1.General Form of N^{th} order linear differential equation with constant coefficient is

$(a_0D+a_1D^{n-1}+....+a_{n-1}D+a_n)y=F(x)$,where D=d/dx

$\Rightarrow \varnothing(D)y=F(x)$, where $\varnothing(D)= a_0D^n+a_1D^{n-1}+....+a_{n-1}D+a_n$

2. $\varnothing(D)y=F(x)$

is called Non-Homogenous LDE with constant coefficients and its general solution is given by,

y(x)=C.F.+P.I.

OR

$y(x)=y_c+y_p$, where CF/y_c- complementary Function

PI/y_p - Particular integral

3.Auxillary Equation of $eq^n(1)$ is given by $\varnothing(D)=0$.

4.***<u>Method for finding the CF $\varnothing(D)y=0$</u>***

a.) Real and Distinct Root:

if roots of $\varnothing(D)=0$ real and distinct,say $m_1,m_2.........m_n$ then

$$C.F.=C_1e^{m_1x} + C_2e^{m_2x}+ C_3e^{m_3x}+......+C_ne^{m_nx}$$

b.)Real and repeated Roots:

i)if roots of Ø(D)=0 are real and repeated, say $m_1, m_2, \ldots\ldots m_n$ then

C.F.=$(C_1x+C_2)\, e^{m_1x}+C_3e^{m_3x}+\ldots\ldots\ldots\ldots\ldots+C_ne^{m_nx}$

ii)if $m_1=m_2=m_3, m_4\ldots\ldots\ldots, m_n$ then

C.F.=$(C_1x^2+C_2x+C_3)\, e^{m_1x}+ C_4e^{m_4x}+\ldots\ldots\ldots+C_ne^{m_nx}$

C)Imaginary roots

1. If roots of Ø(D)=0 are $\alpha\pm\beta i$ then

CF=$e^{\alpha x}(C_1\cos\beta x+C_2\sin\beta x)$

2.If roots $\alpha\pm\beta i$ repeated twice then

CF=$e^{\alpha x}[(C_1x+C_2)\cos\beta x+ (C_3x+C_4)\sin\beta x]$

<u>To find PI of Ø(D) y= f(x)</u>

<u>a.General method:-</u>

1. $\frac{1}{D-m}f(x)=e^{mx}\int e^{-mx}\, dx^r$

2. $\frac{1}{(D-m1)(D-m2)}f(x)=$

$e^{m_1x}\int e^{-m_1x}[e^{m_2x}\int e^{-m_2x}f(x)dx]\, dx$

3. $\frac{1}{(D-m_1)(D-m_2)}f(x) = \frac{1}{(m_1-m_2)}[e^{m_1x}\int e^{-m_1x} fx\,dx - e^{m_2x}\int e^{-m_2x} fx\,dx]$

b.**short cut method:-**

1. If $f(x) = e^{ax}$ then

i) $y_p = \frac{1}{\emptyset(D)}e^{ax} = \frac{1}{\emptyset(a)}e^{ax}$, $\emptyset(a) \neq 0$

ii) $y_p = \frac{1}{\emptyset(D)}k = \frac{1}{\emptyset(0)}k$, $\emptyset(0) \neq 0$

iii) $y_p = \frac{1}{\emptyset(D)}a^x = \frac{1}{\emptyset(a)}e^{x \log a} = \frac{1}{\emptyset(\log a)}a^x$, $\emptyset(\log a) \neq 0$

If $\emptyset(a)=0$ then

i) $y_p = \frac{1}{\emptyset(D)}e^{ax} = x\frac{1}{\emptyset(a)}e^{ax}$, $\emptyset'(a) \neq 0$

ii) $y_p = \frac{1}{(D-a)}e^{ax} = \frac{x^r}{r!}e^{ax}$

iii) $y_p = \frac{1}{(D-a)^r \Psi(D)}e^{ax} = \frac{1}{\Psi(a)}\frac{x^r}{r!}e^{ax}, \Psi(a) \neq 0$

2. $f(x) = \sin(ax + b)$ or $\cos(ax+b)$ then

$$PI = \frac{1}{\emptyset(D^2)}\frac{\sin(ax+b)}{\cos(ax+b)} = \frac{1}{\emptyset(-a^2)}\frac{\sin(ax+b)}{\cos(ax+b)} ,\emptyset(-a^2) \neq 0$$

If $\emptyset(-a^2)=0$ then

$$PI=\frac{1}{\emptyset(D^2)}\begin{matrix}\sin(ax+b)\\ \cos(ax+b)\end{matrix} = x.\ \frac{1}{\emptyset'(-a^2)}\begin{matrix}\sin(ax+b)\\ \cos(ax+b)\end{matrix} \quad ,\emptyset'(-a^2)\neq 0$$

If $\emptyset'(-a^2)=0$ then

$$PI=\frac{1}{\emptyset(D^2)}\begin{matrix}\sin(ax+b)\\ \cos(ax+b)\end{matrix} = x^2.\ \frac{1}{\emptyset''(-a^2)}\begin{matrix}\sin(ax+b)\\ \cos(ax+b)\end{matrix} \quad ,\emptyset''(-a^2)= 0$$

And so on

3.If f(x)=sinh (ax+b) or cosh (ax+b) then

$$PI = \frac{1}{\emptyset(D^2)}\begin{matrix}\sinh(ax+b)\\ \cosh(ax+b)\end{matrix} = \frac{1}{\emptyset(a^2)}\begin{matrix}\sinh(ax+b)\\ \cosh(ax+b)\end{matrix} \quad ,\emptyset(a^2)\neq 0$$

If $\emptyset(a^2)=0$ then

$$PI=\frac{1}{\emptyset(D^2)}\begin{matrix}\sinh(ax+b)\\ \cosh(ax+b)\end{matrix} = x.\ \frac{1}{\emptyset'(a^2)}\begin{matrix}\sinh(ax+b)\\ \cosh(ax+b)\end{matrix} \quad ,\emptyset'(a^2)\neq 0$$

If $\emptyset'(a^2)=0$ then

$$PI=\frac{1}{\emptyset(D^2)}\begin{matrix}\sinh(ax+b)\\ \cosh(ax+b)\end{matrix} = x^2.\ \frac{1}{\emptyset''(a^2)}\begin{matrix}\sinh(ax+b)\\ \cosh(ax+b)\end{matrix} \quad ,\emptyset''(a^2)\neq 0$$

And so on

4 If f(x)=x^m then PI=$\frac{1}{\emptyset(D)}x^m$=$[\emptyset(D)]^{-1}x^m$

a. $(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)x^2}{2!} + \cdots$

b. $\frac{1}{1-x}$ = $1+x+x^2+x^3+$............

c. $\frac{1}{1+x}$= $1-x+x^2-x^3+$............

d. $D^n(x^n)=n!$,$D^{n+1}(x^n)=0$

5. $If\ f(x)$ =$e^{ax}V$, V is function of x then

PI=$\frac{1}{\emptyset(D)}e^{ax}V=e^{ax}\left[\frac{1}{\emptyset(D+a)}\right]V$.

6. $If\ f(x) = x^m$ Sinax or x^m cosax then

PI = $\frac{1}{\emptyset(D)}x^m$ Sinax = IP of $\frac{1}{\emptyset(D)}x^m e^{iax}$ = IP of $e^{iax}\left[\frac{1}{\emptyset(D+ia)}\right]x^m$

PI = $\frac{1}{\emptyset(D)}$ x^m cosax = R.P of $\frac{1}{\emptyset(D)}$ $x^m e^{iax}$ = R.P of $e^{iax}\left[\frac{1}{\emptyset(D+ia)}x^m\right]$

7. If f(x)=xV, V is function of x then

PI=$\frac{1}{\emptyset(D)}\ xV = \left[x - \frac{\emptyset'(D)}{\emptyset(D)}\right]\frac{1}{\emptyset(D)}V$

This formula is application if power of x is one and $\frac{1}{\emptyset(D)}V$ not a case of failure.

c.Method of Variation of parameters:

If Ø(D)y=f(x) is 2^{nd} order DE then

PI=uy_1 +vy_2 where u= $\int \frac{-y_2 f(x)dx}{w}$ and

v= $\int \frac{y_1 f(x)dx}{w}$ where W= $\begin{vmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{vmatrix}$

<u>Cauchy's Differential Equation:</u>

$\left(a_0x^n\frac{d^n}{dx} + a_1x^{n-1}\frac{d^{n-1}}{dx^{n-1}} + a_2x^{n-2}\frac{d^{n-2}}{dx^{n-2}} + \right.$

$\left.\cdots \ldots \ldots . + a_{n-1}\mathrm{x}\frac{d}{dx} + a_n\right)$y=F(x)

To reduce it into LDE with constant coefficient put

$x = e^z \Rightarrow z = \log x$ and

Replace $x\frac{dy}{dx} \to Dy$,

$X^2 = \frac{d^2}{dx} \to D$(D-1)y

....

$x^n\frac{d^n y}{dx} \to D(D-1)(D-2)\ldots\ldots\ldots(D-n+1)y$,

where $D=\frac{d}{dz}$

Legendre's Differential Equation:

$\left(a_0(\mathrm{ax}+\mathrm{b})^n\frac{d^n}{dx} + a_1(\mathrm{ax}+\mathrm{b})^{n-1}\frac{d^{n-1}}{dx} + a_2(\mathrm{ax}+\mathrm{b})^{n-2}\frac{d^{n-2}}{dx} + \cdots + a_{n-1}(\mathrm{ax}+\mathrm{b})\frac{d}{dx} + a_n\right)y=F(x)$

To reduce it into it LDE with constant coefficient put,

$(ax+b)=e^z \Rightarrow z = \log(ax+b)$ and

Replace $(ax+b)\frac{dy}{dx} \to aDy$,

$(ax+b)^2\frac{d^2y}{dx} \to a^2D(D-1)y$

.....

(ax+b)$^n\frac{d^n y}{dx} \rightarrow a^n D(D-1)(D-2) \dots\dots\dots\dots (D-n+1)y, where\ D = \frac{d}{dz}$