Theoretical & computational Neuroscience:

Programming the Brain

(BM 6140)

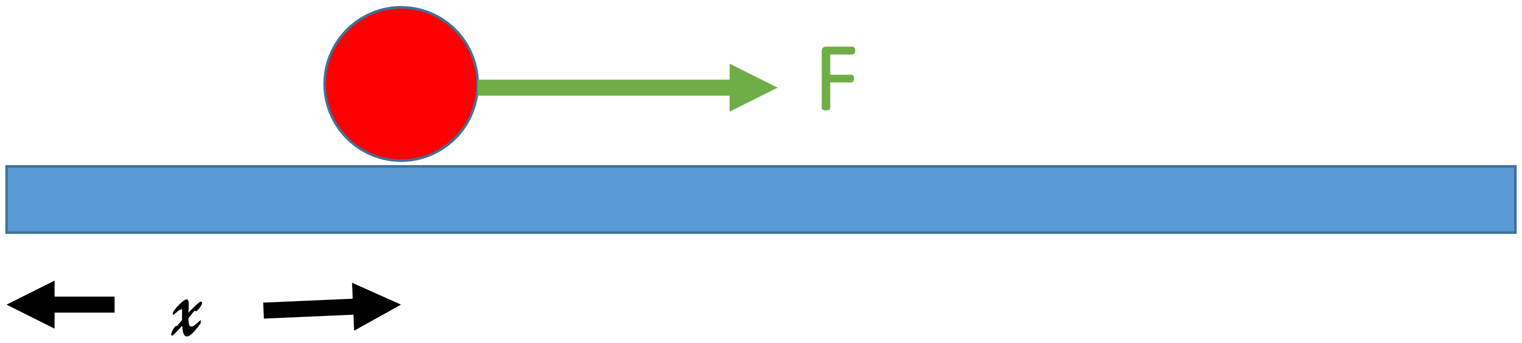
2-credit

Dynamical systems : Systems that change

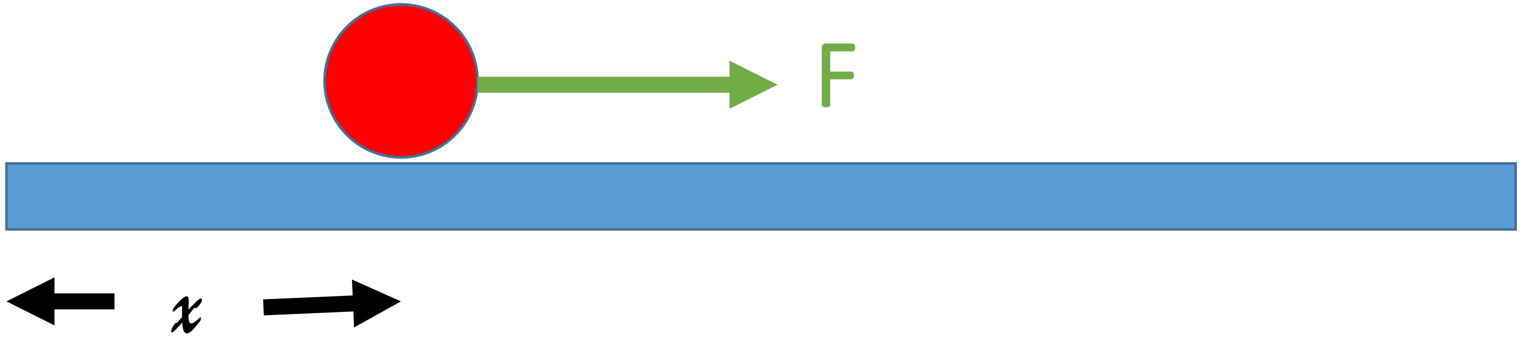
- Capturing the evolution of the system is a primary goal
- Questions addressed in dynamical systems about the system :
  - Will it continue to be in a state of flux ? Will it settle down ? Will it cycle ?
- Neurons, networks and neural systems can be considered to be dynamical systems

Defining the system

- A system consists of a set of interacting players/entities
- The state of the system :
  - Describes the system
  - Set of properties of all entities in the system
  - Mathematically, it is a collection of variables
  - In the system below ‘x’ specifies the state of the system.



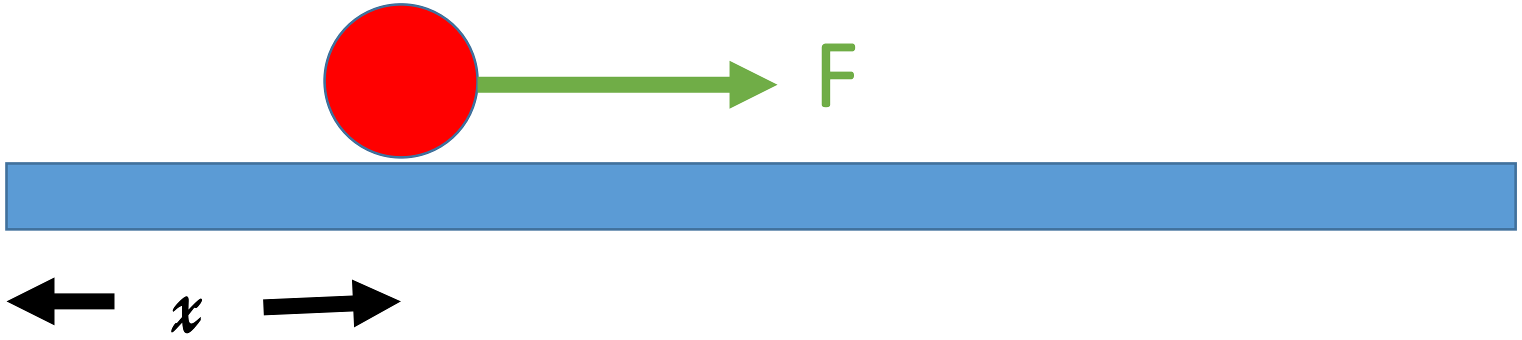
..Defining the system..



- Since we are interested in knowing the changes in the system, we define
- $\frac{dx}{dt} = f(x, F)$
  - f is a function of the state(x) and the external forces acting on the system

- **Is this a complete description of the system ?**

Complete description



- if the set of variables is enough to predict the future values of the same, description is complete.
- Can current location along with external forces determine future position ?
- **You need velocity too !**

A complete description

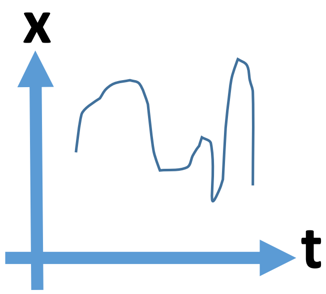
- (x,v) gives the state of the system
- The system can be described by
  - $\frac{dx}{dt} = v$
  - $\frac{dv}{dt} = \frac{F}{m}$

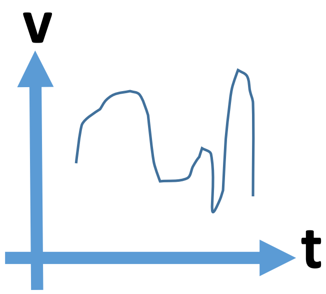
System classification

- Consider the system
  - $\frac{dx}{dt} = v$
  - $\frac{dv}{dt} = \frac{F}{m}$
- Dimension: # variables in the system (Here dimension = 2)
- Order: Highest power of the variables on RHS
  - order = 1 => Linear system
  - If F = -kx as in a spring, system is linear
  - If F = sin(x) or e^-x system is non linear

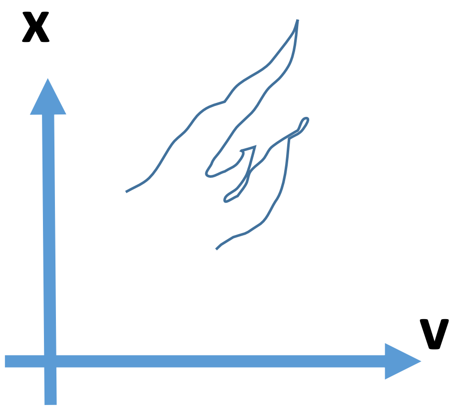
Visualizing the system

- 2 graphs of x vs t , v vs t

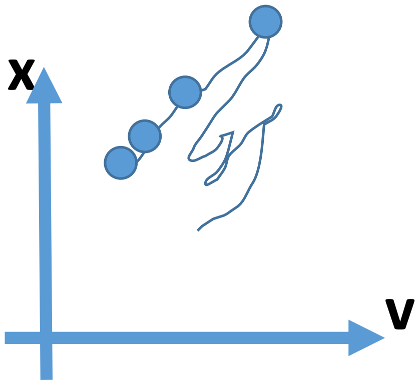




- As a trajectory on phase plane

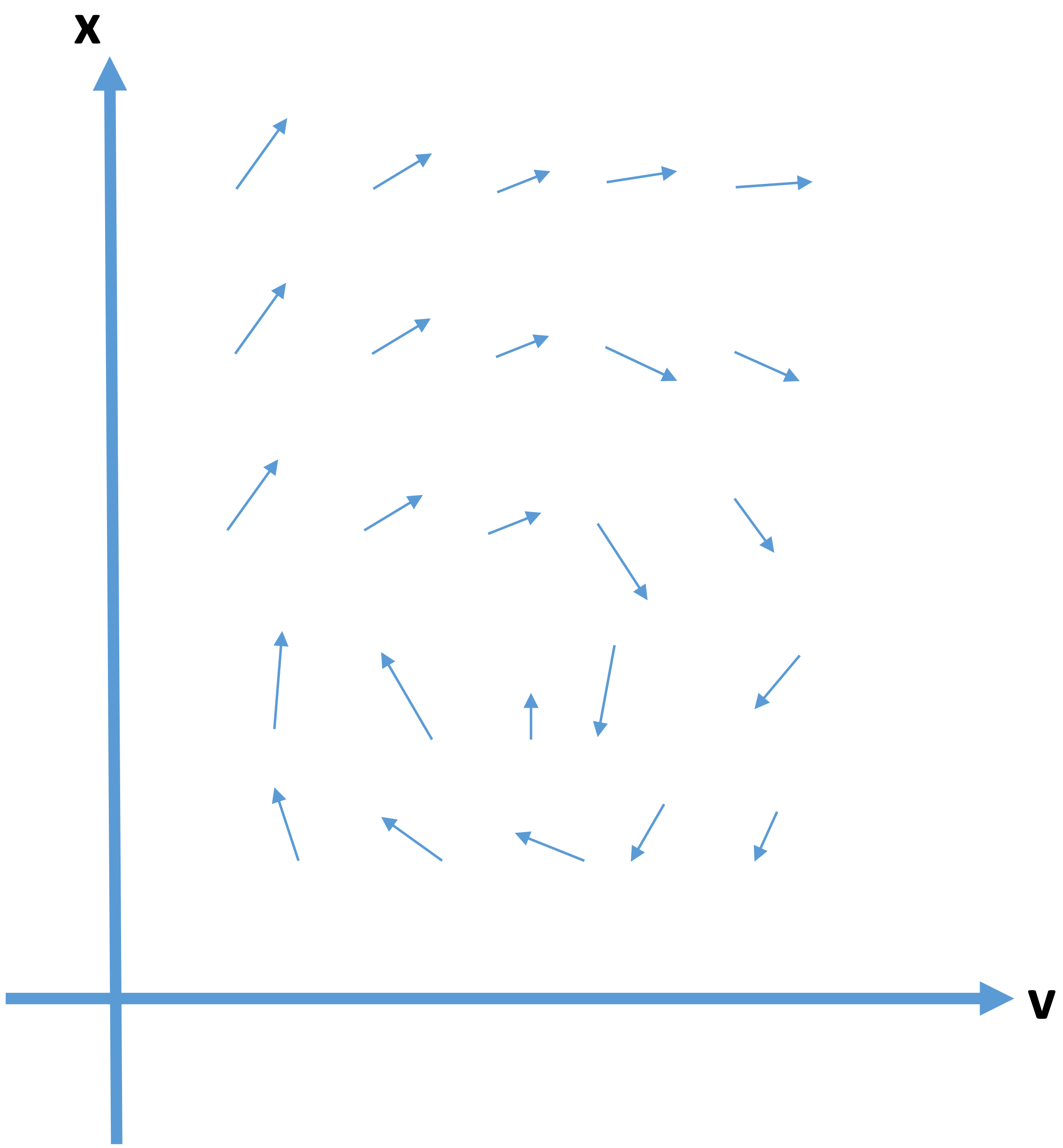


- Time on phase plane
  - Tick density show speed of evolution of the system



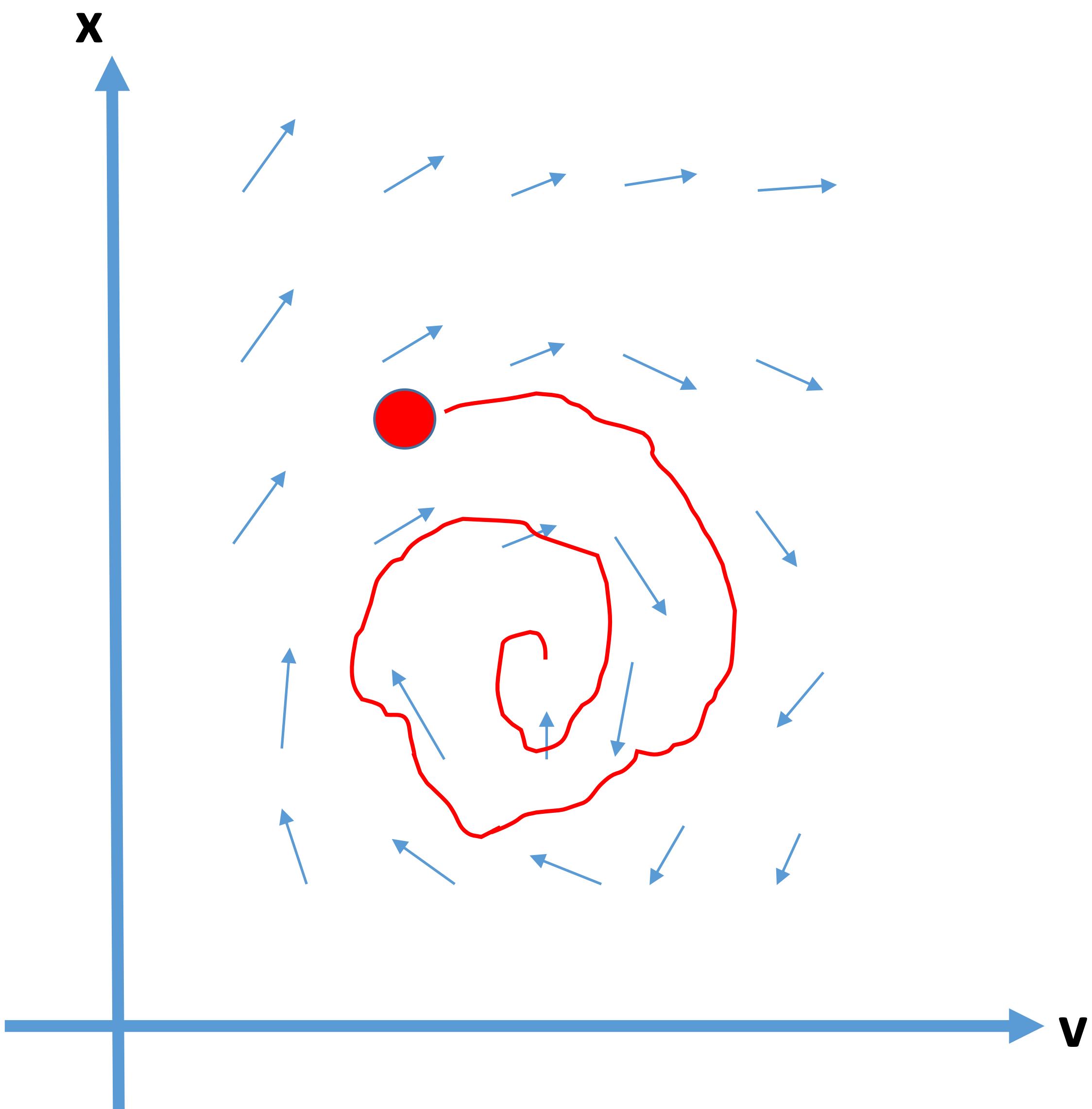
Velocity vector field

- Compute 1-step trajectory starting at all points in the phase plane
- How does this help ? What do we know about the system

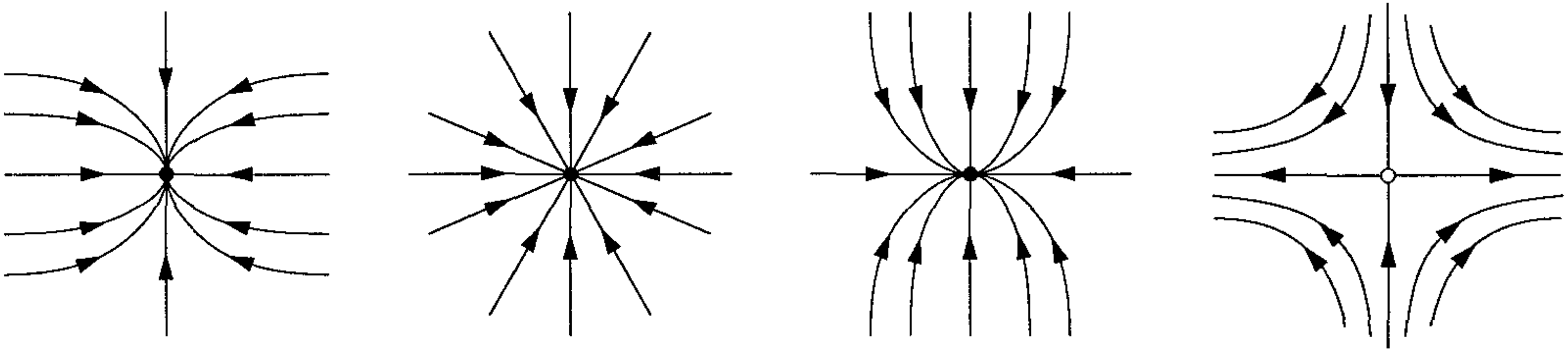


Fate of the system : Qualitative

- Imagine a fluid moving on the plane.
- A particle on phase plane is swept along the currents, determining the evolution of the system



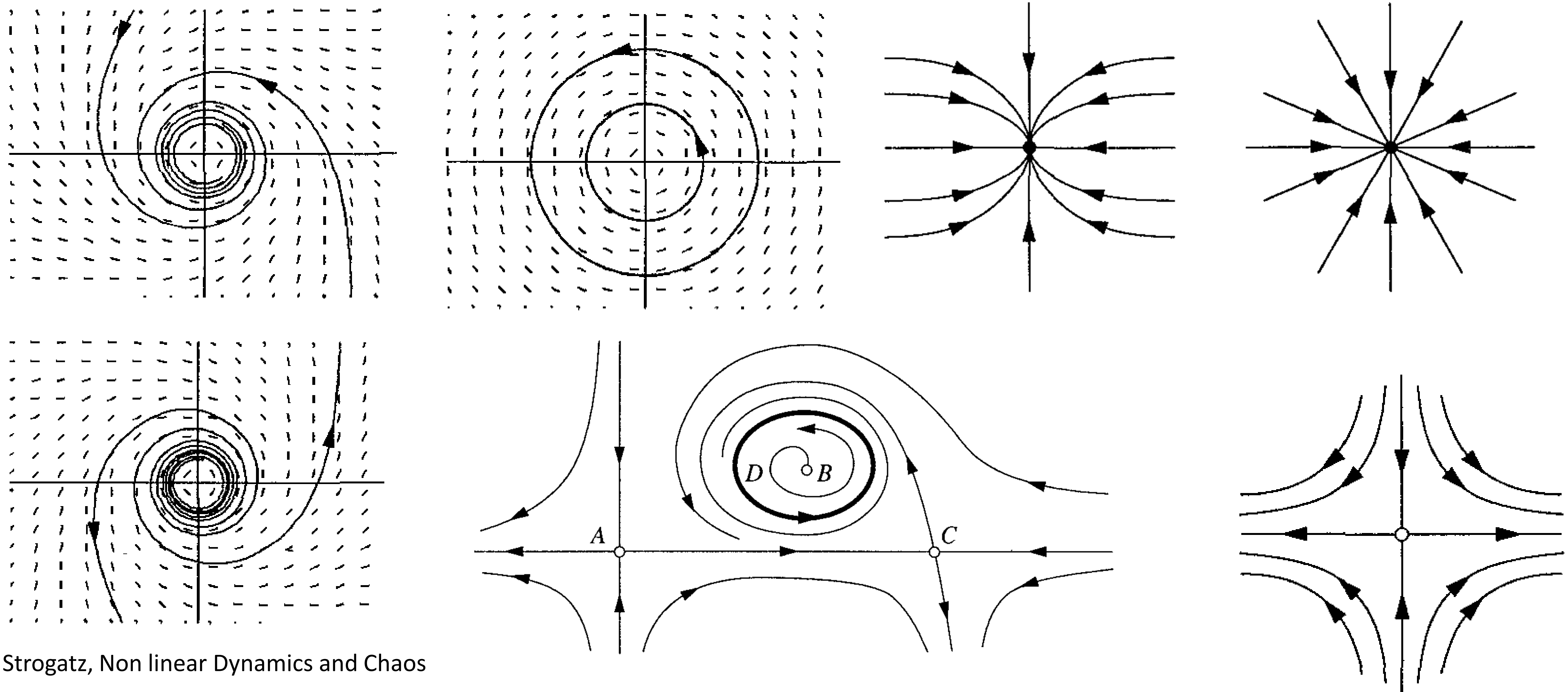
Phase portraits :



- A few typical trajectories and you figure(imagine) the rest by interpolation!

Strogatz, Non linear Dynamics and Chaos

Can we identify where the system will be at rest?



Strogatz, Non linear Dynamics and Chaos

Phase portraits and vector fields exercise only a subset of the phase plane !

- Unexplored areas may have more fixed points
- Explored areas may not uncover the fixed points
- A more rigorous way of identifying f.p required

The vector field must be zero (magnitude)

$If\ the\ system\ is$

- $\frac{dy}{dt} = x + y$
- $\frac{dx}{dt} = x - y$

$setting$ $\frac{dy}{dt}$ = 0 => y = -x

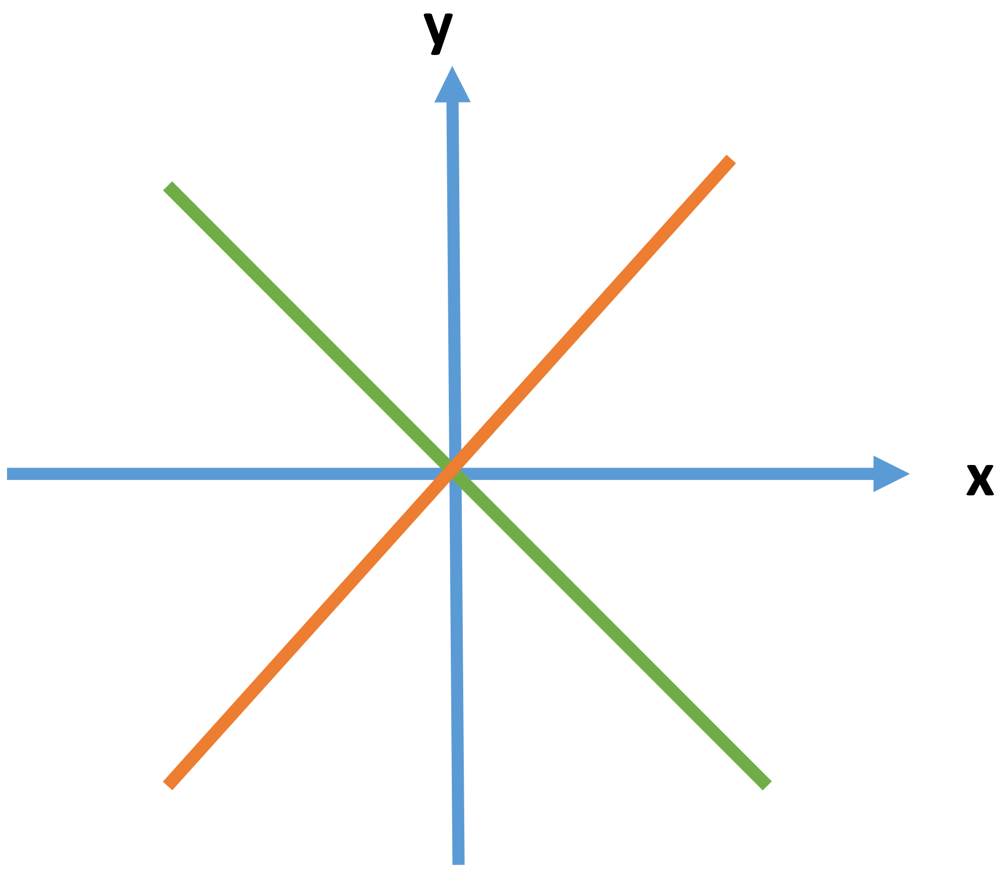
$\frac{dx}{dt}$ = 0 => y = x

- To identify the fixed point solve the simultaneous equations above !

  F.p = (0,0)

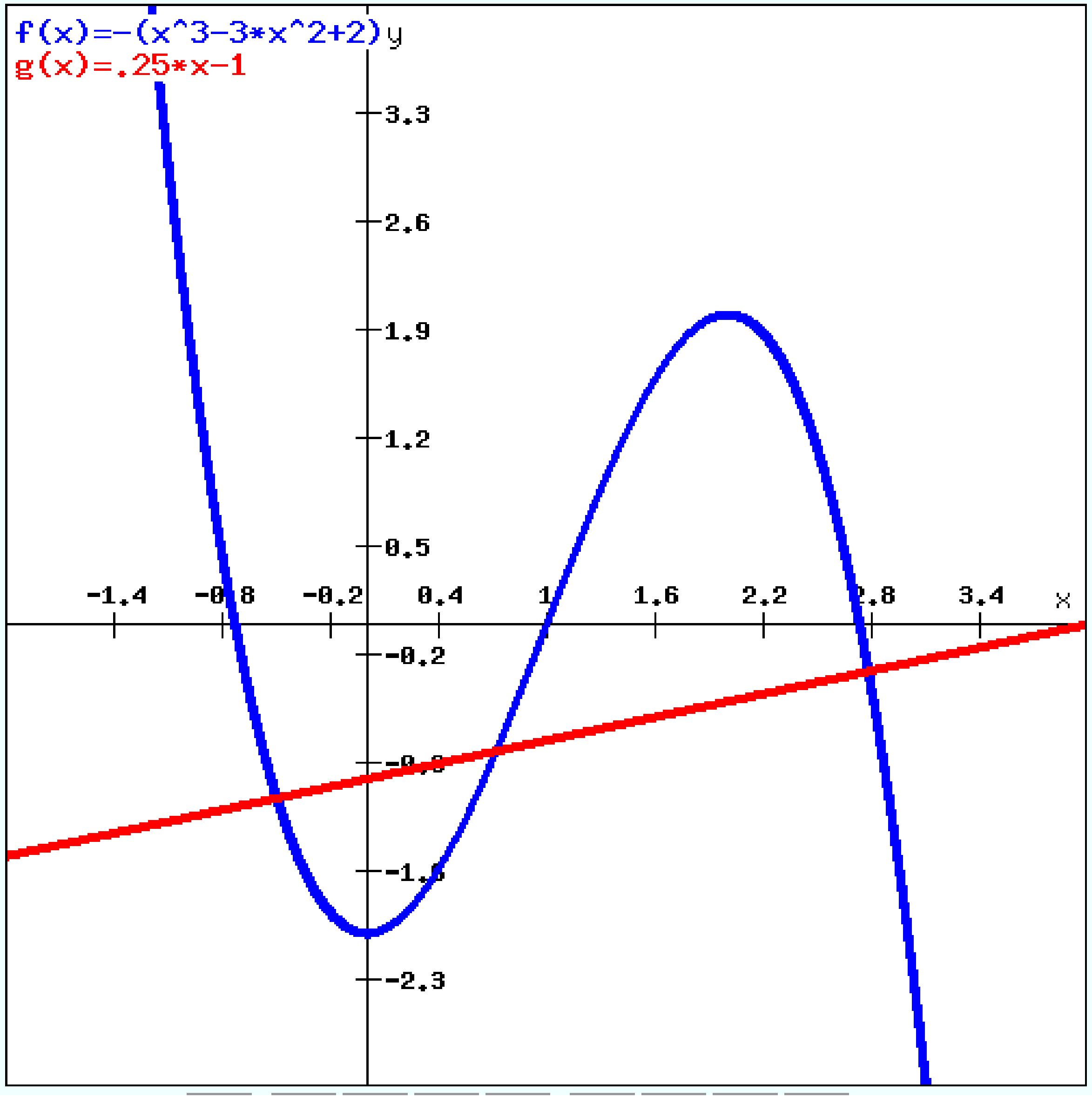
To find the fixed points graphically ?

- Plot $\frac{dx}{dt} = 0, \frac{dv}{dt} = 0.$
- Intersection gives the f.p
- Each of these curves is called a null-cline



Another example…

- $\frac{dy}{dt} = x^3 - 3.x^2 + 2 + y$
- $\frac{dx}{dt} = 0.25.x - 1 - y$
- 3 fixed points

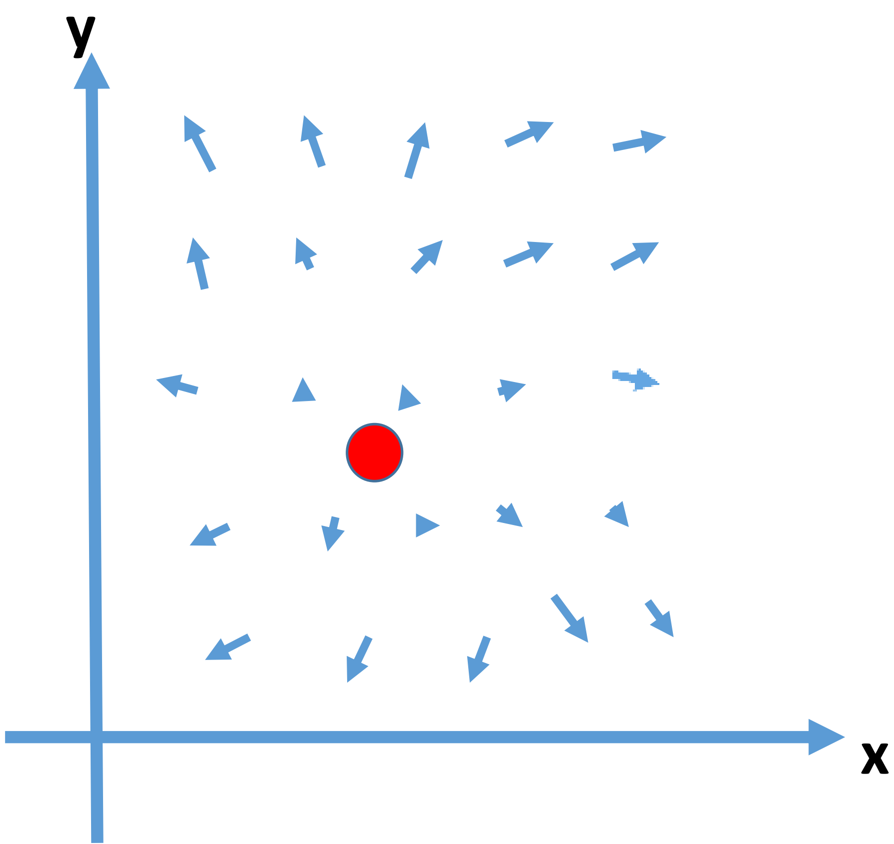


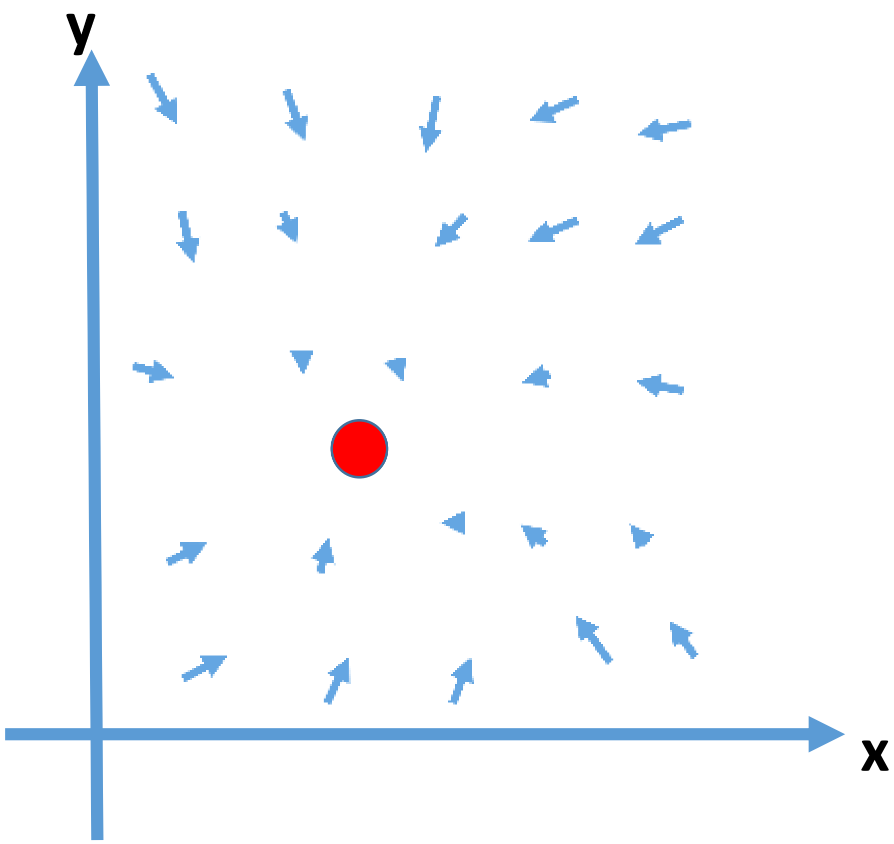
Does the system stay at a fixed point forever?

- $\frac{dy}{dt}$ and $\frac{dx}{dt}$ $are\ zero,$
  $hence\ the\ state\ of\ the\ system\ (x, y) should\ not\ change\ at\ all$

Hint

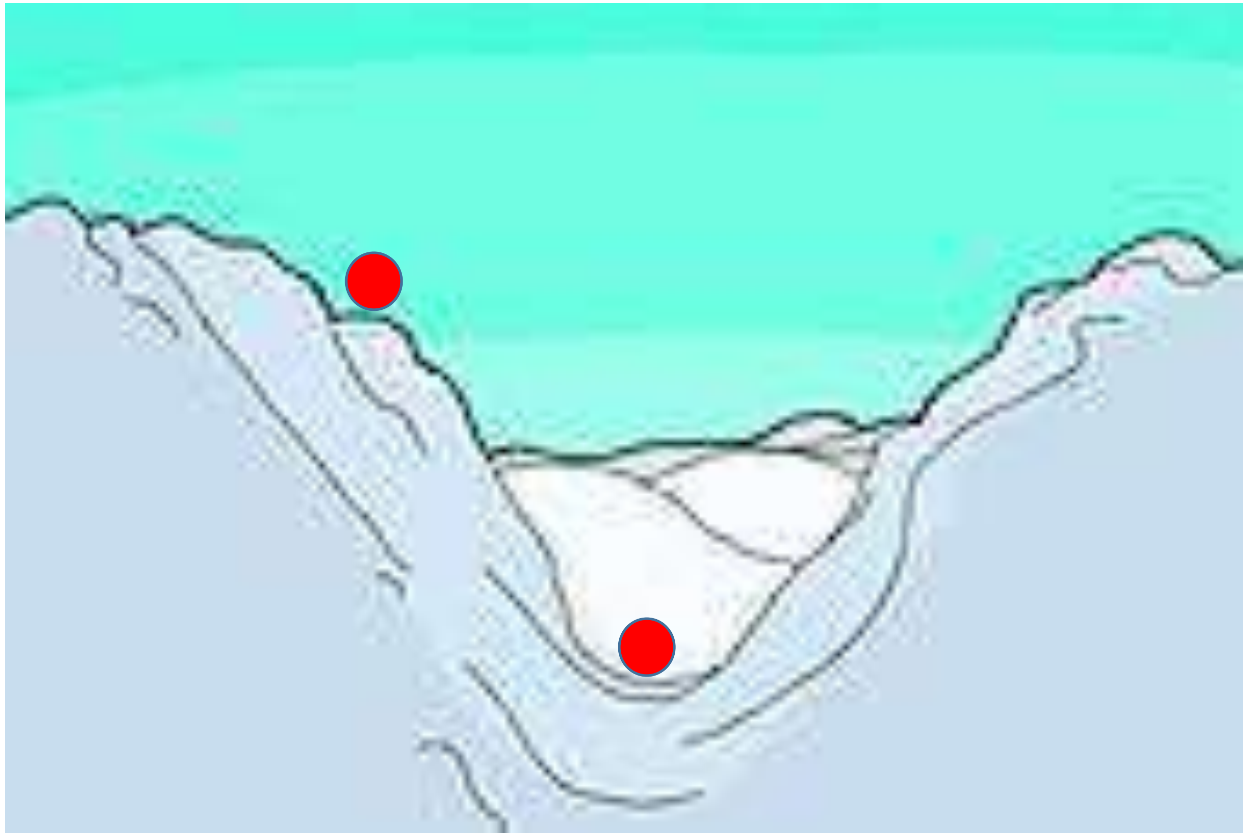
- What is the difference between these 2 systems ?



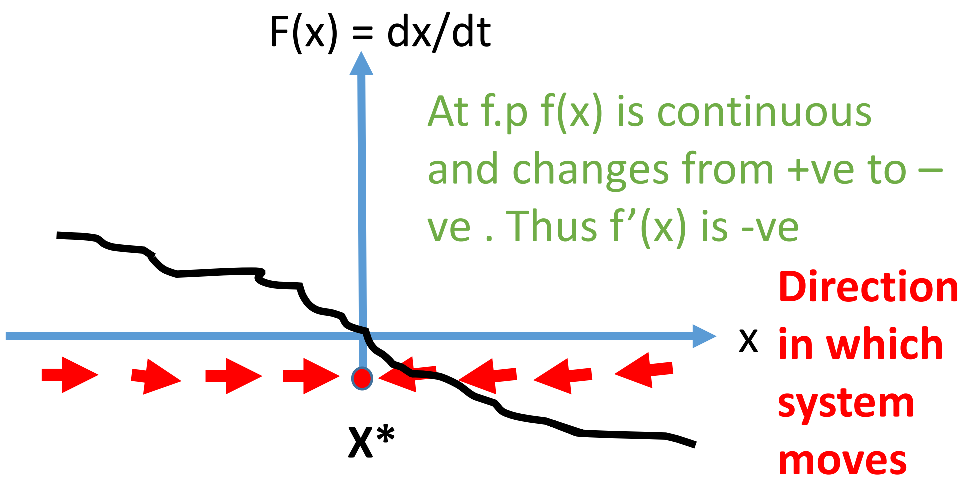


Stability : Resistance to perturbation

- If the system is perturbed from the fixed point, the surrounding gradients should push it back into the fixed point.



Stability : Quantitative



- Consider a 1-dimensional system

$$\frac{dx}{dt} = f(x)$$

- If the state x* is a fixed point, then f(x*) = 0

We want to investigate what happens when the system is perturbed from x* to (x*+h) ?

- For system to be stable, change in the system at x*+h (i.e. $\frac{d(x*+h)}{dt}$ $or f(x * +h)$) should be '-' ve when perturbation(h) is '+'ve and vice versa

- $f(x^* + h) = f(x^*) + hf'(x^*) + \frac{h^2}{2!} f''(x^*) + \frac{h^3}{3!} f'''(x^*) \ldots$

- Neglecting higher order terms

  $f(x^* + h) = hf'(x^*)$ $\quad since\ f(x^*) = 0$

  $For\ stability\ f'(x^*)$ should be negative

For 2-d & higher dimensional systems

- Consider a 2-d system

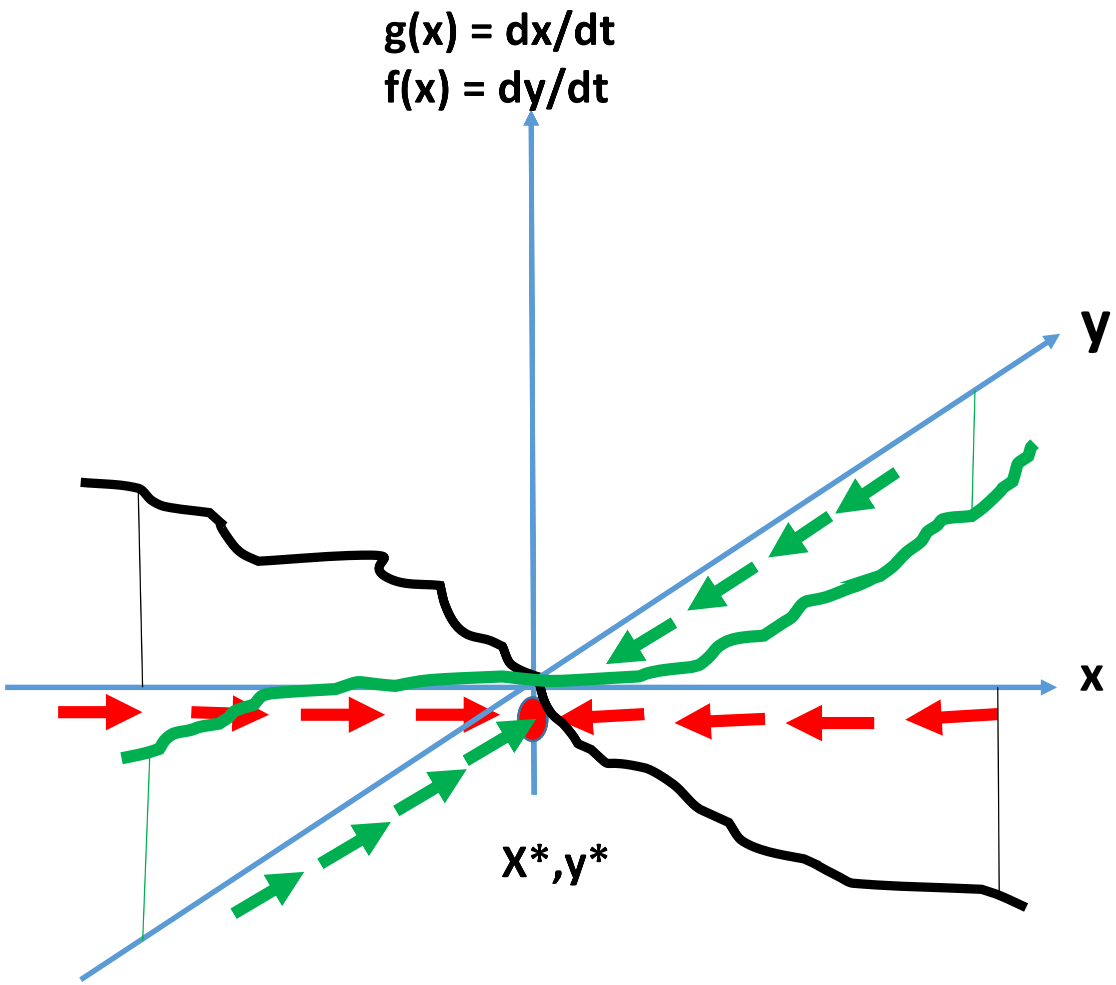
$$\frac{dy}{dt} = f(y)$$

$$\frac{dx}{dt} = g(x)$$

Thus we have a simple correspondence between graphical and mathematical notions of stability !

Except for a small catch !!!

What is that ??



We assumed an uncoupled system !

- i.e dx/dt is a function of x only, while dy/dt is a function of y only
- To consider coupled systems we have to define the system as

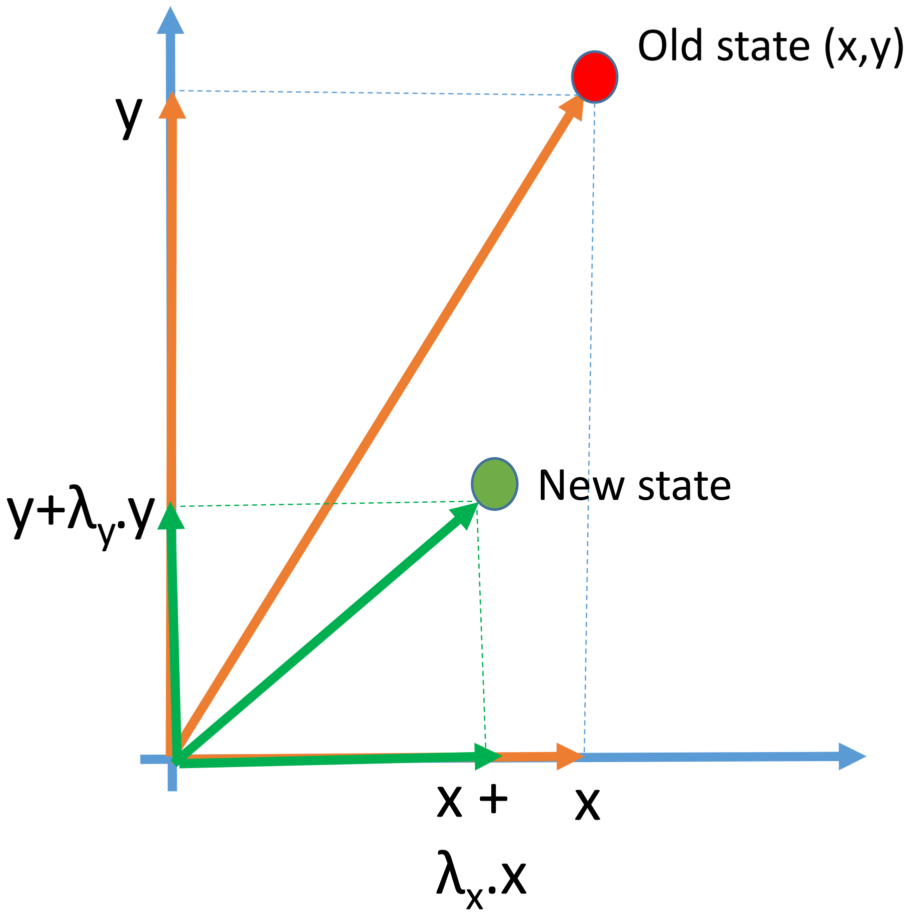
$$\frac{dy}{dt} = f(x, y)$$

$$\frac{dx}{dt} = g(x, y)$$

So how do we deal with this system ?

So what was so special in the uncoupled system ? Try to do the same for coupled systems

- In uncoupled systems, vectors on the x(or y) axis stay there. They just get stretched or compressed.
- Express any perturbed point around the f.p in terms of the x and y axes
- Now stretch or compress the components as per $f'(x)=\lambda_x$ and $g'(y)= \lambda_y$
- Only if both components have negative slopes will the perturbation decay.
- Thus we have only 2 stability conditions, that x and y axes should be compressing.

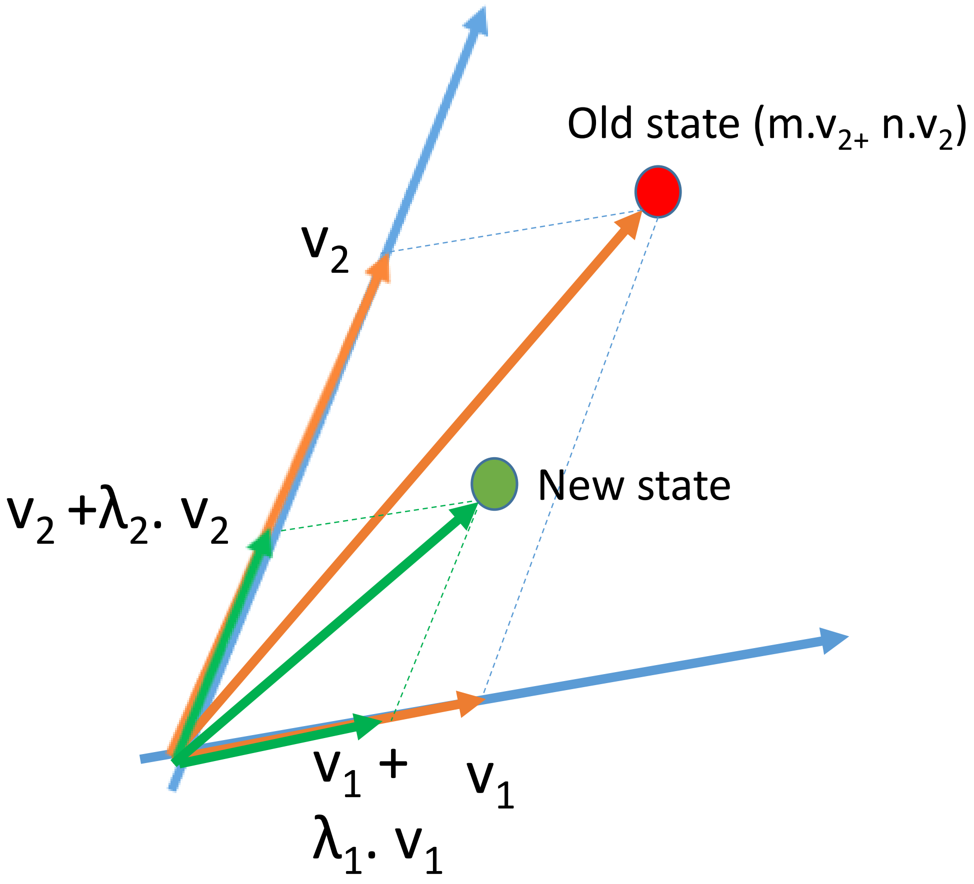


- Find some other vectors other than x and y axes with same property

So how do we tame uncoupled systems ? *- Change of state variables -*

- We want to find vectors, v_{1} & v_{2} which are linear combinations of x and y such that

- $\frac{dv_1}{dt} = f(v_1)$
- $\frac{dv_2}{dt} = g(v_2)$



Some notations….

- Assume a linear system
- => f and g are linear in x,y
- $\frac{dy}{dt} = f(x, y) = ax + by$
- $\frac{dx}{dt} = g(x, y) = cx + dy$
- In matrix notation
- d<u>x</u> = A . <u>X</u> where <u>x</u> = [x y]^T and A = $A = [\begin{matrix} a & b \\ c & d \end{matrix}]$

Change of state variables

- We want to find <u>v such that</u>

- d$\underline{v}$ = Λ . $\underline{v}$ where $\underline{v}$ = $[v_1\ v_2]^T$ and Λ is diagonal

Soln?

These are the Eigen vectors and Eigen values !

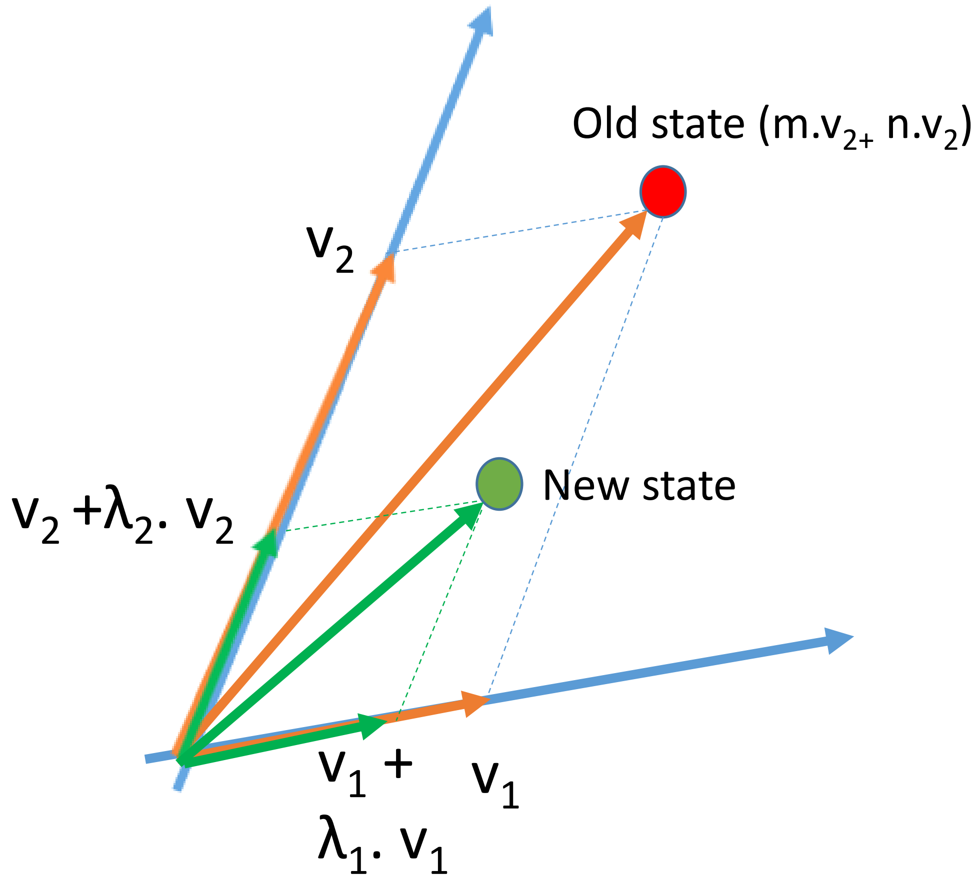
- Λ – Diag (Eigen values)
- $\underline{v}$ – Eigen vectors

Stability in coupled systems

- $\underline{v} = \Lambda \,.\, \underline{v} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \underline{v}$

- $\frac{dv_1}{dt} = f(v_1) = (\lambda_{1.}v_1)$
- $\frac{dv_2}{dt} = g(v_2) = (\lambda_{2.}v_2)$

- f' and g' along v_1 and v_2 are λ_1 and λ_2



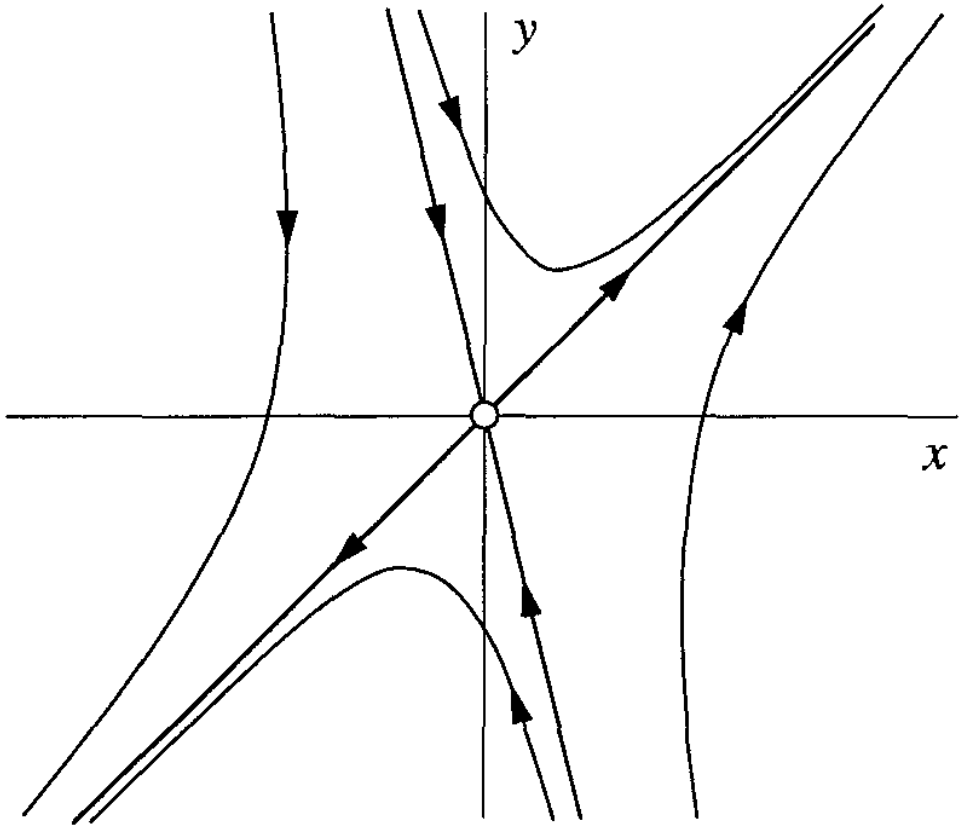
So for coupled systems, stability conditions are :

- $\lambda_i < 0$ for all i

Example

- For a system with A = $\begin{bmatrix} 1 & 1 \\ 4 & -2 \end{bmatrix}$

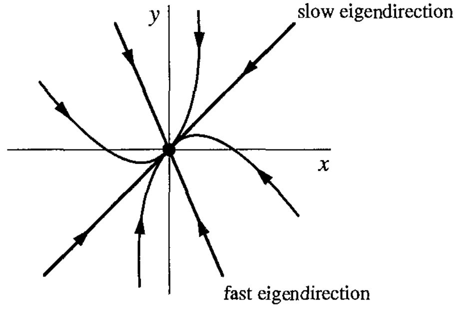
  the phase portrait looks like this
- Eigen values are 2, -3
- Eigen vectors are $[1\ 1]^T$ and $[1\ -4]^T$ respectively
- This f.p is not stable, it is a saddle point

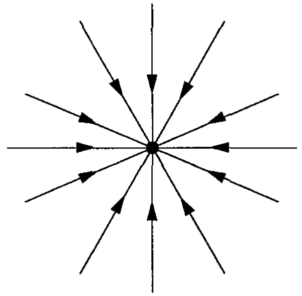


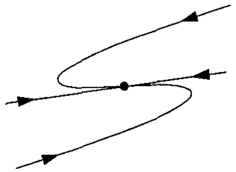
Strogatz, Non linear Dynamics and Chaos

Phase portraits in a stable system

- All initial states in the basin of the f.p fall into the f.p
- Take components of the vector in eigen directions, mpy by eigen vectors. That is your gradient
- Trajectory
  - Faster eigen direction (with larger absolute eigen value) decays faster, at some pt, only component along the slow eigen vector remains… so fall into f.p will curve to become tangential to slow eigen direction
  - If both eigen vectors(distinct) are equal, both components decay equally and the path to f.p is straight (star phase portraits)
  - If both eigen vectors are the same, you get a degenerate node



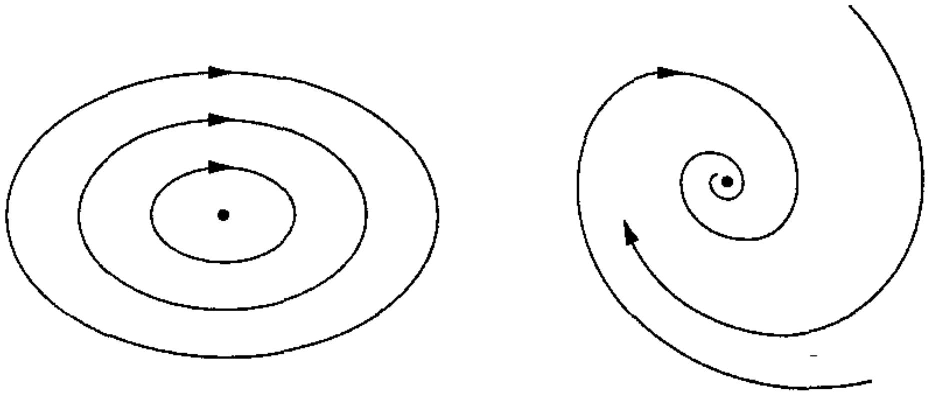




Strogatz, Non linear Dynamics and Chaos

Complex eigenvalues

- Pure imaginary eigen values => centre
- Complex eigen values => spiral
  - Re(lambda) < 0 => Spiral inward (stable)
  - Re(lambda) > 0 => Spiral outward (stable)
  - Im(lambda) decides the frequency of oscillation



(a) center (b) spiral

Strogatz, Non linear Dynamics and Chaos

For a non-linear system

- Function f and g are no more linear functions, so you cannot write the same in the form of a matrix
- But you can linearize the system by using the Jacobian

$$A = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

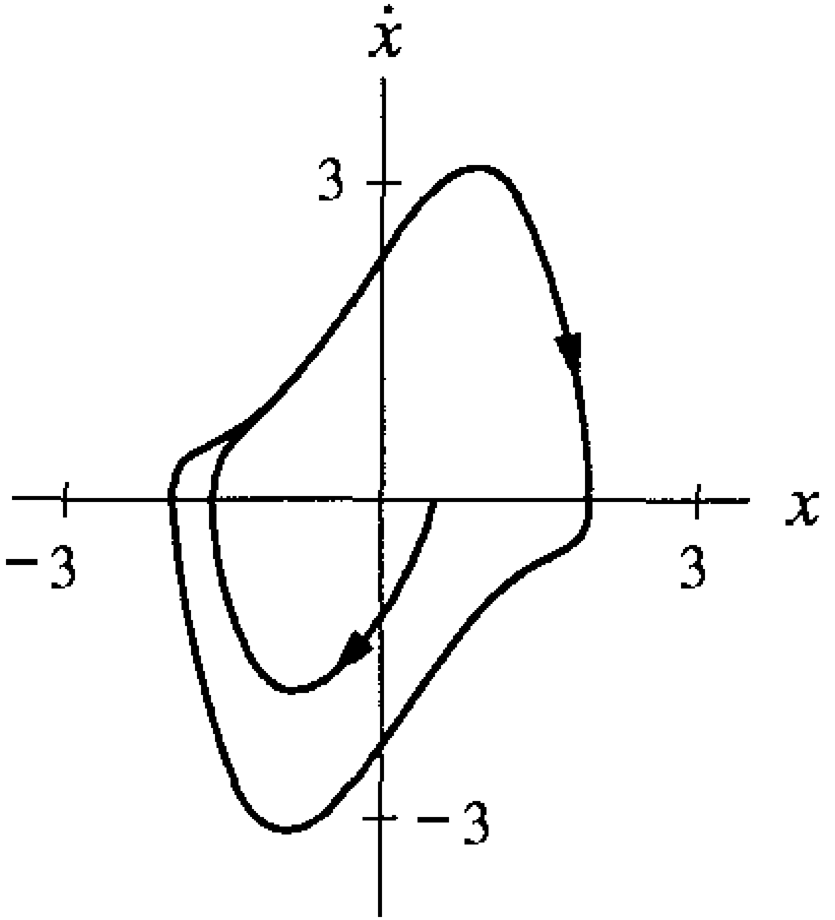
- Note that if the system itself were linear this matrix would automatically degenerate to the usual A as defined earlier.
- Linearization is usually ok when Re(lambda) n.e 0.

Limit cycles

- Uniquely non-linear phenomenon
- We saw the presence of centers d2x/dt2 = -kx, but here there are infinite circles around the f.p
- A limit cycle is an isolated closed orbit. Neighbouring trajectories spiral into the limit cycle or away from it.
- If all neighbouring trajectories fall into L.C then stable , else unstable

E.g. Van Der Pol oscillator

- $\frac{d^2x}{dt} = -x \; - \; \mu(x^2 - 1)\frac{dx}{dt}$



Strogatz, Non linear Dynamics and Chaos

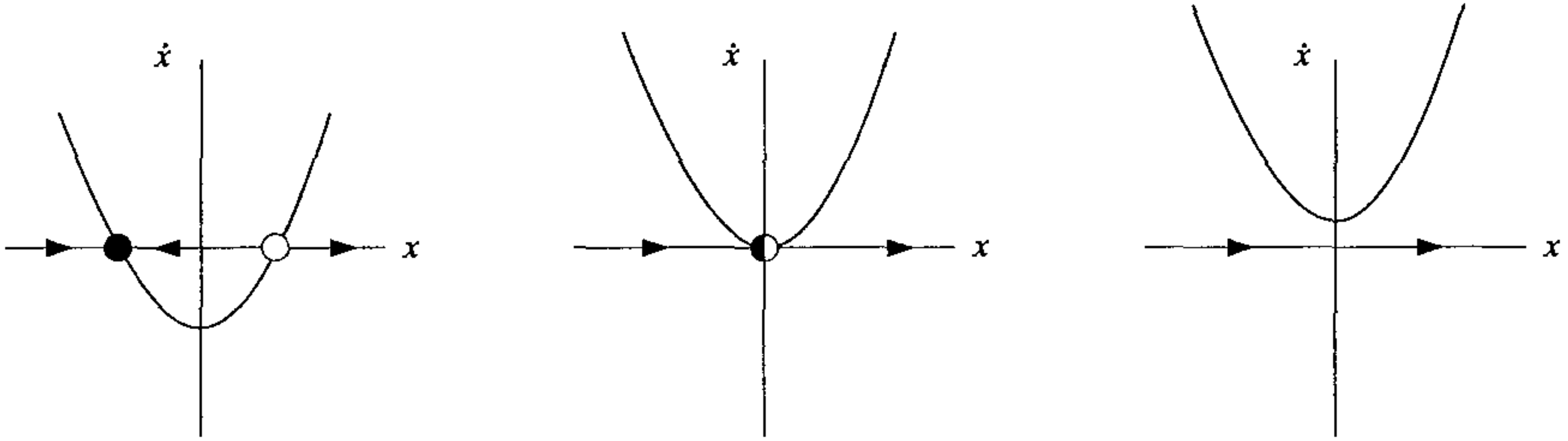
Computer simulations and the velocity vector fields are very good tools for reasoning and understanding the behaviour of non-linear systems !!

Bifurcations

- Qualitative change in dynamics of the system when parameters are varied

Bifurcations : Saddlenode Bifurcation

- As parameter varies, 2 f.p s collide and disappear
- E.g. $\frac{dx}{dt} = r + \ x^2$



(a) $r<0$ (b) $r=0$ (c) $r>0$

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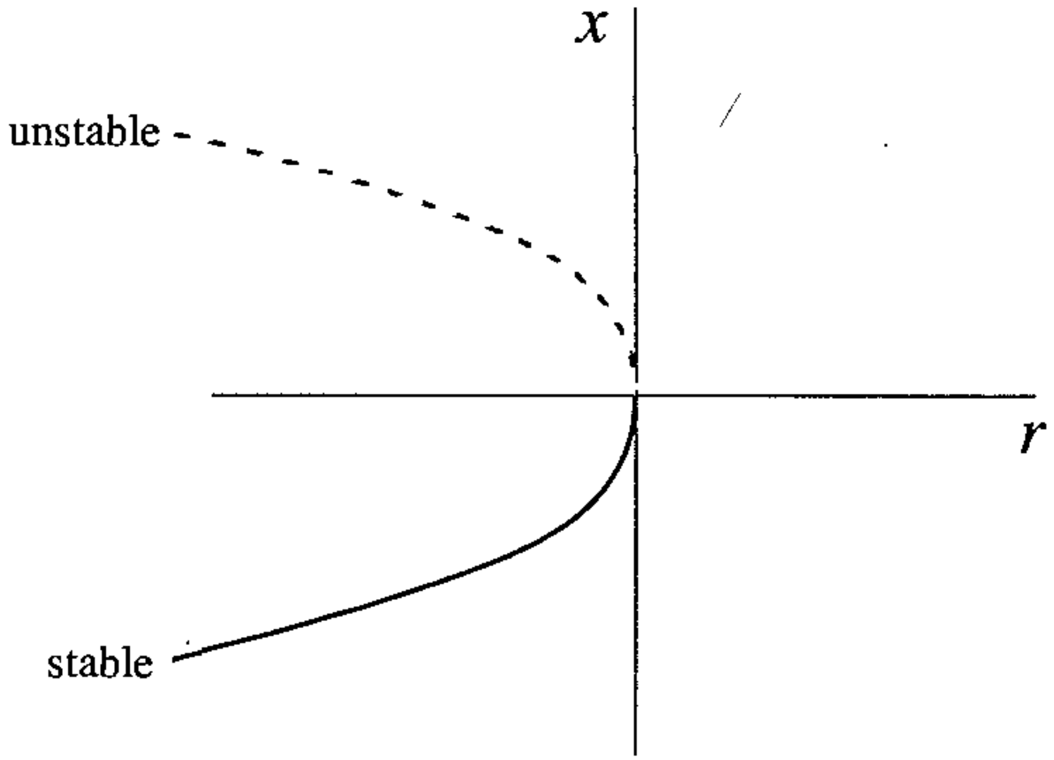
Bifurcation diagram

- Plot of fixed points vs changing parameter



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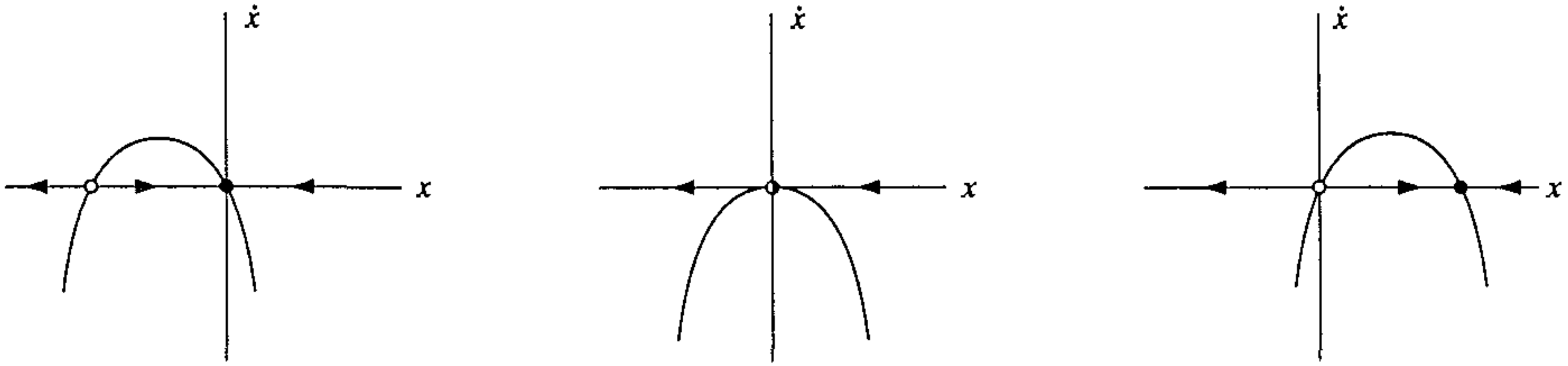
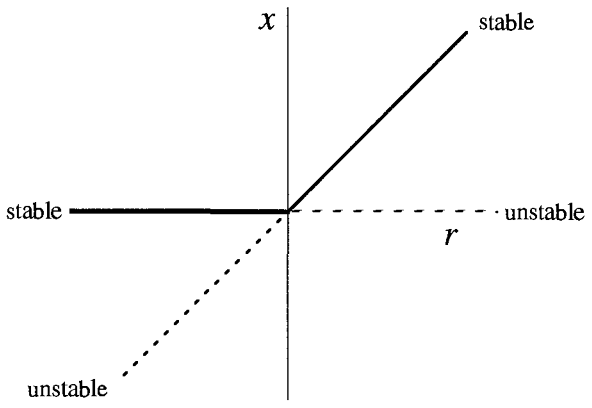
Bifurcations : Transcritical Bifurcation

- As parameter varies, 2 f.p s collide and switch stabilities
- E.g. $\frac{dx}{dt} = rx + \ x^2$



(a) $r < 0$ (b) $r = 0$ (c) $r > 0$



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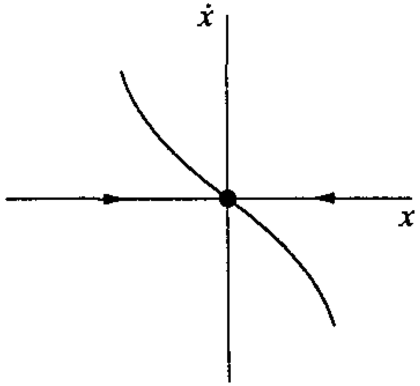
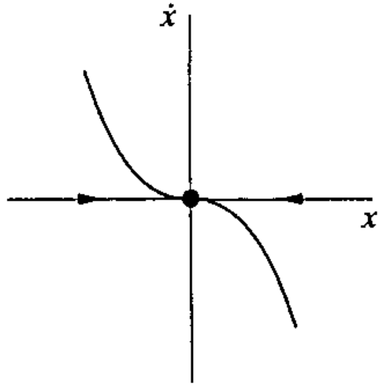
Bifurcations : Supercritical Pitchfork Bifurcation

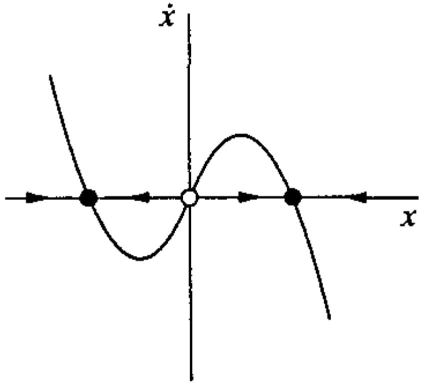
- As parameter varies, f.ps appear and disappear in pairs
- Look at the bifurcation diagram. Name justified ?
- E.g. $\frac{dx}{dt} = rx - \ x^3$



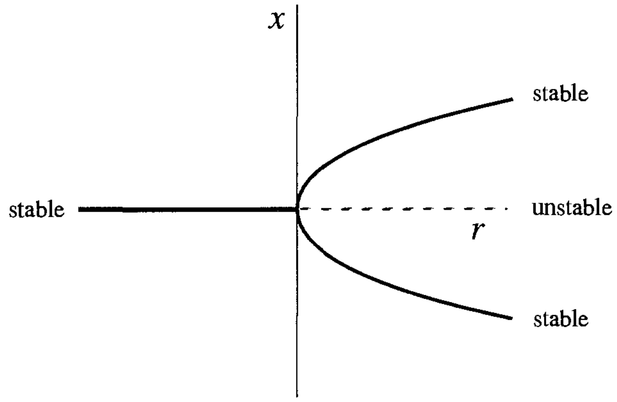
(a) $r < 0$



(b) $r = 0$



(c) $r > 0$



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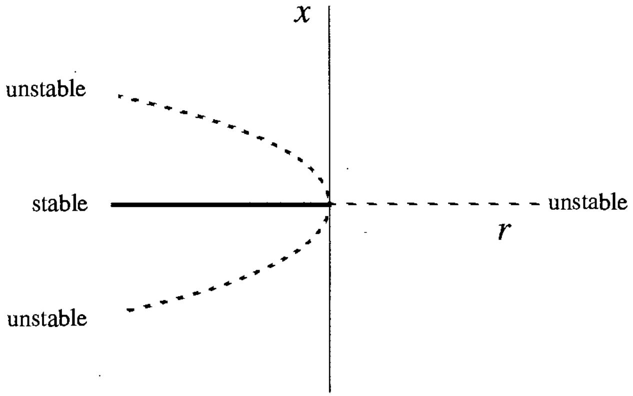
Bifurcations : Subcritical Pitchfork Bifurcation

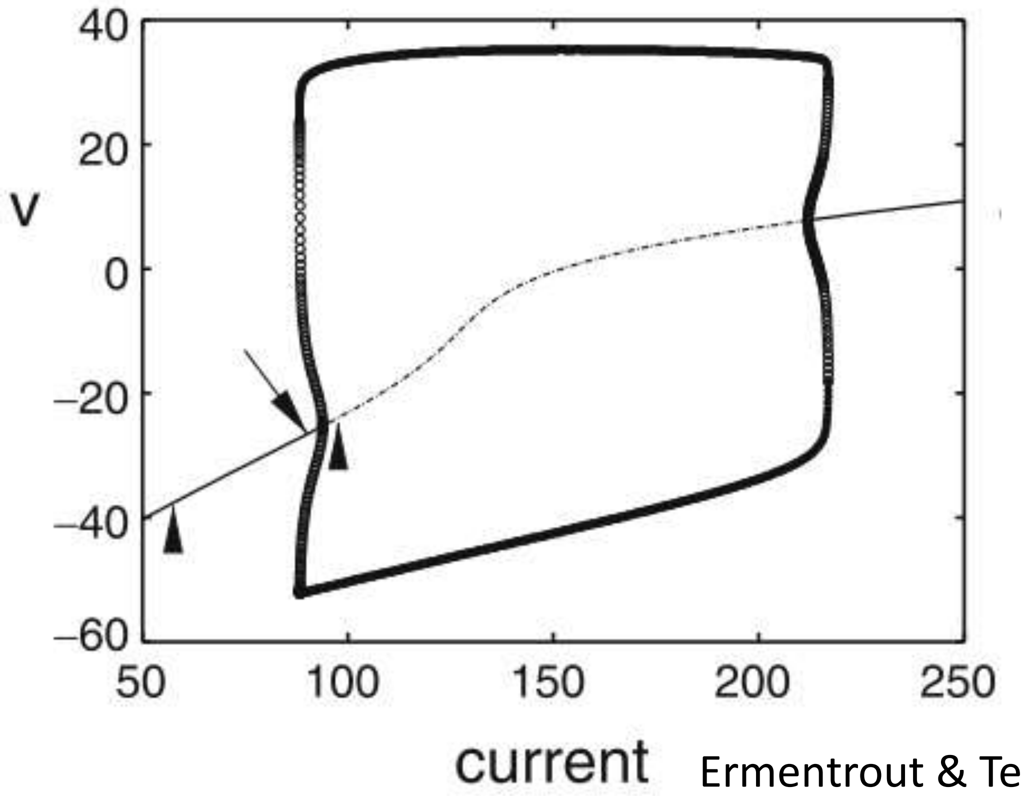
- Similar to supercritical pitchfork, except that you have pair of unstable pairs.
- E.g. $\frac{dx}{dt} = rx + \ x^3$



Strogatz, Non linear Dynamics and Chaos

Hopf bifurcation

- Stable fixed point gives rise to oscillations



Ermentrout & Terman, Mathematical foundations of Neuroscience

Thank you

Stability in a linear system (Real symmteric)

- Using the eigen value decomposition A can be written as
- $A = X.\Delta.X^T$,
  - Where X is a matrix whose columns are the eigen vectors of A
  - Δ is a diagonal matrix with eigen values on the diagonal (in same order as X)
- $d\underline{x} = A.\underline{x}$

  $d\underline{x} = X.\Delta.X^T.\underline{x}$

  $X^T.d\underline{x} = X^T.X.\Delta.X^T.\underline{x}$

  $X^T.d\underline{x} = \Delta.X^T.\underline{x}$ [since eigen vectors are mutually orthogonal, $X^T.X = I$]

  [pre multiplying by X^T = converting the vector to eigen coordinates]

  $d\underline{y} = \Delta.\underline{y}$ [Δ is a diagonal matrix and $\underline{y}$, $d\underline{y}$ are $\underline{x}$ and $d\underline{x}$ converted to eigen coordinates]
- Thus by making the eigen vectors as the new coordinate system, we have an uncoupled system as before.
- **So the new condition for stability is that eigen values should all be -ve**