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Unit-2.

Sequence & Their Limit in $\mathbb{R}$

Sequence :-

Definition :- A sequence of real numbers (or a sequence in $\mathbb{R}$) is a function defined on the set $\mathbb{N} = \{1, 2, ---\}$ of natural numbers whose range is contained in the set $\mathbb{R}$ of real number.

$$f : \mathbb{N} \rightarrow \mathbb{R}$$
$$f(n) = x_n \quad \forall\, n = 1, 2, 3, 4 ---$$

The value x_n are also called the 'terms' or 'elements' of sequence.

Notation :-

Sequence :- $\langle x_n \rangle = \{x_1, x_2, x_3, x_4 ---\}$ Have infinite numbers

Set :- $\{x_n : n \in \mathbb{N}\} =$ Range of func. (May contain single or many number)

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & \cdots & n & n+1 \; \cdots \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow \\ x_1 & x_2 & x_3 & x_4 & x_5 & & x_n & x_{n+1} \end{array}$$

$f(n) = \langle b \rangle = b \quad b \quad b \quad b \quad b \; --- \; b \quad b \; --- \Rightarrow$ A constant sequence

$\langle b \rangle$ = Represents a constant sequence

$\langle 1 \rangle \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \; \cdots \; 1 \quad 1 \; \cdots$

$\langle \frac{1}{n} \rangle \quad 1 \quad \frac{1}{2} \quad \frac{1}{3} \quad \frac{1}{4} \quad \frac{1}{5} \; \cdots \; \frac{1}{n} \quad \frac{1}{n+1} \cdots$

$\langle -1^n \rangle \quad -1 \quad 1 \quad -1 \quad 1 \quad -1 \; \cdots \; (-1)^n \quad (-1)^{n+1} \; \cdots$

$\langle a + (n-1)d \rangle \quad a \quad a+d \quad a+2d \quad a+3d \quad a+4d \; \cdots$ Aisthematic Sequence

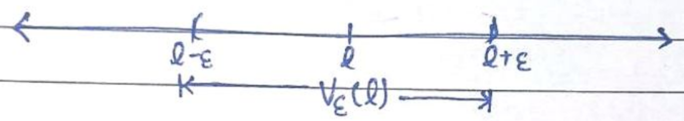
$\langle ar^{n-1} \rangle \quad a \quad ar \quad ar^2 \quad ar^3 \quad ar^4 \; \cdots$ Geometric Sequence

$$\begin{cases} x_1 = 1 \\ x_2 = 1 \\ x_{n+2} = x_n + x_{n+1} \end{cases}$$ Fibonacci Sequence

Limit:-

Definition:- Limit of a sequence $\langle x_n \rangle$ as $n \to \infty$. A sequence $\langle x_n \rangle \to l$ as $n \to \infty$ if "for each $\varepsilon > 0$ $\exists$ $N(\varepsilon) \in \mathbb{N}$ such that $|x_n - l| < \varepsilon \quad \forall \; n \geq N(\varepsilon)$"

$\Leftrightarrow \quad x_n \in V_\varepsilon(l) \quad \forall \; n \geq N(\varepsilon)$



Ques:- Prove that $\lim\limits_{n\to\infty} \left[\frac{1}{n}\right] = 0$

Sol:- Let $\varepsilon > 0$ be an arbitrary real number. Then,

$\left|\frac{1}{n} - 0\right| < \varepsilon \quad \Leftrightarrow \quad \frac{1}{n} < \varepsilon \quad \Leftrightarrow \quad n > \frac{1}{\varepsilon}$

By Archimedean Principle for $\frac{1}{\varepsilon} \in \mathbb{R}$ $\exists$ a natural number $N(\varepsilon)$ such that $N(\varepsilon) \geq \frac{1}{\varepsilon}$

such that $n > N(\varepsilon) \geq \frac{1}{\varepsilon}$

Hence $n > \frac{1}{\varepsilon} \quad \forall \; n \geq N(\varepsilon)$

$\Rightarrow \quad \left|\frac{1}{n} - 0\right| < \varepsilon \quad \forall \; n \geq N(\varepsilon)$

$\lim_{n\to\infty} x_n = l \iff x_n \to l$ as $n \to \infty$

$n \to \infty \iff n$ becomes very large $\iff$ n approaches $\infty \iff$ n tends to ∞
$\iff$ given any $k \in \mathbb{N}$, $n \geq k$

$x_n \to l \iff x_n$ gets very close to $l \iff x_n$ approaches $l \iff x_n$ tends to l
$\iff |x_n - l| < \varepsilon \quad \forall\ \varepsilon > 0$

Def:- $\lim_{n\to\infty} x_n = l$ if for each $\varepsilon > 0 \quad \exists\ N(\varepsilon) \in \mathbb{N}$

such that $|x_n - l| < \varepsilon \quad \forall\ n \geq N(\varepsilon)$

Convergence of a Sequence $\langle x_n \rangle$
We say that a sequence $\langle x_n \rangle$ convergent & converges to 'l' if $\lim_{n\to\infty} x_n$ exists & equal to l.

Divergence of a sequence $\langle x_n \rangle$
We say that a sequence $\langle x_n \rangle$ diverges or divergent if $\lim_{n\to\infty} x_n$ does not exist.

Examples:- Prove that $\lim_{n\to\infty} \left[\frac{1}{n^2+1}\right] = 0$

Proof:- Let $\varepsilon > 0$ be any arbitrary real number.
Now, $\left|\frac{1}{n^2+1} - 0\right| = \left|\frac{1}{n^2+1}\right| = \frac{1}{n^2+1} \quad \forall\ n \in \mathbb{N}$

& $n^2+1 > n^2 \Rightarrow \frac{1}{n^2+1} < \frac{1}{n^2} < \frac{1}{n}$ $\quad [\because n^2 \geq n\ \forall\ n \in \mathbb{N}]$

$$\left|\frac{1}{n^2+1} - 0\right| < \frac{1}{n}$$

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By Archimedean Property, $\exists$ $K \in \mathbb{N}$ such that,

$$K > \frac{1}{\varepsilon} \iff \frac{1}{K} < \varepsilon$$

Choose $N(\varepsilon) = K$

Then, $\forall$ $n \geq N(\varepsilon) = K$ we have,

$$\frac{1}{n} \leq \frac{1}{K} < \varepsilon$$

$$\Rightarrow \left| \frac{1}{n^2+1} - 0 \right| < \frac{1}{n} \leq \frac{1}{K} < \varepsilon \quad \forall \quad n \geq N(\varepsilon)$$

Hence for each $\varepsilon > 0$, $\exists$ $N(\varepsilon)$ such that

$$\left| \frac{1}{n^2+1} - 0 \right| < \varepsilon \quad \forall \quad n \geq N(\varepsilon)$$

Example-2 Prove that $\left\langle \frac{3n+2}{n+1} \right\rangle$ converges & find the real number to which it converges.

Sol:- To show that,

$$\lim_{n\to\infty} \left[\frac{3n+2}{n+1} \right] \text{ exists.}$$

We claim that, $\lim_{n\to\infty} \left[\frac{3n+2}{n+1} \right] = 3$

Let, $\varepsilon > 0$ be any arbitrary real number.

Then, $\left| \frac{3n+2}{n+1} - 3 \right| = \left| \frac{3n+2-3n-3}{n+1} \right| = \left| \frac{-1}{n+1} \right| = \frac{1}{n+1} \quad \forall \; n \in \mathbb{N}$

But $n \geq 1 \iff n+1 > n \iff \frac{1}{n+1} < \frac{1}{n}$

$$\therefore \left| \frac{3n+2}{n+1} - 3 \right| = \frac{1}{n+1} < \frac{1}{n} \quad \forall \; n \in \mathbb{N}$$

By Archimedean Property $\exists$ $K \in \mathbb{N}$ such that

$$K > \frac{1}{\varepsilon} \iff \varepsilon > \frac{1}{K}$$

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Choose $N(\varepsilon) = k$

Then $n \geq N(\varepsilon) = k$ we have,

$$\frac{1}{n} < \frac{1}{k} < \varepsilon$$

$$\left|\frac{3n+2}{n+1} - 3\right| < \frac{1}{n} < \frac{1}{k} < \varepsilon$$

Hence, for each $\varepsilon > 0$ $\exists$ $N(\varepsilon)$ such that

$$\left|\frac{3n+2}{n+1} - 3\right| < \varepsilon \quad \forall \quad n > N(\varepsilon) = k$$

$\therefore$ $\lim_{n\to\infty} \left(\frac{3n+2}{n+1}\right) = 3$

$\Rightarrow$ $\left\langle \frac{3n+2}{n+1} \right\rangle$ converges & converges to 3.

Ques:- Prove that $\langle \sqrt{n+1} - \sqrt{n} \rangle$ is convergent & converges to 0?

Proof:- Let $\varepsilon > 0$ be any arbitrary real number.

Then, $|\sqrt{n+1} - \sqrt{n} - 0| = |\sqrt{n+1} - \sqrt{n}| = \left|\frac{\sqrt{n+1} - \sqrt{n} \times \sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}\right|$

$$= \left|\frac{n+1-n}{\sqrt{n+1} + \sqrt{n}}\right| = \left|\frac{1}{\sqrt{n+1} + \sqrt{n}}\right|$$

$n+1 > n \quad \forall \quad n \in \mathbb{N}$

$\Rightarrow$ $\sqrt{n+1} > \sqrt{n} \quad \forall \quad n \in \mathbb{N}$ $\qquad [\because x > y \Leftrightarrow \sqrt{x} > \sqrt{y}]$

$\Rightarrow$ $\frac{1}{\sqrt{n+1}} < \frac{1}{\sqrt{n}} \quad \forall \quad n \in \mathbb{N}$

[Since $\frac{1}{\sqrt{n+1}} > 0$ $\Rightarrow$ $\sqrt{n+1} + \sqrt{n} > 0 + \sqrt{n}$

$\Rightarrow$ $\frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n}} \quad \forall \quad n \in \mathbb{N}$]

$$\left|\frac{1}{\sqrt{n+1} + \sqrt{n}}\right| = \frac{1}{\sqrt{n+1} + \sqrt{n}} < \frac{1}{\sqrt{n}} \quad \forall \quad n \in \mathbb{N}$$

$\frac{1}{\sqrt{n}} < \varepsilon \iff \frac{1}{n} < \varepsilon^2$

By Archimedean Property, $\exists$ $K \in \mathbb{N}$ such that $K > \frac{1}{\varepsilon^2}$

Choose $N(\varepsilon) = K$ Then $\forall$ $n \geq N(\varepsilon) = K$

We have $n \geq K > \frac{1}{\varepsilon^2}$ $\forall$ $n \geq N(\varepsilon)$

$\iff n > \frac{1}{\varepsilon^2}$ $\forall$ $n \geq N(\varepsilon)$

$\iff \sqrt{n} > \frac{1}{\varepsilon} \iff \frac{1}{\sqrt{n}} < \varepsilon$ $\forall$ $n \geq N(\varepsilon)$

Hence, $|\sqrt{n+1} - \sqrt{n} - 0| < \frac{1}{\sqrt{n}} < \varepsilon$ $\forall$ $n \geq N(\varepsilon)$

$\iff \lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = 0$

Theorem :- Uniqueness of Limit :-

A sequence $\langle x_n \rangle$ of real number can have atmost one limit.

Proof :- Suppose $\lim_{n\to\infty} (x_n) = l_1$ -① & $\lim_{n\to\infty} (x_n) = l_2$ -②

We claim that $l_1 = l_2$

Let $\varepsilon > 0$ be an arbitrary real number

We show that $0 \leq |l_1 - l_2| < \varepsilon$

From -① $\exists$ $N_1 \in \mathbb{N}$ such that

$|x_n - l_1| < \frac{\varepsilon}{2}$ $\forall$ $n \geq N_1$

From -② $\exists$ $N_2 \in \mathbb{N}$ such that

$|x_n - l_2| < \frac{\varepsilon}{2}$ $\forall$ $n \geq N_2$

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Let $N = \text{Max}\{N_1, N_2\}$. Then,

$|x_n - l_1| < \varepsilon/2 \quad \forall \quad n \geq N \geq N_1$

$|x_n - l_2| < \varepsilon/2 \quad \forall \quad n \geq N \geq N_2$

Hence $|x_n - l_1| < \varepsilon/2$ & $|x_n - l_2| < \varepsilon/2 \quad \forall \quad n \geq N$

Thus $0 < |l_1 - l_2| = |l_1 - x_n + x_n - l_2|$

$\leq |l_1 - x_n| + |x_n - l_2|$ [$\because$ Δ-inequality]

$= |x_n - l_1| + |x_n - l_2| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\Rightarrow 0 \leq |l_1 - l_2| < \varepsilon$

$|l_1 - l_2| = 0 \quad \Rightarrow \quad l_1 = l_2$

We claim that a constant sequence is convergent.

i.e if $a_n = a \quad \forall \; n \in \mathbb{N}$ there $\langle a_n \rangle \to a$

Let us take, $\varepsilon > 0$ be any arbitrary $\mathbb{R}$

Then $|a_n - a| = |a - a| \quad \forall \; n \in \mathbb{N}$

$\Rightarrow |a_n - a| = 0 \quad \forall \; n \in \mathbb{N}$

$\Rightarrow |a_n - a| < \varepsilon \quad \forall \; n \in \mathbb{N}$

For each $\varepsilon > 0$, choose $N = 1$

Then $\forall \; n \geq N = 1$, $|a_n - a| < \varepsilon$

$\Rightarrow \langle a_n \rangle \to a$

Example-1 If $0 \leq b < 1$ then $\lim_{n \to \infty} b^n = 0$

$\Leftrightarrow \langle b^n \rangle$ is convergent to 0 if $0 \leq b < 1$

Proof:- $b = 0 \Rightarrow b^n = 0^n = 0 \quad \forall \; n \in \mathbb{N}$

$\Rightarrow \langle b^n \rangle = \langle 0 \rangle$, a constant sequence

It is convergent (by claim)

Let $0 < b < 1$, for $\varepsilon > 0$ we have to find a natural number K such that, $|b^n - 0| < \varepsilon \quad \forall\ n \geq K$

Now, $|b^n - 0| = |b^n| = b^n < \varepsilon$

$\Leftrightarrow \ln(b^n) < \ln(\varepsilon)$ [$\because \ln x$ is a $\uparrow$ func]

$\Leftrightarrow n \ln b < \ln(\varepsilon)$

$\Leftrightarrow n \geq \dfrac{\ln(\varepsilon)}{\ln(b)}$ [$\because \ln(b) < 0$]

$\Leftrightarrow$ choose $K > \dfrac{\ln(\varepsilon)}{\ln(b)}$ where $K \in \mathbb{N}$

By Archimedean Property $\dfrac{\ln(\varepsilon)}{\ln(b)} \in \mathbb{R}$

Hence, $\forall\ n \geq K$ we have $n \geq K > \dfrac{\ln(\varepsilon)}{\ln(b)}$

$\Rightarrow n > \dfrac{\ln(\varepsilon)}{\ln(b)}$

$\Rightarrow b^n < \varepsilon$

$\therefore |b^n - 0| < \varepsilon \quad \forall\ n \geq K$

Theorem:- Let $\langle x_n \rangle$ be a sequence of real numbers and let $x \in \mathbb{R}$. If $\langle a_n \rangle$ is the sequence of real numbers satisfying the following:-

1) $a_n > 0 \quad \forall\ n \in \mathbb{N}$

2) $\lim\limits_{n \to \infty} a_n = 0$

3) $\lim\limits_{n \to \infty} x_n = x$ if $|x_n - x| \leq c \cdot a_n \quad \forall\ n \geq m$ where $c > 0$ is a constant & $m \in \mathbb{N}$

Proof:- By condition 2),

$\lim\limits_{n \to \infty} a_n = 0 \Leftrightarrow \langle a_n \rangle \to 0$

$\Leftrightarrow$ for each $\varepsilon > 0 \ \exists\ K \in \mathbb{N}$ such that

$a_n = |a_n - 0| < \frac{\varepsilon}{c} \quad \forall \; n \geq k \quad -①$

Also, $|x_n - x| \leq c \cdot a_n \quad \forall \quad n \geq m \quad -②$

Let $N = \max\{k, m\}$. Then from - ① we get,

$a_n < \frac{\varepsilon}{c} \quad \forall \quad n \geq N \geq k$

from ②, we get, $|x_n - x| \leq c \cdot a_n \quad \forall \quad n \geq N \geq m$

Hence, $a_n < \frac{\varepsilon}{c}$ & $|x_n - x| < c \cdot a_n \quad \forall \quad n \geq N$

Now, $|x_n - x| < c \cdot a_n$

$< c \cdot \left[\frac{\varepsilon}{c}\right] \quad \forall \quad n \geq N$

$= \varepsilon$

$\Rightarrow \quad |x_n - x| < \varepsilon \quad \forall \quad n \geq N$

Thus, $\lim\limits_{n \to \infty} x_n = x$

Example:- If $a > 0$ then $\left\langle \frac{1}{1+na} \right\rangle \to 0$

Proof:- $\left| \frac{1}{1+na} - 0 \right| \leq c \cdot a_n \quad \Leftrightarrow \quad \frac{1}{1+na} \leq c \cdot a_n$

$a > 0 \quad \Rightarrow \quad na > 0 \quad \forall \quad n \in \mathbb{N}$

$0 < na < 1 + na \quad \forall \quad n \in \mathbb{N}$

$\Rightarrow \quad \frac{1}{1+na} < \frac{1}{na} \quad \forall \quad n \in \mathbb{N}$

Let $c = \frac{1}{a}$, $a_n = \frac{1}{n} \quad \forall \quad n \in \mathbb{N}$

Now, $a_n = \frac{1}{n} > 0 \quad \forall \quad n \in \mathbb{N}$

& $\langle a_n \rangle = \left\langle \frac{1}{n} \right\rangle \to 0$

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$\left|\frac{1}{1+na} - 0\right| = \left|\frac{1}{1+na}\right| = \frac{1}{1+na} < \frac{1}{na} = \left(\frac{1}{a}\right)\left(\frac{1}{n}\right) = c \cdot a_n$

where $c = \frac{1}{a}$ & $a_n = \frac{1}{n}$ $\forall$ $n \in \mathbb{N}$

Hence, $\left\langle \frac{1}{1+na} \right\rangle \to 0$ as $n \to \infty$ [As Previous Theorem]

Example:- If $0 < b < 1$ then $\langle b^n \rangle \to 0$ as $n \to \infty$

Solution:- $0 < b < 1 \iff b = \frac{1}{1+h}$ where $h > 0$

$\Rightarrow$ $b^n = \frac{1}{(1+h)^n}$

$\Rightarrow$ $\frac{1}{(1+h)^n} \le \frac{1}{1+nh} \le \frac{1}{nh}$ [Bernoulli's Inequality]

$\Rightarrow$ $|b^n - 0| = b^n < \left[\frac{1}{h}\right]\left[\frac{1}{n}\right]$ $\forall$ $n \ge 1$

Examples:- If $c > 0$ then $\lim_{n\to\infty} (c^{\frac{1}{n}}) = 1$

$\left|c^{\frac{1}{n}} - 1\right| \le c_1 a_n$

Proof:- Case-1 Let $c = 1$

$c^{\frac{1}{n}} = (1)^{\frac{1}{n}} = 1$ $\forall$ $n \in \mathbb{N}$

& $\langle c^{\frac{1}{n}} \rangle = \langle 1 \rangle = 1$ as $n \to \infty$

Case-2 $0 < c < 1$ Then $0 < c^{\frac{1}{n}} < 1$

Let $c^{\frac{1}{n}} = \frac{1}{1+h_n}$ where $h_n > 0$

$c = \frac{1}{(1+h_n)^n} \le \frac{1}{(1+nh_n)} \le \frac{1}{nh_n}$

$$0 < 1 - c^{\frac{1}{n}} = \frac{h_n}{1+h_n} < h_n < \frac{1}{nc} \quad \forall\ n \in \mathbb{N}$$

$$|c^{\frac{1}{n}} - 1| < \left(\frac{1}{n}\right)\left(\frac{1}{c}\right) \quad \forall\ n \in \mathbb{N}$$

Case-3 $\quad c > 1$ then $c^{\frac{1}{n}} > 1 \quad \forall\ n \in \mathbb{N}$

$$c^{\frac{1}{n}} = 1 + h^n \quad \forall\ n \in \mathbb{N}$$

$$c = (1+h_n)^n \geq 1 + nh_n \quad \forall\ n \in \mathbb{N}$$

$$c - 1 \geq nh_n \quad \forall\ n \in \mathbb{N}$$

$$\left(\frac{c-1}{n}\right) \geq h_n \quad \forall\ n \in \mathbb{N}$$

$$|c^{\frac{1}{n}} - 1| = h_n \leq \frac{(c-1)}{n} \quad \forall\ n \in \mathbb{N}$$

where $C_1 = (c-1)$ $\quad a_n = \left(\frac{1}{n}\right) \quad \forall\ n \in \mathbb{N}$

Example:- $\lim_{n \to \infty} (n^{\frac{1}{n}}) = 1$

Solution:- Case-1 $\stackrel{n=1}{=} \langle n^{\frac{1}{n}} \rangle = \langle 1 \rangle = 1$

Case-2 $\quad$ since $n^{\frac{1}{n}} > 1$ for $n > 1$

$$n^{\frac{1}{n}} = 1 + k_n$$

$$n = (1+k_n)^n \geq 1 + k_n \quad \text{-(1)}$$

$$n - 1 \geq nk_n$$

$$|n - 1| \geq nk_n$$

$$|n^{\frac{1}{n}} - 1| = k_n \leq \left(\frac{n-1}{n}\right) = 1\left(\frac{n-1}{n}\right)$$

where $\lim_{n \to \infty} \left(\frac{n-1}{n}\right) = 1 \neq 0$

So, from eq-(1)

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$n = (1+k_n)^n$

$n \geq 1 + nk_n + \frac{n(n-1)}{2} k_n^2$

$n-1 \geq \frac{1}{2} n(n-1) k_n^2$

$(n^{\frac{1}{n}} - 1) = k_n < \left(\frac{2}{n}\right)^{\frac{1}{2}} \quad \forall\, n \in \mathbb{N}$

where, $c = \sqrt{n}$, $a_n = \frac{1}{\sqrt{n}}$

Hence, by theorem, $\lim_{n\to\infty} (n^{\frac{1}{n}}) = 1$

→ Divergence of a Sequence :-

A sequence $\langle a_n \rangle$ does not converge to l if for some $\varepsilon_0 > 0$, no natural number K, exists such that, $|a_n - l| < \varepsilon \quad \forall\, n \geq k$

That is, no matter what $K \in \mathbb{N}$ is chosen we find a particular $n_k \geq K$ such that

$$|x_{n_k} - l| \geq \varepsilon_0 \iff x_{n_k} \notin (l - \varepsilon_0, l + \varepsilon_0)$$

Section 3.2 Limit Theorems :-

A Bounded Sequence :-

A sequence $\langle x_n \rangle$ of real numbers is said to be bounded if $\exists$ $M > 0$, a real number such that

$|x_n| \leq M \quad \forall n \in \mathbb{N}$

Thus, the set of values,

$\{x_n : n \in \mathbb{N}\}$ is bounded.

$|x_n| \leq M \iff -M \leq x_n < M \quad \forall n \in \mathbb{N}$

$x_n \in [-M, M] \quad \forall n \in \mathbb{N}$

Examples:- $x_n = c \quad \forall n \in \mathbb{N}$

Is $\langle x_n \rangle$ bounded.

$x_n \in [-c-1, c+1] \quad \forall n \in \mathbb{N}$

Theorem:- If $\langle x_n \rangle$ is a convergent sequence of real numbers, then it is bounded.

Proof:- Since $\langle x_n \rangle$ is convergent.

$\lim_{n \to \infty} x_n$ exists & is equal to a unique real number

Suppose $\lim_{n \to \infty} x_n = l$

for each $\varepsilon > 0$ $\exists$ $K \in \mathbb{N}$ (Depending on ε)

such that, $|x_n - l| < \varepsilon \quad \forall n \geq K$

$\iff l - \varepsilon < x_n < l + \varepsilon \quad \forall n \geq K$

$\iff x_n \in (l - \varepsilon, l + \varepsilon)$

$|x_1|, |x_2| \ldots |x_{K-1}|$

Let $M = \max.\{|x_1|, |x_2|, \ldots |x_{K-1}|, l + \varepsilon\}$

Then $|x_i| \leq M \quad \forall i = \{1, 2, \ldots, K-1\}$

& $|x_n| < l + \varepsilon \leq M \quad \forall n \geq K$

Hence $|x_n| < M \quad \forall n \in \mathbb{N}$

$\Rightarrow \langle x_n \rangle$ is bounded

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Consequences :-

i) If $\langle x_n \rangle$ is unbounded then it can't be convergent.

Eg:- 1) $\langle x_n \rangle = \langle n \rangle$ is divergent.

Because it is unbounded.

Suppose $\langle x_n \rangle$ is bounded $\exists$ $M > 0$, a real number such that

$|x_n| = |n| = n \leq M \quad \forall \quad n \in \mathbb{N}$

Which contradict the Archimedean property.

Eg.2) $x_n = \langle n^2 \rangle$ is unbounded.

3) $\langle x_n \rangle = \langle (-1)^n n \rangle$ is unbounded & divergent.

4) $\langle x_n \rangle = \langle \sqrt{n} \rangle$

Theorem :-

Let $\langle x_n \rangle = \langle x_1, x_2, x_3 \dots \rangle$

$\langle y_n \rangle = \langle y_1, y_2, y_3 \dots \rangle$

Definition :-

(1) $\langle x_n + y_n \rangle = \langle x_1 + y_1, x_2 + y_2, \dots \rangle$

(2) $\langle x_n - y_n \rangle = \langle x_1 - y_1, x_2 - y_2, \dots \rangle$

(3) $\langle x_n \cdot y_n \rangle = \langle x_1 \cdot y_1, x_2 \cdot y_2, \dots \rangle$

(4) $\left\langle \frac{x_n}{y_n} \right\rangle = \left\langle \frac{x_1}{y_1}, \frac{x_2}{y_2}, \dots \right\rangle$

If $\langle x_n \rangle \to x$ & $\langle y_n \rangle \to y$ as $n \to \infty$

Then (1) $\langle x_n + y_n \rangle \to x + y$ as $n \to \infty$

(2) $\langle x_n - y_n \rangle \to x - y$ as $n \to \infty$

(3) $\langle x_n \cdot y_n \rangle \to x \cdot y$ as $n \to \infty$

(4) $\left\langle \frac{x_n}{y_n} \right\rangle \to \frac{x}{y}$ as $n \to \infty$ & $y \neq 0$

(5) $\langle c \cdot x_n \rangle \to c \cdot x$ as $n \to \infty$

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Proof:- i)

Let $\varepsilon > 0$ be any real number.

Since $\langle x_n \rangle \to x$ as $n \to \infty$ $\exists$ $K_1 \in \mathbb{N}$ such that, $|x_n - x| < \frac{\varepsilon}{2}$ $\forall$ $n \geq K_1$

Since $\langle y_n \rangle \to y$ as $n \to \infty$ $\exists$ $K_2 \in \mathbb{N}$

$|y_n - y| < \frac{\varepsilon}{2}$ $\forall$ $n \geq K_2$

Let, $K = \max.\{K_1, K_2\}$

$|x_n - x| < \frac{\varepsilon}{2}$ $\forall$ $n \geq K \geq K_1$ — ①

$|y_n - y| < \frac{\varepsilon}{2}$ $\forall$ $n \geq K > K_2$ — ②

From ① & ②

$\Rightarrow |x_n - x| < \frac{\varepsilon}{2}$ & $|y_n - y| < \frac{\varepsilon}{2}$ $\forall$ $n \geq K$

Therefore, we get

$|(x_n + y_n) - (x+y)|$

$\Rightarrow |x_n + y_n - x - y|$

$\Rightarrow |x_n - x + y_n - y| \leq |x_n - x| + |y_n - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ $\forall$

Hence $|(x_n + y_n) - (x+y)| < \varepsilon$ $\forall$ $n \geq K$

Hence, $\langle x_n + y_n \rangle \to x+y$ as $n \to \infty$

Proof:- iii)

Let $\varepsilon > 0$ be any real number.

$\langle x_n \rangle \to x$ as $n \to \infty$

$\Rightarrow \exists$ M_1 such that $M_1 > 0$

$|x_n| < M_1$ $\forall$ $n \in \mathbb{N}$

Let $M = \max.\{M_1, |y|\}$

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Then $M_1 \le M$

$|y| \le M$

$|x_n| \le M_1 \le M \quad \forall\ n \in \mathbb{N}$

Since, $\langle x_n \rangle \to x$ as $n \to \infty$, $\exists\ K_1 \in \mathbb{N}$ such that,

$$|x_n - x| < \frac{\varepsilon}{2M} \quad \forall\ n \ge K_1$$

Since $\langle y_n \rangle \to y$ as $n \to \infty$ $\exists\ K_2 \in \mathbb{N}$

such that

$$|y_n - y| < \frac{\varepsilon}{2M} \quad \forall\ n \ge K_2$$

Let $K = \max\{K_1, K_2\}$ Then we get

$$|x_n - x| < \frac{\varepsilon}{2M} \quad \forall\ n \ge K \ge K_1$$

$$|y_n - y| < \frac{\varepsilon}{2M} \quad \forall\ n \ge K \ge K_2$$

$\Rightarrow \quad |x_n - x| < \frac{\varepsilon}{2M} \quad \& \quad |y_n - y| < \frac{\varepsilon}{2M} \quad \forall\ n \ge K$ — (1)

Therefore,

$$\begin{aligned}
|x_n y_n - xy| &= |x_n y_n + x_n y - x_n y - xy| \\
&\le |x_n y_n - x_n y| + |x_n y - xy| \\
&= |x_n (y_n - y)| + |(x_n - x) y| \\
&= |x_n||y_n - y| + |x_n - x||y| \\
&< M_1\left(\frac{\varepsilon}{2M}\right) + \left(\frac{\varepsilon}{2M}\right)M \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall\ n \ge K
\end{aligned}$$

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$\therefore \quad \langle x_n \rangle \to x \quad$ as $n \to \infty$

$\Rightarrow \quad \exists\ M_1$ such that

$|x_n| \leq M_1 \quad \forall\ n \in \mathbb{N}$

Let $M = \max.\{M_1, |y|\}$

Then $M_1 \leq M$

$|y| \leq M$

Hence $\langle x_n \cdot y_n \rangle \to xy$ as $n \to \infty$

(ii) Let $\varepsilon > 0$ be any real number.

Since, $\langle x_n \rangle \to x$ as $n \to \infty$

$\exists$ $K_1 \in \mathbb{N}$ such that

$|x_n - x| < \frac{\varepsilon}{2}$ $\forall$ $n \geq K_1$ — (1)

Since, $\langle y_n \rangle \to y$ as $n \to \infty$

$\exists$ $K_2 \in \mathbb{N}$ $\ni$ $|y_n - y| < \varepsilon$ $\forall$ $n \geq K_2$ — (2)

Let $K = \text{Max.} \{K_1, K_2\}$

Then we have,

$|x_n - x| < \frac{\varepsilon}{2}$ $\forall$ $n \geq K \geq K_1$

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and $|y_n - y| < \frac{\varepsilon}{2}$ $\forall$ $n > k \geq k_2$

$\Rightarrow$ $|x_n - x| < \frac{\varepsilon}{2}$ and $|y_n - y| < \frac{\varepsilon}{2}$ $\forall$ $n \geq k$

$\therefore$ we get

$|(x_n - y_n) - (x - y)| = |x_n - y_n - x + y| = |x_n - x + y - y_n|$

$\leq |x_n - x| + |y - y_n|$

$\leq |x_n - x| + |y_n - y|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ $\forall$ $n \geq k$

$\Rightarrow$ $|(x_n - y_n) - (x - y)| < \varepsilon$ $\forall$ $n \geq k$

Hence $\langle x_n - y_n \rangle \to x - y$ as $n \to \infty$

(iv) Proof:- Step 1 We show that $\langle \frac{1}{y_n} \rangle \to \frac{1}{y}$ as $n \to \infty$

Given that $\langle y_n \rangle \to y$ as $n \to \infty$, for each $\varepsilon > 0$, $\exists k_1 \in$ N

$\ni$ $|y_n - y| < \frac{\varepsilon}{2} |y|^2$ $\forall$ $n \geq k_1$

In Particular for $\varepsilon = \frac{1}{2}|y| > 0$ $\exists$ $k_2 \in \mathbb{N}$

$\ni$ $|y_n - y| < \frac{1}{2}|y|$ $\forall$ $n \geq k_2$

$\Rightarrow$ $-|y_n - y| > -\frac{1}{2}|y|$ $\forall$ $n \geq k_2$

$\Rightarrow$ $-\frac{1}{2}|y| < -|y_n - y|$ $\forall$ $n \geq k_2$

We know $\forall$ $||a| - |b|| \leq |a - b|$

$\Rightarrow$ $-|a - b| < |a| - |b| \leq |a - b|$ $\forall$ $a, b \in \mathbb{R}$

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So, $-\frac{1}{2}|y| < -|y_n - y| \leq |y_n| - |y| \quad \forall \; n \geq K_2$

$\Rightarrow \; -\frac{1}{2}|y| < |y_n| - |y| \quad \forall \; n \geq K_2$

$\Rightarrow \; |y| - \frac{1}{2}|y| < |y_n| \quad \forall \; n \geq K_2$

$\Rightarrow \; \frac{1}{2}|y| < |y_n| \quad \forall \; n \geq K_2$

$\Rightarrow \; \frac{1}{|y_n|} < \frac{2}{|y|} \quad \forall \; n \geq K_2$

$\Rightarrow \; \left|\frac{1}{y_n}\right| < \frac{2}{|y|} \quad \forall \; n \geq K_2$

Hence, $\left|\frac{1}{y_n} - \frac{1}{y}\right| = \left|\frac{1}{y_n}\right| \frac{1}{|y|} |y_n - y|$

$< \left[\frac{2}{|y|}\right]\left[\frac{1}{|y|}\right]\left(\frac{\varepsilon}{2}|y|^2\right)$

$= \varepsilon \quad \forall \; n \geq K$

Let $K = \max.\{K_1, K_2\}$ Then, both (i) & (ii) hold, $\forall \; n \geq K$

$\Rightarrow \; \left|\frac{1}{y_n} - \frac{1}{y}\right| < \varepsilon \quad \forall \; n \geq K$

$\therefore \; \left\langle \frac{1}{y_n} \right\rangle \rightarrow \frac{1}{y}$ as $n \rightarrow \infty$

Step -2

We show that $\left\langle \frac{x_n}{y_n} \right\rangle \rightarrow \frac{x}{y}$ as $n \rightarrow \infty$

Since $\langle x_n \rangle \rightarrow x$ as $n \rightarrow \infty$ (given.)

and $\left\langle \frac{1}{y_n} \right\rangle \rightarrow \frac{1}{y}$ as $n \rightarrow \infty$ (step-1)

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$\therefore$ by using (iii) we get

$\left\langle \frac{x_n}{y_n} \right\rangle = \left\langle x_n \cdot \frac{1}{y_n} \right\rangle \to x \cdot \frac{1}{y} = \frac{x}{y}$ as $n \to \infty$

Then $\left\langle \frac{x_n}{y_n} \right\rangle$ converges to $\frac{x}{y}$.

Example-1 Show that $\left\langle \frac{2n+1}{n} \right\rangle \to 2$

Proof:- $\frac{2n+1}{n} = \frac{2n}{n} + \frac{1}{n} = 2 + \frac{1}{n}$

Define $x_n = 2 \quad \forall \; n \in \mathbb{N}$
$y_n = \frac{1}{n} \quad \forall \; n \in \mathbb{N}$

$\langle x_n \rangle \to 2$ & $\langle y_n \rangle \to 0$ as $n \to \infty$
$\langle x_n + y_n \rangle \to 2 + 0 = 2$ as $n \to \infty$

Example-2 $\left\langle \frac{2n+1}{n+5} \right\rangle \to 2$ as $n \to \infty$

$\frac{2n+1}{n+5} = \frac{\frac{2n}{n} + \frac{1}{n}}{\frac{n}{n} + \frac{5}{n}} = \frac{2 + \frac{1}{n}}{1 + 5\left(\frac{1}{n}\right)}$

$\left\langle 2 + \frac{1}{n} \right\rangle$ Define $x_n = 2 \quad \forall \; n \in \mathbb{N}$
$y_n = \frac{1}{n} \quad \forall \; n \in \mathbb{N}$

Then $\langle x_n \rangle \to 2$ $\langle y_n \rangle \to 0$ as $n \to \infty$

$\left\langle 2 + \frac{1}{n} \right\rangle \to 2 + 0$ as $n \to \infty$

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$\langle 1 + 5(\frac{1}{n}) \rangle$ Define $x_n = 1$ $\forall\ n \in \mathbb{N}$
$y_n = 1/n$ $\forall\ n \in \mathbb{N}$

Then $\langle x_n \rangle \to 1$ and $\langle y_n \rangle \to 0$ as $n \to \infty$
$\langle 1 + 5(\frac{1}{n}) \rangle \to 1 + 5(0) = 1$ as $n \to \infty$

By (V), we get $\langle \frac{2n+1}{n+5} \rangle = \langle \frac{2 + \frac{1}{n}}{1 + 5(\frac{1}{n})} \rangle \to 2$ as $n \to \infty$

Example-3 $\langle \frac{2n}{n^2+1} \rangle \to 0$ as $n \to \infty$

$\frac{2n}{n^2+1} = \frac{2n/n^2}{n^2/n^2 + 1/n^2} = \frac{2/n}{1 + 1/n^2}$

$\Rightarrow \frac{2(\frac{1}{n})}{1 + (\frac{1}{n})^2} = \frac{2(\frac{1}{n})}{1 + (\frac{1}{n})(\frac{1}{n})}$

$\langle 2(\frac{1}{n}) \rangle$ Define $x_n = 2$ $\forall\ n \in \mathbb{N}$
$y_n = 1/n$ $\forall\ n \in \mathbb{N}$

$\langle x_n \rangle \to 2$ and $\langle y_n \rangle \to 0$ as $n \to \infty$
$\therefore$ $\langle 2(\frac{1}{n}) \rangle = \langle x_n \cdot y_n \rangle \to 2 \cdot 0 = 2$ as $n \to \infty$

$\langle 1 + (\frac{1}{n})(\frac{1}{n}) \rangle$ Define $x_n = 1$ $\forall\ n \in \mathbb{N}$
$y_n = \frac{1}{n}$ $\forall\ n \in \mathbb{N}$

$\langle x_n \rangle = \langle 1 \rangle \to 1$ and
$\langle y_n \rangle = \langle \frac{1}{n} \rangle \to 0$ as $n \to \infty$

$\langle x_n + (y_n)(y_n) \rangle = \langle 1 + (\frac{1}{n})(\frac{1}{n}) \rangle \to 1 + 0 = 1$ as $n \to \infty$

$\Rightarrow \left\langle \dfrac{2(\frac{1}{n})}{1+(\frac{1}{n})(\frac{1}{n})} \right\rangle = \left\langle \dfrac{2}{n^2+1} \right\rangle = \dfrac{0}{1} = 0$ as $n \to \infty$

Theorem 3.2.4

Let $\langle x_n \rangle$ be a sequence of real numbers which satisfies the following:-

1) $x_n \geq 0 \quad \forall \; n \in \mathbb{N}$

2) $\lim\limits_{n \to \infty} x_n = x$

Then $\quad x \geq 0$

Proof:- Suppose $x < 0$ Then $-x > 0$

Let $\varepsilon = -x > 0$

From (ii) $\quad \langle x_n \rangle \to x \iff \forall \; \varepsilon > 0 \quad \exists \; K \in \mathbb{N} \ni$

$|x_n - x| < \varepsilon \quad \forall \; n \geq K$

In particular, for $\varepsilon = -x > 0$, $\exists \; K_1 \in \mathbb{N} \ni$

$|x_n - x| < -x \quad \forall \; n \geq K_1$

$\iff -(-x) < x_n - x < -x \quad \forall \; n \geq K_1$

$\iff \cancel{x+x < x_n} \quad x_n < -x + x \quad \forall \; n \geq K_1$

$\Rightarrow x_n < 0 \quad \forall \; n \geq K_1$

This is the contradiction to (i)

Hence, $x < 0$ is wrong $\Rightarrow x \geq 0$

Theorem 3.2.5 :-

Let $\langle x_n \rangle$ and $\langle y_n \rangle$ be two sequence of real number such that $x_n \leq y_n \quad \forall \; n \in \mathbb{N}$

Then $\quad \lim\limits_{n \to \infty} x_n \leq \lim\limits_{n \to \infty} y_n$

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Proof :- Consider the sequence $\langle z_n \rangle$ of real number defined as $z_n = y_n - x_n \ \forall \ n \in \mathbb{N}$

Then, i) $z_n \geq 0 \quad \forall \ n \in \mathbb{N}$

ii) $\lim_{n\to\infty} z_n = \lim_{n\to\infty} (y_n - x_n) \geq 0$ [By Thm. 3.2.4]

Hence, $\lim_{n\to\infty} (y_n) - \lim_{n\to\infty} (x_n) \geq 0$

$\Rightarrow \lim_{n\to\infty} (y_n) \geq \lim_{n\to\infty} (x_n)$

Theorem :- 3.2.6

If $\langle x_n \rangle$ is a sequence of real numbers such that $a \leq x_n \leq b \ \forall \ n \in \mathbb{N}$ and $a, b \in \mathbb{R}$

Then, $a \leq \lim_{n\to\infty} (x_n) \leq b$

Proof :- First, we prove $a \leq \lim_{n\to\infty} x_n$

Let, $y_n = a \ \forall \ n \in \mathbb{N}$

Then, $\lim_{n\to\infty} (y_n) = \lim_{n\to\infty} (a) = a$

Since, $y_n = a \leq x_n \ \forall \ n \in \mathbb{N}$. Then we have, $\lim_{n\to\infty} y_n \leq \lim_{n\to\infty} x_n$ [By Thm. 3.2.5]

$\Rightarrow a \leq \lim_{n\to\infty} x_n$

Second, we prove $\lim_{n\to\infty} \leq b$

Let $y_n = b \ \forall \ n \in \mathbb{N}$

Then, $\lim_{n\to\infty} (y_n) = \lim_{n\to\infty} (b) = b$

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Since, $x_n \le y_n \ge b \ \forall \ n \in \mathbb{N}$. Then we have $\lim_{n\to\infty} y_n \ge \lim_{n\to\infty} x_n$ [By Theorem 3.2.5]

$\Rightarrow \lim_{n\to\infty} x_n \le b$

$$a \le \lim_{n\to\infty} x_n \le b$$

Theorem 3.2.7 Sequeeze Theorem :-
Let $\langle x_n \rangle$, $\langle y_n \rangle$ & $\langle z_n \rangle$ be three sequences of real numbers such that
i) $x_n \le y_n \le z_n \ \forall \ n \in \mathbb{N}$
ii) $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n$

Then $\langle y_n \rangle$ is convergent &
$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n$$

Proof :- From (ii) $\exists$ a unique $l \in \mathbb{R}$ $\ni$
$$\lim_{n\to\infty} x_n = l = \lim_{n\to\infty} z_n$$

Let $\epsilon > 0$ be any real number. Then,
$\lim_{n\to\infty} x_n = l$ implies $\exists \ k_1 \in \mathbb{N}$ such that

$$|x_n - l| < \epsilon \quad \forall \ n \ge k_1$$

Similarly, since $\lim_{n\to\infty} z_n = l$ $\exists$ $k_2 \in \mathbb{N}$

such that $|z_n - l| < \epsilon \quad \forall \ n \ge k_2$
Let $k = \max.\{k_1, k_2\}$ Then we have,
$|x_n - l| < \epsilon \ \forall \ n \ge k \iff -\epsilon < x_n - l < \epsilon \ \forall \ n \ge k$

$0 < \frac{a+b}{2} < b$

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and $|z_n - l| < \varepsilon \quad \forall\ n \geq K \iff -\varepsilon < z_n - l < \varepsilon \quad \forall\ n \geq k$

from (ii) we have,

$$x_n \leq y_n \leq z_n \quad \forall \quad n \in \mathbb{N}$$

$$\Rightarrow \quad x_n - l \leq y_n - l \leq z_n - l \quad \forall\ n \in \mathbb{N}$$

$$\Rightarrow \quad -\varepsilon \leq x_n - l \leq y_n - l \leq z_n - l < \varepsilon \quad \forall\ n \geq k$$

$$\Rightarrow \quad -\varepsilon < y_n - l < \varepsilon \quad \forall\ n \geq K$$

$$\iff \quad |y_n - l| < \varepsilon \quad \forall\ n \geq K$$

$$\iff \quad \langle y_n \rangle \to l \Rightarrow \lim_{n\to\infty} (y_n) = l$$

Hence $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n$

M.I
Ques. $\lim_{n\to\infty} [\langle {}^1a_n \rangle + \langle {}^2a_n \rangle + \langle {}^3a_n \rangle + \dots + \langle {}^Ka_n \rangle] \quad K \in \mathbb{N}$

$$= \lim_{n\to\infty} \langle {}^1a_n \rangle + \lim_{n\to\infty} \langle {}^2a_n \rangle + \dots \lim_{n\to\infty} \langle {}^Ka_n \rangle \text{ as } n\to\infty$$

Example-1 Let $\langle x_n \rangle \to x$ as $n \to \infty$.
Define $p(t) = a_K t^K + a_{K-1} t^{K-1} + \dots + a_1 t + a_0$
where $K \in \mathbb{N}$ & $a_K \neq 0$, it is a polynomial function of degree K. Show that the sequence
$\langle p(x_n) \rangle \to p(x)$ as $n \to \infty$

Proof:- To show that $\langle p(x_n) \rangle \to p(x)$ as $n \to \infty$
$\iff \lim_{n\to\infty} p(x_n) = p(x)$

Now, $p(x_1) = a_K x_1^K + a_{K-1} x_1^{K-1} + \dots + a_1 x_1 + a_0$

$\vdots$

$p(x_n) = a_K x_n^K + a_{K-1} x_n^{K-1} + \dots + a_1 x_n + a_0$

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$\therefore \lim_{n\to\infty} p(x_n) = \lim_{n\to\infty} [a_K x_n^K + a_{K-1} x_n^{K-1} + \dots + a_1 x_n + a_0]$

$= \lim_{n\to\infty} (a_K x_n^K) + \lim_{n\to\infty} (a_{K-1} x_n^{K-1}) + \dots + \lim_{n\to\infty} (a_1 x_n) + \lim_{n\to\infty} (a_0)$

$\Rightarrow a_K \lim_{n\to\infty} (x_n^K) + a_{K-1} \lim_{n\to\infty} (x_n^{K-1}) + \dots a_1 \lim_{n\to\infty} (x_n) + \lim_{n\to\infty} (a_0)$

$\Rightarrow a_K \left[\lim_{n\to\infty} x_n\right]^K + a_{K-1} \left[\lim_{n\to\infty} x_n\right]^{K-1} + \dots a_1 \left[\lim_{n\to\infty} x_n\right] + a_0$

$\Rightarrow a_K x^K + a_{K-1} x^{K-1} + \dots a_1 x + a_0$ $[\because \langle x_n \rangle \to x$

$= p(x)$

$\Rightarrow \langle \{p(x)\} \rangle \to p(x)$

Example :- 2 Let $\langle x_n \rangle \to x$ as $n \to \infty$

Define $r(t) = \dfrac{p(t)}{q(t)}$, where $p(t)$ & $q(t)$ are polynomial functions.

Suppose $q(x_n) \neq 0$ $\forall n \in \mathbb{N}$ & $q(x) \neq 0$

Then $\langle r(x_n) \rangle \to r(x) = \dfrac{p(x)}{q(x)}$ as $n \to \infty$

Proof :- $r(x_n) = \dfrac{p(x_n)}{q(x_n)}$, $q(x_n) \neq 0$

$\Rightarrow \lim_{n\to\infty} [r(x_n)] = \lim_{n\to\infty} \left[\dfrac{p(x_n)}{q(x_n)}\right] = \dfrac{\lim_{n\to\infty} (p(x_n))}{\lim_{n\to\infty} (q(x_n))} = \dfrac{p(x)}{q(x)}$

$q(x) \neq 0$

$= r(x)$

$\langle r(x_n) \rangle \to r(x)$ as $n \to \infty$

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* If $K \in \mathbb{N}$ & if $\langle a_n \rangle$ is a convergent sequence

$$\lim_{n\to\infty} (a_n^K) = \left[\lim_{n\to\infty} (a_n)\right]^K$$

Proof:- Let $P(K) = \lim_{n\to\infty} (a_n^K) = \left(\lim_{n\to\infty} a_n\right)^K$

For $K = 1$

L.H.S of $P(1) = \lim_{n\to\infty} (a_n^1) = \lim_{n\to\infty} (a_n)$

R.H.S of $P(1) = \left[\lim_{n\to\infty} (a_n)\right]^1 = \lim_{n\to\infty} (a_n)$

$\therefore$ $P(1)$ is true.

Assume $P(K)$ is true, that is,

$$\lim_{n\to\infty} (a_n^K) = \left[\lim_{n\to\infty} a_n\right]^K$$

To prove $P(K+1)$ is true, that is

$$\lim_{n\to\infty} (a_n^{K+1}) = \left[\lim_{n\to\infty} a_n\right]^{K+1}$$

L.H.S $\lim_{n\to\infty} (a_n^{K+1}) = \lim_{n\to\infty} (a^K \cdot a_n)$

$$= \left[\lim_{n\to\infty} (a_n^K)\right]\left[\lim_{n\to\infty} (a_n)\right] \quad \left[\because \lim_{n\to\infty} (x_n \cdot y_n)\right]$$

$$= \left[\lim_{n\to\infty} a_n\right]^K \left[\lim_{n\to\infty} a_n\right] \quad \text{[By Induction Hypotesis]}$$

$$= \left[\lim_{n\to\infty} a_n\right]^{K+1} = \text{R.H.S}$$

Hence, by Induction Principle.
$P(K)$ is true $\forall K \in \mathbb{N}$

Theorem:- 3.2.15 If $\langle x_n \rangle \to x$ as $n \to \infty$
then $\langle |x_n| \rangle \to |x|$ as $n \to \infty$

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Proof:- Let $\epsilon > 0$ be an arbitrary Real number
Given: $\langle x_n \rangle \to x$, $\exists$ $K \in \mathbb{N}$ such that

$$|x_n - x| < \epsilon \quad \forall \; n \geq K$$

$$\Rightarrow \; \big||x_n| - |x|\big| \leq |x_n - x| < \epsilon \quad \forall \; n \geq K \quad (\text{By } \Delta\text{-inequality})$$

$$\Rightarrow \; \big||x_n| - |x|\big| < \epsilon \quad \forall \; n \geq K$$

$$\Leftrightarrow \; \langle |x_n| \rangle \to \langle |x| \rangle \quad \text{as } n \to \infty$$

Theorem 3.2.16 If $\langle x_n \rangle \to x$ as $n \to \infty$, then $\langle \sqrt{x_n} \rangle \to \sqrt{x}$ as $n \to \infty$.

Proof:- Let $\epsilon > 0$ be an arbitrary real number
Since, $x_n \geq 0$ $\forall$ n, $x \geq 0$ [By Theorem]

Case-1 $x > 0$

$$|\sqrt{x_n} - \sqrt{x}| = \left|\frac{(\sqrt{x_n} - \sqrt{x})(\sqrt{x_n} + \sqrt{x})}{\sqrt{x_n} + \sqrt{x}}\right| = \left|\frac{x_n - x}{\sqrt{x_n} + \sqrt{x}}\right|$$

$$= \frac{|x_n - x|}{\sqrt{x_n} + \sqrt{x}} \leq \frac{|x_n - x|}{\sqrt{x}}$$

Now, $|\sqrt{x_n} - \sqrt{x}| \leq \frac{1}{\sqrt{x}} |x_n - x|$

Since, $\langle x_n \rangle \to x$ as $n \to \infty$
$\exists$ $K \in \mathbb{N}$ such that
$|x_n - x| < \sqrt{x} \cdot \epsilon$ $\forall$ $n \geq K$ where $\sqrt{x}$ is a Constant

$$\therefore \; |\sqrt{x_n} - \sqrt{x}| \leq \frac{1}{\sqrt{x}} |x_n - x| < \frac{1}{\sqrt{x}} (\sqrt{x}\,\epsilon) = \epsilon \quad \forall \; n \geq K$$

$$\Rightarrow \; |\sqrt{x_n} - \sqrt{x}| < \epsilon \quad \forall \; n \geq K$$

$$\Leftrightarrow \; \langle \sqrt{x_n} \rangle \to \sqrt{x} \quad \text{as } n \to \infty$$

Case-2 $x = 0$
Since, $\langle x_n \rangle \to 0$ as $n \to \infty$ $\exists$ $K_1 \in \mathbb{N}$
such that, $|x_n - 0| = |x_n| \Rightarrow x_n < \epsilon^2$ $\forall$ $n \geq K_1$

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$\Rightarrow \quad x_n < \varepsilon^2 \quad \forall\ n \geq k_1$

$\Rightarrow \quad \sqrt{x_n} < \sqrt{\varepsilon^2} \quad \forall\ n \geq k_1 \quad [\because\ 0 \leq a < b \Rightarrow \sqrt{a} \leq \sqrt{b}]$

$\Rightarrow \quad \sqrt{x_n} < \varepsilon \quad \forall\ n \geq k_1$

$\Rightarrow \quad |\sqrt{x_n}| < \varepsilon \quad \forall\ n \geq k_1$

$\Rightarrow \quad |\sqrt{x_n} - 0| < \varepsilon \quad \forall\ n \geq k_1$

$\Leftrightarrow \quad \langle \sqrt{x_n} \rangle \to 0 = \sqrt{x} \quad$ as $n \to \infty$

Theorem:- 3.2.17.

Let $\langle x_n \rangle$ be a sequence of real numbers such that

(i) $x_n > 0 \quad \forall\ n \in \mathbb{N}$

(ii) $\lim_{n\to\infty} \left(\frac{x_{n+1}}{x_n} \right) = L$ [Ratio Test]

(iii) If $L < 1$, then $\langle x_n \rangle$ converges to 0.

Proof:- ~~Let~~ first, note that $L \geq 0$ by using theorem 3.2.4

Since there is always a real number between any two real numbers, let $r \in \mathbb{R}$ such that $L < r < 1$.

Let $\varepsilon = r - L$, so that $\varepsilon > 0$. From condition -(ii) we get that there exist a natural number m such that,

$$\left| \frac{x_{n+1}}{x_n} - L \right| < \varepsilon \quad \forall\ n \geq m$$

Therefore, for all $n \geq m$, we have by using triangle inequality.

$$\left| \frac{x_{n+1}}{x_n} \right| = \left| \frac{x_{n+1}}{x_n} - L + L \right| \leq \left| \frac{x_{n+1}}{x_n} - L \right| + L < \varepsilon + L = r$$

That is, $\left| \frac{x_{n+1}}{x_n} \right| < r \quad \forall\ n \geq m \qquad - (1)$

Replacing n by $m, m+1, \ldots, n-1$ successively in (1) above and multiplying the corresponding sides of the result inequalities, we obtain

$$\left| \frac{x_{m+1}}{x_m} \right| , \left| \frac{x_{m+2}}{x_{m+1}} \right| \cdots \left| \frac{x_n}{x_{n-1}} \right| < r^{n-m} \Rightarrow \left| \frac{x_n}{x_m} \right| < r^{n-m}$$

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which implies that,

$$|x_n| < \left(\frac{|x_m|}{r^m}\right) r^n \quad \forall\ n \geq m \quad -(2)$$

Since, $0 < r < 1$ by Example 3.1.11 (c), we get that $r^n \to 0$.

Thus, there exists a natural number p such that

$$|r^n - 0| < \left(\frac{r^m}{|x_m|}\right)\varepsilon \quad \forall\ n \geq p$$

$$\Leftrightarrow \quad r^n < \left(\frac{r^m}{|x_m|}\right)\varepsilon \quad \forall\ n \geq p. \quad -(3)$$

Let $M = \max.\{p, m\}$. Then by using (2) and (3), we get that

$$|x_n| < \varepsilon \quad \forall\ n \geq m$$

Hence, the sequence $\langle x_n \rangle$ converges to 0.

Example:- Let $x_n = \frac{n}{2^n}$ $\forall\ n \in \mathbb{N}$. Test the convergence

Solution:- 1) $x_n = \frac{n}{2^n} > 0 \ \forall\ n \in \mathbb{N}$

2) $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \left(\frac{n+1}{2^{n+1}}\right)\left(\frac{2^n}{n}\right) \Rightarrow \lim_{n\to\infty}\left(\frac{n+1}{n}\right)\left(\frac{1}{2}\right)$

$\Rightarrow \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left(\frac{1}{2}\right) = (1)\left(\frac{1}{2}\right) = \frac{1}{2} < 1$

$\Rightarrow \langle x_n \rangle = \langle \frac{n}{2^n} \rangle \to 0$ by theorem 3.2.17.

Cauchy's Theorem on Limits:

1) First Theorem:-

Let $\langle x_n \rangle \to 0$ as $n \to \infty$

Then $\langle \frac{x_1 + x_2 + \dots + x_n}{n} \rangle \to 0$ as $n \to \infty$

↑ Airthematic Mean of $\langle x_n \rangle$

Proof :- Let $\varepsilon > 0$ be an arbitrary real number. Since the sequence $\langle x_n \rangle$ converges to 0, there exists $m \in \mathbb{N}$ such that

$$|x_n - 0| < \frac{\varepsilon}{2} \quad \forall \; n \geq m \Rightarrow |x_n| < \frac{\varepsilon}{2} \quad \forall \; n \geq m$$

Also since the sequence $\langle x_n \rangle$ is convergent, it is bounded. Therefore, there exists a real number $R > 0$ such that

$$|x_n| \leq R \quad \forall \; n \in \mathbb{N}$$

Now, we have

$$|y_n'| = \frac{|x_1 + x_2 + \dots + x_n|}{n}$$

$$\leq \frac{|x_1| + |x_2| + \dots + |x_n|}{n} \quad \text{[By Triangle-Inequality]}$$

$$= \left(\frac{|x_1|}{n} + \frac{|x_2|}{n} + \dots + \frac{|x_m|}{n}\right) + \left(\frac{|x_{m+1}|}{n} + \dots + \frac{|x_n|}{n}\right)$$

$$\leq \frac{mR}{n} + \left(\frac{n-m}{n}\right)\frac{\varepsilon}{2} \quad \forall \; n > m$$

$$= \frac{mR}{n} + \frac{\varepsilon}{2} - \left(\frac{m}{n}\right)\frac{\varepsilon}{2} \quad \forall \; n > m$$

$$< \frac{mR}{n} + \frac{\varepsilon}{2} \quad \forall \; n \geq m$$

Therefore, we get that

$$|y_n| < \frac{mR}{n} + \frac{\varepsilon}{2} \quad \forall \; n > m$$

By Archimedian property, there exists a natural number

$$K > \frac{2mR}{\varepsilon}$$

So, if $n \geq K$, then $n > \frac{2mR}{\varepsilon}$, which implies that

$$\frac{mR}{n} < \frac{\varepsilon}{2} \quad \forall \; n \geq K$$

Define $M = \max\{K, m\}$. Thus using (3) and (4), we obtain that

$$|y_n| < \frac{mR}{n} + \frac{\varepsilon}{2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall\ n \geq M+1$$

Hence the sequence $\langle y_n \rangle$ converges to 0.

Corollary :-
Let $\langle x_n \rangle \to x$ then the sequence $\langle y_n \rangle \to x$ where $y_n = \frac{x_1 + x_2 + \dots + x_n}{n}$ $\forall\ n \in \mathbb{N}$

Proof:- Given $\langle x_n \rangle \to x$ means $\lim\limits_{n\to\infty} x_n = x$

Define $z_n = x_n - x$ $\forall\ n \in \mathbb{N}$. Then $\langle z_n \rangle$ is a sequence of real number such that,

$$\lim_{n\to\infty} (z_n) = \lim_{n\to\infty} (x_n - x) = \lim_{n\to\infty} (x_n) - x = x - x = 0$$

$$\Rightarrow \langle z_n \rangle \to 0$$

By cauchy's first theorem on limits, we get,

$$\left\langle \frac{z_1 + z_2 + \dots + z_n}{n} \right\rangle \to 0$$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{z_1 + z_2 + \dots + z_n}{n} \right) = 0$$

$$\Rightarrow \lim_{n\to\infty} \left[\frac{(x_1 - x) + (x_2 - x) + \dots + (x_n - x)}{n} \right] = 0$$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{(x_1 + x_2 + \dots + x_n) + (-nx)}{n} \right) = 0$$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{x_1 + x_2 + \dots + x_n}{n} \right) - \lim \frac{nx}{n} = 0$$

$$\Rightarrow \lim_{n\to\infty} \left(\frac{x_1 + x_2 + \dots + x_n}{n} \right) = x$$

$$\Rightarrow \lim_{n\to\infty} (y_n) = x \Rightarrow \langle y_n \rangle \to x$$

★ Cauchy's Second Theorem on Limits :-
Let $\langle x_n \rangle$ be a sequence of positive real number, such that $\langle x_n \rangle \to x$
Then the sequence $\langle y_n \rangle \to x$ where
$y_n = (x_1 \cdot x_2 \cdot \ldots x_n)^{\frac{1}{n}} \quad \forall \; n \in \mathbb{N}$

Proof :- Let $A_n = \dfrac{x_1 + x_2 + \ldots + x_n}{n}$ &

$$H_n = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \cdots + \frac{1}{x_n}} \quad \forall \; n \in \mathbb{N}$$

Then $H_n \le y_n < A_n \quad \forall \; n \in \mathbb{N}$

Since $x_n > 0 \;\; \forall \; n \in \mathbb{N}$, $\lim\limits_{n\to\infty} x_n \ge 0 \Rightarrow x \ge 0$

Case-1 $x = 0$
By cauchy's first theorem on Limit.
$\langle x_n \rangle \to 0 \Rightarrow \left\langle \dfrac{x_1 + x_2 + \ldots + x_n}{n} \right\rangle \to 0$

$\Rightarrow \langle A_n \rangle \to 0$

But $0 \le y_n < A_n \quad \forall \; n \in \mathbb{N}$
By squeeze theorem,
we get, $\lim\limits_{n\to\infty} (0) \le \lim\limits_{n\to\infty} (y_n) \le \lim\limits_{n\to\infty} (A_n)$

$\Rightarrow 0 \le \lim\limits_{n\to\infty} (y_n) \le 0$

$\Rightarrow \lim\limits_{n\to\infty} y_n = 0$

Case-2 $x > 0$
Since $x_n > 0 \quad \forall \; n \in \mathbb{N}$, $\dfrac{1}{x_n} > 0 \quad \forall \; n \in \mathbb{N}$

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Now, $\langle x_n \rangle \to x \Rightarrow \langle \frac{1}{x_n} \rangle \to \frac{1}{x}$ (Quotient Rule)

$\Rightarrow \lim_{n\to\infty} \left(\frac{1}{x_n}\right) = \frac{1}{x}$

$\Rightarrow \lim_{n\to\infty} \left(\frac{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}{n}\right) = \frac{1}{x}$ [By corollary 3.2.10]

$\Rightarrow \lim_{n\to\infty} (H_n) = x$

By corollary 3.2.10 $\lim_{n\to\infty} A_n = x$

Since $H_n \le y_n \le A_n \;\forall\; n \in \mathbb{N}$ we get

$\lim_{n\to\infty} (H_n) \le \lim_{n\to\infty} (y_n) \le \lim_{n\to\infty} (A_n)$

$\Rightarrow x \le \lim_{n\to\infty} (y_n) \le x$

$\Rightarrow \lim_{n\to\infty} (y_n) = x$ By Squeeze Theorem.

$\Rightarrow \lim_{n\to\infty} (x_1 \cdot x_2 \cdot \cdots x_n)^{\frac{1}{n}} = x$

$\Rightarrow \langle y_n \rangle \to x$

Corollary 3.2.13 :-
If $x_n > 0 \;\forall\; n \in \mathbb{N}$ and $\lim_{n\to\infty} \left(\frac{x_{n+1}}{x_n}\right) = x$, then $\lim_{n\to\infty} (x_n)^{\frac{1}{n}} = x$

Proof :- Let $y_n = \frac{x_n}{x_{n-1}} \;\forall\; n \ge 2$ & $y_1 = x_1$

Then $\langle y_n \rangle$ is a positive sequence of real number.

$\lim_{n\to\infty} \left(\frac{x_{n+1}}{x_n}\right) = x$ we get $\lim_{n\to\infty} \left(\frac{x_n}{x_{n-1}}\right) = x$

$\Rightarrow \lim_{n\to\infty} (y_n) = x$

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By cauchy's second theorem on Limits, we have
$\lim_{n\to\infty} (y_1 \cdot y_2 \cdot \cdots y_n)^{\frac{1}{n}} = x$

$\Rightarrow \lim_{n\to\infty} \left[\frac{\cancel{x_1}}{1} \cdot \frac{\cancel{x_2}}{\cancel{x_1}} \cdot \frac{x_3}{\cancel{x_2}} \cdots \frac{x_n}{x_{n-1}}\right]^{\frac{1}{n}} = x$

$\Rightarrow \lim_{n\to\infty} (x_n)^{\frac{1}{n}} = x$

Example 3.2.8 (b) The sequence $((-1)^n)$ is divergent.

Proof :- Suppose $(-1)^n$ is convergent. Let $a \in \mathbb{R}$ such that
$\lim_{n\to\infty} (-1)^n = a$ (unique)

$\Rightarrow \exists\ K \in \mathbb{N}$ such that, $|(-1)^n - a| < \varepsilon \quad \forall\ n \geq K$
in particular for $\varepsilon = 1$ $\quad |(-1)^n - a| < 1 \quad \forall\ n \geq K$

Let n be even,	Let n be odd :-
$|(-1)^n - a| = |1-a| < 1$	$|(-1)^n - a| = |-1-a| \leq 1$
$\Leftrightarrow -1 < 1-a < 1$	$-1 < -1-a < 1$
$\Leftrightarrow -2 < -a < 0$	$0 < -a < 2$
$\Leftrightarrow 0 < a < 2$	$0 > a > -2$

But their is no real number satisfying,
$0 < a < 2$ & $-2 < a < 0$

Hence our assumption is wrong.
$\Rightarrow \langle (-1)^n \rangle$ is divergent.

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Section :- 3.3

Monotone Sequence :-

Definition :- Let $\langle x_n \rangle$ be a sequence of real numbers.

1) $\langle x_n \rangle$ is said to be increasing if it satisfies the following inequalities :-

$$x_1 \le x_2 \le x_3 \le \dots \le x_n \le x_{n+1} \le \dots$$

OR Equivalently, $x_{n+1} - x_n \ge 0$, $\forall\ n \in \mathbb{N}$

2) $\langle x_n \rangle$ is said to be decreasing if it satisfies the following inequalities :-

$$x_1 \ge x_2 \ge x_3 \ge \dots \ge x_n \ge x_{n+1} \ge \dots$$

or equivalently $x_{n+1} - x_n \le 0$

(iii) $\langle x_n \rangle$ is said to be monotone of $\langle x_n \rangle$ is either increasing or decreasing.

Note :- A constant sequence $\langle c \rangle$ is increasing as well as decreasing sequence.

Examples :- $\langle n \rangle$: $1 < 2 < 3 < \dots < n < n+1 < \dots$ $\uparrow$

$\langle \frac{1}{n} \rangle$: $1 > \frac{1}{2} > \frac{1}{3} > \dots > \frac{1}{n} > \frac{1}{n+1} > \dots$ $\downarrow$

$\langle \frac{1}{n^2} \rangle$: $1 > \frac{1}{4} > \frac{1}{9} > \dots > \frac{1}{n^2} > \frac{1}{(n+1)^2} \ge \dots$ $\downarrow$

$a > 1$ $\langle a^n \rangle$: $a < a^2 < a^3 < \dots$ $\uparrow$

$0 < b < 1 \Rightarrow \langle b^n \rangle$

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Theorem :- Monotone Convergence Theorem :-

Let $\langle x_n \rangle$ be a monotone sequence of real numbers. Then $\langle x_n \rangle$ is convergent if and only if $\langle x_n \rangle$ is bounded.

Moreover, we have the following :-

1) if $\langle x_n \rangle$ is bounded & increasing then,

$$\lim_{n\to\infty} (x_n) = \sup \{x_n \mid n \in \mathbb{N}\}$$

2) if $\langle x_n \rangle$ is bounded & decreasing, then

$$\lim_{n\to\infty} (x_n) = \inf \{x_n : n \in \mathbb{N}\}$$

Proof :- Let $\langle x_n \rangle$ be convergent. To show that $\langle x_n \rangle$ is bounded. Since $\langle x_n \rangle$ is convergent, $\lim_{n\to\infty} x_n$ exists and equal to a real number,

Let $\lim_{n\to\infty} x_n = x$

Given $\varepsilon > 0$ $\exists$ a $k \in \mathbb{N}$ such that $|x_n - x| < \varepsilon$ $\forall$ $n \geq k$

$\Rightarrow$ $x - \varepsilon < x_n < x + \varepsilon$ $\forall$ $n \geq k$

In particular, for $\varepsilon = 1$, we have,

$$|x_n - x| < 1 \quad \forall \; n \geq k$$

Define $M = \max\{|x_1|, |x_2|, \ldots |x_{k-1}|, |1+|x||\}$

We claim that $|x_n| \leq M$ $\forall$ $n \in \mathbb{N}$ $\Leftrightarrow$ $\langle x_n \rangle$ is bounded.

Now, $|x_n| = |x_n - x + x| \leq |x_n - x| + |x| < 1 + |x| \leq M$

$\Rightarrow$ $|x_n| \leq M$ $\forall$ $n \geq k$

But $|x_i| \leq M$ $\forall$ $i = 1, 2, \ldots k-1$ (by def of M)

Hence, $|x_n| \leq M$ $\forall$ $n \in \mathbb{N}$

Claim $\Rightarrow$ $\langle x_n \rangle$ is bounded.

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Conversely, suppose $\langle x_n \rangle$ is bounded, to show that $\langle x_n \rangle$ is convergent. Since $\langle x_n \rangle$ is bounded $\exists$ $M \in \mathbb{R}$ such that

$|x_n| \leq M \quad \forall \quad n \in \mathbb{N}$

$-M \leq x_n \leq M \quad \forall \quad n \in \mathbb{N} \quad -(1)$

Define the set $A = \{x_n \mid n \in \mathbb{N}\}$ Then A is bounded by (1), & non-empty of $\mathbb{R}$. By completeness property $\sup A$ and $\inf B$ exist. Let

$\sup A = L$ and $\inf A = l$.

We are given that $\langle x_n \rangle$ is monotone, so it is either increasing or decreasing.

We have two cases:-

Case-1 Let $\langle x_n \rangle$ is bounded and increasing.

We claim that $\lim\limits_{n\to\infty} x_n = L \Rightarrow \langle x_n \rangle \to L$

Since $L = \sup A$, $L - \varepsilon$ cannot be an upper bound of A.

$\Rightarrow \exists$ an element $x_k \in A$ such that,

$L - \varepsilon < x_k \leq L$

But x_n is increasing $x_n \geq x_k \quad \forall \quad n \geq k$

Therefore we have,

$L - \varepsilon < x_k \leq x_n \leq L < L + \varepsilon \quad \forall \quad n \geq k$

$\Rightarrow |x_n - L| < \varepsilon \quad \forall \quad n \geq k$

$\Rightarrow \lim\limits_{n\to\infty} x_n = L$

Case-2 Let $\langle x_n \rangle$ be a bounded decreasing sequence.

Therefore, $x_1 \geq x_2 \geq x_3 \geq \dots \geq x_n \geq x_{n+1} \geq \dots \quad -(2)$

$\Leftrightarrow x_m \geq x_n \quad \forall \quad n \geq m$

Since $\langle x_n \rangle$ is bounded, $-M \leq x_n \leq M \quad \forall \quad n \in \mathbb{N}$

$\therefore$ The set $A = \{x_n : n \in \mathbb{N}\}$ is bounded below. $\left[\because x_n \geq -M \;\; \forall \; n \in \mathbb{N}\right]$

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But $A = \phi \Rightarrow \inf A$ exists (By completeness Property)
let $\inf A = l$

Then $l+\varepsilon$ $(\varepsilon>0)$ is not a lower bound of A.
$\Rightarrow \exists$ an element of A, say x_i such that
$$l < x_i < l+\varepsilon$$
$\Rightarrow \quad x_{i+2} < x_{i+1} < x_i$
$\Rightarrow \quad l \le x_n \le x_i < l+\varepsilon \quad \forall \; n \ge i \quad$ (by-2)
$\Rightarrow \quad l-\varepsilon < l \le x_n \le x_i < l+\varepsilon \quad \forall \; n \ge i$
$\Rightarrow \quad l-\varepsilon < x_n < l+\varepsilon \quad \forall \; n \ge i$
$\Rightarrow \quad |x_n - l| < \varepsilon \quad \forall \; n \ge i$
$\Rightarrow \quad \lim_{n\to\infty} x_n = l \Rightarrow \langle x_n \rangle \to l = \inf\{x_n : n \in \mathbb{N}\}$

Application of MONOTONE CONVERGENCE THEOREM (MCT):-

MCT has the following two steps :-

Step-1 : $\langle x_n \rangle$ is bound.

Step-2 : $\langle x_n \rangle$ is either increasing or decreasing.

Conclusion :- (1) $\langle x_n \rangle$ is convergent
(2) $\langle x_n \rangle \to \sup\{x_n : n \in \mathbb{N}\}$ if $\langle x_n \rangle$ is monotone increasing
$\langle x_n \rangle \to \inf\{x_n : n \in \mathbb{N}\}$ if $\langle x_n \rangle$ is monotone decreasing($\downarrow$)

Note:- If sup & inf are difficult to find then we use, "alternate formula"

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Example-(1) Examine the convergence of $\langle x_n \rangle$ where

$$x_n = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{n+n}$$

Sol:- Step-1 $$|x_n| = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+n}$$

$$< \frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n}$$

$$< \frac{n}{n} = 1 \quad \forall\, n \in \mathbb{N}$$

$\Rightarrow |x_n| < 1 \quad \forall\, n \in \mathbb{N}$

$\Leftrightarrow \langle x_n \rangle$ is bounded.

Step-2 $\langle x_n \rangle$ is either converging or decreasing.

$$x_{n+1} = \frac{1}{(n+1)+1} + \frac{1}{(n+1)+2} + \dots + \frac{1}{(n+1)+(n+1)}$$

$$= \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2(n+1)}$$

$$\therefore x_{n+1} - x_n = \left[\frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2(n+1)}\right] - \left[\frac{1}{n+1} + \frac{1}{n+2} \dots + \frac{1}{2n}\right]$$

$$\Rightarrow \frac{1}{2n+1} + \frac{1}{2(n+1)} - \frac{1}{n+1} \Rightarrow \frac{1}{2n+1} - \frac{1}{2(n+1)}$$

$$\Rightarrow \frac{2n+2-(2n+1)}{(2n+2)(2n+1)} = \frac{1}{(2n+2)(2n+1)} > 0 \quad \forall\, n \in \mathbb{N}$$

$\Rightarrow x_{n+1} > x_n \quad \forall\, n \in \mathbb{N} \Rightarrow \langle x_n \rangle$ is monotone increasing.

Hence combining step-1 & step-2, we get by using MCT, that $\langle x_n \rangle$ is convergent.

Example-2 $x_n = \frac{1}{n^p}, \quad p > 0$

Step-1 $|x_n| = \left|\frac{1}{n^p}\right| = \frac{1}{n^p} < 1$ $\left[\because n \geq 1 \Rightarrow n^p \geq 1^p = 1 \Rightarrow \frac{1}{n^p} \leq 1\right]$

$\Rightarrow$ $\langle x_n \rangle$ is bounded.

Step-2 $\quad x_{n+1} = \dfrac{1}{(n+1)^p}$

$x_{n+1} - x_n = \dfrac{1}{(n+1)^p} - \dfrac{1}{n^p}$

where, $n+1 > n \quad \forall\ n \in \mathbb{N}$

$\Rightarrow (n+1)^p \geq n^p \quad \forall\ p > 0, \ \forall\ n \in \mathbb{N}$

$\Rightarrow \dfrac{1}{n^p} > \dfrac{1}{(n+1)^p}$

$\Rightarrow \dfrac{1}{(n+1)^p} - \dfrac{1}{n^p} < 0$

$\Rightarrow x_{n+1} - x_n < 0 \quad \forall\ n \in \mathbb{N}$

$\Rightarrow x_{n+1} < x_n \quad \forall\ n \in \mathbb{N} \Rightarrow \langle x_n \rangle \downarrow$

Example-3 $\quad x_n = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dots + \dfrac{1}{n}$

Step-1 $\quad x_{n+1} = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dots + \dfrac{1}{(n+1)}$

$\Rightarrow x_{n+1} - x_n = \left(\dfrac{1}{n+1}\right) > 0 \quad \forall\ n \in \mathbb{N}$

$\Rightarrow x_{n+1} > x_n \quad \forall\ n \in \mathbb{N}$

$\Rightarrow \langle x_n \rangle$ is monotone increase.

Step-2

Rough Work $\Rightarrow x_1 = 1 \quad x_2 = \dfrac{3}{2} \quad x_3 = \dfrac{11}{6} \quad x_4 = \dfrac{50}{24} > 2$

Sol:- $\langle x_{2^n} \rangle$.

$\langle x_{2^1} \rangle = \langle x_2 \rangle = 1 + \dfrac{1}{2}$

$\langle x_{2^2} \rangle = \langle x_4 \rangle = \left(1 + \dfrac{1}{2}\right) + \left(\dfrac{1}{3} + \dfrac{1}{4}\right)$

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$\langle x_{2^3} \rangle = \langle x_8 \rangle = \left(1+\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)$

$\langle x_{2^4} \rangle = \langle x_{16} \rangle = \left(1+\frac{1}{2}\right)+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\left(\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}\right)$

$\langle x_{2^4} \rangle = \left(\frac{1}{2^0}+\frac{1}{2^1}\right)+\left(\frac{1}{2^1+1}+\frac{1}{2^2}\right)+\left(\frac{1}{2^2+1}+\cdots+\frac{1}{2^3}\right)+\left(\frac{1}{2^3+1}+\cdots+\frac{1}{2^4}\right)$

$\langle x_{2^n} \rangle \Rightarrow \left(\frac{1}{2^0}+\frac{1}{2^1}\right)+\left(\frac{1}{2^1+1}+\frac{1}{2^2}\right)+\left(\frac{1}{2^2+1}+\cdots+\frac{1}{2^3}\right)+\cdots+\left(\frac{1}{2^{n-1}+1}+\cdots+\frac{1}{2^n}\right)$

$> \left(1+\frac{1}{2}\right)+\left(\frac{1}{4}+\frac{1}{4}\right)+\cdots+\left(\frac{1}{2^n}+\cdots+\frac{1}{2^n}\right)$

$= 1+\frac{n}{2} > \frac{n}{2}$

$\Rightarrow \langle x_{2^n} \rangle \to \infty \Rightarrow \langle x_n \rangle \to \infty$

$\langle x_n \rangle$ is unbounded. Hence by MCT $\langle x_n \rangle$ is divergent.

Example-4 $x_1 = 1$, $x_n = 1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}$ $\forall n \geq 2$

Step-1 $x_{n+1} = 1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{(n+1)!}$

$x_{n+1}-x_n = \frac{1}{(n+1)!}$ $\forall n \geq 2$ $\Rightarrow \langle x_n \rangle$ is $\uparrow$.

Step-1 $n \geq 2$, $n! = 1\cdot 2\cdot 3\cdot 4\cdots n$

$> 1\cdot 2\cdot 2\cdot 2\cdots 2$

$\geq 2^{n-1}$

$\Rightarrow \frac{1}{n!} < \frac{1}{2^{n-1}}$ $\forall n \geq 2$

$x_n = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots\frac{1}{n!}$

$< 1+\left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots+\frac{1}{2^{n-1}}\right)$

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$$= 1 + \frac{\left(1-\frac{1}{2^n}\right)}{1-\frac{1}{2}}$$

$$\Rightarrow 1 + 2\left(1-\frac{1}{2^n}\right) = 1+2-\frac{2}{2^n} = 3-\frac{1}{2^{n-1}} < 3$$

$\forall n \in \mathbb{N}$

$x_n \geq 2 \quad \forall \quad n > 2$

$2 < x_n < 3 \quad \forall \quad n \geq 2$

$\Rightarrow \quad \langle x_n \rangle$ is bounded.

Hence $\langle x_n \rangle \uparrow$ & bounded.

$\Rightarrow$ By MCT, $\langle x_n \rangle$ is convergent

Example:- 5 $\quad x_1 = 1 \quad x_{n+1} = \frac{1}{5}(4x_n + 3) \quad \forall \quad n \geq 1$

Show that x_n is convergent & $\langle x_n \rangle \to 3$.

Sol:- Step-1 $\quad \langle x_n \rangle \uparrow \Leftrightarrow x_{n+1} \geq x_n \quad \forall \; n \in \mathbb{N}$

We use principle of mathematical induction.

Let $P(n) : x_{n+1} \geq x_n \quad \forall \; n \in \mathbb{N}$

$P(1) : x_2 \geq x_1$

$: \frac{1}{5}(4+3) \geq 1$

$x_2 = \frac{7}{5} > 1 = x_1$

$\Rightarrow P(1)$ is true.

Assume $P(K)$ is true, i.e. $x_{K+1} > x_K$

To prove that $x_{K+2} > x_{K+1}$ ie $P(K+1)$ is true.

$\frac{1}{5}(4x_{K+1} + 3)$

We have, $x_{K+1} > x_K$

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$$4x_{k+1} > 4x_k$$

$$\Rightarrow 4x_{k+1} + 3 > 4x_k + 3$$

$$\Rightarrow \frac{1}{5}(4x_{k+1} + 3) > \frac{1}{5}(4x_k + 3)$$

$$\Rightarrow x_{k+2} > x_{k+1}$$

Hence, by principle of induction, $P(x)$ is true $\forall n$.

Step-2 $\langle x_n \rangle$ is bounded.

$\Rightarrow x_1 = 1$, $x_2 = \frac{7}{5} < 2$ $x_3 = \frac{43}{25} < 2$

i.e $x_n \leq 3 \quad \forall \; n \in \mathbb{N}$

Let $P(x) = x_n \leq 3 \quad \forall \; n \in \mathbb{N}$

$P(1) = x_1 = 1 < 3 \Rightarrow P(1)$ is true

Let $P(K)$ be true. i.e. $x_k \leq 3$

To prove that $P(K+1)$ is true i.e. $x_{k+1} \leq 3$

$x_k \leq 3$ (by Hypothesis)

$$\Rightarrow 4x_k \leq 4 \cdot 3 = 12.$$

$$\Rightarrow 4x_k + 3 \leq 15$$

$$\Rightarrow \frac{1}{5}(4x_k + 3) \leq 3$$

$$\Rightarrow x_{k+1} \leq 3$$

Hence, by principle of induction, $P(x)$ is true, $\forall n \in \mathbb{N}$.

$\Rightarrow x_n \leq 3 \quad \forall \; n \in \mathbb{N}$

Hence, by MCT, $\langle x_n \rangle$ is convergent

let $\langle x_n \rangle \to l$

$\langle x_n \rangle \to l \Leftrightarrow \langle x_{n+1} \rangle \to l$

Since, $x_{n+1} = \frac{1}{5}(4x_n + 3)$ we let $n \to \infty$

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$$\lim_{n\to\infty} x_{n+1} = \frac{1}{5}\left(4\left(\lim_{n\to\infty} x_n\right)+3\right)$$

$$l = \frac{1}{5}(4l+3)$$

$$5l = 4l+3$$

$$\boxed{l=3}$$

Example:- Prove that $\langle a_n \rangle$ is convergent, where $a_n = \left(1+\frac{1}{n}\right)^n \; \forall n \in \mathbb{N}$

and $2 < \lim_{n\to\infty} a_n < 3$.

Solution:- $a_n = \left(1+\frac{1}{n}\right)^n \; \forall \; n \in \mathbb{N}$. Using the binomial theorem,

$$(a+b)^n = a^n + {}^nC_1 a^{n-1} b + {}^nC_2 a^{n-2} b^2 + {}^nC_3 a^{n-3} b^3 + \dots + {}^nC_n a^{n-n} b^n.$$

we get, $a_n = 1 + n\left(\frac{1}{n}\right) + \frac{n(n-1)}{2!}\left(\frac{1}{n^2}\right) + \dots + \frac{n(n-1)(n-2)\dots(n-(n-1))}{(n-1)!}\left(\frac{1}{n}\right)^{n-1}$
$+ \left(\frac{1}{n}\right)^n$

$$\Rightarrow 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \dots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{n-1}{n}\right)$$

$$\Rightarrow 2 < a_n < 1+1+\frac{1}{2!}+\frac{1}{3!}+\dots+\frac{1}{n!} \quad \forall n \in \mathbb{N}$$

$$< 1+1+\frac{1}{2}+\frac{1}{2^2}+\dots+\frac{1}{2^{n-1}} \quad \forall n \in \mathbb{N}$$

$$\left(\because n! = 1\cdot2\cdot3\cdot4\dots n \geq 1\cdot2\cdot2\cdot2\dots2 = 2^{n-1}\right) \Rightarrow \frac{1}{n!} \leq \frac{1}{2^{n-1}}$$

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$$\Rightarrow 2 < a_n < 1 + \left[\frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}}\right] = 1 + 2\left[1 - \left(\frac{1}{2}\right)^n\right]$$

$$= 1 + 2 - \left(\frac{1}{2^{n-1}}\right)$$

$$< 3 \quad \left[\because \frac{1}{2^{n-1}} > 0 \ \forall\ n \geq 1\right]$$

$$\Rightarrow 2 < a_n < 3 \quad \forall\ n \in \mathbb{N} \quad \text{— (1)}$$

~~Hence, by Squeeze Theorem, we get~~ Hence, by taking Limits, we get,

$$\lim_{n\to\infty} (2) < \lim_{n\to\infty} a_n < \lim_{n\to\infty} (3)$$

$$2 < \lim_{n\to\infty} a_n < 3.$$

From (1), $\langle a_n \rangle$ is bounded.

We claim that $\langle a_n \rangle$ is monotonic increasing.

$$a_{n+1} - a_n = 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \frac{1}{3!}\left(1 - \frac{2}{n}\right) + \dots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n}{n}\right)$$

$$+ \frac{1}{(n+1)!}\left[\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n}{n+1}\right)\right] - a_n$$

$$= \frac{1}{(n+1)!}\left[\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n}{n+1}\right)\right]$$

$a_{n+1} > a_n \quad \forall\ n \in \mathbb{N} \quad \Rightarrow$ our claim follows.

Note:-

(1) $\lim_{n\to\infty}\left[1 + \frac{1}{n}\right]^n = e$

$$\boxed{(1 + \varepsilon)^{\infty} = e}$$

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② $\lim_{n\to\infty} a_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots \frac{1}{n!} + \cdots = e$

Euler Number $= e$

Examples:- Define the sequence $\langle S_n \rangle$ as follows :-
$S_1 > 0$ is any real number.
$S_{n+1} = \frac{1}{2}\left(S_n + \frac{a}{S_n}\right)$, where $a>0$

Show that $\langle S_n \rangle$ is convergent & $\lim_{n\to\infty} S_n = \sqrt{a}$.

Proof:- Step-1 $\langle S_n \rangle$ is monotonic decreasing
$S_{n+1} - S_n = \frac{1}{2}\left(S_n + \frac{a}{S_n}\right) - S_n \Rightarrow \left(\frac{1}{2} - 1\right) S_n + \frac{a}{2}\left(\frac{1}{S_n}\right)$

$\Rightarrow \frac{-1}{2}(S_n) - \frac{a}{2S_n} = \frac{-S_n^2 + a}{2S_n} \leq 0$

$S_n^2 \geq a \iff S_n \geq \sqrt{a} \quad \forall n \geq 1$

$S_{n+1} = \frac{1}{2}\left(S_n + \frac{a}{S_n}\right) \quad \forall n \geq 1$

$\Rightarrow 2S_n S_{n+1} = S_n^2 + a \quad \forall n \geq 1$

$\Rightarrow S_n^2 - 2S_n S_{n+1} + a = 0 \qquad$ - eq - ①

It is a quadratic equation in S_n = S_n is a root of eq ①

But $S_n > 0 \ \forall n \Rightarrow$ discriminant of ① is ≥ 0

ie $(-2S_{n+1})^2 - 4a > 0$

$= 4S_{n+1}^2 \geq 4a \Rightarrow S_{n+1}^2 \geq a > 0 \quad \forall n \geq 1$

& $S_1 > 0$

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Step-2 $\langle S_n \rangle$ is bounded.

$S_1 > 0$, $S_2 > 0$, ..., $S_n > 0$, ... $\Rightarrow S_n > 0 \ \forall n$

$S_n \geq \sqrt{a} \quad \forall n$

Hence $\langle S_n \rangle$ is monotone decreasing and bounded below.

Therefore, this implies $\rightarrow \langle S_n \rangle$ is convergent. [by MCT Coro-2]

let $\lim_{n\to\infty} S_n = l \Rightarrow \lim_{n\to\infty} S_{n+1} = l$

Take limit as $n \to \infty$ in the equation.

$$S_{n+1} = \frac{1}{2}\left(S_n + \frac{a}{S_n}\right)$$

So, we have $\lim_{n\to\infty}(S_{n+1}) = \frac{1}{2}\left[\lim_{n\to\infty} S_n + \frac{a}{\lim_{n\to\infty}(S_n)}\right]$

$$l = \frac{1}{2}\left(l + \frac{a}{l}\right)$$

$$2l = \frac{l^2 + a}{l}$$

$$2l^2 = l^2 + a$$

$$l^2 = a$$

$$l = \sqrt{a}$$

Example:- Define the sequence $\langle y_n \rangle$ as follows:-

$y_1 = 1$

$y_n = -\sqrt{2|y_{n-1}|} \quad \forall n \geq 2$

Show that $\lim_{n\to\infty}(y_n) = -2$

Proof:- First we show that $\langle y_n \rangle$ is convergent.

Step-1 $\langle y_n \rangle$ is bounded. i.e. $1 \leq |y_n| \leq 2 \quad \forall n$

$|y_n| \geq 1 \quad \forall n$ (by definition)

for $|y_n| \leq 2 \quad \forall n$ we use induction.

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$n=1$, $|y_1| = 1 < 2 \Rightarrow |y_n| \le 2$ for $n=1$
Assume that $|y_K| \le 2$. Then
$2|y_K| \le 2 \cdot 2 \Rightarrow \sqrt{2|y_K|} \le \sqrt{4} = 2$
$= |y_{K+1}| \le 2$

Hence Principle of mathematical induction, $|y_n| \le 2 \quad \forall n \in \mathbb{N}$

Step-2 $\langle y_n \rangle$ is monotone decreasing.
We use induction,
We have $y_1 = 1 > -\sqrt{2} = y_2$
Assume that $y_{K+1} < y_K \Rightarrow 2|y_{K+1}| \ge 2|y_K|$
(As $-y_n \in \mathbb{R}^+ \ \forall n \in \mathbb{N}$) $\Rightarrow \sqrt{2|y_{K+1}|} \ge \sqrt{2|y_K|}$
OR ($y_n < 0 \ \forall n \ge 2$) $\Rightarrow -\sqrt{2|y_{K+1}|} \le \sqrt{2|y_K|}$
$\Rightarrow y_{K+2} \le y_{K+1}$

Hence, by MCT, $\langle y_n \rangle$ is monotone convergent.

let $\lim_{n\to\infty} y_n = l \Rightarrow \lim_{n\to\infty} y_{n-1} = l$.

$\lim_{n\to\infty} y_n = -\sqrt{2\left|\lim_{n\to\infty} [y_{n-1}]\right|}$

$\Rightarrow l = -\sqrt{2|l|}$
$\Rightarrow l^2 = 2|l|$
$\Rightarrow |l|^2 - 2|l| = 0$
$\Rightarrow |l|(|l| - 2) = 0$
$\Rightarrow l = 0, \ l = -2, \ l = 2$

But $y_n < 0 \ \forall n \ge 2$ & $\langle y_n \rangle \downarrow$
$\Rightarrow \underline{l = -2}$

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Section 3.4 Subsequences and the Bolzano-Weiorstrass Theorem

Let $\langle a_n \rangle$ be a sequence of real numbers. Let $n_1 < n_2 < n_3 < \dots < n_K < \dots$ be a strictly increasing sequence of natural number. Then the sequence $\langle a_{n_K} \rangle$ is called a subsequence of $\langle a_n \rangle$

Note :- 1) $\{ a_{n_K} : n_K \in \mathbb{N} \} \subseteq \{ a_n : n \in \mathbb{N} \}$
2) Every sequence is a subsequence of itself.

Example :- (1) Let $x_n = (-1)^n \ \forall \ n \in \mathbb{N}$. Then $\langle x_n \rangle$ is a squence of real numbers which has following subsequence :-

(a) $\langle x_{2n} \rangle = \{x_2, x_4, x_6, x_8, \dots\} = \{1, 1, 1, 1, \dots\} = \langle 1 \rangle$,
$2 < 4 < 6 < 8, \dots \Rightarrow$ strictly increasing.

(b) $\langle x_{2n-1} \rangle = \{x_1, x_3, x_5, x_7, \dots\} = \{-1, -1, -1, -1, \dots\} = \langle -1 \rangle$ is a subsequence because $1 < 3 < 5 < 7, \dots$

(c) Subsequence of prime :- $\langle x_2, x_3, x_5, x_7, x_{11}, x_{13}, \dots \rangle$
$\Rightarrow 2 < 3 < 5 < 7 < 11 < 13 \dots$ increasing and doesn't depend on element of subsequence.

(2) $\langle x_n \rangle = \frac{1}{n} \ \forall \ n \in \mathbb{N}$. Then the following are subsequences of $\langle x_n \rangle$:-

(a) $\langle x_n \rangle$ is also a subsequence of $\langle x_n \rangle$ because $1 < 2 < 3 < 4 \dots$
(b) $\langle x_{2n} \rangle = \langle \frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \dots \rangle$ because subscripts $2 < 4 < 6 < 8, \dots$

(c) $\langle x_{n+1} \rangle = \langle \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots \rangle$ because $2 < 3 < 4 < 5 < \dots$

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ⓓ $\langle x_{2n-1} \rangle = \langle 1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \dots \rangle$ [$\because 1<3<5<7<9 \dots$]

ⓔ $\langle x_{2^n} \rangle = \langle x_2, x_4, x_8, x_{16}, \dots \rangle = \langle \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots \rangle$

③ $x_n = b^n$, $0, b<1$ has the following subsequence.
ⓐ $\langle b^{2n} \rangle$
ⓑ $\langle b^{2n-1} \rangle$
ⓒ $\langle b^{4n} \rangle$

④ $x_n = \left(\sin \frac{1}{n}\right)$ $\forall n \in \mathbb{N}$ has the following subsequence:-

$$x_{n_k} = \sin\left(\frac{1}{1+2^n}\right)$$

Ques:-1 Is $\langle \frac{1}{2}, 1, \frac{1}{4}, \frac{1}{3}, \frac{1}{6}, \dots \rangle$ is a subsequence of $\langle \frac{1}{n} \rangle$?

No, as $\langle x_2, x_1, x_4, x_3, x_6 \rangle$ does not implies, $2<1<4<3$

hence $\langle x_{n_k} \rangle$ is not a subsequence.

Ques:-2 Is $\langle 1, 0, \frac{1}{3}, 0, \frac{1}{5} \rangle$ is a subsequence of $\langle \frac{1}{n} \rangle$?

Ans:- No, $0 \in \langle x_n \rangle$

Lemma:- If $\langle x_{n_k} \rangle$ is a subsequence of $\langle x_n \rangle$ then $n_k \geq K$ $\forall$ $K \in \mathbb{N}$.

Let $P(K) = n_K \geq K$ $\forall$ $K \in \mathbb{N}$

$P(1) = n_1 \geq 1$ as every natural number is greater than or equal to 1.

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Assume that $P(K)$ is true, i.e.

$n_K \geq K$

To show $P(K+1)$ is true i.e.

$n_{K+1} \geq K+1$

By induction hypothesis ① we get,

$n_K \geq K$

Since $\langle x_{n_K} \rangle$ is a sequence of $\langle x_n \rangle$, we must have,

$n_1 < n_2 < n_3 < \dots < n_K < n_{K+1}$

$\Rightarrow n_{K+1} > n_K \geq K \Rightarrow n_{K+1} \geq K$

$\Rightarrow n_{K+1} \geq K+1$

Hence, by PMI, $P(K)$ is true $\forall K \in \mathbb{N}$

Theorem :- If $\langle x_n \rangle \to x$, then every subsequence of $\langle x_{n_K} \rangle$ of $\langle x_n \rangle$ converges to x.

Proof :- Given $\langle x_n \rangle \to x$ for each $\varepsilon > 0$, $\exists K \in \mathbb{N}(\varepsilon)$ such that $|x_n - x| < \varepsilon \quad \forall n \geq K(\varepsilon)$

From lemma, we have $n_K \geq K$ for all natural number, k

$\Rightarrow |x_{n_K} - x| < \varepsilon \quad \forall n_K \geq K \geq K(\varepsilon)$

Corollary :- If $\langle x_n \rangle$ is a sequence of real numbers which satisfies.

either i) $\langle x_n \rangle$ has two convergent sequence $\langle x_{n_K} \rangle$ & $\langle x_{r_K} \rangle$ whose limits are not equal &

or (ii) $\langle x_n \rangle$ is unbounded.

Then $\langle x_n \rangle$ is divergent.

Proof :- In (i) the proof follows by the previous theorem.

In (ii) the proof follows by MCT.

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Example-1 $\langle 1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, \ldots \rangle$ convergent ?

Example-2 $\langle \sin x \rangle$ is divergent.

Theorem 3.4.7 Monotone Subsequence Theorem:-
If $\langle x_n \rangle$ is a sequence of real number then there exists a subsequence of $\langle x_n \rangle$ that is monotone.

'OR'

Every sequence of real number has a monotonic subsequence.

Proof:- First note the following

(i) A term x_k of a sequence $\langle x_n \rangle$ is called the "greatest" term if $\boxed{x_n \leq x_k \quad \forall \; n \in \mathbb{N}}$

→ It is not necessary for a sequence to have a greatest term.
Eg:- $\langle n \rangle$ has no greatest term.

→ Also if a sequence has a greatest term, then it is not necessary that the greatest term is unique.
Eg:- $\langle x_n \rangle = (-1)^n$ has $x_2, x_4, x_6 - -$ as its greatest term.

(ii) If $\langle x_{n_k} \rangle$ is a subsequence of $\langle x_n \rangle$ &

x_{n_p} = Greatest term of $\langle x_{n_k} \rangle$

x_m = Greatest term of $\langle x_n \rangle$

Then, $\boxed{x_{n_p} \leq x_m}$

Now, Let $\langle x_n \rangle$ be the given sequence.

Define $S_0 = (x_1, x_2, x_3, \ldots)$

$S_1 = (x_2, x_3, x_4, \ldots)$

$S_2 = (x_3, x_4, x_5, \ldots)$

$S_n = (x_{n+1}, x_{n+2}, x_{n+3} \ldots)$

and so on.

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Then $S_0, S_1, S_2, \ldots S_n \ldots$ are subsequence of $\langle x_n \rangle$.

<u>Case-1</u> Each subsequence $S_0, S_1, S_2, \ldots$ has a greatest term.
We show that $\langle x_n \rangle$ has an decreasing subsequence.

Let x_{n_1} = 1st greatest term of S_0
x_{n_2} = 1st greatest term of S_{n_1}, (note that $x_{n_2} \geq x_{n_3}$, $n_2 < n_3$)
x_{n_3} = " " " S_{n_2} ($x_{n_3} \geq x_{n_4}$, $x_3 < x_4$)
x_{n_4} =

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Theorem 3.4.8 (Bolzano - Weierstrass Theorem)
A bounded sequence of real numbers has a convergent subsequence.

Proof:- Let $\langle x_n \rangle$ be a sequence of real number. Then by monotone subsequence theorem (state and prove here) $\exists$ a monotone subsequence of $\langle x_n \rangle$. Let $\langle x_{n_k} \rangle$ be a monotonic subsequence of $\langle x_n \rangle$

Let $\langle x_n \rangle$ be bounded. Then $\exists\ M \in \mathbb{R}$ such that,

$$|x_n| \leq M \quad \forall \quad n \in \mathbb{N}$$

Since $\langle x_{n_k} \rangle$ is a subsequence of $\langle x_n \rangle$ every term x_{n_k} is a term of $\langle x_n \rangle$. Therefore,

$$|x_{n_k}| \leq M \quad \forall \quad n_k \in \mathbb{N}$$

$\Rightarrow$ $\langle x_{n_k} \rangle$ is a bounded sequence. Hence we get that $\langle x_{n_k} \rangle$ is a bounded, monotonic sequence.

By MCT, " ______ ", $\langle x_{n_k} \rangle$ is convergent. $\square$

Theorem 3.4.4
Let $\langle x_n \rangle$ be a sequence of real numbers & $x \in \mathbb{R}$
Then the following are equivalent :-

i) $\langle x_n \rangle$ does not converges to x
ii) $\exists$ an $\varepsilon_0 > 0$ such that for any $k \in \mathbb{N}$, $\exists\ n_k \geq k$ which satisfies, $|x_{n_k} - x| \geq \varepsilon_0$
iii) $\exists$ an $\varepsilon_0 > 0$ and a subsequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ such that $|x_{n_k} - x| \geq \varepsilon_0 \quad \forall \quad k \in \mathbb{N}$

Proof :- We show that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i)

(i) $\Rightarrow$ (ii) Given $\langle x_n \rangle$ does not converge to x, i.e
$\langle x_n \rangle \not\to x$

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Let $\varepsilon > 0$ be an arbitrary real number.
If $\langle x_n \rangle \to x$ then $\exists$ $K \in \mathbb{N}$ (depending on ε)
such that, $|x_n - x| < \varepsilon$ $\forall$ $n \geq K$
But $\langle x_n \rangle \not\to x \Rightarrow \exists$ $\varepsilon_0 >$ such that no natural number K can be found such that $|x_n - x| < \varepsilon_0$ $\forall$ $n \geq K$.
This implies that for each $K \in \mathbb{N}$
The inequality $|x_n - x| < \varepsilon_0$ is not true $\forall$ $n \geq K$.
$\Rightarrow$ for each $K \in \mathbb{N}$ $\exists$ $n_K \in \mathbb{N}$ such that
$n_K \geq K$ & $|x_{n_K} - x| \geq \varepsilon_0$ $\Rightarrow$ (2)

(2) $\Rightarrow$ (3)
Suppose (2) holds. Then $\exists$ an $\varepsilon_0 > 0$ $\ni$ for any $K \in \mathbb{N}$ $\exists$ $n_K \geq K$ which satisfies,

$$|x_{n_K} - x| \geq \varepsilon_0$$

Let $n_1 \in \mathbb{N}$ such that $|x_{n_1} - x| \geq \varepsilon_0$ $\forall$ $n_1 \geq 1$ $(K=1)$
let $n_2 \in \mathbb{N}$ be such that $n_2 > n_1$ &
$|x_{n_2} - x| \geq \varepsilon_0$ $\forall$ $n_2 \geq 2$ $(> n_1 \geq 1)$ and
Let $n_3 \in \mathbb{N}$ be such that $n_3 > n_2$
$|x_{n_3} - x| \geq \varepsilon_0$
and so on continuing like this we get a subsequence
$\langle x_{n_K} \rangle = (x_{n_1}, x_{n_2}, x_{n_3}, \ldots)$ where
$n_1 < n_2 < n_3 < \ldots$
satisfying $|x_{n_K} - x| \geq \varepsilon_0$ $\Rightarrow$ (3)

(3) $\Rightarrow$ (1)
Suppose $\langle x_{n_K} \rangle$ is a subsequence of $\langle x_n \rangle$ which satisfies (3)
If $\langle x_n \rangle \to x$ then $\langle x_{n_K} \rangle \to x$ (By Thm 3.4.2)
But this is not possible because by (3)
$|x_{n_K} - x| \geq \varepsilon_0$ $\forall$ $K \in \mathbb{N}$
Hence $\langle x_n \rangle \not\to x$

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Theorem 3.4.9 :-
Let $\langle x_n \rangle$ be a bounded sequence of real numbers and let $x \in \mathbb{R}$. Suppose every convergent subsequence of $\langle x_n \rangle$ converges to x. Then $\langle x_n \rangle$ converges to x.

Proof:- Given that $\langle x_n \rangle$ is bounded, so their exists a real number $M \in \mathbb{R}$ such that,

$$|x_n| \leq M \quad \forall \; n \in \mathbb{N}$$

Suppose $\langle x_n \rangle \not\to x$

From Theorem 3.4.4 (1) $\Rightarrow$ (3), $\exists \; \varepsilon_0 > 0$ & a subsequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ such that

$$|x_{n_k} - x| \geq \varepsilon_0 \quad \forall \; k \in \mathbb{N} \quad - (*)$$

Since $|x_n| \leq M \;\; \forall \; n \in \mathbb{N}$, we have $|x_{n_k}| \leq M \;\; \forall \; k \in \mathbb{N}$
Now $\langle x_{n_k} \rangle$ is a sequence of real number which is bounded. So, by Bolzano Weierstrass Theorem, we obtain that $\langle x_{n_k} \rangle$ has a convergent subsequence.
Let $\langle y_{n_k} \rangle$ be the subsequence of $\langle x_{n_k} \rangle$ which is convergent. By assumption,

$$\langle y_{n_k} \rangle \to x$$

From $(*)$ we have, $|y_{n_k} - x| \geq \varepsilon_0 \quad \forall \; k \in \mathbb{N}$
Because $\{y_{n_k}\}_{k \in \mathbb{N}} \subseteq \{x_{n_k}\}_{k \in \mathbb{N}}$
But this is a contradiction to the fact that

$$\langle y_{n_k} \rangle \to x$$

Hence the assumption that $\langle x_n \rangle \not\to x$ is wrong.
Thus, $\langle x_n \rangle \to x$ □.

Remark:- We know that of $\langle x_n \rangle \to x$. Then,
1) $\langle x_n \rangle$ is bounded &
2) Every subsequence $\langle x_{n_k} \rangle$ of $\langle x_n \rangle$ converges to x.
So, the converse of Theorem 3.4.9 is also true.

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Limit Superior And Limit Inferior :-

Lemma :-

Let $A, B \subseteq \mathbb{R}$ be non-empty such that $A \subseteq B$. Then,

(1) $\sup A \leq \sup B$

(2) $\inf A \geq \inf B$

Let $\langle x_n \rangle$ be a sequence of real number which is bounded above. Then $\exists$ $K \in \mathbb{R}$ such that $x_n \leq K$ $\forall$ $n \in \mathbb{N}$

Consider the set,

$$\{x_n, x_{n+1}, x_{n+2}, \ldots\} \quad \forall \ n \in \mathbb{N}$$

$n=1 \quad = \{x_1, x_2, x_3, \ldots\}$

$n=2 \quad = \{x_2, x_3, x_4, \ldots\}$

$n=3 \quad = \{x_3, x_4, x_5, \ldots\}$ and so on.

Now, $\{x_n, x_{n+1}, x_{n+2}, \ldots\} \subseteq \mathbb{R}$ $\forall$ $n \in \mathbb{N}$ is bounded above. Then, therefore by completeness property, it has a supremum. Let us define,

$$\bar{x}_n = \sup\{x_n, x_{n+1}, x_{n+2}, \ldots\} \quad \forall \ n \in \mathbb{N}$$

Since, $\{x_{n+1}, x_{n+2}, x_{n+3}, \ldots\} \subseteq \{x_n, x_{n+1}, x_{n+2}, \ldots\}$

$$A \subseteq B$$

$\therefore$ By lemma, $\sup\{x_{n+1}, x_{n+2}, x_{n+3}, \ldots\} \leq \sup\{x_n, x_{n+1}, x_{n+2}, \ldots\}$

$\Rightarrow$ $\bar{x}_{n+1} \leq \bar{x}_n$ $\forall$ $n \in \mathbb{N}$

$\Rightarrow$ $\langle \bar{x}_n \rangle$ is a monotonically decreasing sequence of real number.

$\Rightarrow$ Either $\langle \bar{x}_n \rangle$ converges or divergence to '$-\infty$'

We define, $\lim\limits_{n\to\infty} \sup \langle x_n \rangle = \lim\limits_{n\to\infty}(\bar{x}_n)$

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Similarly :-

Let $\langle x_n \rangle$ be a sequence of $\mathbb{R}$ which is bounded below. Then $\exists$ $K' \in \mathbb{R}$ such that,

$$x_n \geq K' \quad \forall \ n \in \mathbb{N}$$

Consider the set, $\{x_n, x_{n+1}, x_{n+2}, \dots\} \ \forall \ n \in \mathbb{N}$

$n=1 \Rightarrow \{x_1, x_2, x_3, \dots\}$

$n=2 \Rightarrow \{x_2, x_3, x_4, \dots\}$

$n=3 \Rightarrow \{x_3, x_4, x_5, \dots\}$ and so on.

Now $\{x_n, x_{n+1}, x_{n+2}, \dots\} \subseteq \mathbb{R} \ \forall \ n \in \mathbb{R}$ is bounded below.
Therefore, by completeness property, it has a infimum.
Let us define,

$$\underline{x}_n = \inf\{x_n, x_{n+1}, x_{n+2}, \dots\} \ \forall \ n \in \mathbb{N}$$

Since $\underbrace{\{x_{n+1}, x_{n+2}, x_{n+3}, \dots\}}_{A} \subseteq \underbrace{\{x_n, x_{n+1}, x_{n+2}, \dots\}}_{B}$

By lemma, $\inf\{x_{n+1}, x_{n+2}, x_{n+3}, \dots\} \geq \inf\{x_n, x_{n+1}, x_{n+2}, \dots\}$

$\Rightarrow \quad \underline{x}_{n+1} \geq \underline{x}_n \quad \forall \ n \in \mathbb{N}$

$\Rightarrow \quad \langle \underline{x}_n \rangle$ is monotonically increasing sequence of real number.

$\Rightarrow$ Either $\langle \underline{x}_n \rangle$ converges or diverges to $+\infty$

We define $\text{Lim inf} \langle x_n \rangle = \lim\limits_{n \to \infty} (\underline{x}_n)$

Definition :-

1) Let $\langle x_n \rangle$ be a sequence of real numbers which is bounded above. Then limit superior of $\langle x_n \rangle$ denoted by $\text{Lim sup} \langle x_n \rangle$ is denoted by the equation.

$$\lim \sup \langle x_n \rangle = \lim_{n\to\infty} (\bar{x}_n) = \begin{cases} l, & l \in \mathbb{R} \\ -\infty \end{cases}$$

If $\langle x_n \rangle$ is not bounded above, then we define,

$$\lim \sup \langle x_n \rangle = +\infty$$

(2) Let $\langle x_n \rangle$ be a sequence of real numbers which is bounded below. Then limit infimum of $\langle x_n \rangle$ denoted by $\lim \inf \langle x_n \rangle$ is defined by the equation.

$$\lim \inf \langle x_n \rangle = \lim_{n\to\infty} (\underline{x_n}) = \begin{cases} l, & l \in \mathbb{R} \\ +\infty \end{cases}$$

If $\langle x_n \rangle$ is not bounded below, then we define

$$\lim \inf \langle x_n \rangle = -\infty$$

~~Note:- If $\langle x_n \rangle$ is not bounded below, then we define~~

Note :- 1) Both $\lim \sup \langle x_n \rangle$ & $\lim \inf \langle x_n \rangle$ belongs to the set $\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$

2) $\langle \bar{x}_n \rangle$ is monotone decreasing.
$\langle \underline{x_n} \rangle$ is monotone increasing.

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Given a sequence $\langle x_n \rangle$ we define :-

$\bar{x}_n = \sup \{x_n, x_{n+1}, x_{n+2}, \dots\} \quad \forall n \in \mathbb{N}$

& $\underline{x}_n = \inf \{x_n, x_{n+1}, x_{n+2}, \dots\} \quad \forall n \in \mathbb{N}$

Then $\langle \bar{x}_n \rangle$ & $\langle \underline{x}_n \rangle$ are two ~~types~~ sequences of real numbers. we define

Note:- $\text{Lim sup } x_n = \lim_{n \to \infty} (\bar{x}_n) = \begin{cases} l \\ -\infty \end{cases}$

$\text{Lim inf } x_n = \lim_{n \to \infty} (\underline{x}_n) = \begin{cases} m \\ +\infty \end{cases}$

Note:- $\text{Lim sup } x_n = \inf \{\bar{x}_1, \bar{x}_2, \bar{x}_3, \dots\}$

$\text{Lim inf } x_n = \sup \{\underline{x}_1, \underline{x}_2, \underline{x}_3, \dots\}$

Examples:- Find limsup & lim inf of $\langle x_n \rangle$ where

(1) $x_n = (-1)^n, \quad n \in \mathbb{N}$

$\bar{x}_1 = \sup \{x_1, x_2, x_3, x_4\} = \sup \{-1, 1, -1, 1, -1, \dots\}$

$= \sup \{-1, 1\} = 1$

$\bar{x}_n = \sup \{x_n, x_{n+1}, x_{n+2}, x_{n+3}, \dots\} = \sup \{+1, -1, 1, -1 \dots\}$

$= \sup \{-1, 1\} = 1 \quad \forall n \in \mathbb{N}$

$\therefore \langle \bar{x}_n \rangle = \langle 1 \rangle \to 1$ Therefore, $\text{Lim sup } x_n = 1$

Similarly,

$\underline{x}_n = \inf \{x_n, x_{n+1}, x_{n+2}, \dots\} = \inf \{1, -1, 1, -1, \dots\}$

$= \inf \{-1, 1\} = 1 \quad \forall n \in \mathbb{N}$

$\therefore \langle \underline{x}_n \rangle = \langle -1 \rangle \to -1$ Therefore $\text{Lim inf } x_n = -1$

(2) $x_1 = 1 = x_4 = x_7 \dots \qquad \langle x_n \rangle = \langle 1, 3, 5 \rangle$

$x_2 = 3 = x_5 = x_8 \dots$

$x_3 = 5 = x_6 = x_9 \dots$

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$\bar{x}_1 = \sup\{x_1, x_2, x_3, \cdots\} = \sup\{1,3,5,1,3,5,\cdots\} = \sup\{1,3,5\} = 5$

$\bar{x}_2 = \sup\{x_2, x_3, x_4 \cdots\} = \sup\{3,5,1,3,5,\cdots\} = \sup\{1,3,5\} = 5$

$\bar{x}_n = \sup\{x_n, x_{n+1}, x_{n+2}, \cdots\} = \max\{1,3,5\} = 5 \ \forall \ n \in \mathbb{N}$

$\underline{x}_1 = \inf\{x_1, x_2, x_3 \cdots\} = \inf\{1,3,5,1,3,5\cdots\} = \min\{1,3,5\} = 1$

$\underline{x}_2 = \inf\{x_2, x_3, x_4 \cdots\} = \inf\{3,5,1,\cdots\} = \min\{1,3,5\} = 1$

$\langle \underline{x}_n \rangle = 1$

$\langle \bar{x}_n \rangle = \langle 5 \rangle \to 5 \Rightarrow \limsup x_n = 5$

$\langle \underline{x}_n \rangle = \langle 1 \rangle = 1 \Rightarrow \liminf x_n = 1$

(3) $x_n = (-2)^{-n}\left(1 + \frac{1}{n}\right)$

$\bar{x}_1 = \sup\{x_1, x_2, x_3, \cdots\} = \sup\left\{-1, \frac{3}{8}, \frac{-1}{6}, \frac{5}{64}, \frac{-3}{80}, \frac{7}{384} \cdots\right\}$

As $\frac{3}{8} > \frac{5}{64} > \frac{7}{384}$, $\bar{x}_1 = \frac{3}{8}$

$\bar{x}_2 = \sup\{x_2, x_3, x_4, \cdots\} = \sup\left\{\frac{3}{8}, \frac{-1}{6}, \frac{5}{64}, \frac{-3}{80}, \cdots\right\} = \frac{3}{8}$

$\bar{x}_3 = \frac{5}{64}$

$\bar{x}_1 = 3/8 = \bar{x}_2$

$\bar{x}_3 = 5/64 = \bar{x}_4$

$\underline{x}_1 = \inf\left\{-1, \frac{3}{8}, \frac{-1}{6}, \frac{5}{64}, \cdots\right\} = -1$

$\underline{x}_2 = \inf\left\{\frac{3}{8}, \frac{-1}{6}, \frac{5}{64}, \frac{-3}{80}, \cdots\right\} = \frac{-1}{6}$

$\underline{x}_1 = -1$

$\underline{x}_2 = \frac{-1}{6} = \underline{x}_3$

$\underline{x}_4 = \frac{-3}{80} = \underline{x}_5$

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$\bar{x}_1 = \bar{x}_2 > \bar{x}_3 = \bar{x}_4 > \bar{x}_5 = \bar{x}_6 > \cdots$

$\underline{x}_1 < \underline{x}_2 = \underline{x}_3 < \underline{x}_4 = \underline{x}_5 < \underline{x}_6 = \underline{x}_7 < \cdots$

$$\limsup x_n = \inf\{\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \ldots\}$$
$$= \inf\left\{\frac{3}{8}, \frac{5}{64}, \frac{7}{384}, \ldots\right\} = 0$$

$$\liminf x_n = \sup\{\underline{x}_1, \underline{x}_2, \underline{x}_3, \underline{x}_4, \ldots\}$$
$$= \sup\left\{-1, \frac{-1}{6}, \frac{-3}{80}, \ldots\right\} = 0$$

4) $x_n = n$

Theorem 3.4.13

Let $\langle a_n \rangle$ be a sequence of real numbers. Then,

$$\lim_{n\to\infty} a_n = l \quad \text{iff} \quad \limsup a_n = l = \liminf a_n$$

(Here, $l \in \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$)

Proof :- Case-1 : $l \in \mathbb{R}$

Suppose $\lim_{n\to\infty} a_n = l$. For each $\varepsilon > 0$, $\exists\ K \in \mathbb{N}$ such that,

$$|a_n - l| < \varepsilon \quad \forall\ n \geq K$$

$$\Leftrightarrow \quad l - \varepsilon < a_n < l + \varepsilon \quad \forall\ n \geq K$$

By definition, $\bar{a}_n = \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\}$

$$\therefore \quad l - \varepsilon < \bar{a}_n < l + \varepsilon \quad \forall\ n \geq K$$

Let $n \to \infty$, Then $\lim_{n\to\infty}(l-\varepsilon) < \lim_{n\to\infty}(\bar{a}_n) < \lim_{n\to\infty} l+\varepsilon$ $\cancel{\forall n \geq K}$

$$= \quad l - \varepsilon < \limsup a_n < l + \varepsilon \quad \cancel{\forall n \geq K}$$

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If $\lim\sup(a_n) = M$, then $l-\varepsilon < M < l+\varepsilon$
$\Rightarrow 0 \le |M-l| < \varepsilon$
Since $\varepsilon > 0$ is arbitrary. $\Rightarrow M-l=0 \Rightarrow M=l$

$$\boxed{\lim\sup a_n = l}$$

Case-2, By definition,

$\underline{a_n} = \inf\{a_n, a_{n+1}, a_{n+2}, \dots\}$

$\therefore\ l-\varepsilon < \underline{a_n} < l+\varepsilon \quad \forall\ n \ge k$

Let $n\to\infty$. Then $\lim_{n\to\infty}(l-\varepsilon) < \lim_{n\to\infty} \underline{a_n} < \lim_{n\to\infty}(l+\varepsilon)$ $\cancel{\forall n \ge k}$

$= l-\varepsilon < \lim\inf a_n < l+\varepsilon$ $\cancel{\forall n \ge k}$

If $\lim\inf a_n = m$, then $l-\varepsilon < m < l+\varepsilon$
$\Rightarrow 0 \le |m-l| < \varepsilon$
Since $\varepsilon > 0$ is arbitrary. $\Rightarrow m-l=0 \Rightarrow m=l$

Hence, $\lim\inf a_n = l$

From both the cases we prove that,

$$\boxed{\lim\sup a_n = l = \lim\inf a_n}$$

Suppose, $\lim\sup a_n = \lim\inf a_n = l$. To prove $\lim_{n\to\infty} a_n = l$

$\lim\sup a_n = l \iff \lim_{n\to\infty}(\bar{a}_n) = l$ —①

$\lim\inf a_n = l \iff \lim_{n\to\infty}(\underline{a_n}) = l$ —②

From —①, we get, for each $\varepsilon>0$, $\exists\ k_1 \in \mathbb{N}$ such that
$|\bar{a}_n - l| < \varepsilon \quad \forall\ n \ge k_1$

From —②, we get, for each $\varepsilon>0$, $\exists\ k_2 \in \mathbb{N}$ such that,
$|\underline{a_n} - l| < \varepsilon \quad \forall\ n \ge k_2$

Let $k = \max\{k_1, k_2\}$. Then for each $\varepsilon>0$ we get

$|\bar{a}_n - l| < \varepsilon \quad \forall\ n \ge k \ge k_1$

$|\underline{a_n} - l| < \varepsilon \quad \forall\ n \ge k \ge k_2$

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$\Rightarrow \quad l-\epsilon < \bar{a}_n < l+\epsilon \quad \forall\, n \geq K$
$\quad \& \quad l-\epsilon < \underline{a}_n < l+\epsilon \quad \forall\, n \geq K$

But $\quad a_n \leq \bar{a}_n \quad \& \quad a_n \geq \underline{a}_n \quad \forall\, n$

$\therefore \quad l-\epsilon < \underline{a}_n \leq a_n \leq \bar{a}_n < l+\epsilon \quad \forall\, n \geq K$

$\Rightarrow \quad l-\epsilon < a_n < l+\epsilon \quad \forall\, n \geq K$

$\Rightarrow \quad |a_n - l| < \epsilon \quad \forall\, n \geq K.$

Case-2 $\quad l = +\infty$

To prove that $\lim\limits_{n\to\infty} a_n = +\infty$ iff $\limsup a_n = \liminf a_n = \infty$

Suppose, $\lim\limits_{n\to\infty} a_n = \infty$. Then $\langle a_n \rangle$ is not bounded above.

$\Rightarrow \quad \boxed{\limsup a_n = \infty}$ [By definition.]

Also, $\langle a_n \rangle$ is not bounded above implies, that for every $M \in \mathbb{R}^+$ $\exists\, K \in \mathbb{N}$ such that $a_n \geq M \quad \forall\, n \geq K$

$\Rightarrow \quad \inf\{a_n, a_{n+1}, a_{n+2}, \ldots\} \geq M \quad \forall\, n \geq K$

$\Rightarrow \quad \underline{a}_n \geq M \quad \forall\, n \geq K$

$\Rightarrow \quad \lim\limits_{n\to\infty} (\underline{a}_n) \geq M \quad \Rightarrow \quad \lim\limits_{n\to\infty} (\underline{a}_n) = +\infty$

$\Rightarrow \quad \boxed{\liminf a_n = +\infty}$

Conversely, suppose $\limsup a_n = \liminf a_n = \infty$

To prove that $\lim\limits_{n\to\infty} a_n = \infty$

As, $\liminf a_n = +\infty \quad \Rightarrow \quad \lim\limits_{n\to\infty} (\underline{a}_n) = +\infty \quad \Rightarrow \quad \lim\limits_{n\to\infty} (\underline{a}_n) \geq M$

where $M \in \mathbb{R}^+$, $\quad \exists\, K \leq n$

$\Rightarrow \quad \underline{a}_n \geq M \quad \forall\, n \geq K$

$\Rightarrow \quad \inf\{a_n, a_{n+1}, a_{n+2}, \ldots\} \geq M \quad \forall\, n \geq K$

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$\Rightarrow \quad a_n \geq M \quad \forall \quad n \geq k$

$\Rightarrow \quad \langle a_n \rangle$ is not bounded above.

$\Rightarrow \quad \lim_{n\to\infty} a_n = \infty$

Case-3 : $l = -\infty$

To prove that, $\lim_{n\to\infty} a_n = +\infty$ iff $\lim\sup a_n = \lim\inf a_n = -\infty$

Suppose $\lim_{n\to\infty} a_n = -\infty$. Then $\langle a_n \rangle$ is not bounded above.