Complex Number

A number of the form, $z = x + iy$ is called a complex number, where $i = \sqrt{-1}$

→ x & y are real numbers.

The set of complex numbers is denoted by $\mathbb{C}$.

Thus $\mathbb{C} = \{x + iy : x, y \in \mathbb{R}, i = \sqrt{-1}\}$

For any $x \in \mathbb{R}$, we have, $x = x + 0i \in \mathbb{C}$.

$\Rightarrow \mathbb{R} \in \mathbb{C}$.

If $z = x + iy \in \mathbb{C}$, then x is the real part of z. & y is the imaginary part of z.

Denote $x = \mathrm{Re}(z)$, $y = \mathrm{Im}(z)$

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ belongs to $\mathbb{C}$. Then

$z_1 = z_2$ iff $x_1 = x_2$ and $y_1 = y_2$

We have,

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$$

$$\frac{1}{z_1} = \frac{1}{x_1 + iy_1} \times \frac{(x_1 - iy_1)}{(x_1 - iy_1)} = \left[\frac{x}{x^2 + y^2}\right] - i\left[\frac{y}{x^2 + y^2}\right]$$

$$\frac{z_1}{z_2} = \left[\frac{x_1 + iy_1}{x_2 + iy_2}\right] \times \left[\frac{x_2 - iy_2}{x_2 - iy_2}\right] = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2^2 + y_2^2)}$$

$$\Rightarrow \left[\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}\right] + i\left[\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}\right]$$

Properties :-

1) $z_1 + z_2 = z_2 + z_1$

2) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$

3) $\exists\ 0 = 0 + i(0) \in \mathbb{C}$ such that $z_1 + 0 = z_1 = 0 + z_1$

4) $\forall\ z \in \mathbb{C}$, $\exists\ (-z) \in \mathbb{C}$ s.t $z+(-z) = 0 = (-z)+z$

5) $z_1 z_2 \in \mathbb{C}$.

6) $z_1(z_2 z_3) = (z_1 z_2) z_3$

7) $\exists\ 1 = 1 + i(0) \in \mathbb{C}$ such that,

$$z \cdot 1 = z = 1 \cdot z$$

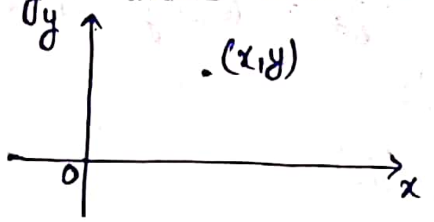
8) $\forall\ z \in \mathbb{C}\ \exists\ \frac{1}{z} \in \mathbb{C}$ such that,

$$z \cdot \frac{1}{z} = 1 = \frac{1}{z} \cdot z \quad , \quad z \neq 0$$

9) $z_1 \cdot z_2 = z_2 \cdot z_1$

10) $z_1 \cdot (z_2 + z_3) = z_1 z_2 + z_1 z_3$

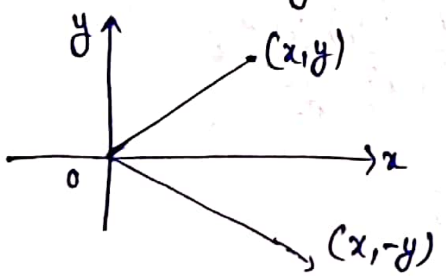
Geometrically, corresponding to every complex number $z = x+iy$, there exists a point (x,y) in the cartesian plane.



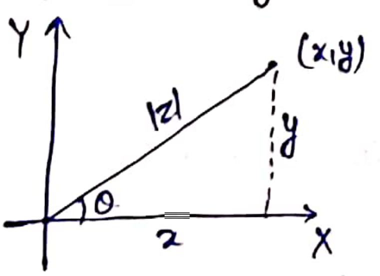
Also, every point in cartesian plane $\exists$ a unique complex number.

→ Conjugate of complex Number :-

Given $z = x+iy \in \mathbb{C}$, its conjugate is, $\bar{z} = x - iy$



→ Modulus And Argument of a complex Number :-



Given a complex number $z = x + iy$ its modulus is denoted by $|z|$ and is given by,

$$|z| = x^2 + y^2 = \text{dist. of } P(x,y) \text{ from origin.}$$

$$|z| \geq 0 \text{ and } |z| = 0 \text{ iff } z = 0$$

$$\cos\theta = \frac{x}{|z|} \qquad \sin\theta = \frac{y}{|z|}$$

$\Rightarrow$ $x = |z|\cos\theta$ $\qquad y = |z|\sin\theta$

$$z = x + iy = |z|(\cos\theta + i\sin\theta)$$

and $\tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{y}{x}$

$$\boxed{\theta = \tan^{-1}\left(\frac{y}{x}\right)} \leftrightarrow \text{Argument of } z$$

where $\theta \in [0, \pi]$ $\rightarrow$ Principal Argument.

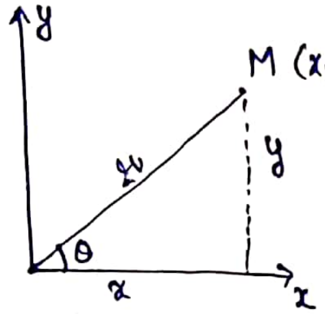
Properties :-

1) $-|z| \leq \text{Re}(z) \leq |z|$, $-|z| \leq \text{Im}(z) \leq |z|$
2) $|z| = |-z| = |\bar{z}|$
3) $|z_1 + z_2| \leq |z_1| + |z_2|$ [Triangle Inequality]
4) $z\bar{z} = |z|^2$
5) $|z_1 z_2| = |z_1||z_2|$
6) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$
7) $|z_1| - |z_2| \leq |z_1 - z_2| \leq |z_1| + |z_2|$

Polar representation of Complex Numbers :-

Let $M(x,y)$ be a point in the co-ordinate plane (or argand plane) different from origin.

The real number $r = \sqrt{x^2+y^2}$ is called polar radius of M, and the angle $\theta, \in [0, 2\pi)$ which ray OM makes with positive direction of x-axis is called polar argument of M.



The pair (r, θ) is called polar co-ordinate of M, and we write $M(r,\theta)$. The point at origin has $r=0$ and the argument of origin is not defined (or in some books say $0°$). From the figure,

$$\frac{x}{r} = \cos\theta \quad , \quad \frac{y}{r} = \sin\theta$$

$$\Rightarrow \boxed{\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}}$$

Given, the polar co-ordinate (r,θ) we can find the cartesian co-ordinate of a point using above formula.

Conversely, given the cartesian co-ordinate (x,y) of a point, we have, $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

Case-1 when $x=0, y \neq 0$

Then $\theta = \tan^{-1}\left(\frac{y}{x}\right) = \begin{cases} \pi/2 & \text{if } y>0 \\ -\pi/2 = 3\pi/2 & \text{if } y<0 \end{cases}$

Case-II : When $x \neq 0$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) + K\pi$$

where

$$K = \begin{cases} 0 & x \geq 0, \; y \geq 0 \\ 1 & x < 0, \; \text{any } y \\ 2 & x > 0, \; y < 0 \end{cases}$$

K=1	K=0
K=1	K=2

Ques:- Find the polar Co-ordinate of the points :-

1) $M_1(2,-2)$; $r = \sqrt{(2)^2+(-2)^2} = \sqrt{8} = 2\sqrt{2}$

$\theta = \tan^{-1}\left(\frac{-2}{2}\right) + 2\pi = \tan^{-1}(-1) + 2\pi$

$= -\frac{\pi}{4} + 2\pi = \frac{7\pi}{4}$

$M_1(2\sqrt{2}, 7\pi/4)$

2) $M_2(-1,0)$, $r = \sqrt{(-1)^2} = 1$

$\theta = \tan^{-1}\left(\frac{0}{-1}\right) + \pi = \pi$

$M_2(1, \pi)$

③ $M_3(-2\sqrt{3},-2)$, $r = \sqrt{12+4} = \sqrt{16} = 4$

$\theta = \tan^{-1}\left[\frac{-2}{-2\sqrt{3}}\right] + \pi = \frac{7\pi}{6}$

$M_3\left[4, \frac{7\pi}{6}\right]$

④ $M_4(\sqrt{3},1)$ $\Rightarrow M_4\left(2, \frac{\pi}{6}\right)$

$r = \sqrt{3+1} = 2$

$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

⑥ $M_6(-2,2)$

$r = 2\sqrt{2}$

$\theta = \tan^{-1}(-1) + \pi$

$= \frac{3\pi}{4}$

$M_6\left(2\sqrt{2}, \frac{3\pi}{4}\right)$

⑤ $M_5(3,0)$ =

$r = 3$ $\theta = \tan^{-1}(0) = 0$

⑦ $M_7(0,1)$ $r = 1$, $\theta = \tan^{-1}\left(\frac{1}{0}\right) = \pi/2$

⑧ $M_8(0,-4)$ = $r = 4$, $\theta = 3\pi/2$

Q- Find the cartesian co-ordinate from the polar co-ordinate?

$M_1\left(2, \frac{2\pi}{3}\right)$ $r=2$, $\theta=\frac{2\pi}{3}$

$x = r\cos\theta = 2\cos\frac{2\pi}{3} = -1$

$y = r\sin\theta = 2\sin\frac{2\pi}{3} = \sqrt{3}$

$M_2\left(3, \frac{7\pi}{4}\right)$ $r=3$, $\theta=\frac{7\pi}{4}$

$x = r\cos\theta = 3\cos\frac{7\pi}{4} = \frac{3}{\sqrt{2}}$

$y = r\sin\theta = 3\sin\frac{7\pi}{4} = \frac{-3}{\sqrt{2}}$

$M_3(1,1)$, $r=1$, $\theta=1$

$x = 1\cos 1 = \cos 1$

$y = \sin 1$

Any complex number $z = x+iy$ can be written as,

$$z = r(\cos\theta) + i(r\sin\theta)$$
$$= r(\cos\theta + i\sin\theta)$$

where $r \in [0,\infty)$ and $\theta \in [0,2\pi)$

The polar argument 'θ' is called argument of z. and is denoted by '$\arg z$'. The polar radius 'r' is equals to $|z|$. For any $\phi = \theta + 2k\pi$ we have,

$$z = r(\cos\theta + i\sin\theta)$$
$$= r[\cos(\phi - 2k\pi) + i\sin(\phi - 2k\pi)]$$
$$= r[\cos\phi + i\sin\phi]$$

The set $\text{Arg}\, z = \{\theta + 2k\pi, k \text{ is integer}\}$ is called extended argument of z

Find the polar representation of the numbers :-

$z_1 = -1-i$ $\quad$ $z_5 = 2i$ $\quad$ Also, find their extended argument.

$z_2 = 2+2i$ $\quad$ $z_6 = -1$

$z_3 = -1+i\sqrt{3}$ $\quad$ $z_7 = 2$

$z_4 = 1-i\sqrt{3}$ $\quad$ $z_8 = -3i$

1) $|z_1| = \sqrt{2}$, $\theta = \tan^{-1}(1) + \pi \Rightarrow \pi/4 + \pi = \frac{5\pi}{4}$

Polar Representation, :- $\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

Extended Argument :- $\left\{\frac{5\pi}{4} + 2k\pi \mid k \in \mathbb{Z}\right\}$

2) $|z_2| = 2\sqrt{2}$ $\quad$ $\theta = \tan^{-1}(1) = \pi/4$

Polar Representation :- $2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

Extended Argument :- $\left\{\frac{\pi}{4} + 2k\pi \mid k \in \mathbb{Z}\right\}$

(3) $|z_3| = 2$ $\quad$ $\theta = \tan^{-1}(-\sqrt{3}) + \pi \Rightarrow -\frac{\pi}{3} + \pi = \frac{2\pi}{3}$

$z_3 = 2\left[\cos\frac{2\pi}{3} + i\sin\left(\frac{2\pi}{3}\right)\right]$

(4) $|z_4| = 2$ $\quad$ $\theta = \tan^{-1}(-\sqrt{3}) + 2\pi \Rightarrow -\frac{\pi}{3} + 2\pi = \frac{5\pi}{3}$

$z_4 = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$

(5) $|z_5| = 2$ $\quad$ $\theta = \tan^{-1}\left(\frac{2}{0}\right) = \frac{\pi}{2}$

$z_5 = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

Extended Argument = $\left\{\frac{\pi}{2} + 2k\pi \mid k \in \mathbb{Z}\right\}$

(6) $|z_6| = 1$ $\quad$ $\theta = \tan^{-1}\left(\frac{0}{-1}\right) + \pi = \pi$

$z_6 = 1\left[\cos\pi + i\sin\pi\right]$

Extended Argument :- $\left\{\pi + 2k\pi \mid k \in \mathbb{Z}\right\}$

7) $|z_7| = 2$ $\quad \theta = \tan^{-1}\left(\frac{0}{2}\right) = 0$

Polar Representation :- $2(\cos 0 + i\sin 0)$

Extended Argument :- $\{0 + 2k\pi \mid k \in \mathbb{Z}\}$

$= \{2k\pi \mid k \in \mathbb{Z}\}$

8) $|z_8| = 3$ $\quad \theta = \tan^{-1}\left(\frac{-3}{0}\right) + 2\pi = \frac{3\pi}{2}$

$z_8 = 3\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$

Extended Argument $= \left\{\frac{3\pi}{2} + 2k\pi \mid k \in \mathbb{Z}\right\}$

Q1:- Find the polar representation of

$z = 1 + \cos a + i\sin a \quad , a \in (0, 2\pi)$

Case-1 :- when $a \in (0, \pi)$

Then, $P(1+\cos a, \sin a)$ lies in first quadrant.

$|z| = \sqrt{(1+\cos a)^2 + \sin^2 a}$

$= \sqrt{1 + \cos^2 a + 2\cos a + \sin^2 a}$

$= \sqrt{2(1+\cos a)} = \sqrt{2\cdot 2\cos^2 a/2} = 2\left|\cos a/2\right|$

$\frac{a}{2} \in \left(0, \frac{\pi}{2}\right)$

$\theta = \tan^{-1}\left(\frac{\sin a}{1+\cos a}\right) = \tan^{-1}\left[\frac{2\sin a/2 \cos a/2}{2\cos^2 a/2}\right]$

$= \tan^{-1}\left(\tan\left(a/2\right)\right) = a/2$

$z = 2\cos\frac{a}{2}\left[\cos\frac{a}{2} + i\sin\frac{a}{2}\right]$

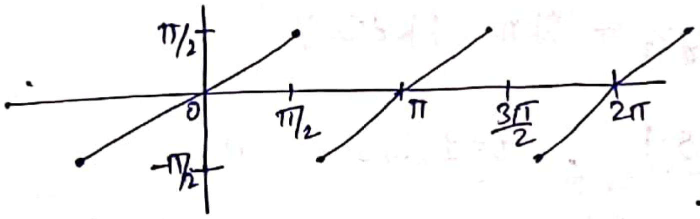
Case-2 When $a \in (\pi, 2\pi)$, then

$P(1+\cos a, \sin a)$ lies in fourth quadrant.

$\Rightarrow \frac{a}{2} \in \left[\frac{\pi}{2}, \pi\right]$

$$\theta = \tan^{-1}\left(\frac{\sin a}{1+\cos a}\right) = \tan^{-1}\left(\tan\frac{a}{2}\right) + 2\pi \Rightarrow \frac{a}{2} - \pi + 2\pi = \frac{a}{2} + \pi$$

$$= \tan^{-1}(\tan x)$$



$$\tan^{-1}(\tan x) = \begin{cases} x, & -\pi/2 < x < \pi/2 \\ x-\pi & \frac{\pi}{2} < x < 3\pi/2 \\ x-2\pi & \frac{3\pi}{2} < x < 2\pi \end{cases}$$

$$z = -2\cos\frac{a}{2}\left[\cos\left(\frac{a}{2}+\pi\right) + i\sin\left(\frac{a}{2}+\pi\right)\right]$$

Case-3 When $a = \pi$, then $z = 0$

Remarks :-

1) $|z_1 z_2| = \sqrt{(r_1 r_2)^2 \cos^2(\theta_1+\theta_2) + (r_1 r_2)^2 \sin^2(\theta_1+\theta_2)}$

$= r_1 \cdot r_2 = |z_1||z_2|$

2) Argument of $z_1 z_2$

$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) - 2k\pi$

where $k = \begin{cases} 0 & \text{for } \arg(z_1) + \arg(z_2) < 2\pi \\ 1 & \text{for } \arg(z_1) + \arg(z_2) > 2\pi \end{cases}$

3) Extended Argument :-

$\arg(z_1 z_2) = \{ \arg z_1 + \arg z_2 + 2k\pi \mid k \in \mathbb{Z} \}$

4) Let $z_i = r_i(\cos\theta_i + i\sin\theta_i)$, $i = 1, 2, 3, ----, n$

Then,

$z_1 z_2 ---- z_n = r_1 r_2 ----r_n \left[\cos(\theta_1+\theta_2+---+\theta_n) + i\sin(\theta_1+\theta_2+---+\theta_n)\right]$

Example :-

Let $z_1 = 1-i$ $\quad z_2 = \sqrt{3}+i$

Then $|z_1| = r_1 = \sqrt{(1)^2+(-1)^2} = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) + 2\pi = \tan^{-1}(-1) + 2\pi = -\tan^{-1}(1) + 2\pi$

$= -\frac{\pi}{4} + 2\pi = \frac{7\pi}{4}$

$z_1 = \sqrt{2}\cos\left[\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right]$

$z_2 = \sqrt{2}\left[\cos\left(\frac{\pi}{6}\right) + i\sin\frac{\pi}{6}\right]$

$z_1 z_2 = 2\sqrt{2}\left[\cos\left(\frac{7\pi}{4}+\frac{\pi}{6}\right) + i\sin\left(\frac{7\pi}{4}+\frac{\pi}{6}\right)\right]$

$= 2\sqrt{2}\left[\cos\left(\frac{23\pi}{12}\right) + i\sin\left(\frac{23\pi}{12}\right)\right]$

De-Moivre's Theorem :-

Let $z = r(\cos\theta + i\sin\theta)$

Then $z^n = r^n(\cos n\theta + i\sin n\theta)$, $n \in \mathbb{N}$

Proof:- Take $z_1 = z_2 = \dots = z_n = z$

Then for $n=1$ $\quad z^1 = r(\cos\theta + i\sin\theta)$

So, result holds for $n=1$.

For $n=2$ $\quad z^2 = [r(\cos\theta + i\sin\theta)][r(\cos\theta + i\sin\theta)]$

$$= r^2[\cos^2\theta - \sin^2\theta + i(\sin\theta\cos\theta + \sin\theta\cos\theta)]$$

$$= r^2[\cos 2\theta + i\sin 2\theta]$$

True for $n=2$ also.

Let the result be true for $n=k$,

i.e, $z^k = r^k(\cos k\theta + i\sin k\theta)$

For $n=k+1$

$$z^{k+1} = z^k \cdot z \Rightarrow [r^k[\cos k\theta + i\sin k\theta]][r(\cos\theta + i\sin\theta)]$$

$$= r^{k+1}(\cos(k+1)\theta + i\sin(k+1)\theta) \quad \text{[By using case for } n=2\text{]}$$

So, the result is true for $n=k+1$ also. Thus, by Mathematical induction, result holds for all n.

Remarks:- 1) $|z^n| = r^n = |z|^n$

2) If $r=1$, then $z^n = \cos n\theta + i\sin n\theta$, $n \in \mathbb{N}$

3) Extended Argument,

$$\text{Arg}(z^n) = \{n\arg(z) + 2k\pi \mid k \in \mathbb{Z}\}$$

Example:- Compute $(1+i)^{1000}$

$$|z| = \sqrt{2} = r$$

$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

$$z = \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]$$

$$z^{1000} = (\sqrt{2})^{1000}\left[\cos\left[1000\frac{\pi}{4}\right] + i\sin\left(1000\frac{\pi}{4}\right)\right]$$

$$= 2^{500}[\cos 250\pi + i\sin 250\pi] \quad \left[\because \begin{array}{l}\cos n\pi = (-1)^n \\ \sin n\pi = 0\end{array}\right]$$

$$= 2^{500}$$

Ques:- Prove that:-

$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

$\cos 5\theta = 16\cos^5\theta - 20\cdot\cos^3\theta + 5\cos\theta$

Let $z = r(\cos\theta + i\sin\theta)$

Take $r = 1$

$z^5 = (\cos\theta + i\sin\theta)^5$

$z^5 = (\cos\theta + i\sin 5\theta)$

$\Rightarrow (\cos 5\theta + i\sin 5\theta) = z^5 = (\cos\theta + i\sin\theta)^5$

$\Rightarrow {}^5C_0\cos^5\theta + {}^5C_1\cos^4\theta(i\sin\theta) - {}^5C_2\cos^3\theta\sin^2\theta - {}^5C_3\cos^2\theta(i\sin^3\theta) + {}^5C_4\cos\theta\sin^4\theta$

$+ {}^5C_5 i\sin^5\theta = (\cos 5\theta + i\sin 5\theta)$

$(\cos 5\theta + i\sin 5\theta) = \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$

Comparing real and imaginary part:-

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$

$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

Now, $\cos 5\theta = \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta + 5\cos^5\theta - 10\cos^3\theta$

$\cos 5\theta = (16\cos^5\theta - 20\cos^3\theta + 5\cos\theta)$

Division of Complex Number:-

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Then $\dfrac{z_1}{z_2} = \dfrac{r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - i\sin\theta_2)}$

$\Rightarrow \dfrac{r_1}{r_2}\left[\cos\theta_1\cos\theta_2 - i\sin\theta_2\cos\theta_1 + i\sin\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\right]$

$\Rightarrow \dfrac{r_1}{r_2}\left[\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \sin\theta_2\cos\theta_1)\right]$

$= \dfrac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) - i\sin(\theta_1-\theta_2)\right]$

Remarks :-

1) $\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$

2) <u>Extended Argument</u> :-

$\text{Arg}\left(\frac{z_1}{z_2}\right) = \{ \arg(z_1) - \arg(z_2) + 2k\pi \mid k \in \mathbb{Z} \}$

3) Let $z_1 = 1$, $z_2 = z = r(\cos\theta + i\sin\theta)$

Then $\frac{1}{z} = z^{-1} = \frac{1}{r}(\cos(0-\theta) + i\sin(0-\theta))$

$= \frac{1}{r}(\cos(-\theta) + i\sin(-\theta)) = \underline{\frac{1}{r}(\cos\theta - i\sin\theta)}$

4) $z^{-n} = r^{-n}(\cos n\theta - i\sin n\theta)$ * Does not depend on n either even or odd

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nth roots of a complex Number :-

Let $n \geq 2$ and let $z_0 \neq 0$ be a complex number

→ Then the equation $\boxed{z^n - z_0 = 0}$ defines the nth roots of z_0.

→ Any solution z of the equation $z^n - z_0 = 0$ is called n^{th} root of z_0.

<u>Theorem</u> :-

Let $z_0 = r(\cos\theta + i\sin\theta)$, where $r > 0$ and $\theta \in [0, 2\pi)$

The number z_0 has n distinct roots given by,

$$z_k = r^{\frac{1}{n}}\left[\cos\left(\frac{2k\pi+\theta}{n}\right) + i\sin\left(\frac{2k\pi+\theta}{n}\right)\right] \quad , k = 0, 1, \ldots, n-1.$$

<u>Ques</u>:- Find the cube roots of $z = 1+i$

i.e. to find $z^{1/3} = (1+i)^{\frac{1}{3}}$

$|1+i| = \sqrt{2}$ $\qquad \theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$

$(1+i) = \sqrt{2}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]$

$(1+i)^{\frac{1}{3}} = (\sqrt{2})^{\frac{1}{3}}\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right]^{\frac{1}{3}}$

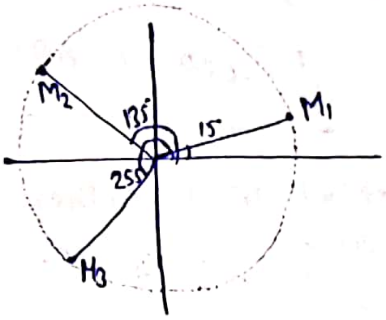
$= (2)^{\frac{1}{6}}\left[\cos\left[\frac{2k\pi + \frac{\pi}{4}}{3}\right] + i\sin\left[\frac{2k\pi + \frac{\pi}{4}}{3}\right]\right]$

If $K=0$, $\Rightarrow (2)^{\frac{1}{6}}\left[\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right] \to M_1$

$K=1$ $\quad (2)^{\frac{1}{6}}\left[\cos\left(\frac{3\pi}{4}\right) + i\sin\frac{3\pi}{4}\right] \to M_2$

$K=2$ $\quad (2)^{\frac{1}{6}}\left[\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right] \to M_3$

$M_1\left(2^{\frac{1}{6}}, \frac{\pi}{12}\right)$ $\quad M_2\left[(2)^{\frac{1}{6}}, \frac{3\pi}{4}\right)$ $\quad M_3\left(2^{\frac{1}{6}}, \frac{17\pi}{12}\right)$



nth roots of unity :-

Let $\quad z=1$

Eq^n $\quad z^n - z_0 = 0$ because,

$\rightarrow z^n - 1 = 0$

$\Rightarrow z^n = 1$

$\Rightarrow z = (1)^{\frac{1}{n}}$

$r = 1$

$\theta = \tan^{-1}\left[\frac{0}{1}\right] = 0$

$1 = \cos 0^\circ + i\sin 0^\circ$

$\Rightarrow (1)^{\frac{1}{n}} = [\cos 0 + i\sin 0]^{\frac{1}{n}} \Rightarrow \cos\left[\frac{2K\pi}{n}\right] + i\sin\left[\frac{2K\pi}{n}\right]$

$K = 0, 1, 2, \ldots (n-1)$

$K=0 \quad = \cos 0 + i\sin 0 = 1$

$K=1 \quad = \cos\left[\frac{2\pi}{n}\right] + i\sin\left[\frac{2\pi}{n}\right] = \varepsilon$ (say)

$K=2 \quad \cos\left[\frac{4\pi}{n}\right] + i\sin\left[\frac{4\pi}{n}\right] = \left[\cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)\right]^2 = \varepsilon^2$

$\vdots$

$K=(n-1) \quad \cos 2(n-1)\frac{\pi}{n} + i\sin(2)(n-1)\frac{\pi}{n} = \varepsilon^{n-1}$

Let $U_n = \{1, \varepsilon, \varepsilon^2, \ldots, \varepsilon^{n-1}\}$

→ This set is generated by ε.

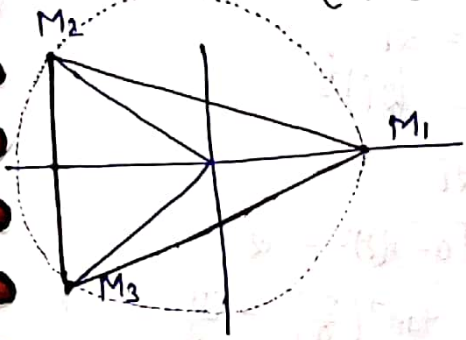
→ The nth roots of unity are the vertices of a regular polygon with n sides inscribed in a circle of radius 1.

For $n=3$

$K=0 \quad = 1$

$K=1 \quad = \cos\left[\frac{2\pi}{3}\right] + i\sin\left(\frac{2\pi}{3}\right) = \frac{-1}{2} + i\frac{\sqrt{3}}{2} \quad \to \omega$

$K=2 \quad \cos\left[\frac{4\pi}{3}\right] + i\sin\left(\frac{4\pi}{3}\right) = \frac{-1}{2} - i\frac{\sqrt{3}}{2} \quad \to \omega^2$



☆ Observation :-

$\varepsilon M_1 = M_2$ verify for $\varepsilon = \left[\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right]$

$\varepsilon^2 M_1 = M_3$

Holds for cube roots of any complex Number.

Solve the following Equations :-

① $z^4 + 16 = 0$

$\Rightarrow z^4 = -16 \quad \Rightarrow z = (-16)^{\frac{1}{4}}$

Let $z_0 = -16$

$|z_0| = |-16| = 16$

Argument of $z_0 = \pi$

$z_0 = 16\left[\cos\pi + i\sin\pi\right)$

$(-16)^{\frac{1}{4}} = 16\left[\cos\pi + i\sin\pi\right]^{\frac{1}{4}}$

$= 16^{\frac{1}{4}}\left[\cos\left(\frac{2k\pi+\pi}{4}\right) + i\sin\left[\frac{2k\pi+\pi}{4}\right]\right]$

$K = 0,1,2,3.$

$k=0 \quad 2\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right] = 2\left[\frac{1}{\sqrt{2}} + i\frac{(1)}{\sqrt{2}}\right] = \frac{1}{\sqrt{2}}(2+2i)$

$k=1 \quad 2\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right] = 2\left(\frac{-1}{\sqrt{2}} + i\frac{(1)}{\sqrt{2}}\right] = \frac{1}{\sqrt{2}}(2-2i)$

$K=2 \quad 2\left[\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right] = 2\left[\frac{-1}{\sqrt{2}} + i\left(\frac{-1}{\sqrt{2}}\right)\right] = \frac{1}{\sqrt{2}}(-2-2i)$

$K=3 \quad 2\left[\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right] = 2\left[\frac{1}{\sqrt{2}} + i\left(\frac{-1}{\sqrt{2}}\right)\right] = \frac{1}{\sqrt{2}}(2-2i)$

Ques:- $z^7 - 2iz^4 - iz^3 - 2 = 0$

$\Rightarrow z^4(z^3 - 2i) - i(z^3 - 2i) = 0 \qquad [\because 2 = -2i^2]$

$\Rightarrow (z^4 - i)(z^3 - 2i) = 0$

$\Rightarrow z^4 - i = 0 \quad , \quad z^3 - 2i = 0$

$\downarrow$ 4 Roots $\qquad \downarrow$ 3 Roots.

$z^4 - i = 0$

$\Rightarrow z^4 = i$

$\Rightarrow z = (i)^{\frac{1}{4}}$

$z_0 = i$

$|z_0| = \sqrt{0^2 + (1)^2} = 1$

$\operatorname{Arg} z_0 = \frac{\pi}{2}$

$z_0 = \left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$z = \left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]^{\frac{1}{4}}$

$= \cos\left[\frac{2k\pi + \frac{\pi}{2}}{4}\right] + i\sin\left[\frac{2k\pi + \frac{\pi}{2}}{4}\right]$

$K=0 \quad = \cos\left[\frac{\pi}{8}\right] + i\sin\left[\frac{\pi}{8}\right]$

$K=1 \quad = \cos\left[\frac{5\pi}{8}\right] + i\sin\left[\frac{5\pi}{8}\right]$

$K=2 \quad = \cos\left[\frac{9\pi}{8}\right] + i\sin\left[\frac{9\pi}{8}\right]$

$K=3 \quad = \cos\left[\frac{13\pi}{8}\right] + i\sin\left[\frac{13\pi}{8}\right]$

$z^3 - 2i = 0$

$\Rightarrow z^3 = 2i$

$\Rightarrow z = (2i)^{\frac{1}{3}}$

$z_0 = 2i$

$|z_0| = \sqrt{0^2 + (2)^2} = 2$

$\operatorname{Arg} z_0 = \tan^{-1}\left(\frac{2}{0}\right) = \frac{\pi}{2}$

$z_0 = 2\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]$

$z = 2\left[\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right]^{\frac{1}{3}}$

$= (2)^{\frac{1}{3}}\left[\cos\left(\frac{2k\pi + \frac{\pi}{2}}{3}\right) + i\sin\left[\frac{2k\pi + \frac{\pi}{2}}{3}\right]\right]$

$K=0 \quad = (2)^{\frac{1}{3}}\left[\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right]$

$K=1 \quad = (2)^{\frac{1}{3}}\left[\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right]$

$K=2 \quad = (2)^{\frac{1}{3}}\left[\cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6}\right]$

$= (2)^{\frac{1}{3}}\left[\cos\frac{3}{2}\pi + i\sin\frac{3\pi}{2}\right]$

$$z^4 = 5(z-1)(z^2-z+1)$$

$$z^4 = 5(z^3-z^2+z-z^2+z-1)$$

$$\Rightarrow z^4-5z^3+5z^2-5z+5z^2-5z+5=0$$

$$\Rightarrow z^4-5z^3+10z^2-10z+5=0$$

$$\Rightarrow \underline{z^4}-5z^3+\underline{10z^2}-10z+\underline{5}=0$$

$$\Rightarrow (z^2+5)^2-5z^2(z-2)=0$$

$$\Rightarrow (z^2+5)^2 = 5z^2(z-2)$$

Binomial Equation:-

A binomial Equation is an equation of the form,

$z^n + a = 0 \quad , \quad a \in \mathbb{C} - \{0\} \quad , n \geq 2$

Example:- $z^3 + 8 = 0$

$z^3 = -8$

$z_0 = -8$

$|z_0| = 8$

$\arg(z_0) = \tan^{-1}\left(\frac{0}{-8}\right) + \pi = \pi$

$z_0 = 8\left[\cos\pi + i\sin\pi\right]$

$(z_0)^{\frac{1}{3}} = (8)^{\frac{1}{3}}\left[\cos\left[\frac{2k\pi + \pi}{3}\right] + i\sin\left[\frac{2k\pi + \pi}{3}\right]\right]$

$k=0 \quad = 2\left[\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right]$

$k=1 \quad = 2\left[\cos\pi + i\sin\pi\right]$

$k=2 \quad = 2\left[\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right]$

② $z^6 - z^3(1+i) + i = 0$

$= z^6 - z^3 - z^3 i + i = 0$

$\Rightarrow z^3(z^3 - i) - 1(z^3 - i) = 0$

$\Rightarrow (z^3 - i)(z^3 - 1) = 0$

$z^3 - i = 0$

$z^3 = i$

$|z_0| = 1$

$\arg(z_0) = \tan^{-1}\left(\frac{1}{0}\right) = \frac{\pi}{2}$

$(z)^{\frac{1}{3}} = \cos\left[\frac{2k\pi + \frac{\pi}{2}}{3}\right] + i\sin\left[\frac{2k\pi + \frac{\pi}{2}}{3}\right]$

$k = 0, 1, 2$

$z^3 - 1 = 0$

$z^3 = 1$

$z_0 = 1$

Roots $= 1 \rightarrow M_4$

$= -\frac{1}{2} + i\frac{\sqrt{3}}{2} \rightarrow M_5$

$= -\frac{1}{2} - i\frac{\sqrt{3}}{2} \rightarrow M_6$

$K=0 \quad = \quad \cos\left[\frac{\pi}{6}\right] + i\sin\left[\frac{\pi}{6}\right] \rightarrow M_1$ R

$K=1 \quad \cos\left[\frac{5\pi}{6}\right] + i\sin\left[\frac{5}{6}\pi\right] \rightarrow M_2$

$K=2 \quad \cos\left[\frac{9\pi}{6}\right] + i\sin\left[\frac{9}{6}\pi\right] - M_3$

Solution of a Cubic Equation.

Consider the general cubic equation,

$$x^3 + bx^2 + cx + d = 0 \quad \text{①}$$

Let $x = y - \frac{b}{3}$

Then ① becomes,

$$\left(y-\frac{b}{3}\right)^3 + b\left(y-\frac{b}{3}\right)^2 + c\left(y-\frac{b}{3}\right) + d = 0$$

$$\Rightarrow y^3 - \frac{b^3}{27} - \frac{3y^2b}{3} + \frac{3yb^2}{9} + by^2 + \frac{b^3}{9} - \frac{2b^2y}{3} + cy - \frac{cb}{3} + d = 0$$

$$\Rightarrow y^3 + \left(\frac{b^2}{3} - \frac{2b^2}{3} + c\right)y + \left(\frac{b^3}{9} - \frac{b^3}{27} - \frac{cb}{3} + d\right) = 0$$

Let, $p = c - \frac{b^2}{3}$, $q = \frac{2b^3}{27} - \frac{cb}{3} + d$ ~~in~~ then

$$\Rightarrow \underline{y^3 + py + q = 0} \quad - \text{eq} - \text{②}$$

If y_1, y_2, y_3 are roots of ② then,

$x_1 = y_1 - \frac{b}{3}$ $\quad x_2 = y_2 - \frac{b}{3}$ $\quad x_3 = y_3 - \frac{b}{3}$ are roots of ①

→ Equation (2) is called <u>Reduced cubic equation.</u>

This method is given by Vieta, so called Method by Vieta

Let $y = z - \frac{p}{3z}$, Then (2) becomes,

$$\left(z - \frac{p}{3z}\right)^3 + p\left(z - \frac{p}{3z}\right) + q = 0$$

$$z^3 - \frac{p^3}{27z^3} + \frac{3zp^2}{9z} - \frac{3zp}{3} + zp - \frac{p^2}{3z} + q = 0$$

$\Rightarrow Z^3 - \frac{p^3}{27Z^3} + q = 0$

$\Rightarrow \quad z^6 + qz^3 - \frac{p^3}{27} = 0$

Let $\quad z^3 = t$

$\Rightarrow \quad t^2 + qt - \frac{p^3}{27} = 0$ Quadratic eqn with variable 't'.

$\Rightarrow \quad t = \frac{-q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2}$

$t = \frac{-q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}$

$t = \frac{-q}{2} \pm \sqrt{R} \quad , \quad R = \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3$

$Z = \sqrt[3]{\frac{-q}{2} + \sqrt{R}} \quad , \quad \sqrt[3]{\frac{-q}{2} - \sqrt{R}} \qquad (\because t = z^3)$

Let $A = \sqrt[3]{\frac{-q}{2} + \sqrt{R}} \qquad B = \sqrt[3]{\frac{-q}{2} - \sqrt{R}}$

$\Rightarrow \quad AB = \sqrt[3]{\left(\frac{-q}{2}\right)^2 - (\sqrt{R})^2} = \sqrt[3]{\left(\frac{-q}{2}\right)^2 - \left(\frac{q}{2}\right)^2 - \left(\frac{p}{3}\right)^3} = \frac{-p}{3}$

The six' value of Z are $A, \varepsilon A, \varepsilon^2 A, B, \varepsilon B, \varepsilon^2 B$

Now $\quad AB = \frac{-p}{3}$

$\varepsilon A \cdot \varepsilon^2 B = \varepsilon^3 AB = AB = \frac{-p}{3}$

$\varepsilon^2 A \cdot \varepsilon B = \varepsilon^3 AB = AB = \frac{-p}{3}$

Hence any root Z is paired with root equals to $\frac{-p}{3Z}$ and the sum of these two values is y.

$y = z + \left(\frac{-p}{3}\right)$

$AB = \frac{-p}{3}$

$\underset{z_1}{A} = \frac{-p}{3B \rightarrow z_2}$

$z_1 = \frac{-p}{3z_2}$

Hence, the 3 values of y are :-

$y_1 = A+B$

$y_2 = \varepsilon A + \varepsilon^2 B$

$y_3 = \varepsilon^2 A + \varepsilon B$

These are called Cardan's formula's.

Solve the equation :-

$y^3 - 15y - 126 = 0$

We have $y = z - \frac{p}{3z}$ — (1)

$= z - \frac{(-15)}{3z} \Rightarrow z + \frac{5}{z}$

Given equation becomes,

$\left(z+\frac{5}{z}\right)^3 - 15\left(z+\frac{5}{z}\right) - 126 = 0$

$z^3 + \frac{125}{z^3} + \cancel{15}z + \frac{\cancel{75}}{z} - \cancel{15}z - \frac{\cancel{75}}{z} - 126 = 0$

$z^3 + \frac{125}{z^3} - 126 = 0$ — (2)

$z^6 - 126z^3 + 125 = 0$

$t^2 - 126t + 125 = 0 \Rightarrow (t-125)(t-1) = 0$

$t^2 - 125t - t + 125 = 0$

$t = 125$ $t = 1$

$z = 5$ $z = 1$

let, $A = 1$ $B = 5$

Six Roots of Eq (2) $1, \varepsilon, \varepsilon^2, 5, 5\varepsilon, 5\varepsilon^2$

Hence the 3 values of y are :-

$y_1 = 1+5 = 6$

$y_2 = \varepsilon + 5\varepsilon^2$

$y_3 = \varepsilon^2 + 5\varepsilon$

$\varepsilon = \frac{-1}{2} + \frac{i\sqrt{3}}{2}$

$\varepsilon^2 = \frac{-1}{2} - \frac{i\sqrt{3}}{2}$

Descarte's solution of biquadratic eqn :-

The general ~~soluti~~ equation is,

$$x^4 + bx^3 + cx^2 + dx + e = 0$$

Let $x = y - \frac{b}{4}$, the above equation reduces to an equation of the form,

$$y^4 + py^2 + qy + r = 0$$

Let $y^4 + py^2 + qy + r = (y^2 + my + n)(y^2 - my + v)$

Ques:- Solve $x^4 - 8x^2 - 24x + 7 = 0$

Let $x^4 - 8x^2 - 24x + 7 = (x^2 + mx + n)(x^2 - mx + v)$

$= x^4 + nx^2 - m^2x^2 - mnx + vx^2 + mvx + nv$

$= x^4 + (n - m^2 + v)x^2 + (mv - mn)x + nv$

Comparing both side, we get

$n - m^2 + v = -8 \Rightarrow n + v = m^2 - 8$

$m(v - n) = -24 \Rightarrow v - n = -24/m$

$nv = 7$

$(v+n)^2 - (v-n)^2 = (m^2-8)^2 - \left(\frac{-24}{m}\right)^2$

$4vn = m^4 + 64 - 16m^2 - \frac{576}{m^2}$

$28m^2 = m^6 - 16m^4 + (64)m^2 - 576$

$\Rightarrow m^6 - 16m^4 + (64 - 28)m^2 - 576 = 0$

$\Rightarrow m^6 - 16m^4 + 36m^2 - 576 = 0$

Let $m^2 = t$

$\Rightarrow t^3 - 16t^2 + 36t - 576 = 0$

~~$\alpha\beta$~~ $\alpha + \beta + \gamma = 16$

$\alpha\beta + \beta\gamma + \alpha\gamma = 36$

$\alpha\beta\gamma = 576$

$\Rightarrow t^2(t-16) + 36(t-16) = 0$

$\Rightarrow (t^2+36)(t-16) = 0$

$t = 16$ $\quad\quad$ $t^2 = -36$

$t = \pm 6i$

$t = 16 \Rightarrow m^2 = 16$

$m = \pm 4$

Let $m = 4$

Now, $v - n = \frac{-24}{m} = -6$

$v + n = m^2 - 8 = 8$

$2v = 2 \Rightarrow v = 1$, $2n = 14 \Rightarrow \underline{n = 7}$

Now, $(x^2+mx+n)(x^2 - mx + v) = (x^2+4x+7)(x^2-4x+1)$

$x = \frac{-4 \pm \sqrt{16-28}}{2}$ $\quad\quad$ $x = \frac{4 \pm \sqrt{16-4}}{2}$

$= -2 \pm i\sqrt{3}$ $\quad\quad$ $= 2 \pm \sqrt{3}$

Four Roots :- $-2+i\sqrt{3}$

$-2-i\sqrt{3}$

$2+\sqrt{3}$

$2-\sqrt{3}$

Ques :- Solve $x^4 - 8x^3 - 12x^2 + 60x + 63 = 0$

Sol :- $x = y - \frac{b}{4} = y - \frac{(-8)}{4} = y + 2$

$(y+2)^4 - 8(y+2)^3 - 12(y+2)^2 + 60(y+2) + 63 = 0$

$\Rightarrow y^4 + 8y^3 + 6y^2(4) + 32y + 16 - 8(y^3 + 8 + 6y^2 + 12y)$

$-12(y^2 + 4 + 4y) + 60y + 120 + 63 = 0$

$\Rightarrow y^4 + 8y^3 + 24y^2 + 32y + 16 - 8y^3 - 64 - 48y^2 - 96y$
$-12y^2 - 48 - 48y + 60y + 120 - 63 = 0$

$\Rightarrow y^4 + (24 - 12 - 48)y^2 + (32 - 96 - 48 + 60)y + 16 + 120 - 64$
$-48 + 63 = 0$

$\Rightarrow y^4 - 36y^2 - 52y + 87 = 0$

$y^4 - 36y^2 - 52y + 87 = (y^2 + my + n)(y^2 - my + v)$
$= y^4 + (n - m^2 + v)y^2 + (mv - mn)y + nv$

$n - m^2 + v = (-36)$
$m(v - n) = -52$
$nv = 87$

$(v+n)^2 - (v-n)^2 = (m^2 - 36)^2 - \left(-\frac{52}{m}\right)^2$

$4vn = m^4 + 1296 - 72m^2 - \frac{2704}{m^2}$

$348 = m^4 + 1296 - 72m^2 - \frac{2704}{m^2}$

$\Rightarrow m^6 + 1296m^2 - 72m^4 - 348m^2 - 2704 = 0$

$\Rightarrow m^6 - 72m^4 + 948m^2 - 2704 = 0$

Put $m^2 = t$

$t^3 - 72t^2 + 948t - 2704 = 0$

By substituting values, we get, $t = 4$

$\Rightarrow m^2 = 4$

$\Rightarrow m = 2$

$$v - n = \frac{-52}{2} = -26$$

$$v + n = 4 - 36 = -32$$

$$2v = -58$$

$$\Rightarrow v = -29$$

$$v + n \Rightarrow -29 + n = -32$$

$$n = -3$$

$$y^4 - 36y^2 - 52y + 87 = (y^2 + 2y - 3)(y^2 - 2y - 29)$$

By using quadratic formula,

1) Solve $y^2 + 2y - 3$

$$y = \frac{-2 \pm \sqrt{4+12}}{2}$$

$$= \frac{-2 \pm 4}{2}$$

$$y = -3, 1$$

2) Solve $(y^2 - 2y - 29)$

$$y = \frac{2 \pm \sqrt{4+116}}{2}$$

$$= \frac{2 \pm \sqrt{120}}{2} = 1 \pm \sqrt{30}$$

Four values of y are $-3, 1, 1+\sqrt{30}, 1-\sqrt{30}$

Then value of $x = y + 2$

$$x_1 = -3 + 2 = -1$$

$$x_2 = 1 + 2 = 3$$

$$x_3 = 1 + \sqrt{30} + 2 = 3 + \sqrt{30}$$

$$x_4 = 1 - \sqrt{30} + 2 = 3 - \sqrt{30}$$

Section 4.1

Division Algorithm :-

Properties of Addition and multiplication :-

Let a, b, c be real numbers. Then

(i) $a+b$ and ab are real numbers.

(ii) $a+b = b+a$ and $ab = ba$

(iii) $a+(b+c) = (a+b)+c$ and $a(bc) = (ab)c$

(iv) $a+0 = a = 0+a$, $a \cdot 1 = a$

(v) ~~a+c~~ $a(b+c) = ab+ac$, $(a+b)c = ac+bc$

vi) $a+(-a) = 0$.

vii) $a \cdot \left(\frac{1}{a}\right) = 1$

viii) $a \leq b \Rightarrow a+c \leq b+c$

ix) $a \leq b$ and $c \geq 0 \Rightarrow ac \leq bc$

x) $a \leq b$ and $c \leq 0 \Rightarrow ac \geq bc$

Theorem :-

Let a and b be natural numbers. Then, there exists unique non-negative integers, with $0 \leq r < b$ such that

$$a = qb + r$$

Proof :- Consider a sequence of non-negative multiples of b i.e. $0b, 1b, 2b, 3b, \ldots$

Case-1 When 'a' is a multiple of b

$\Rightarrow a = bq$ for some $q \in \mathbb{Z}$

$\Rightarrow a = bq + 0$

$\Rightarrow a = bq + r$, with $r = 0$ — (1)

Case-2) When 'a' is not a multiple of b.

Since a is a natural number $\Rightarrow a > 0$

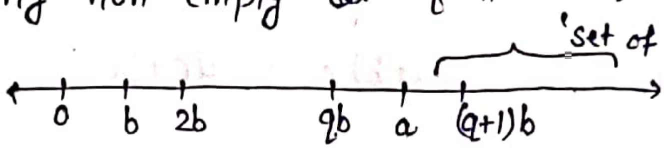
Also, b is a natural number $\Rightarrow 2b > 1$

$\Rightarrow (2b)\,a > a$

Thus $\exists$ multiples of 'b' greater than 'a'. Therefore by, well-ordering principle, the set of multiples of b, that exceeds 'a' has a smallest element, say $(q+1)b$

Remark :- <u>Well-Ordering Principle</u> :-

Any non-empty set of $\mathbb{N}$ has a smallest element.



$\Rightarrow qb < a < (q+1)b$

Now, Let $r = a - qb$

Then $qb \leq a \Rightarrow r \geq 0$

Also, $a < (q+1)b \Rightarrow a < qb + b \Rightarrow a - qb < b$

$\Rightarrow r < b$

$\Rightarrow a = qb + r$ with $0 < r < b$ — (2)

From (1) & (2)

$a = qb + r \quad , \; 0 \leq r < b$.

<u>Uniqueness</u> :-

Let $a = q_1 b + r_1 \quad , \; 0 \leq r_1 < b$

$a = q_2 b + r_2 \quad 0 \leq r_2 < b$

$\Rightarrow 0 = (q_2 - q_1)b + (r_2 - r_1)$

$\Rightarrow (q_1 - q_2)b = r_2 - r_1$

$r_2 - r_1$ is a multiple of b.

Also, $0 \leq r_1 < b \Rightarrow -b < -r_1 \leq 0$

and $0 \leq r_2 < b$

$\left.\right] \; -b < r_2 - r_1 < b$

This is possible when,

$$r_2 - r_1 = 0$$

$$\Rightarrow \quad r_2 = r_1$$

$$\Rightarrow \quad q_1 = q_2$$

The integer q is called quotient and the integer r is called remainder when a is divided by b.

__Remark :-__ When 'a' and 'b' both are +ve and a is divided by b. Then the quotient is the integer part of the number displayed by the calculator and the remainder $a - qb$

__Example :-__ $a = 589621$ & $b = 7893$

when a is divided by b, calculator shows 74.70

$\Rightarrow \quad q = 74$

$\Rightarrow \quad r = a - qb = 589621 - 74(7893)$

__Division Algorithm :-__

Let $a, b \in \mathbb{Z}$ with $b \neq 0$. Then, there exist unique integers q and r with $0 \leq r < |b|$ such that $a = bq + r$.

__Proof :-__ Case-1 When $a > 0$, $b > 0$

Then, by previous theorem $\exists$ unique integers 'q' and 'r' such that $a = bq + r$ with $0 \leq r < b = |b|$

__Case-2__ When $a > 0$, $b < 0$

Then $-b > 0$

By previous theorem, $\exists$ unique non-negative integer 'q' and 'r' ~~with~~ such that $a = q(-b) + r$ $\quad 0 < r < -b$

$$\Rightarrow a = (-q)b + r$$

Let $Q = -q \in \mathbb{Z}$

$\Rightarrow \quad a = Qb + r \quad$ with $\quad 0 \leq r < -b = |b|$

~~Ex~~

Case-3 When $a = 0$

Then $b > 0$ or $b < 0$

Then $0 = 0 \cdot b + 0$

Take $q = 0$, $r = 0$

We have $a = qb + r$

With $0 = r < |b|$

Case-4 When $a < 0$ & $b > 0$

$\Rightarrow \quad -a > 0$

By previous theorem, $\exists$ unique non-negative Integer q and r such that $-a = qb + r$ with $0 \leq r < b$

$\Rightarrow \quad a = -qb - r$

Subcase-1 When $r = 0$

Take $Q = -q$, $R = 0$

$a = Qb + R$ with $0 = R < |b|$

Subcase-2 When $r > 0$

Then $a = -qb - r$

$= -qb - r + b - b$

$= (-q-1)b + (b - r)$

Take $Q = -q-1$ $\quad R = b - r$

$\Rightarrow \quad a = Qb + R$, with $0 < R < |b|$

as $r < b \Rightarrow b - r > 0$

$\Rightarrow R > 0$

as $r<b \Rightarrow b-r>0$

$\Rightarrow R>0$

As $b-r<b \Rightarrow R<b=|b|$

<u>Case-5</u> When $a<0$, $b<0$

$\Rightarrow -b>0$

By case-4), $\exists$ q and r such that

$$a = q(-b)+r$$

with $0 \leq r < -b = |b|$

$\Rightarrow a = (-q)b+r$

Take $Q = -q \in \mathbb{Z}$

$\Rightarrow a = Qb + r, \quad 0 \leq r < |b|$

Floor function denoted by $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

$$\lfloor x \rfloor \leq x$$

Ceiling function denoted by $\lceil x \rceil$ is the least integer greater than or equal to x.

$$x \leq \lceil x \rceil$$