GATE 2011(ME) Problem-19

Sachin Karumanchi (IITH AI)

AI20BTECH11013

Question

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Cars arrive at a service station according to Poisson's distribution with a mean rate of 5 per hour.The service time per car is exponential with a mean of 10 minutes.At a steady state,the average waiting time in the queue is

1. 10 min
2. 20 min
3. 25 min
4. 50 min

Queuing Theory

- Queuing theory is the mathematical study of waiting lines or queues.
- In Queuing theory a model is constructed so that queue lengths and waiting times can be predicted.
- A customer arrives into system for service and gets out of system after service.
- Generally most of the arrival times follow poisson distributions and services follow exponential distributions.
- These queuing models have since seen applications including telecommunications,traffic engineering, computing and the design of factories,shops,offices and hospitals.

Parameters for measuring Queuing performance

Parameters

- $\lambda =$ Average arrival time
- $\mu =$ Average service time
- $\rho =$ Utilization factor
- $L_q =$ Average number in the queue
- $L_s =$ Average number in the system
- $W_q =$ Average waiting time
- $W_s =$ Average time in the system
- $P_n =$ Steady state probability of exactly n customers in the system

Single server model

- From the given question we can say that there is only one queue with no limit of cars in queue,so we can say that it is a single server model.
- Single server model is represented as "**M/M/1/∞/∞/FIFO**" by kendall's notation (or) usually "**M/M/1**"
- Here '**M**' indicates the Markovian property (or) memory less property of the model first **M** is for arrival and second one for service and 1 is the number of servers in the model and '∞' indicates the limit of the queue and second '∞' represent population of jobs to be served and '**FIFO**' represents First-In First-Out service.
- In cases where there is no limit in the queue we only take the cases where $\frac{\lambda}{\mu} < 1$. Otherwise there could be customers who will not get their service.

Deriving formulae

- The memory less property allows us to assume that one event can take place in a small interval of time. The event could be either a arrival or a service.
- For the time interval$(t, t+h)$, where $h \to 0$

$$\Pr(1 \text{ arrival}) = \lambda h \tag{1}$$
$$\Pr(1 \text{ service}) = \mu h \tag{2}$$
$$\Pr(\text{no arrival}) = 1 - \lambda h \tag{3}$$
$$\Pr(\text{no service}) = 1 - \mu h \tag{4}$$

- Probability of n people at time $(t+h)$

$$\begin{aligned} P_n(t+h) = {} & P_{n-1}(t) \times \Pr(1 \text{ arrival}) \times \Pr(\text{no service}) \\ & + P_{n+1}(t) \times \Pr(\text{no arrival}) \times \Pr(1 \text{ service}) \\ & + P_n(t) \times \Pr(\text{no arrival}) \times \Pr(\text{no service}) \end{aligned} \tag{5}$$

contd

$$\implies P_n(t+h) = P_{n-1}(t)(\lambda h)(1-\mu h) + P_{n+1}(t)(\mu h)(1-\lambda h) + P_n(t)(1-\lambda h)(1-\mu h) \quad (6)$$

Now, Neglecting higher order terms of h.

$$\implies P_n(t+h) = P_{n-1}\lambda h + P_{n+1}\mu h + P_n(t)(1-\lambda h-\mu h) \quad (7)$$

$$\implies \frac{P_n(t+h)-P_n(t)}{h} = P_{n-1}(t)\lambda + P_{n+1}(t)\mu - P_n(t)(\lambda+\mu) \quad (8)$$

At steady state, $P_n(t+h) = P_n(t)$

$$\implies \lambda P_{n-1} + \mu P_{n+1} = (\lambda+\mu)P_n \quad (9)$$

contd

Now, calculating $P_0(t+h)$ using (5)

$$P_0(t+h) = P_1(t)(1-\lambda h)(\mu h) + P_0(t)(1-\lambda h) \tag{10}$$

Again, Neglecting higher order terms of h

$$\implies P_0(t+h) = P_1(t)(\mu h) + P_0(t)(1-\lambda h) \tag{11}$$

$$\implies \frac{P_0(t+h) - P_0(t)}{h} = P_1(\mu) - P_0(\lambda) \tag{12}$$

At steady state, $P_0(t+h) = P_0(t)$

$$\implies \mu P_1 = \lambda P_0 \tag{13}$$

$$\implies P_1 = \left(\frac{\lambda}{\mu}\right) P_0 \tag{14}$$

contd

Using (9) by substituting $n = 1$

$$\lambda P_0 + \mu P_2 = (\lambda + \mu)P_1 \tag{15}$$

$$\implies \lambda P_0 + \mu P_2 = \lambda P_1 + \mu P_1 \tag{16}$$

from (13) and (14)

$$\implies \lambda P_0 + \mu P_2 = \lambda P_1 + \lambda P_0 \tag{17}$$

$$\implies P_2 = \left(\frac{\lambda}{\mu}\right) P_1 \tag{18}$$

$$\implies P_2 = \left(\frac{\lambda}{\mu}\right)^2 P_0 \tag{19}$$

We assume $\frac{\lambda}{\mu} = \rho$ and generalize P_n by (14) and (19)

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0 \tag{20}$$

$$\implies P_n = \rho^n P_0 \tag{21}$$

contd

We know that sum of all probabilities equal to 1

$$\sum_{i=1}^{\infty} P_i = 1 \tag{22}$$

$$\implies P_0 + P_1 + P_2 + ... = 1 \tag{23}$$

Using (21)

$$\implies P_0 + \rho P_0 + \rho^2 P_0 + ... = 1 \tag{24}$$

$$\implies P_0 \left(1 + \rho + \rho^2 + ...\right) = 1 \tag{25}$$

$$\implies P_0 \left(\frac{1}{1-\rho}\right) = 1 \tag{26}$$

$$\implies P_0 = 1 - \rho \tag{27}$$

$$\therefore P_n = \rho^n (1 - \rho) \tag{28}$$

Contd

The number of people in the system (L_s) is the expected value

$$L_s = \sum_{i=0}^{\infty} iP_i \tag{29}$$

$$\implies L_s = \sum_{i=0}^{\infty} i\rho^i P_0 = \rho P_0 \sum_{i=0}^{\infty} i\rho^{i-1} \tag{30}$$

$$\implies L_s = \rho P_0 \sum_{i=0}^{\infty} \frac{d}{d\rho}\left(\rho^i\right) \tag{31}$$

$$\implies L_s = \rho P_0 \frac{d}{d\rho} \sum_{i=0}^{\infty} \rho^i \tag{32}$$

$$\implies L_s = \rho P_0 \frac{d}{d\rho}\left(\frac{1}{1-\rho}\right) \tag{33}$$

$$\implies L_s = \rho P_0 \frac{1}{(1-\rho)^2} \tag{34}$$

contd

By using (27)

$$\implies L_s = \rho(1-\rho)\frac{1}{(1-\rho)^2} \tag{35}$$

$$\implies L_s = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda} \tag{36}$$

The average number of people beign served is ρ

$$\therefore L_s = L_q + \text{avg no.of people beign served} \tag{37}$$

$$\implies L_s = L_q + \rho \tag{38}$$

$$\implies L_q = L_s - \rho \tag{39}$$

$$\implies L_q = \frac{\rho}{1-\rho} - \rho \tag{40}$$

$$\implies L_q = \frac{\rho^2}{1-\rho} = \frac{\lambda^2}{\mu(\mu-\lambda)} \tag{41}$$

contd

The relation between L_s,W_s and L_q,W_q are the Little's equation and they are related as

$$W_s = \frac{L_s}{\lambda} = \frac{1}{\mu - \lambda} \tag{42}$$

$$W_q = \frac{L_q}{\lambda} = \frac{\lambda}{\mu(\mu - \lambda)} \tag{43}$$

Results

- $\rho = \frac{\lambda}{\mu}$
- $L_s = \frac{\lambda}{\mu - \lambda}$
- $L_q = \frac{\lambda^2}{\mu(\mu - \lambda)}$
- $W_s = \frac{1}{\mu - \lambda}$
- $W_q = \frac{\lambda}{\mu(\mu - \lambda)}$

Solution

From the question given,

$$\lambda = 5\text{hr}^{-1}$$
$$\mu = \frac{1}{10}\text{min}^{-1} = 6\text{hr}^{-1}$$

also $\rho = \frac{\lambda}{\mu} = \frac{5}{6} < 1$
Therefore, the average waiting time in the queue

$$W_q = \frac{\lambda}{\mu(\mu - \lambda)}$$
$$= \frac{5}{6(6-5)}$$
$$= \frac{5}{6}\text{hr} = 50\text{min}$$

option 4 is correct.