Z- Transform

Defination:-

The Z-transform of a sequence {f(k)} and is defined as:

$$Z\{f(k)\}=F(z)=\sum_{k=-\infty}^{\infty} f(k)z^{-k}=\sum_{k=-\infty}^{\infty} \frac{f(k)}{z^k}$$

Properties of Z-Transform:-

1)Linearity:- $Z\{af(k)+bf(k)\} = aF(z)+bF(z)$

2)Change of scale:- $Z\{a^k f(k)\} = F\left(\frac{z}{a}\right)$

3)Shifting:-

i)Both sided sequence $Z\{f(k\pm n)\} = z^{\pm n}F(z)$

ii)One sided sequence $Z\{f(k\pm n)\} = z^n F(z)-\sum_{r=0}^{n-1} f(r)z^{n-r}$, $k>=0$ and

for $k<0$, $Z\{f(k-n)\} = z^{-n}F(z)+\sum_{r=-n}^{-1} f(r)z^{-(n+r)}$,

iii)Causal sequence

$Z\{f(k+n)\} = z^n F(z)-\sum_{r=0}^{n-1} f(r)z^{n-r}$

$Z\{f(k-n)\} = z^{-n}F(z)$

4)Multiplication by k:-

$Z\{kf(k)\} = \left(-z\frac{d}{dz}\right)F(z)$

$Z\{k^n f(k)\} = \left(-z\frac{d}{dz}\right)^n F(z)$

5)Division by k: $-Z\{\frac{f(k)}{k}\} = \int_z^{\infty} F(z)z^{-1}dz$

6)Initial value theorem:- $f(0) = \lim_{z\to\infty} F(z)\ if\ \{f(k)\}$ is

One sided if sequence k>0

7)final value theorem:-

$\lim_{k\to\infty} f(k) = \lim_{z\to 1}(z-1)\,F(z)\ if\ \{f(k)\}$ is One sided

sequence

i.e.k>=0

8)Partical sum:- $Z\{\sum_{m=-\infty}^{\infty} f(m)\} = \frac{f(z)}{1-z^{-1}}$ $\sum_{m=-\infty}^{\infty} =$

$F(1)$

9) Convolution:-Z{(f(k)*g(k)}=F(z)G(z)

$$h(k) = \sum_{m=-\infty}^{\infty} f(m)g(k-m)$$

If causal then $h(k) = \sum_{m=0}^{k} f(m)g(k-m)$

where h(k)={f(k)}*{(g(k)}

10)$Z\{e^{-ak}f(k)\} = F(e^{a}Z)$

Z transform formulae

1.$Z\{\delta(k)\} = 1\ for\ all\ z$

2.$Z\{U(k)\} = \frac{z}{z-1}$ $|z|>1$

3.$Z\{1\} = \frac{z}{z-1}$ $|z|>1$

4.$Z\{a^k\}=\frac{z}{z-a}$ $k\geq 0$ $|z|>|a|$

5. $Z\{a^k\}=\frac{z}{a-z}$ $k<0$ $|z|<|a|$

6.$Z\{a^{|k|}\}=\frac{az}{1-az} + \frac{z}{z-a}$ $|a|<|z|<\frac{1}{|a|}$

7.$Z\{\cos\alpha k\}=\frac{z(z-\cos\alpha)}{z^2-2z\cos\alpha+1}, k > 0 |z| > 1$

8. $Z\{\sin\alpha k\}=\frac{z\sin\alpha}{z^2-2z\cos\alpha+1}, k \geq 0, |z| > 1$

9. $Z\{\cosh\alpha k\}=\frac{z(z-\cosh\alpha)}{z^2-2z\cosh\alpha+1}, k \geq 0 , |z| > \max(|e^{\alpha}|$ or $(|e^{-\alpha}|)$

10. $Z\{\sinh\alpha k\}=\frac{z\sinh\alpha}{z^2-2z\cosh\alpha+1}, k > 0 |z| >> \max(|e^{\alpha}|$ or $|e^{-\alpha}|)$

11.$Z\{c^k\cos\alpha k\}=\frac{z(z-\cos\alpha)}{z^2-2zc\cos\alpha+c}, k \geq 0 |z| > |c|$

12. $Z\{c^k\sin\alpha k\}=\frac{z\sin\alpha}{z^2-2cz\cos\alpha+c}, k \geq 0, |z| > |c|$

13. $Z\{c^k\cosh\alpha k\}=\frac{z(z-c\cosh\alpha)}{z^2-2cz\cosh\alpha+c}, k \geq 0, |z| >$ $\max(|e^{\alpha}|$ or $(|e^{-\alpha}|)$

14. . $Z\{c^k \sinh\alpha k\} = \frac{cz\sinh\alpha}{z^2 - 2cz\cosh\alpha + c}, k > 0 |z| >$ $\max(|e^{\alpha}|$ or $(|e^{-\alpha}|)$

15. $Z\{{}^{n}c_{k}\} = (1+z^{-1})^{n}$, $0<k<n$, $|z|>0$

16. $Z\{{}^{k}c_{n}\} = z^{-n}(1-z^{-1})^{-(n+1)}$, $k>n$, $|z|>1$

17. $Z\{{}^{(k+n)}c_{n}\} = (1-z^{-1})^{-(n+1)} |z|>1$

18. $Z\{{}^{(k+n)}c_{n}a^{k}\} = (1-az^{-1})^{-(n+1)} |z|>|a|$

19. $Z\{(k+1)a^{k}\} = \frac{z^2}{(z-a)^2}$ $|z|>|a|$

20. $Z\{\frac{(k+1)(k+2)}{2!} a^{k}\} = \frac{z^3}{(z-a)^3}$ $|z|>|a|$

21. $Z\{\frac{(k+1)(k+2)\ldots\ldots\ldots(k+(n-1))}{(n-1)!} a^{k}\} = \frac{z^n}{(z-a)^n}$ $|z|>|a|$

22. $Z\{\frac{a^k}{k!}\} = e^{a/z}$, $k \geq 0$, $\forall z$

Inverse Z Transform

Partial Fraction Term	**Inverse Z Transform f(k) if $\|z\|>\|a\|, k>0$**	**Inverse Z Transform f(k) if $\|z\|<\|a\|, k<0$**
$\frac{z}{z-a}$	$a^k U(k)$	$-a^k$
$\frac{z^2}{(z-a)^2}$	$(k+1)a^k$	$-(k+1)a^k$

$\frac{z^3}{(z-a)^3}$	$\frac{1}{2!}(k+1)(+2)a^kU(k)$	$-\frac{1}{2!}(k+1)z(k+2)a^kU(-k+2)$
$\frac{z^n}{(z-a)^n}$	$\frac{1}{(n-1)!}(k+1)(k+2)\ldots(k+n-1)a^kU(k)$	$-\frac{1}{(n-1)!}(k+1)(k+2)\ldots(k+n-1)a^k$
$\frac{1}{(z-a)}$	$a^{k-1}U(k-1)$	$-a^{k-1}U(-k)$
$\frac{1}{(z-a)^2}$	(k-1) $a^{k-2}U(k-2)$	-(k-1) $a^{k-2}U(-k+1)$
$\frac{1}{(z-a)^3}$	$\frac{1}{2}(k-1)(k-2)\ a^{k-3}U(k-3)$	$-\frac{1}{2}(k-1)(k-2)\ a^{k-3}U(-k+3)$
$\frac{z}{z-1}$	$U(k)$	
$\frac{z(z-\cos\propto)}{z^2-2z\cos\propto+1}$, $\lvert z\rvert>1$	$\cos\propto k$	
$\frac{z\sin\propto}{z^2-2z\cos\propto+1}$, $\lvert z\rvert>1$	$\sin\propto k$	

Integral Method

Formulae:-

$z^{-1}\{F(z)\} = f(k)$

$= \sum [Residues\ of\ F(z)z^{k-1} at\ the\ poles\ of\ F(z)]$

i)Residue for simple pole z=a is=$[(z - a)z^{(k-1)}\ F(z)]_{z=a}$

ii)Residue for n times repeated poles is z=a is=$\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}[(z-a)^n z^{k-1}\ F(z)]_{z=a}$

Difference Equation :-

A relation between f(k) and f(k+1),f(k+2),f(k+3),….. is called difference equation.