Assignment 4

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Download all latex-tikz codes from

https://github.com/sachinkarumanchi/probability_and_random_variables/blob/assignment4/Assignment4.tex

1 Problem

Cars arrive at a service station according to Poisson's distribution with a mean rate of 5 per hour.The service time per car is exponential with a mean of 10 minutes.At a steady state,the average waiting time in the queue is

2 Formulae Derivation

This problem can be solved using Queuing theory.But first we have to understand queuing theory.

- In queuing theory we try to determine what happens when people join in queue.
- **Parameters for measuring Queuing performance**
  1) λ = Average arrival time
  2) μ = Average service time
  3) ρ = Utilization factor
  4) L_q = Average number in the queue
  5) L = Average number in the system
  6) W_q = Average waiting time
  7) W = Average time in the system
  8) P_n = Steady state probability of exactly n customers in the system
- Typically most of the times arrivals follow poisson distribution and services follow exponential distributions.
- The given question only has one queue so we can conclude that it is a single server model and there is no limit for number of cars in the queue so we can say that it is "**M/M/1:/∞/∞/FIFO**" by kendall's notation (or) usually "**M/M/1**"
- Here '**M**' indicates the memory less property of the model first **M** is for arrival and second one for service and 1 is the number of servers in the model and '∞' indicates the limit of the queue and second '∞' represent population and '**FIFO**' represents First-In First-Out service.
- **NOTE:** In cases where there is no limit in the queue we only take the cases where $\frac{\lambda}{\mu} < 1$. Otherwise there could be customers who will not get their service.
  The memory less property allows us to assume that one event can take place in a small interval of time. The event could be either a arrival or a service.
- **Deriving formulas :** For the time interval$(t, t+h)$, where $h \to 0$

$$\Pr(1 \text{ arrival}) = \lambda h \tag{2.0.1}$$
$$\Pr(1 \text{ service}) = \mu h \tag{2.0.2}$$
$$\Pr(\text{no arrival}) = 1 - \lambda h \tag{2.0.3}$$
$$\Pr(\text{no service}) = 1 - \mu h \tag{2.0.4}$$

$$P_n(t+h) = P_{n-1}(t) \times \Pr(1 \text{ arrival}) \times \Pr(\text{no service}) + P_{n+1}(t) \times \Pr(\text{no arrival}) \times \Pr(1 \text{ service}) + P_n(t) \times \Pr(\text{no arrival}) \times \Pr(\text{no service}) \tag{2.0.5}$$

$$\implies P_n(t+h) = P_{n-1}(t)(\lambda h)(1-\mu h) + P_{n+1}(t)(\mu h)(1-\lambda h) + P_n(t)(1-\lambda h)(1-\mu h) \tag{2.0.6}$$

Now, Neglecting higher order terms of h.

$$\implies P_n(t+h) = P_{n-1}\lambda h + P_{n+1}\mu h + P_n(t)(1-\lambda h-\mu h) \tag{2.0.7}$$

$$\implies \frac{P_n(t+h)-P_n(t)}{h} = P_{n-1}(t)\lambda + P_{n+1}(t)\mu - P_n(t)(\lambda+\mu) \tag{2.0.8}$$

At steady state, $P_n(t+h) = P_n(t)$

$$\implies \lambda P_{n-1} + \mu P_{n+1} = (\lambda+\mu)P_n \tag{2.0.9}$$

Now, calculating $P_0(t+h)$ using (2.0.5)

$$P_0(t+h) = P_1(t)(1-\lambda h)(\mu h) + P_0(t)(1-\lambda h) \quad (2.0.10)$$

Again, Neglecting higher order terms of h

$$\implies P_0(t+h) = P_1(t)(\mu h) + P_0(t)(1-\lambda h) \quad (2.0.11)$$

$$\implies \frac{P_0(t+h) - P_0(t)}{h} = P_1(\mu) - P_0(\lambda) \quad (2.0.12)$$

At steady state, $P_0(t+h) = P_0(t)$

$$\implies \mu P_1 = \lambda P_0 \quad (2.0.13)$$

$$\implies P_1 = \left(\frac{\lambda}{\mu}\right) P_0 \quad (2.0.14)$$

Using (2.0.9) by substituting $n = 1$

$$\lambda P_0 + \mu P_2 = (\lambda + \mu) P_1 \quad (2.0.15)$$

$$\implies \lambda P_0 + \mu P_2 = \lambda P_1 + \mu P_1 \quad (2.0.16)$$

from (2.0.13) and (2.0.14)

$$\implies \lambda P_0 + \mu P_2 = \lambda P_1 + \lambda P_0 \quad (2.0.17)$$

$$\implies P_2 = \left(\frac{\lambda}{\mu}\right) P_1 \quad (2.0.18)$$

$$\implies P_2 = \left(\frac{\lambda}{\mu}\right)^2 P_0 \quad (2.0.19)$$

We assume $\frac{\lambda}{\mu} = \rho$ and generalize P_n by (2.0.14) and (2.0.19)

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0 \quad (2.0.20)$$

$$\implies P_n = \rho^n P_0 \quad (2.0.21)$$

We know that sum of all probabilities equal to 1

$$\sum_{i=1}^{\infty} P_i = 1 \quad (2.0.22)$$

$$\implies P_0 + P_1 + P_2 + ... = 1 \quad (2.0.23)$$

Using (2.0.21)

$$\implies P_0 + \rho P_0 + \rho^2 P_0 + ... = 1 \quad (2.0.24)$$

$$\implies P_0 \left(1 + \rho + \rho^2 + ...\right) = 1 \quad (2.0.25)$$

$$\implies P_0 \left(\frac{1}{1-\rho}\right) = 1 \quad (2.0.26)$$

$$\implies P_0 = 1 - \rho \quad (2.0.27)$$

$$\therefore P_n = \rho^n (1-\rho) \quad (2.0.28)$$

The number of people in the system (L_s) is the expected value

$$L_s = \sum_{i=0}^{\infty} i P_i \quad (2.0.29)$$

$$\implies L_s = \sum_{i=0}^{\infty} i \rho^i P_0 \quad (2.0.30)$$

$$\implies L_s = \rho P_0 \sum_{i=0}^{\infty} i \rho^{i-1} \quad (2.0.31)$$

$$\implies L_s = \rho P_0 \sum_{i=0}^{\infty} \frac{d}{d\rho} \left(\rho^i\right) \quad (2.0.32)$$

$$\implies L_s = \rho P_0 \frac{d}{d\rho} \sum_{i=0}^{\infty} \rho^i \quad (2.0.33)$$

$$\implies L_s = \rho P_0 \frac{d}{d\rho} \left(\frac{1}{1-\rho}\right) \quad (2.0.34)$$

$$\implies L_s = \rho P_0 \frac{1}{(1-\rho)^2} \quad (2.0.35)$$

By using (2.0.27)

$$\implies L_s = \rho(1-\rho) \frac{1}{(1-\rho)^2} \quad (2.0.36)$$

$$\implies L_s = \frac{\rho}{1-\rho} \quad (2.0.37)$$

We can also say that the number of people beign served is ρ

$$\therefore L_s = L_q + \text{people beign served} \quad (2.0.38)$$

$$\implies L_s = L_q + \rho \quad (2.0.39)$$

$$\implies L_q = L_s - \rho \quad (2.0.40)$$

$$\implies L_q = \frac{\rho}{1-\rho} - \rho \quad (2.0.41)$$

$$\implies L_q = \frac{\rho^2}{1-\rho} \quad (2.0.42)$$

The relation between L_s and W_s and L_q and W_q are the Little's equation and they are related as

$$L_s = \lambda W_s \tag{2.0.43}$$
$$L_q = \lambda W_q \tag{2.0.44}$$

3 Solution

From the question given,

$$\lambda = 5\text{hr}^{-1} \tag{3.0.1}$$
$$\mu = \frac{1}{10}\text{min}^{-1} = 6\text{hr}^{-1} \tag{3.0.2}$$

Therefore,

$$\text{Utilization rate}(\rho) = \frac{\lambda}{\mu} = \frac{5}{6} \tag{3.0.3}$$

Average number (or) length in queue be L_q

$$L_q = \frac{\rho^2}{1-\rho} \tag{3.0.4}$$
$$= \frac{\left(\frac{5}{6}\right)^2}{1-\frac{5}{6}} \tag{3.0.5}$$
$$= \frac{25}{6} \tag{3.0.6}$$

Let the Average waiting time in queue be W_q

$$W_q = \frac{L_q}{\lambda} \tag{3.0.7}$$
$$= \frac{\frac{25}{6}}{5} \tag{3.0.8}$$
$$= \frac{5}{6}\text{hr} = 50\text{min} \tag{3.0.9}$$

The average waiting time in the queue is 50 min.

Parameter	Value
λ	5hr^{-1}
μ	6hr^{-1}
Utilization rate $(\rho) = \frac{\lambda}{\mu}$	$\frac{5}{6}$
Length in queue $\left(L_q\right) = \frac{\rho^2}{1-\rho}$	$\frac{25}{6}$
Waiting time in queue $\left(W_q\right) = \frac{L_q}{\lambda}$	$\frac{5}{6}\text{hr}$

TABLE 8: Parameters of the given question and values.