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ORIGINAL ARTICLE

An efficient simulation of quantum error correction codes

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Abstract The QCAD which has been technologically advanced during last few years to support basic transformations, simulating multi-dimensional quantum devices and circuits. It is most important to demonstrate the theoretical operation of quantum computer using the simulator. Besides of describing various noise models in terms of quantum gates, a direct support of quantum error correction codes. Using the quantum error correction codes, the effects of applying single bit or phase flip error to the quantum circuits are reported along with the error recovery method for different qubit codes. The results are generated using the QCAD simulator. A simulation model for evaluating single qubit error in the proposed quantum circuit is also given. The working software prototype of an extremely efficient quantum computer simulator which is an attractive and versatile tool for both research and education.

1. Introduction

With the tremendous technological advancement, the quantum information theory has gained rapid and incredible growth in realizing scalable quantum computation and communication. The most crucial requirement is to improve the theoretical details for robust and consistent quantum information processing. Quantum communication needs the error control to protect the quantum information against the effects of decoherence on quantum states. In this case, the quantum error correction code was used to protect against the particular errors in the respective systems, which also shows that even with imperfect encoding and recovery operations, these codes become advantageous. Significant contributions in quantum computing include Shor's algorithm for factorizing a large number in less computation time, when compared to classical computing [1–2]. Thus the quantum computer has the ability to resolve difficult computational problems more efficiently than the present classical computers [1–4]. The seminal paper on quantum error correction by Peter Shor reveals the quantum computation from a practical point of view. Here, he proposes the technique of how to protect the fragile quantum information against decoherence, by encoding the information into the subspace of Hilbert space larger than its own [1]. Calder Bank and Shor gave the asymptotic rates for the existence of codes, and also the upper bounds for such rates [4] (see Table 1).

The QECC is defined as the map that encodes k qubits into a subspace of the Hilbert space of n qubits. The decoding of

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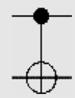
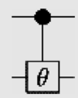
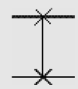
1110-0168

Table 1 Quantum Gates:

Symbol	Name	Transfer Matrix	Symbol	Name	Transfer Matrix
$\boxed{H}$	Hadamard	$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$	$\boxed{\theta}$	Rotation	$\begin{pmatrix}0 & 1\\ 1 & e^{i\theta}\end{pmatrix}$
$\boxed{X}$	Pauli X	$\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$		Controlled NOT	$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\end{pmatrix}$
$\boxed{Y}$	Pauli Y	$\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$		Controlled Rotation	$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & e^{-j\theta}\end{pmatrix}$
$\boxed{Z}$	Pauli Z	$\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$		Swap	$\begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}$

the state can be done correctly on condition that only t or fewer qubits in the encoded state possess errors. The representation for defining such codes is $[[n, k, d]]$, where d signifies the distance of the code, and the code encodes k logical qubits into n physical qubits. Subsequently, many new quantum error correcting codes (QECC) have been discovered [4–7]. Moreover quantum error correction has also been realized using various experiments [8–17]. In this paper, we claim that the simulation of qubit codes has not been done by many researchers, as this is important prior to building quantum computers. The most challenging mission is in building quantum circuits. One of the promising advancements is the reported construction and operation of the three bit quantum computer using the quantum simulator. As the development of the optimal quantum hardware architecture is unveiled, we depend on quantum computer simulators. Qubit circuit simulation is at the forefront, but qubit gates also have their benefits. The reduction in the gate count and the number of circuit levels leads to lowering the errors and the overall cost in quantum circuits.

In the next section, we study the basics of quantum computers. In section 3 we study the theoretical aspects of several qubit codes. In section 4 the results and discussion and finally the summary, are given.

2. Theoretical background

2.1. Quantum bits and register

2.1.1. Qubits and basis

The elementary unit of classical information is a bit, which can take the value of either 0 or 1. Thus, with n bits it has 2^n probable states, which can be characterized by the n-dimensional vector Z_2 (n-bit string). The basic information carriers for quantum systems are the qubits-quantum bits. The typical basis vectors are $|0\rangle$ and $|1\rangle$. The mathematical notation of the physical realization of a particular qubit is

$$|\mu\rangle = s_0|0\rangle + s_1|1\rangle = s_0\begin{pmatrix}1\\0\end{pmatrix} + s_1\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}s_0\\s_1\end{pmatrix}$$

in which the state $|\mu\rangle$ of the qubit is expressed as the superposition state of two orthonormal eigenstate expressed as the vectors $|\mu\rangle = s_0\,|0\rangle + s_1\,|1\rangle$, where s_0 and s_1 are complex coefficients, with $|s_0|^2 + |s_1|^2 = 1$. These complex coefficients within the basis $\{|0, |1\}$ are mostly denoted as computational basis. Any other orthonormal basis such as the Hadamard basis $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ would be useful various other computation.

2.1.2. Quantum register

The multiple qubit state, for the illustration of n qubits, belongs to the Hilbert space of 2^n dimensions, so there are 2^n mutually orthogonal quantum states. The 2^n mutually orthogonal states of n qubits can be expressed as $\{|i\rangle\}$, where i is a n-bit binary number.

The vector $|\mu\rangle$ is a pure state, and that is a way to define the general n qubit state of the quantum register

$$|\mu\rangle = \sum_{a=0}^{2^n-1} s_a|a\rangle = s_0\begin{pmatrix}1\\0\\0\\0\\0\\0\\0\\0\end{pmatrix} + s_1\begin{pmatrix}0\\1\\0\\0\\0\\0\\0\\0\end{pmatrix} + \dots\dots\dots + s_{2^n-1}\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\1\end{pmatrix}$$

is written as the superposition of 2^n basis state. This also indicates 2^n complex numbers s_a which must fulfill $|\mu\rangle = \sum_{a=0}^{2^n-1}|s_a|^2 = 1.$, this is often known quantum parallelism. For example, 3 qubits have $\{|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle\}$ and the states can be expressed as the linear superposition of those basis states as

$$s_0|000\rangle + s_1|001\rangle + s_2|010\rangle + s_3|011\rangle + s_4|100\rangle + s_5|101\rangle + s_6|110\rangle + s_7|111\rangle.$$

Thus, a multiple qubit state normally cannot be expressed as a single qubit state $\frac{1}{\sqrt{2}}(000\rangle + |111\rangle)$. It cannot be decomposed in this way. Such a state is defined as entangled. These entangled states are the ones which offer the quantum system its power. This is one of the key parts of the quantum error correction.

The description of the quantum register in the mixture of states is done using the density operator. For instance the mixed state consists of the state $| \mu_i \rangle$ with the probability p_i that can be denoted as $\rho = \sum_i \rho_k |\mu_i\rangle \mu_k|$. The density operator ρ is hermitian i.e. $\rho^* = \rho$, ρ is positive (all the eigen values of ρ are non-negative) and fulfills the trace condition Tr $(\rho) = 1$. The pure state $| \mu \rangle$ is described as $|\mu\rangle\langle\mu|$ if only Tr $(\rho^2) = 1$.

2.2. Unitary operation and quantum gates

2.2.1. Properties of quantum gates

Property 1. It should be noted that any 2×2 matrix can be represented as a linear combination of Pauli matrices under the identity matrix I .

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \beta_0 I + \beta_1 \sigma_x + \beta_2 \sigma_y + \beta_3 \sigma_z$$

with $\beta_i \in C\ \sigma_x \forall i = 0$ *to* 3

where

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

By expanding and equating the terms,

$$\beta_0 = \frac{b_{11} + b_{22}}{2}; \beta_1 = \frac{b_{12} + b_{21}}{2}; \beta_2 = i\frac{b_{12} - b_{21}}{2}; \beta_3 = \frac{b_{11} + 3b_{22}}{2}$$

Property 2. The qubit and the gates are unitary: preserves rotation and distance

Thus the rotations about the $\vec{x}, \vec{y}, \vec{z}$, respectively are expressed as

$$R_x(\theta) = \exp\left(\frac{-i\theta\sigma_x}{2}\right) = \cos(\theta/2)\sigma_I - \sin(\theta/2)\sigma_x = \begin{bmatrix} \cos\frac{\theta}{2} & -\mathrm{i}\sin\frac{\theta}{2} \\ -\mathrm{i}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix},$$

$$R_y(\theta) = \exp\left(\frac{-i\theta\sigma_y}{2}\right) = \cos(\theta/2)\sigma_I - \sin(\theta/2)\sigma_y = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix},$$

$$R_z(\theta) = \exp\left(\frac{i\theta\sigma_z}{2}\right) = \cos(\theta/2)\sigma_I - \sin(\theta/2)\sigma_z = \begin{bmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{bmatrix},$$

It can be shown that

$$R_x(\theta)R_x^T(\theta) = R_y(\theta)R_y^T(\theta) = R_z(\theta)R_z^T(\theta) = I$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

The quantum gates executing the transformations on these qubits essentially retain this norm. Therefore, when a qubit is acted on by a transformation, this transformation must be unitary. This transformation can be characterized as a matrix; a unitary matrix is defined as follows. A square matrix is a unitary matrix if: $U^H = U^{-1}$ where U^H signifies the conjugate transpose and U^{-1} is the matrix inverse.

It turns out that this example actually has a special name called the Pauli-X matrix. The 'X' is used for historical reasons and the matrix is important, because it is a one of the fundamental and the simplest quantum transformations. The quantum set of operators for a single qubit is the set of Pauli spin matrices

$$\sigma_0 \equiv I \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\sigma_1 \equiv \sigma_x \equiv X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\sigma_2 \equiv \sigma_y \equiv Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$$\sigma_3 \equiv \sigma_z \equiv Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Another gate is the Hadamard gate,

$$H \equiv \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

All the Pauli operators and the Hadamard gate squares to I, i.e.; $\sigma_j^2 = \mathrm{I}$ and $H^2 = \mathrm{I}$. In the other sense $\sigma_j^{\dagger} \equiv \sigma_j$ and $H^{\dagger} \equiv H$. The operators X and Z are correlated by H as $HXH^{\dagger} = Z$ and $HZH^{\dagger} = X$.

3. Quantum error correcting code

3.1. 3-Qubit bit flip code

In the classical codes, cloning and redundancy exist, but in quantum coding the quantum information is dispersed between many qubits. The quantum code uses the mechanism of entanglement. The simplest quantum network of the three-fold repetition code is the quantum bit flip code, [18,19] used to encode a data contained in a single qubit through or distributed among multiple qubits.

$$\begin{array}{l} |0\rangle \xrightarrow{\text{Tensoring}} |0\rangle \otimes |00\rangle = |000\rangle = |0_L\rangle, \\ |1\rangle \xrightarrow{\text{Tensoring}} |1\rangle \otimes |00\rangle = |100\rangle \xrightarrow{\text{CNOT}^{1,2}} |110\rangle \xrightarrow{\text{CNOT}^{1,3}} |111\rangle = |1_L\rangle \end{array} \quad (1)$$

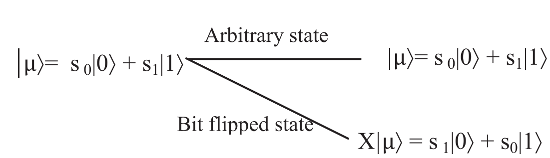


Figure 1 Quantum bit flip channel.

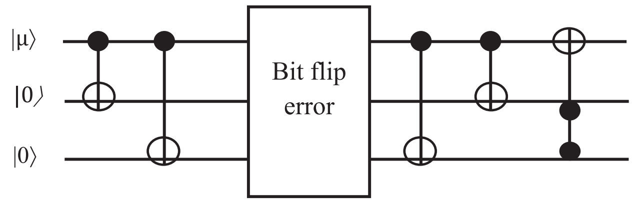


Figure 2 Quantum bit flip code circuit.

The operator $\mathrm{CNOT}^{i,j}$ is the CNOT gate from i to j defined as when the control qubit is $|1\rangle$, the target qubit is flipped. The quantum scheme with the characteristics of the bit flip channel is shown in Fig. 1. The circuit for protecting the qubits against the bit flip error is shown in Fig. 2.

The arbitrary state of the three qubits can be expressed as follows:

$$|\mu\rangle_{123} = s_0|0\rangle + s_1|1\rangle$$

$$|\mu\rangle_{123} = s_0|000\rangle + s_1|001\rangle + s_2|010\rangle + s_3|011\rangle + s_4|100\rangle + s_5|101\rangle + s_6|110\rangle + s_7|111\rangle$$

In QEC encoding, the initial states of all the qubits are the input state $|0\rangle$ or $|1\rangle$, so that the superposed state is $s_0|0\rangle + s_1|1\rangle$ is acceptable as the input state:

$$\left.\begin{array}{l}\text{qubit 1: } s_0|0\rangle + s_1|1\rangle \\ \text{qubit 2: } |0\rangle \\ \text{qubit 3: } |0\rangle\end{array}\right\} \rightarrow \text{3 qubit state: } s_0|000\rangle + s_1|100\rangle$$

$$\mathrm{base}|\mu_1\rangle|\mu_2\rangle|\mu_3\rangle = \begin{bmatrix}|000\rangle\\|001\rangle\\|010\rangle\\|011\rangle\\|100\rangle\\|101\rangle\\|110\rangle\\|111\rangle\end{bmatrix}; \quad |\mu_{123}\rangle = |\mu_1\rangle|0\rangle|0\rangle$$

$$= \begin{pmatrix}s_0\\s_1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{bmatrix}s_0\\0\\0\\0\\s_1\\0\\0\\0\end{bmatrix}$$

The procedure involved in the encoding of the three qubit bit flip code is to entangle the qubit with two other qubits, using two *CNOT* gates with input $|0\rangle$. The result will be an entanglement of $s_0|000\rangle + s_1|111\rangle$. This is just a tensor product of three qubits, and different from cloning a state.

$$\mathrm{CNOT}_{1,2} \bullet CNOT_{1,3} \bullet |\mu_{123}\rangle$$

$$= CNOT_{1,2} \bullet \begin{bmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&1&0\end{bmatrix}\begin{bmatrix}s_0\\0\\0\\0\\0\\s_1\\0\\0\end{bmatrix}$$

$$= \begin{bmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&0&0&1&0\\0&0&0&0&0&0&0&1\\0&0&0&0&1&0&0&0\\0&0&0&0&0&1&0&0\end{bmatrix}\begin{bmatrix}s_0\\0\\0\\0\\0\\s_1\\0\\0\end{bmatrix} = \begin{bmatrix}s_0\\0\\0\\0\\0\\0\\0\\s_1\end{bmatrix}$$

$$CNOT_{1,2} = (s_0|000\rangle + s_1|100\rangle) = (s_0|000\rangle + s_1|101\rangle)$$

$$CNOT_{1,3} = (s_0|000\rangle + s_1|101\rangle) = (s_0|000\rangle + s_1|111\rangle)$$

Just before the decoding part, the three qubits have the following probabilities.

$$p^3 : s_0|000\rangle + s_1|111\rangle$$

$$p^2(1-p) : s_0|100\rangle + s_1|011\rangle$$

$$p^2(1-p) : s_0|010\rangle + s_1|101\rangle$$

$$p^2(1-p) : s_0|001\rangle + s_1|110\rangle$$

The emitter guides its qubit through the error or noisy channel, where the entanglement between the qubit and the environment occurs. This process involves all possible linear combinations of the bit flip error. The group of the correctable errors is constructed by the tensor products comprising of three elements (the identity, identity and the bit flip error).

$$\mathrm{Q} = \{I \otimes I \otimes I \equiv X_0, X \otimes I \otimes I \equiv X_1, I \otimes X \otimes I \equiv X_2, I \otimes I \otimes X \equiv X_3\}$$

The subscript specifies the attacked qubit. This code reveals that it can correct only one bit error, i.e., the bit flip error.

The channel can be described by the operation

$$\varphi(\rho) = (1-\rho)\rho + \rho X \rho X$$

Each bit flip error X is unitary operation acting on the encoded state $|\mu\rangle$ as $X\,|\mu\rangle = |\mu'\rangle$ including the no-error case. The output state $|\mu'\rangle$ is given by

$$|\mu'\rangle = \begin{cases} s_0|000 + s_1|111\rangle \text{ for} X_0 \\ s_0|100 + s_1|011\rangle \text{ for } X_1 \\ s_0|010 + s_1|101\rangle \text{ for } X_2 \\ s_0|001 + s_1|110\rangle \text{ for } X_3 \end{cases}$$

When error has occurred, it can be corrected by error correction. The process of error correction is done by identifying the output state $|\mu'$ with the use of ancilla qubits. To diagnose bit flips in any of the three possible qubits, the syndrome analysis is needed. The output after the syndrome analysis is given by

$|00\rangle$ for $s_0|000\rangle + s_1|111\rangle$

$|11\rangle$ for $s_0|100\rangle + s_1|011\rangle$

$|10\rangle$ for $s_0|010\rangle + s_1|101\rangle$

$|01\rangle$ for $s_0|001\rangle + s_1|110\rangle$

respectively. The state of the ancilla can be any of the four with some probability and not the superposed state. The ancilla measurement is shown in Table 2.

However, it must be implemented carefully to avoid destroying quantum coherence in the stored information, using only unitary operations to transfer the stored information to the output qubit. The error correction is shown in Fig. 3.

The reverse process of the encoding is the decoding, where the disentanglement occurs $s_0|000\rangle + s_1|111\rangle$ if no error occurs. This result is obtained after two *CNOT* operations on the error state and can be easily found as

$$\begin{cases} s_0|000\rangle + s_1|111\rangle \to s_0|000\rangle + s_1|011\rangle \\ s_0|010\rangle + s_1|101\rangle \to s_0|010\rangle + s_1|110\rangle \\ s_0|001\rangle + s_1|110\rangle \to s_0|001\rangle + s_1|101\rangle \end{cases}$$

After the decoding the three qubits states are given as

$p^3 : (s_0|0\rangle + s_1|1\rangle)|00$

$p^2(1-p) : (s_0|1\rangle + s_1|0\rangle)|11\rangle$

$p^2(1-p) : (s_0|0\rangle + s_1|1\rangle)|10\rangle$

$p^2(1-p) : (s_0|0\rangle + s_1|1\rangle)|01\rangle$

The decoded states have no entanglement but the first qubit is reversed in the second line. This mismatch can be corrected by the use of the Toffoli gate: if and only if two ancilla qubits are in an excited state $|11\rangle$, flip the first qubit.

Thus the recovery operator is composed with the *CNOT* and *CCNOT* gate and is denoted as U_R. The corrected state is given by the tensor product form,

$\rho = U_R \rho' U_R^{-1} = |\mu\rangle\langle\mu| \otimes \rho'_a$

where

$\rho'_a = p_0|00\rangle\langle 00| + p_1|11\rangle 11 + p_2|10\rangle\langle 10 + p_3|01\rangle\langle 01|$

Table 2 Ancilla measurement for single bit flip error.

Ancilla	State	Error
00	$\{\lvert 000\rangle, \lvert 111\rangle\}$	No error
11	$\{\lvert 100\rangle, \lvert 011\rangle\}$	Error on qubit1
10	$\{\lvert 010\rangle, \lvert 101\rangle\}$	Error on qubit2
01	$\{\lvert 001\rangle, \lvert 110\rangle\}$	Error on qubit3

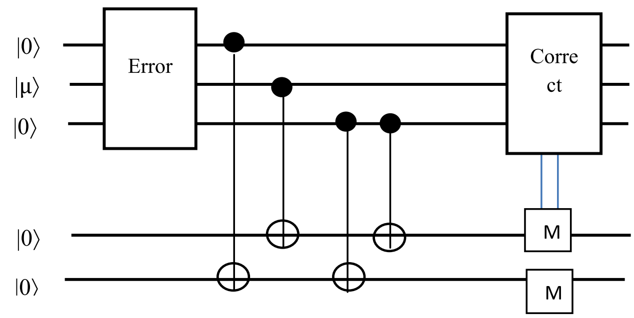


Figure 3 Error correction circuit for the 3-qubit flip code.

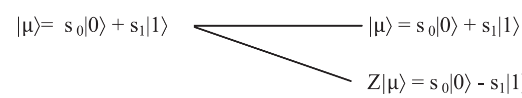


Figure 4 Quantum phase flip channel.

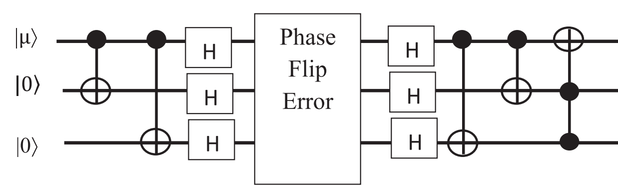


Figure 5 Three-qubit phase flip code circuit.

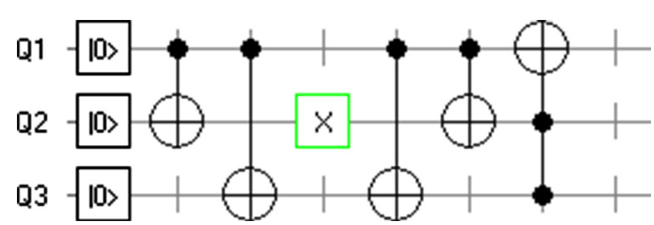


Figure 6 Bit flip error in Second Qubit (Q2).

$$= \begin{bmatrix} p_0 & 0 & 0 & 0 \\ 0 & p_3 & 0 & 0 \\ 0 & 0 & p_2 & 0 \\ 0 & 0 & 0 & p_1 \end{bmatrix}$$

This extracting of the data qubit state is computing the partial trace over the ancilla states using the $Tr\rho'_a = 1$.

The figure of merit used to decide the performance of the code is the fidelity. Hence this occurs with probability $(1-p)^3 + 3p(1-p)^2 = 1 - 3p^2 + 2p^3$. As a result the probability of an error that remains uncorrected is $3p^2 - 2p^3$. This code has the ability to correct a single bit flip error. The distance between the two codeword states is d and the number of errors that can be corrected is $t = [(d-1)/2]$. The above code shown has $d = 3$, and hence $t = 1$.

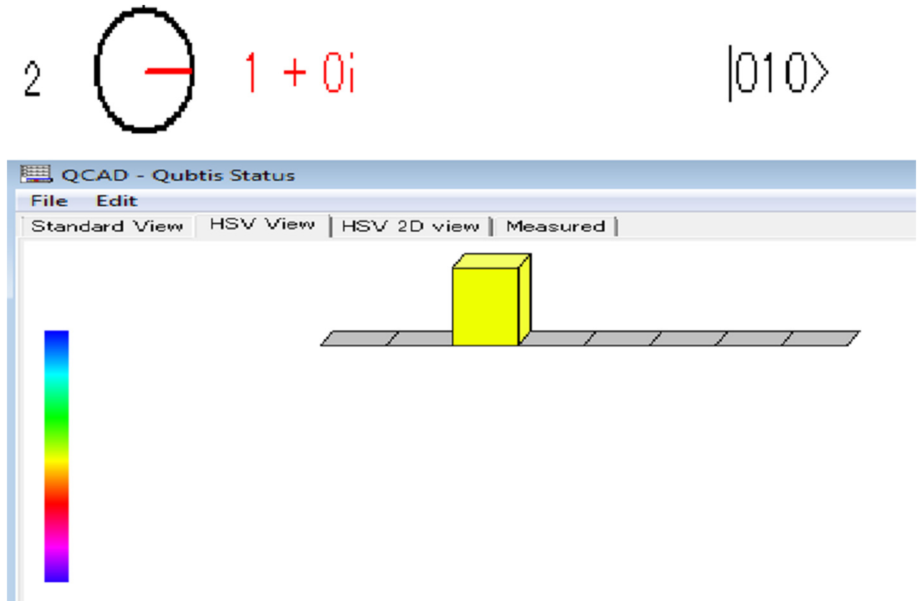


Figure 7 Output of Fig. 6.

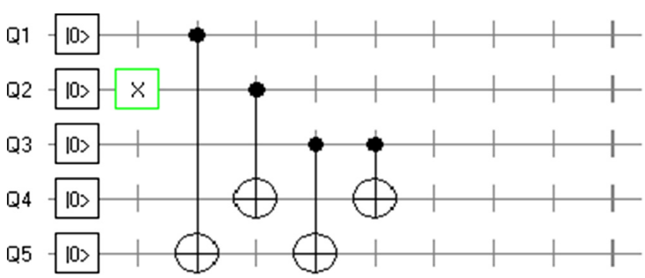


Figure 8 Bit flip error in the middle Qubit (Q2).

3.2. Quantum phase flip code

Bit flip is the only kind of error in classical coding. Yet a different error which is the phase flip error is distinctive in the quantum computer, whose error correction is associated with the bit flip error. It is merely interchanging the roles of the X and Z bases (duality between the roles of the bit and the phase flip error). The single-qubit basis states $|0\rangle \leftrightarrow |+\rangle$ and $|1\rangle \leftrightarrow |-\rangle$ interchange the roles of the Pauli operators X $\leftrightarrow$ Z.

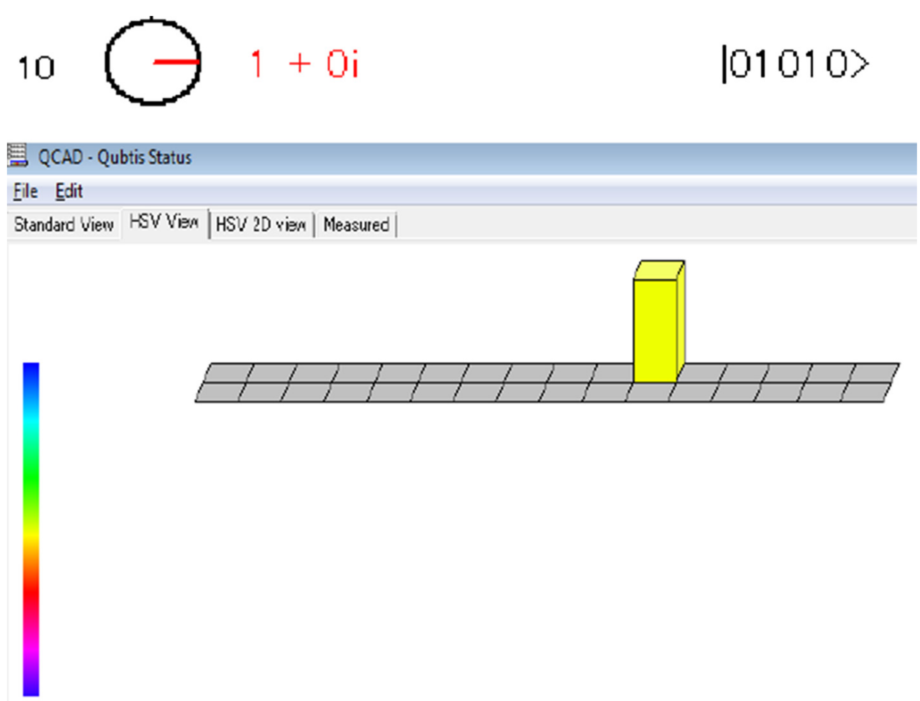


Figure 9 Output of Fig. 8.

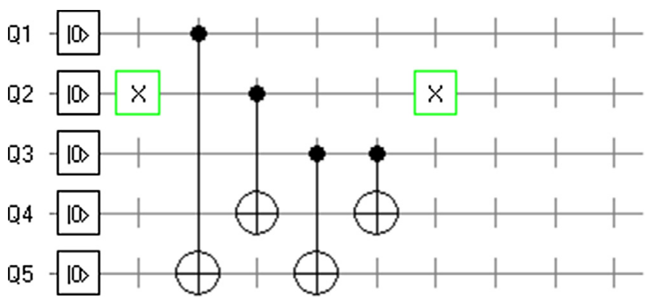


Figure 10 Bit flip error in the middle Qubit (Q2) is corrected.

The duality is done using the Hadamard transformation $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$|0\rangle = |+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}; \quad |1\rangle = |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

The quantum phase flip channel is shown in Fig. 4. The phase- flip code is defined as a qubit encoded in the 3-qubit and is described as:

$$Z|\mu\rangle = (s_0|0\rangle + s_1|1\rangle) \rightarrow s_0|+++\rangle + s_1|---\rangle$$

The quantum circuit that accomplishes the encoding, decoding and error correction steps is shown in Fig. 5. which comprises of the circuit for the bit- flip code, followed by a Hadamard gate H applied to each qubit (since H makes the appropriate transformations between X and Z bases, H$|0\rangle = |+\rangle$ and H$|1\rangle = |-\rangle$). The role of the Hadamard gate is to rotate the phase flip into the bit flip $HZH = X$ [15,17].

The encoded states are

$$|0\rangle_L = |000\rangle = \frac{1}{2\sqrt{2}}\{|000\rangle + |001\rangle + |010\rangle + |100\rangle + |011\rangle + |101\rangle + |110\rangle + |111\rangle\}$$

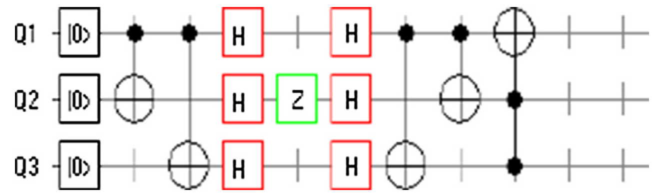


Figure 12 Sign flip error in the middle Qubit (Q2).

$$|1\rangle_L = |111\rangle = \frac{1}{2\sqrt{2}}\{|000\rangle - |001\rangle - |010\rangle - |100\rangle + |011\rangle + |101\rangle + |110\rangle - |111\rangle\}$$

In this basis, the phase flip operator Z performs as an ordinary bit flip operator. Therefore the ordinary bit flip error correction code can be used. The three Hadamards at the right are responsible for the transformation of the basis.

After the encoding, the qubits are transmitted through the noisy channel where the phase flip errors occur whose noise characteristics are the same as the bit flip error. The attack causes a phase decoherence: entanglement between the environment and the qubits.

The group of S errors that can be corrected are as follows:

$$Q = \{I \otimes I \otimes I \equiv Z_{000}, Z \otimes I \otimes I \equiv Z_{100}, I \otimes Z \otimes I \equiv Z_{010}, I \otimes I \otimes Z \equiv Z_{001}\}$$

The subscript refers to the qubits which have been attacked by the errors. The error detection and correction procedure are the same as those of the bit flip code. The syndrome measurement is analogous to the bit flip code except that the three Hadamard gates are placed before and after the error correction.

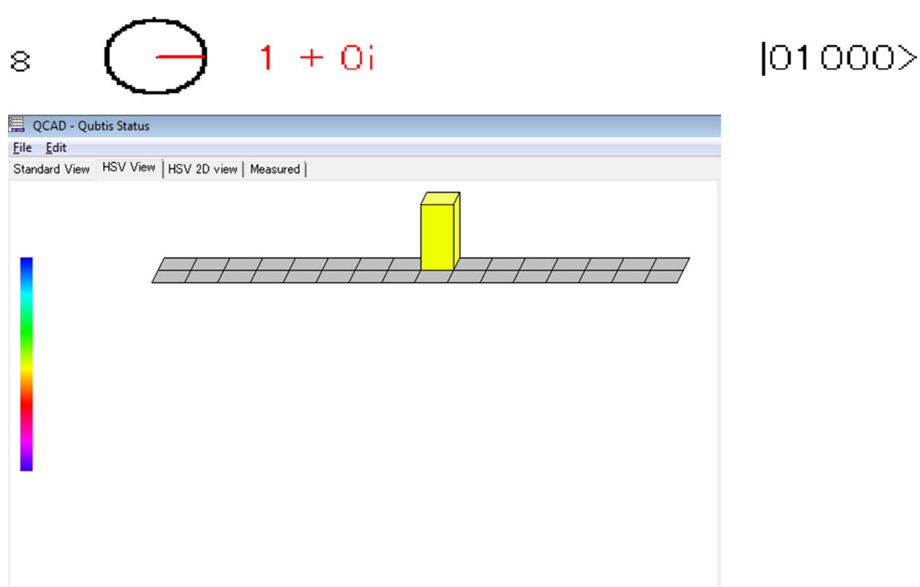


Figure 11 Output of Fig. 10.

4. Implementation work using the QCAD: examples

To demonstrate the use of QCAD procedures, we discuss two quantum error correction codes: bit flip and phase flip.

4.1. Bit flip code

Bit Flip is defined as

$X : X|0\rangle = |1\rangle, X|1\rangle = |0\rangle,$

Implement the bit flip code circuits using *CNOT* and *CCNOT* gates in Q-CAD as given below. In these circuits we can measure the error, i.e., only one qubit at a time and the error is introduced in the second position (qubit 2) using *X* Pauli gate which is shown in Fig. 6. The output of this circuit is shown in Fig. 7. The error can be induced in various other positions whose operations are same as the above.

The output of Fig. 7. is equal to P2,

$P_2 = |010\rangle\langle 010| + |101\rangle\langle 101|.$

4.1.1. Syndrome measurement

The syndrome conveys how to correct the received state, in order to recover the initial state. In this example, the syndrome conveys, that the first qubit was flipped, therefore, simply flip it back by applying *X* gate to it. This is implemented by coupling the two supplementary ancilla qubits to the data block. The Q4 and Q5 are the ancilla qubits, while Q1–Q3 are the data. The error was induced in the bit flip channel at the position i.e. Q2 by simply introducing the *X* Pauli gate to it, which is shown in Fig. 8. whose QCAD output is shown in Fig. 9. As mentioned already, the output in QCAD is read from right to left; therefore, the ancillas 01, 10 and 11 indicate that the error occurred in the first, middle and the last qubit. The induced errors at various positions have been corrected by applying *X* gate, after all the *CNOT* gates, whose circuit is shown in Fig. 10. The corresponding output is shown in Fig. 11.

4.2. Sign flip code

Sign Flip is defined as

$Z : Z|0\rangle = |0\rangle, Z|1\rangle = -|1\rangle$

The role of the Hadamard gate is to rotate the phase flip to the bit flip. Let Z-Pauli gate be a quantum channel that can cause at the most one phase flip at various positions. The phase flip error is induced in the second position which is shown in Fig. 12. The corresponding output is shown in Fig. 13.

5. Summary and outlook

In the present work, we have used the QCAD to deliver simplified support for the simulation of quantum error correction codes. For these codes, the results are shown in the figures. The results show the simulation of the theoretical work of the earlier works of recurrence of the error correction procedure [18], in a structure of three qubits. Repetition error correction codes in a quantum system of three qubits are

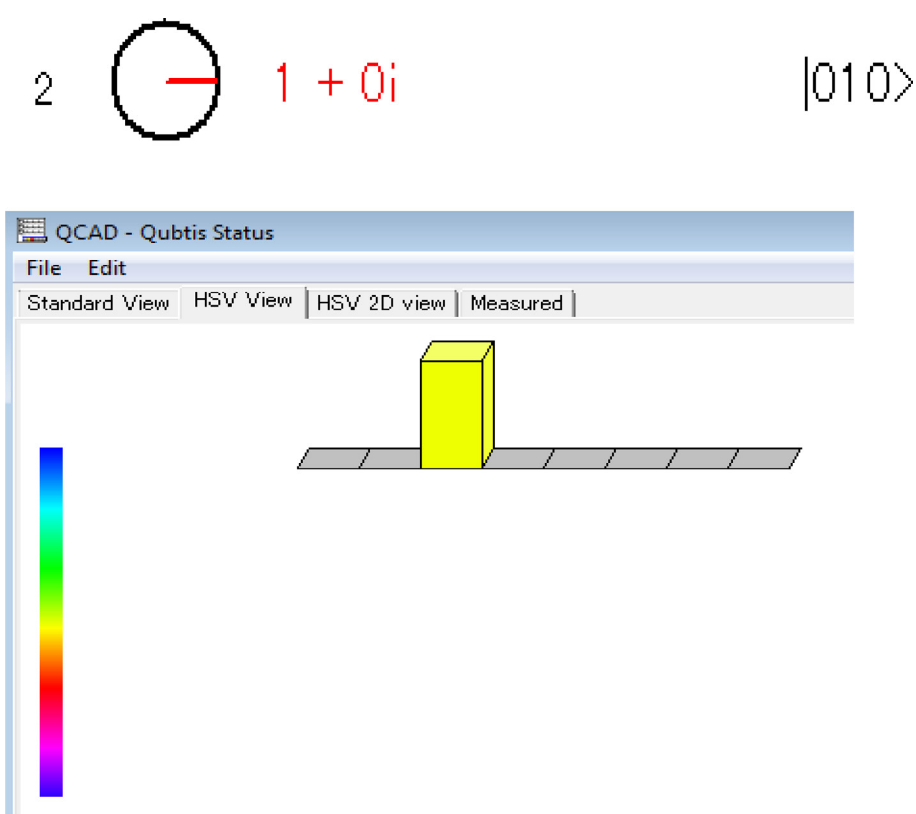


Figure 13 Output of Fig. 12.

simulated. Our analysis with the simulation starts with the case where no error, errors in various positions and the positioned errors are corrected. These codes can be simulated using any other quantum simulator in the near future. This has become a versatile tool for simulation and analysis of quantum gates, circuits and devices. QCAD can be used for both, in investigating well known or new scenarios in quantum information theory as well as for teaching the basic elements.

In future there are several directions which provide a direct support for quantum error correction codes, quantum cryptography, and teleportation. Also any qubit codes can be simulated using the universal quantum gates in QCAD.

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