s_2$ (non-basic variables) with

$$Z_{max} = 5 \times \frac{15}{4} + 3 \times \frac{3}{4}$$

$$= \frac{75}{4} + \frac{9}{4} = \frac{84}{4} = 21.$$

Thus, the optimal solution is given by :

$$x_1 = \frac{15}{4}, \quad x_2 = \frac{3}{4} \quad \text{and } Z_{max} = 21.$$

Summary :

(i) Express the given L.P.P. in standard form and represent it in table form.

(ii) Start with an I.B.F.S. and set the initial table.

(iii) Calculate relative profit coefficients and enter them in $Z_j - C_j$ row.

(iv) If all the $Z_j - C_j$ coefficients are non-negative, the current basic feasible solution is optimal; otherwise the non-basic variable with largest magnitude with negative sign of $Z_j - C_j$ enters the basis. The column corresponding to this coefficient is called *pivot* column.

(v) Apply the minimum ratio test to determine the basic variable to leave the basis. The corresponding row is called pivot row.

(vi) Perform the pivot operations to get the new table and the new basic feasible solution.

(vii) Calculate relative profit coefficients.

Return to step (iv).

Each sequence of steps (iv) and (vii) is called an iteration, and obtain a new table and improved solution.

1.10.2 Minimization Problem

The coefficients in the $(Z_j - C_j)$ row give the net change in the value of Z per unit increase in the value of a non-basic variable. In maximization case, the negative coefficient in $(Z_j - C_j)$ row indicated, the corresponding non-basic variable, when increased, increased the value of the objective function. In minimization problem, the positive coefficient in $(Z_j - C_j)$ row indicates, the corresponding non-basic variable, when increased reduces the value of the objective function. So, in the case of minimization problem only those non-basic variables with positive $(Z_j - C_j)$ values are eligible to enter the basis and improve the objective function in the expected sense. The optimal solution is achieved when all coefficients in the $(Z_j - C_j)$ row are non-positive i.e. ≤ 0.

Thus all the seven steps of the simplex method outlined earlier can be used for minimization problem with a minor modification (as suggested above) in step (iv).

Modified step (iv) : If all the relative profit coefficients in $(Z_j - C_j)$ row are non-positive, the current b.f.s. is optimal; otherwise, select the non-basic variable with largest positive $(Z_j - C_j)$ coefficient to enter the basis.

Note : Another method to solve minimization problem is to convert it to an equivalent maximization problem and then use simplex method as outlined for a maximization problem. Such conversion is done by multiplying the objective function of the minimization problem by (–1).

1.10.3 Use of Artificial Variables

In case of problems either maximization or minimization if there are constraints of the type $\geq$, use of artificial variables is must. Let us consider a problem of minimization, in which $\geq$ type constraint is involved and how to solve it by using *Big-M Method*.

Illustrative Examples

Example 1.16 : Minimize $Z = 4x_1 + x_2$

Subject to

$$\begin{aligned} 3x_1 + x_2 &= 3 \\ 4x_1 + 3x_2 &\geq 6 \\ x_1 + 2x_2 &\leq 4; \\ x_1, x_2 &\geq 0. \end{aligned}$$

Solution : First we convert the problem into standard form by augmenting a surplus variable s_1 and adding a slack variable s_2 to the left of the constraints second and third. Thus we have,

Minimize : $Z = 4x_1 + x_2 + 0.s_1 + 0.s_2$

Subject to

$$\begin{aligned} 3x_1 + x_2 &= 3 \\ 4x_1 + 3x_2 - s_1 &= 6 \\ x_1 + 2x_2 + s_2 &= 4 \\ x_1, x_2, s_1, s_2 &\geq 0. \end{aligned}$$

The BIG M-Method :

In above problem, even in standard form I.B.F. solution is not available at first sight, so instead of going for pivot operations, use of artificial variables is helpful to expedite the solution process. So introducing artificiable variables, we have,

Minimize $Z = 4x_1 + x_2 + 0.s_1 + 0.s_2 + MR_1 + MR_2$

Subject to

$$\begin{aligned} 3x_1 + x_2 + R_1 &= 3 \\ 4x_1 + 3x_2 - s_1 + R_2 &= 6 \\ x_1 + 2x_2 + s_2 &= 4 \end{aligned}$$

where, R_1, $R_2 \geq 0$ are artificiables and M is positive and sufficiently large, so that it helps to remove artificiables from basis in case of minimization problem. In case of maximization problem, M is taken sufficiently large with negative sign. Because of use of M, the method carried out to solve the above problem is called Big M-technique or method.

Here, we have to take $Z_0 = -Z$, so minimum of Z_0 = minimum of $(-Z)$, so that maximum of $Z = -$ minimum of Z_0.

Table 1.5

C_B	C_j / X_B basis	4 x_1	1 x_2	0 s_1	0 s_2	M R_1	M R_2	Const.	Ratio
M	R_1	$\boxed{3}$	1	0	0	1	0	3	$\frac{3}{3} = 1$
M	R_2	4	3	–1	0	0	1	6	$\frac{6}{4} = \frac{3}{2}$
0	s_2	1	2	0	1	0	0	4	$\frac{4}{1} = 4$
$Z_j - C_j$		7M – 4	4M – 1	– M	0	0	0	Z = 9M	

The I.B.F. solution is $R_1 = 3$, $R_2 = 6$, $s_2 = 4$ and value of Z = 9 M. Now we calculate $Z_j - C_j$ to check, whether the present solution is optimal. Since not all $Z_j - C_j$ are non-positive, the present solution is not optimal (this is minimization problem). The largest positive of all $Z_j - C_j$ is $Z_1 - C_1 = 7M - 4$, which corresponds to first column, hence x_1 is entering variable. Now to decide leaving variable, we calculate the ratios : with constant to that of coefficients of x_1 in the three constraints. We see that out of these the ratio 1 is minimum, hence first row is pivot row, so that R_1 is leaving variable. Performing pivot operations we obtain the following table 1.6.

Table 1.6

C_B	C_j / X_B	4 x_1	1 x_2	0 s_1	0 s_2	M R_1	M R_2	Const.	Ratio
4	x_1	1	$\frac{1}{3}$	0	0	$\frac{1}{3}$	0	1	$\frac{1}{1/3} = 3$
M	R_2	0	$\boxed{\frac{5}{3}}$	–1	0	$-\frac{4}{3}$	1	2	$\frac{2}{5/3} = \frac{6}{5}$
0	s_2	0	$\frac{5}{3}$	0	1	$-\frac{1}{3}$	0	3	$\frac{3}{5/3} = \frac{9}{5}$
$Z_j - C_j$		0	$\frac{5}{3}M + \frac{1}{3}$	– M	0	$\frac{4}{3} - \frac{7}{3}M$	0	Z = 2M + 4	

The I.B.F. solution is given by $x_1 = 1$, $R_2 = 2$, $s_2 = 3$, $x_2 = 0$, $R_1 = s_1 = 0$ with Z = 2M + 4.

Since not all $Z_j - C_j$ are non-positive, the present solution is not optimal. Since $\frac{5}{3}$ M + $\frac{1}{3}$ is the largest among positive $Z_j - C_j$, x_2 is entering variable. Looking at the ratios, we see that second row is pivot row, hence R_1 is leaving variable. Thus x_2 is entering variable and R_2 is leaving variable. Performing the pivot operations, we obtain the following table 1.7.

Table 1.7

C_B	C_i / X_B	4 x_1	1 x_2	0 s_1	0 s_2	M R_1	M R_2	Const.	Ratio
4	x_1	1	0	$\frac{1}{5}$	0	$\frac{3}{5}$	$-\frac{1}{5}$	$\frac{3}{5}$	$\frac{3/5}{1/5} = 3$
1	x_2	0	1	– 3/5	0	–4/5	3/5	$\frac{6}{5}$	$\frac{6/5}{-3/5} = -\frac{6}{3}$
0	s_2	0	0	$\boxed{1}$	1	1	–1	1	$\frac{1}{1} = 1$
$Z_j - C_j$		0	0	$\frac{1}{5}$	0	$\frac{8}{5} - M$	$-\frac{1}{5} - M$	$Z = \frac{18}{5}$	

The I.B.F. solution is given by $x_1 = \frac{3}{5}$, $x_2 = \frac{6}{5}$, $s_2 = 1$, $R_1 = R_2 = s_1 = 0$, with $Z = \frac{18}{5}$, since again not all $Z_j - C_j$ are non-positive, the present solution yet can be improved. Since $\frac{1}{5}$ is largest positive among all $Z_j - C_j$, the s_1 is entering variable and looking at the ratio we see that 1 is the smaller positive ratio so, s_2 is leaving. [Note that the negative ratios to be neglected from considering minimum ratio.] Thus s_2 is leaving and s_1 is entering variable. Performing the pivot operations, we obtain table 1.8.

Table 1.8

C_B	C_i / X_B	4 x_1	1 x_2	0 s_1	0 s_2	M R_1	M R_2	Constant	Ratio
4	x_1	1	0	0	$-\frac{1}{5}$	$\frac{2}{5}$	0	$\frac{2}{5}$	
1	x_2	0	1	0	$\frac{3}{5}$	$-\frac{1}{5}$	0	$\frac{9}{5}$	
0	s_1	0	0	1	1	1	– 1	1	
$Z_j - C_j$		0	0	0	$-\frac{1}{5}$	$\frac{7}{5} - M$	– M	$Z = \frac{17}{5}$	

The I.B.F. solution is $x_1 = \frac{2}{5}$, $x_2 = \frac{9}{5}$, $s_1 = 1$, $R_1 = R_2 = s_1 = 0$, with $Z = \frac{17}{5}$. Since all $Z_j - C_j \geq 0$, the *present solution is optimal.*

Note : Let us see a problem which is *maximization* problem and involves *artificial* variables. We use Big M-method to solve such type of problems with appropriate changes.

Example 1.17 : Maximize $Z = -2x_1 - 4x_2 - x_3$

Subject to
$$x_1 + 2x_2 - x_3 \leq 5$$
$$2x_1 - x_2 + 2x_3 = 2$$
$$-x_1 + 2x_2 + 2x_3 \geq 1 \quad \text{with } x_1, x_2, x_3 \geq 0.$$

Solution : Converting the problem into standard form, we have,

Maximize $Z = -2x_1 - 4x_2 - x_3 + 0.s_1 + 0.s_2$.

Subject to

$$x_1 + 2x_2 - x_3 + s_1 = 5$$
$$2x_1 - x_2 + 2x_3 = 2$$
$$-x_1 + 2x_2 + 2x_3 - s_2 = 1$$

with $x_1, x_2, x_3, s_1, s_2 \geq 0$, and where s_1 and s_2 are slack and surplus variables.

The I.B.F.S. is not available at this stage, instead of performing pivot operations, we introduce artificial variables and convert problem to

Maximize $Z = -2x_1 - 4x_2 - x_3 + 0.s_1 + 0.s_2 - MR_1 - MR_2$

Subject to $x_1 + 2x_2 - x_3 + s_1 = 5$

$$2x_1 - x_2 + 2x_3 + R_1 = 2$$
$$-x_1 + 2x_2 + 2x_3 - s_2 + R_2 = 1$$

with $x_1, x_2, x_3, s_1, s_2, R_1, R_2 \geq 0$.

Here R_1 and R_2 are artificial variables. The proficient in objective function for R_1 and R_2 is to be taken to be – M, where M is sufficiently large positive number.

Now we have I.B.F.S. as $s_1 = 5$, $R_1 = 2$ and $R_2 = 1$ with $Z = -3M$. Since I.B.F.S. involves artificial variables at positive level, using Big M we reduce them to zero as soon as possible.

We have the following initial table 1.9.

Table 1.9

C_B	C_i / X_B	**– 2** x_1	**– 4** x_2	**– 1** x_3	**0** s_1	**0** s_2	**– M** R_1	**– M** R_2	**Const.**	**Ratio**
0	s_1	1	2	–1	1	0	0	0	5	$\frac{5}{-1} = -5$
– M	R_1	2	–1	2	0	0	1	0	2	$\frac{2}{2} = 1$
– M	R_2	–1	2	$\boxed{2}$	0	–1	0	1	1	$\frac{1}{2}$
$z_j - C_j$		– M + 2	– M + 4	– 4M + 1	0	M	0	0	Z = – 3M	

Since the problem is of maximization type, to have optimal solution all coefficients $Z_j - C_j$ should be positive. In this case we look at the most negative coefficient $Z_j - C_j$, which is $-4M + 1$, and corresponds to third column, so it is pivot column. Taking ratios of right hand side constants with the coefficients of x_3 in three constraints, we see that $\frac{1}{2}$ is the minimum ratio, hence R_2 is leaving variable and x_3 is entering variable. Performing pivot operations, we obtain next table 1.10.

Table 1.10

C_B	C_i / X_B	**– 2** x_1	**– 4** x_2	**– 1** x_3	**0** s_1	**0** s_2	**– M** R_1	**– M** R_2	**Const.**
0	s_1	$\frac{1}{2}$	3	0	–1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{11}{2}$
– M	R_1	$\boxed{3}$	– 3	0	0	1	1	– 1	1
– 1	x_3	$-\frac{1}{2}$	1	1	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$
$z_j - C_j$		– 3M +5/2	3M+3	0	0	$-M+\frac{1}{2}$	0	$2M-\frac{1}{2}$	$Z = -M-\frac{1}{2}$

The new I.B.F. solution is $s_1 = \frac{11}{2}$, $R_1 = 1$, $x_3 = \frac{1}{2}$ with $Z = -M - \frac{1}{2}$. Looking at the coefficients $Z_j - C_j$, since not all $Z_j - C_j$ are positive, the present solution is not optimal. The most negative $Z_j - C_j$ coefficient is $-3M + \frac{5}{2}$ which corresponds to first column, hence x_1 is entering variable. Applying minimum ratio test, we see that R_1 is leaving variable. Thus x_1 is entering variable and R_1 is leaving variable. Performing pivot operations we obtain the following table 1.11.

Table 1.11

C_B	X_B \ C_i	– 2 x_1	– 4 x_2	– 1 x_3	0 s_1	0 s_2	– M R_1	– M R_2	Const.	Ratio
0	s_1	0	7/2	0	– 1	– 2/3	$-\frac{1}{6}$	$\frac{2}{3}$	5	$-\frac{15}{2}$ neglect
– 2	x_1	1	–1	0	0	$\boxed{\frac{1}{3}}$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	1
– 1	x_3	0	$\frac{1}{2}$	1	0	$-\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{2}{3}$	– 2 neglect
$Z_j - C_j$		0	$\frac{11}{2}$	0	0	$-\frac{1}{3}$	–5/6 + M	$\frac{1}{3}$ + M	$Z = -\frac{4}{3}$	

The new basic feasible solution is $s_1 = 5$, $x_1 = \frac{1}{3}$, $x_3 = \frac{2}{3}$ with $Z = -\frac{4}{3}$. Again, since not all $Z_j - C_j$ coefficients are positive, the present solution is not optimal. Applying minimum ratio test we see that s_2 is entering variable and x_1 is leaving variable. Performing pivot operations we obtain the following table 1.12.

Table 1.12

C_B	X_B \ C_i	– 2 x_1	– 4 x_2	– 1 x_3	0 s_1	0 s_2	– M R_1	– M R_2	Const.	Ratio
0	s_1	2	3/2	0	–1	0	$\frac{1}{2}$	0	$\frac{17}{3}$	
0	s_2	3	–3	0	0	1	1	–1	1	
–1	x_3	1	$-\frac{1}{2}$	1	0	0	$\frac{1}{2}$	0	1	
		1	9/2	0	0	0	$-\frac{1}{2}$ + M	M	Z = – 1	

Since M is sufficiently large we see that all the coefficients $Z_j - C_j$ in this table are all positive, hence the new solution $s_1 = \frac{17}{3}$, $s_2 = 1$, $x_3 = 1$ with $Z = -1$ is optimal solution.

Thus the solution to the problem is $x_3 = 1$, $s_1 = \frac{17}{3}$, $s_2 = 1$, $x_2 = x_1 = R_1 = R_2 = 0$ with $Z_{max.} = -1$.

Example 1.18 : Minimize $Z_0 = 2x_1 + 4x_2 + x_3$

Subject to
$$x_1 + 2x_2 - x_3 \le 5$$
$$2x_1 - x_2 + 2x_3 = 2$$
$$-x_1 + 2x_2 + 2x_3 \ge 1$$

with $x_1, x_2, x_3 \ge 0$.

Here we have to take $Z_{-0} = -Z$, so minimum of Z_0 = Minimum of $(-Z)$, so that maximum of $Z = -$ minimum of Z_0.

Let us verify this fact solving example (2).

Solution : First let us convert the problem into standard form. We have

Minimize $Z_0 = 2x_1 + 4x_2 + x_3 + 0.s_1 + 0.s_2$

Subject to
$$x_1 + 2x_2 - x_3 + s_1 = 5$$
$$2x_1 - x_2 + 2x_3 = 2$$
$$-x_1 + 2x_2 + 2x_3 - s_2 = 1$$

with $x_1, x_2, x_3, s_1, s_2 \ge 0.$

Since the I.B.F.S. is not available at this stage, we proceed by using aritificial variables as follows.

Minimize $Z_0 = 2x_1 + 4x_2 + x_3 + 0.s_1 + 0.s_2 + MR_1 + MR_2$

Subject to
$$x_1 + 2x_2 - x_3 + s_1 = 5$$
$$2x_1 - x_2 + 2x_3 + R_1 = 2$$
$$-x_1 + 2x_2 + 2x_3 - s_2 + R_2 = 1$$

with $R_1, R_2 \ge 0$ and M sufficiently large positive number.

Now we have I.B.F. solution $s_1 = 5, R_1 = 2, R_2 = 1$ with $Z_0 = 3$ M. We form initial table.

Note : Since this is minimization problem we take M sufficiently large positive number.

Table 1.13

C_B	C_j / X_B	**2** x_1	**4** x_2	**1** x_3	**0** s_1	**0** s_2	**M** R_1	**M** R_2	**Const.**	**Ratio**
0	s_1	1	2	–1	1	0	0	0	5	$\frac{5}{-1} = -5$ neglect
M	R_1	2	–1	2	0	0	1	0	2	$\frac{2}{2} = 1$
M	R_2	–1	2	$\boxed{2}$	0	–1	0	1	1	$\frac{1}{2}$
$Z_j - C_j$		M-2	M-4	4M-1	0	–M	0	0	$Z_0 = 3M$	

Since not all $Z_j - C_j$ coefficients are non-positive, the present I.B.F. solution is not optimal. As this is minimization problem we look at the maximum positive $Z_j - C_j$ coefficient, which is 4M-1, corresponds to column three, hence x_3 is entering variable. Using minimum ratio test, we see that R_2 is leaving variable. Thus R_2 is leaving and x_3 is entering variable. Performing pivot operations, we have a new table 1.14.

Table 1.14

C_B	C_i / X_B	2 x_1	4 x_2	1 x_3	0 s_1	0 s_2	M R_1	M R_2	Const.	Ratio
0	s_1	$\frac{1}{2}$	3	0	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{11}{2}$	11
M	R_1	$\boxed{3}$	– 3	0	0	1	1	– 1	1	$\frac{1}{3}$
1	x_3	$-\frac{1}{2}$	1	1	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	–1 neglect
$Z_j - C_j$		$3M - \frac{5}{2}$	–3M –3	0	0	$M - \frac{1}{2}$	0	$-2M + \frac{1}{2}$	$Z_0 = M - \frac{1}{2}$	

The new basic feasible solution is $s_1 = \frac{11}{2}$, $R_1 = 1$, $x_3 = 1$, with $Z_0 = M - \frac{1}{2}$. Calculating the coefficients $Z_j - C_j$, we see that not all $Z_j - C_j$ are non-positive, hence the present solution is *not optimal.*

As M is sufficiently large positive real number, $3M - \frac{5}{2}$ is the largest positive $Z_j - C_j$ coefficient, which corresponds to first column, hence x_1 is entering variable. Applying minimum ratio test we see that R_1 is *leaving* variable. Thus R_1 is leaving and x_3 is entering. Performing the pivot operations, we obtain new table 1.15.

Table 1.15

C_B	C_i / X_B	2 x_1	4 x_2	1 x_3	0 s_1	0 s_2	M R_1	M R_2	Const.	Ratio
0	s_1	0	$\frac{7}{2}$	0	1	$-\frac{2}{3}$	$-\frac{1}{6}$	$\frac{2}{3}$	5	$-\frac{15}{2}$ neglect
2	x_1	1	– 1	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	1
1	x_3	0	$\frac{1}{2}$	1	0	$-\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{2}{3}$	– 2 neglect
$Z_j - C_j$		0	$\frac{11}{2}$	0	0	$\frac{1}{3}$	$\frac{5}{6} - M$	$-\frac{1}{3}$	$Z_0 = \frac{4}{3}$	

The new basic feasible solution is $s_1 = 5$, $x_1 = \frac{1}{3}$, $x_3 = \frac{2}{3}$ with $Z_0 = \frac{4}{3}$. Calculations of the coefficients $Z_j - C_j$, show that the present solution is not optimal since $\frac{1}{3}$, $Z_j - C_j$ coefficient corresponding s_2 is at positive level, hence s_2 is entering variable. The minimum ratio test shows that x_1 is leaving variable. Thus x_1 is leaving and s_2 is entering variable. Performing pivot operations we obtain new table 1.15.

Table 1.16

C_B	C_i / X_B	2 x_1	4 x_2	1 x_3	0 s_1	0 s_2	M R_1	M R_2	Const.
0	s_1	2	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	$\frac{17}{3}$
0	s_2	3	– 3	0	0	1	1	– 1	1
1	x_3	1	$-\frac{1}{2}$	1	0	0	$\frac{1}{2}$	0	1
$Z_j - C_j$		– 1	$-\frac{9}{2}$	0	0	0	$\frac{1}{2}$ – M	– M	$Z_0 = 1$

The new basic feasible solution is $s_1 = \frac{17}{3}$, $s_2 = 1$, $x_3 = 1$, with $Z_0 = 1$. Since all $Z_j - C_j$ coefficients are non-positive, the current b.f.s. is optimal solution. Thus the optimal solution to the problem is $s_1 = \frac{17}{3}$, $s_2 = 1$, $x_3 = 1$ with $Z_0 = 1$, showing maximum of Z = – min. Z_0 = – (1).

Remark : Students are requested to observe the corresponding steps in examples 2.3 and 2.4.

Example 1.19 : Minimize $Z = x_1 + x_2 + x_3$

Subject to

$$\begin{aligned} x_1 - 3x_2 + 4x_3 &= 5 \\ x_1 - 2x_2 &\leq 3 \\ 2x_1 - x_3 &\geq 4, \end{aligned}$$

$x_1, x_2, x_3 \geq 0$.

Solution : Converting the problem in standard form by using slack and surplus variables, we have

Minimize $Z = x_1 + x_2 + x_3 + 0.s_1 + 0.s_2$

Subject to

$$\begin{aligned} x_1 - 3x_2 + 4x_3 &= 5 \\ x_1 - 2x_2 + s_1 &= 3 \\ 2x_1 - x_3 - s_2 &= 4 \end{aligned}$$

$x_1, x_2, x_3, s_1, s_2 \geq 0$.

Since the I.B.F. solution is not available, at this stage, we introduce artificial variables. So we have

Minimize $Z = x_1 + x_2 + x_3 + 0.s_1 + 0.s_2 + MR_1 + MR_2$

Subject to

$$\begin{aligned} x_1 - 3x_2 + 4x_3 + R_1 &= 5 \\ x_1 - 2x_2 + s_1 &= 3 \\ 2x_1 - x_3 - s_2 + R_2 &= 4 \end{aligned}$$

$x_1, x_2, x_3, s_1, s_2, R_1, R_2 \geq 0$, and where, M is large positive number. This gives us the I.B.F. solution as : $R_1 = 5$, $s_1 = 3$, $R_2 = 4$ with Z = 9 M. We prepare initiable table for the problem.

Table 1.17

C_B	C_i / X_B	1 x_1	1 x_2	1 x_3	0 s_1	0 s_2	M R_1	M R_2	Const.	Ratio
M	R_1	1	– 3	4	0	0	1	0	5	5/1 = 5
0	s_1	1	– 2	0	1	0	0	0	3	3/1 = 3
M	R_2	$\boxed{2}$	0	– 1	0	– 1	0	1	4	4/2 = 2
$Z_j - C_j$		3M-1	–3M-1	3M-1	0	–M	0	0	Z = 9M	

Since not all $Z_j - C_j$ coefficients are non-positive, the present I.B.F.S. is not optimal, so we choose the one which is largest positive among $Z_j - C_j$'s. Here 3M – 1 is the largest positive $Z_j - C_j$ coefficient, but it corresponds two columns, namely first and third, there we break the tie randomly and we choose x_1 to be entering variable corresponding to the first column. Using minimum ratio test we observe that R_2 is leaving variable. Thus x_1 is entering variable and R_2 is leaving variable. Applying pivot operations we get the next table 1.18 as follows :

Table 1.18

C_B	C_i / X_B	1 x_1	1 x_2	1 x_3	0 s_1	0 s_2	M R_1	M R_2	Const.	Ratio
M	R_1	0	– 3	$\boxed{\frac{9}{2}}$	0	$\frac{1}{2}$	1	$-\frac{1}{2}$	3	$\frac{3}{9/2} = \frac{2}{3}$
0	s_1	0	– 2	$\frac{1}{2}$	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	1	$\frac{1}{1/2} = 2$
1	x_1	1	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	2	$\frac{2}{-1/2} = -4$
		0	–3M–1	$\frac{9}{2}$ M $-\frac{3}{2}$	0	$\frac{1}{2}$ M $-\frac{1}{2}$	0	$\frac{3}{-2}$ M $+\frac{1}{2}$	Z = 3M + 2	

At this stage, the I.B.F. solution is given by $R_1 = 3$, $s_1 = 1$, $x_2 = 2$, with Z = 3M + 2. We observe that not all $Z_j - C_j$ coefficients are non-positive, the present solution is not optimal. Hence we proceed further, as the largest positive $Z_j - C_j$ coefficient in Table 2.14 is 9/2 M – 3/2, which corresponds to third column, hence the variable x_3 is entering variable. Applying minimum ratio test we see that R_1 is leaving variable. Thus x_3 is entering variable and R_1 is leaving variable. Applying pivot operations, we obtain next table 1.19.

Table 1.19

	C_i / X_B	1 x_1	1 x_2	1 x_3	0 s_1	0 s_2	M R_1	M R_2	Const.	Ratio
1	x_3	0	$-\frac{2}{3}$	1	0	$\frac{1}{9}$	$\frac{2}{9}$	$-\frac{1}{9}$	$\frac{2}{3}$	
0	s_1	0	$-\frac{1}{2}$	0	1	$\frac{4}{9}$	$-\frac{1}{9}$	$-\frac{3}{8}$	$\frac{2}{3}$	
1	x_1	1	$-\frac{1}{3}$	0	0	$-\frac{4}{9}$	$\frac{1}{9}$	$\frac{4}{9}$	$\frac{7}{3}$	
$Z_j - C_j$		0	– 1	0	0	$-\frac{3}{9}$	$\frac{1}{3}$ – M	$\frac{1}{3}$ – M	$Z = \frac{2}{3} + \frac{7}{3} = 3$	

From this table 2.15 we see that the basic feasible solution is given by

$$x_1 = \frac{7}{3},\ x_3 = \frac{2}{3},\ s_1 = \frac{2}{3}, \quad \text{with } Z = \frac{2}{3} + \frac{7}{3} = 3.$$

Since all the $Z_j - C_j$ coefficients are non-positive, the present solution is *optimal* solution. Thus the solution of the problem is $x_1 = \frac{7}{3}$, $x_2 = 0$, $x_3 = \frac{2}{3}$, with $Z_{min} = \frac{7}{3} + \frac{2}{3} = 3$.

Example 1.20 : Solve the following L.P.P.

Maximize $Z = 3x_1 + 2x_2$

Subject to

$$\begin{aligned} 2x_1 + x_2 &\le 2 \\ 3x_1 + 4x_2 &\ge 12 \\ x_1, x_2 &\ge 0. \end{aligned}$$

Solution : Converting the problem to standard form, we have

Maximize $Z = 3x_1 + 2x_2 + 0.s_1 + 0.s_2$

Subject to

$$2x_1 + x_2 + s_1 = 2$$
$$3x_1 + 4x_2 - s_2 = 12$$

$x_1, x_2, s_1, s_2 \geq 0$, where s_1 and s_2 are slack and surplus variables.

At this stage, the I.B.F. solution is not available, so we use artificial variables and obtain :

Maximize $Z = 3x_1 + 2x_2 + 0.s_1 + 0.s_2 - MR$

Subject to

$$2x_1 + x_2 + s_1 = 2$$
$$3x_1 + 4x_2 - s_2 + R = 12$$

where, $R > 0$ is artificial variable and M is large positive number. Here the basic feasible solution is given by $s_1 = 2$, $R = 12$, with $Z = -12M$. We prepare the initial table.

Table 1.20

C_B	C_i / X_B	**3** x_1	**2** x_2	**0** s_1	**0** s_2	**– M** R	**Const.**	**Ratio**
0	s_1	2	$\boxed{1}$	1	0	0	2	$\frac{2}{1} = 2$
– M	R	3	4	0	– 1	1	12	$\frac{12}{4} = 3$
$Z_j - C_j$		– 3M – 3	– 4M – 2	0	M	0	Z = – 12M	

As we see from the table not all $Z_j - C_j$ coefficients are negative, the present solution is not optimal. We look for the most negative $Z_j - C_j$ coefficient, in this case it is $-4M - 2$, which corresponds to x_2, hence x_2 is entering variable. Applying minimum ratio rule we observe that s_1 is leaving variable. Thus x_2 is entering variable and s_1 is leaving variable. Performing pivot operations we obtain next table 1.21.

Table 1.21

C_B	C_i / X_B	**3** x_1	**2** x_2	**0** s_1	**0** s_2	**– M** R	**Const.**
2	x_2	2	1	1	0	0	2
– M	R	– 5	0	– 4	– 1	1	4
$Z_j - C_j$		5M + 1	0	4M+2	M	0	Z = – 4M + 4

Since all the $Z_j - C_j$ coefficients are positive the current basic feasible solution : $x_2 = 2$, $R = 4$, with $Z = -4M + 4$ is *optimal*. However, the artificial variable R is at positive level (R = 4) which indicates that the problem has *no feasible solution.*

Example 1.21 : Maximize $Z = 3x_1 + 2x_2 + 5x_3$

Subject to

$$x_1 + 2x_2 + x_3 \leq 430$$
$$3x_1 + 2x_3 \leq 460$$
$$x_1 + 4x_2 \leq 120$$
$$x_1, x_2, x_3 \geq 0.$$

Solve the problem by simplex method.

Solution : Converting the problem into standard form we have,

Maximize $Z = 3x_1 + 2x_2 + 5x_3 + 0.s_1 + 0.s_2 + 0.s_3$

Subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 + s_1 &= 430 \\ 3x_1 + 2x_3 + s_2 &= 460 \\ x_1 + 4x_2 + s_3 &= 120 \end{aligned}$$

$s_1, s_2, s_3 \geq 0$ are slack variables.

The I.B.F.S. is given by $s_1 = 430$, $s_2 = 460$, $s_3 = 120$, with Z = 0. Preparing the initial table (See table 1.22), we have

Table 1.22

C_B	C_i / X_B	3 x_1	2 x_2	5 x_3	0 s_1	0 s_2	0 s_3	Const.	Ratio
0	s_1	1	2	1	1	0	0	430	$\frac{430}{1} = 430$
0	s_2	3	0	$\boxed{2}$	0	1	0	460	$\frac{460}{2} = 230$
0	s_3	1	4	0	0	0	1	120	neglect
$Z_j - C_j$		– 3	– 2	– 5	0	0	0	Z = 0	

As not all the $Z_j - C_j$ coefficients are positive, the current solution is not optimal. The most negative $Z_j - C_j$ is – 5, which corresponds to x_3, hence x_3 is entering variable. Applying minimum ratio test we see that s_2 is leaving variable. Thus x_3 is entering and s_2 is leaving variable. Performing pivot operations, we have a next table 1.23.

Table 1.23

C_B	C_i / X_B	3 x_1	2 x_2	5 x_3	0 s_1	0 s_2	0 s_3	Const.	Ratio
0	s_1	$-\frac{1}{2}$	2	0	1	$-\frac{1}{2}$	0	200	$\frac{200}{2} = 100$
5	x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230	neglect
0	s_3	1	$\boxed{4}$	0	0	0	1	120	$\frac{120}{4} = 30$
$Z_j - C_j$		$\frac{9}{2}$	– 2	0	0	$\frac{5}{2}$	0	Z = 1150	

The table 1.23 shows that the basic feasible solution is $s_1 = 200$, $x_3 = 230$, $s_3 = 120$ with Z = 1150. After calculating the $Z_j - C_j$ coefficients we see that one $Z_j - C_j$ coefficient is negative; namely – 2, which corresponds to x_2, hence x_2 is entering variable. Applying minimum ratio test we see that s_3 is leaving variable. Thus x_2 is entering and s_3 is leaving variable. Performing pivot operations, we obtain the following table 1.24.

Table 1.24

C_B	C_i / X_B	3 x_1	2 x_2	5 x_3	0 s_1	0 s_2	0 s_3	Const.	Ratio
0	s_1	– 1	0	0	1	$-\frac{1}{2}$	$-\frac{1}{2}$	140	
5	x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230	
2	x_2	$\frac{1}{4}$	1	0	0	0	$\frac{1}{4}$	30	
$Z_j - C_j$		5	0	0	0	$\frac{5}{2}$	$\frac{1}{2}$	$Z = 5 \times 230 + 2 \times 30 = 1210$	

This table 2.20 gives the basic feasible solution as : $s_1 = 140$, $x_2 = 30$, $x_3 = 230$, with $Z = 5 \times 230 + 2 \times 30 = 1210$. After calculating $Z_j - C_j$ coefficients, we observe that all $Z_j - C_j$ coefficients are positive. Therefore the current solution is *optimal*. Thus the solution to the problem is :

$x_1 = 0$, $x_2 = 30$, $x_3 = 230$, with maximum $Z = 1210$.

Example 1.22 : Solve the following L.P.P. by simplex method.

Maximize $Z = 30x_1 + 16x_2 + 25x_3$

Subject to

$$0.8x_1 + 0.4x_2 + 0.5x_3 \le 100$$
$$0.5x_1 + 0.3x_2 + 0.3x_3 \le 65$$
$$0.9x_1 + 0.6x_2 + 0.9x_3 \le 126$$

$x_1, x_2, x_3 \ge 0$.

Solution : The coefficients in decimal form are difficult to carry out the calculations in pivot operations. Multiplying each constraint by fixed constant through makes no difference. Therefore we multiply each constraint in the problem by 10 throughout, so that we get rid of the fractions. Then converting the L.P.P. into standard form, we have

Maximize $Z = 30x_1 + 16x_2 + 25x_3 + 0.s_1 + 0.s_2 + 0.s_3$

Subject to

$$8x_1 + 4x_2 + 5x_3 + s_1 = 1000$$
$$5x_1 + 3x_2 + 3x_3 + s_2 = 650$$
$$9x_1 + 6x_2 + 9x_3 + s_3 = 1260$$

where, $s_1, s_2, s_3 \ge 0$ are slack variables.

This shows that the I.B.F.S. is $s_1 = 1000$, $s_2 = 650$, $s_3 = 1260$, with $Z = 0$. We prepare the initial table 1.25 as shown below.

Table 1.25

C_B	C_i / X_B	30 x_1	16 x_2	25 x_3	0 s_1	0 s_2	0 s_3	Const.	Ratio
0	s_1	$\boxed{8}$	4	5	1	0	0	1000	$\frac{1000}{8} = 125$
0	s_2	5	3	3	0	1	0	650	$\frac{650}{5} = 130$
0	s_3	9	6	9	0	0	1	1260	$\frac{1260}{9} = 140$
$Z_j - C_j$		– 30	– 16	– 25	0	0	0	Z = 0	

The calculations of the coefficients $Z_j - C_j$ show that not all $Z_j - C_j$ are positive, hence the present solution is not optimal. As – 30 is the most negative $Z_j - C_j$ coefficient and it corresponds to x_1, hence x_1 enters the basis. Applying minimum ratio test we see that s_1 is leaving variable. Performing pivot operations, we have the following table 1.26.

Table 1.26

C_B	C_i / X_B	30 x_1	16 x_2	25 x_3	0 s_1	0 s_2	0 s_3	Const.	Ratio
30	x_1	1	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{1}{8}$	0	0	125	$\frac{125}{5/8} = 200$
0	s_2	0	$\frac{1}{2}$	$-\frac{1}{8}$	$-\frac{5}{8}$	1	0	25	neglect
0	s_3	0	$\frac{3}{2}$	$\boxed{\frac{27}{8}}$	$-\frac{9}{8}$	0	1	135	$\frac{135}{27/8} = 40$
$Z_j - C_j$		0	– 1	$-\frac{25}{4}$	$\frac{30}{8}$	0	0	Z = 30 × 125 = 3750	

From the table we see that basic feasible solution is given by $x_1 = 125$, $s_2 = 25$, $s_3 = 135$ with $Z = 30 \times 125 = 3750$. After calculating $Z_j - C_j$ coefficients, we see that the current solution is *not optimal.* As $-\frac{25}{4}$ is the most negative $Z_j - C_j$ coefficient, x_3 enters the basis. Applying minimum ratio test we find that s_3 leaves the basis. Thus s_3 is leaving and x_3 is entering variable. Performing pivot operations, we obtain the following table 1.27.

Table 1.27

C_B	C_i / X_B	30 x_1	16 x_2	25 x_3	0 s_1	0 s_2	0 s_3	Const.
30	x_1	1	$\frac{2}{9}$	0	$\frac{1}{3}$	0	$-\frac{5}{27}$	100
0	s_2	0	$\frac{5}{9}$	0	$-\frac{2}{3}$	1	$\frac{1}{27}$	30
25	x_3	0	$\frac{4}{9}$	1	$-\frac{1}{3}$	0	$\frac{8}{27}$	40
$Z_j - C_j$		0	$\frac{16}{9}$	0	$\frac{5}{3}$	0	$\frac{50}{27}$	Z = 4000

From the table we observe that the basic feasible solution is given by :

$x_1 = 100,\ x_2 = 30,\ x_3 = 40$, with

$Z = 30 \times 100 + 25 \times 40 = 4000.$

After calculating the $Z_j - C_j$ coefficients, we observe that all the $Z_j - C_j$ coefficients are positive, hence the current solution is *optimal solution.* Thus, the solution of the problem is given by

$$x_1 = 100,\ x_2 = 0,\ x_3 = 40, \text{ with } Z = 4000.$$

Example 1.23 : Minimize $Z = 4x_1 + 2x_2 + 3x_3$

Subject to

$$\begin{aligned} 2x_1 + 4x_3 &\geq 5 \\ 2x_1 + 3x_2 + x_3 &\geq 4 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

Solution : Converting the problem into standard form, we have

Minimize $Z = 4x_1 + 2x_2 + 3x_3 + 0.s_1 + 0.s_2$

Subject to

$$2x_1 + 4x_3 - s_1 = 5$$
$$2x_1 + 3x_2 + x_3 - s_2 = 4$$

$s_1, s_2 \geq 0$ are surplus variables. Since the I.B.F. solution is not available at this stage, we use artificial variable, so that problem reduces to

Minimize $Z = 4x_1 + 2x_2 + 3x_3 + 0.s_1 + 0.s_2 + MR_1 + MR_2$

Subject to

$$2x_1 + 4x_3 - s_1 + R_1 = 5$$
$$2x_1 + 3x_2 + x_3 - s_2 + R_2 = 4$$

where, $R_1, R_2 \geq 0$ are artificial variables and M is sufficiently large positive number.

The I.B.F.S. is given by

$R_1 = 5$, $R_2 = 4$ with Z = 9M. We prepare a simplex table 1.28 for checking the optimality of the present solution.

Table 1.28

C_B	C_j / X_B	4 x_1	2 x_2	3 x_3	0 s_1	0 s_2	M R_1	M R_2	Const.	Ratio
M	R_1	2	0	$\boxed{4}$	–1	0	1	0	5	5/4
M	R_2	2	3	1	0	– 1	0	1	4	4/1 = 4
$z_j - C_j$		4M – 4	3M – 2	5M – 3	– M	– M	0	0	Z = 9M	

Since not all $Z_j - C_j$ coefficients are negative or zero, the present solution is not optimal. The largest positive $Z_j - C_j$ coefficient is 5M – 3, which corresponds to x_3, so x_3 is entering variable. Using minimum ratio test we see from table 1.28 that R_1 is leaving variable. Performing pivot operations, we obtain the next table 1.29 as follows :

Table 1.29

C_B	C_j / X_B	4 x_1	2 x_2	3 x_3	0 s_1	0 s_2	M R_1	M R_2	Const.
3	x_3	$\frac{1}{2}$	0	1	$-\frac{1}{4}$	0	$\frac{1}{4}$	0	$\frac{5}{4}$
M	R_2	$\frac{3}{2}$	3	0	$\frac{1}{4}$	– 1	$-\frac{1}{4}$	1	$\frac{11}{4}$
$Z_j - C_j$		$\frac{3M}{2} + \frac{5}{2}$	3M – 2	0	$\frac{1}{4}M - \frac{3}{4}$	– M	$-\frac{3}{4}M - \frac{3}{4}$	0	$Z = \frac{15}{4} + \frac{11}{4}M$

From table 1.28,

The basic feasible solution is given by : $x_3 = \frac{5}{4}$, $R_2 = \frac{11}{4}$, with $Z = \frac{15}{4} + \frac{11}{4}M$.

Since there are positive $Z_j - C_j$ coefficients, the current solution is *not optimal*. The largest positive $Z_j - C_j$ coefficient is 3M – 2, which corresponds to x_2, so x_2 is entering variable. Applying minimum ratio test, we see from table 2.28 that R_2 is leaving variable. Performing the pivot operations, we obtain table 1.30.

Table 1.30

C_B	C_i / X_B	4 x_1	2 x_2	3 x_3	0 s_1	0 s_2	M R_1	M R_2	Const.	Ratio
3	x_3	$\frac{1}{2}$	0	1	$-\frac{1}{4}$	0	$\frac{1}{4}$	0	$\frac{5}{4}$	
2	x_2	$\frac{1}{2}$	1	0	$\frac{1}{12}$	$-\frac{1}{3}$	$-\frac{1}{12}$	$\frac{1}{3}$	$\frac{11}{12}$	
$Z_j - C_j$		$-\frac{3}{2}$	0	0	$-\frac{7}{12}$	$-\frac{2}{3}$	$-M-\frac{7}{4}$	$-M+\frac{2}{3}$	$Z = \frac{67}{12}$	

From table 2.29, the basic feasible solution is given by $x_2 = \frac{11}{12}$, $x_3 = \frac{5}{4}$ with $Z = \frac{67}{12}$, since all the $Z_j - C_j$ coefficients are non-negative, the current solution *is optimal*. Thus the solution of the problem is $x_1 = 0$, $x_2 = \frac{11}{12}$, $x_3 = \frac{5}{4}$, with $Z = \frac{67}{12}$.

Example 1.24 : Maximize $Z = 2x_1 + 3x_2 + 4x_3$

Subject to

$$3x_1 - 2x_3 \leq 41$$
$$2x_1 + x_2 + x_3 \leq 35$$
$$2x_2 + 3x_3 \leq 30$$

$x_1, x_2, x_3 \geq 0.$

Solution : Converting the problem into standard form, we have :

Maximize $Z = 2x_1 + 3x_2 + 4x_3 + 0.s_1 + 0.s_2 + 0.s_3$

Subject to

$$3x_1 - 2x_3 + s_1 = 41$$
$$2x_1 + x_2 + x_3 + s_2 = 35$$
$$2x_2 + 3x_3 + s_3 = 30$$

$x_1, x_2, x_3; s_1, s_2, s_3 \geq 0$, where s_1, s_2 and s_3 are slack variables. From this we find that the I.B.F. solution is given by $s_1 = 41$, $s_2 = 35$, $s_3 = 30$ with $Z = 0$. We prepare the initial simplex table 2.30 to check the optimality.

Table 1.31

C_B	C_i / X_B	2 x_1	3 x_2	4 x_3	0 s_1	0 s_2	0 s_3	Const.	Ratio
0	s_1	3	0	– 2	1	0	0	41	neglect
0	s_2	2	1	1	0	1	0	35	$\frac{35}{1} = 35$
0	s_3	0	2	$\boxed{3}$	0	0	1	30	$\frac{30}{3} = 10$
$Z_j - C_j$		– 2	– 3	– 4	0	0	0	Z = 0	

From table 1.32, we see that there are $Z_j - C_j$ coefficients, which indicate that the I.B.F.S. given before table 1.32 is *not* optimal. Since this is maximization problem, we look at the most negative $Z_j - C_j$ coefficient; which is – 4, hence x_3 is entering variable. Applying minimum ratio test we find that s_3 is leaving variable. Thus x_3 is entering and s_3 is leaving variable. Performing pivot operations, we obtain next table 1.32.

Table 1.32

C_B	C_i / X_B	2 x_1	3 x_2	4 x_3	0 s_1	0 s_2	0 s_3	Const.
0	s_1	3	$\frac{4}{3}$	0	1	0	$\frac{2}{3}$	61
0	s_2	$\boxed{2}$	$\frac{1}{3}$	0	0	1	$-\frac{1}{3}$	25
4	x_3	0	$\frac{2}{3}$	1	0	0	$\frac{1}{3}$	10
$Z_j - C_j$		-2	$-\frac{1}{3}$	0	0	0	$\frac{1}{3}$	Z = 40

From table 1.32, we observe that the basic feasible solution to the problem is given by $s_1 = 61$, $s_2 = 25$, $x_3 = 10$, with Z = 40. Again, as there are negative $Z_j - C_j$ coefficients, the current solution is *not optimal*. We look at the most negative $Z_j - C_j$ coefficient, which is – 2 here and corresponds to x_1, hence x_1 is entering variable. Applying minimum ratio test we also find that s_2 is leaving variable. Thus x_1 is entering and s_2 is leaving variable. Performing pivot operations, we obtain the following table 1.33.

Table 1.33

C_B	C_i / X_B	2 x_1	3 x_2	4 x_3	0 s_1	0 s_2	0 s_3	Const.
0	s_1	0	$\frac{5}{6}$	0	1	$-\frac{3}{2}$	$\frac{7}{6}$	$\frac{47}{2}$
2	x_1	1	$\frac{1}{6}$	0	0	$\frac{1}{2}$	$-\frac{1}{6}$	$\frac{25}{2}$
4	x_3	0	$\frac{2}{3}$	1	0	0	$\frac{1}{3}$	10
$Z_j - C_j$		0	0	0	0	1	1	Z = 65

From table 2.32 we see that the basic feasible solution is given by $s_1 = \frac{47}{2}$, $x_1 = \frac{25}{2}$, $x_3 = 10$, with Z = 65. Also we see that all the $Z_j - C_j$ coefficients are non-negative, hence the present solution *is optimal.* Thus the solution to the given problem is given by;

$$x_1 = \frac{25}{2}, \quad x_2 = 0, \quad x_3 = 10, \text{ with } Z = 65.$$

Example 1.25 : Maximize $Z = 40x_1 + 35x_2$

Subject to
$$2x_1 + 3x_2 \le 60$$
$$4x_1 + 3x_2 \le 96$$
$$x_1, x_2 \ge 0.$$

Solution : Converting the problem into standard form, we have

Maximize $Z = 40x_1 + 35x_2 + 0.s_1 + 0.s_2$

Subject to
$$2x_1 + 3x_2 + s_1 = 60$$
$$4x_1 + 3x_2 + s_2 = 96$$
$$x_1, x_2; s_1, s_2 \ge 0.$$

We prepare the initial simplex table 1.34.

Table 1.34

C_B	C_j / X_B	40 x_1	35 x_2	0 s_1	0 s_2	Const.
0	s_1	2	3	1	0	60
0	s_2	$\boxed{4}$	3	0	1	96
$Z_j - C_j$		– 40	– 35	0	0	Z = 0

From the table 2.33 we see that the I.B.F.S. $s_1 = 60$, $s_2 = 96$, with Z = 0 is *not optimal*, as there are negative $Z_j - C_j$ coefficients. We consider the most negative $Z_j - C_j$ coefficient, which is – 40 and corresponds to x_1, hence x_1 is entering variable. Applying minimum ratio test we see that s_2 is leaving variable. Thus x_1 is entering variable and s_2 is leaving variable. Performing the pivot operations, we obtain the next table 1.35.

Table 1.35

C_B	C_j / X_B	40 x_1	35 x_2	0 s_1	0 s_2	Const.	Ratio
0	s_1	0	$\boxed{\frac{3}{2}}$	1	$-\frac{1}{2}$	12	$\frac{12}{3/2} = 8$
40	x_1	1	$\frac{3}{4}$	0	$\frac{1}{4}$	24	$\frac{24}{3/4} = 32$
$Z_j - C_j$		0	– 5	0	10	Z = 960	

From table 2.34, the basic feasible solution is given by $s_1 = 12$, $x_1 = 24$, with Z = 960. Since there are negative $Z_j - C_j$ coefficients, the current solution *is not optimal*. From table 2.34 it is clear that x_2 is entering variable and s_1 is leaving variable. Performing pivot operations, we obtain the next table 1.36.

Table 1.36

C_B	C_j / X_B	40 x_1	35 x_2	0 s_1	0 s_2	Const.
35	x_2	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	8
40	x_1	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	18
$Z_j - C_j$		0	0	$\frac{10}{3}$	$\frac{25}{3}$	Z = 1000

From table 1.36, the basic feasible solution is given by $x_1 = 18$, $x_2 = 8$ with Z = 1000, which *is optimal* solution, because all the $Z_j - C_j$ coefficients are non-negative.

Think Over It

1. What are the various areas in industry where operations research techniques can be used as an effective tool in decision-making by the management ?

Points to Remember

1. A linear programming problem (L.P.P.) consists of three parts :
   (a) Objective function.
   (b) Problem constraints.
   (c) Non-negativity constraints.
2. The objective function and problem constraints are linear in decision variables x, y.
3. In 2-dimensional plane the region bounded by problem constraints and non-negativity of L.P.P. is convex region, called feasible region.
4. The optimum solution of L.P.P. occurs at the corner point of the convex polygon bounding the feasible region.
5. L.P.P. can have more than one optimum solutions.
6. Any redundent constraint do not take part in the optimum solution of L.P.P.
7. An L.P. problem is either maximization or minimization i.e. optimization type.
8. To solve the problem by using simplex method first we write the problem in standard form, by converting all inequalities into equations.
   (a) Less than or equal to ≤ constraints are converted into equations by addition of slack variables.
   (b) Greater than or equal to i.e. ≥ constraints are converted into equations by subtracting surplus variables and adding artificial variables.
   (c) For equations only artificial variables are added.
9. After preparing initial simplex table we get initial feasible solution.
   It is examined for any further improvement and accordingly make the necessary improvements in the table.
10. The improvement in the solution is further checked and if possible the above procedure is repeated, till we arrived at optimum solution.
11. In the whole process at any intermediate step we observe some special cases like degeneracy, choice of entering variable unbounded solution, infeasible solution etc. we follow the proper rules to resolve the degeneracy and to break the ties. We study this in next chapter.

Miscellaneous Exercise

(A) True/False Questions

State whether the following is True or False :

1. The constraints in L.P.P. specify upper limits on the availability of resources or lower limit on their requirements.
2. Every L.P.P. has unique solution.
3. The reddundent constraints in L.P.P. do not take any part in the solution of L.P.P.

4. Graphical solution of L.P.P. is useful only when L.P.P. involves only two decision variables.
5. The decision variables in L.P.P. can assume negative values.
6. In the simplex table the key column corresponds to the largest negative entry in the $Z_j - C_j$ row.
7. The constraints involving $\geq$ signs are transformed into equations with the help of surplus variables and artificial variables.
8. An optimum solution is reached when all index numbers $Z_j - C_j$ are positive.
9. In case of degeneracy L.P.P. has alternate optimum solutions.
10. In the simplex method, initial solution contains only slack variables in the product mix.

(B) Multiple Choice Questions

1. In an L.P.P. …………

   (A) Objective function is linear and all constraints are linear.

   (B) Objective function is linear but some constraints may not be linear.

   (C) Constraints are linear but objective function may or may not be linear.

   (D) Neither linearity of objective nor linearity of constraints is required.

2. The constraints of a certain L.P.P. are :

$$x + y \leq 14 \quad \ldots (i)$$
$$3x + 2y \geq 36 \quad \ldots (ii)$$
$$2x + y \leq 24 \quad \ldots (iii)$$
$$x \geq 0, \; y \geq 0 \quad \ldots (iv)$$

   Then …………

   (A) (i) is redundent.

   (B) (ii) is redundent.

   (C) (iii) is redundent.

   (D) There is no redundent constraint.

3. In the problem

   Maximize $Z = 5x + 2y$

   Subject to
$$x + 2y \leq 2$$
$$4x + 3y \geq 12$$
$$x \geq 0, \; y \geq 0$$

   (A) There are infinitely many solutions.

   (B) There is unique solution.

   (C) There is no solution.

   (D) No conclusion can be drawn about the solution.

4. In the L.P.P.

   Maximize $Z = 6x - 2y$

   Subject to
$$x - y \leq 1$$
$$3x - y \leq 6$$
$$x \geq 0, \; y \geq 0$$

The optimum value is …………

(A) 6 (B) 12 (C) 14 (D) 18

5. The maximum and minimum values of

$$Z = 3x + 2y$$

Subject to
$$\begin{aligned} 5x + 10y &\leq 50 \\ x + y &\geq 1 \\ x &\leq 4 \\ x &\geq 0, \quad y \geq 0 \end{aligned}$$

are …………

(A) 3, 2 (B) 18, 3 (C) 12, 10 (D) 18, 2

6. LP is a …………

(A) Constrained optimization technique

(B) technique for economic allocation of limited resources

(C) mathematical techniques

(D) all of the above

7. Constrains in LP model represents …………

(A) limitations (B) requirements

(C) balancing limitations (D) all of the above

8. The feasible region represented by the constraints $x_1 + x_2 \leq 1$, $3x_1 + x_2 \geq 3$, $x_1, x_2 \geq 0$ of the objective function $z = x_1 + 2x_2$ is …………

(A) limitations (B) requirements

(C) balancing limitations (D) all of the above

9. In a L.P.P. with m restrictions in n variables the maximum number of basic fesible solutions are …………

(A) $^nC_{m+1}$ (B) $^nC_{m-2}$

(C) ^nC_m (D) $^nC_{m-1}$

10. Which of the following statements is true with respect to the optimal solution of an L.P.P. …………

(A) every L.P.P. has an optimal solution

(B) optimal solution of L.P.P. always occurs at an extreme point.

(C) at optimal solution all resources are used completely.

(D) if an optimal solution exists, there will always be at least one at a corner.

11. In the optimum simplex table $Z_j - C_j = 0$ implies that there is …………

(A) cycling (B) alternative solution

(C) infeasible solution (D) unbounded solution

12. In the L.P. problem of maximization the simplex iteration terminates when …………

(A) $Z_j - C_j \geq 0$ (B) $Z_j - C_j \leq 0$

(C) $Z_j - C_j = 0$ (D) $Z_j = 0$

13. Which situation arises when there is no solution satisfying all constraints of L.P.P. ?

    (A) unbounded solution (B) tie among rows

    (C) tie among columns (D) infeasibility

14. In the Big-M method, which value is assigned to the original solution, in order to achieve the initial basic feasible solution ?

    (A) M (B) – M (C) 0 (D) none

15. In the maximization L.P.P. the objective function coefficient for artificial variable is …………

    (A) 0 (B) M (C) – M (D) none

(C) Theory Questions

1. Write a not on linear programming problem.
2. What are the limitations of L.P.P. ?
3. Explain various areas in industry where operations research techniques can be used as an effective tool in decision-making by the management.
4. Explain the following terms :
   (i) Objective functions.
   (ii) Linearity constraints.
   (iii) Feasible region.
   (iv) Feasible solution.
   (v) Non-negativity of constraints.
   (vi) Optimum solution.
   (vii) Decision variables.
5. Explain various steps in graphical method of solution of L.P.P.
   What one the limitations of this method ?
6. Discuss the essential characteristics of problems for which linear programming technique is useful.
7. Write note on :
   (i) Assumptions in Linear Programming Problem
   (ii) Limitation of Linear Programming
8. Explain the terms : feasible solution, slack variable, surplus variable artificial variable.
9. What is degeneracy in L.P.P. solution ? Explain how degeneracy is resolved.
10. Explain the term infeasibility. How is it identified in the simple table ?

(D) Numerical Problems

Solve the following graphically :

1. Maximize $Z = 5x - y$

   Subject to
$$\begin{aligned} x + y &\geq 2 \\ x + 2y &\leq 2 \\ 2x + y &\leq 2 \\ x \geq 0, y &\geq 0 \end{aligned}$$

2. Minimize $Z = x + 3y$

Subject to
$$\begin{aligned} x + y &\geq 3 \\ -x + y &\leq 2 \\ x - 2y &\leq 2 \\ x \geq 0,\ y &\geq 0 \end{aligned}$$

3. Minimize $Z = 3x + y$

Subject to
$$\begin{aligned} 2x + 3y &\geq 2 \\ x + y &\geq 1 \\ x \geq 0,\ y &\geq 0 \end{aligned}$$

4. Minimize $Z = 20x + 40y$

Subject to
$$\begin{aligned} 36x + 6y &\geq 108 \\ 3x + 12y &\geq 36 \\ 20x + 10y &\geq 100 \\ x \geq 0,\ y &\geq 0 \end{aligned}$$

5. Maximize $Z = 15x + 10y$

Subject to
$$\begin{aligned} 4x + 6y &\leq 360 \\ 3x &\leq 180 \\ 5y &\leq 200 \\ x \geq 0,\ y &\geq 0 \end{aligned}$$

6. Maximize $Z = 5x_1 + 3x_2$, subject to
$$\begin{aligned} 3x_1 + 5x_2 &\leq 15 \\ 5x_1 + 2x_2 &\leq 10 \\ x_1,\ x_2 &\geq 0. \end{aligned}$$

7. Minimize $Z = 10x_1 + 6x_2 + 2x_3$, subject to
$$\begin{aligned} -x_1 + x_2 + x_3 &\geq 1 \\ 3x_1 + x_2 - x_3 &\geq 2 \\ x_1,\ x_2,\ x_3 &\geq 0. \end{aligned}$$

8. Maximize $Z = 2x_1 + 3x_2$, subject to
$$\begin{aligned} x_1 + 2x_2 &\leq 3 \\ 3x_1 + 4x_2 &\geq 12 \\ x_1, x_2 &\geq 0. \end{aligned}$$

9. Maximize $Z = 3x_1 + 5x_2 + 4x_3$, subject to
$$\begin{aligned} 2x_1 + 3x_2 &\leq 8 \\ 2x_1 + 2x_2 + 4x_3 &\leq 15 \\ 2x_2 + 5x_3 &\leq 10 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

10. Minimize $Z = 4x_1 + 2x_2$, subject to

$$\begin{aligned} 3x_1 + x_2 &\geq 27 \\ x_1 + x_2 &\geq 21 \\ x_1 + 2x_2 &\geq 30 \\ x_1, x_2 &\geq 0. \end{aligned}$$

11. Maximize $Z = x_1 + 2x_2$, subject to

$$\begin{aligned} x_1 &\leq 4 \\ x_2 &\leq 3 \\ x_1 + 2x_2 &\leq 8 \\ x_1, x_2 &\geq 0. \end{aligned}$$

12. Minimize $Z = 2x_1 + 4x_2 + x_3$, subject to

$$\begin{aligned} x_1 + 2x_2 - x_3 &\leq 5 \\ 2x_1 - x_2 + 2x_3 &= 2 \\ -x_1 + 2x_2 + 2x_3 &\geq 1 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

13. Maximize $Z = 3x_1 - x_2$, subject to

$$\begin{aligned} 2x_1 + x_2 &\geq 2 \\ x_1 + 3x_2 &\leq 3 \\ x_2 &\leq 4 \\ x_1, x_2 &\geq 0. \end{aligned}$$

14. Minimize $Z = 2x_1 + 3x_2$, subject to

$$\begin{aligned} x_1 + x_2 &\geq 5 \\ x_1 + 2x_2 &\geq 6 \\ x_1, x_2 &\geq 0. \end{aligned}$$

15. Minimize $Z = x_1 + x_2$, subject to

$$\begin{aligned} 2x_1 + x_2 &\geq 4 \\ x_1 + 7x_2 &\geq 7 \\ x_1, x_2 &\geq 0. \end{aligned}$$

Answers

(A) True or False :

1. True	2. False	3. True	4. True	5. False
6. True	7. True	8. False	9. False	5. False

(B) Multiple Choice Questions :

1. (A)	2. (C)	3. (C)	4. (B)	5. (D)
6. (D)	7. (D)	8. (A)	9. (C)	10. (B)
11. (B)	12. (A)	13. (D)	14. (C)	15. (C)

(D) Problems :

1. $Z_{max} = 10.$ When $x = 2, y = 2.$
2. $Z_{min} = \frac{11}{3}.$ When $x = \frac{8}{3}, \; y = \frac{1}{3}.$
3. $Z_{min} = 1.$ When $x = 0, y = 1.$
4. $Z_{min} = 160.$ When $x = 4, \; y = 2.$
5. $Z_{max} = 1100.$ When $x = 60, \; y = 20.$
6. $x_1 = \frac{20}{19}, \; x_2 = \frac{45}{19}$, with $Z_{max.} = \frac{235}{19}$ (Three simplex tables) .
7. $x_1 = \frac{1}{4}, \; x_2 = \frac{5}{4}, \; x_3 = 0$, with $Z_{min} = 10.$

(Big M-technique, 3-simplex tables).

8. The problem has no feasible solution, since one artificial variable is at positive level. (optimal solution in 3-simplex tables).
9. $x_1 = \frac{89}{41}, \; x_2 = \frac{50}{41}, \; x_3 = \frac{62}{41}$ with $Z = \frac{765}{41}$.

   (Four simplex tables.)
10. $x_1 = 3, \; x_2 = 18$, with $Z = 48$ (*four* simplex tables – Big M-technique.)
11. $x_1 = 2, \; x_2 = 3$, with $Z = 8$

    (Three simplex tables)
12. $x_1 = 0 = x_2, \; x_3 = 1$, with $Z = 1$. (four simplex tables)
13. $x_1 = 3, \; x_2 = 0$, with $Z = 9$.
14. $x_1 = 4, \; x_2 = 1$ with $Z = 11$.
15. $x_1 = \frac{21}{13}, \; x_2 = \frac{10}{13}, \; Z = \frac{31}{13}$.

☆☆☆

Chapter 2...

Linear Programming Problem II

George Dantzig

George Bernard Dantzig, professor emeritus of operations research and of computer science who devised the "simplex method" and invented linear programming (which is not related to computer programming), died May 13 at his Stanford home of complications from diabetes and cardiovascular disease. He was 90 years old. A funeral service has been the mathematicians and held. "George B. Dantzig is regarded by most experts as having been the initiator of and leading figure in the revolutionary scientific development of mathematical programming as a powerful method for optimally managing resources in literally thousands of applications in industry and government in the last three decades," said Arthur F. Veinott Jr., professor of management science and engineering. "So pervasive is the influence of Dantzig's simplex method that experts have estimated that from 10 percent to 25 percent of all scientific computation is devoted to it. Indeed, that method is probably the single most widely used algorithm originated in the last six decades."

2.1 Special Cases in Simplex Method

In the application of simplex method, there arise some special cases.

They are :

(i) Tie among the key rows (Degeneracy) i.e. departing variable.

(ii) Tie among key columns i.e. entering variable.

(iii) Alternative optimal solution.

(iv) Unbounded solution.

(v) Non-existing or infeasible solution.

2.1.1 Degeneracy

In the simplex table, at any particular stage the least of a set of non-negative ratios is responsible to decide which variable is entering variable and which one is leaving variable. A tie occurs when two or more ratios are equal and the least. If we choose arbitrarily any one of the tied up the others continue to remain in the basis also become zero. This results in the situation that, one or more of the basic variables have zero value. Such a situation is called degeneracy.

Now in the next iteration we see that the entering variable and leaving variable are both zero.

In order to resolve the degeneracy and selection of the Key row we proceed as below :

(a) All elements in the body and identity in the rows with identical entries one divided by the corresponding key elements of the rows.

(b) The quotients in step (a) are compared column by column; in the order columns of slack variables followed by columns of artificial variables followed by columns of decision variables.

(c) After the selection of key row, we proceed to regular simplex procedure.

Illustrative Example

Example 2.1 : Consider the problem

Maximize $Z = 30x + 20y$

Subject to $6x + 8y \leq 480$

$3x + 3y \leq 240$

$x \geq 0, y \geq 0$

Solution : We introduce slack variables s_1 and s_2.

Then problem is

Maximize $Z = 30x + 20y + 0s_1 + 0s_2$

Subject to $6x + 8y + s_1 = 480$

$3x + 3y + s_2 = 240$

The initial simplex table is as below :

C_j			30	20	0	0		
Objective	**Product mix**	**Solution**	**x**	**y**	s_1	s_2	**Ratio**	
0	s_1	480	6	8	1	0	80	
0	s_2	240	3	3	0	1	80	Tie
Z_j		0	0	0	0	0		
$Z_j - C_j$		0	– 30	– 20	0	0		
			↑ Key column					

Now the ratio column suggests that tie has occurred.

In order to resolve degeneracy, in the two rows are divide the entries in the column of s_1 and s_2 and entries in the column of x, y by 6 and 3.

	s_1	s_2	x	y
First row	$\frac{1}{6}$	0	1	$\frac{8}{6} = \frac{4}{3}$
Second row	0	$\frac{1}{3}$	$\frac{2}{3} = 1$	$\frac{3}{3} = 1$

In this table column wise. Comparison suggests that in the second row there is smallest ratio viz. 0. Therefore, second row is key row.

Thus first column is the key column and second row is the key row.

This means x is entering variable and s_2 is leaving variable.

C_j			**30**	**20**	**0**	**0**	
Objective	**Product mix**	**Solution**	**x**	**y**	**s_1**	**s_2**	**Ratio**
0	s_1	480	6	8	1	0	80
0	s_2	240	3	3	0	1	80 ← key row
Z_j		0	0	0	0	0	
$Z_j - C_j$		0	– 30	– 20	0	0	

↑ Key column

Intersecting element

Key element

The next improvement is as shown in the table below.

First row : In the first row s_1 remains as it is.

Solution : $480 - \frac{6}{3} \times 240 = 0$,

$x : 6 - \frac{6}{3} \times 3 = 0$ $\qquad$ $y : 8 - \frac{6}{3} \times 3 = 2$

$s_1 : 1 - \frac{6}{3} \times 0 = 1$ $\qquad$ $s_2 : 0 - \frac{6}{3} \times 1 = -2$

Second row : Solution : $\frac{240}{3} = 80$

$x : \frac{3}{3} = 1$ $\qquad$ $y : \frac{3}{3} = 1$

$s_1 : \frac{0}{3} = 0$ $\qquad$ $s_2 : \frac{1}{3} = \frac{1}{3}$

Improved table is as below :

C_j			**30**	**20**	**0**	**0**
Objective	**Product mix**	**Solution**	**x**	**y**	**s_1**	**s_2**
0	s_1	0	0	2	1	– 2
30	x	80	1	1	0	1/3
Z_j		2400	30	30	0	10
$Z_j - C_j$		2400	0	10	0	10

In this table all $Z_j - C_j$ values (index numbers) are non-negative. Thus we reached at the optimum solution.

$\therefore \quad x = 80, \quad y = 0$

and $\quad Z_{max} = 2400$

2.1.2 Tie Among Key Columns

At any intermediate application of simplex method it may happen that two or more columns have the same $Z_j - C_j$ value.

In this situation we break the tie by following the rules below :

(a) If there is a tie between decision variables, we select the entering variable arbitrarily.

(b) If there is a tie between slack (or surplus) variables, we select the entering variable arbitrarily.

(c) If there is a tie between decision variable and slack (surplus) variable, then the first choice is given to decision variable for entering into the basis.

Illustrative Example

Example 2.2 :

Maximize $Z = 10x + 6y + 6z$

Subject to
$$3x + 2y + 2z \le 240$$
$$2x + 3y + 3z \le 270$$
$$x \le 60$$
$$x \ge 0, \quad y \ge 0, \quad z \ge 0$$

Solution : In standard form given L.P.P. is

Maximize $Z = 10x + 6y + 6z.$

Subject to
$$3x + 2y + 2z + s_1 = 240$$
$$2x + 3y + 2z + s_2 = 270$$
$$x + s_3 = 60$$
$$x \ge 0, \; y \ge 0, \; z \ge 0$$

Initial simplex table

			C_j						**Ratio**
Objective	**Mix**	**Solution**	10	6	6	0	0	0	
			x	y	z	s_1	s_2	s_3	
0	s_1	240	3	2	2	1	0	0	80
0	s_2	270	2	3	3	0	1	0	135
0	s_3	60	1	0	0	0	0	1	← 60 key row
Z_j		0	0	0	0	0	0	0	
$Z_j - C_j$		0	– 10	– 6	– 6	0	0	0	
			↑						

Key element Key column

s_3 is departing variable and x is entering variable.

Objective	Mix	Solution	10	6	6	0	0	0	Ratio
			x	y	z	s_1	s_2	s_3	
0	s_1	60	0	2	2	1	0	– 3	30
0	s_2	150	0	3	3	0	1	– 2	50
10	x	60	1	0	0	0	0	1	∞
Z_j		600	10	0	0	0	0	10	
$Z_j – C_j$		600	0	– 6	– 6	0	0	10	

↑

Tie key column

A tie has occurred among the columns of y and z. We select any one of them say y as entering variable.

The row corresponding to s_1 is key row.

Hence s_1 is leaving variable and y is entering variable.

Improved table

Objective	Mix	Solution	10	6	6	0	0	0	Ratio
			x	y	z	s_1	s_2	s_3	
6	y	30	0	1	1	1/2	0	–3/2	
0	s_2	60	0	0	0	–3/2	1	5/2	
10	x	60	1	0	0	0	0	1	
Z_j		780	10	6	6	3	0	1	
$Z_j – C_j$		780	0	0	0	3	0	1	

We note that in this table all index numbers are non-negative.

We reached at the optimum solution

optimum solution is x = 60, y = 30, z = 0 and Z_{max} = 780.

2.2 Alternative Optimum Solution

We know that in the maximization problem an optimum solution is reached when all $Z_j – C_j$ entries are greater than or equal to 0. Consider the situation in which $Z_j – C_j = 0$ in the column of some **non-basic** variable.

Each entry in the $Z_j – C_j$ row indicates the contribution loss per unit of that particular variable in the objective function value if it enters into the basis.

Hence, when a non-basic variable associated with $Z_j – C_j = 0$ enters into the basis, a new solution is obtained and in this case the value of the objective function remains unchanged. This is illustrated in the following example.

Illustrative Example

Example 2.3 : Consider the L.P.P.

Maximize $Z = 6x + 3y$

Subject to
$$2x + y \leq 8$$
$$3x + 3y \leq 18$$
$$y \leq 3$$
$$x \geq 0, \quad y \geq 0$$

Solution : In its standard form L.P.P. is

Maximize $Z = 6x + 3y + 0s_1 + 0s_2 + 0s_3$

Subject to
$$2x + y + s_1 = 8$$
$$3x + 3y + s_2 = 18$$
$$y + s_3 = 3$$
$$x \geq 0, \quad y \geq 0$$

Initial simplex table

Objective	**Product mix**	**Solution**	6	3	0	0	0	
			x	y	s_1	s_2	s_3	**Ratio**
0	s_1	8	2	1	1	0	0	4
0	s_2	18	3	3	0	1	0	6
0	s_3	3	0	1	0	0	1	∞
Z_j		0	0	0	0	0	0	
$Z_j - C_j$		0	– 6	– 3	0	0	0	

↑

Key elements Key column

Improved simplex table is as below :

Objective	**Product mix**	**Solution**	6	3	0	0	0
			x	y	s_1	s_2	s_3
6	x	4	1	1/2	1/2	0	0
0	s_2	6	0	3/2	– 3/2	1	0
0	s_3	3	0	1	0	0	1
Z_j		24	6	3	3	0	0
$Z_j - C_j$		24	0	0	3	0	0

All entries in $Z_j - C_j$ row are non-negative.

Therefore, we reached at the optimum solution.

Optimum solution is x = 4, y = 0; $Z_{max} = 24$.

In the above table y is non-basic variable and $Z_j - C_j = 0$ in the column of y.

This suggests that there is alternative optimum solution.

Let us reproduce the above table and then improve it.

Objective	**Product mix**	**Solution**	6	3	0	0	0	
			x	y	s_1	s_2	s_3	**Ratio**
6	x	4	1	1/2	1/2	0	0	8
0	s_2	6	0	3/2	– 3/2	1	0	4 ←
0	s_3	3	0	1	0	0	1	3
Z_j		24	6	3	3	0	0	8
$Z_j - C_j$		24	0	0 ↑	3	0	0	∞

The variable y is entering variable and s_2 is departing variable.

The key element is $\frac{3}{2}$ which is shown as circled.

Improved table

Objective	**Product mix**	**Solution**	6	3	0	0	0
			x	y	s_1	s_2	s_3
6	x	2	1	0	1	– 1/3	0
3	y	4	0	1	– 1	2/3	0
0	s_3	– 1	0	0	1	– 2/3	0
Z_j		24	6	3	3	0	0
$Z_j - C_j$		24	0	0	3	0	0

Now, optimum solution is reached. x = 2, y = 4, $Z_{max} = 24$.

2.3 Unbounded Solution

In the solution of L.P.P. if it happens that any one or more decision variables can increase indefinitely without violating any constraint, such a situation is described by saying that L.P.P. has unbounded solution.

In the simplex table suppose that the $Z_j - C_j$ row contains negative entries, which means the optimum solution is not reached. However, the replacement ratios have negative or '∞' values.

This situation suggests that entering variable exists but not departing variable. Hence, the entering variable is allowed to increase infinitely without departing any variable from the current basis.

Illustrative Example

Example 2.4 : Consider L.P.P.

Maximize $Z = 2x + y$

Subject to
$$x - y - z \le 1$$
$$x - 2y + z \le 2$$
$$x \ge 0,\ y \ge 0,\ z \ge 0$$

Solution : In the standard form this L.P.P. is

Maximize $Z = 2x + y$

Subject to
$$x - y - z + s_1 = 1$$
$$x - 2y + z + s_2 = 2$$
$$x \ge 0,\quad y \ge 0,\quad z \ge 0$$

Initial simplex table

C_j			2	1	0	0	0	
Objective	**Product mix**	**Solution**	x	y	z	s_1	s_2	**Ratio**
0	s_1	1	1	– 1	– 1	1	0	← 1
0	s_2	2	1	– 2	1	0	1	2
Z_j		0	0	0	0	0	0	
$Z_j - C_j$		0	– 2	– 1	0	0	0	

key element

The entering variable is x and departing variable is s_1.

Improved table

			2	1	0	0	0	
Objective	**Product mix**	**Solution**	x	y	z	s_1	s_2	**Ratio**
2	x	1	1	– 1	– 1	1	0	– 1
0	s_2	1	0	– 1	2	– 1	1	– 1
Z_j		2	2	– 2	– 2	2	0	
$Z_j - C_j$		2	0	– 3	– 2	2	0	

↑

At this stage we note that y is entering variable. However, the replacement ratios are negative i.e. – 1, – 1. This means there is no departing variable. We conclude that the value of y can be increased without allowing one of the basic variables to zero. Therefore, L.P.P. has unbounded solution.

2.4 Infeasibility

In the solution of L.P.P. the infeasibility occurs when two or more of the constraints contradict each other or the problem itself is not properly formulated. Suppose in the final simplex table the values of $Z_j - C_j$ suggest that optimum solution is reached but at least one of the artificial variables appears as a basis variable. This situation suggests that the solution is infeasible.

Illustrative Example

Example 2.5 : Given L.P.P.

Maximize $Z = x + 4y$

Subject to
$$x + 2y \leq 2$$
$$4x + 3y \geq 12$$
$$x \geq 0, \quad y \geq 0$$

Solution : We have to

Maximize $Z = x + 4y + 0s_1 + 0s_2 - MA_1$

Subject to
$$x + 2y + s_1 = 2$$
$$4x + 3y - s_2 + A_1 = 12$$
$$x \geq 0, \quad y \geq 0$$

Initial simplex table is as follow.

			1	4	0	0	– M	
Objective	**Product mix**	**Solution**	x	y	s_1	s_2	A_1	**Ratio**
0	s_1	2	1	2	1	0	0	← 1 key row
– M	A_1	12	4	3	0	– 1	1	3
Z_j		– 12 M	– 4 M	– 3 M	0	M	– M	
$Z_j - C_j$		– 12 M	– 4 M – 1	– 3M – 4	0	M	0	

Key element ↑ Key column

x is entering variable and s_1 is departing variable.

Improved simplex table

			1	**4**	**0**	**0**	**– M**
Objective	**Product mix**	**Solution**	**x**	**y**	**s_1**	**s_2**	**A_1**
1 – M	x A_1	2 4	1 0	2 – 5	1 – 4	0 – 1	0 1
Z_j		– 4M + 2	1	5M + 2	4M+ 1	M	– M
$Z_j – C_j$		– 4M + 2	0	5M – 2	4M + 1	M	0

In the row of $Z_j - C_j$ there are non-negative entries for all variables. Therefore, optimum solution is reached. However, one of the basis variables viz. A_1 is artificial variable. Therefore, the conclusion is that the given L.P.P. has no feasible solution.

2.5 Duality

Every linear programming problem is associated with another linear programming problem which is called the *'dual'* of the problem. The original problem is called *'primal'*, while the other is called its *'dual'*. The optimal solution of either problem reveals information concerning the optimal solution of the other. If the optimal solution of either problem is known then the optimal solution of the other is also available.

2.5.1 Dual Problem

1. Consider a L.P.P. in canonical form :

$$\left.\begin{array}{l}
\textbf{Primal : } \text{Maximize } Z = c_1 x_1 + \ldots + c_n x_n \\
\text{Subject to} \\
\quad a_{11} x_1 + a_{12} x_2 + \ldots\ldots\ldots + a_{1n} x_n \le b_1 \\
\quad a_{21} x_1 + a_{22} x_2 + \ldots\ldots\ldots + a_{2n} x_n \le b_2 \\
\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
\quad a_{m1} x_1 + a_{m2} x_2 + \ldots\ldots\ldots + a_{mn} x_n \le b_m, \\
\quad x_1, x_2, \ldots\ldots, x_n \ge 0.
\end{array}\right\} \qquad \ldots \text{(I)}$$

The problem in (I) can be written in matrix form as :

Maximize $Z = CX^t$

Subject to

$$\begin{aligned} AX^t &\le B, \\ X^t &\ge 0, \text{ where} \end{aligned} \qquad \ldots \text{(II)}$$

$$X^t = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$C = [c_1, c_2 \ldots c_n]$$

and $$A = \begin{bmatrix} a_{11} & a_{12} & \ldots\ldots & a_{1n} \\ a_{21} & a_{22} & \ldots\ldots & a_{2n} \\ \vdots & \vdots & \ldots\ldots & \vdots \\ a_{m1} & a_{m2} & \ldots\ldots & a_{mn} \end{bmatrix}$$

the matrix of coefficients of the constraints.

Now the *dual* of the problem (I) is given by :

$$\left.\begin{array}{l} \text{Minimize } \; Z^* = b_1 y_1 + b_2 y_2 + \ldots + b_m y_m \\ \text{Subject to} \\ \quad a_{11} y_1 + a_{21} y_2 + \ldots\ldots + a_{m1} y_m \ge c_1 \\ \quad a_{12} y_1 + a_{22} y_2 + \ldots\ldots + a_{m2} y_m \ge c_2 \\ \\ \quad a_{1n} y_1 + a_{2n} y_2 + \ldots\ldots + a_{mn} y_n \ge c_n \\ \quad y_1, \; y_2, \; \ldots, \; y_m \ge 0. \end{array}\right\} \quad \ldots \text{(III)}$$

The problem in (III) can be written in matrix form as :

$$\left.\begin{array}{l} \text{Minimize } \; Z^* = Y.B \\ \text{Subject to} \\ \qquad A^t Y^t \;\ge\; C^{t,} \;,\; Y^t \ge 0 \end{array}\right\} \quad \ldots \text{(IV)}$$

where , $$Y^t = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ \vdots \\ y_m \end{bmatrix}, \quad C^t = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ \vdots \\ c_n \end{bmatrix}$$

Note :

(i) The operations involved in (II) and (IV) are matrix operations.

(ii) 't' on right hand top of the letter denotes the transpose of the matrix.

Thus observe that :

(i) The number of constraints in one problem is equal to number of variables in other problem.

(ii) The number 'b_i' on the right hand side of the i^{th} constraint in one problem corresponds the coefficient of the i^{th} variable in objective function of the other problem.

(iii) If one problem is of the maximization type, other problem is of minimization type.

(iv) If the constraints in one are of the type $\le$ ($\ge$) then the constraints in other problem are of the type $\ge$ ($\le$) except the feasibility constraints.

(v) If the matrix of coefficients of the constraints in other problem is its transpose A^t.

Illustrative Example

Example 2.6 : Consider the problem

$$\left.\begin{aligned} &\text{Maximize } Z = 5x_1 + 3x_2 \\ &\text{Subject to} \\ &\qquad x_1 + x_2 \le 2 \\ &\qquad 5x_1 + x_2 \le 10 \\ &\qquad 3x_1 + 8x_2 \le 12 \end{aligned}\right\} \qquad \ldots (*)$$

$x_1, x_2 \ge 0.$

Find the dual of this problem.

Solution : Here,

$$C = [5, 3]$$

$$X^t = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 10 \\ 12 \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & 1 \\ 5 & 1 \\ 3 & 8 \end{bmatrix}$$

Therefore the dual problem in matrix form is given by;

Dual : Minimize $Z^* = YB$

Subject to

$$A^t Y^t \ge C^t,$$

$y^t \ge 0$, where, $Y = [y_1, y_2, y_3]$

The problem can be expressed in explicit form as :

$$\left.\begin{aligned} &\text{Minimize } Z^* = 2y_1 + 10y_2 + 12y_3 \\ &\text{Subject to} \\ &\quad y_1 + 5y_2 + 3y_3 \ge 5 \\ &\quad y_1 + y_2 + 8y_3 \ge 3, \\ &y_1, y_2, y_3 \ge 0. \end{aligned}\right\} \qquad \ldots (**)$$

Theorem

Dual of the dual is primal.

We can illustrate this fact by considering particular example.

For example, consider the primal problem (*) given in 2.6.

Considering the problem in 2.6 given in (**) as primal problem, we have

$$C = [2, 10, 12]$$

$$Y^t = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \text{ and } A = \begin{bmatrix} 1 & 5 & 3 \\ 1 & 1 & 8 \end{bmatrix}$$

with $B = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$

Therefore the dual of the problem in (**) is given by;

Maximize $Z_1 = XB$

Subject to

$$A^t X^t \leq C^t, \quad \text{with}$$

$X^t \geq 0$. That is, explicitly this is written as :

Maximize $Z_1 = 5x_1 + 3x_2$

Subject to

$$\begin{aligned} x_1 + x_2 &\leq 2 \\ 5x_1 + x_2 &\leq 10 \\ 3x_1 + 8x_2 &\leq 12 \end{aligned}$$

$x_1, x_2 \geq 0$, which is primal problem given in (*). Thus dual of a dual is primal.

2.5.2 Dual of the standard L.P.P.

How to write the dual of the problem in which some or all constraints are equalities ? Let us illustrate this fact by an example.

Illustrative Example

Example 2.7 : Write down the dual of the following L.P.P.

Maximize $Z = x_1 + 3x_2$

Subject to

$$\begin{aligned} 3x_1 + 2x_2 &\leq 6 \\ 3x_1 + x_2 &= 4 \end{aligned}$$

$x_1, x_2 \geq 0$.

Solution : The equality $3x_1 + x_2 = 4$ in the problem can be written in the form of two inequalities as follows.

$$\begin{aligned} 3x_1 + x_2 &\leq 4 \\ \text{and} \quad 3x_1 + x_2 &\geq 4 \end{aligned}$$

Multiplying second inequality by (–1), we have equivalent inequalities corresponding to the given equality.

$$\begin{aligned} 3x_1 + x_2 &\leq 4 \\ -3x_1 - x_2 &\leq -4 \end{aligned}$$

Thus, the given problem can be written in canonical form as :

$$\left.\begin{aligned} &\text{Maximize } Z = 5x_1 + 3x_2 \\ &\text{Subject to} \\ &\quad 3x_1 + 2x_2 \leq 6 \\ &\quad 3x_1 + x_2 \leq 4 \\ &\quad -3x_1 - x_2 \leq -4 \\ &x_1, x_2 \geq 0. \end{aligned}\right\} \quad \dots (*)$$

Now the dual of the problem in (*) can be worked out by usual method. Since there are three constraints in dual there will be three variables in objective function say y_1, y_2, y_3. Then the dual is given by :

Minimize $Z^* = 6y_1 + 4y_2 - 4y_3$

Subject to

$$\begin{bmatrix} 3 & 3 & -3 \\ 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \geq \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

That is;

Minimize $Z^* = 6y_1 + 4(y_2 - y_3)$

Subject to

$$3y_1 + 3y_2 - 3y_3 \geq 5$$
$$2y_1 + y_2 - y_3 \geq 3$$

$y_1, y_2, y_3 \geq 0$.

OR Minimize $Z^* = 6y_1 + 4(y_2 - y_3)$

Subject to

$$3y_1 + 3(y_2 - y_3) \geq 5$$
$$2y_1 + (y_2 - y_3) \geq 3$$

$y_1, y_2, y_3 \geq 0$.

Replacing $y_2 - y_3$ by y_2^*, we write dual :

Minimize $Z^* = 6y_1 + 4y_2^*$

Subject to

$$3y_1 + 3y_2^* \geq 5$$
$$2y_1 + y_2^* \geq 3$$

where, $y_1 \geq 0$ and y_2^* is unrestricted variable, as it is difference of two non-negative variables.

Observe : In the above problem we see that the variable in dual corresponding to equality in primal problem is unrestricted in sign. Thus we can generalise this fact as. In case of a problem in standard form, the dual variables related to standard primal problem are unrestricted variables. More precisely if,

Primal Maximize $Z = c_1 x_1 + c_2 x_2 + \ldots\ldots + c_n x_n$

Subject to

$$a_{11} x_1 + a_{12} x_2 + \ldots\ldots + a_{1n} x_n \leq b_1$$
$$a_{21} x_1 + a_{22} x_2 + \ldots\ldots + a_{2n} x_n \leq b_2$$

$$a_{m1} x_1 + a_{m2} x_2 + \ldots\ldots + a_{mn} x_n \leq b_m$$

$x_1, x_2, \ldots, x_n \geq 0$. Then

Dual : Minimize $Z^* = b_1 y_1 + b_2 y_2 + \dots + b_m y_m$

Subject to

$$a_{11} y_1 + a_{21} y_2 + \dots\dots + a_{m1} y_m \geq c_1$$
$$a_{12} y_1 + a_{22} y_2 + \dots\dots + a_{m2} y_m \geq c_2$$

$$a_{1n} y_1 + a_{2n} y_2 + \dots\dots + a_{mn} y_m \geq c_n$$

where, $y_1, y_2, \dots, y_m$ are unrestricted variables.

2.5.3 Duality and Simplex Method

Let us consider the solved example 1.25 in previous chapter.

Maximize $Z = 40x_1 + 35x_2$

Subject to

$$2x_1 + 3x_2 \leq 60$$
$$4x_1 + 3x_2 \geq 96$$

$x_1, x_2 \geq 0.$

The optimal simplex table for this problem, we have obtained it in the following form.

Table 2.1

C_B	C_i / X_B	40 x_1	35 x_2	0 s_1	0 s_2	Constant
35	x_2	0	1	$\frac{2}{3}$	$-\frac{1}{3}$	8
40	x_1	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	18
$Z_j - C_j$		0	0	$\frac{10}{3}$	$\frac{25}{3}$	Z = 1000

Here the slack variables s_1, s_2 have provided the I.B.F. solution as :

$s_1 = 60, \quad s_2 = 96, \quad$ with $Z = 0$.

Now, let us solve the dual of this (primal) problem by simplex method.

The dual of the problem under consideration is given by :

Minimize $Z^* = 60y_1 + 96y_2$

Subject to

$$2y_1 + 4y_2 \geq 40$$
$$3y_1 + 3y_2 \geq 35$$

$y_1, y_2 \geq 0.$

Adding surplus and artificial variables, we have :

Minimize $Z^* = 60y_1 + 96y_2 + 0.s_1 + 0.s_2 + MR_1 + MR_2$

Subject to

$$2y_1 + 4y_2 - s_1 + R_1 = 40$$
$$3y_1 + 3y_2 - s_2 + R_2 = 35$$

$y_1, y_2, s_1, s_2, R_1, R_2 \geq 0.$

The initial simplex table 2.2 is

Table 2.2

C_B	C_i / X_B	60 y_1	96 y_2	0 s_1	0 s_2	M R_1	M R_2	Const.	Ratio
M	R_1	2	$\boxed{4}$	–1	0	1	0	40	$\frac{40}{4} = 10$
M	R_2	3	3	0	– 1	0	1	35	$\frac{35}{3}$
$Z_j - C_j$		5M–60	7M–96	–M	–M	0	0	Z = 75M	

From the table we see that initial basic feasible solution is given by $R_1 = 40$, $R_2 = 35$ with Z = 75. Since there are positive $Z_j - C_j$ coefficients, the present solution is *not optimal*. 7M – 96 is the largest positive among $Z_j - C_j$ coefficients. Applying ratio test, R_1 is leaving variable and x_2 is entering variable. Performing pivot operations, we obtain the table 2.3.

Table 2.3

C_B	C_i / X_B	60 y_1	96 y_2	0 s_1	0 s_2	M R_1	M R_2	Const.	Ratio
96	y_2	$\frac{1}{2}$	1	$-\frac{1}{4}$	0	$\frac{1}{4}$	0	10	$\frac{10}{1/2} = 20$
M	R_2	$\frac{3}{2}$	0	$\frac{3}{4}$	– 1	$-\frac{3}{4}$	1	5	$\frac{5}{3/2} = \frac{10}{3}$
$Z_j - C_j$		$\frac{3}{2}$ M – 12	0	$\frac{3}{4}$ M – 24	–M	$-\frac{3}{4}$ M + 24	0	Z = 5M + 960	

We proceed in usual way and obtain table 2.4 :

Table 2.4

C_B	C_i / X_B	60 y_1	96 y_2	0 s_1	0 s_2	M R_1	M R_2	Const.
96	y_2	0	1	$-\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{2}$	$-\frac{1}{3}$	$\frac{25}{3}$
60	y_1	1	0	$\frac{1}{2}$	$-\frac{2}{3}$	$-\frac{1}{2}$	$\frac{2}{3}$	$\frac{10}{3}$
$Z_j - C_j$		0	0	– 18	– 8	18 – M	8 – M	Z = 1000

From table 2.4 we observe that the basic feasible solution for the problem (dual) is given by; $y_1 = \frac{10}{3}$, $y_2 = \frac{25}{3}$, with Z = 1000. This solution is *optimal* since all $Z_j - C_j$ coefficients are negative.

From this problem we observe the following :

The variables of starting solution in the primal problem are s_1 and s_2. The dual variables y_1 and y_2 correspond to *primal* equations containing s_1 and s_2 respectively. Look at the coefficients of s_1 and s_2 in $Z_j - C_j$ row of the optimal primal table. These are given by :

• Starting solution variables of primal problem	s_1	s_2
• Corresponding $Z_j - C_j$ coefficients of s_1 and s_2 in optimal solution of primal problem	$\frac{10}{3}$	$\frac{25}{3}$
• The corresponding dual variables	y_1	y_2

The coefficients $\frac{10}{3}$ and $\frac{25}{3}$ of s_1 and s_2 in the primal optimal solution in $Z_j - C_j$ row directly give the optimal solution of the dual problem : $y_1 = \frac{10}{3}$ and $y_2 = \frac{25}{3}$, which is the same as obtained by solving the dual problem independently.

On the other hand consider the coefficients of the variables in starting solution of the dual problem, that is R_1 and R_2 in the *optimal dual* table in $Z_j - C_j$ row. We have

• Starting variables of solution of dual problem.	R_1	R_2
• Corresponding $Z_j - C_j$ coefficients of R_1 and R_2 in optimal dual solution.	$18 - M$	$8 - M$
• Corresponding primal variables	x_1	x_2

Thus, we observe that, ignoring the constant M, the resulting coefficients 18 and 8 directly give the optimal solution to the primal problem as $x_1 = 18$ and $x_2 = 8$, which we have already obtained.

From above illustration, we see the two problems (dual and primal) are connected with each other very closed in the sense that if the optimal solution of one problem is given, the complete information about the other problem follows immediately.

Illustrative Examples

Example 2.8 : Obtain the dual of the following L.P.P. and find optimal solution of the dual problem.

Maximize $Z = 3x_1 + 2x_2 + 5x_3$

Subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 &\leq 430 \\ 3x_1 + 2x_3 &\leq 460 \\ x_1 + 4x_2 &\leq 120 \end{aligned}$$

$x_1, x_2, x_3 \geq 0.$

Solution : Since there are three constraints, let y_1, y_2, y_3 be dual variables. Then the dual of the problem is given by

$$\text{Minimize } Z^* = [y_1 \;\; y_2 \;\; y_3] \begin{bmatrix} 430 \\ 460 \\ 120 \end{bmatrix}$$

Subject to

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 0 & 4 \\ 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \geq \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$$

$y_1, \; y_2, \; y_3 \geq 0.$

That is,

Dual : Minimize $Z^* = 430y_1 + 460y_2 + 120y_3$

Subject to

$$\begin{aligned} y_1 + 3y_2 + y_3 &\geq 3 \\ 2y_1 + 4y_3 &\geq 2 \\ y_1 + 2y_2 + 4y_3 &\geq 5 \end{aligned}$$

$y_1, y_2, y_3 \geq 0.$

We have obtained optimal solution of the primal problem in example (2.8) of previous chapter. Using the interrelation of primal and dual problems, the optimal solution of the dual problem can be completely described from the optimal solution of the primal.

The coefficients of the starting variables in primal s_1, s_2 and s in $Z_j - C_j$ row of the optimal solution of the primal problem are 0, $\frac{5}{2}$ and $\frac{1}{2}$ respectively. Hence the optimal solution of the dual is given by $y_1 = 0$, $y_2 = \frac{5}{2}$ and $y_3 = \frac{1}{2}$, with $Z^* = 1210$.

Remark : The students are advised to solve actually the above dual problem, and verify the interrelation of primal and dual.

Example 2.9 : Solve the following L.P.P. by using its dual.

Minimize $Z = 10x_1 + 6x_2 + 2x_3$

Subject to

$$\begin{aligned} -x_1 + x_2 + x_3 &\geq 1 \\ 3x_1 + x_2 - x_3 &\geq 2 \end{aligned}$$

$x_1, \; x_2, \; x_3 \geq 0.$

Solution : Since there are two constraints, dual problem has two variables, say y_1 and y_2. The dual of the given problem is written as :

Maximize $Z^* = [y_1, y_2] \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Subject to

$$\begin{bmatrix} -1 & 3 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \geq \begin{bmatrix} 10 \\ 6 \\ 2 \end{bmatrix}$$

Dual : Or, Maximize $Z^* = y_1 + 2y_2$

Subject to

$$-y_1 + 3y_2 \leq 10$$

$$y_1 + y_2 \leq 6$$
$$y_1 - y_2 \leq 2$$

$y_1,\ y_2 \geq 0.$

Adding slack variables, we have

Maximize $Z^* = y_1 + 2y_2 + 0.s_1 + 0.s_2 + 0.s_3$

Subject to

$$-y_1 + 3y_2 + s_1 = 10$$
$$y_1 + y_2 + s_2 = 6$$
$$y_1 - y_2 + s_3 = 2$$

$y_1, y_2, y_3,\ s_1,\ s_2,\ s_3 \geq 0.$

Preparing initial simplex table 2.5, we have,

Table 2.5

C_B	C_i / X_B	**1** y_1	**2** y_2	**0** s_1	**0** s_2	**9** s_3	**Const.**	**Ratio**
0	s_1	– 1	$\boxed{3}$	1	0	0	10	$\frac{10}{3}$
0	s_2	1	1	0	1	0	6	$\frac{6}{1}$
0	s_3	1	– 1	0	0	1	2	$\frac{2}{-1}$ neglect
$Z_j - C_j$		–1	–2	0	0	0	$Z^* = 0$	

From table 2.5, we see that y_2 enters and s_1 leaves, so applying pivot operations we obtain :

Table 2.6

C_B	C_i / X_B	**1** y_1	**2** y_2	**0** s_1	**0** s_2	**9** s_3	**Const.**	**Ratio**
2	y_2	$-\frac{1}{3}$	1	$\frac{1}{3}$	0	0	$\frac{10}{3}$	neglect
0	s_2	$\frac{4}{3}$	0	$-\frac{1}{3}$	1	0	$\frac{8}{3}$	2
0	s_3	$\frac{2}{3}$	0	$\frac{1}{3}$	0	1	$\frac{16}{3}$	8
$Z_j - C_j$		$-\frac{5}{3}$	0	$\frac{2}{3}$	0	0	$Z^* = \frac{20}{3}$	

From table 2.6, we observe that y_1 enters and s_2 leaves. Applying pivot operations, we obtain :

Table 2.7

C_B	C_j / X_B	y_1	y_2	s_1	s_2	s_3	Const.
2	y_2	0	1	$\frac{1}{4}$	$\frac{1}{4}$	0	4
1	y_1	1	0	$-\frac{1}{4}$	$\frac{3}{4}$	0	2
0	s_3	0	0	$\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{20}{3}$
$Z_j - C_j$		0	0	$\frac{1}{4}$	$\frac{5}{4}$	0	$Z^* = 10$

From table 2.7, we see that the basic solution is $y_1 = 2$, $y_2 = 4$ with $Z^* = 10$. Also, since all $Z_j - C_j$ coefficients are non-negative, the present solution is *optimal*. Therefore, the *optimal* solution of the *primal* problem (since the variables in the starting solution of the dual are s_1, s_2 and s_3 which correspond to the variables x_1, x_2 and x_3 of the primal problem respectively) is given by $x_1 = \frac{1}{4}$, $x_2 = \frac{5}{4}$, $x_3 = 0$ with minimum $Z = 10 \times \frac{1}{4} + 6 \times \frac{5}{4} + 2 \times 0 = 10$.

Thus the optimal solution to the given primal problem is;

$x_1 = \frac{1}{4}$, $x_2 = \frac{5}{4}$, $x_3 = 0$ with $Z = 10$.

Example 2.10 : Show that the following L.P.P. has unbounded by using its dual problem.

Minimize $Z = 8x_1 - 2x_2 - 4x_3$

Subject to

$$\begin{aligned} x_1 - 4x_2 - 2x_3 &\geq 2 \\ x_1 + x_2 - 3x_3 &\geq -1 \\ -3x_1 - x_2 + x_3 &\geq 1 \end{aligned}$$

$x_1, x_2, x_3 \geq 0.$

Solution : Since there are three constraints, the dual problem has three variables, say y_1, y_2 and y_3. The dual of the given problem is

$$\text{Maximize } Z^* = [2 \ -1 \ 1] \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Subject to

$$\begin{bmatrix} 1 & 1 & -3 \\ -4 & 1 & -1 \\ -2 & -3 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \leq \begin{bmatrix} 8 \\ -2 \\ -4 \end{bmatrix}$$

OR, Maximize $Z^* = 2y_1 - y_2 + y_3$

Subject to

$$y_1 + y_2 - 3y_3 \le 8$$
$$-4y_1 + y_2 - y_3 \le -2$$
$$-2y_1 - 3y_2 + y_3 \le -4$$

$y_1, y_2, y_3 \ge 0.$

Now, to solve the dual problem, we add slack variables and convert the problem in standard form as :

Maximize $Z^* = 2y_1 - y_2 + y_3$

Subject to

$$y_1 + y_2 - 3y_3 + s_1 = 8$$
$$-4y_1 + y_2 - y_3 + s_2 = -2$$
$$-2y_1 - 3y_2 + y_3 + s_3 = -4$$

$y_1, y_2, y_3;\ s_1, s_2, s_3 \ge 0.$

We prepare initial table 2.8 of simplex method.

Table 2.8

C_B	C_i / X_B	**2** y_1	**–1** y_2	**1** y_3	**0** s_1	**0** s_2	**0** s_3	**Const.**	**Ratio**
0	s_1	$\boxed{1}$	1	– 3	1	0	0	8	$\frac{8}{1}$
0	s_2	– 4	1	– 1	0	1	0	– 2	
0	s_3	– 2	– 3	1	0	0	1	– 4	neglect
$Z_j - C_j$		–2	1	–1	0	0	0	Z = 0	

From table 2.8, we observe that y_1 enters and s_1 leaves, then applying pivot operations, we obtain :

Table 2.9

C_B	C_i / X_B	**2** y_1	**–1** y_2	**1** y_3	**0** s_1	**0** s_2	**0** s_3	**Const.**
2	y_1	1	1	– 3	1	0	0	8
0	s_2	0	5	– 13	4	1	0	30
0	s_3	0	– 1	– 5	2	0	1	12
$Z_j - C_j$		0	3	– 5	2	0	0	Z = 16

The table 2.9 shows that $Z_j - C_j$ for y_3 indicates that the improvement in the value of objective function is possible, however, no coefficient for y_3 in the pivot column is positive. This shows that the value of y_3 can be increased indefinitely. This shows that the problem has *unbounded* solution. Therefore, the given *primal* problem has unbounded solution.

Remark : The unbounded solution can be detected if, at any iteration any of the entering variable has all its coefficients negative or zero in the constraints.

Example 2.11 : Solve the following L.P.P. by using its dual.

Maximize $Z = 5x_1 - 2x_2 + 3x_3$

Subject to

$$\begin{aligned} 2x_1 + 2x_2 - x_3 &\geq 2 \\ 3x_1 - 4x_2 &\leq 3 \\ x_2 + 3x_3 &\leq 5 \end{aligned}$$

$x_1, x_2, x_3 \geq 0.$

Solution : We write the given problem in canonical form as follows :

Maximize $Z = 5x_1 - 2x_2 + 3x_3$

Subject to

$$\begin{aligned} -2x_1 - 2x_2 + x_3 &\leq -2 \\ 3x_1 - 4x_2 + 0x_3 &\leq 3 \\ x_2 + 3x_3 &\leq 5 \end{aligned}$$

and $x_1, x_2, x_3 \geq 0.$

The dual of the above problem is given by

Minimize $Z^* = -2y_1 + 3y_2 + 5y_3$

Subject to

$$\begin{aligned} -2y_1 + 3y_2 + 0.y_3 &\geq 5 \\ -2y_1 - 4y_2 + y_3 &\geq -2 \\ y_1 + 0.y_2 + 3y_3 &\geq 3 \end{aligned}$$

$y_1, y_2, y_3 \geq 0.$

To solve above dual problem by simplex method, we convert it to a maximization problem. The requirement in second constraint is made positive by multiplying both sides of this constraint by – 1. Thus the problem becomes

Maximize $Z_1^* = -\text{Minimize } Z^* = 2y_1 - 3y_2 - 5y_3$

Subject to

$$\begin{aligned} -2y_1 + 3y_2 + 0.y_3 &\geq 5 \\ 2y_1 + 4y_2 - y_3 &\leq 2 \\ y_1 + 0.y_2 + 3y_3 &\geq 3 \end{aligned}$$

$y_1, y_2, y_3 \geq 0.$

Introducing surplus variables s_1 and s_3 in first and third constraints; slack variable s_2 in second constraint; R_1 and R_2 artificial variables in second and third constraints, we have

Maximize $Z_1^* = 2y_1 - 3y_2 - 5y_3 + 0.s_1 + 0.s_2 + 0.s_3 + R_1 + R_2$

Subject to

$$\begin{aligned} -2y_1 + 3y_2 + 0.y_3 - s_1 + R_1 &= 5 \\ 2y_1 + 4y_2 - y_3 + s_2 &= 2 \\ y_1 + 0.y_2 + 3y_3 - s_3 + R_2 &= 3 \\ y_1, y_2, y_3, s_1, s_2, s_3, R_1, R_2 &\geq 0 \end{aligned}$$

Now, taking $y_1 = y_2 = y_3 = s_1 = s_3 = 0$, we obtain I.B.F. solution as $R_1 = 5$, $s_2 = 2$, $R_2 = 3$, with $Z_1^* = -8$ M, where M is sufficiently large positive number.

We prepare first simplex table as follows :

Table 2.10

C_B	C_i / X_B	2 y_1	–3 y_2	–5 y_3	0 s_1	0 s_2	0 s_3	–M R_1	–M R_2	Co.	Ra.
–M	R_1	–2	3	0	–1	0	0	1	0	5	$\frac{5}{3}$
0	s_2	2	$\boxed{4}$	–1	0	1	0	0	0	2	$\frac{2}{4}$
–M	R_2	1	0	3	0	0	–1	0	1	3	neg.
$Z_j - C_j$		–M–2	–3M +3	–3M +5	–M	0	M	0	0	$Z_1^* = -8M$	

From table 2.10 we observe that y_2 enters and s_2 leaves. Performing pivot operations, we obtain :

Table 2.11

C_B	C_i / X_B	2 y_1	–3 y_2	–5 y_3	0 s_1	0 s_2	0 s_3	–M R_1	–M R_2	Co.	Ra.
–M	R_1	$-\frac{7}{2}$	0	$\frac{3}{4}$	–1	$-\frac{3}{4}$	0	1	0	$\frac{7}{2}$	$\frac{14}{3}$
–3	y_2	$\frac{1}{2}$	1	$-\frac{1}{4}$	0	$\frac{1}{4}$	0	0	0	$\frac{1}{2}$	
–M	R_2	1	0	$\boxed{3}$	0	0	–1	0	1	3	1
$Z_j - C_j$		$\frac{5M-7}{2}$	0	$\frac{-15M+23}{4}$	M	$\frac{3M-3}{4}$	M	0	0	$Z_1^* = -\frac{13}{2}M - \frac{3}{2}$	

From table 2.12, we see that y_3 is entering and R_2 is leaving variable. Performing pivot operations we obtain :

Table 2.12

C_B	C_i / X_B	2 y_1	–3 y_2	–5 y_3	0 s_1	0 s_2	0 s_3	–M R_1	–M R_2	Co.	Ra.
–M	R_1	$-\frac{15}{4}$	0	0	–1	$-\frac{3}{4}$	$\boxed{\frac{1}{4}}$	1	$-\frac{1}{4}$	$\frac{11}{4}$	
–3	y_2	$\frac{7}{12}$	1	0	0	$\frac{1}{4}$	$-\frac{1}{12}$	0	$\frac{1}{12}$	$\frac{3}{4}$	
–5	y_3	$\frac{1}{3}$	0	1	0	0	$-\frac{1}{3}$	0	$\frac{1}{3}$	1	
$Z_j - C_j$		$\frac{15}{4}M - \frac{65}{12}$	0	0	M	$\frac{3}{4}M - \frac{3}{4}$	$-M/4 + \frac{23}{12}$	0	$\frac{5}{4}M + \frac{23}{12}$	$Z_1^* = \frac{-11}{4}M - \frac{29}{4}$	

From table 2.13 we observe that s_3 is entering variable and R_1 is leaving variable. Applying pivot operations, we obtain :

Table 2.13

C_B	C_i / X_B	2 y_1	–3 y_2	–5 y_3	0 s_1	0 s_2	0 s_3	–M R_1	–M R_2	Co.	Ra.
0	s_3	– 15	0	0	–4	–3	1	4	–1	11	
–3	y_2	$-\frac{2}{3}$	1	0	$-\frac{1}{3}$	0	0	$\frac{1}{3}$	0	$\frac{5}{3}$	
–5	y_3	$-\frac{14}{3}$	0	1	$-\frac{4}{3}$	–1	0	$\frac{4}{3}$	0	$\frac{14}{3}$	
$Z_j - C_j$		$\frac{70}{3}$	0	0	$\frac{23}{3}$	5	0	$M - \frac{23}{3}$	M	$Z = -\frac{85}{3}$	

We see that the basic feasible solution is given by $s_3 = 11$, $y_2 = \frac{5}{3}$, $y_3 = \frac{14}{3}$ with $Z_1^* = -\frac{85}{3}$. Also we observe that all the $Z_j - C_j$ coefficients are non-negative, hence the present solution is optimal. Thus, the solution to the dual problem is $y_1 = 0$, $y_2 = \frac{5}{3}$, $y_3 = \frac{14}{3}$ with max. $Z_1^* = -$ min. $Z^* = -\frac{85}{3}$; that is $y_1 = 0$, $y_2 = \frac{5}{3}$, $y_3 = \frac{14}{3}$, with $Z^* = \frac{85}{3}$. This is solution of the dual problem. The solution to the primal problem is obtained as : The surplus and slack variables s_1, s_2 and s_3 of the starting solution correspond to x_1, x_2 and x_3 respectively of the primal problem. Observing table 2.13, we have solution to the primal problem is given by :

$$x_1 = \frac{23}{3}, \quad x_2 = 5, \quad x_3 = 0 \quad \text{with} \quad Z = \frac{85}{3}.$$

Example 2.12 : Find the dual of the following L.P.P.

Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4$

Subject to

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 15 \\ 2x_1 + x_2 + 5x_3 &\le 20 \\ x_1 + 2x_2 + x_3 + x_4 &= 10 \\ x_1,\ x_2,\ x_3,\ x_4 &\ge 0. \end{aligned}$$

Solution : The equalities can be written in the form of inequalities as :

$$x_1 + 2x_2 + 3x_3 \leq 15$$
$$x_1 + 2x_2 + 3x_3 \geq 15$$
$$x_1 + 2x_2 + x_3 + x_4 \leq 10$$
$$x_1 + 2x_2 + x_3 + x_4 \geq 10.$$

Again multiplying second and last inequalities among these by (–1), we obtain all the inequalities in the form ≤. Using this the given problem can be written in canonical form as :

Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4$

Subject to

$$x_1 + 2x_2 + 3x_3 \leq 15$$
$$-x_1 - 2x_2 - 3x_3 \leq -15$$
$$2x_1 + x_2 + 5x_3 \leq 20$$
$$x_1 + 2x_2 + x_3 + x_4 \leq 10$$
$$-x_1 - 2x_2 - x_3 - x_4 \leq -10$$

$x_1, x_2, x_3, x_4 \geq 0.$

Now, as there are five inequalities there are five dual variables, say y_1, y_2, y_3, y_4 and y_5. So the dual problem can be written as :

Dual :

Minimize $Z^* = 15y_1 - 15y_2 + 20y_3 + 10y_4 - 10y_5$

Subject to

$$\begin{bmatrix} 1 & -1 & 2 & 1 & -1 \\ 2 & -2 & 1 & 2 & -2 \\ 3 & -3 & 5 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} \geq \begin{bmatrix} 1 \\ 2 \\ 3 \\ -4 \end{bmatrix}$$

OR Minimize $Z^* = 15(y_1 - y_2) + 20y_3 + 10(y_4 - y_5)$

Subject to

$$(y_1 - y_2) + 2y_3 + (y_4 - y_5) \geq 1$$
$$2(y_1 - y_2) + y_3 + 2(y_4 - y_5) \geq 2$$
$$3(y_1 - y_3) + 5y_3 + (y_4 - y_5) \geq 3$$
$$(y_4 - y_5) \geq -1.$$

OR Minimize $Z^* = 15y_1^{'} + 20y_3 + 10y_2^{'}$

Subject to

$$y_1' + 2y_3 + y_2' \geq 1$$

$$2y_1' + y_3 + 2y_2' \geq 2$$

$$3y_1' + 5y_3 + y_2' \geq 3$$

$$y_2' \geq -1$$

where y_1' and y_2' are unrestricted variables and $y_3 \geq 0$.

Exercise (2.1)

1. Write the dual of the problem

   Minimize $Z = 3x_1 + x_2$, subject to

   $$2x_1 + 3x_2 \geq 2$$

   $$x_1 + x_2 \geq 1.$$

   $$x_1, x_2 \geq 0.$$

2. Minimize $Z = 2x_2 + 5x_3$, subject to

   $$x_1 + x_2 \geq 2$$

   $$2x_1 + x_2 + 6x_3 \leq 6$$

   $$x_1 - x_2 + 3x_3 = 4$$

   $$x_1, x_2, x_3 \geq 0.$$

   Write the dual of this problem.

3. Minimize $Z = x_1 + x_2 + x_3$, subject to

   $$x_1 - 3x_2 + 4x_3 = 5$$

   $$2x_1 - 2x_2 \leq 3$$

   $$2x_2 - x_3 \geq 5$$

   $x_1, x_2 \geq 0$ and x_3 is unrestricted in sign. Write down the dual of this problem.

4. Maximize $Z = x_1 + 2x_2 - x_3$, subject to

   $$2x_1 - 3x_2 + 4x_3 \leq 5$$

   $$2x_1 - 2x_2 \leq 6$$

   $$3x_1 - x_3 \geq 4$$

   $$x_1, x_2, x_3 \geq 0.$$

   Write down the dual of this problem and further show that dual of the dual is primal.

5. Write down the dual of the following L.P.P.

   Maximize $Z = 2x_1 + 3x_2 + 4x_3$, subject to

$$3x_1 - 2x_3 \leq 41$$
$$2x_1 + x_2 + x_3 \leq 35$$
$$2x_2 + 3x_3 \leq 30$$
$$x_1, x_2, x_3 \geq 0.$$

Solve the dual problem by using simplex method, hence or otherwise find the solution of the primal problem.

6. Solve the following L.P.P. using Big M. Simplex Method

Maximize : $Z = -3x_1 + x_2 + x_3$

Subject to : $x_1 - 2x_2 + x_3 \leq 11$

$$-4x_1 + x_2 + 2x_3 \geq 3$$
$$2x_1 - x_3 = -1$$

$x_1, x_2, x_3 \geq 0.$ **(March 2010)**

7. Solve the following L.P.P. (**Hint :** Use dual)

Maximize : $Z = 2x_1 + 3x_2$

Subject to : $5x_1 + 7x_2 \leq 10$

$x_1, x_2 \geq 0$ **(March 2010)**

Answers (2.1)

1. Maximize $Z^* = 2y_1 + y_2$, subject to

$$2y_1 + y_2 \leq 3$$
$$3y_1 + y_2 \leq 1$$
$$y_1, y_2 \geq 0.$$

2. Maximize $Z^* = 2y_1 - 6y_2 - 4y_3$, subject to

$$y_1 - 2y_2 - y_3 \leq 0$$
$$y_1 - y_2 + y_3 \leq 2$$
$$0.y_1 - 6y_2 - 3y_3 \leq 5$$

$y_1, y_2 \geq 0$ and y_3 unrestricted variables.

3. Maximize $Z = -5y_1 + 5y_2 + 3y_3 + 5y_4$, subject to

$$-y_1 + y_2 - 2y_3 + 0\cdot y_4 \leq 1$$
$$3y_1 - 3y_2 + 2y_3 + 2y_4 \leq 1$$
$$-4y_1 + 4y_2 + 0.y_3 - y_4 \leq 1$$
$$4y_1 - 4y_2 + 0.y_3 + y_4 \leq -1$$
$$y_1, y_2, y_3, y_4 \geq 0.$$

4. Minimize $Z^* = 5y_1 + 6y_2 - 4y_3$, subject to

$$\begin{aligned} 2y_1 + 2y_2 - 3y_3 &\geq 1 \\ -3y_1 - 2y_2 + 0.y_3 &\geq 2 \\ 4y_1 + 0.y_2 + y_3 &\geq -1 \\ y_1, y_2, y_3 &\geq 0. \end{aligned}$$

5. Minimize $Z^* = 41y_1 + 35y_2 + 30y_3$, subject to

$$\begin{aligned} 3y_1 + 2y_2 &\geq 2 \\ y_2 + 2y_3 &\geq 3 \\ -2y_1 + y_2 + 3y_3 &\geq 4 \\ y_1, y_2, y_3 &\geq 0. \end{aligned}$$

Solution of the dual is : $y_1 = 0,\ y_2 = 1,\ y_3 = 1$ with $Z = 65$.

Solution of the primal : $x_1 = \frac{25}{2}$, $x_2 = 0$, $x_3 = 10$, with $Z = 65$.

Think Over It

1. How the dual if kth constraint of the primal is an equality ? Explain.
2. Dual of dual is primal problem. Explain.

Points to Remeber

1. An L.P. problem is either maximization or minimization i.e. optimization type.
2. To solve the problem by using simplex method first we write the problem in standard form, by converting all inequalities into equations.
   (a) Less than or equal to $\leq$ constraints are converted into equations by addition of slack variables.
   (b) Greater than or equal to i.e. $\geq$ constraints are converted into equations by subtracting surplus variables and adding artificial variables.
   (c) For equations only artificial variables are added.
3. After preparing initial simplex table we get initial feasible solution.
   It is examined for any further improvement and accordingly make the necessary improvements in the table.
4. The improvement in the solution is further checked and if possible the above procedure is repeated, till we arrived at optimum solution.
5. In a given L.P.P. called primal problem, suppose the number of constraints exceeds the number of decision variables. In such a case we can solve its dual problem with less efforts.
6. Dual of dual is primal problem.

7. If primal problem is of maximization (minimization) then dual problem is minimization (maximization).
8. The constants in right side of the constraints in the primal problem take place of coefficients of the objective function in dual problem and the coefficients in the objective function of the primal problem take place of constants in right side of the constraints in the dual problem.
9. The constraints of ≤ type (≥ type) in primal problem become the constraints of ≥ type (≤ type) in dual problem.
10. The column coefficients of the primal constraints take place of two coefficients in the dual constraints.
11. The values of slack variables in the index row of dual problem correspond to the values of decision variables in the primal problem.
12. If the primal constraint is an equation, then the corresponding dual variable is unrestricted in sign. Conversely, if the primal variable is unrestricted in sign, then the corresponding dual constraint is an equation.
13. The optimal solution of primal problem and that of dual problem match.

Miscellaneous Exercise

(A) True or False

State whether the following is true or false :

1. The dual of dual is a primal problem.
2. While writing the dual of given L.P.P. it is necessary that given L.P.P. is of maximization type.
3. If the objective function in primal L.P.P. is maximized then the objective function in its dual L.P.P. is minimized.
4. Given L.P.P.

   Maximize $Z = 4x_1 + 10x_2 + 3x_3$

   Subject to $x_1 + 2x_2 + x_3 \leq 10$

   $2x_1 - x_2 + 3x_3 = 8$

   $x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$

   Its dual is

   Minimize $u = 10y_1 + 8y_2$

   Subject to $y_1 + 2y_2 \geq 4$

   $2y_1 - y_2 \geq 10$

   $y_1 + 3y_2 \geq 3$

   $y_1 \geq 0,\ y_2$ unresitricted

(B) Multiple Choice Questions

Choose the correct alternative for the following :

1. If dual problem has unbounded solution, then primal has …………
   (A) unbounded solution (B) no feasible solution
   (C) feasible solution (D) bounded solution
2. If a primal L.P.P. has a finite solution, then dual L.P.P. has …………
   (A) finite solution (B) infeasible solution
   (C) unbounded solution (D) none of the above
3. For a given L.P.P. and its dual …………
   (A) optimal values of objective function equal
   (B) primal possesses optimal value if and only if dual possesses optimal value
   (C) both primal and dual cannot be infeasible
   (D) A, B, C all hold
4. If k^{th} constraint of the primal is an equality, then the dual variable …………
   (A) y_{k-1} is unrestricted in sign (B) y_{k+1} is unrestricted in sign
   (C) y_k is unrestricted in sign (D) none of the above

(C) Theory Questions

1. What is key column key row, key element ? How they are used in improving the simplex table ?
2. Explain the procedure to change the rows in the simplex table, after deciding the key row and key column.
3. By preparing a brief table explain how a tie occurs along (i) key columns (ii) key columns.
   How you break the tie ?
4. Explain briefly duality in linear programming.
5. Explain how a given problem is converted into its dual.
6. Explain clearly the significance of duality of considering suitable example.
7. What are the general rules while formulating dual L.P.P. from primal problem ?
8. If a given L.P.P. possesses optimal solution then its dual possesses optimal solution. Discuss.

(D) Numerical Problems

Solve the following L.P.P. by using *simplex method.*

1. Maximize $Z = 2x_1 + 3x_2$,

   Subject to
$$\begin{aligned} -3x_1 + x_2 &\leq 4 \\ x_1 - x_2 &\leq 2 \\ x_1, x_2 &\geq 0. \end{aligned}$$

2. Maximize $Z = x_1 + x_2$,

subject to
$$\begin{aligned} x_1 + 2x_2 &\le 20 \\ x_1 + x_2 &\le 15 \\ x_1, x_2 &\ge 0. \end{aligned}$$

Show that this problem has alternate solution.

3. Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4$,

subject to
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 15 \\ 2x_1 + x_2 + 5x_3 &= 20 \\ x_1 + 2x_2 + x_3 + x_4 &= 10 \\ x_1, x_2, x_3, x_4 &\ge 0. \end{aligned}$$

4. Maximize $Z = 4x_1 + 3x_2$,

subject to
$$\begin{aligned} x_1 + x_2 &\le 50 \\ x_1 + 2x_2 &\ge 80 \\ 3x_1 + 2x_2 &\ge 140 \\ x_1, x_2 &\ge 0. \end{aligned}$$

5. Maximize $Z = 3x_1 + 2x_2 + x_3$,

subject to
$$\begin{aligned} -3x_1 + 2x_2 + 2x_3 &= 8 \\ -3x_1 + 4x_2 + x_3 &= 7 \\ x_1, x_2, x_3 &\ge 0. \end{aligned}$$

6. Maximize $Z = 6x_1 + 4x_2$,

subject to
$$\begin{aligned} 2x_1 + 3x_2 &\le 30 \\ 3x_1 + 2x_2 &\le 24 \\ x_1 + x_2 &\ge 3 \\ x_1, x_2 &\ge 0. \end{aligned}$$

Show that this problem has alternate solution. Find three different solutions.

7. Maximize $Z = 3x_1 + 4x_2 + 5x_3 + 4x_4$,

subject to
$$\begin{aligned} 2x_1 + 5x_2 + 4x_3 + 3x_4 &\le 224 \\ 5x_1 + 4x_2 - 5x_3 + 10x_4 &\le 280 \\ 2x_1 + 4x_2 + 4x_3 - 2x_4 &\le 184 \\ x_1, x_2, x_3, x_4 &\ge 0. \end{aligned}$$

8. Maximize $Z = 4x_1 + 10x_2$, subject to
$$\begin{aligned} 2x_1 + x_2 &\le 10 \\ 2x_1 + 5x_2 &\le 20 \\ 2x_1 + 3x_2 &\le 18 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Show that the problem has alternate solution. Find two solutions.

9. Minimize $Z = x_1 - 3x_2 + 2x_3$,

subject to
$$3x_1 - x_2 + 2x_3 \le 7$$
$$-2x_1 + 4x_2 \le 12$$
$$-4x_1 + 3x_2 + 8x_3 \le 10$$
$$x_1, x_2, x_3 \ge 0.$$

10. Maximize $Z = 2x_1 + 5x_2 + 7x_3$,

subject to
$$3x_1 + 2x_2 + 4x_3 \le 100$$
$$x_1 + 4x_2 + 2x_3 \le 100$$
$$x_1 + x_2 + 3x_3 \le 100$$
$$x_1, x_2, x_3 \ge 0.$$

11. Solve the following problem by simplex method.

Maximize $Z = 3x + 2y$,

subject to
$$x + y \le 15$$
$$2x - y \le 5$$
$$x, y \ge 0$$

12. Below are given four successive simplex tables for a certain L.P.P. Also five statements (a), (b), (c), (d), (e) are given. Match one or more statements for each table.

Table 2.14 : Basis solution

		x_1	x_2	x_3	x_4
x_1	2	1	1	0	2
x_3	5	0	2	1	1
Z	– 10	0	3	0	∞

Table 2.15

x_1	10	1	0	1	2
x_2	2	0	1	2	1
Z	– 20	0	0	– 3	1

Table 2.16

x_2	20	1	1	– 2	0
x_4	– 10	– 3	0	0	1
Z	0	3	0	10	0

Table 2.17

x_3	0	– 1	0	1	3
x_2	10	3	1	0	2
Z	– 20	12	0	0	– 3

Statements :

(a) The solution is not basic feasible.
(b) Optimum solution is reached.
(c) Optimum solution is reached, but it is not unique.
(d) Further improvement in the objective function is possible.
(e) Solution is degenerate.

13. Use simplex method to solve

$$\begin{aligned} \text{Maximize} \quad Z &= 5x_1 + 3x_2 + x_3 \\ \text{Subjected to} \quad 2x_1 + x_2 + x_3 &= 3 \\ -x_1 + 2x_3 &= 4 \\ x_1 &\geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0 \end{aligned}$$

14. Solve using Big-M method

$$\begin{aligned} \text{Minimize} \quad Z &= 2x_1 - 3x_2 + 6x_3 \\ \text{Subjected to} \quad 3x_1 - 4x_2 - 6x_3 &\leq 2 \\ 2x_1 + x_2 + 2x_3 &\geq 11 \\ x_1 + 3x_2 - 2x_3 &\leq 5 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

15. Maximize $Z = 4x_1 + 2x_2$

$$\begin{aligned} \text{Subject to} \quad x_1 + x_2 &\leq 8 \\ x_1 &= 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(a) $x_1 = 4$ replaced by $x_1 \leq 4$. Then optimal solution will not change.
(b) $x_1 = 4$ replaced by $x_1 \geq 4$ then optimal solution will change.

16. Show that optimum solution to the following L.P.P. is unbounded.

$$\begin{aligned} \text{Maximize} \quad Z &= 20x_1 + x_2 + 10x_3 \\ \text{Subject to} \quad x_1 + 4x_2 - x_3 &\leq 20 \\ x_1 + x_2 &\leq 10 \\ 3x_1 + 5x_2 - 3x_3 &\leq 50 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

17. Set-up initial simplex table for the following L.P.P.

$$\begin{aligned} \text{Maximize} \quad Z &= 5x + 6y \\ \text{Subject to} \quad x + y &< 100 \\ 2x + 3y &= 170 \\ 4x + 5y &\geq 430 \\ x, y &\geq 0 \end{aligned}$$

18. Use simplex method and show that the L.P.P.

$$\begin{aligned} \text{Maximize} \quad Z &= 2x_1 + x_2 \\ \text{Subject to} \quad x_1 - x_2 - x_3 &\leq 1 \\ x_1 - 2x_2 + x_3 &\leq 2 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Does not possess finite optimum solution.

From the final simplex table construct a feasible solution with value of the objective function greater than 2000.

19. A small manufacturing company produces one-band pocket and two-band table radios. Each two-band model requires twice as much time as one one-band model. If the company were to produce only two-band models, it could manufacture 150 units per week. The company is licensed to produce in all not more than 250 units per week. The market survey has shown that no more than 100 pieces of two-band model per week could be sold. The company is also committed to supply at least 50 pieces of one-band model per week.

    If the net profit on the sale of one-hand model is ₹ 10 per piece and on the two-band model ₹ 15 per piece, how should the company plan its production to maximize the profit ?

    Formulate the L.P.P. and solve any using simplex method.

20. The manager of an agricultural farm of 80 hectares finds that for effective protection against insects, he should spray at least 15 units of chemical A and 20 units of chemical B per hectare. Three bands of insecticides are available in the market which contain these chemicals one brand contains 4 units of A and 8 units of B per kg and costs ₹ 5 per kg, the second brand contains 12 and 8 units respectively and costs ₹ 8 per kg and the third band contains 8 and 4 units respectively and costs ₹ 6 per kg. It is also learnt that more than 2.5 kg per hectare of insecticides will be harmful to the crops. Determine the quantity of each insecticide he should buy to minimize the total cost for the whole farm.

    Formulate L.P.P. and solve it by using simplex method.

21. Write the dual of

    Maximize $Z = x - y + 3z$

    Subject to
    $$x + y + z \leq 10$$
    $$2x - y - z \leq 2$$
    $$2x - 2y - 3z \leq 6$$
    $$x \geq 0, \ y \geq 0, \ z \geq 0$$

22. Solve the following L.P.P. graphically and show that it has unbounded solution. Write the dual problem. Show that the dual problem has no solution

    Maximize $Z = 3x + 4y$

    Subject to
    $$y - x \leq 1$$
    $$x + y \geq 4$$
    $$x - 3y \leq 3$$
    $$x \geq 0, \ y \geq 0$$

23. Solve the following problem by dual simplex method

    Minimize $Z = 2x + 3y$

    Subject to
    $$2x + 3y \leq 30$$
    $$x + 2y \geq 10$$
    $$x \geq 0, y \geq 0$$

Answers

(A) True/False Questions :

1. True 2. False 3. True 4. True

(B) Multiple Choice Questions :

1. (B) 2. (A) 3. (B) 4. (C)

(D) Numerical Problems :

1. The problem has unbounded solution. In second simplex table improvement is indicated but coefficients of entering variable are all negative.
2. $x_1 = 10,\ x_2 = 5$ (or $x_1 = 15,\ x_2 = 0$) with $Z = 15$.

   (The profit coefficient $Z_j - C_j$ of x_2 is zero even it is not member of basis, indicates alternate solution, first solution with two simplex tables.)
3. $x_1 = \frac{5}{2},\ x_2 = \frac{5}{2},\ x_3 = \frac{5}{2},\ x_4 = 0$ with $Z = 15$.

   (Big-M-technique, *four* simplex tables).
4. The problem has no feasible solution.

   (Three-simplex tables)
5. The problem has unbounded solution indicated in *third* simplex table.
6. $x_1 = 8,\ x_2 = 0$ with $Z = 48$.

   $x_1 = \frac{12}{5},\ x_2 = \frac{42}{5},\quad Z = 48$

   $x_1 = \frac{26}{5},\ x_2 = \frac{2}{5},\ Z = 48.$

   (Three simplex tables)
7. $x_1 = 60,\ x_2 = 0,\ x_3 = 20,\ x_4 = 8,\ Z = 312.$

   (four simplex tables)
8. $x_1 = 0,\ x_2 = 4,\ Z = 40.$

   $x_1 = \frac{15}{4},\ x_2 = \frac{5}{2},\quad Z = 40.$
9. $x_1 = 4, x_2 = 5,\ x_3 = 0$, with $Z = -11$.
10. $x_1 = 0,\ x_2 = \frac{50}{3},\ x_3 = \frac{50}{3},\quad Z = 200.$
11. $Z_{max} = \frac{110}{3}$.

12.

Table	1	2	3	4
Statement(s)	b, c	d	a	e

13. $x_1 = 0$, $x_2 = 1$, $x_3 = 2$; $Z_{max} = 5$.

14. $x_1 = 0$, $x_2 = 4$, $x_3 = 7/2$; $Z_{min} = 9$.

17.

			x_1	x_2	s_1	s_2	A_1	A_2
Objective	**Product mix**	**Solution**	5	6	0	0	$-m$	$-m$
0	s_1	100	1	1	1	0	0	0
$-m$	A_1	170	2	3	0	0	1	0
$-m$	A_2	430	4	5	0	-1	0	1
Z_j		$-600m$	$-6m$	$-8m$	0	m	$-m$	$-m$
$C_j - Z_j$			$5 + 6m$	$6 + 8m$	0	$-m$	0	0

18. $x_1 = 1$, $x_2 = 2000$, $x_3 = 0$.

19. $x_1 = 200$, $x_2 = 50$, $Z_{max} = 2750$.

20. $x_1 = \frac{15}{8}$, $x_2 = \frac{5}{8}$, $x_3 = 0$, $Z_{min} = 1150$.

21. Minimum $P = 10u + 2v + 6w$

Subject to
$$u + 2v + 2w \geq 1$$
$$u - v - 2w \geq -1$$
$$u - v - 3w \geq 3$$

23. $x = 0$, $y = 5$, $z = 15$.

☆☆☆

Chapter **3**...

Transportation Models

Daniel Henry Gottlieb

He received his Ph. D. in Mathematics from S.T. Hu in 1962 at UCLA. My specialty is Topology. His interests have broadened to Mathematical Physics and the History of Mathematics in recent years. His conviction that the topological invariants of the degree of a map and index of a vector field must describe basic physical ideas led me to Physics. I turned to the History of Mathematics in order to test my assertion that Mathematics is the study of Well-defined concepts. He also believe that mathematics are the guardians and exemplars of critical reasoning. Just as English Professors speak out against shoddy reasoning. Nowhere in our society is the lack of careful reasoning more prevalent than in the legal profession. Therefore he will include essays on Mathematics and the Law.

3.1 Introduction

As discussed in Chapter 1, the simplex algorithm can be regarded as the most generalized method for solving any linear programming problem. A transportation problem is a special type of linear programming problems which can be solved by simplex algorithm (procedure). But even a small transportation problems contains large number of variables and linear constraints (conditions). In such a cases, simplex algorithm becomes laborious and so this method is not preferred. However, a transportation problem has a special mathematical structure which permits it to be solved by convenient and fairly efficient procedure known as **transportation method.** Two contributions are mainly responsible for development of transportation problem, these are number of shipping sources (origins) and the number of destinations (jobs).

If there are more than one **sources** from where the goods need to be shipped to more than one **destinations** and the costs of shipping from each of the **origins** to each of the **destinations** are different and known, the problem is to ship the goods from different **origins** to different **destinations** in such a way that the cost of shipping (or transportation) is minimum.

Thus the transportation problem is to transport different amounts of a single homogeneous commodity, that are initially stored at different origins, to different destinations in such a way that the total transportation cost is minimum.

The transportation problem may be regarded as machine assignment, plant location, product mix problem and many others.

3.2 Terminology of Transportation Problem

1. Balanced Transportation Problem : A transportation problem is said to be *balanced* if the total supply from all the sources (origins) is equal to the total demand (requirement) in all the destinations.

2. Unbalanced Transportation Problem : A transportation problem is said to be *unbalanced* if the total supply from all the sources is not equal to the total demand in all the destinations.

3. Feasible Solution : A set of non-negative values x_{ij}, $i = 1, 2, ..., m$, $j = 1, 2, ..., n$ which satisfies the constraints in the transportation problem is called *feasible solution*.

4. Basic Feasible Solution : A feasible solution of m by n transportation problem is said to be a *basic feasible solution* if the total number of positive allocations x_{ij} is exactly equal to $m + n - 1$.

5. Optimal solution : A feasible solution (not necessarily basic) is said to be *optimal* if it minimizes the total transportation cost.

6. An Initial Feasible Solution : A solution that satisfies supply and demand conditions but it may or may not be optimal is called an *initial feasible solution*. There are three methods of finding initial feasible solutions. These methods are :

(i) North-West Corner Method,

(ii) Least Cost Entry Method (Matrix Minima Method),

(iii) Vogel's Approximation Method (VAM).

7. Degenerate Basic Feasible Solution : A basic feasible solution to a $m \times n$ transportation problems is said to be *degenerate* if the number of stone squares (occupied cells) are less than $(m + n - 1)$, where m represents supply and n represents destinations.

8. Non-degenerate Basic Feasible Solution : A basic feasible solution to a $m \times n$ transportation problems is said to be *non-degenerate* if the number of stone squares (occupied cells) are exactly $(m + n - 1)$, where m represents supply (origins) and n represents destinations.

9. Dummy Source/Destination : Unbalanced transportation problem can be converted into balanced transportation problem by adding extra row/column with zero cost in each of its cell called *dummy source/destination*.

3.2.1 Tabular Representation of T.P.

Suppose that the factories F_i $(i = 1, 2, ..., m)$ called the **origins** or **sources** produce the non-negative quantities a_i $(i = 1, 2, ..., m)$ of the product and the non-negative quantities b_j $(j = 1, 2, ... , n)$ of the same product are required at other n places, called the **destinations** such that the total quantity produced is equal to the total quantity required.

i.e. $$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

Suppose, c_{ij} = The cost of the transportation of an unit from i^{th} source to the j^{th} destination.

and x_{ij} = The quantity transported from the i^{th} source to the j^{th} destination.

The transportation problem as discussed above can be represented in tabular form as under.

Destination (Warehouse) → / Sources (Factory) ↓	W_1	W_2	...	W_j	...	W_n	Factory capacities (Supply) ↓
F_1	c_{11}	c_{12}	...	c_{1j}	...	c_{1n}	a_1
F_2	c_{21}	c_{22}	...	c_{2j}	...	c_{2n}	a_2
...	...	...	...	...	...	...	...
F_i	c_{i1}	c_{i2}	...	c_{ij}	...	c_{in}	a_i
...	...	...	...	...	...	...	...
F_m	c_{m1}	c_{m2}	...	c_{mj}	...	c_{mn}	a_m
Warehouse (Demand) requirements →	b_1	b_2	...	b_j	...	b_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

Destination (Warehouse) → / Sources (Factory) ↓	W_1	W_2	...	W_j	...	W_n	Factory capacities (Supply) ↓
F_1	x_{11}	x_{12}	...	x_{1j}	...	x_{1n}	a_1
F_2	x_{21}	x_{22}	...	x_{2j}	...	x_{2n}	a_2
...	...	...	...	...	...	...	...
F_i	x_{i1}	x_{i2}	...	x_{ij}	...	x_{in}	a_i
...	...	...	...	...	...	...	...
F_m	x_{m1}	x_{m2}	...	x_{mj}	...	x_{mn}	a_m
Warehouse (Demand) requirements →	b_1	b_2	...	b_j	...	b_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

In general, the above two tables are combined by inserting each unit cost c_{ij} together with the corresponding amount x_{ij} into the cell (i, j). The product (x_{ij}) (c_{ij}) gives us the net cost of shipping x_{ij} units from factory F_i to warehouse W_j. In order to avoid the confusion of entries in the cell, the quantities in the square bracket $\square$ will denote the unit cost c_{ij}. It may be written at the upper corners of the cell to the right or to the left.

Destination (Warehouse) → / Sources (Factory) ↓	W_1	W_2	...	W_j	...	W_n	Factory capacities (Supply) ↓
F_1	c_{11} x_{11}	c_{12} x_{12}	...	c_{1j} x_{1j}	...	c_{1n} x_{1n}	a_1
F_2	c_{21} x_{21}	c_{22} x_{22}	...	c_{2j} x_{2j}	...	c_{2n} x_{2n}	a_2
...	...	...	...	...	...	...	...
F_i	c_{i1} x_{i1}	c_{i2} x_{i2}	...	c_{ij} x_{ij}	...	c_{in} x_{in}	a_i
...	...	...	...	...	...	...	...
F_m	c_{m1} x_{m1}	c_{m2} x_{m2}	...	c_{mj} x_{mj}	...	c_{mn} x_{mn}	a_m
Warehouse (Demand) requirements →	b_1	b_2	...	b_j	...	b_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

The total cost of transportation of x_{ij} units from F_i to W_j is

$c_{11}\, x_{11} + c_{12}\, x_{12} + ... + c_{1n}\, x_{1n} + c_{21}\, x_{21} + ... + c_{mn}\, x_{mn}$

$$\Rightarrow \quad \text{Total cost} = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x_{ij}$$

Our struggle is to determine the value x_{ij} in such a way that the total transportation cost is minimized.

i.e. Total transportation cost $= \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x_{ij}$ is minimized.

3.2.2 Mathematical Formulation of T.P.

Mathematically the transportation problem can be stated as follows :

Find x_{ij} ($i = 1, 2, ..., m$; $j = 1, 2, ..., n$) in order to

$$\text{minimize } z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij}\, x_{ij} \qquad \text{(Objective function)}$$

Subject to

(i) the supply constraints :

$$\sum_{j=1}^{n} x_{ij} = a_i \quad ; \quad \forall\ i = 1, 2, ..., m$$

and (ii) the demand constraints :

$$\sum_{i=1}^{m} x_{ij} = b_j \quad ; \quad \forall\ j = 1, 2, ..., n$$

It is observed that the objective function and the constraints (i) and (ii) are all linear in x_{ij}, so it is just equivalent to linear programming problems.

This special type of L.P.P. will be called **transportation problem (T.P.).**

Note : (i) For balanced T.P. : $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$ [See Definition (1)].

(ii) For unbalanced T.P. : $\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$ [See Definition (2)].

3.2.3 Solution of a Transportation Problem

The solution to a transportation problem basically consists of following steps :

Step 1 : Define the **objective function** to be minimized with the constraints imposed on the problem.

Step 2 : Set-up a transportation table with m rows and n columns.

Step 3 : Find an initial feasible solution (I.F.S.) ($x_{ij} \geq 0$) from the data.

Step 4 : Test whether the initial feasible solution (I.F.S.) is a basic feasible solution (B.F.S.). A feasible solution is basic if the number of positive allocations is one less than the sum of origins and destinations i.e. $m + n - 1$. The cell having allocations is known as occupied cell (stone-square). The cell which do not having allocations is known as unoccupied cell (water-square).

Step 5 : Test the solution for optimality.

Step 6 : If the solution is not optimal, get an improved transportation schedule.

Step 7 : Repeat the steps 5 and 6 till the solution becomes optimal (i.e. no further improvement is possible in the solution.)

3.2.4 Initial Feasible Solution

There are number of methods available for obtaining an initial feasible solution for the transportation problem. We shall discuss here only the following three methods :

1. North-West Corner Method (Upper left corner method),
2. Matrix-Minima Method (Lowest cost entry method),
3. Vogel's Approximation Method.

1. North-West Corner Method (NWCM) : (April 1999, Oct. 1999)

Various steps for finding the initial feasible solution of this method can be summarized as under :

Step 1 : Construct an empty $m \times n$ matrix (where, m is the origin and n is the destination) with c_{ij}, a_i, b_j.

Step 2 : Start with the North-West corner cell i.e. (1, 1) cell. Allocate the x_{11} quantity as much as possible in this cell. Note that $x_{11} = \min(a_1, b_1)$. Enclose the allocated quantity x_{11} in a circle with (1, 1) cell.

Step 3 : Next allocate the quantity x_{12} in the cell (1, 2) and so on, till the first row total is completely allocated.

Step 4 : Next allocate the quantity x_{22} in the cell (2, 2) and subsequently in the remaining cells of the second row till the second row total is completely allocated.

Step 5 : Continue such allocation in subsequent rows till all rows and columns total are completely allocated. The solution so obtained is I.F.S.

Step 6 : Evaluate $z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$

Illustrative Examples

Example 3.1 : Find an initial basic feasible solution of the following transportation problem.

Destination → / Source ↓	W_1	W_2	W_3	W_4	Capacity
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Demand	5	8	7	14	34

Solution :

Destination → / Source ↓	W_1	W_2	W_3	W_4	Capacity
F_1	19 5	30 2	50	10	7
F_2	70	30 6	40 3	60	9
F_3	40	8	70 4	20 14	18
Demand (b_j)	5	8	7	14	34

Explanation : We start with cell (1, 1). Here capacity is 7 units and demand is 5 units, so we allocate 5 units. This allocation leaves the surplus capacity of 2 units for row 1, so we allocate 2 units to cell (1, 2). Now, allocations of first row and first column are complete, but there is a deficiency of 6 units in column 2. So we allocate 6 units in the cell (2, 2). Column 1 and column 2 requirements are satisfied, leaving a surplus capacity of 3 units for row 2. So we allocate 3 units in the cell (2, 3) and column 3 still requires 4 units. So we allocate 4 units in the cell (3, 3). Column 3 and column 4 requirements are satisfied leaving a surplus capacity of 14 units for row 3. So we allocate 14 units in the cell (3, 4).

Since all capacity and demand are satisfied, the initial solution is obtained.

Here number of allocations = 6 = m + n – 1 = 3 + 4 – 1

⇒ Initial feasible solution (IFS) is a basic feasible solution (BFS).

∴ Total transportation cost = 5 (19) + 2 (30) + 6 (30) + 3 (40) + 4 (70) + 14 (20) = ₹ 1015

Example 3.2 : Find an initial feasible solution for the following transportation problem.

Destination → / Origin ↓	D_1	D_2	D_3	D_4	Availability
O_1	23	27	16	18	30
O_2	12	17	20	51	40
O_3	22	28	12	32	53
Requirement	22	35	25	41	123

Solution :

Destination → / Origin ↓	D_1	D_2	D_3	D_4	a_i
O_1	23 22	27 8	16	18	30
O_2	12	17 27	20 13	51	40
O_3	22	28	12 12	32 41	53
b_j	22	35	25	41	123

Explanation : We start with cell (1, 1). Here availability is 30 units and requirement is 22 units. So we allocate 22 units. This allocation leaves the surplus amounts of 8 units for row I, so we allocate 8 units to cell (1, 2). Now allocation of first row and first column is complete, but there is a deficiency of 27 units in column 2. So, we allocate 27 units in the cell (2, 2). Column I and column II requirements are satisfied leaving surplus amount of 3 units for row II. So we allocate 13 units in the cell (2, 3) and column III still requires 12 units. So we allocate 12 units in the cell (3, 3). Column III and column IV requirements are satisfied leaving a surplus amount of 41 units for row III. So we allocate 41 units in the cell (3, 4).

Since all availability and requirement are satisfied, we can get an initial feasible solution.

It should be noted that, here the total number of allocations = x_{ij} = 6 = m + n – 1 = 3 + 4 – 1 = 6, so the initial feasible solution (I.F.S.) is also a basic feasible solution (B.F.S.).

∴ Total transportation cost

= 22 (23) + 8 (27) + 27 (17) + 13 (20) + 12 (12) + 41 (32) = ₹ 2897

Example 3.3 : Find a feasible solution of the following T. P.

D / O	D_1	D_2	D_3	D_4	Availability
O_1	1	2	1	4	30
O_2	3	3	2	1	50
O_3	4	2	5	9	20
Requirement	20	40	30	10	100

Solution :

D / O	D_1	D_2	D_3	D_4	a_i
O_1	1 20	2 10	1	4	30
O_2	3	3 30	2 20	1	50
O_3	4	2	5 10	9 10	20
b_j	20	40	30	10	100

Explanation : We start with cell (1, 1). Here, availability is 30 and requirement is 20. So we allocate 20 units. This allocation leaves surplus amount of 10 units for row I. So we allocate 10 units in the cell (1, 2). Now, allocation of row I and column I is complete, but there is a deficiency of 30 units in column II. So we allocate 30 units in the cell (2, 2). Column I and column II are satisfied, leaving a surplus amount of 20 units for row II, so we allocate 20 units in the cell (2, 3) and column III still requires 10 units. So we allocate 10 units in the cell (3, 3). Column III and column IV are satisfied leaving a surplus amount of 10 units for row III. So we allocate 10 units in the cell (3, 4).

Since all availability and requirement are satisfied, we can get an initial feasible solution. Furthermore,

$$\text{No. of allocations} = 6$$

and

$$m + n - 1 = 6$$

So, the initial feasible solution (I.F.S.) is a basic feasible solution (B.F.S.).

$\therefore$ Total transportation cost

$= 20 (1) + 10 (2) + 30 (3) + 20 (2) + 5 (10) + 9 (10) =$ ₹ 310

Example 3.4 : Find an initial feasible solution of the following transportation problem.

Destination → / Origin ↓	D_1	D_2	D_3	D_4	Supply
O_1	1	2	3	4	6
O_2	4	3	2	0	8
O_3	0	2	2	1	10
Demand	4	6	8	6	24

Solution :

Destination → / Origin ↓	D_1	D_2	D_3	D_4	a_i
O_1	1 (4)	2 (2)	3	4	6
O_2	4	3 (4)	2 (4)	0	8
O_3	0	2	2 (4)	1 (6)	10
b_j	4	6	8	6	24

Explanation : We start with cell (1, 1). Here supply is 6 and demand is 4, so we allocate 4 units. This allocation leaves surplus amount of 2 units for row I. So we allocate 2 units in the cell (1, 2). Now, allocation of row I and column I is complete, but there is a deficiency of 4 units in column II. So, we allocate 4 units in the cell (2, 2). Column I and column II are satisfied, leaving a surplus amount of 4 units for row II. So, we allocate 4 units in the cell (2, 3) and column III still requires 4 units. So we allocate 4 units in the cell (3, 3). Column III and column IV are satisfied, leaving surplus amount of 6 units for row III. So we allocate 6 units in the cell (3, 4).

Since all supply and demand are satisfied, we can get an initial feasible solution.

Furthermore,

$$\text{No. of allocations} = 6 \quad \text{and} \quad m + n - 1 = 6$$

So, the initial feasible solution (I.F.S.) is a basic feasible solution (B.F.S.).

$\therefore$ Total transportation cost $= 4(1) + 2(2) + 4(3) + 4(2) + 4(2) + 6(1) =$ ₹ 42

Example 3.5 : Obtain initial basic feasible solution of the following transportation problem.

	D_1	D_2	D_3	Supply
O_1	13	15	16	17
O_2	7	11	2	12
O_3	19	20	9	16
Demand	14	8	23	45

Solution :

Destination → / Origin ↓	D_1	D_2	D_3	a_i
O_1	13 (14)	15 (3)	16	17
O_2	7	11 (5)	2 (7)	12
O_3	19	20	9 (16)	16
b_j	14	8	23	45

Explanation : We start with cell (1, 1). Here supply is 17 and demand is 14. So, we allocate 14 units. This allocation leaves surplus amount of 3 units for row I. So we allocate 3 units in the cell (1, 2). Now, allocation of row I and column I is complete, but there is a deficiency of 5 units in column II. So we allocate 5 units in the cell (2, 2). Column I and column II are satisfied, leaving a surplus amount of 7 units for row II. So we allocate 7 units in the cell (2, 3). Column III still requires 16 units. So we allocate 16 units in the cell (3, 3).

Since, all supply and demand are satisfied, we get an initial feasible solution.

Furthermore,

$$\text{No. of allocations} = 5$$

and $$m + n - 1 = 6 - 1 = 5$$

So, the initial feasible solution (I.F.S.) is an initial basic feasible solution (I.B.F.S.).

$\therefore$ Total transportation cost

$$= 14\,(13) + 3\,(15) + 5\,(11) + 7\,(2) + 16\,(9) = ₹\,440$$

Example 3.6 : Find an initial basic feasible solution of the following transportation problem.

To → / From ↓	W_1	W_2	W_3	W_4	Supply
F_1	30	25	40	20	100
F_2	29	26	35	40	250
F_3	31	33	37	30	150
Demand	90	160	200	50	500

Solution :

To → / From ↓	W_1	W_2	W_3	W_4	Supply
F_1	30 (90)	25 (10)	40	20	100
F_2	29	26 (150)	35 (100)	40	250
F_3	31	33	37 (100)	30 (50)	150
Demand	90	160	200	50	500

Explanation : We start with cell (1, 1). Here supply is 100 and demand is 90. So, we allocate 90 units. This allocation leaves surplus amount of 10 units for row I. So, we allocate 10 units in the cell (1, 2). Now, allocation of row I and column I is complete, but there is a deficiency of 150 units in column II. So, we allocate 150 units in the cell (2, 2). Column I and column II are satisfied, leaving surplus amount of 100 units for row II. So, we allocate 100 units in the cell (2, 3). Column III still requires 100 units. So we allocate 100 units in the cell (3, 3). Column III and column IV are satisfied leaving surplus amount of 50 units for row III. So we allocate 50 units in the cell (3, 4).

Now, all demand and supply are satisfied, so we can get initial feasible solution.

Furthermore, number of allocations $= 6$

and $m + n - 1 = 3 + 4 - 1 = 6$

So, the initial feasible solution is the initial basic feasible solution.

∴ Total transportation cost

$= 90\,(30) + 10\,(25) + 150\,(26) + 100\,(35) + 100\,(37) + 50\,(30)$

$= 12040$

2. Matrix-Minima Method (Lowest Cost Entry Method) :

Various steps for obtaining an initial basic feasible solution to a transportation problem can be summarized as follows :

Step 1 : Construct an empty $m \times n$ matrix with c_{ij}, a_i, b_j.

Step 2 : Find the cell with the lowest cost among all rows and columns. Allocate the maximum possible quantity in this cell, keeping in mind the corresponding row, column totals.

Step 3 : Next, allocate the maximum possible quantity in the next lowest cost cell, again keeping in mind the corresponding row, column totals.

Step 4 : Repeat the above steps 2 and 3 till the total rows, columns requirements have been satisfied completely. The solution so obtained is I.F.S.

Step 5 : Evaluate the total cost of transportation.

$$\text{i.e.} \qquad z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

Illustrative Examples

Example 3.7 : Find an initial basic feasible solution of the following transportation problem.

Destination → / Source ↓	D_1	D_2	D_3	D_4	Capacity
S_1	19	30	50	10	7
S_2	70	30	40	60	9
S_3	40	8	70	20	18
Demand	5	8	7	14	34

Solution :

Destination → / Source ↓	D_1	D_2	D_3	D_4	a_i
S_1	19	30	50	10 7	7
S_2	70 2	30	40 7	60	9
S_3	40 3	8 8	70	20 7	18
b_j	5	8	7	14	34

Explanation : Here lowest cost is 8.

First allocation is made in the cell (3, 1) = min (18, 8) = 8.

This satisfies the demand of D_2 and so we cross-off the column II.

Second allocation is made in the cell (1, 4) = min (14, 7) = 7.

This satisfies the capacity of S_1 and so we cross-off the row I.

Third allocation is made in the cell (3, 4) = min (14 – 7, 7) = 7.

This satisfies the demand of D_4 and so we cross-off the column IV.

Fourth allocation is made in the cell (2, 3) = min (9, 7) = 7.

This satisfies the demand of D_3 and so we cross-off the column III.

Fifth allocation is made in the cell (3,1) = min (18 – 15, 5) = min (3, 5) = 3.

This satisfies the capacity of S_3 and so we cross-off the row III.

Sixth and last allocation is made in the cell (2, 1) = min (9 – 7, 5 – 3) = 2.

This satisfies the demand of D_1 as well as capacity of S_2.

Here all capacity and demand satisfied, therefore, we get an initial feasible solution.

Furthermore, therefore the total number of allocations = 6

and $\quad m + n - 1 = 3 + 4 - 1 = 6$

$\Rightarrow$ An initial feasible solution is also basic feasible solution.

$\therefore$ The transportation cost

$= 7 (10) + 2 (70) + 7 (40) + 3 (40) + 8 (8) + 7 (20)$

$=$ ₹ 814

This cost is less by (1015 – 814 = 201) as compared to the cost obtained by north-west cost method.

Example 3.8 : Find an initial basic feasible solution of the following T. P.

Destination → / **Origin ↓**	D_1	D_2	D_3	D_4	**Availability**
O_1	23	27	16	18	30
O_2	12	17	20	51	40
O_3	22	28	12	32	53
Requirement	22	35	25	41	123

Solution :

Destination → / **Origin ↓**	D_1	D_2	D_3	D_4	a_i
O_1	23	27	16	18 (30)	30
O_2	12 (22)	17 (18)	20	51	40
O_3	22	28 (17)	12 (25)	32 (11)	53
b_j	22	35	25	41	123

Explanation :

The lowest cost is 12 which lies in the cell (2, 1) and (3, 3). Therefore,

First allocation is made in the cell (2, 1) = min (40, 22) = 22. This satisfies requirement of D_1 and so we cross-off the column I.

Second allocation is made in the cell (3, 3) = min (25, 53) = 25. This satisfies requirement of D_3 and so we cross-off the column IV.

Third allocation is made in the cell (1, 3). But the requirement in D_3 is already over.

Fourth allocation is made in the cell (2, 2) = min (35, 18) = 18.

Fifth allocation is made in the cell (1, 4) = min (41, 30) = 30.

Sixth allocation is made in the cell (3, 2) = min (17, 28) = 17.

Now, the last allocation is made in the cell (3, 4) = min (11, 11) = 11. Thus we get the initial basic feasible solution as given in the above table.

The transportation cost according to the above route is

$$z = (30 \times 18) + (22 \times 12) + (18 \times 27) + (17 \times 28) + (25 \times 12) + (11 \times 32)$$
$$= ₹\ 2238$$

The initial feasible solution is also a basic feasible solution as m + n – 1 = 6 = 3 + 4 – 1.

The cost of T.P. is less as compared to the north cost method by ₹ 659.

Example 3.9 : Find an initial basic feasible solution of the following transportation problem.

Destination → / **Origin ↓**	**D_1**	**D_2**	**D_3**	**D_4**	**Supply**
O_1	11	13	17	14	250
O_2	16	18	14	10	300
O_3	21	24	13	10	400
Demand	200	225	275	250	

Solution :

Destination → / **Source ↓**	**D_1**	**D_2**	**D_3**	**D_4**	**a_i**
O_1	11 200	13 50	17	14	250
O_2	16	18 175	14 125	10	300
O_3	21	24	13 150	10 250	400
b_j	200	225	275	250	

Explanation :

The lowest cost is 10 which lies in the cell (2, 4) and (3, 4).

First allocation is made in the cell (3, 4) = min (400, 250) = 250. This satisfies demand of D_4 and we cross-off column IV.

Second allocation is made in the cell (1, 1) = min (250, 200) = 200. This satisfies the demand of D_1 and so we cross-off the column I.

Third allocation is made in the cell (1, 2) = min (250 – 200, 225) = min (50, 225) = 50.

Fourth allocation is made in the cell (3, 3) = min (400 – 250, 275) = min (150, 275) = 150. This satisfies the supply of O_3 and so we cross-off row III.

Fifth allocation is made in the cell (2, 3) = min (300, 275 – 150) = min (300, 125) = 125.

Sixth allocation obviously made at (2, 2) = min (175, 175) = 175.

Rows and columns supply and demand have been satisfied completely and we get an initial feasible solution.

Also, number of allocations = 6

and $m + n - 1 = 3 + 4 - 1 = 6$

So, the initial feasible solution is also the basic feasible solution.

$\therefore$ Total transportation cost

$z = (200 \times 11) + (50 \times 13) + (175 \times 18) + (125 \times 14) + (150 \times 13) + (250 \times 10)$

$= ₹\ 12{,}200$

Example 3.10 : Obtain the basic feasible solution of the following transportation problem.

Destination → / **Source ↓**	**D_1**	**D_2**	**D_3**	**D_4**	**Supply**
S_1	10	0	20	11	15
S_2	12	7	9	20	25
S_3	0	14	16	18	5
Demand	5	15	15	10	45

Solution :

Destination → / **Source ↓**	**D_1**	**D_2**	**D_3**	**D_4**	**a_i**
S_1	10 0	0 15	20	11 0	15
S_2	12	7	9 15	20 10	25
S_3	0 5	14	16	18	5
b_j	5	15	15	10	

Explanation : Lowest cost is 10 which lies in the cell (3, 1) and (1, 2).

First allocation is made in the cell (3, 1) = min (5, 5) = 5. This satisfies the demand of D_1 and supply of S_3. So we cross-off column I and row III.

Second allocation is made in the cell (1, 2) = min (15, 15) = 15 which also satisfies the demand of D_2 and supply S_1. Therefore, we cross-off column II and row I.

Third allocation is made in the cell (2, 3) = min (25, 15) = 15.

Fourth allocation is made in the cell (2, 4) = min (10, 25) = 10. Rows and columns demand and supply are satisfied, and we get the initial feasible solution (also initial basic feasible solution).

∴ Total transportation cost

= 0 (10) + 15 (0) + 15 (9) + 10 (20) + 5 (0) + 0 (11)

= 135 + 200 = ₹ 335

Example 3.11 : Find the initial feasible solution of the following T.P.

Destination → / Origin ↓	D_1	D_2	D_3	Supply
O_1	13	15	16	17
O_2	7	11	2	12
O_3	19	20	9	16
Demand	14	8	23	

Solution :

Destination → / Origin ↓	D_1	D_2	D_3	Supply
O_1	13 (14)	15 (3)	16	17
O_2	7	11 (0)	2 (12)	12
O_3	19	20 (5)	9 (11)	16
Demand	14	8	23	

Explanation : Here lowest cost is 2, which lies in the cell (2, 3).

First allocation is made in the cell (2, 3) = min (12, 23) = 12. This satisfies supply of O_2, so we cross-off row II.

Second allocation is made in the cell (3, 3) = min (23 – 12, 16) = min (11, 16) = 11.

Third allocation is made in the cell (2, 2) = min (0, 8) = 0.

Fourth allocation is made in the cell (1, 1) = min (14, 17) = 14.

Fifth allocation is made in the cell (1, 2) = min (17 – 14, 8) = 3.

Sixth allocation is made in the cell (3, 2) = min (16 – 11, 8) = 5.

Rows and columns demand and supply are satisfied. So we get an initial basic feasible solution.

∴ Total transportation cost

$= (14 \times 13) + (3 \times 15) + (0 \times 11) + (12 \times 2) + (5 \times 20) + (11 \times 9) = ₹\ 450$

Example 3.12 : Solve the following transportation problem.

Destination → / Origin ↓	D_1	D_2	D_3	D_4	Supply
O_1	1	2	3	4	6
O_2	4	3	2	0	8
O_3	0	2	2	1	10
Demand	4	6	8	6	24

Solution :

Destination → / Origin ↓	D_1	D_2	D_3	D_4	Supply
O_1	1	2 (6)	3	4	6
O_2	4	3	2 (2)	0 (6)	8
O_3	0 (4)	2 (0)	2 (6)	1	10
Demand	4	6	8	6	24

Explanation : Here lowest cost is 0, which lies in the cell (2, 4) and (3, 1).

First allocation is made in the cell (2, 3) = min (6, 8) = 6.

Second allocation is made in the cell (3, 1) = min (4, 6) = 4. First column is exhausted.

Third allocation is made in the cell (1, 2) = min (6, 6) = 6. First column is exhausted.

Fourth allocation is made in the cell (2, 3) = min (8 – 6, 8) = (2, 8) = 2. Second row is exhausted.

Fifth allocation is made in the cell (2, 3) = min (6 – 6, 10) = 0.

Sixth allocation is made in the cell (3, 3) = min (10 – 4, 8) = 6.

Rows and columns demand and supply are satisfied and we get an initial basic feasible solution.

∴ Total transportation cost

$= (6 \times 2) + (2 \times 2) + (6 \times 0) + (4 \times 0) + (0 \times 2) + (6 \times 2)$

$= ₹\ 28/-$

(It is initial basic feasible solution because total allocation = 3 + 4 – 1 = 6 = m + n – 1)

3. Vogel's Approximation Method (VAM) :

(Unit Cost Penalty Method) :

VAM is not quite as simple as the North-West-Corner method and Matrix-Minima method for an initial assignment but it gives a much better initial feasible solution and sometimes the first solution turns out to be an optimal assignment. This is due to the fact that VAM finds a good initial assignment by taking into consideration the costs associated with each alternative route and selecting the best route, which was not done in the previous methods.

This method usually provides a better starting solution than the above two methods (N. W.C.M. and L.C.E.M.). In fact, most of the time, VAM yields an optimum or close to optimum, starting (initial) solution.

Various steps obtaining the initial feasible solution to a transportation problem can be summarized as below.

Step 1 : Determine the difference between the smallest cost and the second smallest cost, for each row and for each column. These differences are called **penalties**. For row put these differences along the side of the transportation table against the respective rows. For column put these differences below the corresponding columns. It should be noted that the difference may be zero.

Step 2 : Identify the row or column with the maximum difference among all the rows and columns. If the maximum difference occurs in the row then we mark $\leftarrow$. If the maximum difference occurs in the column then we mark $\uparrow$. Choose the cell which has the smallest cost and allocate the maximum possible quantity to the lowest cost cell in that row or column so as to exhaust the supply (capacity, availability) at a particular destination.

If a tie occurs in the differences select that row or column which has maximum cost. If there is a tie in the minimum cost also, select that rows or columns which has maximum possible assignments. This reduces the computational work.

Step 3 : Reduce the row supply (capacity, availability) or the column demand (requirement) by the amount assigned to the cell.

Step 4 : If the supply (capacity, availability) satisfy, cross-out that row. If the demand (requirement) satisfy, cross-out that column. If both satisfy, cross-out both row and column.

Step 5 : Repeat step 1, step 2, step 3, and step 4 for the reduced transportation table until all capacity (supply, availability) and demand (requirement) satisfied.

Step 6 : Now, all these allocations (or assignments) that we have made in the above tables can now be combined into a single transportation table. From this single transportation table, we can obtain the initial feasible solution (I.F.S.) by evaluating

$$z = \sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij}\, x_{ij}$$

Illustrative Examples

Example 3.13 : Find the initial basic feasible solution of the following transportation problem.

Warehouse → Factory ↓	W_1	W_2	W_3	W_4	Capacity
F_1	19	30	50	10	7
F_2	70	30	40	60	9
F_3	40	8	70	20	18
Requirement	5	8	7	14	34

Solution :

Warehouse → Factory ↓	W_1	W_2	W_3	W_4	Capacity	Difference
F_1	19	30	50	10	7	9
F_2	70	30	40	60	9	10
F_3	40	8 8	70	20	18 10	12
Requirement	5	8	7	14	34	
Difference	21	22 ↑	10	10		

Maximum difference (22) lies in the column W_2 with minimum cost 8. Here, we allocate 8 units in the cell (3, 2). Here requirement is satisfied. So we cross-out column W_2. The reduced T.P. is given below.

Warehouse → Factory ↓	W_1	W_3	W_4	Capacity	Difference
F_1	19 5	50	10	~~7~~ 2	9
F_2	70	40	60	9	20
F_3	40	70	20	10	20
Requirement	5	7	14		
Difference	21 ↑	10	10		

Maximum difference (21) lies in the column W_1 with minimum cost 5. Here, we allocate 5 units in the cell (1, 1). Here requirement is satisfied. So we cross-out column W_1. The reduced T. P. is given below.

Warehouse → Factory ↓	W_3	W_4	Capacity	Difference
F_1	50	10	2	40
F_2	40	60	9	20
F_3	70	20 10	~~10~~ 0	50 ←
Requirement	7	~~7~~4		
Difference	10	10		

Maximum difference (50) lies in the row W_3 with minimum cost 20. Here, we allocate 10 units in the cell (3, 2). Here capacity is satisfied. So we cross out row F_3. The reduced T. P. is

Warehouse → Factory ↓	W_3	W_4	Capacity	Difference
F_1	50	10 2	~~2~~ 0	40
F_2	40	60	9	20
Requirement	7	~~4~~ 2		
Difference	10	50 ↑		

Maximum difference (50) lies in the column W_4 with minimum cost 10. Here we allocate 2 units in the cell (1, 2). Here capacity is satisfied, so we cross out F_1. The reduced T.P. is given below.

Warehouse → Factory ↓	W_3	W_4	Capacity	Difference
F_2	40 7	60 2	9	20 ←
Requirement	7	2	9	

Here maximum difference (20) lies in the row F_2 with minimum cost 40. Hence we allocate 2 units in the cell (1, 2) and 7 units in the cell (1, 1).

Now, all requirement and capacity have been satisfied.

All the above allocations that we have done is combined in the following single transportation table.

Warehouse → / Factory ↓	W_1	W_2	W_3	W_4	Capacity
F_1	19 (5)	30	50	10 (2)	7
F_2	70	30	40 (7)	60 (2)	9
F_3	40	8 (8)	70	20 (10)	18
Requirement	5	8	7	14	34

Total transportation cost

$= (5 \times 19) + (2 \times 10) + (7 \times 40) + (2 \times 60) + (8 \times 8) + (10 \times 20)$

$=$ ₹ 779

which is less as compared to the cost obtained by the previous two methods.

Thus,

(i) By **North-West Corner method :** Total cost = ₹ 1015/-

(ii) By **Matrix-Minima method :** Total cost = ₹ 814/-

(iii) By **Vogel's Approximation method :** Total cost = ₹ 779/-

If we compare the total transportation cost, we find that Vogel's approximation method often gives the better initial feasible solution to start with. This total transportation is very close to the optimal solution.

Example 3.14 : Find an initial feasible solution by VAM rule for the following transportation problem.

Destination → / Origin ↓	D_1	D_2	D_3	D_4	Availability
O_1	23	27	16	18	30
O_2	12	17	20	51	40
O_3	22	28	12	32	53
Requirement	22	35	25	41	123

Solution :

Destination → / Origin ↓	D_1	D_2	D_3	D_4	Availability	Difference
O_1	23	27	16	18 30	~~30~~ 0	2
O_2	12	17	20	51	40	5
O_3	22	28	12	32	53	10
Requirement	22	35	25	~~41~~ 11		
Difference	10	10	4	14		

↑

Maximum difference 14 lies in the column D_4 with minimum cost 18. Here, we allocate min (30, 41) = 30 units in the cell (1, 4). Here availability is satisfied. So we cross out row F_1. The reduced T.P. is given below.

Destination → / Origin ↓	D_1	D_2	D_3	D_4	Availability	Difference
O_2	12 1	17	20	51	40	5
O_3	22	28	12	32 11	~~53~~ 42	10
Requirement	22	35	25	~~11~~ 0		
Difference	10	11	8	19		

↑

Maximum difference 19 lies in the column D_4 with minimum cost 32. Here, we allocate min (11, 53) = 11 units in the cell (2, 4). Here requirement is satisfied, so we cross-out D_4. The reduced T.P. is given below.

Destination → / Origin ↓	D_1	D_2	D_3	Availability	Difference
O_2	12	17 (35)	20	~~40~~ 5	5
O_3	22	28	12	42	10
Requirement	22	~~35~~ 0	25		
Difference	10	11	8		

Maximum difference 11 lies in the column D_2 with minimum cost 17. Here, we allocate min (40, 35) = 35 units in the cell (1, 2). Here requirement is satisfied. So we cross-out D_2. The reduced T.P. is given below.

Destination → / Origin ↓	D_1	D_3	Availability	Difference
O_2	12	20	5	8
O_3	22	12 (25)	~~42~~ 17	10
Requirement	22	25		
Difference	10	8		

Here tie occurs. So for this, see step 2 in the procedure.

Maximum difference 10 lies in the row O_3 with minimum cost 12. Here we allocate min (42, 25) = 25 units in the cell (2, 2). Now requirement is satisfied, so cross-out D_3. The reduced T.P. is given below.

Destination → / Origin ↓	D_1	Availability
O_2	12 (5)	5
O_3	22 (17)	17
Requirement	22	
Difference	10	

Here the maximum difference (10) lies in column D_1 with minimum cost 12. Hence, we allocate 5 units in cell (1, 1) and 17 units in the cell (2, 1).

Now, all requirement and availability have been satisfied.

All the above allocations that we have made is combined in the following single transportation table as under

Destination → / Origin ↓	D_1	D_2	D_3	D_4	Availability
O_1	23	27	16	18 (30)	30
O_2	12 (5)	17 (35)	20	51	40
O_3	22 (17)	28	12 (25)	32 (11)	53
Requirement	22	35	25	41	

Total transportation cost

$= (30 \times 18) + (5 \times 12) + (35 \times 17) + (17 \times 22) + (25 \times 12) + (11 \times 32) = ₹\ 2221$

This cost is less as compared to the cost obtained by the previous methods.

Thus,

(i) By **North-West Corner method :** Total cost = ₹ 2897

(ii) By **Matrix-Minima method :** Total cost = ₹ 2238

(iii) By **Vogel's Approximation method :** Total cost = ₹ 2221/-

Total cost obtained by VAM is very close to the optimal solution.

Example 3.15 : Find an initial basic feasible solution by VAM (Vogel's Approximation method).

To → / From ↓	W_1	W_2	W_3	W_4	Supply
F_1	30	25	40	20	100
F_2	29	26	35	40	250
F_3	31	33	37	30	150
Demand	90	160	200	50	500

Solution :

T → / F ↓	W_1	W_2	W_3	W_4	Supply	Difference
F_1	30	25	40	20 (50)	~~100~~ 50	5
F_2	29	26	35	40	250	3
F_3	31	33	37	30	150	1
Demand	90	160	200	~~50~~ 0		
Difference	1	1	2	10		

Maximum difference 10 lies in the column W_4 with minimum cost 20. We allocate min (100, 50) = 50 units in the cell (1, 4). Demand is satisfied, so we cross-out column W_4. The reduced T.P. is given below :

T → / F ↓	W_1	W_2	W_3	Supply	Difference
F_1	30	25 (50)	40	50	5
F_2	29	26	35	250	3
F_3	31	33	37	150	2
Demand	90	~~160~~ 110	200		
Difference	1	1	2		

Maximum difference 5 lies in the row F_1 with minimum cost 25. We allocate min (160, 50) = 50 units in the cell (1, 2). Supply is satisfied, so we cross-off row F_1. The reduced T.P. is given below :

T → / F ↓	W_1	W_2	W_3	Supply	Difference
F_2	29	26 (110)	35	~~250~~ 140	3
F_3	31	33	37	150	2
Demand	90	~~110~~ 0	200		
Difference	2	7	2		

Maximum difference 7 lies in the column W_2 with minimum cost 26. We allocate min (110, 250) = 110 units in the cell (1, 2). Demand is satisfied, so we cross-off column W_2. The reduced T.P. is given below :

T → / F ↓	W_1	W_3	Supply	Difference
F_2	29 90	35	~~140~~ 50	6 ←
F_3	31	37	150	6
Demand	~~90~~ 0	200		
Difference	2	2		

(If tie occurs select that column/row which has minimum cost). Maximum difference 6 lies in the row F_2 and F_3. We have selected F_2 with minimum cost 29. We allocate min (90, 140) = 90 units in the cell (1, 1). Demand is satisfied, so we cross-out column W_1. Thus reduced T.P. is as under.

T → / F ↓	W_3	Supply
F_2	35 50	~~50~~ 0
F_3	37 150	~~150~~ 0
Demand	200	
Difference	2 ↑	

Finally we allocate 50 units in the cell (1, 1) and 150 units in the cell (2, 1).

All demand and supply are satisfied. We summarize the above allocation as under :

T → / F ↓	W_1	W_2	W_3	W_4	Supply
F_1	30	25 50	40	20 50	100
F_2	29 90	26 110	35 50	40	250
F_3	31	33	37 150	30	150
Demand	90	160	200	50	500

Total transportation cost

$= (50 \times 25) + (50 \times 20) + (90 \times 29) + (110 \times 26) + (50 \times 35) + (150 \times 37)$

$= 1250 + 1000 + 2610 + 2860 + 1750 + 5550$

= ₹ 15020

Example 3.16 : Obtain initial basic feasible solution by VAM. **(Oct. 99)**

	D_1	D_2	D_3	Supply
O_1	13	15	16	17
O_2	7	11	2	12
O_3	19	20	9	16
Demand	14	8	23	45

Solution :

D → / O ↓	D_1	D_2	D_3	Supply	Difference
O_1	13	15	16	17	2
O_2	7	11	2	12	5
O_3	19	20	9 16	~~16~~ 0	10 ←
Demand	14	8	~~23~~ 7	45	
Difference	6	4	7		

Maximum difference 10 lies in the row O_3 with minimum cost 9. We allocate min (23, 16) = 16 units in the cell (3, 3). Supply is satisfied, so we cross-out row O_3. The reduced T.P. is as under :

D → / O ↓	D_1	D_2	D_3	Supply	Difference
O_1	13	15	16	17	2
O_2	7	11	2 7	~~12~~ 5	5
Demand	14	8	7		
Difference	6	4	14 ↑		

Maximum difference 14 is in the column D_3 having minimum cost 2. Therefore, allocate min (7, 12) = 7 units in the cell (2, 3). Here demand is already over, so we cross-off column D_3. The reduced T.P. will be as under :

D → / O ↓	D_1	D_2	Supply	Difference
O_1	13	15	17	2
O_2	7 (5)	11	~~5~~ 0	4
Demand	~~14~~ 9	8		
Difference	6 ↑	4		

Maximum difference 6 lies in the column D_1 having minimum cost 7. Therefore, allocate min (14, 5) = 5 units in the cell (2, 1). Here supply is already over. So ignore row O_2. The reduced T.P. will be as under :

D → / O ↓	D_1	D_2	Supply	Difference
O_1	13 (9)	15 (8)	17	2
Demand	~~9~~ 0	~~8~~ 8		

Finally we allocate 9 units in the cell (1, 1) and 8 units in the cell (1, 2). All demand and supply have been satisfied.

All assignments that have been made are combined in a single table as under :

D → / O ↓	D_1	D_2	D_3	Supply
O_1	13 (9)	15 (8)	16	17
O_2	7 (5)	11	2 (7)	12
O_3	19	20	9 (16)	16
Demand	14	8	23	45

Total transportation cost

$= (9 \times 13) + (8 \times 15) + (5 \times 7) + (7 \times 2) + (16 \times 9)$

$=$ ₹ 430

Note : Number of allocations = 5 and $m + n - 1 = 3 + 3 - 1 = 5$. Therefore, initial feasible solution is also a initial basic feasible solution.

Example 3.17 : Find the initial basic feasible solution of the following T.P. by VAM.

(March 2010)

Destination → / Origin ↓	D_1	D_2	D_3	D_4	Supply
O_1	2	3	11	7	6
O_2	1	0	6	1	1
O_3	5	8	15	9	10
Demand	7	5	3	2	

Solution :

Destination → / Origin ↓	D_1	D_2	D_3	D_4	Supply	Difference
O_1	2	3	11	7	6	1
O_2	1	0	6	1 (1)	~~1~~ 0	1
O_3	5	8	15	9	10	3
Demand	7	5	3	~~2~~ 1		
Difference	1	3	5	6 ↑		

Maximum difference 6 lies in the column D_4 having minimum cost 1. We allocate min (2, 1) = 1 unit in the cell (2, 4) which satisfied the supply at O_2 and therefore, we cross-off the row O_2. This yields table as under :

Destination → / Origin ↓	D_1	D_2	D_3	D_4	Supply	Difference
O_1	2	3 (5)	11	7	~~6~~ 1	1
O_3	5	8	15	9	10	3
Demand	7	~~5~~ 0	3	~~2~~ 1		
Difference	3	5 ↑	4			

Maximum difference 5 lies in the column D_2, with minimum cost 3. We allocate (5, 6) = 5 units in the cell (1, 2) which satisfied the demand at D_2 and therefore, we cross-off the column D_2. After crossing-off the column D_2, we get the following reduced T.P.

Destination → **Origin ↓**	**D_1**	**D_3**	**D_4**	**Supply**	**Difference**
O_1	2 1	11	7	~~1~~ 0	5
O_3	5	15	9	10	4
Demand	~~7~~ 6	3	1		
Difference	3	4	2		

Maximum difference 5 lies in the row O_1 having minimum cost 2. Therefore, we allocate min (1, 7) = 1 unit in the cell (1, 1) which satisfied supply at O_1. So, we cross-off the row O_1. We get the required T.P. as under :

Destination → **Origin ↓**	**D_1**	**D_3**	**D_4**	**Supply**	**Difference**
O_3	5 6	15 3	9 1	~~10~~ 0	4 ←
Demand	~~6~~ 0	~~3~~ 0	~~1~~ 0		

Here, we allocate demand 6 at (1, 1), 2 at (1, 2) and 2 at (1, 3), so that all demand and supply have been satisfied.

All these assignments in the above tables can now be combined into a single transportation table with the assignments as established above. This table is shown below.

Destination → **Origin ↓**	**D_1**	**D_2**	**D_3**	**D_4**	**Supply**
O_1	2 1	3 5	11	7	6
O_2	1	0	6	1 1	1
O_3	5 6	8	15 3	9 1	10
Demand	7	5	3	2	17

∴ Total transportation cost

$= (1 \times 2) + (5 \times 3) + (1 \times 1) + (6 \times 5) + (3 \times 15) + (1 \times 9) = ₹\ 102$

It should be noted that the number of allocations = 6 and $m + n - 1 = 3 + 4 - 1 = 6$. Therefore, the initial feasible solution (I.F.S.) is also a initial basic feasible solution (I.B.F.S.).

Example 3.18 : Solve the following transporation problem using (a) North-West corner method. (b) VAM **(April 2009)**

5	1	8	12
2	4	0	14
3	6	7	4
9	10	11	

Solution : (a)

To

From		P	Q	R	Supply
	A	5 9	1 3	8	12
	B	2	4 7	0 7	14
	C	3	6	7 4	4
	Demand	9	10	11	30

Transportation schedule

From	**To**	**Units**	**Cost**
A	P	9	45
A	Q	3	3
B	Q	7	28
B	R	7	0
C	R	4	28
Total cost			104

(b)

To

From		P	Q	R	Supply	Row diff.	
	A	5 2	1 10	8 X	12	4	4
	B	2 3	4 X	0 11	14	2	2
	C	3 4	6 X	7 X	4	3	3
Demand		9	10	11	30		
Column difference		1	3	7 ↑			
		1	3	–			

Transportation schedule

From	To	Units	Cost
A	P	2	10
A	Q	10	10
B	Q	3	6
B	R	11	0
C	P	4	12
Total cost			38

Example 3.19 : Find I.B.F.S. of the following transporation problem by VAM. **(Oct. 2008)**

9	6	0	5
5	1	0	20
3	2	4	10
7	5	2	15
25	10	15	

Solution :

From		To: P	Q	R	Supply			
	A	X [9]	X [6]	5 [0]	5	6 ←	–	–
	B	10 [5]	10 [1]	X [0]	20	1	1	4 ←
	C	10 [3]	X [2]	X [4]	10	1	1	1
	D	5 [7]	X [5]	10 [2]	15	3	3 ←	2
Demand		25	10	15	50			
		2	1	0				
		2	1	2				
		2	1	–				

Transportation schedule

From	To	Units	Cost
A	R	5	0
B	P	10	50
B	Q	10	10
C	P	10	30
D	P	5	35
D	R	10	20
Total cost			145

Example 3.20 For the following transportation problem, obtain the initial basic feasible solution by VAM. **(Oct. 2007)**

	To					
From	**1**	**2**	**3**	**4**	**5**	**Supply**
1	80	69	103	64	61	12
2	47	100	72	65	40	16
3	16	103	87	36	94	20
4	86	15	57	19	25	8
5	27	20	72	94	19	8
Demand	16	14	18	6	10	

Solution :

From \ To	1	2	3	4	5	Supply						
1	80 X	69 X	103 12	64 X	61 X	12	3	3	3	8	34	34
2	47 X	100 X	172 6	65 X	40 10	16	7	25	25	32 ←	28	28
3	16 16	103 X	87 X	36 4	94 X	20	20 ←	51	–	–	–	–
4	86 X	15 6	57 X	19 2	25 X	8	4	4	4	10	42	42
5	27 X	29 8	72 X	94 X	19 X	8	1	1	1	1	52 ←	–
Demand	16	14	18	6	10	64						
	11	5	15	17	6							
	–	5	15	17	6							
	–	5	15	↑45	6							
	–	5	15	–	6							
	–	5	15	–	–							
	–	↑ 54	15	–	–							

Transsportation schedule

From	To	Units	Cost
1	3	12	1236
2	3	6	432
2	5	10	400
3	1	16	256
3	4	4	144
4	2	6	90
4	4	2	38
5	2	8	160
Total Cost			2756

3.2.5 Optimum Solution

After getting an initial basic feasible solution to the transportation problem, our next step is to find optimum solution. An optimum solution is obtained by making successive improvements in the initial basic feasible solution (I.B.F.S.) until no further decrease in the transportation cost is possible.

The basic steps for reaching the optimum solution are given below :

Step 1 : Examine the initial basic feasible for non-degeneracy. If it is degenerate, we do some modification to make the transportation problem non-degenerate.

Step 2 : Determine the cost difference for empty cells (non-basic variables).

Step 3 : Take the optimal test of current solution.

Step 4 : Determine the entering variable (provided that step 3 indicates that the current solution can be improved).

Step 5 : Select the leaving variable.

Step 6 : Repeat the above from step 1 to step 5 until an optimum solution is obtained.

Note : A basic feasible solution of $m \times n$ transportation problem is said to be non-degenerate if,

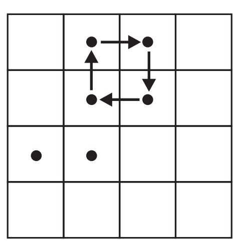
(i) The number of allocations in initial basic feasible solution are $m + n - 1$.

(ii) The allocations should be such that there is no closed loop through these allocations i.e. these allocations must be in '**independent positions'.**

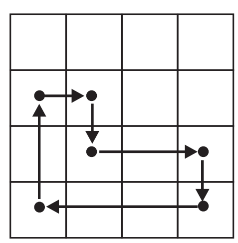
The independent and non-independent positions of allocations are shown below. In the below allocation patterns, the dotted lines constitute loops. (We will discuss the properties of loops in the next article.) A loop may or may not contain all allocations. It consists of horizontal and vertical lines with an allocation at each corner.

Following figures indicate the non-independent and independent positions of the loops.

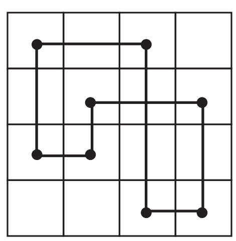
Non-independent positions



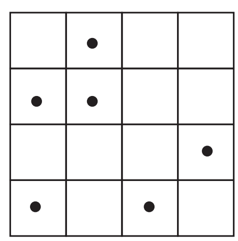
Independent positions



Non-independent positions



Independent positions



3.2.6 Properties of Loops

In a transportation table, an ordered set of four or more cells is said to form a **loop**, if

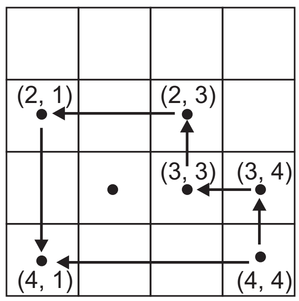
(i) any two adjacent cells in the ordered set lie in the same row or in the same column.

(ii) any three or more adjacent cells in the ordered set do not lie in the same row or in the same column. The first cell is considered to follow the last cell in the set.

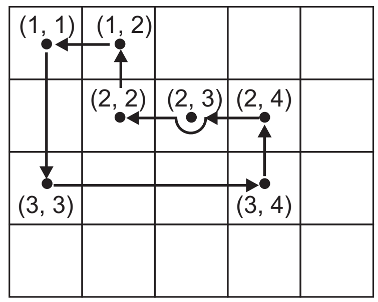
Note :

(i) Every loop has an even number of cells.

(ii) A feasible solution is basic if the corresponding cells do not contain a loop.



Loop



Non-loop

3.2.7 Optimal Test

For finding the optimal solution, we start with an empty cell (the cell which do not have allocation) and we allocate + 1 unit to this cell. We make necessary adjustment in the solution in order to maintain the row and column sum unchanged.

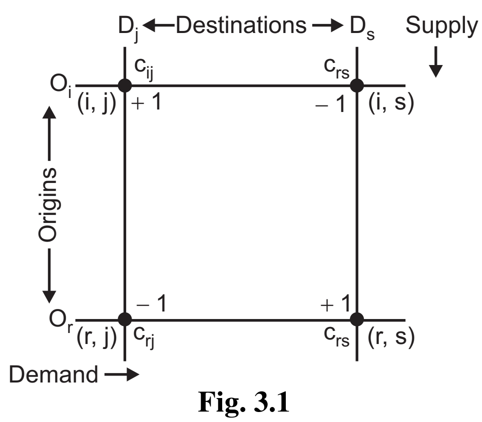


Fig. 3.1

For example, if + 1 unit is allocated to the empty cell (i, j) and in order to balance the total demand of destinations D_j add – 1 unit to occupied cell (i, j). Due to this, the total amount supply from origin O_r will be balanced by adding + 1 unit to occupied cell (r, s), which in turn causes column O_s to become unbalanced. For balancing O_s add – 1 unit to the occupied cell (i, s).

The net change in the total cost resulting from these adjustments, is called the **evaluation (cost difference)** for the empty cell.

If the cell evaluation is positive, then the adjustments would increase the total cost of each empty cell.

If the cell evaluation is negative, then the adjustments would decrease the total cost of each empty cell.

If there are $(m \times n) - (m + n - 1) = (m - 1)(n - 1)$ empty cells then there would be $(m - 1)(n - 1)$ cell evaluation.

If all the cell evaluations are positive or zero, then we cannot decrease the total transportation cost and hence the solution under test is the required optimal solution.

3.2.8 Procedure of Optimal Test

The optimality test for given basic feasible solution (B.F.S.) of the transportation problem may be summarized as under :

Step 1 : Determine the set of $(m + n)$ numbers,
u_i $(i = 1, 2, ..., m)$ and v_j $(j = 1, 2, ..., n)$ such that for each occupied cell (r, s) $c_{rs} = u_r + v_s$

Step 2 : Calculate cell evaluations (unit cost differences) d_{ij} for each empty cell (unoccupied cell) (i, j), which is given by

$$d_{ij} = c_{ij} - (u_i + v_j)$$

Step 3 : Examine the matrix of cell evaluations and conclude that

(a) If all $d_{ij} > 0$, then the solution under test is ***optimal and unique***.

(b) If all $d_{ij} \geq 0$, with at least one $d_{ij} = 0$, then the solution under test is *optimal* and ***alternative optimal solution*** exists.

(c) *If at least one $d_{ij} < 0$,* then the solution ***is not optimal***.

Step 4 : For case (c) in step 3, we form a new basic feasible solution (B.F.S.). In the next B.F.S. we give maximum allocation to the cell such that d_{ij} is minimum and negative, by making an occupied cell empty.

Step 5 : Repeat the steps 1, 2 and 3 to test the optimality for this new B.F.S.

3.2.9 Modi Method (Modified Distribution Method)

In Modi method, cell evaluations of all the unoccupied cells are calculated simultaneously and only one closed path for the most negative cell is traced. This method determines an optimum solution of a minimization transportation problem.

The various steps involved in obtaining the optimum solution from initial basic feasible solution of a transportation problem by Modi method are summarized as under :

Step 1 : Construct a transportation table entering source capacities a_i, the destination requirements b_j and the cost c_{ij}.

Step 2 : Find an initial basic feasible solution by any one of the previous methods. Enter the allocations at the centres of the cells in the circle.

Step 3 : For occupied (assigned) cells, solve the system of equations

$$u_i + v_j = c_{ij} \quad \forall \; i, j$$

starting with $u_i = 0$. Thus we get the values of all u_i and v_j. Enter these values at the upper right corner of the corresponding cell (i, j).

Cell (i, j)

	c_{ij}
x_{ij}	

Step 4 : For unoccupied cells, find the cell evaluations

i.e. $d_{ij} = c_{ij} - (u_i + v_j)$

Enter these values at the lower left corner of the corresponding cell (i, j).

Cell (i, j)

	c_{ij}
d_{ij}	

Step 5 : Apply optimality test by examining the sign of each d_{ij}. If all $d_{ij} \geq 0$, the current solution is optimal. If some $d_{ij} < 0$, select the cell with the maximum negative d_{ij}.

Step 6 : Shift some quantity from the occupied (allocated) cells to the cell with maximum negative d_{ij}, keeping in view that row and column totals are maintained. For this, we proceed as under.

Call the cell with maximum negative d_{ij} at home cell. From this cell draw a loop passing through occupied cells and every time, when it passes through an occupied cell it changes the path through 90^o to pass through another occupied cell till it comes back to the home cell. While making loop, at a time it passes through only one occupied cell in one direction (neglecting in between cells in that direction and then changes path in 90^o through another occupied cell).

Assign the quantity θ in the home cell and accordingly $-\theta$, $+\theta$, $-\theta$, $+\theta$, ... quantities in the occupied cells of the loop keeping in view row total and column total.

Consider the allocated cells with $-\theta$ in the loop. Of these cells (with $-\theta$) select the cell with the maximum x_{ij} and hence the value of θ should be the value of the minimum quantity x_{ij} in such cell with $(-\theta)$. Thus we get the revised transportation schedule.

Step 7 : Evaluate the transportation cost

$$z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij}$$

for the improved solution obtained in step 6.

Step 8 : Repeat the same process from step 2 to step 6 until an optimal solution is obtained. i.e. $d_{ij} \geq 0$, for unoccupied cells.

Illustrative Examples

Example 3.21 : The following is initial basic feasible solution of a transportation problem. Check whether it is optimal. If no, find optimal solution.

Warehouse → / Factory ↓	W_1	W_2	W_3	W_4	u_i
F_1	19 (5)	30	50	10 (2)	
F_2	70	30	40 (7)	60 (2)	
F_3	40	8 (8)	70	20 (10)	
v_j					

Figures in the right hand top corners of each cell denote transportation cost per unit and the number in the circle of the cells denote assignment made. **(April 1998)**

Solution : (1) The transportation cost, according to the above given route is (It is obtained by VAM, students may refer example 1 on VAM) = $(5 \times 19) + (2 \times 10) + (7 \times 40) + (2 \times 60) + (8 \times 8) + (10 \times 20) = ₹\ 779$

Further, the initial feasible solution is basic feasible, as the number of allocations in independent positions = 6 = m + n – 1 = 3 + 4 – 1.

(2) Calculation of u_i and v_j : For this we select $u_i = 0$ (any arbitrary value) (convenient rule is to select u_i which has the largest number of allocations in its row). Here all rows contain the same number of allocations. Take $u_3 = 0$ (say).

For occupied cells : Use $u_i + v_j = c_{ij}$

$$\begin{array}{llllll} u_1 + v_1 = 19 & \Rightarrow & v_1 = 29 & & u_1 = -10 & v_1 = 29 \\ u_1 + v_4 = 10 & \Rightarrow & u_1 = -10 & & u_2 = 40 & v_2 = 8 \\ u_2 + v_3 = 40 & \Rightarrow & v_3 = 0 & & u_3 = 0 & v_3 = 0 \\ u_2 + v_4 = 60 & \Rightarrow & u_2 = 40 & \Rightarrow & & v_4 = 20 \\ u_3 + v_2 = 8 & \Rightarrow & v_2 = 8 & & & \\ u_3 + v_4 = 20 & \Rightarrow & v_4 = 20 & & & \end{array}$$

(3) Calculation of d_{ij} :

For unoccupied cells : Use $d_{ij} = c_{ij} - (u_i + v_j)$

$$d_{12} = 30 - (-10 + 8) = 32$$

$$d_{13} = 50 - (-10 + 0) = 60$$

$$d_{21} = 70 - (40 + 29) = 1$$
$$d_{22} = 30 - (40 + 8) = -18 \text{ (Negative)}$$
$$d_{31} = 40 - (0 + 29) = 11$$
$$d_{33} = 70 - (0 + 0) = 70$$

(4) Now, d_{ij} and u_i and v_j are entered in the following table.

19 5	30 32	50 60	10 2	$u_1 = -10$
70 1	30 –18	40 7	60 2	$u_2 = 40$
40 11	8 8	70 70	20 10	$u_3 = 0$
$v_1 = 29$	$v_2 = 8$	$v_3 = 0$	$v_4 = 20$	

Here, all d_{ij} are not $\geq 0 \Rightarrow$ solution is not optimal.
Allocate θ to cell (2, 2).

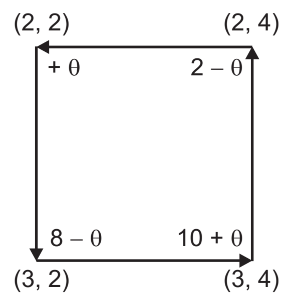


Fig. 3.2

In general, θ is obtained by equating to zero the minimum of the allocations which contains $-\theta$.

$\Rightarrow$ $\min(8 - \theta, 2 - \theta) = 0 \Rightarrow \theta = 2$

(5) **Improved basic feasible solution** is given below.

19 5	30	50	10 2	$u_1 = -10$
70	30 2	40 7	60	$u_2 = 22$
40	8 6	70	20 12	$u_3 = 0$
$v_1 = 29$	$v_2 = 8$	$v_3 = 18$	$v_4 = 20$	

Total transportation cost

$= (5 \times 19) + (2 \times 10) + (2 \times 30) + (7 \times 40) + (6 \times 8) + (12 \times 20)$

$=$ ₹ 743

which is less than the previous cost (₹ 779) by ₹ 36.

Again, repeat the same steps discussed above.

(6) Calculation of u_i and v_j :

For occupied cells : Use $u_i + v_j = c_{ij}$ (Take $u_3 = 0$).

$$\left.\begin{array}{lcl} u_1 + v_1 = 19 & \Rightarrow & v_1 = 29 \\ u_1 + v_4 = 10 & \Rightarrow & u_1 = -10 \\ u_2 + v_2 = 30 & \Rightarrow & u_2 = 22 \\ u_2 + v_3 = 40 & \Rightarrow & v_3 = 18 \\ u_3 + v_2 = 8 & \Rightarrow & v_2 = 8 \\ u_3 + v_4 = 20 & \Rightarrow & v_4 = 20 \end{array}\right\} \Rightarrow \begin{array}{l|l} u_1 = -10 & v_1 = 29 \\ u_2 = 22 & v_2 = 8 \\ u_3 = 0 & v_3 = 18 \\ & v_4 = 20 \end{array}$$

(7) Calculation of d_{ij} :

For unoccupied cells : Use $d_{ij} = c_{ij} - (u_i + v_j)$

$$d_{12} = 30 - (-10 + 8) = 32$$
$$d_{13} = 50 - (-10 + 18) = 42$$
$$d_{21} = 70 - (22 + 29) = 29$$
$$d_{24} = 60 - (22 + 20) = 18$$
$$d_{31} = 40 - (0 + 29) = 11$$
$$d_{33} = 70 - (0 + 18) = 52$$

Since all $d_{ij} > 0$, thus the improved solution is optimal and unique.

Thus, total transportation cost = ₹ 743

Example 3.22 : The following is an initial basic feasible solution of a transportation problem. Is this problem an optimal solution ? If not, obtain the optimal solution.

Destination → / Origin ↓	D_1	D_2	D_3	D_4	Supply
O_1	23	27	16	18 (30)	30
O_2	12 (5)	17 (35)	20	51	40
O_3	22 (17)	28	12 (25)	32 (11)	53
Demand	22	35	25	41	123

Solution : (1) The transportation cost according to the above given route is (obtained by VAM, see example 2 on VAM)

$= (30 \times 18) + (5 \times 12) + (35 \times 17) + (17 \times 22) + (25 \times 12) + (11 \times 32)$

$=$ ₹ 2221

Note that the number of allocations are $6 = m + n - 1$ and the allocations are in independent positions. So the solution is basic feasible solution (B.F.S) and non-degenerate solution.

(2) Calculation of u_i and v_j : $(i = 1, 2, 3)$ $(j = 1, 2, 3, 4)$

For occupied cells : Use $u_i + v_j = c_{ij}$ (Choose $u_1 = 0$)

$$\left.\begin{aligned} u_1 + v_4 &= 18 &\Rightarrow\quad v_4 &= 18 \\ u_2 + v_1 &= 12 &\Rightarrow\quad u_2 &= 4 \\ u_2 + v_2 &= 17 &\Rightarrow\quad v_2 &= 13 \\ u_3 + v_1 &= 22 &\Rightarrow\quad v_1 &= 8 \\ u_3 + v_3 &= 12 &\Rightarrow\quad v_3 &= -2 \\ u_3 + v_4 &= 32 &\Rightarrow\quad u_3 &= 14 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} u_1 &= 0 \\ u_2 &= 4 \\ u_3 &= 14 \\ & \end{aligned} \;\Bigg|\; \begin{aligned} v_1 &= 8 \\ v_2 &= 13 \\ v_3 &= -2 \\ v_4 &= 18 \end{aligned}$$

(3) Calculation of d_{ij} :

For unoccupied cells : Use $d_{ij} = c_{ij} - (u_i + v_j)$

$$\begin{aligned} d_{11} &= c_{11} - (u_1 + v_1) = 23 - (0 + 8) = 15 \\ d_{12} &= c_{12} - (u_1 + v_2) = 27 - (0 + 13) = 14 \\ d_{13} &= c_{13} - (u_1 + v_3) = 16 - (0 - 2) = 18 \\ d_{23} &= c_{23} - (u_2 + v_3) = 20 - (4 - 2) = 18 \\ d_{24} &= c_{24} - (u_2 + v_4) = 51 - (4 + 18) = 29 \\ d_{32} &= c_{32} - (u_3 + v_2) = 28 - (14 + 13) = 1 \end{aligned}$$

Here all $d_{ij} > 0$.

(4) d_{ij} for unoccupied cells, and u_i and v_j are entered in the following table.

23 / 15	27 / 14	16 / 18	18 / (30)	$u_1 = 0$
12 / (5)	17 / (35)	20 / 18	51 / 22	$u_2 = 4$
22 / (17)	28 / 1	12 / (25)	32 / (11)	$u_3 = 14$
$v_1 = 8$	$v_2 = 13$	$v_3 = -2$	$v_4 = 18$	

Since $d_{ij} > 0$, the given solution is an optimal solution and it is unique. Hence total transportation cost = 2221 (Calculated above in step 1).

Note : If $d_{ij} > 0$, then no need to proceed for step 4 and prepare the table as given in the above example 2. Because by optimality test $d_{ij} > 0$ itself indicates that the given solution is optimal and it is unique. See article 3.8.3. Step 3 (a).

Example 3.23 : The following is a solution of a transportation problem.

1 (20)	2	1 (10)	4
3	3 (20)	2 (20)	1 (10)
4	2 (20)	5	9

Show that it is an optimal solution and there exists one more optimal solution. Find the alternative optimal solution. The transportation cost per unit is written in the top right corners of the cells. The number in the circle of the cells denote assignment made. **(April 1998, 1999)**

Solution : (1) The transportation cost according to the above given route

$$= (20 \times 1) + (10 \times 1) + (20 \times 3) + (20 \times 2) + (10 \times 1) + (20 \times 2) = ₹\ 180$$

Note that the number of allocations are 6 = m + n – 1 and the allocations are in independent positions. So the solution is basic feasible and non-degenerate solution.

(2) Calculation of u_i and v_j :

For occupied cells : Use $u_i + v_j = c_{ij}$ (Choose $u_1 = 0$)

$$\left.\begin{array}{lclcl} u_1 + v_1 & = & 1 & \Rightarrow & v_1 = 1 \\ u_1 + v_3 & = & 1 & \Rightarrow & v_3 = 1 \\ u_2 + v_2 & = & 3 & \Rightarrow & v_2 = 2 \\ u_2 + v_3 & = & 2 & \Rightarrow & u_2 = 1 \\ u_2 + v_4 & = & 1 & \Rightarrow & v_4 = 0 \\ u_3 + v_2 & = & 2 & \Rightarrow & u_3 = 0 \end{array}\right\} \Rightarrow \begin{array}{l|l} u_1 = 0 & v_1 = 1 \\ u_2 = 1 & v_2 = 2 \\ u_3 = 0 & v_3 = 1 \\ & v_4 = 0 \end{array}$$

(3) Computation of d_{ij} :

For unoccupied cells : Use $d_{ij} = c_{ij} - (u_i + v_j)$

$$\begin{aligned} d_{12} &= c_{12} - (u_1 + v_2) = 2 - (0 + 2) = 0 \\ d_{14} &= c_{14} - (u_1 + v_4) = 4 - (0 + 0) = 4 \\ d_{21} &= c_{21} - (u_2 + v_1) = 3 - (1 + 1) = 1 \\ d_{31} &= c_{31} - (u_3 + v_1) = 4 - (0 + 1) = 3 \\ d_{33} &= c_{33} - (u_3 + v_3) = 5 - (0 + 1) = 4 \\ d_{34} &= c_{34} - (u_3 + v_4) = 9 - (0 + 0) = 9 \end{aligned}$$

(4) Entering d_{ij}, u_i and v_j in the table, thus we get

1 20	2 0	1 10	4 4	$u_1 = 0$
3 1	3 20	2 20	1 10	$u_2 = 1$
4 3	2 20	5 4	9 9	$u_3 = 0$
$v_1 = 1$	$v_2 = 2$	$v_3 = 1$	$v_4 = 0$	

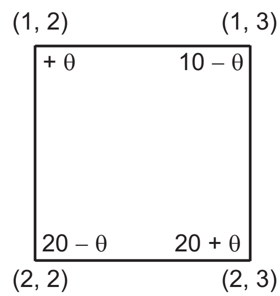


Fig. 3.3

$\min(10-\theta, 20-\theta) = 0 \Rightarrow \theta = 10$

(5) Improved basic feasible solution :

1 20	2 10	1	4	$u_1 = 0$
3	3 10	2 30	1 10	$u_2 = 1$
4	2 20	15	9	$u_3 = 0$
$v_1 = 1$	$v_2 = 2$	$v_3 = 1$	$v_4 = 0$	

Total transportation cost

$= (20 \times 1) + (10 \times 2) + (10 \times 3) + (30 \times 2) + (10 \times 1) + (20 \times 2)$

$=$ ₹ 180

(6) Computation of u_i and v_j :

For occupied cells : Use $u_i + v_j = c_{ij}$ (Take $u_1 = 0$)

$$\left.\begin{array}{rclcrcl} u_1 + v_1 & = & 1 & \Rightarrow & v_1 & = & 1 \\ u_1 + v_2 & = & 2 & \Rightarrow & v_2 & = & 2 \\ u_2 + v_2 & = & 3 & \Rightarrow & u_2 & = & 1 \\ u_2 + v_3 & = & 2 & \Rightarrow & v_3 & = & 1 \\ u_2 + v_4 & = & 1 & \Rightarrow & v_4 & = & 0 \\ u_3 + v_2 & = & 2 & \Rightarrow & u_3 & = & 0 \end{array}\right\} \Rightarrow \begin{array}{rcl|rcl} u_1 & = & 0 & v_1 & = & 1 \\ u_2 & = & 1 & v_2 & = & 2 \\ u_3 & = & 0 & v_3 & = & 1 \\ & & & v_4 & = & 0 \end{array}$$

(7) Computation of d_{ij} :

For unoccupied cells : Use $d_{ij} = c_{ij} - (u_i + v_j)$

$$\begin{aligned} d_{13} &= c_{13} - (u_1 + v_3) = 1 - (0 + 1) = 0 \\ d_{14} &= c_{14} - (u_1 + v_4) = 4 - (0 + 0) = 4 \\ d_{21} &= c_{21} - (u_2 + v_1) = 3 - (1 + 1) = 1 \\ d_{31} &= c_{31} - (u_3 + v_1) = 4 - (0 + 1) = 3 \\ d_{33} &= c_{33} - (u_3 + v_3) = 5 - (0 + 1) = 4 \\ d_{34} &= c_{34} - (u_3 + v_4) = 9 - (0 + 0) = 9 \end{aligned}$$

Here $d_{13} = 0$, alternate solution exist and again we get the same solution i.e. 180.

Example 3.24 : Test the following solution for optimality. If not, find the optimal solution.

1	2 (6)	3	4
4	3	2 (2)	0 (6)
0 (4)	2 (0)	2 (6)	1

Solution : The initial solution to this transportation problem can be easily obtained. In order to make the number of allocations = m + n – 1, we must assign 0 amount in the cell (3, 2), otherwise it is not possible to test for optimality. (We will discuss this matter in the degeneracy of transportation problem).

(1) Total transportation cost according to the above given route

$= (6 \times 2) + (2 \times 2) + (6 \times 0) + (4 \times 0) + (0 \times 2) + (6 \times 2) =$ ₹ 28

(2) Calculation of u_i and v_j :

For occupied cells : Use $u_i + v_j = c_{ij}$ (Choose $u_1 = 0$)

$$\left.\begin{array}{rclcrcl} u_1 + v_2 & = & 2 & \Rightarrow & v_2 & = & 2 \\ u_2 + v_3 & = & 2 & \Rightarrow & u_2 & = & 0 \\ u_2 + v_4 & = & 0 & \Rightarrow & v_4 & = & 0 \\ u_3 + v_1 & = & 0 & \Rightarrow & v_1 & = & 0 \\ u_3 + v_2 & = & 2 & \Rightarrow & u_3 & = & 0 \\ u_3 + v_3 & = & 2 & \Rightarrow & v_3 & = & 2 \end{array}\right\} \Rightarrow \begin{array}{rcl|rcl} u_1 & = & 0 & v_1 & = & 0 \\ u_2 & = & 0 & v_2 & = & 2 \\ u_3 & = & 0 & v_3 & = & 2 \\ & & & v_4 & = & 0 \end{array}$$

(3) Computation of d_{ij} : For unoccupied cells : Use $d_{ij} = c_{ij} - (u_i + v_j)$

$$d_{11} = c_{11} - (u_1 + v_1) = 1 - (0 + 0) = 1$$

$$d_{13} = c_{13} - (u_1 + v_3) = 3 - (0 + 2) = 1$$

$$d_{14} = c_{14} - (u_1 + v_4) = 4 - (0 + 0) = 4$$

$$d_{21} = c_{21} - (u_2 + v_1) = 4 - (0 + 0) = 4$$

$$d_{22} = c_{22} - (u_2 + v_2) = 3 - (0 + 2) = 1$$

$$d_{34} = c_{34} - (u_3 + v_4) = 1 - (0 + 0) = 1$$

(4) Here, all $d_{ij} > 0$, the solution under test is optimal and that solution is unique and is given in step 1 i.e. total cost = ₹ 28.

Example 3.25 : Find the initial basic feasible solution by N–W Corner rule. Obtain its optimal solution by Modi's method for the following transportation problem. **(April 2000)**

Destination → **Origin ↓**	**D_1**	**D_2**	**D_3**	**D_4**	**Supply**
O_1	1	2	3	4	6
O_2	4	3	3	0	8
O_3	0	2	2	1	10
Demand	4	6	8	6	24

Solution : This example is already solved by N-W Corner rule (See previous article).

Table by N-W Corner rule :

Destination → **Origin ↓**	**D_1**	**D_2**	**D_3**	**D_4**	**Supply**
O_1	1 4	2 2	3	4	6
O_2	4	3 4	3 4	0	8
O_3	0	2	2 4	1 6	10
Demand	4	6	8	6	24

(1) By North-West Corner rule, the total cost

$= (4 \times 1) + (2 \times 2) + (4 \times 3) + (4 \times 3) + (4 \times 2) + (6 \times 1)$

$= 4 + 4 + 12 + 12 + 8 + 6 = ₹ 46$

An optimal solution is obtained by using Modi's method as displaced in the following table (Refer previous example).

1	2 6	3	4
4	3	2 2	0 6
0 4	2 0	2 6	1

∴ The transportation cost according to the optimum route is

$= (6 \times 2) + (2 \times 2) + (6 \times 0) + (4 \times 0) + (0 \times 2) + (6 \times 2) = 12 + 4 + 12 = ₹\ 28$

(Students are requested to solve this example completely like previous examples.)

Example 3.26 : Solve the transportation problem with the cost coefficient demands and supplies as given in the following table.

Origin	W_1	W_2	W_3	W_4	Supply
O_1	1	2	– 2	3	70
O_2	2	4	0	1	38
O_3	1	2	– 2	5	32
Demand	40	28	30	42	140

Solution : Basic feasible solution obtained by VAM as displaced in the following table (See previous Example 3.21 under the article VAM) is

$(40 \times 1) + (26 \times 2) + (4 \times 3) + (38 \times 1) + (2 \times 2) + (30 \times -2) = ₹\ 86.$

1 40	2 26	– 2	3 4
2	4	0	1 38
1	2 2	– 2 30	5

By Modi : (1) Number of allocations = $m + n - 1 = 3 + 4 - 1 = 6$

So the solution is basic and non-degenerate as allocations are in independent positions.

(2) Computation of u_i and v_j : For occupied cells : Use $u_i + v_j = c_{ij}$ (set $u_1 = 0$)

$$\left.\begin{array}{lllll} u_1 + v_1 = 1 & \Rightarrow & v_1 = 1 \\ u_1 + v_2 = 2 & \Rightarrow & v_2 = 2 \\ u_1 + v_4 = 3 & \Rightarrow & v_4 = 3 \\ u_2 + v_4 = 1 & \Rightarrow & u_2 = -2 \\ u_3 + v_2 = 2 & \Rightarrow & u_3 = 0 \\ u_3 + v_3 = -2 & \Rightarrow & v_3 = -2 \end{array}\right\} \Rightarrow \begin{array}{l|l} u_1 = 0 & v_1 = 1 \\ u_2 = -2 & v_2 = 2 \\ u_3 = 0 & v_3 = -2 \\ & v_4 = 3 \end{array}$$

(3) Computation of d_{ij} :

For unoccupied cells : Use $d_{ij} = c_{ij} - (u_i + v_j)$

$$
\begin{aligned}
d_{13} &= -2-(-2) = 0 \\
d_{21} &= 2-(-1) = 3 \\
d_{22} &= 4-0 = 4 \\
d_{23} &= 0-(-2+2) = 0 \\
d_{31} &= 1-1 = 0 \\
d_{34} &= 5-3 = 2
\end{aligned}
$$

$\Rightarrow \quad d_{ij} = 0$

$\Rightarrow$ It has alternative optimal solution.

(4) Entering the values of u_i, v_j and d_{ij} in the transportation table, we get,

1 40	2 26	– 2 0	13 4	$u_1 = 1$
2 3	4 4	0 0	1 38	$u_2 = -2$
1 0	2 2	– 2 30	5 2	$u_3 = 0$
$v_1 = 1$	$v_2 = 2$	$v_3 = -2$	$v_4 = 3$	

(5) Allocating θ in the cell (1, 3) and completing the loop as prepared in the previous examples assigning the values – θ, + θ alternately, we get θ = 0, so we get the same alternate optimal solution. Thus the total transportation cost = $(40 \times 1) + (26 \times 2) + (4 \times 3) + (38 \times 1) + (2 \times 2) - (30 \times 2)$ = ₹ 86

3.2.10 Degeneracy in Transportation Problem (Oct. 1999)

We know that for an initial basic feasible solution in a general transportation problem, the number of occupied cells must be $(m + n - 1)$, where m is the number of supply and n is the number of demand.

If for a given transportation problem, the number of occupied cells is less than $(m + n - 1)$, then the solution is known as **degenerate solution.** In such a position the current solution cannot be improved, because it is not possible to draw a closed path for every occupied cell.

Also for number of occupied cells less than $(m + n - 1)$, the values of a_i and b_j $(i = 1, 2, ..., m;$ $j = 1, 2, ..., n)$ which are used in the test of optimality, cannot be computed.

Degeneracy in transportation problem occurs in two stages :

(i) At initial stage.

(ii) At intermediate stage.

Resolution of degeneracy :

(i) To resolve degeneracy at the initial stage, a very small quantity ε (epsilon) is added in the unoccupied cell with the lowest transportation cost. The cell is now considered to be occupied. This would make the number of occupied cells to be (m + n – 1). The quantity ε is so small that it does not affect the rim conditions. Also, if ε is transferred to an occupied cell, it does not affect the quantity of allocations in the cell.

$$\text{i.e.} \quad \left.\begin{aligned} x_{ij} + \varepsilon &= x_{ij} \\ x_{ij} - \varepsilon &= x_{ij} \\ a_i + \varepsilon &= a_i \\ a_i - \varepsilon &= a_i \\ b_j + \varepsilon &= b_j \\ b_j - \varepsilon &= b_j \\ \text{and} \quad k_\varepsilon &= 0 \end{aligned}\right\} \text{For all i, j and } k \in N$$

The value of ε is assumed to be zero. It does not change the total transportation cost of the allocation.

(ii) To remove degeneracy during the test of optimality at any intermediate stage, ε is assigned to one or more unoccupied cells, so that the total number of occupied cells is equal to (m + n – 1). In the intermediate stage previously occupied cell becomes unoccupied due to the same minimum allocation and hence these cells become simultaneously unoccupied.

Illustrative Examples

Example 3.27 : Solve the following transportation problem with the unit cost matrix as follows :

Destination → / Origin ↓	D_1	D_2	D_3	D_4	a_i
O_1	2	3 (6)	4	5	6
O_2	5	4	3 (2)	1 (6)	8
O_3	1 (4)	3	3 (6)	2	10
b_j	4	6	8	6	24

Solution : I.F.S. = 52

It should be noted that the number of allocated cells are 5 ≠ (m + n – 1). For degeneracy, number of allocated cells should be 6 = (3 + 4 – 1). So the solution is degenerate. We make it non-degenerate by allocating small value ε in one of the unoccupied cell say (2, 2), so that the solution is non-degenerate and the allocations are at independent positions.

As $a_2 + \varepsilon = a_2$, $b_2 + \varepsilon = b_2$, $k.\varepsilon = 0$

$\therefore$ I.F.S. is $x_{12} = 6$, $x_{22} = \varepsilon\ (= 0)$, $x_{23} = 2$, $x_{24} = 6$, $x_{31} = 4$, $x_{32} = 6$

Therefore total transportation cost is

$$= (6 \times 3) + (\varepsilon \times 4) + (2 \times 3) + (6 \times 1) + (4 \times 1) + (6 \times 3) = 52 \quad (\text{as } \varepsilon \to 0)$$

Now we use **MODI method**, for optimality test.

(1) Total transportation cost = 52

Number of allocations are 6 = m + n – 1 and the allocations are in independent positions, so the solution is basic feasible solution and non-degenerate.

(2) Computation of u_i and v_j : For occupied cells : Use $u_i + v_j = c_{ij}$ (Choose $u_i = 0$)

We get, $u_1 = 0$, $u_2 = 1$, $u_3 = 1$; $v_1 = 0$, $v_2 = 2$, $v_3 = 2$, $v_4 = 0$

(3) Computation of d_{ij} : For unoccupied cells : Use $d_{ij} = c_{ij} - (u_i + v_j)$

We get,
$d_{11} = 2$
$d_{13} = 2$
$d_{14} = 5$
$d_{21} = 4$
$d_{32} = -1$ (negative)
$d_{34} = 1$

Since at least one d_{ij} (i.e. $d_{32} = -1$) is negative, the solution is not optimal.

(4) Entering u_i, v_j and d_{ij} in the following table, we get,

2 2	3 (6)	4 2	5 5	$u_1 = 0$
5 4	4 (ε)	3 (2)	1 (6)	$u_2 = 1$
1 (4)	3 (6)	3 – 1	2 1	$u_3 = 1$
$v_1 = 0$	$v_2 = 2$	$v_3 = 2$	$v_4 = 0$	

(5)

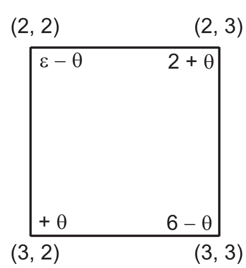


Fig. 3.4

Min $(\varepsilon - \theta, 6 - \theta) = 0 \Rightarrow \min(6, \varepsilon) = \varepsilon$ choose further $\varepsilon = 0$.

(6) Improved allocations :

2	3 (6)	4	5	$u_1 = 0$
5	4	3 (2)	1 (6)	$u_2 = 1$
1 (4)	3 (0)	3 (6)	2	$u_3 = 1$
$v_1 = 0$	$v_2 = 2$	$v_3 = 2$	$v_4 = 0$	

Here, number of allocations are m + n – 1 = 6 and are at independent positions. So the solution is non-degenerate. Again testing for optimality using Modi method, we get,

For occupied cells : $u_i + v_j = c_{ij}$ (Take $u_1 = 0$).

$\Rightarrow$ $u_1 = 0$, $u_2 = 0$, $u_3 = 0$; $v_1 = 1$, $v_2 = 3$, $v_3 = 3$, $v_4 = 0$

For unoccupied cells : $d_{ij} = c_{ij} - (u_i + v_j)$

$$\Rightarrow \quad \begin{aligned} d_{11} &= 1 \\ d_{13} &= 1 \\ d_{14} &= 4 \\ d_{21} &= 4 \\ d_{22} &= 1 \\ d_{34} &= 1 \end{aligned}$$

As, $d_{ij} \geq 0$, $\Rightarrow$ the above solution is optimal and unique.

Therefore, total transportation cost = $(6 \times 3) + (2 \times 3) + (6 \times 1) + (4 \times 1) + (0 \times 3) + (6 \times 3)$

= 52 (which is same as previous transportation cost).

Example 3.28 : Obtain the optimal solution for the following T.P. using Modi method.

	X	Y	Z	Available
A	8	7	3 (60)	60
B	3 (50)	8	9 (20)	70
C	11	3 (80)	5	80
Requirement	50	80	80	210

Solution : Here, the number of allocations = 4 ≠ (m + n – 1 = 5). So the transportation problem is degenerate at very beginning. It is possible to resolve the degeneracy by the addition of small quantity ε to suitable unoccupied cell.

(We choose such empty cell with careful judgement. The allocations of ε in unoccupied cell (1, 1), (1, 3), (2, 1) and (2, 3) will become in non-independent positions.)

We can resolve the degeneracy of the problem by adding ε to any one of the unoccupied cells (1, 2), (2, 2), (3, 1) and (3, 3). Resolve the degeneracy by allocating ε to least cost independent unoccupied cell say (3, 3). The above table becomes

	X	Y	Z	Available
A	8	7	3 60	60
B	3 50	8	9 20	70
C	11	3 80	5 ε	80 + ε
Requirement	50	80	80 + ε	

I.F.S. is $x_{13} = 60$, $x_{21} = 50$, $x_{23} = 20$, $x_{32} = 80$, $x_{33} = \varepsilon\ (= 0)$

∴ Total transportation cost is = $(60 \times 3) + (50 \times 3) + (20 \times 9) + (80 \times 3) + (\varepsilon \times 5) = ₹\ 750$

Now we have to test the solution for optimality by **MODI Method**.

For occupied cells : Use $u_i + v_j = c_{ij}$ (Take $u_3 = 0$)

$$\left.\begin{aligned} u_1 + v_3 &= 3 \;\Rightarrow\; u_1 = -2 \\ u_2 + v_1 &= 3 \;\Rightarrow\; v_1 = -1 \\ u_2 + v_3 &= 9 \;\Rightarrow\; u_2 = 4 \\ u_3 + v_2 &= 3 \;\Rightarrow\; v_2 = 3 \\ u_3 + v_3 &= 5 \;\Rightarrow\; v_3 = 5 \end{aligned}\right\} \Rightarrow \begin{array}{l|l} u_1 = -2 & v_1 = -1 \\ u_2 = 4 & v_2 = 3 \\ u_3 = 0 & v_3 = 5 \end{array}$$

For unoccupied cells : Use $d_{ij} = c_{ij} - (u_i + v_j)$

$$d_{11} = 8 - (u_1 + v_1) = 8 - (-3) = 11$$
$$d_{12} = 7 - (u_1 + v_2) = 7 - (1) = 6$$
$$d_{22} = 8 - (u_2 + v_2) = 8 - (7) = 1$$
$$d_{31} = 11 - (u_3 + v_1) = 11 - (-1) = 12$$

Entering all these values of u_i, v_j and d_{ij} in the table, we get,

8 11	7 6	3 60	$u_1 = -2$
3 50	8 1	9 20	$u_2 = 4$
11 12	3 80	5 0	$u_3 = 0$
$v_1 = -1$	$v_2 = 3$	$v_3 = 5$	

Here, we find that all $d_{ij} > 0$, so the solution under test is optimal and it is unique, hence the final answer is

$$(60 \times 3) + (50 \times 3) + (20 \times 9) + (80 \times 3) = ₹\ 750 \text{ (which is to be expected)}$$

It should be noted that the addition of quantity ε do not change the real total cost of the transportation problem. In many of the cases infinitesimal quantity ε plays only an auxiliary role and has no significance, it is removed when the optimal solution is obtained.

Example 3.29 : Find the initial allocation of the following transportation problem by VAM and optimize it using **MODI method**.

	D_1	D_2	D_3	D_4	D_5	D_6	Available
O_1	9	12	9	6	9	10	5
O_2	7	3	7	7	5	5	6
O_3	6	5	9	12	3	11	2
O_4	6	8	11	2	2	10	9
Required	4	4	6	2	4	2	22

Solution : Using VAM, the initial basic feasible solution is given below. (Students are requested to solve the given example and obtain the following table.)

9	12	9 (5)	6	9	10	5
7	3 (4)	7	7	5	5 (2)	6
6 (1)	5	9 (1)	12	3	11	2
6 (3)	8	11	2 (2)	2 (4)	10	9
4	4	6	2	4	2	22

Here the number of allocations are 8 which is less than (m + n – 1) = 9. This transportation problem is degenerate. We can resolve the degeneracy of the problem by the addition of ε to any one of the unoccupied independent cell (2, 3) (say). The above transportation table becomes

9	12	9 (5)	6	9	10	5
7	3 (4)	7 ε	7	5	5 (2)	6 + ε
6 (1)	5	9 (1)	12	3	11	2
6 (3)	8	11	2 (2)	2 (4)	10	9
4	4	6 + ε	2	4	2	22

Total transportation cost

= $(5 \times 9) + (4 \times 3) + (\varepsilon \times 7) + (2 \times 5) + (1 \times 6) + (1 \times 9) + (3 \times 6) + (2 \times 2) + (4 \times 2)$

= ₹ 112

Now, out next task is to test the solution for optimality by **MODI** method.

Computation of u_i and v_j :

For occupied cells : Use $u_i + v_j = c_{ij}$ (Take $u_2 = 0$)

$$\left.\begin{array}{lll} u_1 + v_3 = 9 & \Rightarrow & u_1 = 7 \\ u_2 + v_2 = 3 & \Rightarrow & v_2 = 3 \\ u_2 + v_3 = 7 & \Rightarrow & v_5 = 7 \\ u_2 + v_6 = 5 & \Rightarrow & v_6 = 5 \\ u_3 + v_1 = 6 & \Rightarrow & v_1 = 4 \\ u_3 + v_3 = 9 & \Rightarrow & u_3 = 2 \\ u_4 + v_1 = 6 & \Rightarrow & u_4 = 2 \\ u_4 + v_4 = 2 & \Rightarrow & v_4 = 0 \\ u_4 + v_5 = 2 & \Rightarrow & v_5 = 0 \end{array}\right\} \Rightarrow \begin{array}{l} u_1 = 2 \\ u_2 = 0 \\ u_3 = 2 \\ u_4 = 2 \end{array} \left| \begin{array}{l} v_1 = 4 \\ v_2 = 3 \\ v_3 = 7 \\ v_4 = 0 \\ v_5 = 0 \\ v_6 = 5 \end{array}\right.$$

Computation of d_{ij} : For unoccupied cells : Use $d_{ij} = c_{ij} - (u_i + v_j)$

$$\begin{aligned}
d_{11} &= c_{11} - (u_1 + v_1) = 9 - 6 = 3 \\
d_{12} &= c_{12} - (u_1 + v_2) = 12 - 7 = 5 \\
d_{14} &= c_{14} - (u_1 + v_4) = 6 - 2 = 4 \\
d_{15} &= c_{15} - (u_1 + v_5) = 9 - 2 = 7 \\
d_{16} &= c_{16} - (u_1 + v_6) = 10 - 7 = 3 \\
d_{21} &= c_{21} - (u_2 + v_1) = 7 - 4 = 3 \\
d_{24} &= c_{24} - (u_2 + v_4) = 7 - 0 = 7 \\
d_{25} &= c_{25} - (u_2 + v_5) = 5 - 0 = 5 \\
d_{32} &= c_{32} - (u_3 + v_2) = 5 - 5 = 0 \\
d_{34} &= c_{34} - (u_3 + u_4) = 12 - 2 = 10 \\
d_{35} &= c_{35} - (u_3 + v_5) = 3 - 2 = 1 \\
d_{36} &= c_{36} - (u_3 + v_6) = 11 - 7 = 4 \\
d_{42} &= c_{42} - (u_4 + v_2) = 8 - 5 = 3 \\
d_{43} &= c_{43} - (u_4 + v_3) = 11 - 9 = 2 \\
d_{46} &= c_{46} - (u_4 + v_6) = 10 - 7 = 3
\end{aligned}$$

As at least one d_{ij} (i.e. $d_{32} = 0$) = 0, the solution is optimal and has alternate optimal solution.

9	12	9 (5)	6	9	10	$u_1 = 2$
7	3 (4)	7	7	5	5 (2)	$u_2 = 0$
6 (1)	5	9 (1)	12	3	11	$u_3 = 2$
6 (3)	8	11	2 (2)	2 (4)	10	$u_4 = 2$
$v_1 = 4$	$v_2 = 3$	$v_3 = 7$	$v_4 = 0$	$v_5 = 0$	$v_6 = 5$	

Optimal solution is given by

$(5 \times 9) + (4 \times 3) + (2 \times 5) + (1 \times 6) + (1 \times 9) + (3 \times 6) + (2 \times 2) + (4 \times 2) = ₹\ 112$

3.2.11 Unbalanced Transportation Problems (Oct. 1997, April 1998)

Uptil now we have discussed the balanced type of transportation problems where the total supply equals to total demand (i.e. $\Sigma\ a_i = \Sigma\ b_j$). But sometimes in practical situations, supply may be greater than demand (i.e. $\Sigma\ a_i > \Sigma\ b_j$) or demand may be greater than supply (i.e. $\Sigma\ b_j > \Sigma\ a_i$). These situations are known as **unbalanced transportation problems.**

Thus, if in a transportation problem, total supply is not equal to total demand,

i.e. $$\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$$

then such problem is called an **unbalanced transportation problem.**

Case 1 : If $\Sigma\ a_i \geq \Sigma\ b_j$

The dummy destination column in the given transportation problem is introduced. The unit transportation costs to this dummy destination column are all set equal to zero. The requirement at this dummy destination column is assumed to be equal to $\Sigma\ a_i - \Sigma\ b_j$.

For example :

Destination → / **Source ↓**	**D_1**	**D_2**	**D_3**	**D_4**	**D_5**	**D_6**	**Capacities = a_i**
O_1	4	2	3	2	6	0	8
O_2	5	4	5	2	1	0	12
O_3	6	5	4	7	3	0	14
Required = b_j	4	4	6	8	8	4	34

Case 2 : If $\Sigma\ a_i \leq \Sigma\ b_j$

The dummy source row in the given transportation problem is introduced. The unit transportation costs to this dummy source row are all set equal to zero. The availability at this dummy source row is assumed to be equal to $\Sigma\ b_j - \Sigma\ a_i$.

For example :

Destination → / **Source ↓**	**D_1**	**D_2**	**D_3**	**D_4**	**D_5**	**Availability**
O_1	5	8	6	6	3	800
O_2	4	7	7	6	5	500
O_3	8	4	6	6	4	900
O_4	0	0	0	0	0	300
Requirement	400	400	500	400	800	2500

Illustrative Examples

Example 3.30 : Find the initial basic feasible solution of the following T.P. and obtain the optimal solution by using Modi method.

Destination → / Origin ↓	D_1	D_2	D_3	D_4	D_5	Capacity
O_1	4	2	3	2	6	8
O_2	5	4	5	2	1	12
O_3	6	5	4	7	3	14
Requirement	4	4	6	8	8	34 / 30

Solution : For balancing add column destination D_6.

Destination → / Source ↓	D_1	D_2	D_3	D_4	D_5	D_6	Capacity
O_1	4	2	3	2	6	0	8
O_2	5	4	5	2	1	0	12
O_3	6	5	4	7	3	0	14
Requirement	4	4	6	8	8	4	34

By using Vogel's approximation method (VAM), the initial feasible solution is obtained which is given in the table below. (Students are requested to solve it completely.)

Destination → / Source ↓	D_1	D_2	D_3	D_4	D_5	D_6	a_i
O_1	4	2 (4)	3	2 (4)	6	0	8
O_2	5	4	5	2 (4)	1 (8)	0	12
O_3	6 (4)	5	4 (6)	7	3	0 (4)	14
b_j	4	4	6	8	8	4	34

From the above allocations, the total transportation cost

$= (4 \times 2) + (4 \times 2) + (4 \times 2) + (4 \times 2) + (8 \times 1) + (4 \times 6) + (4 \times 0) + (6 \times 4)$

$=$ ₹ 80

For optimality test (Modi method) : Here the total number of allocations is 7 ($\neq m + n - 1 = 8$). The solution, thus obtained is a degenerate basic feasible solution. Allocate ε to cell (1, 1). We have the following table.

Destination → / Source ↓	D_1	D_2	D_3	D_4	D_5	D_6	a_i
O_1	4 ε	2 ④	3	2 ④	6	0	8
O_2	5	4	5	2 ④	1 ⑧	0	12
O_3	6 ④	5	4 ⑥	7	3	0 ④	14
b_j	4	4	6	8	8	4	34

Computation of u_i and v_j : For occupied cells : Use $u_i + v_j = c_{ij}$ (Take $u_1 = 0$)

$$\left.\begin{array}{lllll} u_1 + v_1 = 4 & \Rightarrow & v_1 = 4 \\ u_1 + v_2 = 2 & \Rightarrow & v_2 = 2 \\ u_1 + v_4 = 2 & \Rightarrow & v_4 = 2 \\ u_2 + v_4 = 2 & \Rightarrow & u_2 = 0 \\ u_2 + v_5 = 1 & \Rightarrow & v_5 = 1 \\ u_3 + v_1 = 6 & \Rightarrow & u_3 = 2 \\ u_3 + v_3 = 4 & \Rightarrow & v_3 = 2 \\ u_3 + v_6 = 0 & \Rightarrow & v_6 = -2 \end{array}\right\} \Rightarrow \begin{array}{l|l} u_1 = 0 & v_1 = 4 \\ u_2 = 0 & v_2 = 2 \\ u_3 = 2 & v_3 = 2 \\ & v_4 = 2 \\ & v_5 = 1 \\ & v_6 = -2 \end{array}$$

Computation of d_{ij} : For unoccupied cells : Use $d_{ij} = c_{ij} - (u_i + v_j)$

$$d_{13} = 3 - 2 = 1$$
$$d_{15} = 6 - 1 = 5$$
$$d_{16} = 0 - (-2) = 2$$
$$d_{21} = 5 - 4 = 1$$
$$d_{22} = 4 - 2 = 2$$
$$d_{23} = 5 - 2 = 3$$
$$d_{26} = 0 - (-2) = 2$$
$$d_{32} = 5 - 3 = 2$$
$$d_{34} = 7 - 4 = 3$$
$$d_{35} = 3 - 3 = 0$$

Since $d_{ij} \geq 0$, the solution under test is optimal and at least one i.e. $d_{35} = 0$ indicates that alternative solutions will also exist.

Example 3.31 : Find the initial basic feasible solution of the following T.P. by Vogel's approximation method and obtain the optimal solution by using Modi method.

Destination → / Origin ↓	D_1	D_2	D_3	D_4	a_i
O_1	9	5	8	5	225
O_2	9	10	13	7	75
O_3	14	5	3	7	100
b_j	225	80	95	100	400 / 500

Solution : For balancing, adding row-source O_4, we have

	D_1	D_2	D_3	D_4	a_i
O_1	9	5	8	5	225
O_2	9	10	13	7	75
O_3	14	5	3	7	100
O_4	0	0	0	0	100
b_j	225	80	95	100	500

By using Vogel's approximation method (VAM), the initial feasible solution is obtained which is given in the table below (students are requested to get the allocations given in the table.)

	D_1	D_2	D_3	D_4	a_i
O_1	9 125	5 75	8	5 25	225
O_2	9	10	13	7 75	75
O_3	14	5 5	3 95	7	100
O_4	0 100	0	0	0	100
b_j	225	80	95	100	500

(1) The number of allocations = 7 = (m + n – 1) and the allocations are at independent positions, so the solution is non-degenerate.

Total transportation cost

= $(125 \times 9) + (75 \times 5) + (25 \times 5) + (7 \times 75) + (5 \times 5) + (95 \times 3) + (100 \times 0)$

= ₹ 2460

Optimal Test by Modi Method :

(2) Determination of u_i and v_j : For occupied cells : Use $u_i + v_j = c_{ij}$ (Take $u_1 = 0$)

$$\left.\begin{aligned} u_1 + v_1 &= 9 \\ u_1 + v_2 &= 5 \\ u_4 + v_1 &= 0 \\ u_3 + v_2 &= 5 \\ u_3 + v_3 &= 3 \\ u_4 + v_4 &= 5 \\ u_3 + v_4 &= 7 \end{aligned}\right\} \Rightarrow \begin{array}{ll|ll} v_1 &= 0 & v_1 &= 9 \\ u_2 &= 2 & v_2 &= 5 \\ u_3 &= 0 & v_3 &= 3 \\ u_4 &= -9 & v_4 &= 5 \end{array}$$

(3) Determination of d_{ij} :

For unoccupied cells : Use $d_{ij} = c_{ij} - (u_i + v_j)$

$$\begin{array}{ll|ll} d_{13} &= 5 & d_{34} &= 2 \\ d_{21} &= -2 \text{ (negative)} & d_{42} &= 4 \\ d_{22} &= 3 & d_{43} &= 6 \\ d_{23} &= 8 & d_{44} &= 4 \\ d_{31} &= 5 & & \end{array}$$

At least one $d_{ij} < 0$ i.e. $d_{21} = -2 < 0$ indicates that the solution is not optimal.

(4) We have to improve the optimal solution by adding small quantity θ in the cell (2, 1) which we call home-cell.

Draw loop from that home-cell (2, 1) as under :

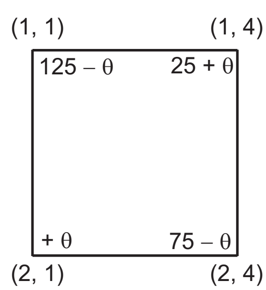


Fig. 3.5

Min $(125 - \theta, 75 - \theta) = 0 \Rightarrow \theta = 75$

(5) The revised transportation table is

	D_1	D_2	D_3	D_4	a_i
O_1	9 (50)	5 (75)	8	5 (100)	225
O_2	9 (75)	10	13	7	75
O_3	14	5 5	3 (95)	7	100
O_4	0 (100)	0	0	0	100
b_j	225	80	95	100	500

The new revised solution is non-degenerate (since $m + n - 1 = 4 + 4 - 1 = 7$ = no. of allocations made). Again,

(6) Determination of u_i and v_j :

For occupied cells : Use $u_i + v_j = c_{ij}$ (Set $u_1 = 0$).

$$\left.\begin{aligned} u_1 + v_1 &= 9 \\ u_1 + v_2 &= 5 \\ u_1 + v_4 &= 5 \\ u_2 + v_1 &= 9 \\ u_3 + v_2 &= 5 \\ u_4 + v_1 &= 0 \\ u_3 + v_3 &= 3 \end{aligned}\right\} \Rightarrow \begin{aligned} u_1 &= 0 \\ u_2 &= 2 \\ u_3 &= 0 \\ u_4 &= -9 \end{aligned} \quad \left| \quad \begin{aligned} v_1 &= 9 \\ v_2 &= 5 \\ v_3 &= 3 \\ v_4 &= 5 \end{aligned}\right.$$

(7) Determination of d_{ij} :

For unoccupied cells : Using $d_{ij} = c_{ij} - (u_i + v_j)$, we get

$$\begin{aligned} d_{13} &= 5 \\ d_{22} &= 5 \\ d_{23} &= 10 \\ d_{24} &= 2 \\ d_{31} &= 4 \end{aligned} \quad \left| \quad \begin{aligned} d_{34} &= 2 \\ d_{42} &= 4 \\ d_{43} &= 6 \\ d_{44} &= 4 \end{aligned}\right.$$

Since all $d_{ij} > 0$, the improved (revised) solution which is given in step 5 is optimal solution and it is the final solution. Therefore, the total optimal cost obtained from the assignment given in the table in step 5 is $(50 \times 9) + (75 \times 5) + (100 \times 5) + (75 \times 9) + (5 \times 5) + (95 \times 3) + (100 \times 0)$ = ₹ 2310.

Example 3.32 : Find initial allocation of the following transportation problem by VAM and optimize it using Modi method. **(Oct. 1997)**

	D_1	D_2	D_3	Capacity
O_1	8	16	16	152
O_2	32	48	32	164
O_3	16	32	48	154
Requirement	144	204	82	

Solution :

	D_1	D_2	D_3	Capacity
O_1	8	16	16	152
O_2	32	48	32	164
O_3	16	32	48	154
Requirement	144	204	82	470 / 430

We first balance this transportation problem by adding the column D_4 with all units 0 and requirement 40.

	D_1	D_2	D_3	D_4	Capacity
O_1	8	16	16	0	152
O_2	32	48	32	0	164
O_3	16	32	48	0	154
Requirement	144	204	82	40	470

By VAM, initial basic solutions are :

8	16 (70)	16 (82)	0
32	48 (124)	32	0 (40)
16 (144)	32 (10)	48	0

Total cost = ₹ 11008/-

Test for Optimality : (Modi Method) :

(1) Calculation of u_i and v_j :

For occupied cells : Use $u_i + v_j = c_{ij}$ (Take $u_1 = 0$)

$$\left.\begin{aligned} u_1 + v_2 &= 16 \Rightarrow & v_2 &= 16 \\ u_1 + v_3 &= 16 \Rightarrow & v_3 &= 16 \\ u_2 + v_2 &= 48 \Rightarrow & u_2 &= 32 \\ u_2 + v_4 &= 0 \Rightarrow & v_4 &= -32 \\ u_3 + v_1 &= 16 \Rightarrow & v_1 &= 0 \\ u_3 + v_2 &= 32 \Rightarrow & u_3 &= 16 \end{aligned}\right\} \Rightarrow \begin{array}{l|l} u_1 = 0 & v_1 = 0 \\ u_2 = 32 & v_2 = 16 \\ u_3 = 16 & v_3 = 16 \\ & v_4 = -32 \end{array}$$

(2) Calculations of d_{ij} :

For unoccupied cells : Use $d_{ij} = c_{ij} - (u_i + v_j)$

$$\begin{aligned} d_{11} &= 8 - 0 = 8 \\ d_{14} &= 0 - (-32) = 32 \\ d_{21} &= 32 - 32 = 0 \\ d_{23} &= 32 - 48 = -16 \text{ (negative)} \\ d_{33} &= 48 - 32 = 16 \\ d_{34} &= 0 - (-16) = 16 \end{aligned}$$

At least one d_{ij} i.e. $d_{23} < 0$, the solution is not optimal, we have to improve the solution.

(3) Entering the values of d_{ij}, u_i and v_j in the table, we get,

8 8	16 70	16 82	0 32	$u_1 = 0$
32 0	48 124	32 –16	0 40	$u_2 = 32$
16 144	32 10	48 16	0 32	$u_3 = 16$
$v_1 = 0$	$v_2 = 16$	$v_3 = 16$	$v_4 = -32$	

We have to improve the optimal solution by adding small quantity θ in the home-cell (2, 3). Draw loop from (2, 3).

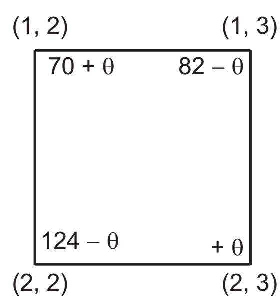


Fig. 3.6

Min $(82 - \theta, 124 - \theta) = 0 \Rightarrow \theta = 82$.

So, improved assignments are :

8	16 (152)	16	0
32	48 (42)	32 (82)	0 (40)
16 (144)	32 (10)	48	0

Again, test for optimality using Modi method. Here, we find that the solution given in the above improved table is optimal.

∴ Optimal solution

$= (152 \times 16) + (42 \times 48) + (82 \times 32) + (40 \times 0) + (144 \times 16) + (10 \times 32) =$ ₹ 9696

which is less than the previous cost (₹ 11008) by ₹ 1312.

3.2.12 Maximization in Transportation Problems

In certain types of business problems, the objective function is to be maximized rather than minimized. However, such types of problems may belong to the category of transportation problems. These problems can be solved by transportation model. Some steps involved in such problems are given under.

1. Convert the given maximization problems into a minimization problem by subtracting each of the profit element associated with the transportation route from the largest profit element.
2. The minimization problem is solved as usual. The optimal solution to minimization problem is also the optimal solution to the original maximization problem.
3. The optimal value of the objective function for the original maximization problem is obtained by referring the matrix of maximization problem.

Illustrative Examples

Example 3.33 : There are three manufacturing plants located at A (Amravati), B (Mumbai), C (Kolkata), transporting the product to two market centres who sell the product. The following data is given.

Market centres	**X** **S.P. /unit = ₹ 44**	**Y** **S.P. /unit = ₹ 47**	**Production**
Plants/ Production cost/unit (₹)	**Transportation cost/unit (₹)**		**Capacity** a_i
A 15	10	15	20
B 16	20	30	30
C 17	25	36	40
Requirement b_j	35	36	90 / 71

Solution : Profit (per unit) = S.P. – Production cost – Transportation cost.

Profit per unit is given for each cell in the table below.

	X	**Y**	**a_i**
A	19	17	20
B	8	1	30
C	12	5	40
b_j	35	36	90 / 71

Here, the T.P. is unbalanced as $\Sigma a_i > \Sigma b_j$. We first make it balance by adding one additional column D_3 with $b_3 = 19$. We get the following balanced table.

D / **O**	**D_1**	**D_2**	**D_3**	**a_i**
O_1	19	17	0	20
O_2	8	1	0	30
O_3	12	5	0	40
b_j	35	36	19	90

Next, for minimization problem, subtracting each element in the cell from largest element 19 in the cell, we get the following table.

D / **O**	**D_1**	**D_2**	**D_3**	**a_i**
O_1	0	2	19	20
O_2	11	18	19	30
O_3	17	14	19	40
b_j	35	36	19	90

Now, we get the initial basic feasible solution by VAM. Note that the number in the circle of the cells denote the assignment made by this method. (Students are requested to get these assignments by VAM). The assignments are given below.

0	2 20	19	20
11 30	18	19	30
17 5	14 16	19 19	40
35	36	19	90

The total transportation cost

$$= (20 \times 2) + (30 \times 11) + (5 \times 17) + (16 \times 14) + (19 \times 19)$$

$$= ₹\ 960$$

Here, the number of allocations made = (m + n – 1) = 5 = (3 + 3 – 1) and are at independent positions. Therefore, the solution is non-degenerate.

Now, it remains to check the optimality by Modi method. For this we have,

(1) Determination of u_i and v_j :

For occupied cells : Use $u_i + v_j = c_{ij}$ (Set $u_1 = 0$).

$$\left.\begin{array}{lcllcl} u_1 + v_2 & = & 2 & \Rightarrow & v_2 & = 2 \\ u_3 + v_2 & = & 14 & \Rightarrow & u_3 & = 12 \\ u_3 + v_1 & = & 7 & \Rightarrow & v_1 & = -5 \\ u_3 + v_3 & = & 19 & \Rightarrow & v_3 & = 7 \\ u_2 + v_1 & = & 11 & \Rightarrow & u_2 & = 16 \end{array}\right\} \Rightarrow \begin{array}{ll|ll} u_1 & = 0 & v_1 & = -5 \\ u_2 & = 16 & v_2 & = 2 \\ u_3 & = 32 & v_3 & = 7 \end{array}$$

(2) Determination of d_{ij} :

For unoccupied cells : Use $d_{ij} = c_{ij} - (u_i + v_j)$

$$d_{11} = u_{11} - (u_1 + v_1) = 0 - (-5) = 5$$

$$d_{13} = u_{13} - (u_1 + v_3) = 19 - (7) = 12$$

$$d_{22} = u_{22} - (u_2 + v_2) = 18 - (18) = 0$$

$$d_{23} = u_{23} - (u_2 + v_3) = 19 - (23) = -4.$$

Enter all the values of u_i , v_j and d_{ij} in the table below.

0 5	2 20	19 12	$u_1 = 0$
11 30	18 0	19 – 4	$u_2 = 16$
7 5	14 16	19 19	$u_3 = 12$
$v_1 = -5$	$v_2 = 2$	$v_3 = 7$	

(3) Still, our work is not completed, as at least one $d_{ij} < 0$ i.e. $d_{23} < 0$. Therefore, we have to improve (revise) our solution. For this we draw a loop from the home-cell (2, 3) and assign θ to cell (2, 3) as under.

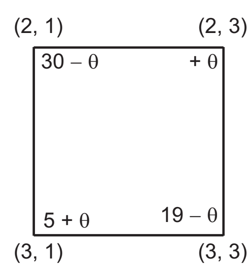


Fig. 3.7

$\theta = \min(30 - \theta, 19 - \theta) = 0 \Rightarrow \theta = 19$

So, improved (revised) transportation schedule is as follows :

O \ D	D_1	D_2	D_3
O_1	0	2 (20)	19
O_2	11 (11)	18	19 (19)
O_3	7 (24)	14 (16)	19

Here, number of allocations = m + n – 1 = 5 and are at independent positions. So the solution is non-degenerate.

Again, we have to test this solution for optimality by Modi method. For this, we have,

(4) Determination of u_i and v_j :

For occupied cells : Using $u_i + v_j = c_{ij}$ (Set $u_1 = 0$), we have,

$$\left.\begin{array}{lll} u_1 + v_2 = 2 & \Rightarrow & v_2 = 2 \\ u_2 + v_1 = 11 & \Rightarrow & u_2 = 16 \\ u_2 + v_3 = 19 & \Rightarrow & v_3 = 3 \\ u_3 + v_1 = 7 & \Rightarrow & v_1 = -5 \\ u_3 + v_2 = 14 & \Rightarrow & u_3 = 12 \end{array}\right\} \Rightarrow \begin{array}{l|l} u_1 = 0 & v_1 = -5 \\ u_2 = 16 & v_2 = 2 \\ u_3 = 12 & v_3 = 3 \end{array}$$

(5) Determination of d_{ij} :

For unoccupied cells : Use $d_{ij} = c_{ij} - (u_i + v_j)$

$$d_{11} = u_{11} - (u_1 + v_1) = 0 - (-5) = 5$$

$$d_{13} = u_{13} - (u_1 + v_3) = 19 - (3) = 16$$

$$d_{22} = u_{22} - (u_2 + v_2) = 18 - 18 = 0$$

$$d_{33} = u_{33} - (u_3 + v_3) = 19 - 15 = 4$$

Here, each $d_{ij} \geq 0$, shows that the solution is optimal and at least one d_{ij} i.e. $d_{22} = 0$ indicates that alternative solutions will also exist.

Using improved (revised) table given in step 3, and second balanced table, we get the optimal total profit

$= (20 \times 17) + (11 \times 8) + (19 \times 0) + (24 \times 12) + (16 \times 5) = ₹\,796.$

(Note that the values 17, 8, 0, 12, 5 are taken from balanced table.)

Think Over It

1. What is the purpose of a transportation model 1 which is initially imbalance may require the addition of dummy supply destination to be used.
2. How the degeneracy in a transportation problem is resolved ?

Miscellaneous Exercise

(A) True/False Questions

1. For the following transportation problem, using least cost method, the cost of transportation is 115.

					Supply
	2	3	11	7	6
	1	0	6	1	1
	5	8	15	9	10
Demand	7	5	3	2	

2. A transportation model 1 which is initially imbalanced may require the addition of a dummy supply station or dummy destination for the purpose of effective balancing.
3. In an $m \times n$ transportation matrix the degeneracy occurs when at any stage the number of stone squares is less than $m + n - 1$.
4. There always exists an optimal solution to a balanced TP.
5. The number of basic variable in $m \times n$ transportation matrix are at the most $m + n - 1$.

(B) Multiple Choice Questions

1. Consider the following transportation table.

						Supply
	3	4	6	8	9	20
	2	10	1	5	8	30
	7	11	20	40	3	15
	2	1	9	14	16	13
Demand	40	6	8	18	6	

By using North-West corner rule method the transportation cost is ………

(A) 972 (B) 878

(C) 799 (D) 887

2. Degeneracy in a transportation problem is resolved by ………

(A) Stepping stone method.

(B) MODI method.

(C) Introducing infinitesimally small quantity to water squares (s).

(D) Introducing very large quantity to water square (s).

3. The dummy source or destination in TP is added to ....
   (A) satisfy rim condition
   (B) prevent solution from becoming degenerate
   (C) ensure that total cost does not exceed a limit
   (D) none of the above
4. The solution to a TP with m-rows (supplies) and n-columns (destinations) is feasible if number of positive allocations are ....
   (A) $m + n$ (B) $m \times n$
   (C) $m + n - 1$ (D) $m + n + 1$
5. The number of non-negative variables in basic feasible solution to $m \times n$ TP is ....
   (A) $m \times n$ (B) $m + n$
   (C) $m + n + 1$ (D) $m + n - 1$

(C) Theory Questions

1. What is transportation problem ?
2. Obtain the mathematical formulation of transportation problem.
3. Explain how a transportation problem is a special case of a simplex problem.
4. Define the following terms in T.P.
   (i) Feasible solution,
   (ii) Basic feasible solution,
   (iii) Optimal solution.
   (iv) Non-degenerate basic feasible solution.
5. Define 'loop' in T.P. What is the role of loop ? Give some properties.
6. Explain the following methods for obtaining an initial basic feasible solution of a transportation problem.
   (i) North-West Corner method (NWCM)
   (ii) Lowest cost entry method (Matrix-minima method)
   (iii) Vogel's approximation method (VAM).
7. Write short note on 'Optimum solution'.
8. What is optimal test ? When and how it is used ? Explain the steps involved in optimal test.
9. Explain steps involved in obtaining the optimum solution from initial basic feasible solution of T.P. by Modi method (Modified distribution method or u-v method).
10. Explain degeneracy in a transportation problem. How it can be removed ?
11. What do you understand by unbalanced transportation problem ? How would you convert it into balanced transportation problem ?

(D) Numerical Problems

1. Determine an initial basic feasible solution to the following T.P. ($O_i \equiv$ origin, $D_i \equiv$ destination)

	D_1	D_2	D_3	D_4	Supply
O_1	6	4	1	5	14
O_2	8	9	2	7	16
O_3	4	3	6	2	5
Demand	6	10	15	4	

2. Determine an initial basic feasible solution to following T.P.

	I	II	III	IV	
A	13	11	15	20	2
B	17	14	12	13	6
C	18	18	15	12	7
	3	3	4	5	

3. Obtain an initial basic feasible solution to the following T.P. by matrix-minima method.

	D_1	D_2	D_3	D_4	Capacity
O_1	1	2	3	4	6
O_2	4	3	2	0	8
O_3	0	2	2	1	10
Demand	4	6	8	6	

4. Obtain the initial basic feasible solution by VAM.

	D_1	D_2	D_3	D_4	Supply
S_1	21	16	15	13	11
S_2	17	18	14	23	13
S_3	32	27	18	41	19
Demand	6	10	12	15	

5. Determine an initial basic feasible solution to the following T.P. by

(i) North-West Cost Rule, (ii) Vogel's Approximation Method.

	D_1	D_2	D_3	D_4	D_5	Supply
O_1	2	11	10	3	9	4
O_2	1	4	7	2	1	8
O_3	3	9	4	8	12	9
Demand	3	3	4	5	6	

6. Obtain the IBFS of the following T.P. by VAM.

	D_1	D_2	D_3	Supply
O_1	13	15	16	17
O_2	7	11	2	12
O_3	19	20	9	16
Demand	14	8	23	

7. Obtain the IBFS by Vogel's method to the following T.P.

	D_1	D_2	D_3	Supply
O_1	50	30	200	1
O_2	90	45	170	3
O_3	250	200	50	4
Demand	4	2	2	

8. Solve the following T.P. (Obtain allocations by VAM).

	D_1	D_2	D_3	Available
O_1	2	7	4	5
O_2	3	3	1	8
O_3	5	4	7	7
O_4	1	6	2	14
Required	7	9	18	

9. Obtain IBFS by LCEM and hence solve the following T.P.

	D_1	D_2	D_3	D_4	Available
O_1	1	2	1	4	30
O_2	3	3	2	1	50
O_3	4	2	5	9	20
Required	20	40	30	10	

10. Find the optimum solution of the following T.P. (Obtain IBFS by VAM).

	D_1	D_2	D_3	D_4	D_5	a_i
O_1	3	4	6	8	8	20
O_2	2	10	1	5	30	30
O_3	7	11	20	40	15	15
Q_4	2	1	9	14	18	13
b_j	40	6	8	18	6	

11. Find the initial solution by VAM and then find optimal solution to the following T.P.

	D_1	D_2	D_3	D_4	Supply
O_1	1	2	– 2	3	70
O_2	2	4	0	1	38
O_3	1	2	– 2	5	32
Demand	40	28	30	32	

12. Find the initial solution by VAM and obtain optimal solution to the following T.P.

	D_1	D_2	D_3	D_4	Supply
O_1	5	3	6	4	30
O_2	3	4	7	8	15
O_3	9	6	5	8	15
Demand	10	25	18	7	

13. Obtain initial basic solution by VAM and hence find the minimum cost of the following T.P.

	D_1	D_2	D_3	D_4	Supply
O_1	23	27	16	16	30
O_2	12	17	20	51	40
O_3	22	28	12	32	53
Demand	22	35	25	41	

14. Find the initial allocation of the following T.P. by VAM and optimize it using Modi method.

(i)

	D_1	D_2	D_3	D_4	a_i
O_1	5	3	6	2	19
O_2	4	7	9	1	37
O_3	3	4	7	5	34
b_j	16	18	31	25	

(ii)

	D_1	D_2	D_3	D_4	a_i
O_1	3	8	9	16	8
O_2	6	11	14	6	9
O_3	5	13	10	12	13
b_j	6	7	7	10	

15. Find the initial solution by VAM, then find optimal cost.

	D_1	D_2	D_3	D_4	D_5	D_6	a_i
O_1	9	12	9	6	9	10	5
O_2	7	3	7	7	5	5	6
O_3	6	5	9	12	3	11	2
O_4	6	8	11	2	2	10	9
b_j	4	4	6	2	4	2	

16. A company has 4 warehouses and 6 stores, the cost of shipping one unit from warehouse i to store j is c_{ij} if

$$c_{ij} = \begin{bmatrix} 7 & 10 & 7 & 4 & 7 & 8 \\ 5 & 1 & 5 & 5 & 3 & 3 \\ 4 & 3 & 7 & 9 & 1 & 9 \\ 4 & 6 & 9 & 0 & 0 & 8 \end{bmatrix}$$

The requirements of stores and quantities at warehouses are 4, 4, 6, 2, 4, 2 and 5, 6, 2, 9 respectively. Find IBFS by LCEM and the minimum cost solution.

17. Using North-West Corner rule, find initial solution and obtain an optimum solution to the following degenerate T.P.

7	3	4	2
3	1	3	3
2	4	6	5
4	1	5	

18. Use NWCM to obtain initial basic feasible solution. Obtain also optimum solution to the following degenerate transportation problem.

	D_1	**D_2**	**D_3**	
O_1	0	2	1	5
O_2	2	1	5	10
O_3	3	4	3	5
	4	12	4	

19. Solve the following unbalanced T.P.

	D_1	**D_2**	**D_3**	
O_1	7	3	6	5
O_2	4	6	8	7
O_3	5	8	4	7
O_4	5	4	3	3
	5	8	10	

20. A company has three factories at A, B, C which supply to warehouses at D, E, F and G respectively. Monthly production capacities of three factories are 250, 300, and 400 units respectively. If overtime production is utilized, factories A and B can produce 50 and 75 additional units with an incremental cost of ₹ 4 and ₹ 5 respectively. The current warehouse requirements are 200, 225, 275 units respectively. The unit transportation costs in rupees from factories to warehouses are as under.

	D	**E**	**F**	**G**
A	11	13	17	14
B	16	18	14	10
C	21	24	13	10

Determine the optimum distribution for this company, to minimize the cost.

21. Find IBS by VAM and find an optimal solution of the following T.P.

	D_1	D_2	D_3	D_4	D_5	
O_1	50	80	60	60	30	800
O_2	40	70	70	60	50	600
O_3	80	40	60	60	40	2500
	400	400	500	400	800	

22. Find the optimum cost transportation schedule for the following T.P.

	Distributors				
Plants	D_1	D_2	D_3	D_4	**Availability**
P_1	19	30	50	12	7
P_2	70	30	40	60	10
P_3	40	10	60	20	18
Requirements	5	8	7	15	

23. Following is the initial basic feasible solution of the transportation problem. Is the solution optimal ? If not find the optimal solution. **(March 2010)**

5	8	6	6 (5)	3 (3)
4 (4)	7	7	6 (1)	5
8	4 (4)	6	6	4 (5)

24. A transportation problem has a feasible solution as given below :

10			10
	20	20	40
10	20	20	

25. Given the following cost matrix.

		Person			
		I	**II**	**III**	**IV**
	1	17	20	13	21
Job	**2**	15	21	14	18
	3	17	18	17	21
	4	14	22	12	22

Assign the jobs to the persons so as to minimize the total cost.

Answers

(A) True or False :

1. False 2. True 3. True 4. True 5. True

(B) Multiple Choice Questions :

1. (B) 2. (C) 3. (A) 4. (C) 5. (D)

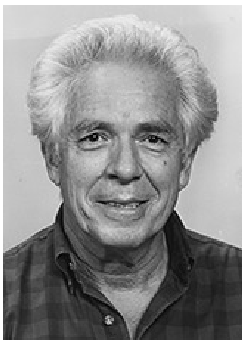
(D) Numerical Problems :

1. $x_{11} = 6$, $x_{12} = 8$, $x_{22} = 2$, $x_{23} = 14$, $x_{33} = 1$, $x_{34} = 4$, Cost = ₹ 128
2. $x_{11} = 2$, $x_{21} = 1$, $x_{22} = 3$, $x_{23} = 2$, $x_{33} = 2$, $x_{34} = 5$, Cost = ₹ 199/-
3. $x_{12} = 6$, $x_{23} = 2$, $x_{24} = 6$, $x_{31} = 4$, $x_{32} = 0$, $x_{33} = 6$
5. (i) $x_{11} = 3$, $x_{12} = 1$, $x_{22} = 2$, $x_{23} = 4$, $x_{24} = 2$, $x_{34} = 3$, $x_{35} = 6$, Cost = ₹ 153
   (ii) $x_{14} = 4$, $x_{22} = 2$, $x_{35} = 6$, $x_{34} = 3$, $x_{32} = 1$, $x_{33} = 4$, $x_{34} = 1$, Cost = ₹ 68/-
6. $x_{11} = 9$, $x_{12} = 3$, $x_{21} = 5$, $x_{23} = 7$, $x_{33} = 16$
7. $x_{11} = 1$, $x_{21} = 3$, $x_{31} = 0$, $x_{32} = 2$, $x_{33} = 2$
8. Optimal solution = ₹ 76/-
9. $x_{11} = 20$, $x_{13} = 10$, $x_{22} = 20$, $x_{23} = 20$, $x_{33} = 10$, $x_{32} = 20$
10. $x_{11} = 14$, $x_{15} = 6$, $x_{21} = 4$, $x_{23} = 8$, $x_{24} = 18$, $x_{31} = 15$, $x_{41} = 7$, $x_{42} = 6$, min cost = ₹ 321/-
11. $x_{11} = 40$, $x_{12} = 26$, $x_{14} = 4$, $x_{24} = 38$, $x_{32} = 2$, $x_{35} = 30$,
    Optimal solution = ₹ 86/-
12. $x_{12} = 20$, $x_{13} = 3$, $x_{14} = 7$, $x_{21} = 10$, $x_{22} = 5$, $x_{33} = 15$
    Optimal solution = ₹ 231/-
13. $x_{14} = 30$, $x_{21} = 5$, $x_{22} = 35$, $x_{31} = 17$, $x_{33} = 25$, $x_{34} = 11$,
    Cost = ₹ 1921/-
14. (i) $x_{11} = 18$, $x_{13} = 1$, $x_{21} = 12$, $x_{24} = 25$, $x_{31} = 4$, $x_{33} = 33$. Min cost = ₹ 355/-
    (ii) $x_{11} = 4$, $x_{12} = 4$, $x_{24} = 9$, $x_{31} = 2$, $x_{32} = 3$, $x_{33} = 7$, $x_{34} = 1$. Min cost = ₹ 256/-
15. $x_{13} = 5$, $x_{22} = 4$, $x_{26} = 2$, $x_{31} = 1$, $x_{41} = 3$, $x_{44} = 2$, $x_{45} = 4$. Optimal cost = ₹ 112/-
16. min cost = ₹ 68/-
17. $x_{13} = 2$, $x_{22} = 1$, $x_{23} = 2$, $x_{31} = 4$, $x_{33} = 1$, min cost = ₹ 29/-
18. $x_{11} = 4$, $x_{12} = 1$, $x_{22} = 10$, $x_{32} = 1$, $x_{33} = 4$, min cost = ₹ 28/-
19. ₹ 84/-
21. By VAM : $x_{15} = 800$, $x_{21} = 400$, $x_{24} = 200$, $x_{32} = 400$,
    $x_{33} = 500$, $x_{34} = 200$, Optimal solution = ₹ 110000/-
22. Optimum cost = ₹ 799/-
25. $1 \to$ III, $2 \to$ IV, $3 \to$ II, $4 \to$ I, Minimum cost = 63.

☆☆☆

Chapter 4...

Assignment Models



Harold William Kuhn

Harold William Kuhn (July 29, 1925 – July 2, 2014) was an American mathematician who studied game theory. He won the 1980 John von Neumann Theory Prize along with David Gale and Albert W. Tucker. A former Professor Emeritus of Mathematics at Princeton University, he is known for the Karush–Kuhn–Tucker conditions, for Kuhn's theorem, for developing Kuhn poker as well as the description of the Hungarian method for the assignment problem. Recently, though, a paper by Carl Gustav Jacobi, published posthumously in 1890 in Latin, has been discovered that anticipates by many decades the Hungarian algorithm.

4.1 Assignment Problem

Assignment problems is a special type of transportation problem. In assignment problem, the objective is to find the optimum allocation of a number of tasks (jobs or origins or sources) to an equal number of facilities (persons or machines or designations).

Associated to each assignment problem there is a matrix called cost matrix $[c_{ij}]$, where c_{ij} is the cost of assigning i^{th} facility to j^{th} job.

When the number of jobs is equal to the number of facilities, the problem under consideration is said to be balanced assignment problem.

Obviously, in this case the cost matrix is a square matrix.

When the number of jobs is not equal to the number of facilities, the problem is unbalanced problem. Here the cost matrix is not square matrix but we can make it a square matrix by introducing dummy row or dummy column.

All the elements in a dummy row or dummy column are kaken as zero.

Mathematical Formulation of Assignment Problem :

Let c_{ij} denote the cost of assigning i^{th} job to j^{th} facility where

$$i = 1, 2, 3, \ldots, n$$

$$j = 1, 2, 3, \ldots, n$$

The situation is now as shown in the $n \times n$ matrix as under.

Facilities

	1	2	3	…	n
	1	c_{11}	c_{13}	…	c_{1n}
	2	c_{21}	c_{23}	…	c_{2n}
	3		…	…	
	–		…		
Jobs	–		…		
	–				
	–				
	–				
	n	c_{n1}	c_{n3}	…	c_{nn}

The assignment model can be expressed mathematically as follows :

Let x_{ij} = 0, if i^{th} facility is not assigned to j^{th} job

= 1, if i^{th} facility is assigned to j^{th} job

The model is thus given by,

$$\text{Minimize, } Z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

4.2 Basic Theorems in Assignment Problem

The basic theorems in the assignment problem were proved by Hungarian mathematician 'Konig'. It states that :

(I) The optimal solution of the assignment model remains the same if a constant is added or subtracted to any row or column of the cost matrix, i.e. in an assignment problem if a constant is added or subtracted to every element of a row (or column) of the cost matrix, then an assignment which minimizes the total cost for one matrix, also minimize the total cost for the other matrix.

(II) If all $c_{ij} \geq 0$ and there exists a solution $x_{ij} = x_{ij}^*$ such that

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}^* = 0$$

then this solution is an optimal solution (i.e. this solution minimizes Z).

4.3 Assignment Algorithm

From the above basic theorems, we get a powerful method termed as "*assignment algorithm*" for solving an assignment problem. This method is also known as **Hungarian method** or **Reduced matrix method.** The steps involved in this method are as under :

Step I : Prepare the cost matrix, if not given.

Step II : Subtract the minimum element in a column of the matrix from all the elements in that column. Repeat the process for all the columns.

Step III : Subtract the minimum element in a row of the matrix from all the elements in that row. Repeat the process for all the rows.

Thus, there will be at least zero in each column and each row of the reduced matrix.

Step IV : The optimal assignment for the reduced matrix can be obtained as follows :

(a) Examine the rows successively until a row with exactly one unmarked zero observed. Enclose this zero in a square $\square$. An assignment is made at this point. Cross $\times$ other zeros in that column as they cannot be considered for assignment in future. Continue the process, until all the rows have been examined.

(b) Examine the columns successively until a column with exactly one unmarked zero observed. Enclose this zero in a square $\square$. Cross $\times$ all the other zeros in that row. Continue this process, until all the columns have been examined.

(c) Steps (a) and (b) are repeated successively until one of the following conditions satisfied.

(i) All the zeros are either marked $\square$ or crossed $\times$ and there is one assignment in each row and in each column. This is the optimal solution of the assignment problem.

(ii) There are at least two unmarked zeros in each row and each column. Here we assign one of the unmarked zeros arbitrarily and cross all other zeros in that row and column. Repeat the process until no unmarked zero is left.

(iii) There may be some row or column which have no assignment. Here we follow the next step.

Step V : Draw the minimum number of lines to cover all the zeros of the reduced matrix as follows :

(i) Tick (√) the rows for which the assignment has not been made.

(ii) Tick (√) the columns (not already ticked) but having zero in the ticked rows.

(iii) Tick (√) the rows (not already ticked) but having assigned zeros in the ticked column.

(iv) Repeat (i), (ii), (iii) until no more rows or columns are left.

(v) Draw horizontal lines through all unticked rows and vertical lines through all ticked columns.

Step VI : (a) Let the number of horizontal and vertical lines drawn be equal to the order n of the cost matrix. Here the solution is optimal.

(b) Let the number of horizontal and vertical lines drawn be less than n. Here we go to the next step.

Step VII : Examine the elements which are not covered by the lines. Select the minimum of all such elements and subtract it from all the elements not covered by the lines. Add this element to every element at the intersection of the lines drawn through rows and columns. Then we get a modified cost matrix.

Step VIII : Repeat the procedure in steps IV to VI until an optimal solution is obtained.

Step IX : The optimal assignment schedule corresponding to the assigned zeros obtained is noted down.

Step X : Optimal cost, if needed can be obtained by adding the costs at the assigned points in the cost matrix.

We illustrate the procedure by some examples.

Illustrative Examples

Example 4.1 : Four professors are capable of teaching any one of four different courses. The average weekly preparation time for each subject by each professor (in hours) is given below :

	I	II	III	IV
A	2	10	9	7
B	15	4	14	8
C	13	14	16	11
D	4	15	13	9

How to assign each professor, one and only one course so as to minimize the total course preparation time for all four courses ?

Solution : Consider the cost matrix.

	I	II	III	IV
A	2	10	9	7
B	15	4	14	8
C	13	14	16	11
D	4	15	13	9

Subtracting 2 from each element in the first column, 4 from each element in the second column, 9 from each element in the third column, 7 from each element in the fourth column, we get the following matrix :

	I	II	III	IV
A	0	6	0	0
B	13	0	5	1
C	11	10	7	4
D	2	11	4	2

Subtracting 0 from each element in the first row, 0 from each element in the second row, 4 from each element in the third row, 2 from each element in the fourth row, we get the following matrix :

	I	II	III	IV
A	0	6	0	0
B	13	0	5	1
C	7	6	3	0
D	0	9	1	0

Start examining the rows with only one unmarked zero. Second row contains only one unmarked zero. It is enclosed in a square $\square$. Cross $\times$ if any (here none) unmarked zero in the corresponding column (second column). Similarly enclose only one marked zero observed in the third row. Cross × all other unmarked zeros in the corresponding column. The procedure is repeated for all rows and we get the following assignment :

	I	II	III	IV
A	~~0~~	6	0	0
B	13	$\boxed{0}$	5	1
C	7	6	3	$\boxed{0}$
D	$\boxed{0}$	9	1	~~0~~

Start examining the columns from the column, since the third column contains only one unmarked zero. Enclose it in a square $\boxed{}$. Cross X all zeros in the row containing the marked zero (here none).

	I	II	III	IV
A	~~0~~	6	$\boxed{0}$	~~0~~
B	13	$\boxed{0}$	5	1
C	7	6	3	$\boxed{0}$
D	$\boxed{0}$	9	1	~~0~~

Note that at this stage, all zeros are enclosed in a square or crossed. Also note that every row and every column contains exactly one assigned zero, $\left(\text{enclosed in a square } \boxed{}\right)$. Hence the solution is optimal. The optimal assignment schedule is

Professor	Course	Preparation time(in hours)
A	III	9
B	II	4
C	IV	11
D	I	4

Thus, the total (minimum) time required $= 9 + 4 + 11 + 4 = 28$ hours.

Example 4.2 : Solve the following assignment problem.

	Job				
Person	**1**	**2**	**3**	**4**	**5**
A	8	4	2	6	1
B	0	9	5	5	4
C	3	8	9	2	6
D	4	3	1	0	3
E	9	5	8	9	5

(Oct. 99)

Solution : Subtracting 0 from each element in the first column, 3 from each element in the second column, 1 from each element in the third column, 0 from each element in the fourth column, 1 from each element in the fifth column, we get the following matrix :

	Job				
Person	**1**	**2**	**3**	**4**	**5**
A	8	1	1	6	0
B	0	6	4	5	3
C	3	5	8	2	5
D	4	0	0	0	2
E	9	2	7	9	4

Subtracting 0 from each element in the first row, 0 from each element in the second row, 2 from each element in the third row, 0 from each element in the fourth row, 2 from each element in the fifth row, we get the following matrix :

	Job				
Person	**1**	**2**	**3**	**4**	**5**
A	8	1	1	6	0
B	0	6	4	5	3
C	1	3	6	0	3
D	4	0	0	0	2
E	7	0	5	7	2

Start examining the rows with only one unmarked zero. Here first row contains only one unmarked zero. Enclose it in a square $\square$. Cross × all the zeros in the column containing the marked zero (here none). Second row contains only one unmarked zero. Enclose it in a square $\square$. Cross × all zeros in the column containing the marked zero (here none). Third row contains only one unmarked zero. Enclose it in a square $\square$. Cross × all zeros in the column containing the marked zero. Fifth row contains only one unmarked zero. Enclose it in a square $\square$. Crossing all other zeros in the column containing the marked zero, we get the following matrix :

	Job				
Person	**1**	**2**	**3**	**4**	**5**
A	8	1	1	6	$\boxed{0}$
B	$\boxed{0}$	6	4	5	3
C	1	3	6	$\boxed{0}$	3
D	4	$\cancel{0}$	0	0	2
E	7	$\boxed{0}$	5	7	2

Start examining the columns from the first, the third column contains only one unmarked zero. Enclose it in a square $\square$. Crossing all zeros in the row containing the marked zero, we get the following matrix :

	Job				
Person	**1**	**2**	**3**	**4**	**5**
A	8	1	1	6	$\boxed{0}$
B	$\boxed{0}$	6	4	5	3
C	1	3	6	$\boxed{0}$	3
D	4	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	2
E	7	$\boxed{0}$	5	7	2

The optimum assignment schedule is

Person	Job	Cost
A	5	1
B	1	0
C	4	2
D	3	1
E	2	5

$\therefore$ The minimum cost of assignment = 1 + 0 + 2 + 1 + 5 = ₹ 9

Example 4.3 : Jobs A, B, C are to be assigned to three machines X, Y, Z. The processing costs (in ₹) are given below. Find the allocation that minimizes the overall processing cost.

	Machine		
Job	**X**	**Y**	**Z**
A	19	28	31
B	11	17	16
C	12	15	13

Solution : Subtracting 11 from each element in the first column, 15 from each element in the second column, 13 from each element in the third column, we get the matrix as under

	Machine		
Job	**X**	**Y**	**Z**
A	8	13	18
B	0	2	3
C	1	0	0

Subtracting 8 from each element in the first row, 0 from each element in the second row, 0 from each element in the third row, we get the matrix as under

	Machine		
Job	**X**	**Y**	**Z**
A	0	5	10
B	0	2	3
C	1	0	0

Start examining the rows. The first row contains only one unmarked zero. Hence enclose it in a square $\boxed{}$. Crossing the other zeros in the first column, we get the matrix as under

	Machine		
Job	**X**	**Y**	**Z**
A	$\boxed{0}$	5	10
B	~~0~~	2	3
C	1	0	0

Start examining the columns. To the first column, already zero is assigned. In the second column, there is only one unmarked zero. Enclose it in a square $\boxed{}$. Cross × other zeros in that row.

	Machine		
Job	**X**	**Y**	**Z**
A	$\boxed{0}$	5	10
B	~~0~~	2	3
C	1	$\boxed{0}$	~~0~~

Here all the zeros are not marked $\boxed{}$ and cross ×. Hence this is not an optimal solution. Tick the unassigned second row. There is a crossed zero in the first column and second row. Hence tick first column. This column contains assigned zero in the first row. Hence tick first row. Since first row does not contain any other zero, hence ticking is complete.

Draw lines through unticked rows and ticked columns.

	Machine				
Job	**X**	**Y**	**Z**		
A	$\boxed{0}$	5	10	√	3
B	0	2	3	√	1
C	1	$\boxed{0}$	~~0~~	L_1	
	√				
	2				
	L_2				

Since 2 is minimum of the elements not on the lines. Subtracting 2 from each of these elements and adding 2 to the element at the intersection of the lines, we get the following reduced matrix :

	X	Y	Z
A	0	3	8
B	0	0	1
C	3	0	0

Examine the rows, starting with first. First row contains only one unmarked zero, hence enclose it in a square ☐ and across other zeros in the first column.

Second row contains only one unmarked zero, hence enclose it in a square ☐ and cross other zeros in the second column.

Third row contains only one unmarked zero, hence enclose it in a square ☐ and cross other zeros in the third column. The required matrix is

	Machine		
Job	**X**	**Y**	**Z**
A	$\boxed{0}$	3	8
B	$\cancel{0}$	$\boxed{0}$	1
C	3	$\cancel{0}$	$\boxed{0}$

Each row and each column contains an assigned zero, hence the assignment is optimum.

Job	**Machine**	**Cost**
A	X	19
B	Y	17
C	Z	13

The optimum cost (minimum cost) = 19 + 17 + 13 = ₹ 49

Example 4.4 : Solve the following assignment problem for minimum cost.

	I	**II**	**III**	**IV**	**V**
A	11	17	8	16	20
B	9	7	12	6	15
C	13	16	15	12	16
D	21	24	17	28	26
E	14	10	12	11	15

Solution : Subtracting 8 from each element in the first row, 6 from each element in the second row, 12 from each element in the third row, 17 from each element in the fourth row, 10 from each element in the fifth row, we get the matrix, as under

	I	**II**	**III**	**IV**	**V**
A	3	9	0	8	12
B	3	1	6	0	9
C	1	4	3	0	4
D	4	7	0	11	9
E	4	0	2	1	5

Subtracting 1 from each element in the first column, 0 from each element in the second column, 0 from each element in the third row, 0 from each element in the fourth row, 4 from each element in the fifth row, we get the matrix, as under

	I	**II**	**III**	**IV**	**V**
A	2	9	0	8	8
B	2	1	6	0	5
C	0	4	3	0	0
D	3	7	0	11	5
E	3	0	2	1	1

Start examining the rows. First row contains only one unmarked zero, hence enclose it in a square $\square$, cross the other zeros in the third column. Second row contains only one unmarked zero, hence enclose it in a square $\square$, cross the other zeros in the fourth column. Fifth row contains only one unmarked zero, hence enclose it in a square $\square$, cross the other zeros in second column (here none). Start examining the columns. First column contains only one unmarked zero, hence enclose it in a square $\square$, cross the other zeros in the third row.

	I	**II**	**III**	**IV**	**V**
A	2	9	$\boxed{0}$	8	8
B	2	1	6	$\boxed{0}$	5
C	$\boxed{0}$	4	3	~~0~~	~~0~~
D	3	7	~~0~~	11	5
E	3	$\boxed{0}$	2	1	1

Here all the zeros are not marked $\square$ and cross ×. Hence this is not optimal solution.

Tick the unassigned fourth row, there is a cross zero in the third column. Hence tick third column. This column contains assigned zero in the first row. Hence tick first row. Draw lines through unticked rows and ticked columns.

	I	**II**	**III**	**IV**	**V**	
A	2	9	$\boxed{0}$	8	8	√ 3
B	2	1	6	$\boxed{0}$	5	
C	$\boxed{0}$	4	3	0	0	
D	3	7	0	11	5	√ 1
E	3	$\boxed{0}$	2	1	1	
			√ 2			

We modify the reduced matrix by subtracting the element 2 from all the elements not covered by the lines and adding 2 at the intersection of two lines, we get the following matrix :

	I	II	III	IV	V
A	0	7	0	6	6
B	2	1	8	0	5
C	0	4	5	0	0
D	1	5	0	9	3
E	3	0	4	1	1

Now, repeating the same process starting from the first step, we have,

	I	II	III	IV	V
A	$\boxed{0}$	7	$\cancel{0}$	6	6
B	2	1	8	$\boxed{0}$	5
C	$\cancel{0}$	4	5	$\cancel{0}$	$\boxed{0}$
D	1	5	$\boxed{0}$	9	3
E	3	$\boxed{0}$	4	1	1

Each row and each column contains an assigned zero, hence the assignment is optimum.

A	I	11
B	IV	6
C	V	16
D	III	17
E	II	10

The minimum assignment cost = 11 + 6 + 16 + 17 + 10 = 60

Example 4.5 : Solve the following assignment problem for the minimum cost.

	I	II	III	IV	V
1	3	8	2	10	3
2	8	7	2	9	7
3	6	4	2	7	5
4	8	4	2	3	5
5	9	10	6	9	10

Solution : Subtracting 2 from each element in the first row, 2 from each element in the second row, 2 from each element in the third row, 2 from each element in the fourth row, 6 from each element in the fifth row, we get,

	I	II	III	IV	V
1	1	6	0	8	1
2	6	5	0	7	5
3	4	2	0	5	3
4	6	2	0	1	3
5	3	4	0	3	4

Subtracting 1 from each element in the first column, 2 from each element in the second column, 1 from each element in the fourth column, 1 from each element in the fifth column, we get,

	I	II	III	IV	V
1	0	4	0	7	0
2	5	3	0	6	4
3	3	0	0	4	2
4	5	0	0	0	2
5	2	2	0	2	3

Start examining the rows. Second row contains only one unmarked zero, enclose it in a square $\square$, cross other zeros in the third column. Third row contains only one unmarked zero, enclose it in a square $\square$, cross other zero in the second column, fourth row contains only one unmarked zero, enclose it in a square $\square$, cross the other zeros (here none).

	I	II	III	IV	V
1	0	4	0	7	0
2	5	3	$\boxed{0}$	6	4
3	3	$\boxed{0}$	$\cancel{0}$	4	2
4	5	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	2
5	2	2	$\cancel{0}$	2	3

Start examining columns. First column contains only one unmarked zero, enclose it in a square $\square$, cross the other zero in the first row.

	I	II	III	IV	V
1	$\boxed{0}$	4	$\cancel{0}$	7	0
2	5	3	$\boxed{0}$	6	4
3	3	$\boxed{0}$	0	4	2
4	5	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	2
5	2	2	$\cancel{0}$	2	3

Here all the zeros are not marked $\boxed{}$ and cross ×. Hence this is not optimal solution.

Tick the unassigned fifth row, there is a cross zero in the third column. Hence tick third column. This column contains assigned zero in the second row. Hence tick the second row. Draw lines through unticked rows and columns.

	I	II	III	IV	V		
1	$\boxed{0}$	4	0	7	0		
2	5	3	$\boxed{0}$	6	4	√	3
3	3	$\boxed{0}$	0	4	2		
4	5	0	0	$\boxed{0}$	2		
5	2	2	0	2	3	√	1
			√				
			2				

Modify the matrix by subtracting the minimum element 2 from all the elements not covered by the lines and adding the same element 2 at the intersection of two lines.

	I	II	III	IV	V
1	0	4	2	7	0
2	3	1	0	4	2
3	3	0	2	4	2
4	5	0	2	0	2
5	0	0	0	0	1

Now, repeating the same process, starting from the third step, we have,

	I	II	III	IV	V
1	$\cancel{0}$	4	2	7	$\boxed{0}$
2	3	1	$\boxed{0}$	4	2
3	3	$\boxed{0}$	2	4	2
4	5	$\cancel{0}$	2	$\boxed{0}$	2
5	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	1

Each row and each column contains an assigned zero, hence the assignment is optimum.

1	V	3
2	III	2
3	II	4
4	IV	3
5	I	9

The minimum cost according to the above assignment

$$= 3 + 2 + 4 + 3 + 9 = ₹\ 21$$

Example 4.6 : Six wagons A, B, C, D, E and F are available at six stations S_1, S_2, S_3, S_4, S_5 and S_6. The mileages between various stations are given below.

Stations → / Wagons ↓	S_1	S_2	S_3	S_4	S_5	S_6
A	30	33	28	20	26	30
B	60	30	27	26	25	21
C	70	40	50	65	18	17
D	16	17	20	30	110	19
E	28	29	38	27	70	80
F	19	20	30	40	50	65

How should the wagons be transported so as to minimize the total mileage covered ?

Solution : Subtracting 16 from each element in the first column, 17 from each element in the second column, 20 from each element in the third column, 20 from each element in the fourth column, 18 from each element in the fifth column, 19 from each element in the sixth column, we get the reduced matrix as under :

Stations / Wagons	S_1	S_2	S_3	S_4	S_5	S_6
A	14	16	8	0	8	13
B	44	13	7	6	7	4
C	54	23	30	45	0	0
D	0	0	0	10	92	2
E	12	12	18	7	52	63
F	3	3	10	20	32	48

Similarly, repeating the same process for rows, we get the following reduced matrix :

Stations / Wagons	S_1	S_2	S_3	S_4	S_5	S_6
A	14	16	8	0	8	13
B	40	9	3	2	3	0
C	54	23	30	45	0	0
D	0	0	0	10	92	2
E	5	5	11	0	45	56
F	0	0	7	17	29	45

Start examining the rows. First row contains only one unmarked zero, hence enclose it in a square $\boxed{}$, cross other zeros in the fourth column. Second row contains only one unmarked zero, hence enclose it in a square $\square$, and crossing other zeros in the sixth column, we get the following reduced matrix :

Wagons \ Stations	S_1	S_2	S_3	S_4	S_5	S_6
A	14	16	8	$\boxed{0}$	8	13
B	40	9	3	2	3	$\boxed{0}$
C	54	23	30	45	0	$\cancel{0}$
D	$\cancel{0}$	$\cancel{0}$	0	10	92	2
E	5	5	11	$\cancel{0}$	45	56
F	0	$\cancel{0}$	7	17	29	45

Start examining the columns. The third column contains only one unmarked zero, hence enclose it in a square $\square$, cross other zeros in the fourth row. First column contains only one unmarked zero hence enclose it in a square $\square$, cross other zeros in the sixth row. Fifth column contains only one unmarked zero, hence enclose it in a square $\square$, cross the other zeros (here none).

Wagons \ Stations	S_1	S_2	S_3	S_4	S_5	S_6
A	14	16	8	$\boxed{0}$	8	13
B	40	9	3	2	3	$\boxed{0}$
C	54	23	30	45	$\boxed{0}$	$\cancel{0}$
D	0	0	$\boxed{0}$	10	92	2
E	5	5	11	$\cancel{0}$	45	56
F	$\boxed{0}$	$\cancel{0}$	7	17	29	45

It should be noted that fifth row and second column does not contain an assigned zero. Hence the solution is not optimal.

Tick the unassigned fifth row, there is a cross zero in the fourth column. Hence tick fourth column. This column contains assigned zero in the first row. Hence tick first row.

Draw lines through unticked rows and ticked columns.

Stations / Wagons	S_1	S_2	S_3	S_4	S_5	S_6	
A	14	16	8	$\boxed{0}$	8	13	√ 3
B	40	9	3	2	3	$\boxed{0}$	
C	54	23	30	45	$\boxed{0}$	$\cancel{0}$	
D	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	10	92	2	
E	5	5	11	$\cancel{0}$	45	56	√ 1
F	$\boxed{0}$	$\cancel{0}$	7	17	29	45	
				√ 2			

We modify the reduced matrix by subtracting the minimum element 5 from all the elements not covered by the lines and adding 5 at the intersection of the two lines, we get the following matrix :

Stations / Wagons	S_1	S_2	S_3	S_4	S_5	S_6
A	9	11	3	0	3	8
B	40	9	3	7	3	0
C	54	23	30	50	0	0
D	0	0	0	15	92	2
E	0	0	6	0	40	51
F	0	0	7	22	29	45

Now, repeating the same process, starting from the third step, we have,

Stations / Wagons	S_1	S_2	S_3	S_4	S_5	S_6
A	9	11	3	$\boxed{0}$	3	8
B	40	9	3	7	3	$\boxed{0}$
C	54	23	30	50	$\boxed{0}$	$\cancel{0}$
D	$\cancel{0}$	0	$\boxed{0}$	15	92	2
E	0	$\cancel{0}$	6	$\cancel{0}$	40	51
F	0	0	7	22	29	45

Note that there are two unmarked zeros in fifth row and sixth row. We assign one of the zeros in the fifth row, enclose it in a square $\boxed{}$, cross all other zeros in the second column. Next assign zero in the sixth row, enclose it in a square $\boxed{}$, cross all other zeros in the first column.

Wagons \ Stations	S_1	S_2	S_3	S_4	S_5	S_6
A	9	11	3	$\boxed{0}$	3	8
B	40	9	3	7	3	$\boxed{0}$
C	54	23	30	50	0	0
D	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	15	92	2
E	$\boxed{0}$	$\cancel{0}$	6	0	40	51
F	$\cancel{0}$	$\boxed{0}$	7	22	29	45

Now, each row and each column contains exactly one assigned zero. So the solution is optimal. The optimal assignment schedule is as follows :

Wagons	Stations	Mileage
A	S_4	20
B	S_6	21
C	S_5	18
D	S_3	20
E	S_1	28
F	S_2	20

$\therefore$ The total minimum mileage covered = 20 + 21 + 18 + 20 + 28 + 20 = 127

Example 4.7 : Solve the assignment problem represented by the following matrix.

	I	**II**	**III**	**IV**	**V**	**VI**
A	9	22	58	11	19	27
B	43	78	72	50	63	48
C	41	28	91	37	45	33
D	74	42	27	49	39	32
E	36	11	57	22	25	18
F	3	56	53	31	17	28

Solution : Subtracting the smallest element of each row from every element of the corresponding row, we get the following reduced matrix :

	I	**II**	**III**	**IV**	**V**	**VI**
A	0	13	49	2	10	18
B	0	35	29	7	20	5
C	13	0	63	9	17	5
D	47	15	0	22	12	5
E	25	0	46	11	14	7
F	0	53	50	28	14	25

Now, subtracting the smallest element of each column from every element of the corresponding column, we get the following reduced matrix :

	I	**II**	**III**	**IV**	**V**	**VI**
A	0	13	49	0	0	13
B	0	35	29	5	10	0
C	13	0	63	7	7	0
D	47	15	0	20	2	0
E	25	0	46	9	4	2
F	0	53	50	26	4	20

Start examining the rows. Fifth row contains only one unmarked zero, hence enclose it in a square $\square$, cross the other zeros in the second column. Sixth row contains only one unmarked zero, hence enclose it in a square $\square$, cross the other zeros in the first column. Second row contains only one unmarked zero, hence enclose it in a square $\square$, cross the other zeros in the sixth column.

	I	**II**	**III**	**IV**	**V**	**VI**
A	$\cancel{0}$	13	49	0	0	13
B	$\cancel{0}$	35	29	5	10	$\boxed{0}$
C	13	$\cancel{0}$	63	7	7	$\cancel{0}$
D	47	15	$\boxed{0}$	20	2	$\cancel{0}$
E	25	$\boxed{0}$	46	9	4	2
F	$\boxed{0}$	53	50	26	4	20

Start examining the column. Third column contains only one unmarked zero, hence enclose it in a square $\square$, cross the other zeros (here none). Fourth column contains only one unmarked zero, hence enclose it in a square $\square$, cross the other zeros in the first row.

	I	**II**	**III**	**IV**	**V**	**VI**
A	$\cancel{0}$	13	49	$\boxed{0}$	$\cancel{0}$	13
B	$\cancel{0}$	35	29	5	10	$\boxed{0}$
C	13	$\cancel{0}$	63	7	7	$\cancel{0}$
D	47	15	$\boxed{0}$	20	2	$\cancel{0}$
E	25	$\boxed{0}$	46	9	4	2
F	$\boxed{0}$	53	50	26	4	20

Here, third row does not contain an assigned zero. Hence the solution is not optimal.

Tick the unassigned third row, there is cross in the second and sixth column. Hence tick second and sixth column. Second column contains assigned zero in the fifth row. Hence tick fifth row. Again sixth column contains assigned zero in the second row. Hence tick second row. Then we tick sixth row which has assignment in the tick column one.

We draw lines through all marked columns I, II, III. Then we draw lines through unmarked row I and IV. Thus, we get five lines (minimum number) to cover all the zeros.

	I	**II**	**III**	**IV**	**V**	**VI**	
A	$\cancel{0}$	13	49	0	$\cancel{0}$	13	
B	$\cancel{0}$	35	29	5	10	$\boxed{0}$	√ 5
C	13	$\cancel{0}$	63	7	7	$\cancel{0}$	√ 1
D	47	15	$\boxed{0}$	20	2	$\cancel{0}$	
E	25	$\boxed{0}$	46	9	4	2	√ 4
F	$\boxed{0}$	53	50	26	4	20	√
	√ 6	√ 2				√ 3	

The smallest of the elements which do not have a line through them is 4. Subtracting the element 4 from all the elements which do not have a line through them and adding 4 to every element that lies at the intersection of two lines, we get the following reduced matrix :

	I	II	III	IV	V	VI
A	4	17	49	0	0	17
B	0	35	25	1	6	0
C	13	0	59	3	3	0
D	51	19	0	20	2	4
E	25	0	42	5	0	2
F	0	53	46	22	0	20

Repeating again the same process, starting from step 3, we get the following reduced matrix :

	I	II	III	IV	V	VI
A	4	17	49	$\boxed{0}$	$\cancel{0}$	17
B	$\boxed{0}$	35	25	1	6	$\cancel{0}$
C	13	$\cancel{0}$	59	3	3	$\boxed{0}$
D	51	19	$\boxed{0}$	20	2	4
E	25	$\boxed{0}$	42	5	$\cancel{0}$	2
F	$\cancel{0}$	53	46	22	$\boxed{0}$	20

The optimal assignment is

A	IV	11
B	I	43
C	VI	33
D	III	27
E	II	11
F	V	17

Minimum cost = 11 + 43 + 33 + 27 + 11 + 17 = 142

4.4 Maximization in Assignment Problems

Sometimes we want the assignment to be such that the total optimum value is maximum. for example, outputs, profits, sales etc. To solve the maximization problem it has to be first converted into minimization problem. The method of solving minimization problem is already studied in chapter 2.

The steps in solving the maximization problems are as follows :

(1) Select the largest element of the cost matrix.

(2) Subtract each element of the given matrix from this largest element.

(3) The resulting new matrix is the matrix for minimization in the assignment problem.

The another method for converting the maximization problem to minimization problem is to multiply each element of the cost matrix by – 1, which gives the modified reduced matrix.

Illustrative Examples

Example 4.8 : Three different aeroplanes are to be assigned to handle three cargo consignments with a view to maximize profit (in lakh rupees). The profit matrix is as under :

		Cargo consignment		
		C_1	**C_2**	**C_3**
Aeroplanes	**A_1**	1	4	5
	A_2	2	3	3
	A_3	3	1	2

Work out the optimal assignment plane.

Solution : This is the problem of maximizing the total profit. We can convert this problem to a problem of minimization.

The largest element in the matrix is 5. Hence subtracting all the elements of the given matrix from 5, we get

4	1	0
3	2	2
2	4	3

Now, we proceed as minimization problem as follows :

Subtracting 2 from each element in the first column, 1 from each element in the second column, 0 from each element in the third column, we get the following reduced matrix :

2	0	0
1	1	2
0	3	3

Subtracting 1 from each element in the second row, we get the following reduced matrix :

2	0	0
0	0	1
0	3	3

Start examining row. The third row contains only one unmarked zero, hence enclose it in a square $\boxed{}$, crossing the other zeros in the first column, we get the following reduced matrix :

2	0	0
$\cancel{0}$	0	1
$\boxed{0}$	3	3

Start examining column. The third column contains only one unmarked zero, hence enclose it in a square $\boxed{}$, cross the other zeros in the first row. The second column contains only one unmarked zero, hence enclosing it in a square, we get the following reduced matrix :

2	$\cancel{0}$	$\boxed{0}$
$\cancel{0}$	$\boxed{0}$	1
$\boxed{0}$	3	3

Now, each row, each column has one assignment. So the solution is optimal. The optimal assignment schedule is as follows :

Aeroplane	**Cargo consignment**	**Profit (in lakh ₹)**
A_1	C_3	5
A_2	C_2	3
A_3	C_1	3

The optimal (maximum) profit = 5 + 3 + 3 = ₹ 11 lakhs.

Example 4.9 : A marketing manager wished to assign four zones Z_1, Z_2, Z_3 and Z_4 to four salesmen S_1, S_2, S_3 and S_4. These salesmen differ in different zones. The yield matrix is given below.

Zone / Salesmen	$\mathbf{Z_1}$	$\mathbf{Z_2}$	$\mathbf{Z_3}$	$\mathbf{Z_4}$
$\mathbf{S_1}$	7	12	37	18
$\mathbf{S_2}$	25	27	18	25
$\mathbf{S_3}$	16	3	17	23
$\mathbf{S_4}$	10	25	14	9

Give the assignment for maximizing the sale.

Solution : The problem is converted to minimization. By subtracting each element of the matrix from the matrix-maximum element 37, we get the following reduced matrix :

30	25	0	19
12	10	19	12
21	34	20	14
27	12	23	28

Subtracting the column minimum for respective column elements, we get the following reduced matrix :

18	15	0	7
0	0	19	0
9	24	20	2
15	2	23	16

Subtracting the row minimum from respective row elements, we get the following reduced matrix :

18	15	0	7
0	0	19	0
7	22	18	0
13	0	21	14

Start examining the rows. The first row contains only one unmarked zero. Hence enclose it in a square $\square$, cross the other zeros (here none). The third row contains only one unmarked zero, hence enclose it in a square $\square$, cross the other zeros in the fourth column. The fourth row contains only one unmarked zero, hence enclose it in a square $\square$, crossing the other zeros in the second column, we get the following reduced matrix :

18	15	$\boxed{0}$	7
0	$\cancel{0}$	19	$\cancel{0}$
7	22	18	$\boxed{0}$
13	$\boxed{0}$	21	14

Start examining the columns. The first column contains only one unmarked zero. Hence enclose it in a square $\square$, crossing the other zeros (here none), we get the following reduced matrix.

18	15	$\boxed{0}$	7
$\boxed{0}$	$\cancel{0}$	19	$\cancel{0}$
7	22	18	$\boxed{0}$
13	$\boxed{0}$	21	14

Here we observe that all zeros are either crossed or enclosed in a square. Also each row and each column contains only one assigned zero. So solution is optimal. The optimal assignment schedule is as follows.

Salesmen	Zone	Yield
S_1	Z_3	37
S_2	Z_1	25
S_3	Z_4	23
S_4	Z_2	25

The maximum yield = 37 + 25 + 23 + 25 = 110 units.

Example 4.10 : A student has to select one and only one elective in each semester and the same elective should not be selected in different semesters. Due to various reasons, the expected grades in each subject, if selected in different semesters vary and they are given below.

Subjects → / Semesters ↓	OR	Statistics	Graph Theory	Discrete Mathematics
I	F	E	D	C
II	E	E	C	C
III	C	D	C	A
IV	B	A	H	H

The grade points are : H = 10, A = 9, B = 8, C = 7, D = 6, E = 5 and F = 4.

How will the student select the electives in order to maximize the total expected points and what will be his maximum expected total points ?

Solution : The cost matrix from the above given table is

4	5	6	7
5	5	7	7
7	6	7	9
8	9	10	10

Since the problem is to maximize the total expected points, we convert the problem into that of minimization by subtracting all the elements of the grade points matrix from the highest grade points H (= 10). The reduced matrix is given below :

6	5	4	3
5	5	3	3
3	4	3	1
2	1	0	0

Subtracting the minimum element of each row, from all the elements of the respective row, we get the following reduced matrix :

3	2	1	0
2	2	0	0
2	3	2	0
2	1	0	0

Subtracting the minimum element of each column from all the elements of the respective column, we get the following reduced matrix :

1	1	1	0
0	1	0	0
0	2	2	0
0	0	0	0

Start examining the rows. First row contains only one unmarked zero, hence enclose it in a square $\boxed{}$, cross the other zeros in the fourth column. The reduced matrix is given below.

1	1	1	$\boxed{0}$
0	1	0	$\cancel{0}$
0	2	2	$\cancel{0}$
0	0	0	$\cancel{0}$

Start examining the columns. Second column contains only one unmarked zero, hence enclose it in a square $\boxed{}$, cross the other zeros in fourth row. The third column contains only one unmarked zero, hence enclose it in a square $\boxed{}$, cross the other zeros in the second row. The first column contains only one unmarked zero, hence enclose it in a square $\boxed{}$, crossing the other zeros (here none), we get the following reduced matrix :

1	1	1	$\boxed{0}$
$\cancel{0}$	1	$\boxed{0}$	$\cancel{0}$
$\boxed{0}$	2	2	$\cancel{0}$
$\cancel{0}$	$\boxed{0}$	0	$\cancel{0}$

Here, we observe that all zeros are either crossed or enclosed in a square. Also each row and each column contains only one assigned zero. So the solution is optimal. The optimal assignment is as follows :

Semester	**Subjects**	**Points**
I	Discrete Mathematics	7
II	Graph Theory	7
III	O.R.	7
IV	Statistics	9

Maximum expected total points = 7 + 7 + 7 + 9 = 30.

Example 4.11 : Five different machines can process any of the five required jobs as under, with different profits resulting from each assignment.

		Machine				
		A	**B**	**C**	**D**	**E**
	1	30	37	40	28	40
	2	40	24	27	21	36
Job	**3**	40	32	33	30	35
	4	25	38	40	36	36
	5	29	32	41	34	39

Find the maximum profit through optimum assignment.

Solution : The problem is maximization problem. So we can convert it to a minimization problem. Here the maximum element is 41. Subtracting each element of the matrix from 41, the reduced matrix is given below.

11	4	1	13	1
1	17	14	20	5
1	9	8	11	6
16	3	1	5	5
12	9	0	7	2

Subtracting the minimum element of each column from all the elements of the respective columns, we get the following reduced matrix :

10	1	1	8	0
0	14	14	15	4
0	6	8	6	5
15	0	1	0	4
11	6	0	2	1

Start examining the rows. First row contains a single unmarked zero. Enclose it in a square $\boxed{}$. There is no other zero in the column containing this marked zero. The second row contains only one unmarked zero. Enclose it in a square $\boxed{}$, cross other zeros in the first column. Fifth row contains only one unmarked zero. Enclose it in a square $\boxed{}$, crossing the other zeros in the column (here none), we get the following reduced matrix :

10	1	1	8	$\boxed{0}$
$\boxed{0}$	14	14	15	4
~~0~~	6	8	6	5
15	0	1	0	4
11	6	$\boxed{0}$	2	1

Start examining the columns. Second column contains only one unmarked zero. Enclose it in a square $\boxed{}$, cross the other zeros in the fourth row. The reduced matrix is given below.

10	1	1	8	$\boxed{0}$
$\boxed{0}$	14	14	15	4
~~0~~	6	8	6	5
15	$\boxed{0}$	1	~~0~~	4
11	6	$\boxed{0}$	2	1

Here all the zeros are not marked $\square$ and cross ×. Hence the assignment is not optimum.

Tick the unassigned third row, there is a cross zero in the first column. Hence tick first column. First column contains assigned zero in the second row. Tick second row. Now ticking is complete.

Draw minimum number of lines through ticked columns and unticked rows.

10	1	1	8	$\boxed{0}$		
$\boxed{0}$	14	14	15	4	√	3
0	6	8	6	5	√	1
15	$\boxed{0}$	1	0	4		
11	6	$\boxed{0}$	2	1		
√						
2						

Consider the elements not on the lines. Here 4 is the minimum of all these elements. Subtract 4 from all the elements not on the lines. Adding 4 to the elements at the intersection of two lines, we get the reduced matrix as under :

14	1	1	8	0
0	10	10	11	0
0	2	4	2	1
19	0	1	0	4
15	6	0	2	1

Start examining the rows. First row contains only one unmarked zero. Enclose it in a square $\square$. Cross the other zeros in the fifth column. The second row contains only one unmarked zero. Enclose it in a square. Cross the other zeros in the first column. Fifth row contains only one unmarked zero. Enclose it in a square $\square$, cross the other zeros in the column (here none).

Start examining columns. Second column contains only one unmarked zero. Enclose it in a square $\square$. Crossing the other zeros in the fourth row, we get the following reduced matrix :

14	1	1	8	$\boxed{0}$
$\boxed{0}$	10	10	11	$\times$
$\times$	2	4	2	1
19	$\boxed{0}$	1	$\times$	4
15	6	$\boxed{0}$	2	1

This is not optimal solution. Tick unmarked third row. It contains crossed zero in the first column. Hence tick first column. First column contains marked zero in the second row. Tick second row. Second row contains crossed zero in the fifth column. Tick fifth column. Fifth column contains marked zero in the first row. Tick first row. Ticking is complete.

14	1	1	8	$\boxed{0}$	√	5
$\boxed{0}$	10	10	11	$\cancel{0}$	√	3
$\cancel{0}$	2	4	2	1	√	1
19	$\boxed{0}$	1	$\cancel{0}$	4		
15	6	$\boxed{0}$	2	1		
√				√		
2				4		

Draw minimum number of lines through ticked columns and unticked rows.

Consider all the elements not on these lines, of which 1 is the least element. Subtract 1 from each of these elements. Adding 1 to the elements at the point of intersections of two lines, we obtain the reduced matrix as under :

14	0	0	7	0
0	9	9	10	0
0	1	3	1	1
20	0	1	0	5
16	6	0	2	2

Start examining the rows. Third row contains a single unmarked zero. Enclose it in a square $\square$. Cross the other zeros in the first column. Fifth row contains a single unmarked zero. Enclose it in a square $\square$. Cross other zeros in the third column.

Start examining the columns. Fourth column contains only one unmarked zero. Enclose it in a square $\square$. Crossing other zeros in the fourth row, we obtain the following reduced matrix :

14	$\cancel{0}$	0	7	0
$\cancel{0}$	9	9	10	0
$\boxed{0}$	1	3	1	1
20	$\cancel{0}$	1	$\boxed{0}$	5
16	6	$\boxed{0}$	2	2

Start examining the rows. Second row contains only one unmarked zero. Enclose it in a square $\square$. Cross other zeros in the fifth column.

Start examining the columns. Second column contains only one unmarked zero. Enclosing it in a square $\square$, we get the following reduced matrix :

14	$\boxed{0}$	$\cancel{0}$	7	0
$\cancel{0}$	9	9	10	$\boxed{0}$
$\boxed{0}$	1	3	1	1
20	$\cancel{0}$	1	$\boxed{0}$	5
16	6	$\boxed{0}$	2	2

Here all zeros are either crossed or marked. Each row and each column contains one assigned zero. Hence the assignment is optimum. The optimum assignment schedule is as follows :

Job	Machine	Profit
1	B	37
2	E	36
3	A	40
4	D	36
5	C	41

The maximum profit = 37 + 36 + 40 + 36 + 41 = 190 units.

Example 4.12 : A company has four zeros open and four salesman available for assignment. The zones are not equally rich in their sales potential. It is estimated that a typical salesman operating in each zone would bring the following annual sales.

Zones	I	II	III	IV
Annual sales	60,000	50,000	40,000	30,000

The four salesmen are also considered to differ in ability. It is estimated that, working under the same condition, their yearly sales would be proportionally as under.

Salesman	A	B	C	D
Proportion	7	5	5	4

If the criterion is maximum expected total sales, the intuitive answer is to assign the best salesman to the richest zone, the next best salesman to the second richest zone and so on. Verify this answer by the assignment technique.

Solution : The sum of the proportions of sales of four salesmen

$$= 7 + 5 + 5 + 4 = 21$$

Taking ₹ 1000 as one unit,

For A : $\frac{7}{21} \times 6, \quad \frac{7}{21} \times 5, \quad \frac{7}{21} \times 4, \quad \frac{7}{21} \times 3$

For B : $\frac{5}{21} \times 6, \quad \frac{5}{21} \times 5, \quad \frac{5}{21} \times 4, \quad \frac{5}{21} \times 3$

For C : $\frac{5}{21} \times 6, \quad \frac{5}{21} \times 5, \quad \frac{5}{21} \times 4, \quad \frac{5}{21} \times 3$

For D : $\frac{4}{21} \times 6, \quad \frac{4}{21} \times 5, \quad \frac{4}{21} \times 4, \quad \frac{4}{21} \times 3$

In order to avoid the fractional values of the annual sales; it will be convenient to consider the sales for 21 years. The problem, therefore, can be rewritten as :

		Zone			
		I	**II**	**III**	**IV**
	A	42	35	28	21
Salesman	**B**	30	25	20	15
	C	30	25	20	15
	D	24	20	16	12

This is a maximization problem. We can convert it to minimization problem by – 1. Thus, the resulting matrix is as follows :

– 42	– 35	– 28	– 21
– 30	– 25	– 20	– 15
– 30	– 25	– 20	– 15
– 24	– 20	– 16	– 12

Subtracting the minimum element of each row from that row, we get the following matrix :

0	7	14	21
0	5	10	15
0	5	10	15
0	4	8	12

Subtracting the minimum element of each column from that column, we get the following matrix :

0	3	6	9
0	1	2	3
0	1	2	3
0	0	0	0

Draw the minimum number of lines to cover all the zeros of the reduced matrix.

$\boxed{0}$	3	6	9
$\cancel{0}$	1	2	3
$\cancel{0}$	1	2	3
0	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$

Subtracting the lowest element 1 from all the elements not covered by the lines and adding the same at the intersection of two lines, we obtain the following matrix :

$\boxed{0}$	2	5	8	√	4
0	$\boxed{0}$	1	2	√	5
0	0	1	2	√	1
1	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$		
√	√				
2	3				

We again modify the above table by subtracting the lowest element 1 from all the elements not covered by the lines and adding at the intersection of two lines, we get the reduced table as follows :

(I)

$\boxed{0}$	2	4	7
$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	1
$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	1
2	1	$\cancel{0}$	$\boxed{0}$

OR

(II)

$\boxed{0}$	2	4	7
$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	1
0	$\boxed{0}$	$\cancel{0}$	1
2	1	$\cancel{0}$	$\boxed{0}$

For I : Optimum assignments are :

A	I	42
B	II	25
C	III	20
D	IV	12

Maximum sales = 42 + 25 + 20 + 12 = 99 units = 99000

For II : Optimum assignments are :

A	I	42
B	III	20
C	II	25
D	IV	12

Maximum sales = 42 + 20 + 25 + 12 = 99 units = 99000

4.5 Unbalanced Assignment Problems

When the number of jobs is not equal to the number of facilities, the assignment problem is said to be unbalanced assignment problem. In this case, the given cost matrix is not square matrix (i.e. number of rows ≠ number of columns).

In such a situation, to make the assignment problem balanced, the matrix has to be made square matrix. This is done by adding dummy rows or columns. The cost of these dummy elements is taken to be sufficiently large so that the assignment would be least preferred.

Example 4.13 : Solve the following problem to minimize overall effectiveness. The effectiveness matrix is given below.

		Jobs		
		I	**II**	**III**
	A	7	3	5
Persons	**B**	2	7	4
	C	6	5	3
	D	3	4	7

Solution : The cost matrix is not square matrix. We make it square by introducing a dummy job with sufficiently large effectiveness say 50 for each person. We obtain the following matrix.

7	3	5	50
2	7	4	50
6	5	3	50
3	4	7	50

Subtracting column minimum from each column the following reduced matrix is obtained.

5	0	2	0
0	4	1	0
4	2	0	0
1	1	4	0

Since each row, each column contains at least one zero, start examining the rows. Fourth row contains only one unmarked zero. Enclose it in a square $\square$. Cross the other zeros in the fourth column. The following matrix is obtained.

5	0	2	$\xcancel{0}$
0	4	1	$\xcancel{0}$
4	2	0	$\xcancel{0}$
1	1	4	$\boxed{0}$

Start examining the columns. The first column contains only one unmarked zero. Enclose it in a square $\square$. Cross the other zeros (here none). Second column contains only one unmarked zero. Enclose it in a square $\square$. Cross the other zeros (here none). Third column contains only one unmarked zero. Enclose it in a square $\square$. Cross the other zeros (here none).

5	$\boxed{0}$	2	$\xcancel{0}$
$\boxed{0}$	4	1	$\xcancel{0}$
4	2	$\boxed{0}$	$\xcancel{0}$
1	1	4	$\boxed{0}$

It should be noted that each row and each column contains one assigned zero. So the solution is optimal. Neglecting the assignment of person D to dummy job IV, we get the following optimal assignment :

Person	**Job**	**Effectiveness**
A	II	3
B	I	2
C	III	3

The total minimum effectiveness = $3 + 2 + 3 = 8$ units.

Example 4.14 : A company is faced with the problem of assigning 4 machines to 6 different jobs (one machine to one job only). The profits are estimated as below.

		Machine			
		A	**B**	**C**	**D**
	1	3	6	2	6
	2	7	1	4	4
Job	**3**	3	8	5	8
	4	6	4	3	7
	5	5	2	4	3
	6	3	7	8	4

Solve the problem to maximize the total profits.

Solution : The problem is unbalanced problem. We make it balanced by adding two dummy columns. The reduced matrix thus obtained is as under.

3	6	2	6	0	0
7	1	4	4	0	0
3	8	5	8	0	0
6	4	3	7	0	0
5	2	4	3	0	0
3	7	8	4	0	0

Again, this is the assignment problem of maximization. We convert it into minimization by subtracting all the elements of the profit matrix from maximum element viz. 8.

5	2	6	2	0	0
1	7	4	4	0	0
5	0	3	0	0	0
2	4	5	1	0	0
3	6	4	5	0	0
5	1	2	4	0	0

Subtracting minimum element of each column from that column, we get the following matrix.

4	2	4	2	0	0
0	7	2	4	0	0
4	0	1	0	0	0
1	4	3	1	0	0
2	6	2	5	0	0
4	1	0	4	0	0

Making proper assignment in rows and columns, draw minimum number of lines to cover all the zeros.

4	2	4	2	$\boxed{0}$	$\cancel{0}$	√
$\boxed{0}$	7	2	4	$\cancel{0}$	$\cancel{0}$	
4	$\boxed{0}$	1	$\cancel{0}$	0	$\cancel{0}$	
1	4	3	1	$\cancel{0}$	$\boxed{0}$	√
2	6	2	5	$\cancel{0}$	$\cancel{0}$	√
2	1	$\boxed{0}$	4	$\cancel{0}$	$\cancel{0}$	
				√	√	

Next, subtract smallest element '1' from all the elements not covered by the lines and adding the same element (i.e. 1) at the intersection of two lines; we obtain the following reduced matrix.

3	1	3	1	0	0
0	7	2	4	1	1
4	0	1	0	1	1
0	3	2	0	0	0
1	5	1	4	0	0
2	1	0	4	1	1

Start examining the columns. Second column contains only one unmarked zero. Enclose it in a square $\boxed{}$. Cross the other zeros in third row. Third row contains only one unmarked zero. Enclose it in a square $\boxed{}$. Cross other zeros (here none). Fourth columns contains only one unmarked zero. Enclose it in a square $\boxed{}$. Cross the other zeros in the fourth row. Next start examining the rows and proceed as above. We get the following reduced matrix.

3	1	3	1	0	$\boxed{0}$
$\boxed{0}$	7	2	4	1	1
4	$\boxed{0}$	1	$\cancel{0}$	1	1
$\cancel{0}$	3	2	$\boxed{0}$	0	$\cancel{0}$
1	5	1	4	$\boxed{0}$	$\cancel{0}$
2	1	$\boxed{0}$	4	1	1

Here all zeros are either crossed or marked. Each row and each column contains one assigned zero. Hence the assignment is optimum. The optimum assignment schedule is as follows :

Job	**Machine**	**Profit**
1	–	–
2	A	7
3	B	8
4	D	7
5	–	–
6	C	8

Maximum profit according to this assignment schedule

$$= 7 + 8 + 7 + 8 = 30$$

(Since Job 1 and Job 5 are not assigned to any of the machines).

Example 4.15 : A company has one surplus truck in each of the cities A, B, C, D and E and one deficit truck in each of the cities 1, 2, 3, 4, 5 and 6. The distance between the cities in km is shown in the matrix below. Find the assignment of trucks from cities in surplus to cities in deficit so that the total distance covered by vehicle is minimum.

	1	**2**	**3**	**4**	**5**	**6**
A	12	10	15	22	18	8
B	10	18	25	15	16	12
C	11	10	3	8	5	9
D	6	14	10	13	13	12
E	8	12	11	7	13	10

Solution : Here the number of rows $\neq$ the number of columns. Therefore, it is the unbalanced assignment problem. We add dummy city (i.e. dummy row) with surplus vehicle. Since there is no distance associated with it, we take all the elements in the dummy row to be zero. We obtain the following reduced matrix.

12	10	15	22	18	8
10	18	25	15	16	12
11	10	3	8	5	9
6	14	10	13	13	12
8	12	11	7	13	10
0	0	0	0	0	0

Subtracting the minimum element of each row from that row, we get the following reduced matrix.

4	2	7	14	10	0
0	8	15	5	6	2
8	7	0	5	2	6
0	8	4	7	7	6
1	5	4	0	6	3
0	0	0	0	0	0

Start examining the columns. Second column contains only one unmarked zero. Enclose it in a square $\boxed{}$. Cross the other zeros in the sixth row. Third columns contains only one unmarked zero. Enclose it in a square $\boxed{}$. Cross the other zeros (here none). Fourth column contains only one unmarked zero. Enclose it in a square $\boxed{}$. Cross the other zeros (here none). Six column contains only one unmarked zero. Enclose it in a square $\boxed{}$. Cross the other zeros (here none). We obtain the following reduced matrix.

4	2	7	14	10	$\boxed{0}$
0	8	15	5	6	2
8	7	$\boxed{0}$	5	2	6
0	8	4	7	7	6
1	5	4	$\boxed{0}$	6	3
$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$

Start examining the rows. The second row contains only one unmarked zero. Enclose it in a square $\square$. Cross the other zeros in the first column. We obtain the following reduced matrix.

4	2	7	14	10	$\boxed{0}$
$\boxed{0}$	8	15	5	6	2
8	7	$\boxed{0}$	5	2	6
$\cancel{0}$	8	4	7	7	6
1	5	4	$\boxed{0}$	6	3
0	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$

It should be noted that fourth row has no assigned zero. Hence the solution is not optimal.

Tick unassigned fourth row, there is a cross in the first column, hence tick first column. This column contains assigned zero in the second row, hence tick second row.

Draw lines through unticked rows and ticked columns.

4	2	7	14	10	$\boxed{0}$		
$\boxed{0}$	8	15	5	6	2	√	3
8	7	$\boxed{0}$	5	2	6		
$\cancel{0}$	8	4	7	7	6	√	1
1	5	4	$\boxed{0}$	6	3		
$\cancel{0}$	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$		
√							
2							

We modify the reduced matrix by subtracting the minimum element 2 from all the elements not covered by the lines and adding 2 at the intersection of two lines, we get the following reduced matrix :

6	2	7	14	10	0
0	6	13	3	4	0
10	7	0	5	2	6
0	6	2	5	5	4
3	5	4	0	6	3
2	0	0	0	0	0

Now, repeat the same process, starting from the third step. Start examining the columns, second column contains only one unmarked zero. Enclose it in a square $\square$. Cross the other zeros in the sixth row. Third column contains only one unmarked zero. Enclose it in a square $\square$. Cross the other zeros (here none). Fourth column contains only one unmarked zero. Enclose it in a square $\square$. Crossing the other zeros (here none), we obtain the following reduced matrix :

6	2	7	14	10	0
0	6	13	3	4	0
10	7	$\boxed{0}$	5	2	6
0	6	2	5	5	4
3	5	4	$\boxed{0}$	6	3
2	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$

Start examining the rows. First row contains only one unmarked zero. Enclose it in a square $\square$. Cross the other zeros in the sixth column. Second row contains only one unmarked zero. Enclose it in a square $\square$. Crossing other zeros in the first column, we obtain the following reduced matrix.

6	2	7	14	10	$\boxed{0}$
$\boxed{0}$	6	13	3	4	$\cancel{0}$
10	7	$\boxed{0}$	5	2	6
$\cancel{0}$	6	2	5	5	4
3	5	4	$\boxed{0}$	6	3
2	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$

It should be noted that fourth row do not contain an assigned zero. Hence the solution is not optimal.

Tick the unassigned fourth row, there is a cross in the first column. Hence tick first column. This column contains assigned zero in the second row, hence tick second row. There is a cross in the sixth column, hence tick sixth column. This column contains assigned zero in the first row, hence tick first row.

Draw lines through unticked rows and ticked columns.

6	2	7	14	10	$\boxed{0}$	√	5
$\boxed{0}$	6	13	3	4	0	√	3
10	7	$\boxed{0}$	5	2	6		
0	6	2	5	5	4		
3	5	4	$\boxed{0}$	6	3	√	1
2	$\boxed{0}$	0	0	0	0		
√ 2					√ 4		

Next, subtract lowest element 2 from all the elements not covered by the lines and adding the same element (i.e. 2) at the intersection of the two lines, we obtain the following reduced matrix :

6	0	5	12	8	0
0	4	11	1	2	0
12	7	0	5	2	8
0	4	0	3	3	4
5	5	4	0	6	5
4	0	0	0	0	2

Again, repeating the same process starting from the third step, we have :

Fifth column contains only one unmarked zero. Enclose it in a square $\square$. Cross the other zeros in the sixth row. Fourth column contains only one unmarked zero. Enclose it in a square $\square$. Cross the other zeros (here none). Second column contains only one unmarked zero. Enclose it in a square $\square$. Cross the other zeros (here none). Third row contains only one unmarked zero. Enclose it in a square $\square$. Cross the other zeros in the third row. Fourth row contains only one unmarked zero. Enclose it in a square $\square$. Cross the other zeros in the first column. Second row contains only one unmarked zero. Enclose it in a square $\square$. Cross the other zeros (already done). The reduced matrix thus obtained is as under.

6	$\boxed{0}$	5	12	8	$\cancel{0}$
$\cancel{0}$	4	11	1	2	$\boxed{0}$
12	7	$\boxed{0}$	5	2	8
$\boxed{0}$	4	$\cancel{0}$	3	3	4
5	5	4	$\boxed{0}$	6	5
4	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	2

Here, we observed that all zeros are either crossed or enclosed in a square. Also each row and each column contains only one assigned zero. So solution is optimal.

The optimal assignment is as follows :

City	Vehicle	Distance
A	2	10
B	6	12
C	3	3
D	1	6
E	4	7

$\therefore$ Minimum distance travelled $= 10 + 12 + 3 + 6 + 7 = 38$ km.

4.6 Assignment Problems with Restrictions

In a general assignment problem, there may be restriction(s) on the assignment(s) of job(s) to the facilities. In other words, a particular job cannot be assigned to a particular facility. This difficulty can be solved by assigning a very high cost (∞) to the corresponding element in the matrix, so that possibility of assignment of such restricted element will be automatically excluded from the optimal assignment schedule.

Example 4.16 : Consider the problem of assigning four operators to four machines. The assignment costs in ₹ are given below. Operator 1 cannot be assigned to machine 3. Also operator 3 cannot be assigned to machine 4. Find the optimal assignment so that the total cost will be minimum.

		Machines			
		1	**2**	**3**	**4**
Operators	**1**	5	5	–	2
	2	7	4	2	3
	3	9	3	5	–
	4	7	2	6	7

Solution : A very high cost (say ∞) is assigned to the restricted element of the given matrix. Therefore, the cost matrix is

5	5	∞	2
7	4	2	3
9	3	5	∞
7	2	6	7

Subtracting 5 from the elements in the first column, 2 from elements in the second column, 2 from the elements in the third column and 2 from the elements in the fourth column, the reduced matrix is as under

0	3	∞	0
2	2	0	1
3	0	2	∞
2	0	4	5

Start examining the rows. Second row contains only one unmarked zero. Enclose it in a square $\square$. Cross the other zeros in the column (here none). Third row contains only one unmarked zero. Enclose it in a square $\square$. Cross the other zeros in the second column containing this marked zero.

0	3	∞	0
2	2	$\boxed{0}$	1
3	$\boxed{0}$	2	∞
2	$\times$	4	5

Examine the columns. First column contains only one unmarked zero. Enclose it in a square $\square$. Cross the other zeros in the first row. The reduced matrix is as under

$\boxed{0}$	3	∞	$\times$
2	2	$\boxed{0}$	1
3	$\boxed{0}$	2	∞
2	$\times$	4	5

Tick the unassigned fourth row. There is a crossed row in the second column. Tick second column. This column contains assigned zero in the third row. Tick third row. Ticking is complete.

Draw minimum number of lines through unticked rows and ticked columns.

$\boxed{0}$	3	∞	$\cancel{0}$		
2	2	$\boxed{0}$	1		
3	$\boxed{0}$	2	∞	√	3
2	$\cancel{0}$	4	5	√	1
	√				
	2				

Consider the elements not on the lines. Here 2 is the minimum of all these elements. Subtract 2 from all the elements not on the lines. Adding 2 to the elements at the intersection of two lines, we get the reduced matrix as follows :

0	5	∞	0
2	4	0	1
1	0	0	∞
0	0	2	3

Start examining rows. Second row contains only one unmarked zero. Enclose it in a square □. Cross the other zeros in the third column. Third row contains only one unmarked zero. Enclose it in a square □. Cross the other zeros in the second column. Fourth row contains only one unmarked zero. Enclose it in a square □. Crossing the other zeros in the first column, we obtain the following reduced matrix.

$\cancel{0}$	5	∞	0
2	4	$\boxed{0}$	1
1	$\boxed{0}$	$\cancel{0}$	∞
$\boxed{0}$	$\cancel{0}$	2	3

Now, examine the columns. Fourth column contains only one unmarked zero. Enclose it in a square □. Cross other zeros in the column (here none). The modified matrix is given below :

$\cancel{0}$	5	∞	$\boxed{0}$
2	4	$\boxed{0}$	1
1	$\boxed{0}$	$\cancel{0}$	∞
$\boxed{0}$	$\cancel{0}$	2	3

All the zeros are either marked or crossed. Each row and each column contains exactly one marked zero. Hence the solution is optimal. The optimal assignment schedule is as under.

Operators	**Machines**	**Cost**
1	4	2
2	3	2
3	2	3
4	1	7

The total minimum cost $= 2 + 2 + 3 + 7 =$ ₹ 14/-

Example 4.17 : Four new machines M_1, M_2, M_3, and M_4 are to be installed in a machine shop. There are five vacant places A, B, C, D and E available. Due to limited space, machine M_2 cannot be placed at C and M_3 cannot be placed at A. The cost of locating of various machines in various places is given in the following cost matrix (in ₹). Find the optimal assignment schedule.

	A	**B**	**C**	**D**	**E**
M_1	9	11	15	10	11
M_2	12	9	–	10	9
M_3	–	11	14	11	7
M_4	14	8	12	7	8

Solution : The given problem is unbalanced and restricted assignment problem.

There are 5 places and 4 machines. Here we add dummy machines having cost 30 (say) so that the problem will be balanced. Further there is a restriction on the elements. So on the place of restriction put highest cost ∞. The reduced matrix thus obtained is as under.

9	11	15	10	11
12	9	∞	10	9
∞	11	14	11	7
14	8	12	7	8
30	30	30	30	30

Subtracting the minimum element of each column from that column, we get the following matrix.

0	3	3	3	4
3	1	∞	3	2
∞	3	2	4	0
5	0	0	0	1
21	22	18	23	23

Subtracting the minimum element of each row from that row, we get the following matrix.

0	3	3	3	4
2	0	∞	2	1
∞	3	2	4	0
5	0	0	0	1
3	4	0	5	5

Start examining the rows. First row contains only one unmarked zero. Enclose it in a square $\square$. Cross other zeros in the column (here none). Second row contains only one unmarked zero. Enclose it in a square $\square$. Cross the other zeros in the second column. Third row contains only one unmarked zero. Enclose it in a square $\square$. Cross the other zeros in the column (here none). Fifth row contains only one unmarked zero. Enclose it in a square $\square$. Crossing the other zeros in the third column, we get the following reduced matrix.

$\boxed{0}$	3	3	3	4
2	$\boxed{0}$	∞	2	1
∞	3	2	4	$\boxed{0}$
5	$\cancel{0}$	$\cancel{0}$	0	1
3	4	$\boxed{0}$	5	5

Start examining the columns. Fourth column contains only one unmarked zero. Enclose it in a square $\boxed{}$. Crossing the other zeros in the rows (here none), we get the following reduced matrix.

$\boxed{0}$	3	3	3	4
2	$\boxed{0}$	∞	2	1
∞	3	2	4	$\boxed{0}$
5	$\cancel{0}$	$\cancel{0}$	$\boxed{0}$	1
3	4	$\boxed{0}$	5	5

It should be noted that all the zeros are either crossed or marked. Each row and each column contains at least one assigned zero. Hence the solution is optimum.

The optimal assignment schedule is as follows.

Machines	**Places**	**Cost (₹)**
M_1	A	9
M_2	B	9
M_3	C	7
M_4	D	7

Total (minimum) cost = 9 + 9 + 7 + 7 = ₹ 32/-

(Assignment M_5 to C is ignored, as M_5 being dummy machine.)

Example 4.18 : Given the following matrix of set-up costs. Show how to sequence production so as to minimize set-up cost per cycle.

		To				
		A	**B**	**C**	**D**	**E**
	A	∞	2	5	7	1
	B	6	∞	2	8	2
From	**C**	8	7	∞	4	7
	D	12	4	6	∞	5
	E	1	3	2	8	∞

Solution : Subtracting the minimum element of each row from that row, we get the following matrix.

∞	1	4	6	0
4	∞	0	6	0
4	3	∞	0	3
8	0	2	∞	1
0	2	1	7	∞

Since each row and each column contains single zero, hence making assignments in rows and columns, we get

∞	1	4	6	$\boxed{0}$
4	∞	$\boxed{0}$	6	$\cancel{0}$
4	3	∞	$\boxed{0}$	3
8	$\boxed{0}$	2	∞	1
$\boxed{0}$	2	1	7	∞

The optimum assignment is : A $\rightarrow$ E, E $\rightarrow$ A, B $\rightarrow$ C, C $\rightarrow$ D, D $\rightarrow$ B with minimum cost as 13.

For the following salesman problem, we make assignment at (1, 2) instead of a zero assignment at (1, 5).

∞	$\boxed{1}$	3	6	$\cancel{0}$
4	∞	$\boxed{0}$	6	0
4	3	∞	$\boxed{0}$	3
8	$\cancel{0}$	$\cancel{0}$	∞	$\boxed{1}$
$\boxed{0}$	2	0	7	∞

The optimum assignment schedule to the travelling salesman problem is A $\rightarrow$ B $\rightarrow$ C $\rightarrow$ D $\rightarrow$ E $\rightarrow$ A with total minimum cost = 2 + 3 + 4 + 5 + 1 = 15.

Example 4.19 : Solve the following assignment problem. **(Oct. 2008)**

	I	II	III	IV
A	1	4	6	3
B	9	7	10	9
C	4	5	11	7
D	8	7	8	5

Solution : (A) In each row we subtract the smallest element from each element of that row. The resulting matrix is as below :

	I	II	III	IV
A	0	3	5	2
B	2	0	3	2
C	0	1	7	3
D	3	2	3	0

(B) In each column, we subtract the smallest element from each element of that column. The resulting matrix is as below :

	I	II	III	IV
A	0	3	2	2
B	2	0	0	2
C	0	1	4	3
D	3	2	0	0

(C) We cover all zeros by drawing minimum number of horizontal and vertical lines.

	I	II	III	IV
A	0	3	2	2
B	2	0	0	2
C	0	1	4	3
D	3	2	0	0

The number of lines required to cover all zeros is 3; which is less than 4, the order of the cost matrix.

(D) The smallest uncovered element is 1 in the C-II cell. It is subtracted from each uncovered element; it is added to those elements which are covered by two lines. The elements covered by only one line are reproducted as they are the resulting matrix is as below.

	I	II	III	IV
A	0	2	1	1
B	3	0	0	2
C	0	0	3	2
D	4	2	0	0

(E) Repetition of step (C) gives

	I	II	III	IV
A	0	2	1	1
B	3	0	0	2
C	0	0	3	2
D	4	2	0	0

The minimum number of lines required to cover all zeros is 4, which is equal to the order of the cost matrix.

Therefore, optimal solution is reached.

(F) We make the assignment as below. First we reproduce the matrix in step (E).

	I	II	III	IV
A	$\boxed{0}$	2	1	1
B	3	0	0	2
C	$\cancel{0}$	0	3	2
D	4	2	0	0

The first row contains only one zero. The zero in the A-I cell is put into the square and then zero in the C-I cell is crossed out.

	I	II	III	IV
A	$\boxed{0}$	2	1	1
B	3	$\cancel{0}$	$\boxed{0}$	2
C	$\cancel{0}$	$\boxed{0}$	3	2
D	4	2	$\cancel{0}$	$\boxed{0}$

Now third row contains only one zero. The zero in the C-II cell is put into the square and then zero in B-II cell is crossed out.

Now, second row contains only one zero. The zero in B-III cell is put into the square and then zero in D-III cell is crossed out.

Finally the zero in D-IV cell is put into the square.

The following table gives the assignment.

Worker	**Job**	**Cell cost**
A	I	1
B	III	10
C	II	5
D	IV	5
Total cost		21

Example 4.20 : Consider the problem of assigning four operators to four machines. The assignment costs in dollars are given. Operator 1 cannot be assigned to machine 3. Also, operator 3 cannot be assigned to machine 4. Find the optimal assignment. **(April 2008)**

Machine

		1	2	3	4
	1	5	5	–	2
	2	7	4	2	3
Operator	3	8	3	5	–
	4	7	2	6	7

Solution : When a certain operator cannot be assigned to particular machine, we take the corresponding cell cost as ∞.

Now, the cost matrix is

		Machine 1	2	3	4
	1	5	5	∞	2
	2	7	4	2	3
Operator	3	9	3	5	∞
	4	7	2	6	7

(A) In each row, we subtract the smallest element of the row from each element of that row.

		Machine 1	2	3	4
	1	3	3	∞	0
	2	5	2	0	1
Operator	3	6	0	2	∞
	4	5	0	4	5

(B) In each column, we subtract the smallest element of the column from each element of that column.

		Machine 1	2	3	4
	1	0	3	∞	0
	2	2	2	0	1
Operator	3	3	0	2	∞
	4	2	0	4	5

(C) We cover all zeros by drawing minimum number of horizontal and vertical lines.

		Machine 1	2	3	4
	1	0	3	∞	0
	2	2	2	0	1
Operator	3	3	0	2	∞
	4	2	0	4	5

The number of lines required to cover all zeros is 3, which is less than 4; order of the cost matrix.

(D) The smallest uncovered element is subtracted from each uncovered element; it is added to those elements which are covered by two lines. The elements covered by only one line are reproduced as they are.

Machine

Operator		1	2	3	4
	1	0	4	∞	0
	2	1	2	0	0
	3	2	0	2	∞
	4	1	0	4	4

(E) Repetition of step in (C)

Machine

Operator		1	2	3	4
	1	0	4	∞	0
	2	1	2	0	0
	3	2	0	2	∞
	4	1	0	4	4

(F) Repetition of step in (D).

Machine

Operator		1	2	3	4
	1	0	5	∞	0
	2	1	3	0	0
	3	1	0	1	∞
	4	0	0	3	3

(G) Repetition of step in (C).

Machine

Operator		1	2	3	4
	1	0	5	∞	0
	2	1	3	0	0
	3	1	0	1	∞
	4	0	0	3	3

(H) The number of lines required to cover all zeros is 4; which is equal to the order of the cost matrix. Therefore, optimal solution is reached.

(I) We make the assignment as below. First we reproduce the matrix of step (G).

Machine

Operator		1	2	3	4
	1	0	5	∞	0
	2	1	3	0	0
	3	1	$\boxed{0}$	1	∞
	4	0	0	3	3

The third row contains exactly one zero. The zero in the (3, 2) cell is put into the square and zero in the (4, 2) cell is crossed out.

Machine

		1	2	3	4
	1	~~0~~	5	∞	0
	2	1	3	0	0
Operator	3	1	$\boxed{0}$	1	∞
	4	$\boxed{0}$	~~0~~	3	3

Now, fourth row contains exactly one zero.

The zero in the (4, 1) cell is put into the square and then zero in the (1, 1) cell is crossed out.

Machine

		1	2	3	4
	1	~~0~~	5	∞	0
	2	1	3	0	0
Operator	3	1	$\boxed{0}$	1	∞
	4	$\boxed{0}$	~~0~~	3	3

Now, third column contains exactly one zeros.

The zero in the (2, 3) cell is put into the square and then zero in the (2, 4) cell is crossed out.

Machine

		1	2	3	4
	1	~~0~~	5	∞	0
	2	1	3	$\boxed{0}$	~~0~~
Operator	3	1	$\boxed{0}$	1	∞
	4	$\boxed{0}$	~~0~~	3	3

Finally, zero in the (1, 4) cell is put into the square.

Machine

		1	2	3	4
	1	~~0~~	5	∞	$\boxed{0}$
	2	1	3	$\boxed{0}$	~~0~~
Operator	3	1	$\boxed{0}$	1	∞
	4	$\boxed{0}$	0	3	3

The assignment is as shown in the following table.

Operator	**Machine**	**Cell cost**
1	4	2
2	3	2
3	2	3
4	1	7
Total cost		14

Example 4.21 : A company has to assign four workers A, B, C and D to four jobs W, X, Y and Z. The cost matrix is given below :

	W	X	Y	Z
A	1000	1200	400	900
B	600	500	300	800
C	200	300	400	500
D	600	700	300	1000

Find an optiaml assignment for minimization. **(Oct. 2007)**

Solution : (A) We subtract the smallest element of each row from each element of that row. The resulting matrix is

	W	X	Y	Z
A	600	800	0	500
B	300	200	0	500
C	0	100	200	300
D	300	400	0	700

(B) We subtract the smallest element of each column from each element of that column. The resulting matrix is

	W	X	Y	Z
A	600	700	0	200
B	300	100	0	200
C	0	0	200	0
D	300	300	0	400

(C) Next we cover all zero by minimum number of horizontal and vertical lines.

	W	X	Y	Z
A	600	700	0	200
B	300	100	0	200
C	0	0	200	0
D	300	300	0	400

All zeros are covered by only 2 lines and 2 < 4; where 4 is the order of cost matrix.

(D) The smallest uncovered element is 100. Now, it is subtracted from all uncovered elements which are covered twice and the elements covered by only one line are reproduced as they are. The resulting matrix is as given below :

	W	X	Y	Z
A	500	600	0	100
B	200	0	0	100
C	0	0	300	0
D	200	200	0	300

(E) Repetition of step (C) gives the matrix :

	W	X	Y	Z
A	500	600	0	100
B	200	0	0	100
C	0	0	300	0
D	200	200	0	300

The number of lines is 3 < 4.

(F) Repetition of step (D).

	W	X	Y	Z
A	400	500	0	0
B	200	0	100	0
C	0	0	400	0
D	100	100	0	200

(G) Repetition of step (C).

	W	X	Y	Z
A	400	500	0	0
B	200	0	100	0
C	0	0	400	0
D	100	100	0	200

The number of lines required to cover all zeros is 4; which is equal to the order of the cost matrix.

Therefore, the optimal solution is reached.

(H) Reproduced the matrix in step (G).

	W	X	Y	Z
A	400	500	0	0
B	200	$\boxed{0}$	100	$\cancel{0}$
C	$\boxed{0}$	$\cancel{0}$	400	$\cancel{0}$
D	100	100	$\boxed{0}$	200

In this matrix the last row contains only one zero.

	W	X	Y	Z
A	400	500	0	0
B	200	0	100	0
C	0	0	400	0
D	100	100	0	200

The zero in DY cell is put into the square and then zero in AY cell is crossed.

Now, first column contains only one zero.

The zero in CW cell is put into the square and then remaining zeros in the row of C are crossed.

Now, first row contains only one zero.

The zero in AZ cell is put into the square and the zero in BZ cell is crossed.

The second row contains only one zero. It is in the BX cell and it is put into the square.

The job assignment is as below :

Worker	Job	Cell cost
A	Z	900
B	X	500
C	W	200
D	Y	300
Total cost		**1900**

Example 4.22 : A cricket team captain wishes to allot five middle batting positions III, IV, V, VI, VII to five batsmen A, B, C, D, E on an average from the past experience, the runs scored by these batsmen at the given positions is as shown in the following table :

	Number of runs scored at the position				
Batsmen	**III**	**IV**	**V**	**VI**	**VII**
A	54	52	44	64	54
B	62	64	63	59	57
C	46	34	20	29	31
D	44	44	39	29	54
E	24	23	24	22	29

Solve this assignment problem in order to achieve the score as maximum as possible. What is the maximum possible score ?

Solution : We transform the maximization problem into minimization problem by changing the signs of all entries.

	III	**IV**	**V**	**VI**	**VII**
A	– 54	– 52	– 44	– 64	– 54
B	– 62	– 64	– 63	– 59	– 57
C	– 46	– 34	– 20	– 29	– 31
D	– 44	– 44	– 39	– 29	– 54
E	– 24	– 23	– 24	– 22	– 29

In each row we subtract the smallest entry of that row from each entry of that row.

	III	IV	V	VI	VII
A	10	12	20	0	10
B	2	0	1	5	6
C	0	12	26	17	15
D	10	10	15	25	0
E	5	6	5	7	0

In each column we subtract the smallest entry of that column from each entry of the same.

	III	IV	V	VI	VII
A	10	12	19	0	10
B	2	0	0	5	7
C	0	12	25	17	15
D	10	10	14	25	0
E	5	6	4	7	0

In this table all zeros can be covered by 4 lines (horizontal/vertical). So the optimum solution is not achieved.

	III	IV	V	VI	VII
A	10	12	19	0	10
B	2	0	0	5	7
C	0	12	25	17	15
D	10	10	14	25	0
E	5	6	4	7	0

In the above table the smallest uncovered element is 4.

Next 4 is subtracted from each uncovered element 4 is added to each doubly covered element signally covered elements will remain as they are.

The next table is as below.

	III	IV	V	VI	VII
A	10	12	19	0	14
B	2	0	0	5	11
C	0	12	25	17	19
D	6	6	10	21	0
E	1	2	0	3	0

Again we draw minimum number of horizontal and vertical lines to cover all zeros.

	III	**IV**	**V**	**VI**	**VII**
A	10	12	19	0	14
B	2	0	0	5	11
C	0	12	25	17	19
D	6	6	10	21	0
E	1	2	0	3	0

The minimum number of lines required to cover all zeros is 5, which is equal to order of the assignment matrix.

Hence optimum solution is achieved.

We proceed to the actual assignment now.

	III	**IV**	**V**	**VI**	**VII**
A	10	12	19	$\boxed{0}$	14
B	2	$\boxed{0}$	0	5	11
C	$\boxed{0}$	12	25	17	19
D	6	6	10	21	$\boxed{0}$
E	1	2	$\boxed{0}$	3	0

The optimum assignment of positions to batsmen is as under :

Batsman	**Position**	**Runs**
A	VI	64
B	IV	64
C	III	46
D	VII	54
E	V	24
		252

In this assignment the maximum possible score of five batsmen is 252.

Think Over It

1. Explain why the assignment problem can be solved by using transportation algorithm.
2. How an assignment problem with maximization of objective function is solved ?

Miscellaneous Exercise

(A) True/False Questions

1. Assignment techniques always gives unique optimal solution.
2. In an assignment technique the main constraint is that the number of jobs and the number of machines must be equal.
3. The assignment problem can be solved by using transportation algorithm.

(B) Multiple Choice Questions

1. The four job operators A, B, C, D are to be assigned to four machine P, Q, R, S.
   The operator A cannot operate on machine R.
   The operator C cannot operate on machine Q.
   The assignment costs are given in the table below :

	P	Q	R	S
A	7	4	–	7
B	9	5	4	6
C	11	–	7	5
D	9	9	8	4

   The minimum costs of assignment is ………..
   (A) 20 (B) 22
   (C) 24 (D) 18

2. Three workers A, B, C are to be assigned three jobs I, II, III. The time taken by each worker on each job is given in the following table :

		Job		
		I	II	III
Worker	A	5	3	8
	B	9	6	4
	C	5	6	7

   The total minimum time of assigning the jobs to the workers is ………..
   (A) 10 (B) 11
   (C) 12 (D) 13

(C) Theory Questions

1. Explain the difference between the transportation problem and assignment problem.
2. Give the mathematical formulation and difference between transportation and assignment problem.
3. What is assignment problem ? Give two areas of its application.
4. Write an algorithm to solve assignment problem for optimal cost.
5. What is an unbalanced assignment problem ? How to make such problem balanced ?
6. Explain how an assignment problem with maximization of objective function is solved.
7. Describe how an assignment problem with certain restrictions can be solved.

(D) Numerical Problems

1. Three customers in a certain sales territory have requested technical assistance, three technicians are available for assignment with the distance in km for each technician to each customer as follows :

Customer

		A	B	C
Technician	**1**	470	580	410
	2	385	920	740
	3	880	550	430

If it costs on Re per km for travel, find the assignment of technicians to customers that will result in minimum travel costs.

2. A departmental head has four subordinates and four tasks to be performed. The subordinates differ in efficiency and the tasks differ in their intrinsic difficulty. His estimate, of the time each man would take to perform each task, is given in the matrix below :

		Men			
		E	**F**	**G**	**H**
Tasks	**A**	18	26	17	11
	B	13	28	14	26
	C	38	19	18	15
	D	19	26	24	10

How should the tasks be allotted, one to a man, so as to minimize the total man-hours ?

3. A computer centre has got three expert programmers. The centre needs three application programmes to be developed. The Head of the Computer Centre, after studying carefully the programmes to be developed, estimates the computer time in minutes required by the experts to the application programmes as follows :

Programmes

		A	B	C
Programmers	**1**	120	100	80
	2	80	90	110
	3	110	140	120

Assign the programmers to the programmes in such a way that the total computer time is least.

4. Solve the following assignment problem for minimum cost :

	1	2	3	4
A	10	12	19	11
B	5	10	7	8
C	12	14	13	11
D	8	15	11	9

5. The Head of the department has five jobs A, B, C, D, E and five sub-ordinates V, W, X, Y and Z. The number of hours each man would take to perform each job is as follows :

	V	W	X	Y	Z
A	3	5	10	15	8
B	4	7	15	18	8
C	8	12	20	20	12
D	5	5	8	10	6
E	10	10	15	25	10

How should the jobs be allocated to minimize the total time ?

6. Five jobs are to be processed and five machines are available. Any machine can process any job with the resulting profit (in ₹) as follows :

	Machines				
Jobs	**A**	**B**	**C**	**D**	**E**
1	32	38	40	28	40
2	40	24	28	21	36
3	41	27	33	30	37
4	22	38	41	36	36
5	29	33	40	35	39

What is the maximum profit that may be expected, if an optimum assignment is made ?

7. Five different machines can do any of the five required jobs with different profits resulting from each assignment as shown below.

	Machines				
Jobs	**A**	**B**	**C**	**D**	**E**
1	30	37	40	28	40
2	40	24	27	21	36
3	40	32	33	30	35
4	25	38	40	36	36
5	29	62	41	34	39

Find out the maximum profit possible through optimum assignment.

8. The owner of a small machine shop has four machinists available to assign to jobs for the days. Five jobs are offered with expected profit for each machinist on each job as follows :

	A	B	C	D	E
1	62	78	50	101	82
2	71	84	61	73	59
3	87	92	111	71	81
4	48	64	87	77	80

Find the assignment of machinists to jobs that will result in a maximum profit. Which job should be declined ?

9. A company is faced with the problem of assigning 4 machines to 6 different jobs (one machine to one job only). The profits are estimated as follows :

	Machines			
Jobs	**A**	**B**	**C**	**D**
1	3	6	2	6
2	7	1	4	4
3	3	8	5	8
4	6	4	3	7
5	5	2	4	3
6	5	7	6	4

Solve the problem to maximize the total profits.

10. Solve the following unbalanced assignment problem of minimizing total time for doing all the jobs.

	Job				
Operator	**1**	**2**	**3**	**4**	**5**
1	6	2	5	2	6
2	2	5	8	7	7
3	7	8	6	9	8
4	6	2	3	4	5
5	9	3	8	9	7
6	4	7	4	6	8

11. A machine operator processes five types of items on his machine each week, and must choose a sequences for them. The set-up cost per change depends on the item presently on the machine and the set-up to be made, according to the following table.

If he processes each type of item once and only once each week, how should he sequence the items on his machine in order to minimize the total set-up cost ?

	To item				
For item	**A**	**B**	**C**	**D**	**E**
A	∞	4	7	3	4
B	4	∞	6	3	4
C	7	6	∞	7	5
D	3	3	7	∞	7
E	4	4	5	7	∞

12. Solve the following travelling salesman problem so as to minimize the cost per cycle.

	To				
From	**A**	**B**	**C**	**D**	**E**
A	–	3	6	2	3
B	3	–	5	2	3
C	6	5	–	6	4
D	2	2	6	–	6
E	3	3	4	6	–

13. Solve the travelling salesman problem given by the following data :

$C_{12} = 20$, $C_{13} = 4$, $C_{14} = 10$, $C_{23} = 5$, $C_{34} = 6$, $C_{25} = 10$, $C_{35} = 6$, $C_{45} = 20$, where $C_{ij} = C_{ji}$ and there is no route between cities i and j, if a value for C_{ij} is not shown.

14. Solve the following travelling salesman problem :

	To						
From	**1**	**2**	**3**	**4**	**5**	**6**	**7**
1	–	6	12	6	4	8	1
2	6	–	10	5	4	3	3
3	8	7	–	11	3	11	8
4	5	4	11	–	5	8	6
5	5	2	7	8	–	4	7
6	6	3	11	5	4	–	2
7	2	3	9	7	4	3	–

15. Consider the problem of assigning four operators to five machines. The assignment costs in rupees are given. Operator 1 cannot be assigned to machine 3. Also, operator 3 cannot be assigned to machine 4. Find optimal assignment.

		Machine				
		1	2	3	4	5
	1	5	5	–	2	2
Operator	2	7	4	2	3	1
	3	9	3	5	–	2
4	7	7	2	6	7	8

(March 2009)

16. A marketing manager has 5 salesman and 5 districts. Considering the capabilities of the salesmen and the nature of districts, the marketing manager estimates that sale per month (in hundred ₹) for each salesman in each district would be as follows :

		Districts				
		A	B	C	D	E
	1	32	38	40	28	40
Salesmen	2	40	24	28	21	36
	3	41	27	33	30	37
	4	22	38	41	36	36
	5	29	33	40	35	39

Find an optical assignment to maximize the profit. **(Oct. 2007)**

17. Solve the following assignment problem :

		Machines			
		1	**2**	**3**	**4**
	A	1	3	5	2
Jobs	**B**	8	6	9	8
	C	3	4	10	6
	D	7	6	7	4

(March 2009)

Answers

(A) True or False :

1. True 2. True

(B) Multiple Choice Questions :

1. (B) 2. (C)

(D) Numerical Problems :

1. $2 \to A$, $3 \to B$, $1 \to C$ min cost = 1345 ₹
2. $A \to G$, $B \to E$, $C \to F$, $D \to H$, Minimum time = 59 man hours.
3. $1 \to C$, $2 \to B$, $3 \to A$, Minimum computer time = 280 min.
4. $A \to 2$, $B \to 3$, $C \to 4$, $D \to 1$, Minimum cost = 38.
5. $A \to X$, $B \to W$, $C \to V$, $D \to Y$, $E \to Z$, Minimum cost time = 45.
6. $1 \to B$, $2 \to A$, $3 \to E$, $4 \to C$, $5 \to D$,
   Minimum profit = ₹ 191/- There are also another assignments with same minimum profit = 191.
7. $1 \to C$, $2 \to E$, $3 \to A$, $4 \to D$, $5 \to B$, Maximum profit = 214.
8. $1 \to D$, $2 \to B$, $3 \to C$, $4 \to E$, Maximum profit = 376.
9. $2 \to A$, $3 \to B$, $4 \to D$, $6 \to C$, Maximum profit = 28.
10. $1 \to 4$, $2 \to 1$, $3 \to 6$, $4 \to 5$, $5 \to 2$, $6 \to 3$, Minimum time = 16.
11. $A \to E$, $E \to C$, $C \to B$, $B \to D$, $D \to A$, Total set-up cost = 21.
12. $A \to D \to B \to C \to E \to A$ or $A \to E \to C \to B \to D \to A$, Total minimum cost = 16).
13. $I \to IV \to V \to II \to III \to I$, Total set-up cost = 49.
14. $1 \to 7 \to 2 \to 3 \to 5 \to 6 \to 4 \to 1$, Minimum cost = 31.
15. **Hint :** Introduce dummy operator 5; with zero cell costs in its row.
    Job 1 remains unassigned.

Operator	Job	Cell cost
1	4	2
2	3	2
3	5	2
4	2	2
	Total cost	8

16. **Hint :** Convert to minimization problem by changing the signs of cell costs.
    Multiple optimal assignment maximum profit : 191.
17. Optimal solution, cost = 18.

☆☆☆

APPENDIX

PRACTICAL NO. 01 : FORMULATION OF LINEAR PROGRAMMING PROBLEMS

Problem 1 : A manufacturer produced 3 models I, II and III of a certain product using raw material A and B. The following table gives the data for the problem.

Raw material	Requirement/Unit			Availability
	I (x_1)	**II (x_2)**	**III (x_3)**	
A	2	3	5	4,000
B	4	2	7	6,000
Minimum demand	200	200	150	
Profit/Unit (in ₹)	30	20	50	

The labour time per unit of model I is twice that of II and III times as that of III.

The entire labour force of the factory can produce the equivalent of 1500 units of model I. The market requirement specified the ratios 3 : 2 : 5 for the production of three respective models. Formulate the problem as L.P.

Solution :

Step I : Determination of Decision variables

We have to determine the number of units of production of three models to maximize profit

$\therefore$ Let x_1, x_2, x_3 denotes number of units produced of model I, II, III respectively.

Step II : Determination of objective function:

Let Z denote the total profit on the sale of units of three models

Hence, Z = (profit on units of Model I) + (profit on units of Model II)
+ (profit on units of Model III)

Now, Profit for 1 unit of model I is 30.

$\therefore$ Profit for x_1 units of model I is $30x_1$.

Similarly,

Profit for unit 1 of model II is 20.

$\therefore$ Profit for x_2 unit of model is $20x_2$.

Also,

Profit for 1 unit of model III is ₹ 50.

$\therefore$ Profit for x_3 unit of model III is ₹ $50x_3$.

$\therefore$ $Z = 30x_1 + 20x_2 + 50x_3$.

Step III: Determination of constraints:

We know

We have (Requirement of Raw material A) $\leq$ (Available units of raw material A)

Now, Model I requires 2 units of A corresponding to 1 unit.

$\therefore$ x_1 units of model I require $2x_1$ units of A.

x_2 units of model II require $3x_2$ units of A.

x_3 units of model III requires $5x_3$ units of A.

$\therefore$ Requirement of raw material A is $2x_1 + 3x_2 + 5x_3$.

Available units of A are 4000. $\therefore$ $2x_1 + 3x_2 + 5x_3 \le 4000$. … (1)

Also, $\therefore$ (Requirement of raw material B) $\le$ 6000

$4x_1 + 2x_2 + 7x_3 \le 6{,}000$. … (2)

Minimum demand for model I is 200 units $\therefore x_1 \ge 200$ … (3)

Minimum demand for model II is 200 units $\therefore$ $x_2 \ge 200$ … (4)

Also, Minimum demand for model III is 150 units. $\therefore$ $x_3 \ge 150$. … (5)

Let total labour force require for time t hours for production of 1 unit of model I.

$\therefore$ Time require for 1 unit of mode II is $\frac{t}{2}$ units and time require for 1 unit of model III is $\frac{t}{3}$ units.

$\therefore$ Time require to produce x_1, x_2, x_3 units of model I, II, III is $tx_1 + \frac{t}{2}x_2 + \frac{t}{3}x_3$.

Total labour force produces 1500 units of model I.

$\therefore$ Total available labour time is 1500 t.

$$\therefore \quad tx_1 + \frac{t}{2}x_2 + \frac{t}{3}x_3 \le 1500t$$

$$x_1 + \frac{x_2}{2} + \frac{x_3}{3} \le 1500$$

$$\therefore \quad \frac{6x_1 + 3x_2 + 2x_3}{6} \le 1500$$

$$6x_1 + 3x_2 + 2x_3 \le 9000 \qquad \ldots (6)$$

The market requirement is in the ratio 3 : 2 : 5 of model I, II and III.

$$\frac{x_1}{3} = \frac{x_2}{2} \quad , \quad \frac{x_2}{2} = \frac{x_3}{5}$$

$$\therefore \quad 2x_1 = 3x_2 \quad , \quad 5x_2 = 2x_3$$

$$\therefore \quad 2x_1 - 3x_2 = 0 \qquad \ldots (7)$$

$$\text{and} \quad 5x_2 - 2x_3 = 0 \qquad \ldots (8)$$

$$x_1 \ge 0, \; x_2 \ge 0, \; x_3 \ge 0$$

$\therefore$ Formulation of L.P.P. is,

Maximize $Z = 30x_1 + 20x_2 + 50x_3$

Subject to

$$\begin{aligned} 2x_1 + 3x_2 + 5x_3 &\le 4000 \\ 4x_1 + 2x_2 + 7x_3 &\le 6000 \\ x_1 &\ge 200, \; x_2 \ge 200, \; x_3 \ge 150 \\ 6x_1 + 3x_2 + 2x_3 &\le 9000 \\ 2x_1 - 3x_2 &= 0 \\ 5x_2 - 2x_3 &= 0 \end{aligned}$$

Non-negativity restrictions are $x_1, x_2, x_3 \ge 0$.

Problem 2 : Solve graphically

Maximize $Z = 0.07x_1 + 0.10x_2$

Subject to, $x_1 + x_2 \le 30{,}000$

$$x_1 \geq 6,000$$
$$x_2 \geq 12,000$$
$$x_1 - x_2 \geq 0; x_1, x_2 \geq 0$$

Solution :

To sketch the feasible region , first we will sketch the lines

$x_1 + x_2 = 30,000$ … (1)

$x_1 = 6000$ … (2)

$x_2 = 12000$ … (3)

x_1-x_2=0 … (4)

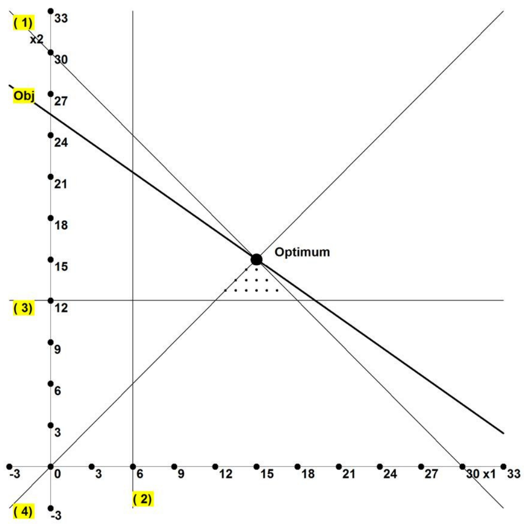


Fig. A.1

Here scale is used as 1 cm = 1000 units.

Since optimum value of linear functional on closed convex set occure at corner points of that set. We give the corner points of feasible region and the value of objective function at that points in following table.

Corner point	Equations of line whose intersection is given corner point	Value of objective
(12000,12000)	(3) and (4)	2040
(15000, 15000)	(1) and (2)	2550
(18000, 12000)	(1) and (3)	2460

∴ Optimum solution $x_1 = 15000$, $x_2 = 15000$ and Z = 2550.

Problem 3. : Solve graphically the following L.P.P.

Minimize $Z = 3x_1 + 5x_2$

Subject to, $-3x_1 + 4x_2 \leq 12$

$$2x_1 + 3x_2 \geq 12$$
$$2x_1 - x_2 \geq -2$$
$$x_1 \leq 4, x_2 \geq 2 \text{ and } x_1, x_2 \geq 0$$

Solution : We sketch the lines

$3x_1 + 4x_2 = 12$ … (1)

$x_1 + 3x_2 = 1.$ … (2)

$x_1 - x_2 = -2$ … (3)

$x_1 = 4$ … (4)

$x_2 = 2$ … (5)

The feasible region is as shown below

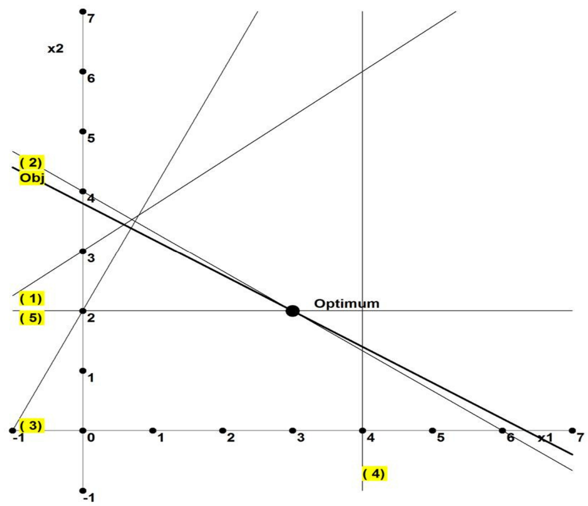


Fig. A.2

Corner point	Equations of line whose intersection is given corner point	Value of objective
(3, 2)	(2) and (5)	19
(0.75, 14/4)	(2) and (3)	19.75
(4, 6)	(1) and (4)	42
(4, 2)	(4) and (5)	22
(4/5, 18/5)	(1) and (3)	20.4

$\therefore$ Z is minimum at (3, 2).

$\therefore$ Optimum solution is $x_1 = 3$, $x_2 = 2$, $Z = 19$.

Problem 4 : Chemlabs produces two domestic cleaning solution A and B by processing two raw materials I and II. The processing of 1 unit of raw material I costs ₹ 80 and produces 0.5 unit of solution A and 0.5 unit of solution B. Moreover, the processing of 1 unit of raw material II costs ₹ 50 and produces 0.4 units of A and 0.6 units of solution B. The daily demand for solution A lies between 10 and 15 units and for solution B lies between 12 and 20 units. Use the graphical method to find the optimal mix of A and B that Chemlab should produce.

Solution :

		Cleaning solutions	
	Cost	**A**	**B**
Raw material I (x_1)	80	0.5	0.5
Raw material II (x_2)	50	0.4	0.6
Daily demand lies between		10 and 15	12 and 20

Step (I):Determination of decision variables:

Let x_1 : Denote number of units of raw material I and

x_2 : Denote number of units of raw material II.

Step (II): Determination of objective function:

Let Z be total cost on production.

Therefore

Z = cost on production of raw material I + cost on production of raw material II.

The cost of 1 unit of raw material I is 80.

$\therefore$ Cost of x_1 units of raw material I is $80x_1$.

Similarly,

The cost of 1 unit of raw material II is 50.

$\therefore$ Cost of x_2 units of raw material II is $50x_2$.

$\therefore$ $Z = 50x_1 + 50x_2$

Our objective to minimize Z.

Step (III) Determination of costraints

1 unit of raw material I produces 0.5 unit of solution A.

$\therefore$ x_1 units of raw material I produces $0.5x_1$ units of solution A.

Also,

1 unit of raw material I produces 0.5 unit of B.

$\therefore$ x_1 units of raw material I produces $0.5x_1$ units of B.

Similarly, 1 unit of raw material II produces 0.4 unit of A.

$\therefore$ x_2 units of raw material II produces $0.4x_2$ units of A.

and 1 units of raw material II produces 0.6 units of B.

$\therefore$ x_2 units of raw material II produces $0.6x_2$ units of B.

$\therefore$ Total production of solution A is $0.5x_1 + 0.4x_2$.

and Demand for A lies between 10 and 15.

$\therefore$ $10 \le 0.5x_1 + 0.4x_2 \le 15$

$\therefore$ $0.5x_1 + 0.4x_2 \ge 10$ … (1)

and $0.5x_1 + 0.4x_2 \le 15$ …(2)

Also, Total production of solution B is $0.5x_1 + 0.6x_2$. Daily demand of solution B lies between 12 and 20.

$\therefore$ $12 \le 0.5x_1 + 0.6x_2 \le 20$

$\therefore$ $0.5x_1 + 0.6x_2 \ge 12$ … (3)

$0.5x_1 + 0.6x_2 \le 20$ …(4)

$x_1, x_2 \ge 0$

$\therefore$ Formulation of L.P.P. is

Minimize $Z = 80x_1 + 50x_2$

Subject to :

$0.5x_1 + 0.4x_2 \ge 10 \qquad 0.5x_1 + 0.4x_2 \le 15$

$0.5x_1 + 0.6x_2 \ge 12 \qquad 0.5x_1 + 0.6x_2 \le 20$

non-negative restriction $x_1, x_2 \ge 0$.

Feasible region is given by

$0.5x_1 + 0.4x_2 \ge 10 \quad 0.5x_1 + 0.4x_2 \le 15$ … (1)

$0.5x_1 + 0.6x_2 \ge 12$ … (2)

$$0.5x_1 + 0.6x_2 \le 20 \qquad \dots (3)$$

$$x_1 = 0,\ x_2 = 0 \qquad \dots (4)$$

Now feasible region is as shown

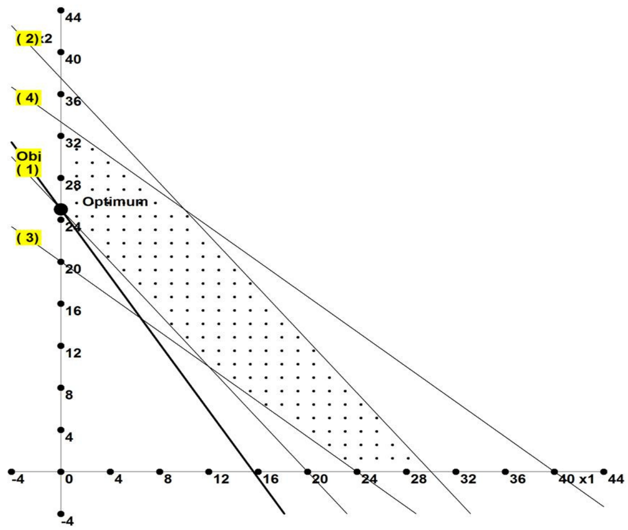


Fig. A.3

Corner point	Equations of line whose intersection is given corner point	Value of objective
(24, 0)	(3) and x_1 axis	1920
(0, 25)	(1) and x_2 axis	1250
(0, 33.33)	(4) and x_2 axis	1666.5
(30, 0)	(2) and x_1 axis	2400
(12, 10)	(1) and (3)	1460
(10, 25)	(4) and (2)	2050

Optimum solution is at $x_1 = 0$, $x_2 = 25$ and Z = 1250.

Problem 5 : A company produces two products A and B. The sale volume for A is atleast 80% of the total sales of both A and B. However, the company cannot sell more than 100 units of A per day. Both products use one raw material whose maximum availability is limited to 240 *l*b a day. The usage rates of the raw material are 2 *l*b per unit of A and 4 *l*b per unit of B. The unit prices of A and B are ₹ 200 and ₹ 500 respectively.

Use the graphical method to find the optimal product mix for the company.

Solution :

Step (I) Determination of decision variables:

Let x_1, x_2 denotes number of units of product A and product B respectively.

Step (II) Determination of objective function:

Let Z denote total price of selled units of products

Price of 1 unit of product A is ₹ 200.

∴ Price of x_1 units of product A is ₹ $200x_1$.

Also, Price of 1 unit of product B is ₹ 500.

∴ Price of 2 unit of product B is ₹ $500x_2$.

$$\therefore \quad Z = 200x_1 + 500x_2$$

Our objective is to maximize $Z = 200x_1 + 500x_2$

Step (III) Determination of constraints:

One unit of product A require 2 *l*b of raw material.

∴ x_1 units of product A require $2x_1$ *l*b of raw material.

Similarly,

1 unit of product B require 4 *l*b of raw material.

∴ x_2 units of product B require $4x_2$ *l*b of raw material.

∴ Total required of raw material is $2x_1 + 4x_2$.

Maximize availability of raw material is 240.

$$\therefore \quad 2x_1 + 4x_2 \leq 240 \qquad \dots (1)$$

Total sale of product A and B is $x_1 + x_2$. Sales volume for A is atleast 80% of the total sales.

$$\therefore \quad x_1 \geq \frac{80}{100}(x_1 + x_2)$$

$$\therefore \quad 100x_1 \geq 80x_2 + 80x_1$$

$$\therefore \quad -20x_1 + 80x_2 \leq 0 \qquad \dots (2)$$

Since, company cannot sale more than 100 units of A per day.

Hence $$x_1 \leq 100 \qquad \dots (3)$$

and $$x_1, x_2 \geq 0$$

Formulation of L.P.P. is,

Maximize $Z : 200x_1 + 500x_2$

Subject to : $2x_1 + 4x_2 \leq 240$ … (1)

$-20x_1 + 80x_2 \leq 0$ … (2)

$x_1 \leq 100$ … (3)

Non-negative restriction, $x_1, x_2 \geq 0$.

Feasible region is as shown below

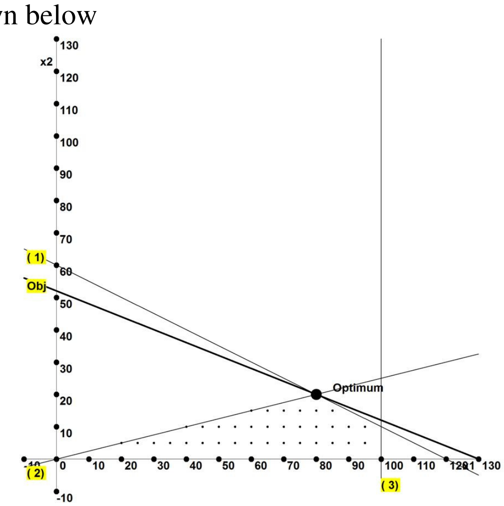


Fig. A.4

Corner point	Equations of line whose intersection is given corner point	Value of objective
(0, 0)	x_1, x_2 axis	0
(100, 10)	(1) and (3) axis	25000
(80, 20)	(1) and (2) axis	26000
(100, 0)	(3) and x_1 axis	20000

$\therefore$ Z is maximum at (80, 20).

$\therefore$ Optimal solution is at $x_1 = 80$, $x_2 = 20$ and Z = 26000.

Problem 6 : An assembly line consisting of the consecutive stations produces two ratio models. Hi Fi 1 and Hi Fi 2. The following table provides the assembly times for the three work stations.

Work station	Minutes per unit	
	Hi Fi 1	**Hi Fi 2**
w_1	**6**	**4**
w_2	**5**	**5**
w_3	**4**	**6**

The daily maintenance for stations w_1, w_2, w_3 consumes 10%, 14% and 12% respectively of the maximum 480 minutes available for each station each day. Use the graphical method to determine the optimal product mix that will minimize the idle (or used) times in the three work stations.

Solution :

Step (I) Determination of objective function:

Let x_1 and x_2 denotes number of units of Hi Fi 1 and Hi Fi 2 respectively per day.

Step (II) Determination of objective function:

Let Z denote total iddle time. Therefore

Z = (iddle time at work station W_1) + (Total idle at work station W_2) + (iddle time at work station W_3)

Iddle time at each work station is given by

(Total available time at the workstation) – (Total maintenance time) – (Total assembley time)

Now, Assembly times for work stations are

$$w_1 : 6x_1 + 4x_2 \leq 480 \qquad w_2 : 5x_1 + 5x_2 \leq 480 \qquad w_3 : 4x_1 + 6x_2 \leq 480$$

But three work stations consumers 10%, 14%, 12% of maximum 480 min for their daily maintenance for there.

Daily maintenance time for,

$$w_1 = \frac{10}{100} \times 480 = 48 \text{ min.} \qquad w_2 = \frac{14}{100} \times 480 = 67.2 \qquad w_3 = \frac{12}{100} \times 480 = 57.6$$

We have to subtract this daily maintenance time from total available time 480 min. for each work station.

$\therefore$ Idle time for, $w_1 = 432 - 6x_1 - 4x_2$; $w_2 = 412.8 - 5x_1 - 5x_2$ and $w_3 = 422.4 - 4x_1 - 6x_2$

$$\therefore \qquad Z = 1267.2 - 15x_1 - 15x_2$$

Our objective is to minimize Z.

Step (III) Determination of constraints:

$\therefore$ The constraints for available time are

$$w_1 : 6x_1 + 4x_2 \le 480 - 48 = 432$$
$$w_2 : 5x_1 + 5x_2 \le 480 - 67.2 = 412.8$$
$$w_3 : 4x_1 + 6x_2 \le 480 - 57.6 = 422.4$$

$\therefore$ Formulation of L.P.P is,

$$\text{Minimize } Z = 1267.2 - 15x_1 + 5x_2$$

That is Maximize $Z' = 15x_1 + 5x_2$

Hence $Z = 1267.2 - Z'$

Subject to $6x_1 + 4x_2 \le 432$ … (1)

$5x_1 + 5x_2 \le 412.8$ … (2)

$4x_1 + 6x_2 \le 422.4$ … (3)

$x_1, x_2 \ge 0$

The maximum value of objective function is

Z'=1238.40 and hence minimum value of Z=1267.2-1238.40=28.8

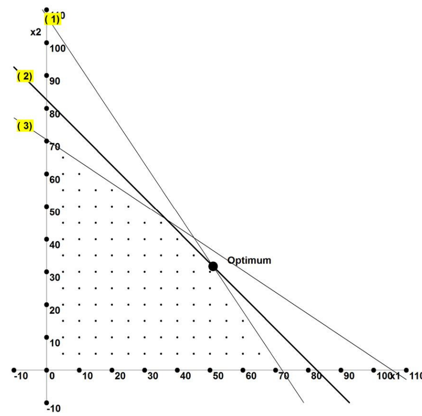


Fig. A.5

PRACTICAL NO. 02 : TRANSITION FROM GRAPHICAL TO SIMPLEX METHOD

Problem 1 : Consider following linear programming problem

Maximize $Z = 2x_1 + 3x_2$

Subject to $x_1 + 3x_2 \le 6$, $3x_1 + 2x_2 \le 6$ and $x_1 \ge 0$, $x_2 \ge 0$

(a) Express the problem in equation form.

(b) Determine the all basic solutions and classify them.

(c) Use the direct substitution in the objective function to determine the optimum basic feasible solution.

(d) Verify graphically that the solution obtained in (c) is the optimum solution – hence conclude that the optimum solution can be determined algebraically by considering the basic feasible solution only.

(e) Show how the infeasible basic solutions are represented on the graphical solution space.

Solution :

(a) Adding the slak variables s_1, s_2, we get the equation form of the given problem

Maximize $Z = 2x_1 + 3x_2 + 0s_1 + 0s_2$

Subject to $x_1 + 3x_2 + s_1 = 6$, $3x_1 + 2x_2 + s_2 = 6$ And $x_1 \ge 0$, $x_2 \ge 0$, $s_1 \ge 0$, $s_2 \ge 0$

(b) Here the number of equations are 2 in two variables x_1, x_2, s_1, s_2

Any basic solution is obtained by putting any two variables zero and solving equations in remaining variables. The variables are kept zero are called as current non-basis and other variables are called as current basis

If current solution obtained has values of basic variables nonnegative then that solution is called as feasible basic solution. We summarize all basic solutions in following table:

Non-Basic Variable	Basic Variables	Equations obtained	Basic Solution obtained	Is it Feasible?	Value of objective function	Is it optimal?
s_1, s_2 B	x_1, x_2	$x_1 + 3x_2 = 6$ $3x_1 + 2x_2 = 6$	$x_1 = 6/7$, $x_2 = 12/7$	Yes	$Z = 48/7$	Yes
x_2, s_1 D	x_1, s_2	$x_1 = 6$ $3x_1 + s_2 = 6$	$x_1 = 6$, $s_2 = -12$	No	-	-
x_1, x_2 O	s_1, s_2	$s_1 = 6$ $s_2 = 6$	$s_1 = 6$ $s_2 = 6$	Yes	$Z = 0$	No
x_1, s_1 A	x_2, s_2	$3x_2 = 6$, $2x_2+s_2= 6$	$x_2 = 2$, $s_2 = 2$	Yes	$Z = 6$	No
x_1, s_2 E	x_2, s_1	$2x_2+s_1= 6$, $2x_2= 6$	$x_2 = 3$, $s_1 = -3$	No	-	-
x_2, s_2 C	x_1, s_1	$x_1+s_1= 6$, $3x_1= 6$	$x_1 = 2$, $s_1 = 4$	Yes	$Z = 4$	No

(d), (e): Now we will see the Feasible region graphically

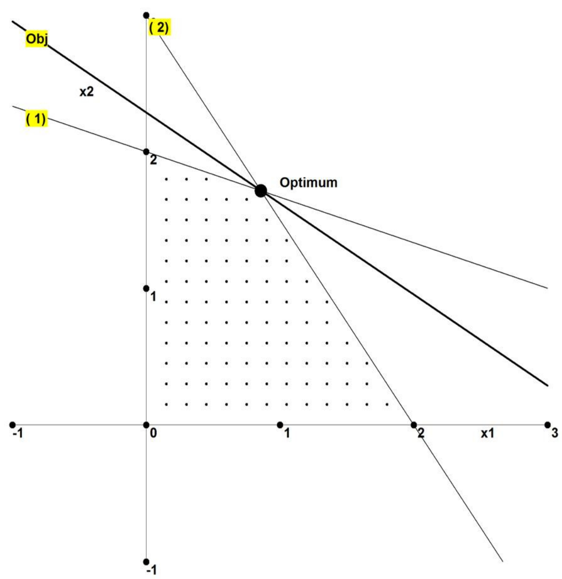


Fig. A.6

Problem 2 : Consider following linear programming problem

Maximize $Z = x_1 + 3x_2$

Subject to $x_1 + x_2 \leq 2$,

$-x_1 + x_2 \leq 4$ and x_1 is unrestricted, $x_2 \geq 0$

(a) Determine all the basic feasible solution of the problem.

(b) Use the direct substitution in the objective function to determine the best basic solution.

(c) Solve the problem graphically, and verify the solution obtained in (b) is optimum

Solution:

Here the variable x_1 is unrestricted in sign means, we can write

$x_1 = x_3 - x_4$ with $x_3 \geq 0$, $x_4 \geq 0$

Hence the given problem becomes

Maximize $Z = x_3 - x_4 + 3x_2$

Subject to

$x_3 - x_4 + x_2 \leq 2$,

$-x_3 + x_4 + x_2 \leq 4$ and $x_2, x_3, x_4 \geq 0$

Now we write this problem in equation form by adding slak variables

Maximize $Z = x_3 - x_4 + 3x_2 + 0s_1 + 0s_2$

Subject to $x_3 - x_4 + x_2 + s_1 = 2$,

$-x_3 + x_4 + x_2 + s_2 = 4$ and $x_2, x_3, x_4\ s_1, s_2 \geq 0$

Now we have two equations in 5 unknowns.

We can obtained all basic solutions by putting any three variables to zero and solving equations in remaining variables as below

Non-Basis	**Basis**	**Equations obtained**	**Basic solution**	**Is it feasible?**	**Value Of Z**	**Is it optimal?**
x_2, x_3, x_4	s_1, s_2	$s_1 = 2$, $s_2 = 4$	$s_1 = 2$, $s_2 = 4$	Yes	Z=0	No
x_2, x_3, s_1	x_4, s_2	$-x_4 = 2$, $x_4 + s_2 = 4$	$x_4 = -2$, $s_2 = 6$	No	-	
x_2, s_1, x_4	x_3, s_2	$x_3 = 2$, $-x_3 + s_2 = 4$	$x_3 = 2$, $s_2 = 6$	Yes	Z = 2	No
s_1, x_3, x_4	x_2, s_2	$x_2 = 2$, $x_2 + s_2 = 4$	$x_2 = 2$, $s_2 = 2$	Yes	Z = 6	Yes
x_2, x_3, s_2	x_4, s_1	$x_4 + s_1 = 2$, $x_4 = 4$	$s_1 = -2$, $x_4 = 4$	No	-	
x_2, s_2, x_4	x_3, s_1	$x_3 + s_1 = 2$, $-x_3 = 4$	$s_1 = 6$, $x_3 = -4$	No	-	
s_2, x_3, x_4	x_2, s_1	$x_2 + s_1 = 2$, $x_2 = 4$	$s_1 = -2$, $x_2 = 4$	No	-	
x_2, s_2, s_1	x_3, x_4	$x_3 - x_4 = 2$, $-x_3 + x_4 = 4$	Does Not exist	No		- -
s_1, x_3, s_2	x_2, x_4	$-x_4 + x_2 = 2$, $x_4 + x_2 = 4$	$x_2=3$, $x_4=1$	Yes	Z = 2	No
s_1, x_4, s_2	x_2, x_3	$x_3 + x_2 = 2$, $-x_3 + x_2 = 4$	$x_2=3$, $x_3=-1$	No	-	

Graphical solution is as shown on next page

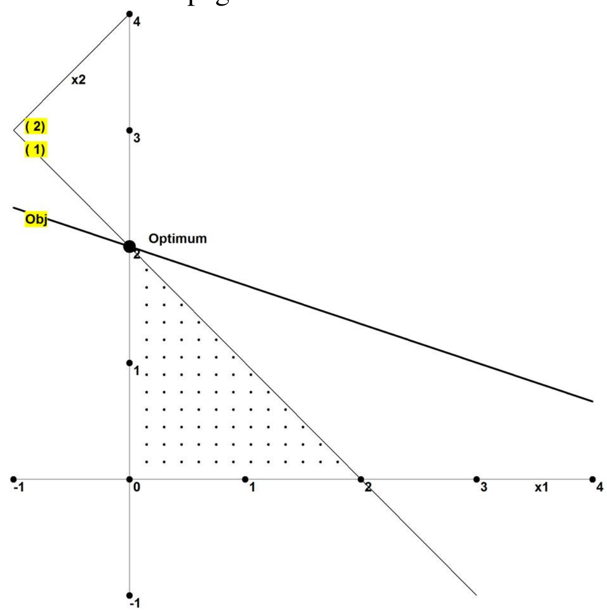


Fig. A.7

Problem 3: Consider the following two dimensional solution space.

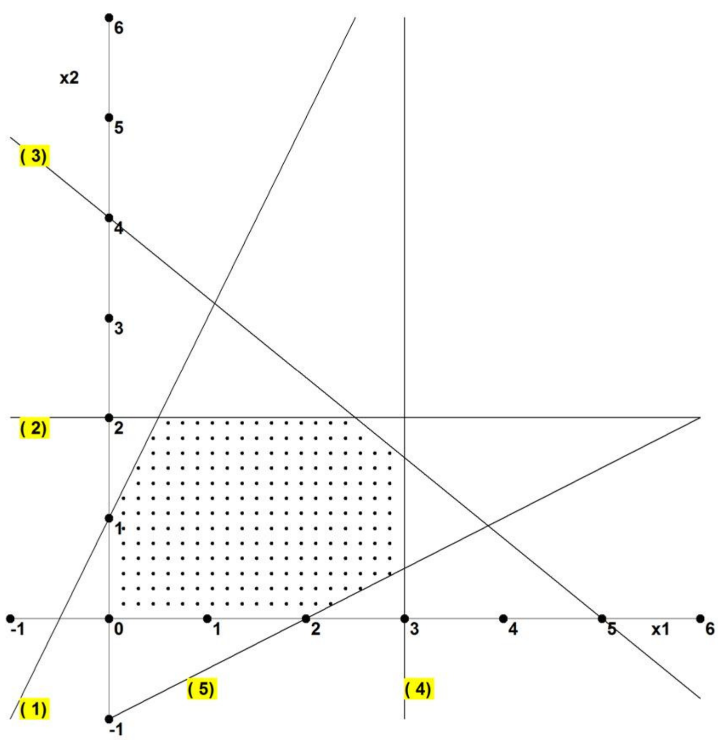


Fig. A.8

(a) Suppose that the objective function is given as

Maximize $Z = 3x_1 + 6x_2$

If the simplex iterations starts at A, identify the path to the optimum point E.

(b) Determine the entering variable, the corresponding ratios Of the feasibility condition, and the change in the value of Z, assuming that the objective function is given as $Z = 4x_1 + x_2$

Solution :

(a) Since $\frac{\partial Z}{\partial x_1} = 3, \frac{\partial z}{\partial x_2} = 6$, the rate of increase of Z is greater in the direction of x_2 than that of x_1.

We move along x_2 axis from the A (origin), to get optimum value of Z, until we just leave out the feaible region.

For this we will find the x_2 intercept of each constraint line and we move upto the point of minimum positive intercept which is G.

Now at G we must have to move towards point F, because along

GF, rate of change of Z is $\nabla Z = (3i + 6j) \cdot \frac{\overrightarrow{GF}}{||GF||} > 0$.

After reaching point F, we move towards E so that we cannot leave out feasible region to reach point E.

So path obtained is A-G-F-E.

(b) Here $Z = 4x_1 + x_2$.

Since $\frac{\partial Z}{\partial x_1} = 4, \frac{\partial Z}{\partial x_2} = 1$, the rate of increase of Z is greater in the direction of x_1 than that of x_2.

We move along x_1 axis from the A (origin), to get optimum value of Z, until we just leave out the feasible region.

For this we will find the x_1 intercept of each constraint line and we move upto the point B of minimum positive intercept equal to 2.

Hence entering variable is x_1. Now change in value of Z required is

$Z_B - Z_A = (4(2) + 0) - (4(0) + 0) = 8$

Problem 4 : Solve the following LPP using simplex method

Maximize $Z = 3x_1 + 2x_2 + 5x_3$

Subject to $x_1 + 2x_2 + 2x_3 \leq 8$, $3x_1 + 2x_2 + 6x_3 \leq 12$, $2x_1 + 3x_2 + 4x_3 \leq 12x_1, x_2, x_3 \geq 0$

Solution : Given L.P. in standard form is

Maximize $Z = 3x_1 + 2x_2 + 5x_3 + 0s_1 + 0s_2 + 0s_3$

Subject to :

$$x_1 + 2x_2 + 2x_3 + s_1 = 8$$
$$3x_1 + 2x_2 + 6x_3 + s_2 = 12$$
$$2x_1 + 3x_2 + 4x_3 + s_3 = 12$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

$\therefore$ Objective function is $Z - 3x_1 - 2x_2 - 5x_3 - 0s_1 - 0s_2 - 0s_3 = 0$.

Initial simplex table.

$m = 3, n = 6 \therefore n - m = 3$ are non-basic variable.

Basic variable	Z	x_1	x_2	$\downarrow x_3$	s_1	s_2	s_3	Solution	Ratio
Z-row	1	– 3	– 2	– 5	0	0	0	0	
s_1	0	1	2	2	1	0	0	8	4
$\leftarrow s_2$	0	3	2	$\boxed{6}$	0	1	0	12	$\sqrt{2}$
s_3	0	2	3	4	0	0	1	12	3

Basic variable	Z	x_1	x_2	$\downarrow x_3$	s_1	s_2	s_3	Solution	Ratio
Z-row	1	– 1/2	– 1/3	0	0	5/6	0	10	
s_1	0	0	4/3	0	1	– 1/3	0	4	∞
x_3	0	1/2	1/3	1	0	1/3	0	2	4
s_3	0	0	5/3	0	0	2/3	1	4	∞

Basic variable	Z	x_1	x_2	$\downarrow x_3$	s_1	s_2	s_3	Solution	Ratio
Z-row	1	0	0	1	0	1	0	12	
s_1	0	0	4/3	0	1	– 1/3	0	4	
x_1	0	1	2/3	2	0	1/3	0	4	
s_3	0	0	5/3	0	0	2/3	1	4	

∴ All Z-row coefficient are non-negative.

∴ The solution is optimal and optimal solution is

$$x_1 = 4, x_2 = 0, x_3 = 0, s_1 = 4, s_3 = 4, s_2 = 0$$

∴ $Z_{max} = 12$

Problem 5: Solve the following LPP using simplex method

Maximize $Z = 8x_1 - 4x_2$

Subject to $4x_1 + 5x_2 \leq 20$, $-x_1 + 3x_2 \geq -23$, $x_1 \geq 0$, x_2 unrestricted in sign

Solution : Standard form is :

Maximize $Z = 8x_1 - 4x_2$

Subject to $4x_1 + 5x_2 - 5x_2 + s_1 = 20$

$$+ x_1 - 3x_2^- + 3x_2^+ + s_2 = 23$$

$$x_1, x_2^-, x_2^+, s_1, s_2 \geq 0$$

Since x_2 is unrestricted in sign, we can represent x_2 as $x_2 = x_2^+ - x_2^-$

Objective function is $Z - 8x_1 + 4x_2^- - 4x_2^+ + 0s_1 + 0s_2 = 0.$

Initial simplex table is

Basic variable	Z	$x_1 \downarrow$	x_2	x_3	s_1	s_2	Solution	Ratio
Z-row	1	– 8	+ 4	– 4	0	0	0	
$\leftarrow s_1$	0	$\boxed{4}$	5	– 5	1	0	20	5
s_2	0	1	– 3	+ 3	0	1	23	23

Basic variable	Z	x_1	x_2^-	x_2^+	s_1	s_2	Solution	Ratio
Z-row	1	0	14	– 14	2	0	40	
x_1	0	1	5/4	– 5/4	1/4	0	5	– 4
$\leftarrow s_2$	0	0	– 17/4	$\boxed{17/4}$	– 1/4	1	18	72/17

Basic variable	Z	$x_1 \downarrow$	x_2	x_3	s_1	s_2	Solution	Ratio
Z-row	1	0	0	0	20/7	52/17	1008/17	
x_1	0	1	0	– 15/28	22/108	85	175/17	
x_2^+	0	0	– 1	1	– 1/17	4/17	72/17	

$\therefore$ All Z-row coefficient are non-negative.

$\therefore$ Solution is optimal and optimal solution is $x_1 = \frac{175}{17}$, $x_2^+ = \frac{72}{17}$, $Z = \frac{1008}{17}$.

$x_2 = 0$, $s_1 = s_2 = 0$

$\therefore$ $x^- = x_2^- - x_2^+ = -\frac{72}{17}$

$\therefore$ $x_2 = \frac{175}{17}$, $x_2 = -\frac{72}{17}$, $s_1 = 0$, $s_2 = 0$, $Z = \frac{1008}{17}$

Problem 6 : Solve the LPP by using big M-method

Minimize $Z = 4x_1 + x_2$, $3x_1 + x_2 = 3$, $4x_1 + 3x_2 \geq 6$, $x_1 + 2x_2 \leq 4$ and $x_1, x_2 \geq 0$

Solution : Standard form is,

Minimize $Z = 4x_1 + x_2$

Subject to

$$\begin{aligned} 3x_1 + x_2 &= 3 \\ 4x_1 + 3x_2 - s_1 &= 6 \\ x_1 + 2x_2 + s_2 &= 4 \\ x_1, x_2, s_1, s_2 &\geq 0 \end{aligned}$$

We cannot find initial basic feasible solution in terms of slaks. Hence we add artificial variables R_1 and R_2 in first and second constraints. And to remove artificial some iterations from initial basis we give large positive coefficient M called as penalty, to R_1 and R_2 in objective function.

Minimize $Z = 4x_1 + x_2 + MR_1 + MR_2$

Subject to $3x_1 + x_2 + R_1 = 3$

$$4x_1 + 3x_2 - s_1 + R_2 = 6$$

$$x_1 + 2x_2 + s_2 = 4$$

$$x_1, x_2, s_1, s_2 \geq 0$$

Objective function is, $Z - 4x_1 - x_2 - MR_1 - MR_2 = 0$.

Initial basic solution is :

$x_1 = 0, x_2 = 0, s_1 = 0, R_1 = 3, R_2 = 6, s_2 = 4, z = 9\text{ M}$

Basic variable	**Z**	x_1	x_2	s_1	s_2	R_1	R_2	**Solution**	**Ratio**
Z-row	1	– 4	– 1	0	0	– M	– M	0	
R_1	0	3	1	0	0	1	0	3	
R_2	0	4	3	– 1	0	0	1	6	
s_2	0	1	2	0	1	0	0	4	

∴ Initial basic feasible solution is $x_1 = 6, x_2 = 0, s_1 = 0, R_1 = 3, R_2 = 6, s_2 = 4, Z = 9M$. But the table shows solution $Z = 0$ so to make consistency. We have to make substitution using R_1 and R_2 row to get Z-row coefficient of R_1 and R_2 to be zero.

∴ New Z-row, old Z-row = MR_1 row + MR_2 row.

Basic variable	**Z**	$x_1 \downarrow$	x_2	s_1	s_2	R_1	R_2	**Solution**	**Ratio**
Z-row	1	7M–4	4M–1	–M	0	0	0	9M	
$\leftarrow R_1$	0	$\boxed{3}$	1	0	0	1	0	3	1
R_2	0	4	3	– 1	0	0	1	6	3/2
s_2	0	1	2	0	1	0	0	4	4

Basic variable	**Z**	x_1	$x_2 \downarrow$	R_1	R_2	s_1	s_2	**Solution**	**Ratio**
Z-row	1	0	$\frac{1}{3} + \frac{5}{3}M$	$\frac{4 - 7M}{3}$	0	– M	0	2M + 4	
x_1	0	1	1/3	0	0	1/3	0	1	3
$\leftarrow R_2$	0	0	$\boxed{5/3}$	– 1/3	1	– 1	0	2	6/5
s_2	0	0	5/3	– 1/3	0	0	1	3	9/5

Basic variable	Z	x_1	x_2	R_1	R_2	$s_1 \downarrow$	s_2	Solution	Ratio
Z-row	1	0	0	$\frac{24-15M}{15}$	$-M - \frac{1}{5}$	1/5	0	18/5	
x_1	0	1	0	3/15	– 1/5	1/5	0	3/5	3
x_2	0	0	1	4/15	3/5	– 3/5	0	6/5	– 2
$\leftarrow s_2$	0	0	0	1	– 1	$\boxed{1}$	1	1	1

Basic variable	Z	x_1	x_2	R_1	R_2	s_1	s_2	Solution	Ratio
Z-row	1	0	0	8 – M	– M	0	– 1/5	17/5	
x_1	0	1	0	2/5	0	0	0	– 1/5	
x_2	0	0	1	– 1/5	0	0	0	9/15	
s_1	0	0	0	1	– 1	1	1	1	

$\therefore$ All Z-row coefficient are non-positive.

$\therefore$ Solution is optimal.

$\therefore$ Optimal solution is $x_1 = -15$, $x_2 = 915$, $s_1 = 1$, $s_2 = 0$, Minimize $Z = 1715$.

PRACTICAL NO. : 03 CASES OF SOLUTION IN LPP

Problem 1 : Show by graphical method and also by the simplex method the following LPP has an alternate optimum solution

Maximize $Z = 2x_1 + 4x_2$

Subject to $x_1 + 2x_2 \leq 5$, $x_1 + x_2 \leq 4$ and $x_1, x_2 \geq 0$.

Solution : Standard form of given L.P. is

Maximize $Z = 2x_1 + 4x_2 + 0s_1 + 0s_2$

Subject to
$$x_1 + 2x_2 + s_1 = 5$$
$$x_1 + x_2 + s_2 = 4$$
$$x_1, x_2, s_1, s_2 \geq 0$$

Objective function is $Z - 2x_1 - 4x_2 - 0s_1 - 0s_2 = 0$.

Basic variables are s_1, s_2 and non-basic variables are x_1, x_2.

Basic variables	Z	x_1	$x_2 \downarrow$	s_1	s_2	Solution	Ratio
Z-row	1	– 2	– 4	0	0	0	
$\leftarrow s_1$	0	1	$\boxed{2}$	1	0	5	5/2
s_2	0	1	1	0	1	4	4

Basic variable	Z	$x_1 \downarrow$	x_2	s_1	s_2	Solution	Ratio
Z-row	1	0	0	2	0	10	
x_2	0	½	1	1/2	0	5/2	5
$\leftarrow s_2$	0	½	0	– 1/2	1	3/2	3

∴ All coefficients of Z-row are non-negative.

∴ Solution is optimal and optimal solution is $x_1 = 0$, $x_2 = 5/2$, $s_1 = 0$, $s_2 = 3/2$, Z = 10.

Basic variables are x_2, s_2 and non-basic variables are x_1, s_1.

Here x_1 is non-basic variable and its z-row coefficient is zero.

∴ x_1 enters the basic without changing objective Z-value.

∴ Given L.P.P. has alternate solution.

To find alternate solution enters s_1 in basis.

Basic variables	**Z**	x_1	x_2	s_1	s_2	**Solution**	**Ratio**
Z-row	1	0	0	2	0	10	
x_2	0	0	1	1	– 1	1	
x_1	0	1	0	– 1	2	3	

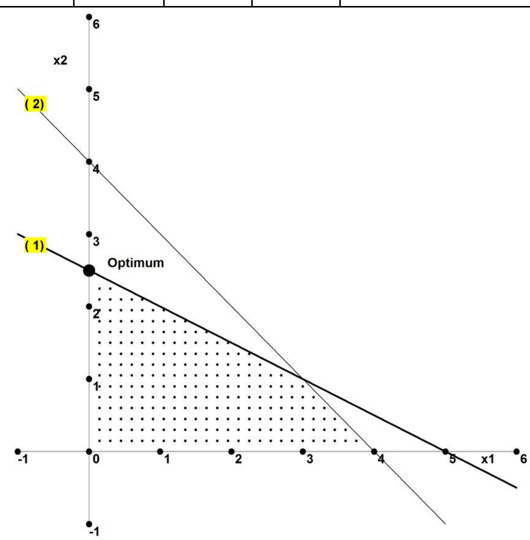


Fig. A.9

∴ All coefficients of Z-row are non-negative.

∴ Solution is optimal and optimal solution is $x_1 = 3$, $x_2 = 1$, $s_1 = 0$, $s_2 = 0$, Z = 10.

Entering s_2 in basis do not alter the objective value, because Z-row coefficient of s_2 is zero.

Problem 2 : Show by graphical method and also by the simplex method the following L.P.P. has unbounded solution

Maximize $Z = 2x_1 + x_2$

Subject to $x_1 - x_2 \leq 10$, $2x_1 \leq 40$, and $x_1, x_2 \geq 0$.

Solution : Standard form of given L.P. is,

Maximize $Z = 2x_1 + x_2 + 0s_1 + 0s_2$

Subject to
$$x_1 - x_2 + s_1 = 10$$
$$2x_1 + s_2 = 40$$
$$x_1, x_2, s_1, s_2 \geq 0$$

∴ Objective function is $Z - 2x_1 - x_2 - 0s_1 - 0s_2 = 0$.

Basic variables	Z	$x_1 \downarrow$	x_2	s_1	s_2	Solution	Ratio
Z-row	1	− 2	− 1	0	0	0	
$\leftarrow s_1$	0	$\boxed{1}$	− 1	1	0	10	10
s_2	0	2	0	0	1	40	20

Basic variables	Z	x_1	$\downarrow x_2$	s_1	s_2	Solution	Ratio
Z-row	1	0	− 3	2	0	20	
x_1	0	1	− 1	1	0	10	− 10
$s_2 \leftarrow$	0	0	$\boxed{-2}$	2	− 1	− 20	10

Basic variable	Z	x_1	x_2	$s_1 \downarrow$	s_2	Solution	Ratio
Z-row	1	0	0	− 1	3/2	50	
x_1	0	1	0	0	1/2	20	N.D.
x_2	0	0	1	− 1	1/2	10	− 10

Here, s_1 is entering variable but there is no leaving variable.

Hence, the solution is unbounded. Refer Fig. A.10 on next page.

Problem 3 : Show by graphical method and also by the simplex method the following L.P.P. has infeasible solution

Maximize $Z = 3x_1 + 2x_2$

Subject to

$2x_1 + x_2 \leq 2,$

$3x_1 + 4x_2 \geq 12,$

and $x_1, x_2 \geq 0.$

Solution : Standard form of given L.P.P. is,

Fig. of Problem 2

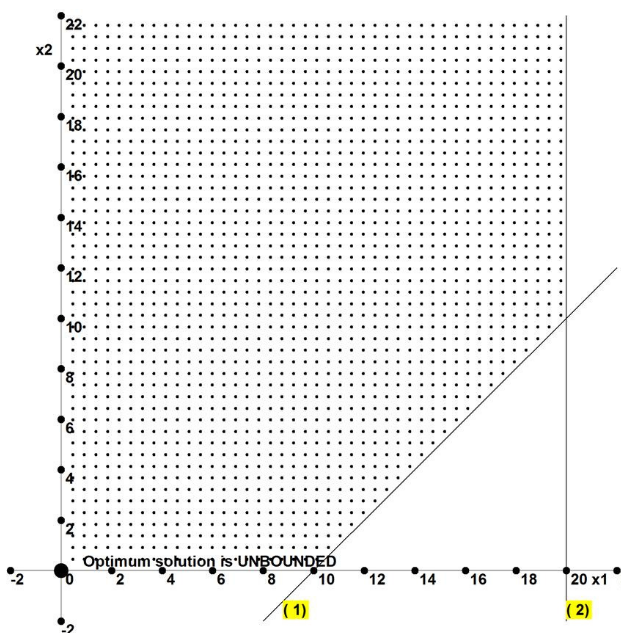


Fig. A.10

Problem 3 cont…..

Maximize Z : $3x_1 + 2x_2 - MR_1 + 0s_1 + 0s_2$

Subject to : $2x_1 + x_2 + s_1 = 2$

$3x_1 + 4x_2 + R_1 + s_2 = 12$

Objective function is $Z - 3x_1, - 2x_2 + MR_1 + 0s_1 + 0s_2 = 0$.

Basic variable	**Z**	x_1	x_2	s_1	s_2	R_1	**Solution**	**Ratio**
Z-row	1	– 3	– 2	0	0	M	0	
s_1	0	2	1	1	0	0	2	
R_1	0	3	4	0	– 1	1	12	

Penalise R_1 in objective function.

Basic variable	**Z**	x_1	$\downarrow x_2$	s_1	s_2	R_1	**Solution**	**Ratio**
Z-row	1	– 3 – M	– 2 – 4M	0	M	0	– 12 M	
$\leftarrow s_1$	0	2	1	1	0	0	2	2
R_1	0	3	4	0	– 1	1	12	3

Basic variable	**Z**	x_1	x_2	s_1	s_2	R_1	**Solution**	**Ratio**
Z-row	1	5M + 1	0	4 M + 2	M	0	4 – 4M	
x_2	0	2	1	1	0	0	2	
R_1	0	– 5	0	– 4	– 1	1	4	

All Z-row coefficient are non-negative.

$\therefore$ Solution is optimal $x_1 = 0$, $x_2 = 2$, $R_1 = 4 > 0$.

$\therefore$ Artificial variable R_1 remains in the basis at positive value $R_1 = 4$.

$\therefore$ Problem has no feasible solution.

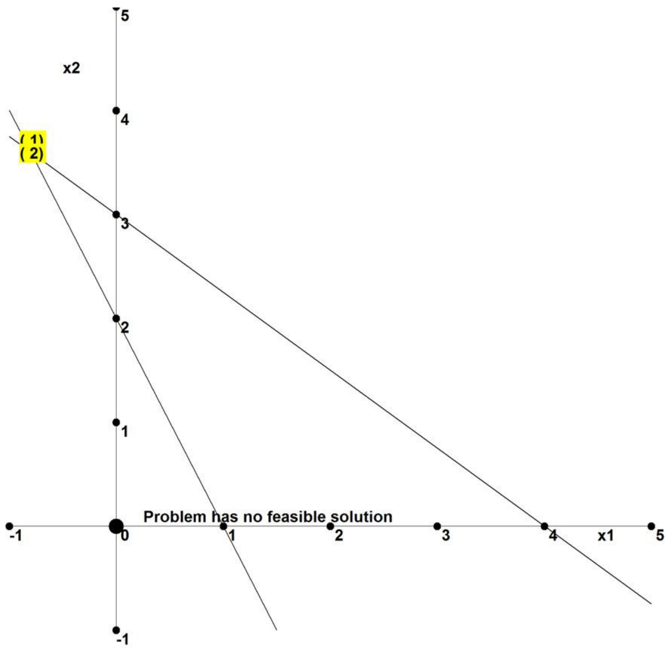


Fig. A.11

Problem 4 : Solve the following L.P.P. by two phase method:

Minimize $Z = x_1 + x_2$

Subject to

$2x_1 + 4x_2 \geq 4$,

$x_1 + 7x_2 \geq 7$,

and $x_1, x_2 \geq 0$.

Solution : Standard form of given L.P. is

Minimize $Z = x_1 + x_2 + MR_1 + MR_2$

Subject to : $2x_1 + 4x_2 - s_1 + R_1 = 4$

$x_1 + 7x_2 - s_2 + R_2 = 7$

Objective function : $Z - x_1 - x_2 - MR_1 - MR_2 + 0s_1 + 0s_2 = 0$.

Phase (1) : Minimize $r = R_1 + R_2$

Subject to : $2x_1 + 4x_2 - s_1 + R_1 = 4$

$x_1 + 7x_2 - s_2 + R_2 = 7$

Basic variables are R_1, R_2.

Non-basic variables are $x_1, x_2, s_1, s_2 = 0$.

$R_1 = 4, R_2 = 7, Z = 7M, r = 7$.

Initial table

Basic variables	**r**	x_1	x_2	s_1	s_2	R_1	R_2	**Solution**	**Ratio**
r-row	1	0	0	0	0	1	1	0	
R_1	0	2	4	– 1	0	1	0	4	
R_2	0	1	7	0	– 1	0	1	7	

Penalise R_1 and R_2 in r-row

Basic variables	r	x_1	$x_2 \downarrow$	s_1	s_2	R_1	R_2	Solution	Ratio
r-row	1	– 3	– 11	1	1	0	0	– 11	
$\leftarrow R_1$	0	2	4	– 1	0	1	0	4	1
R_2	0	1	7	0	– 1	0	1	7	1

Basic variable	r	x_1	x_2	$s_1 \downarrow$	s_2	R_1	R_2	Solution	Ratio
r-row	1	5/2	0	7/4	– 1	–11/4	0	0	
x_2	0	1/2	1	– 1/4	0	1/4	0	0	– 4
$\leftarrow R_2$	0	– 5/2	0	7/4	– 1	– 7/4	1	1	0

Basic variables	r	x_1	x_2	s_1	s_2	R_1	R_2	Solution	Ratio
r-row	1	0	0	0	0	– 1	– 1	0	
x_2	0	1/7	1	0	– 1/7	0	1/7	1	
s_1	0	– 1/7	0	1	– 4/7	– 1	4/7	0	

Minimize r = 0 is not positive.

$\therefore$ Phase produces basic feasible solution and it is $x_1 = 0, x_2 = 1, s_1 = 0, s_2 = 0$.

At this stage artificial variables R_1, R_2 completed their role.

$\therefore$ We can delete their columns and move to phase II.

After deleting R_1, R_2 column, we write original problem as

$$Z = x_1 + x_2$$

$$\frac{x_1}{7} + x_2 - \frac{1}{7} s_2 = 1$$

$$-\frac{10}{7} x_1 + s_1 - \frac{4}{7} s_2 = 0$$

$$x_1, x_2, s_1, s_2 \geq 0$$

$\therefore$ The initial table associated with phase II is as follows :

Basic variable	Z	x_1	x_2	s_1	s_2	Solution	Ratio
z-row	1	– 1	– 1	0	0	0	
x_2	0	1/7	1	0	– 1/7	1	
s_1	0	– 10/7	0	1	– 4/7	0	

$\therefore$ Basic variables x_2 have non-zero coefficient in Z-row it must be zero.

$\therefore$ New Z-row = old Z-row + x_2 row.

Basic variables	Z	x_1	x_2	s_1	s_2	Solution	Ratio
r-row	1	6/7	0	0	1/7	1	
x_2	0	1/7	1	0	– 1/7	1	
s_1	0	– 10/7	0	1	– 4/7	0	

∴ All Z-row coefficients are non-negative.

∴ Solution is optimal.

Optimal solution is $x_1 = 0$, $x_2 = 1$, $s_1 = 0$, $s_2 = 0$, $Z = 1$.

Problem 5 : Using Two phase method solve the following LPP:

Minimize $Z = 150x_1 + 150x_2 + 100x_3$

Subject to

$2x_1 + 3x_2 + x_3 \geq 4,$

$3x_1 + 2x_2 + x_3 \geq 3,$

and $x_1, x_2, x_3 \geq 0$

Solution : The standard form of given L.P.P. is

Minimize $Z = 150x_1 + 150x_2 + 100x_3 + MR_1 + MR_2 + 0s_1 + 0s_2 = 0$

$2x_1 + 3x_2 + x_3 - s_1 + R_1 = 4$

$3x_1 + 2x_2 + x_3 - s_1 + R_2 = 3$

Objective function is $Z = 150x_1 - 150x_2 - 100x_3 - MR_1 - MR_2 + 0s_1 + 0s_2 = 0$

Phase 1 : Minimize $r = R_1 + R_2$

Subject to : $2x_1 + 3x_2 + x_3 - s_1 + R_1 = 4$

$3x_1 + 2x_2 + x_3 - s_2 + R_2 = 3$

Basic variables are R_1, R_2.

Non-basic are x_1, x_2, s_1, s_2 which are kept zero.

$R_1 = 4$, $R_2 = 3$, $r = 7$, $Z = 7M$.

∴ $r - R_1 - R_2 = 0$.

Initial Table

Basic variable	r	x_1	x_2	x_3	s_1	s_2	R_1	R_2	Solution	Ratio
r-row	1	0	0	0	0	0	– 1	– 1	0	
R_1	0	2	3	1	– 1	0	1	0	4	
R_2	0	3	2	1	0	– 1	0	1	3	

Penalise R_1, R_2 in row

Basic variable	r	$\downarrow x_1$	x_2	x_3	s_1	s_2	R_1	R_2	Solution	Ratio
r-row	1	5	5	2	– 1	– 1	0	0	7	
R_1	0	2	3	1	– 1	0	1	0	4	2
$\leftarrow R_2$	0	$\boxed{3}$	2	1	0	– 1	0	1	3	1

Basic variable	r	x_1	$x_2\downarrow$	x_3	s_1	s_2	R_1	R_2	Solution	Ratio
r-row	1	0	5/3	1/3	– 1	2/3	0	– 5/3	2	
$\leftarrow R_1$	0	0	$\boxed{5/3}$	1/3	– 1	2/3	1	– 2/3	2	6/5
x_1	0	1	2/3	1/3	0	– 1/3	0	1/3	1	3/2

Basic variable	r	x_1	x_2	x_3	s_1	s_2	R_1	R_2	Solution	Ratio
r-row	1	0	0	0	0	0	– 1	– 1	0	
x_2	0	0	1	1/5	– 3/5	2/5	3/5	– 2/5	6/5	
x_1	0	1	0	1/5	2/5	– 3/5	– 2/15	1/15	1/5	

$\therefore$ Minimize r = 0 is not positive.

Phase (1) : Produces the basic feasible solution and it is $x_1 = \frac{1}{5}$, $x_2 = \frac{6}{5}$, $s_1 = 0$, $s_2 = 0$ at this stage.

The artificial variables have completed their role or their mission and hence we can eliminate their columns together and have to phase.

Phase 2 : After deleting the artificial columns together. We write the original problem as

Minimize $Z = 150x_1 + 180x_2 + 100x_3$

Subject to : $x_2 + \frac{1}{5}x_3 - \frac{3}{5}s_1 + \frac{2}{5}s_2 = \frac{6}{5}$

$x_1 + \frac{1}{5}x_3 + \frac{2}{5}s_1 - \frac{3}{5}s_2 = \frac{1}{5}$

$x_1, x_2, x_3, s_1, s_2 \geq 0.$

The initial table associated with phase II is

Basic variable	Z	x_1	x_2	x_3	s_1	s_2	Solution	Ratio
Z-row	1	– 150	– 150	– 100	0	0	0	
x_2	0	0	1	1/5	– 3/5	2/5	6/5	
x_1	0	1	0	1/5	2/5	– 3/5	1/5	

$\therefore$ Basic variables x_1 and x_2 have non-zero coefficient in the Z-row they must be zero.

$\therefore$ New Z row = old Z row + $150x_1$ row + $150x_2$ row.

Basic variable	**Z**	x_1	x_2	x_3	s_1	s_2	**Solution**
Z-row	1	0	0	– 40	– 30	– 30	210
x_2	0	0	1	1/5	– 3/5	2/5	6/5
x_1	0	1	0	1/5	2/5	– 3/5	1/5

$\therefore$ All coefficient of Z row are non-negative.

$\therefore$ Solution is optimal and optimal solution is

$Z = 210, x_2 = 615, x_1 = 1/5.$

PRACTICAL NO. 4 : DUALITY IN L.P.P.

Problem 1 : Write the dual of the following primal problem:

Minimize $Z = 5x_1 + 2x_2 + x_3$

Subject to

$$2x_1 + 3x_2 + x_3 \geq 20,$$
$$6x_1 + 8x_2 + 5x_3 \geq 30,$$
$$7x_1 + x_2 + 3x_3 \geq 40,$$
$$x_1 + 2x_2 + 4x_3 \geq 50$$
and $x_1, x_2, x_3 \geq 0.$

Solution : Given L.P.P. in the standard form.

Minimize $Z = 5x_1 + 2x_2 + x_3 + 0s_1 + 0s_2 + 0s_3 + 0s_4$

Subject to :
$$2x_1 + 3x_2 + x_3 - s_1 = 20$$
$$6x_1 + 8x_2 + 5x_3 - s_2 = 30$$
$$7x_1 + x_2 + 3x_3 - s_3 = 40$$
$$x_1 + 2x_2 + 4x_3 - s_4 = 50$$
$$x_1, x_2, x_3, s_1, s_2, s_3, s_4 \geq 0.$$

Let y_1, y_2, y_3, y_4 are dual variables corresponding to the first, second, third, fourth constraints respectively. Since primal objective is of minimization type, dual objective function is of maximization type and all constraints are written as $\leq$ type.

Dual is written as,

Maximize $W = 20y_1 + 30y_2 + 40y_3 + 50y_4$

Subject to :
$$2y_1 + 6y_2 + 7y_3 + y_4 \leq 5$$
$$3y_1 + 8y_2 + y_3 + 2y_4 \leq 2$$
$$y_1 + 5y_2 + 3y_3 + 4y_4 \leq 1$$
$$-y_1 \leq 0, -y_2 \leq 0, -y_3 \leq 0, -y_4 \leq 0$$

i.e. $y_1, y_2, y_3, y_4 \geq 0.$

Problem 2 : Write the dual of the following primal problem:

Minimize $Z = 4x_1 + 2x_2$

Subject to
$$x_1 - 2x_2 \geq 2,$$
$$x_1 + 2x_2 = 8,$$
$$x_1 - x_2 \leq 10$$ and $x_1 \geq 0$, x_2 is unrestricted in sign.

Solution : Given L.P. P. in standard form :

Minimize $Z = 4x_1 + 2x_2$

Subject to : $x_1 - 2x_2 - s_1 = 2$

$x_1 + 2x_2 = 8$

$x_1 - x_2 - s_2 = 10,\ x_1 \geq 0$

$\therefore$ x_2 is unrestricted hence replace x_2 by $\bar{x}_2 - \overset{+}{x}_2$.

Minimize $Z = 4x_1 + 2\ \bar{x}_2 - \overset{+}{x}_2$

Subject to : $x_1 - 2\bar{x}_2 + 2\overset{+}{x}_2 - s_1 = 2$

$x_1 - 2\bar{x}_2 - 2\overset{+}{x}_2 = 8$

$x_1 - \bar{x}_2 + \overset{+}{x}_2 + s_2 = 10$

Let y_1, y_2, y_3 be dual variables corresponding to first, second, third constraints respectively. Since primal objective is of minimization type, dual objective is of maximization type and each dual constraint is written as $\leq$ type

Dual is written as,

Maximize $W = 2y_1 + 8y_2 + 10y_3$

Subject to : $y_1 + y_2 + y_3 \leq 4$

$-2y_1 - 2y_2 - y_3 \leq 2$

$+2y_1 - 2y_2 + y_3 \leq -2$

$-2y_1 + 2y_2 - y_2 \leq 2$

$-y_1 \leq 0$, that is $y_1 \geq 0$

y_2 is unrestricted in sign, (since there is no non-negativity constraint for y_2.) $y_3 \leq 0$.

Problem 3 : Consider the following LPP :

Write the dual of the following primal problem:

Minimize $Z = 5x_1 + 2x_2 + 3x_3$

Subject to

$x_1 + 5x_2 + 2x_3 = 30,$

$1x_1 - 5x_2 + 6x_3 \leq 40,$

and $x_1, x_2, x_3 \geq 0.$

The optimal solution yields the following objective equation:

$Z + 0x_1 - 23x_2 + 7x_3 + (5 + M)\ x_4 + x_5 = 150.$

Where artifitial x_4 and slak x_5 are the starting basic variables. Write the associated dual problem and determine its optimal solution from the optimal Z equation.

Solution : Given L.P.P. in standard form :

Minimize $Z = 5x_1 + 2x_2 + 3x_3 - M\ x_4 + 0x_5$

Subject to : $x_1 + 5x_2 + 2x_3 + x_4 = 30$

$x_1 - 5x_2 - 6x_3 + x_5 = 40$

$x_1, x_2, x_3, x_4, x_5 \geq 0$

Let y_1 and y_2 are dual variables corresponding to first and second constraints in primal problem.

Dual is written as,

Maximize $W = 30y_1 + 40y_2$

Subject to : $y_1 + y_2 \le 5$

$5y_1 - 5y_2 \le 2$

$2y_1 - 6y_2 \le 3$

y_1 is unrestricted in sign; $y_2 \le 0$; $y_1 \to x_4$, $y_2 \to x_5$

$$\text{Optimal value of } y_1 = \begin{pmatrix}\text{Optimal primal Z-coefficient}\\ \text{of starting variable } x_4\end{pmatrix} + \begin{pmatrix}\text{Original objective}\\ \text{coefficinet of } x_4\end{pmatrix}$$

$$= 5 + M - M = 5$$

$$\text{Optimal value of } y_2 = \begin{pmatrix}\text{Optimal primal Z-coefficient}\\ \text{of starting variable } x_5\end{pmatrix} + \begin{pmatrix}\text{Original objective}\\ \text{coefficient of } x_5\end{pmatrix}$$

$$= 0 + 0 = 0$$

$$W = 30\,(5) + 40\,(0) = 150$$

Problem 4 : Solve the dual of the following problem, then find its optimal solution

From the solution of the dual. Does the solution of the dual offer computational advantages over solving the primal directly?

Minimize $Z = x_1 + 1/2\ x_2$

Subject to

$6x_1 - 2x_2 \le 24,$

$3x_1 + 2x_2 \le 18,$

$x_1 + 3x_2 \le 12$ and $x_1, x_2 \ge 0$

Solution : Given L.P.P. in standard form,

Minimize $Z = x_1 + \frac{1}{2} x_2$

Subject to : $6x_1 - 2x_2 - s_1 = 24$

$3x_1 + 2x_2 - s_2 = 18$

$x_1 + 3x_2 - s_3 = 12$; $x_1, x_2 \ge 0$

Let y_1, y_2, y_3 be dual variables corresponding to first, second, third constraints respectively

Dual is written as

Maximize $W = 24y_1 + 18y_2 + 12y_3$

Subject to : $6y_1 + 3y_2 + y_3 \le 1$

$-2y_1 + 2y_2 + 3y_3 \le \frac{1}{2}$

$-y_1 \le 0, -y_2 \le 0, -y_3 \le 0$.that is $y_1, y_2, y_3 \ge 0$

Standard form,

Maximize $W = 24y_1 + 18y_2 + 12y_3$

Subject to : $6y_1 + 3y_2 + y_3 + s_1 = 1$

$-2y_1 + 2y_2 + y_3 + s_2 = \frac{1}{2}$

$y_1, y_2, y_3, s_1, s_2 \ge 0$

Initial simplex table

Basic variable	$y_1 \downarrow$	y_2	y_3	s_1	s_2	Solution	Ratio
W-row	– 24	– 18	– 12	0	0	0	
s_1	6	3	1	1	0	1	1/6 →
s_2	– 2	2	3	0	1	1/2	– 1/4

First iteration table

Basic variable	y_1	y_2	$y_3 \downarrow$	s_1	s_2	Solution	Ratio
W-row	0	– 6	– 8	4	0	4	
y_1	1	1/2	1/6	1/6	0	1/6	1
s_2	0	3	10/3	1/3	1	5/6	1/4 →

Second iteration table

Basic variable	y_1	y_2	y_3	s_1	s_2	Solution	Ratio
W-row	0	6/5	0	24/5	24/10	6	
y_1	1	7/20	0	3/20	– 1/40	1/8	
y_3	0	9/10	1	1/10	3/20	1/4	

Since in the objective function row there is non-negative coefficient we have found optimal point.

Optimal point : $y_1 = 1/8$, $y_2 = 0$, $y_3 = 1/4$

$$W_{max} = 6$$

$$x_1 \to s_1,\ x_2 \to s_2$$

$$\text{Optimal value of } x_1 = \begin{pmatrix}\text{Optimal primal Z-coefficient} \\ \text{of starting variable } s_1\end{pmatrix} + \begin{pmatrix}\text{Original objective} \\ \text{coefficient of } s_1\end{pmatrix}$$

$$= \frac{48}{10} + 0 = \frac{24}{5}$$

$$\text{Optimal value of } x_2 = \begin{pmatrix}\text{Optimal primal} \\ \text{Z-coefficient of starting} \\ \text{variable } s_2\end{pmatrix} + \begin{pmatrix}\text{Original objective} \\ \text{coefficient of } s_2\end{pmatrix}$$

$$= \frac{24}{10} + 0 = \frac{12}{5}$$

$$\text{Minimum value of } Z = \frac{24}{5} + \frac{1}{2}\left(\frac{12}{5}\right) = 6$$

Yes, there are computational advantages in simplex tables due to less number of dual constraints.

Problem 5 : Wild west produces two types of cowboy hats. Type I hat requires twice as much labour time as Type II, If all the available labour time is dedicated to Type II alone, the company can produce a total of 400 Type II hats a day. The respective market limits for the two types are 150 and 200 hats per day. The revenue is ₹ 80 per Type I hat and ₹ 50 per Type II hat.

(a) Use the graphical solution to determine the number of hats of each type that maximize revenue.

(b) Determine the dual price of the production capacity (in terms of the type II hats) and the range for which it is applicable.

(c) If the daily demand limit of the Type I hat is decreased to 120, use the dual price to determine the corresponding effect on the optimal revenue.

(d) What is the dual price of the market share of the Type II hat ?

By how much the market share be increased while yielding the computed worth per unit ?

Solution :

Let x_1 = Number of hats type I produced per day.

x_2 = Number of hats type I produced per day.

$\therefore$ L.P.P. is as follows :

Maximize $Z = 80x_1 + 50x_2$

Subject to : $2x_1 + x_2 \le 400$

$x_1 \le 150$

$x_2 \le 200$

(a) The graphical solution is a given below :

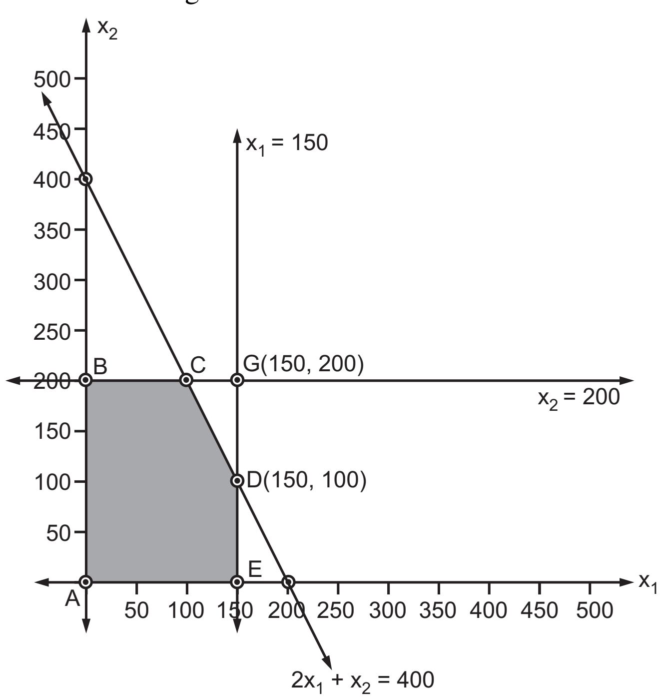


Fig. A.12

Corner points	(x_1, x_2)	Z
A	(0, 0)	0
B	(0, 200)	10,000
C	(100, 200)	18,000 (optimum)
D	(150, 100)	17,000
E	(150, 0)	12,000

Point C (100, 200) gives optimal solution for given problem.

i.e. There should be production of 100 hats of type I and 200 of type II.

(b) Consider production capacity is $2x_1 + x_2 \le 400$.

If we change capacity from 400 to 500. The capacity constraint becomes $2x_1 + x_2 \le 500$.

Now the optimal occurs at G with optimal value of Z as $Z_G = 80 \times 150 + 50 \times 200 = 12000 + 10000 = 22000$.

Hence change in optimal value of objective is $Z_G - Z_C = 22000 - 18000 = 4000$.

Hence rate of change of revenue change with respect to change in capacity is

$(Z_G - Z_C)$/(change in capacity) = 4000/100 = 40.

Therefore the dual price of production capacity = 40.

The minimum capacity at (0, 200) and the maximum capacity at (150, 200).

Therefore dual price is applicable in the range 200 to 500.

(c) If we change market limit for type I hat from 150 to 100 then the constraint becomes

$x_1 \le 100$

$x_2 \le 200$

The optimal occurs at C (100, 200).

$\therefore$ Dual prize Z = 0. $\therefore$ There is no change in objective optimum value is 100 and it is at C (100, 200).

Minimum market limit for type I hat.

Maximum market limit at (200, 0) and it is 200.

$\therefore$ The dual price rupees zero is applicable in the range 100 to 200.

Since 120 falls in the range 100 to 200 the dual prize i.e. there will be no change in optimal revenue.

(d) $x_2 \le 200$. If we change capacity of x_2 by 50 units then optimal value of Z changes by 500 units. $\therefore$ Dual price = $\frac{500}{50}$ = 10.

Minimum market limit is at point (150, 100) = 100.

Maximum market limit at point (0, 400) = 400.

∴ Range for market limit of type II hat is 100 to 400.

Dual price 10 is applicable in range (100, 400).

Maximum increase = 400 − 200 = 200

∴ Market share can be increased by maximum of 200 hats per day for type II.

Problem 6 : Reddy Mikks produces both, interior and exterior paints from two raw materials M_1 and M_2. The following table provides the basic data of the problem.

Raw Material Type	Tons of raw material per ton of *Exterior paint*	*Interior paint*	Maximum availability
M_1	6	4	24 tons
M_2	1	2	6 tons
Profit in ₹ per ton	5000	4000	

A market survey indicates that the daily demand for the interior paint cannot exeed that of exterior paint by more than 1 ton. Also the maximum daily demand of the interior paint is 2 tons.

(a) Use the graphical solution to determine the optimum product mix of the interior and exterior paints that maximizes the total daily profit.

(b) Determine the range for the ratio of the unit revenue of the exterior paint to the unit revenue of the interior paint.

(c) If the revenue per ton of the exterior paint remains constant at ₹ 5000 per ton, determine the maximum unit revenue of the interior paid that will keep the present optimum solution unchanged.

(d) If for marketing reasons the unit revenue of the interior paint must be reduced to 3000, will the current optimum production mix change?

Solution :

Let x_1 = Tons produced daily of exterior point

x_2 = Tons produced daily for interior point.

Maximize $Z = 5000\,x_1 + 4000x_2$

Subject to :

$$6x_1 + 4x_2 \le 24$$
$$x_1 + 2x_2 \le 6$$
$$-x_1 + x_2 \le 1$$
$$x_2 \le 2,\ x_1, x_2 \ge 0$$

We first use graphical method to determine optimal product maximum.

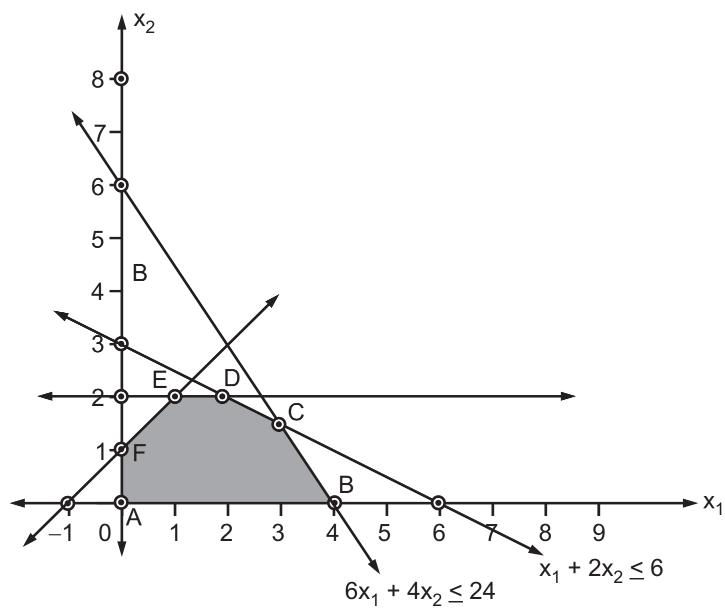


Fig. A.13

Corner points	(x_1, x_2)	Z
A	(0, 0)	0
B	(4, 0)	20
C	(3, 1.5)	21 (optimal)
D	(2, 2)	18
E	(1, 2)	13
F	(0, 1)	4

(b) Here $$\frac{C_1}{C_2} = \frac{5000}{4000} = \frac{5}{4}$$

$Z = C_1x_1 + C_2x_2$ lies between two lines $x_1 + 2x_2 = 6$ and $6x_1 + 4x_2 = 24$.

This means that ratio C_1/C_2 can vary between $\leftrightarrow \frac{1}{2}$ and $\frac{3}{2}$.

i.e. $$\frac{1}{2} \leq \frac{C_1}{C_2} \leq \frac{3}{2}$$

(c) If we fix C_1 at 5000 then

$$\frac{1}{2} \leq \frac{5000}{C_2}$$

$$C_2 = 10,000$$

We can increase unit revenue of interior point upto 10,000 keeping optimum solution unchanged.

(d) Keeping a fixed at 5000. We can find rage for C_2.

Since $$\frac{1}{2} \le \frac{C_1}{C_2} \le \frac{3}{2}$$

$$C_1 = 5000$$

$$\frac{1}{2} \le \frac{5000}{C_2} \le \frac{3}{2}$$

$$10000 \le 3C_2$$

$$\frac{10000}{3} \le C_2$$

To keep current optimal solution unchanged.

C_2 must be greater or equal to $\frac{10000}{3}$.

But here $$C_2 = 3000$$

$$C_2 \le \frac{10000}{3}$$

$\therefore$ Current optimum production maximum will change for that until revenue value of interior point.

PRACTICAL NO. 5 : TRANSPORTATION PROBLEMS

Problem 1: Find the basic feasible solution of the following transportation problem by (a) North West corner (b) Least cost Method. The entries I the matrix indicate the cost in rupees of the transportation a unit from a particular source to a particular destination.

Sources	**Destinations**				**Availability**
	D_1	D_2	D_3	D_4	
S_1	10	8	11	7	20
S_2	9	12	14	6	40
S_3	8	9	12	10	35
Requirement	16	18	31	30	95

Solution : NWCM : North West Corner Method.

Destination / Source	D_1	D_2	D_3	D_4	Availability
s_1	$\boxed{10}$ 16	$\boxed{8}$ 4	$\boxed{11}$	$\boxed{7}$	~~20~~ ~~4~~ X
s_2	$\boxed{9}$	$\boxed{12}$ 14	$\boxed{14}$ 26	$\boxed{6}$	~~40~~ ~~26~~ X
s_3	$\boxed{8}$	$\boxed{9}$	$\boxed{12}$ 5	$\boxed{10}$ 30	~~35~~ ~~30~~ X
Requirement	~~16~~ X	~~18~~ ~~14~~ X	~~31~~ ~~5~~ X	~~30~~ X	95

$$\sum a_i = \sum b_j$$

∴ Given transportation problem is balanced.

∴ Solution is $x_{11} = 16$, $x_{12} = 4$, $x_{22} = 14$, $x_{23} = 26$, $x_{33} = 5$, $x_{34} = 30$.

∴ Transportation cost $= 16 \times 10 + 4 \times 8 + 14 \times 12 + 26 \times 14 + 5 \times 2 + 30 \times 10$

$= $ ` 1084

(b) LCM : Least Cost Method :

∴ $$\sum a_i = \sum b_j$$

∴ Given transportation problem is balanced.

Destination / Source	D_1	D_2	D_3	D_4	Availability
s_1	$\boxed{10}$	$\boxed{8}$ 18	$\boxed{11}$ 2	$\boxed{7}$	~~20~~ ~~2~~ X
s_2	$\boxed{9}$		$\boxed{14}$ 10	$\boxed{6}$ 30	~~40~~ ~~10~~ X
$\boxed{12}$	$\boxed{8}$ 16		$\boxed{12}$ 19	$\boxed{10}$	~~35~~ ~~19~~ X
$\boxed{9}$					
	~~16~~ X		~~31~~ ~~29~~ ~~10~~ X	~~30~~ X	95
~~18~~ X					

∴ Solution is $x_{12} = 18$, $x_{13} = 2$, $x_{23} = 10$, $x_{24} = 30$, $x_{31} = 16$, $x_{33} = 19$.

∴ Transportation cost $= 18 \times 8 + 2 \times 11 + 10 \times 14 + 30 \times 6 + 16 \times 8 + 19 \times 12$

$=$ ₹ 842

Problem 2 : Find initial basic feasible solutions of the following transportation problem by VAM the entries in the matrix indicate the cost in rupees of the transporting a unit from a particular factory to a particular warehouse.

	Warehouse				**Supply**
Factory	**X**	**Y**	**Z**	**W**	
A	25	55	40	60	60
B	35	30	50	40	140
C	36	45	26	26	150
D	35	30	41	50	50
Demand	90	100	120	140	

Solution :

∴ $\sum a_i \le \sum b_j$ i.e. Supply < Demand.

∴ Add dummy source in factory E.

Warehouse / Factory	X	Y	Z	W	Supply	Penalty
A	25 60	55	40	60	~~60~~ X	15 15
B	35	30 50	50	40 90	~~140~~ ~~50~~ X	5 5 5 5 5 30
C	36 30	45	26 120	66	~~150~~ ~~30~~ X	10 10 10 9 9
D	35	30 50	41	50	~~50~~ X	5 5 5 5 5 30
6	0	0	0	0 50	~~50~~ X	
Demand	~~90~~ ~~30~~ X	~~100~~ ~~50~~ X	~~120~~ X	~~140~~ ~~90~~ X		
Penalty	25 10 0 0 0	30 0 0 0 0 0	26 14 15	40 10 10 10		

$\therefore$ Solution is $x_{11} = 60$, $x_{22} = 50$, $x_{24} = 90$, $x_{31} = 30$, $x_{33} = 120$, $x_{42} = 50$, $x_{54} = 50$.

$\therefore$ The transportation cost $= 60 \times 25 + 50 \times 30 + 90 \times 40 + 30 \times 36 + 120 \times 26 + 50 \times 30 + 0 \times 50 = 12300$

Problem 3 : The table given below has been taken from the solution procedure of a transportation problem. (minimization transportation cost)

	Warehouse			**Capacity**
Factory	**X**	**Y**	**Z**	
A	***31*** (4)	***25*** (8)		56
B	***41*** (16)	(24)	***41*** (16)	82
C	(8)	***77*** (16)	(24)	77
	72	102	41	

(a) Show that the solution is not optimal (b) Find optimal solution (c) Does the problem have multiple optimal solution ? Give reasons. If so, find more optimal solutions.

Solution : $\therefore$ $\sum a_i = \sum b_i$ $\therefore$ Given problem is balanced.

Factory \ Warehouse	X	Y	Z	Capacity	
A	4 31	8 25	8 4	56	$u_1 = 4$
B	16 41	24 4	16 41	82	$u_2 = 16$
C	18 – 4	16 77	24 12	77	$u_3 = 12$
	72	102	41		
	$v_1 = 0$	$v_2 = 4$	$v_3 = 0$		

For occupied cell, $c_{ij} = u_i + v_j$

For non-occupied cell, $d_{ij} = c_{ij} - (u_i + v_j)$

$\therefore$ Number of allocation $= 5 = m + n - 1$

$\therefore$ Solution is non-degenerate.

∴ Some $d_{ij} < 0$ i.e. $d_{31} < 0$.

∴ Solution is not optimal.

∴ From a loop corresponding to cell (3, 1). We get

$\boxed{4}$ 4	$\boxed{8}$ 56	$\boxed{8}$ 8	$u_1 = -16$
$\boxed{16}$ 0 – 41	$\boxed{24}$ + 41	$\boxed{16}$ 41	$u_2 = 0$
$\boxed{8}$ 12 – 0	$\boxed{16}$ 46 – 5	$\boxed{24}$	$u_3 = -8$
$v_1 = 16$	$v_2 = 24$	$v_3 = 16$	

∴ Number of allocation = 5 = m + n − 1.

∴ Solution is non-degenerate.

∴ All $d_{ij} \geq 0$.

∴ The solution is optimal.

∴ Optimum solution is $x_{12} = 56$, $x_{22} = 41$, $x_{23} = 41$, $x_{31} = 16$, $x_{32} = 46$.

∴ Optimum solution is

Minimum T.C. = 2744.

∴ $d_{22} = 0$

∴ The given system has alternate solution.

∴ The alternate solution is given by from a loop corresponding to cell (2, 2).

$\boxed{4}$ 4	$\boxed{8}$ 56	$\boxed{4}$ 8	$u_1 = 8$
$\boxed{16}$ 0	$\boxed{24}$ 41	$\boxed{16}$ 41	$u_2 = 24$
$\boxed{8}$ 72	$\boxed{16}$ 5	$\boxed{24}$ 16	$u_3 = 16$
$v_1 = -8$	$v_2 = 0$	$v_3 = -8$	

∴ All $d_{ij} > 0$.

∴ The solution is optimal.

∴ The alternate solution is $x_{12} = 56$, $x_{22} = 41$, $x_{23} = 41$, $x_{31} = 72$, $x_{32} = 5$.

∴ Minimum T.C. = 2744 .

Problem 4 : A company has four warehouses and five stores the warehouses have total surplus of 430 units of a given commodity that is divided among them as follows

Warehouse	W_1	W_2	W_3	W_4
Surplus	150	30	120	130

The five stores, in all a requirement of 450 units of the commodity individual requirements are :

Store	S_1	S_2	S_3	S_4	S_5
Requirement	80	60	20	210	80

Cost of shipping one unit from i^{th} warehouse to the j^{th} store is displayed in the following table:

	Store				
Warehouse	S_1	S_2	S_3	S_4	S_5
$\mathbf{W_1}$	9	12	10	10	6
$\mathbf{W_2}$	5	18	12	11	2
$\mathbf{W_3}$	10	M	7	3	20
$\mathbf{W_4}$	5	6	2	M	8

M indicates that the route is not available. How should the company arrange to transport the units so that the transportation cost is minimized?

Solution : $\therefore\ \sum a_i = \sum b_j$ i.e.

Requirement < Surplus. $\therefore$ Add dummy row i.e. dummy warehouse penalty.

	s_1	s_2	s_3	s_4	s_5	Req.	Penalty
w_1	9 10	12	10	10 90	6 50	~~150~~ ~~60~~ 50	33333333
w_2	5	18	12	11	2 30	~~30~~ X	33333
w_3	10	M	7	3 120	20	~~120~~ X	44
w_4	5 70	6 40	2 20	M	8	~~130~~ ~~110~~ ~~70~~ X	333 1 333
w_5	0	0	0	0	0	~~20~~ X	~~0~~~~0~~~~0~~~~0~~0
Surplus	~~80~~ ~~10~~ X	~~60~~ ~~40~~ X	~~20~~ X	~~210~~ ~~90~~ X	~~80~~ ~~50~~ X		
Penalty	5 0 0 0 0 4 4 4	6 6 6 6	2 5 8	3 7 1 1 1	2 4 6 6 6 2 2 2		

9 10	12 2	10 4	10 90	6 50	$u_1 = 4$
5 0	18 12	12 10	11 5	2 30	$u_2 = 0$
10 8	M M – 3	7 8	3 120	20 19	$u_3 = –3$
5 70	6 40	2 20	M M – 6	8 6	$u_4 = 0$
0	0 20	0 4	0 0	0 4	$u_5 = –6$
$v_1 = 5$	$v_2 = 6$	$v_3 = 2$	$v_4 = 6$	$v_5 = 2$	

Number of allocation = 9 = m + n – 1.

∴ Solution is non-degenerate.

∴ All $d_{ij} \geq 0$.

∴ Solution is optimal.

∴ Optimal solution is $x_{11} = 10$, $x_{14} = 90$, $x_{15} = 50$, $x_{25} = 30$, $x_{34} = 120$, $x_{41} = 70$, $x_{42} = 40$, $x_{43} = 20$, $x_{52} = 20$.

∴ Minimum T.C. = 2340 ∴ Some $d_{ij} = 0$. Here $d_{21} = 0$

∴ Given T.P. has alternate solution to find that from a loop corresponding to cell (2, 1).

9 0	12 2	10 4	10 90	6 60	$u_1 = 4$
5 10	18 12	12 10	11 5	2 20	$u_2 = 0$
10 8	M M – 3	7 8	3 120	20 21	$u_3 = –3$
5 70	6 40	2 20	M M – 6	8 6	$u_4 = 0$
0 1	0 20	0 4	0 0	0 4	$u_5 = –6$
$v_1 = 5$	$v_2 = 6$	$v_3 = 2$	$v_4 = 6$	$v_5 = 2$	

∴ All $d_{ij} \geq 0$. ∴ Solution is optimal.

∴ Optimal solution is $x_{14} = 90$, $x_{15} = 60$, $x_{21} = 10$, $x_{25} = 20$, $x_{34} = 120$, $x_{41} = 70$, $x_{42} = 40$, $x_{43} = 20$, $x_{51} = 20$. ∴ Minimum T.C.= 2340

Problem 5 : A company manufacturing television sets has four plants I , II, III and IV with capacities 125, 250, 175 and 100 units respectively. The company supplies TV sets to its four showrooms I, II, III and IV which have demands of 100, 400, 90 and 60 units respectively. Due to the differences in the raw material cost and the transportation cost, the profit per unit in ₹ Differ, which are given below in the table.

	Showroom			
Plant	I	II	III	IV
I	90	100	120	110
II	100	105	130	117
III	111	109	110	120
IV	130	125	108	113

Determine the distribution programme so as to maximize the profit.

Also determine the maximum total profit.

Solution : The given T.P. maximum profit is 130.

∴ Subtract all profits from 130.

∴ Resulting matrix become minimization.

						Penalty
	40	30 (125)	10	20	~~125~~ X	10 10 10 30
	30	25 (100)	0 (90)	13 (60)	~~250~~ ~~100~~ ~~100~~ X	13 13 12 25 25
	19	21 (175)	20	10	~~175~~ X	9 10 11 21
	10 (100)	5	22	17	~~100~~ X	5
	~~100~~ X	~~400~~ ~~275~~ ~~175~~ X	~~90~~ X	~~60~~ X		
Penalty	19	16 4 4 4 4	10 10	3 3 3		

Solution : $x_{12} = 125$, $x_{22} = 100$, $x_{23} = 90$, $x_{24} = 60$, $x_{32} = 175$, $x_{41} = 100$.

$\therefore$ Profit is 10705 .

40 15	30 125	10 5	20 2	$u_1 = 30$
30 10	25 100	0 90	13 60	$u_2 = 25$
19	21 175	20 24	10	$u_3 = 21$
0 100	5 ε	22 42	17 24	$u_4 = 5$
$v_1 = -5$	$v_2 = 0$	$v_3 = -25$	$v_4 = -12$	

Number of allocation = 6 < m + n − 1.

$\therefore$ Solution is degenerate.

$\therefore$ Add (ε) allocations to cell (4, 2).

$\therefore$ Add $d_{ij} \geq 0$.

$\therefore$ Solution is optimal.

Optimal solution is $x_{12} = 125$, $x_{22} = 100$, $x_{23} = 90$, $x_{32} = 175$, $x_{24} = 60$, $x_{41} = 100$.

$\therefore$ Maximum profit = $125 \times 100 + 100 \times 105 + 109 \times 175 + 90 \times 130 + 60 \times 117 + 100 \times 130 = 73795$.

Problem 6 : The following table shows all necessary information on the availability of supply to each warehouse, the requirement of each market and the unit transportation cost from each warehouse to each market.

	Market				
Warehouse	P	Q	R	S	**Supply**
A	6	9	5	4	22
B	5	3	2	7	15
C	5	7	8	6	80
Requirement	7	12	17	9	

The shipping clerk has worked the following schedule from experience :

12 units from A to Q, 1 unit from A to R, 9 units from A to S, 15 units from B to R, 7 units from C to P and 1 unit from C to R.

(a) Check and see, if the clerk has optimal schedule.

(b) Find the optimal schedule and the minimum transportation cost

(c) If the clerk is approached by a courier to route C to Q, who offers to reduce his rate in the hope of getting some business, by how much the rate should be reduced that the clerk will offer him the business?

Solution :

	P	Q	R	S	Supply
A	6	3; 12	5; 1	4; 9	22
B	5	9	2; 15	7	15
C	5; 7	7	8; 1	6	8
Required	7	12	17	9	45/45

	6; 4	3; 12	5; 1	4; 9	$u_1 = 5$
	5; 4	9; 9	2; 15	7; 6	$u_2 = 2$
	5; 7	7; 1	8; 1	6	$u_3 = 8$
	$v_1 = -3$	$v_2 = -2$	$v_3 = 0$	$v_4 = -1$	

Number of allocation = 6 = m + n − 1.

∴ Solution is non-degenerate. ∴ Some $d_{ij} < 0$. ie. $d_{34} < 0$

∴ The solutions is not optimal. ∴ From a loop corresponding to cell (3, 4).

	6; 4	3; 12	5; 1	4; 9	$u_1 = 0$
	5; 4	9; 9	2; 15	7; 6	$u_2 = -3$
	5; 7	7; 2	8; 1	6; 1	$u_3 = 2$
	$v_1 = 3$	$v_2 = 3$	$v_3 = 5$	$v_4 = 4$	

∴ Number of allocations = 6 = m + n − 1. ∴ Solution is non-degenerate.

∴ All $d_{ij} > 0$ and optimal solution is $x_{12} = 12$, $x_{13} = 2$, $x_{14} = 8$, $x_{23} = 15$, $x_{31} = 7$, $x_{34} = 1$.

∴ Minimum T.C. = 149.

(c)

6	3 12	5 2	4 8	$u_1 = 0$
5	9	2 15	7 6	$u_2 = -3$
5 7	7	8	6 1	$u_3 = 2$
$v_1 = 3$	$v_2 = 3$	$v_3 = 5$	$v_4 = 4$	

If clerk is approached by a curvier to root CQ who offers to reduced its rate.

The optimal value will change it $d_{32} < 0$ let the cost of root CQ reduced by u units.

$\therefore$ The clerk will get some business if

$$d_{32} = (7 - x) - (5 + 2) < 0$$

i.e. $$7 - x - 5 < 0$$

$\therefore$ $$x > 2 \text{ i.e. } x < x$$

$\therefore$ The currier reduced the cost by 2 or more units then clerk will get some business.

PRACTICAL NO. 6 ASSIGNMENT PROBLEMS

Problem 1: Five employees of a company are to be assigned to five jobs which can be done by any of them. The amount of the time (in hours) taken by each employee to do a assigned job is give in the following table :

	Employee				
Job	**A**	**B**	**C**	**D**	**E**
J_1	7	9	3	3	2
J_2	6	1	6	6	5
J_3	3	4	9	10	7
J_4	1	5	2	2	4
J_5	6	6	9	4	2

Determine the optimal assignment pattern of jobs to the employees that minimizes the total time.

Solution : Since, Number of jobs = Number of employee.

$\therefore$ Given assignment problem is balanced.

$$\begin{bmatrix} 7 & 9 & 3 & 3 & 2 \\ 6 & 1 & 6 & 6 & 5 \\ 3 & 4 & 9 & 10 & 7 \\ 1 & 5 & 2 & 2 & 4 \\ 6 & 6 & 9 & 4 & 2 \end{bmatrix}$$

For each row, we find minimum element of a row and subtracting it from all entries of that row.

$$\begin{bmatrix} 5 & 7 & 1 & 1 & 0 \\ 5 & 0 & 5 & 5 & 4 \\ 0 & 1 & 6 & 7 & 4 \\ 0 & 4 & 1 & 1 & 3 \\ 4 & 4 & 7 & 2 & 0 \end{bmatrix}$$

For each column, we find minimum element of a column and subtracting it from all entries of that column.

$$\begin{bmatrix} 5 & 7 & 0 & 0 & 0 \\ 5 & 0 & 4 & 4 & 4 \\ 0 & 1 & 5 & 6 & 4 \\ 0 & 4 & 0 & 0 & 3 \\ 4 & 4 & 6 & 1 & 0 \end{bmatrix}$$

Now we do assignments of the jobs to employees so as to minimize total working hours and each employee should get job. For this we make assignment ($\boxed{}$) in a row or a column which has exactly one zero. We call this as compulsory assignment

Now make assignment at other zeros. Once we make an assignment in a row or a column and we cross-off other zeros the respective column and respective row.

$$\begin{bmatrix} 5 & 7 & \boxed{0} & \cancel{0} & \cancel{0} \\ 5 & \boxed{0} & 4 & 4 & 4 \\ \boxed{0} & 1 & 5 & 6 & 4 \\ \cancel{0} & 4 & \cancel{0} & \boxed{0} & 3 \\ 4 & 4 & 6 & 1 & \boxed{0} \end{bmatrix}$$

OR

$$\begin{bmatrix} 5 & 7 & \cancel{0} & \boxed{0} & \cancel{0} \\ 5 & \boxed{0} & 4 & 4 & 4 \\ \boxed{0} & 1 & \boxed{5} & \cancel{0} & 4 \\ \cancel{0} & 4 & 0 & 0 & 3 \\ 4 & 4 & 6 & 1 & \boxed{0} \end{bmatrix}$$

Number of assignment = 5 = Order of matrix.

That is each employee get exactly one job

∴ Solution is optimal.

Jobs	Employee Or		Time Or	
1	C	D	3	2
2	B	B	1	1
3	A	A	3	3
4	D	C	2	3
5	E	E	2	2
–		–	11	hours

∴ Minimum total time is 11 hours.

Problem 2 : A company solicits bids on each of four projects from five contarctors. The bids received (in thousands of rupees) are given in the accompanying table. Contractor D feels unable to carry out project 3 and therefore, submits no bid.

	Contractor				
Project	**A**	**B**	**C**	**D**	**E**
P_1	18	25	22	26	25
P_2	26	29	26	27	24
P_3	28	31	30	-	31
P_4	26	28	27	30	29

Determine the optimal assignment schedule of the projects to the contractors that minimizes the total cost.

Solution: Number of projects = 4. Number of contractor = 5.

∴ Number of projects < Number of contractors.

∴ Add dummy project with zero costs. We get,

$$\begin{bmatrix} 18 & 25 & 22 & 26 & 25 \\ 26 & 29 & 26 & 27 & 24 \\ 28 & 31 & 30 & \infty & 31 \\ 26 & 28 & 27 & 30 & 29 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now given A.P. is balanced. ∴For each row, find minimum of a row or column substract it from all entries of the respective row. Further similar steps for columns we get;

$$\begin{bmatrix} 0 & 7 & 4 & 8 & 7 \\ 2 & 5 & 2 & 3 & 0 \\ 0 & 3 & 2 & \infty & 3 \\ 0 & 2 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Do assignments, first compulsory and then optional as shown below:

$$\begin{bmatrix} \boxed{0} & 7 & 4 & 8 & 7 \\ 2 & 5 & 2 & 3 & \boxed{0} \\ \cancel{0} & 3 & 2 & \infty & 3 \\ \cancel{0} & 2 & 1 & 4 & 3 \\ \cancel{0} & \boxed{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{bmatrix}$$

Draw minimum number of vertical and horizontal lines to cover all zeros.

Number of assignments = 3 < order of the matrix.

(or equivalently, number of lines which covers all zeros are less than order of matrix)

$\therefore$ Solution obtained is not optimal.

Minimum element uncovered by lines is 1.

$\therefore$ Substract from uncovered element and add 1 to intersection of lines.

$$\begin{bmatrix} \boxed{0} & 6 & 3 & 7 & 6 \\ 3 & 5 & 2 & 3 & \boxed{0} \\ \cancel{0} & 2 & \boxed{1} & \infty & 2 \\ \cancel{0} & 1 & \boxed{0} & 3 & 2 \\ 1 & \boxed{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{bmatrix}$$

Number of assignment = 4 < order of matrix.

$\therefore$ Solution is not optimal.

Minimum uncovered element is 1.

$\therefore$ Add 1 to elements where intersection of lines and subtract from uncovered elements.

$$\begin{bmatrix} \boxed{0} & 5 & 2 & 6 & 5 \\ 4 & 5 & 2 & 3 & \boxed{0} \\ \cancel{0} & 1 & \boxed{0} & \infty & 1 \\ 1 & 1 & \cancel{0} & 3 & 2 \\ 2 & \boxed{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{bmatrix}$$

Number of assignments = 4 < order of matrix.

$\therefore$ Solution is not optimal.

Minimum uncovered element is 1.

$\therefore$ Add 1 to intersection of lines and subtract 1 from all uncovered elements.

$$\begin{bmatrix} \boxed{0} & 4 & 2 & 5 & 4 \\ 5 & 5 & 2 & 3 & \boxed{0} \\ \cancel{0} & \boxed{0} & \cancel{0} & \infty & \cancel{0} \\ 1 & \cancel{0} & \boxed{0} & 2 & 1 \\ 3 & \cancel{0} & 1 & \boxed{0} & \cancel{0} \end{bmatrix}$$

$\therefore$ Number of assignment = 5 = order of matrix. $\therefore$ Solution is optimal.

Project	**Contractor**	**Cost**
P_1	A	18
P_2	E	24
P_3	B	31
P_4	C	27
P_5	D	0
		100 Thousands of Rupees

$\therefore$ Total minimum cost = 100 (thousands of rupees).

Problem 3 : Suggest optimal assignment for four salesman to sale territories where the estimated sales (in lakh rupees) to be made by each of the salesman in different sales territories are given below:

	Sales position				
Salesman	**I**	**II**	**III**	**IV**	**V**
A	16	15	17	10	8
B	16	16	20	15	12
C	12	8	10	13	15
D	18	16	17	12	10

Also find the maximum sale.

Solution:

We will apply the Hungerian method to solve this assignment problem.

We convert this problem of maximization into problem of minimization by computing relative losses at each cell.

Maximum profit is 20.

Relative loss at the cell = Maximum profit – the profit at each cell

$\therefore$ Subtract all entries from 20, we get

$$\begin{bmatrix} 4 & 5 & 3 & 10 & 12 \\ 4 & 4 & 0 & 5 & 8 \\ 8 & 12 & 10 & 7 & 5 \\ 2 & 4 & 3 & 8 & 10 \end{bmatrix}$$

Given assignment problem is not balanced, because

Number of salesman < Number of sales territories.

That is number of rows not equal to number of columns.

$\therefore$ Add dummy salesman E .That is add row 5 with each entry equal to 0.

$$\begin{bmatrix} 4 & 5 & 3 & 10 & 12 \\ 4 & 4 & 0 & 5 & 8 \\ 8 & 12 & 10 & 7 & 5 \\ 2 & 4 & 3 & 8 & 10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Subtract minimum element from each row. And making assignments we have

$$\begin{bmatrix} 1 & 2 & \boxed{0} & 7 & 9 \\ 4 & 4 & \cancel{0} & 5 & 8 \\ 3 & 7 & 5 & 2 & \boxed{0} \\ 0 & 2 & 1 & 6 & 8 \\ \cancel{0} & \boxed{0} & \cancel{0} & \cancel{0} & \cancel{0} \end{bmatrix}$$

$\therefore$ Number of assignments = 4 < order of matrix.

$\therefore$ Solution is not optimal.

Minimum uncovered element is 1.

Hence add 1 to intersection of lines and subtract 1 from all uncovered elements.

$$\begin{bmatrix} 0 & 1 & \cancel{0} & 6 & 8 \\ 3 & 3 & \boxed{0} & 4 & 7 \\ 3 & 7 & 6 & 2 & \boxed{0} \\ 0 & 2 & 2 & 6 & 8 \\ \cancel{0} & \cancel{0} & 1 & \boxed{0} & \cancel{0} \end{bmatrix}$$

$\therefore$ Number of assignements < order of matrix.

$\therefore$ Solution is not optimal.

Minimum uncovered element is 1.

Hence substract 1 from all uncovered elements and add 1 to intersection of lines (elements).

$$\begin{bmatrix} \cancel{0} & \boxed{0} & \cancel{0} & 5 & 7 \\ 3 & 2 & \boxed{0} & 3 & 6 \\ 4 & 7 & 7 & 2 & \boxed{0} \\ \boxed{0} & 1 & 2 & 5 & 7 \\ 1 & \cancel{0} & 1 & \boxed{0} & \cancel{0} \end{bmatrix}$$

Number of assignments = order of matrix.

$\therefore$ Solution is optimal.

Optimal solution is

Salesman	Sales territories	Sale
A	II	15
B	III	20
C	V	15
D	I	18
E	IV	0
		68 lakhs rupees

∴ Total maximum sale is 68 lakh rupees.

Problem 4 : The captain of a cricket team has to allot five middle batting positions to five batsmen. The average runs scored by each batsman at these positions are as below:

	Batting Position				
Batsman	**I**	**II**	**III**	**IV**	**V**
P	40	40	35	25	50
Q	42	30	16	25	27
R	50	48	40	60	50
S	20	19	20	18	25
T	58	60	59	55	53

(a) Find the optimal assignment of batsmen to positions, which would give the maximum number of runs.

(b) If another batsman U with the following average runs in batting positions as given below:

Bating position	I	II	III	IV	V
Average Runs	45	52	38	50	49

is added to the team, should he be included to play in the team?

If so, who will be replaced by him?

Solution :

(a) Number of batsman = Bating positions.

∴ Given A.P. is balanced.

Minimum runs are 60.

∴ Subtract all entries from 60, we get

$$\begin{bmatrix} 20 & 20 & 25 & 35 & 10 \\ 18 & 30 & 44 & 35 & 33 \\ 10 & 12 & 20 & 0 & 10 \\ 40 & 41 & 40 & 42 & 35 \\ 2 & 0 & 1 & 5 & 7 \end{bmatrix}$$

Subtract minimum element from each row.

$$\begin{bmatrix} 10 & 10 & 15 & 25 & 0 \\ 0 & 12 & 26 & 17 & 15 \\ 10 & 12 & 20 & 0 & 10 \\ 5 & 6 & 5 & 7 & 0 \\ 2 & 0 & 1 & 5 & 7 \end{bmatrix}$$

Subtract minimum element from each columns. Make assignments

$$\begin{bmatrix} 10 & 10 & 14 & 25 & \boxed{0} \\ \boxed{0} & 12 & 25 & 17 & 15 \\ 10 & 12 & 18 & \boxed{0} & 10 \\ 5 & 6 & 4 & 7 & \cancel{0} \\ 2 & \boxed{0} & \cancel{0} & 5 & 7 \end{bmatrix}$$

Number of assignments = 4 < order of matrix.

∴ Solution is not optimal.

Minimum uncovered element = 4.

∴ Substract 4 from all uncovered elements and add 4 to intersection of lines.

$$\begin{bmatrix} 6 & 6 & 10 & 21 & 0 \\ 0 & 12 & 25 & 17 & 19 \\ 10 & 12 & 18 & 0 & 14 \\ 1 & 2 & 0 & 3 & 0 \\ 2 & 0 & 0 & 5 & 11 \end{bmatrix}$$

Number of assignments = 5 = order of matrix.

∴ Solution is optimal.

∴

Batsman	Batting position	Runs
P	V	50
Q	I	42
R	IV	60
S	III	20
T	II	60
		232 runs

∴ Maximum number of runs = 232 runs.

(b) Now another batsman V is added. Hence given A.P. becomes

$$\begin{bmatrix} 40 & 40 & 35 & 25 & 50 \\ 42 & 30 & 16 & 25 & 27 \\ 50 & 48 & 40 & 60 & 50 \\ 20 & 19 & 20 & 18 & 25 \\ 58 & 60 & 59 & 55 & 53 \\ 45 & 52 & 38 & 50 & 49 \end{bmatrix}$$

∴ Number of batsman < Bating positions.

∴ Add dummy batting position i.e. VI.

∴ Given A.P. becomes.

$$\begin{bmatrix} 40 & 40 & 35 & 25 & 50 & 0 \\ 42 & 30 & 16 & 25 & 27 & 0 \\ 50 & 48 & 40 & 60 & 50 & 0 \\ 20 & 19 & 20 & 18 & 25 & 0 \\ 58 & 60 & 59 & 55 & 53 & 0 \\ 45 & 52 & 38 & 50 & 49 & 0 \end{bmatrix}$$

Maximum runs are 60.

∴ Subtract all elements from 60, we get,

$$\begin{bmatrix} 20 & 20 & 25 & 35 & 10 & 0 \\ 18 & 30 & 44 & 35 & 33 & 0 \\ 10 & 12 & 20 & 0 & 10 & 0 \\ 40 & 41 & 40 & 42 & 35 & 0 \\ 2 & 0 & 1 & 5 & 7 & 0 \\ 15 & 8 & 22 & 10 & 11 & 0 \end{bmatrix}$$

Subtract minimum element from each row.

$$\begin{bmatrix} 20 & 20 & 25 & 35 & 10 & 0 \\ 18 & 30 & 44 & 35 & 33 & 0 \\ 10 & 12 & 20 & 0 & 10 & 0 \\ 40 & 41 & 40 & 42 & 35 & 0 \\ 2 & 0 & 1 & 5 & 7 & 0 \\ 15 & 8 & 22 & 10 & 11 & 0 \end{bmatrix}$$

Subtract minimum element from each column.

$$\begin{bmatrix} 18 & 20 & 24 & 35 & 3 & \cancel{0} \\ 16 & 30 & 43 & 35 & 26 & \cancel{0} \\ 8 & 12 & 19 & \boxed{0} & 3 & \cancel{0} \\ 38 & 41 & 38 & 42 & 28 & \boxed{0} \\ \boxed{0} & \cancel{0} & \cancel{0} & 5 & \cancel{0} & \cancel{0} \\ 13 & 8 & 21 & 10 & 4 & \cancel{0} \end{bmatrix}$$

Number of assignments = 3 < order of matrix.

∴ Solution is not optimal and minimum uncovered element is 3.

Hence add 3 to intersection of lines and subtract 4 from all uncovered elements.

We get,

$$\begin{bmatrix} 15 & 17 & 21 & 32 & 0 & \cancel{0} \\ 13 & 27 & 40 & 32 & 23 & \cancel{0} \\ 8 & 12 & 19 & \boxed{0} & 3 & 3 \\ 35 & 38 & 35 & 39 & 25 & \cancel{0} \\ \boxed{0} & \cancel{0} & \cancel{0} & 5 & \cancel{0} & 3 \\ 10 & 5 & 8 & 7 & 1 & \boxed{0} \end{bmatrix}$$

Number of assignments = 4 < order of matrix.

∴ Solution is not optimal and minimum uncovered element is 1.

∴ Hence add 1 to intersection of lines and substract 1 from all uncovered elements.

$$\begin{bmatrix} 15 & 17 & 21 & 32 & \boxed{0} & \cancel{1} \\ 12 & 26 & 39 & 31 & 22 & \cancel{0} \\ 8 & 12 & 19 & \boxed{0} & 3 & 4 \\ 34 & 37 & 34 & 38 & 24 & \cancel{0} \\ \cancel{0} & \cancel{0} & \boxed{0} & 5 & \cancel{0} & 4 \\ 9 & 4 & 7 & 6 & \cancel{0} & \boxed{0} \end{bmatrix}$$

Number of assignments = 4 < order of matrix.

∴ Solution is not optimal and minimum uncovered element is 4.

∴ Subtract 4 from all uncovered element and add 4 to intersection of lines.

$$\begin{bmatrix} 11 & 13 & 17 & 28 & \boxed{0} & \cancel{1} \\ 8 & 22 & 35 & 27 & 22 & \boxed{0} \\ 8 & 12 & 19 & \boxed{0} & 7 & 8 \\ 30 & 33 & 30 & 34 & 24 & \cancel{0} \\ \cancel{0} & \cancel{0} & \boxed{0} & 5 & 4 & 8 \\ 5 & \boxed{0} & 3 & 2 & \cancel{0} & \cancel{0} \end{bmatrix}$$

Number of assignments = 5 < order of matrix.

∴ Solution is not optimal and minimum uncovered element is 8.

∴ Substract from all uncovered elements and add 8 to intersection of lines. Further we make assignments

$$\begin{bmatrix} 3 & 6 & 9 & 28 & \boxed{0} & 1 \\ \boxed{0} & 14 & 27 & 27 & 22 & \cancel{0} \\ \cancel{0} & 4 & 11 & \boxed{0} & 7 & 8 \\ 22 & 25 & 22 & 34 & 24 & \boxed{0} \\ \cancel{0} & \cancel{0} & \boxed{0} & 13 & 12 & 16 \\ 5 & \boxed{0} & 3 & 10 & 8 & 8 \end{bmatrix}$$

Number of assignments = 6 = order of matrix.

∴ Solution is optimal. ∴Total number of runs obtained for this optimal assignments is

Batsman	**Batting position**	**Runs**
P	V	50
Q	I	42
R	IV	60
S	VI	0
T	III	59
V	II	52
		263 runs

By including batsman V are runs are increases. Hence batsman 5 will replace by V.

Problem 5 : A transportation problem has the supplies at four sources and requirements at five destinations. The following table shows the cost of shipment one unit from particular source to a particular destination.

	Destination				
Source	D_1	D_2	D_3	D_4	D_5
S_1	12	4	9	5	9
S_2	8	1	6	6	7
S_3	1	12	4	7	7
S_4	10	15	6	9	1

The following feasible transportation pattern is proposed : $x_{11} = 25$, $x_{14} = 30$, $x_{22} = 20$, $x_{23} = 25$, $x_{31} = 15$, $x_{33} = 15$, $x_{43} = 10$, $x_{45} = 140$ and all other $x_{ij} = 0$. You are required to test wether this pattern has the least possible transportation cost. If so, how ? If not, determine the optimal transportation pattern.

Solution :

12 / 25	4 / –6	9 / 6	5 / 30	9	$u_1 = 15$
12 / 5	1 / 20	12 / 25	12 / 10	12 / 6	$u_2 = 6$
1 / 15	12 / 13	4 / 15	7 / 3	7 / 8	$u_3 = 4$
10 / 7	15 / 14	16 / 10	9 / 13	1 / 40	$u_4 = 6$
$v_1 = -3$	$v_2 = -5$	$v_3 = 0$	$v_4 = -10$	$v_5 = -5$	

Number of all allocates = 8 = m + n − 1. ∴ Solution is non-degenerate.

∴ Some $d_{ij} < 0$. ∴ Solution is not optimal. ∴ Form a loop corresponding to cell (1, 2).

12 / 10	4 / 15	9 / 0	5 / 30	9	$u_1 = 0$
8 / –1	1 / 5	6 / 40	6 / 4	7 / 6	$u_2 = -3$
1 / 30	12 / 5	4 / 2	7 / 13	7 / 0	$u_3 = -11$
10 / 1	15 / 14	16 / 10	9 / 7	1 / 40	$u_4 = -3$
$v_1 = 12$	$v_2 = 4$	$v_3 = 9$	$v_4 = 5$	$v_5 = 4$	

Number of allocations = 8 = m + n − 1.

∴ Solution is non-degenerate.

∴ $d_{21} < 0$.

∴ Solution is not optimal.

∴ Form a loop corresponding to cell (2, 1).

12 / 5	4 / 20	9 / –1	5 / 30	9 / 4	$u_1 = 12$
8 / 5	1 / 1	6 / 40	6 / 5	7 / 6	$u_2 = 8$
1 / 30	12 / 19	4 / 5	7 / 13	7 / 13	$u_3 = 1$
10 / 2	15 / 15	16 / 10	9 / 8	1 / 40	$u_4 = 8$
$v_1 = 0$	$v_2 = -8$	$v_3 = -2$	$v_4 = -7$	$v_5 = -7$	

Number of allocations = 8 = m + n − 1.

∴ Solution is non-degenerate.

∴ $d_{13} < 0$.

∴ Solution is not optimal.

∴ Form a loop corresponding to (1, 3).

12 / 5	4 / 20	9 / 5	5 / 30	9 / 5	$u_1 = 9$
8 / 10	1 / 0	6 / 35	6 / 4	7 / 6	$u_2 = 6$
1 / 30	12 / 6	4 / 5	7 / 12	7 / 13	$u_3 = -1$
10 / 2	15 / 14	6 / 10	9 / 7	1 / 40	$u_4 = 6$
$v_1 = 2$	$v_2 = 5$	$v_3 = 0$	$v_4 = -4$	$v_5 = -5$	

Number of allocations = 8 = m + n − 1.

∴ Solution is non-degenerate. All $d_{ij} \geq 0$

∴ Solution is optimal.

∴ Optimal solution is $x_{12} = 20$, $x_{13} = 5$, $x_{14} = 30$, $x_{21} = 10$, $x_{23} = 35$, $x_{31} = 30$, $x_{43} = 10$, $x_{45} = 40$.

∴ Minimum T.C. = 80 + 45 + 150 + 80 + 210 + 30 + 6 + 40 = 695 units

= 695

∴ Minimum T.C. is 695 units.

Problem 6 : The departmental store wishes to purchase the following quantities of sarees:

Type of saree	A	B	C	D	E
Quantity	150	100	75	250	200

Tenders are submitted by four different manufacturers who undertake supply not more than the quantities mentioned below:

Manufacturer	W	X	Y	Z
Total quality	300	250	150	200

The stores estimates that its profit per saree will vary with the manufacturer as shown in the following matrix:

	Saree				
Manufacturer	**A**	**B**	**C**	**D**	**E**
W	275	350	425	225	150
X	300	325	450	175	100
Y	250	350	475	200	125
Z	325	275	400	250	175

How the orders be placed?

Solution : Here total demand of sarees = 150 + 100 + 75 + 250 + 200 = 775

and Total supply of sarres = 300 + 250 + 150 + 200 = 900

$\therefore$ Total supply excess the total demand by 900 – 775 = 125

$\therefore$ Given transportation problem is not balanced. To solve the given transportation problem by VAM method we have to make it balanced. For this we add new fictious destination (Department store) F in the transportation table with demand of 125 sarres.

Destination

Sources		**A**	**B**	**C**	**D**	**E**	**F**	**Supply**
	W	275	350	425	225	150	0	300
	X	300	325	450	175	100	0	250
	Y	250	350	475	200	125	0	150
	Z	325	275	400	250	175	0	200
	Demand	150	100	75	250	200	125	

Now we have to maximize total profit and according to that we have to place the orders. The entries in tables are estimate of profit per saree. For example, the entry in cell (W, A) is 275 which represent profit per saree if store A purchase it from manufacturer W.

Now we will convert our problem of maximization in minimization. Here maximum profit per saree is 475. $\therefore$We will compute losses per saree in each cell as 475 – profit/saree for each cell.

Destination

Sources		**A**	**B**	**C**	**D**	**E**	**F**	**Supply**
	W	200	125	50	250	325	475	300
	X	175	150	25	300	375	475	250
	Y	225	125	0	275	350	475	150
	Z	150	200	75	225	300	475	200
	Demand	150	100	75	250	200	125	

Now we will find initial basic feasible solution by VAM method.

Further iterate by MODI method Iteration 01:

	D1	**D2**	**D3**	**D4**	**D5**	**D6**	**supply**	
W	200 ***-25***	125 ***-25***	50 ***00***	250 **250**	325 ***00***	475 **50**	300	$u_1 = 0$
X	175 **150**	150 ***-25***	25 **75**	300 ***-50***	375 ***-50***	475 **25**	250	$u_2 = 0$
Y	225 ***-50***	125 **100**	0 ***25***	275 ***-25***	350 ***-25***	475 **50**	150	$u_3 = -25$
Z	150 ***00***	200 ***-100***	75 ***-75***	225 ***00***	300 **200**	475 ***-25***	200	$u_4 = -25$
Demand	150	100	75	250	200	125		
	$v_1 = 175$	$v_2 = 125$	$v_3 = 25$	$v_4 = 250$	$v_5 = 325$	$v_6 = 475$		

Objective value :222500.

Iteration 02:

	D1	**D2**	**D3**	**D4**	**D5**	**D6**	**supply**	
W	200 ***-25***	125 ***25***	50 ***-25***	250 **250**	325 ***00***	475 **50**	300	$u_1 = 0$
X	175 **150**	150 ***00***	25 **25**	300 ***-50***	375 ***00***	475 **75**	250	$u_2 = 0$
Y	225 ***-75***	125 **100**	0 **50**	275 ***-50***	350 ***-50***	475 ***-25***	150	$u_3 = -25$
Z	150 ***00***	200 ***-75***	75 ***-75***	225 ***00***	300 **200**	475 ***-25***	200	$u_4 = -25$

Contdi….

	D1	**D2**	**D3**	**D4**	**D5**	**D6**	**supply**	
Demand	150	100	75	250	200	125		
	$v_1=$ 175	$v_2=$ 150	$v_3=25$	$v_4=250$	$v_5=325$	$v_6=$ 475		

Objective value : 221250.

Iteration 03:

	D1	D2	D3	D4	D5	D6	supply	
W	200 ***-25***	125 **25**	50 ***-50***	250 **250**	325 **0**	475 **25**	300	$u_1 = 0$
X	175 **150**	150 ***-25***	25 ***-25***	300 ***-50***	375 ***-50***	475 **75**	250	$u_2 = 0$
Y	225 ***-50***	125 **75**	0 **75**	275 ***-25***	350 ***-25***	475 ***00***	150	$u_3 = 0$
Z	150 ***00***	200 ***-100***	75 -***100***	225 ***00***	300 **200**	475 ***-25***	200	$u_4 = -25$
Demand	150	100	75	250	200	125		
	$v_1=$ 175	$v_2=$ 125	$v_3=0$	$v_4=25$ 0	$v_5=32$ 5	$v_6=$ 475		

Objective value : 220625.So total max profit = 475 × 775 – 220625 = 147500

☆☆☆

MODEL QUESTION PAPER

OPERATIONS RESEARCH : MTC - 242

Time : 2 Hours **Maximum Marks : 35**

Instructions to the candidates :

(i) All questions are compulsory.

(ii) Figures to the right indicate full marks.

---

1. Attempt any five of the following : **(5)**

(i) What are the limitation of L.P.P. ?

(ii) Explain the term infeasibility.

(iii) Explain duality in linear programming.

(iv) What is transportation problem ?

(v) Write short note on 'Optimum Solution'.

(vi) What is assignment problem ?

(vii) The assignment problem is solved by using transportation algorithm. Is it true ?

2. Attempt any two of the following : **(10)**

(i) A firm manufactures headache pills in two sizes A and B. Size A contains 2 grains of aspirin, 5 grains of bicarbonate and 1 grain of codeine, Size B contains 1 grain of aspirin, 8 grains of bicarbonate and 6 grains of codeine. It is found by users that it requires at least 12 grains of aspirin, 74 grains of bicarbonate and 24 grains of codeine for providing immediate effect. It is required to determine the least number of pills a patient should take to get immediate relief. Formulate the problem as a standard L.P.P.

(ii) Maximize $z = 3x_1 + 2x_2$

subject to the constraints :

$x_1 + x_2 \leq 4,$

$x_1 - x_2 \leq 2$ and $x_1 , x_2 \geq 0.$

(iii) Solve by using big-M method the following L.P.P. :

Max $z = -2x_1 - x_2$

subject to $3x_1 + x_2 = 3,$

$4x_1 + 3x_2 \geq 6,$

$x_1 + 2x_2 \leq 4$ and $x_1, x_2 \geq 0.$

(iv) Find the dual of the following primal problem :

Min $z = 2x_2 + 5x_2$

subject to $x_1 + x_2 \geq 2,$

$2x_1 + x_2 + 6x_3 \geq 6,$

$x_1 - x_2 + 3x_3 = 4$ and $x_1, x_2, x_3 \geq 0.$

3. Attempt any two of the following : (10)

(i) Obtain an initial basic feasible solution to the following transportation problem :

	Stores				Availability
	I	**II**	**III**	**IV**	
Warehouse A	7	3	5	5	34
B	5	5	7	6	15
C	8	6	6	5	12
D	6	1	6	4	19
Demand	21	25	17	17	80

(ii) Determine an initial feasible solution to the following transportation problem using North-west corner rule :

Origin	**Destination**				**Supply**
	$\mathbf{D_1}$	$\mathbf{D_2}$	$\mathbf{D_3}$	$\mathbf{D_4}$	
A	11	13	17	14	250
B	16	18	14	10	300
C	21	24	13	10	400
Demand	200	225	275	250	950

(iii) A certain equipment needs five repair jobs which have to be assigned to five machines. The estimated time (in hours) that each mechanic requires to complete the repair job is given in the following table :

Job / **Machine**	$\mathbf{J_1}$	$\mathbf{J_2}$	$\mathbf{J_3}$	$\mathbf{J_4}$	$\mathbf{J_5}$
M_1	7	5	9	8	11
M_2	9	12	7	11	10
M_3	8	5	4	6	9
M_4	7	3	6	9	5
M_5	4	6	7	5	11

Assuming that each machine can be assigned to only one job, determine the minimum time assignment.

(iv) The owner of a small machine shop has four machinists available to do jobs for the day. Five jobs are offered with expected profit for each machinist on each job as follows :

	1	2	3	4
A	32	41	57	18
B	48	54	62	34
C	20	31	81	57
D	71	43	41	57
E	52	29	51	50

Find by using the assignment method, the assignment of machinists to jobs that will result in a maximum profit. Which jobs should be declined.

4. Attempt any one of the following : (10)

(i) Solve the following L.P.P. by graphical method

Minimize $z = 20x_1 + 10x_2$

Such that : $x_1 + 2x_2 \geq 40$

$3x_1 + x_2 \geq 30$

$4x_1 + 3x_2 \geq 60$ and

$x_1 \geq 0, \; x_2 \geq 0.$

(ii) Solve the following problem by simplex method

Max $z = 5x_1 + 3x_2$

Such that : $3x_1 + 5x_2 \leq 15$

$5x_1 + 2x_2 \leq 10$ and

$x_1 , x_2 \geq 0.$

(ii) The following is I.B.F.S. of the transportation problem :

1 20	2	1 10	4
3	3 20	2 20	1 10
4	2 20	5	9

Show that it is optimal solution. Also find alternate optimal solution.

(iv) Solve the following assignment problem to maximize production. The data given in the table refers to production in certain units :

Machine

		A	B	C	D
Operator	1	10	5	7	8
	2	11	4	9	10
	3	8	4	9	7
	4	7	5	6	4
	5	8	9	7	5

☆☆☆