Poisson process martingale exercises

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2021-05-04

Poisson process compensator

Let $N(t)$ be the number of events in $[0, t]$ where $N(t) - N(s) \sim \text{Poisson}((t - s)\lambda)$ for $s < t$ and N has independent increments.

Does the Doob-Meyer decomposition apply to $N(t)$? If so, identify the compensator of $N(t)$.

Solution

Problem statement: Find the unique predictable process $X(t)$ such that $N(t) - X(t)$ is a martingale.

Steps (WAG method):

1. I think $X(t) = \lambda t$.
2. Is λt predictable? Yes, it is left-continuous because

$$\lim_{s \uparrow t} \lambda s = \lambda t$$

   and all left-continuous processes are predictable.
3. Is $N(t) - \lambda t$ a martingale? Let's check the definition: for $s < t$, we have

$$\begin{aligned} E(N(t) - \lambda t | \mathcal{F}_s) = E(N(t) - N(s) + N(s) - \lambda t | \mathcal{F}_s) = \\ E(N(t) - N(s) | \mathcal{F}_s) + E(N(s) | \mathcal{F}_s) - \lambda t = \\ E(N(t) - N(s)) + N(s) - \lambda t = \\ (t - s)\lambda + N(s) - \lambda t = N(s) - \lambda s. \end{aligned}$$

Steps (start at the end method):

1. $N(t) - X(t)$ is a martingale iff $E(N(t) - X(t)|\mathcal{F}_s) = N(s) - X(s)$ for $s < t$ and a predictable process X. Hence,

$$\begin{aligned} E(N(t) - X(t) - N(s) + X(s)|\mathcal{F}_s) = 0 \Leftrightarrow \\ E((N(t) - N(s)) + (X(s) - X(t))|\mathcal{F}_s) = 0 \Leftrightarrow \\ E(N(t) - N(s)|\mathcal{F}_s) = E(X(t) - X(s)|\mathcal{F}_s) \Leftrightarrow \\ (t - s)\lambda = E(X(t) - X(s)|\mathcal{F}_s). \end{aligned}$$

   Since $X(t)$ is predictable, we must have $E(X(t) - X(s)|\mathcal{F}_s) = X(t) - X(s)$. If not, then we could find a t^* such that $E(X(t^*)|\mathcal{F}_{t^*-}) \neq X(t^*)$, which would violate the definition of predictability. The result follows.

Predictable and optional variation processes

In the Poisson process example, we showed earlier that $M(t) = N(t) - \lambda t$ is a martingale. Show that $\langle M \rangle = \lambda t$.

What is $[M]$? Hint: $[M](t) = \sum_{s \leq t} (M(s) - M(s-))^2$ for processes with finite variation.

Solution

By the Doob Meyer, and the fact that λt is predictable, we have to show that $M^2(t) - \lambda t$ is a martingale. We do this by checking the definition:

$$\begin{aligned} E((N(t) - \lambda t)^2 - \lambda t | \mathcal{F}_s) = \\ E(N^2(t) - 2N(t)\lambda t + \lambda^2 t^2 - \lambda t | \mathcal{F}_s). \end{aligned}$$

Above we showed that $E(N(t)|\mathcal{F}_s) = N(s) + \lambda(t-s)$ by adding and subtracting $N(s)$, so we have now

$$\begin{aligned} E(N^2(t)|\mathcal{F}_s) - 2(N(s) + \lambda(t-s))\lambda t + \lambda^2 t^2 - \lambda t = \\ E(N^2(t)|\mathcal{F}_s) - 2N(s)\lambda t - 2\lambda^2 t^2 + 2\lambda^2 st + \lambda^2 t^2 - \lambda t = \\ E(N^2(t)|\mathcal{F}_s) - 2N(s)\lambda t - \lambda^2 t^2 + 2\lambda^2 st - \lambda t. \end{aligned}$$

Looking now at the first term, it equals

$$\begin{aligned} E((N(s) + N(t) - N(s))^2 | \mathcal{F}_s) = \\ N(s)^2 + 2N(s)E(N(t) - N(s)) + E((N(t) - N(s))^2) = \\ N(s)^2 + 2N(s)\lambda(t-s) + Var(N(t) - N(s)) + (E(N(t) - N(s)))^2 = \\ N(s)^2 + 2N(s)\lambda(t-s) + \lambda(t-s) + \lambda^2(t-s)^2. \end{aligned}$$

Plugging back into the last line of the previous display:

$$\begin{aligned} N(s)^2 + 2N(s)\lambda(t-s) + \lambda(t-s) + \lambda^2(t-s)^2 - 2N(s)\lambda t - \lambda^2 t^2 + 2\lambda^2 st - \lambda t = \\ N(s)^2 - 2N(s)\lambda s - \lambda s + \lambda^2(t-s)^2 - \lambda^2 t^2 + 2\lambda^2 st = \\ N(s)^2 - 2N(s)\lambda s - \lambda s - \lambda^2 s^2 = \\ (N(s) - \lambda s)^2 - \lambda s, \end{aligned}$$

which shows that the martingale property is satisfied.

What about $[M](t)$? For this we use the definition of the optional variation process, and observe that it equals

$$\begin{aligned} \sum_{s \leq t} (M(s) - M(s-))^2 = \sum_{s \leq t} (N(s) - N(s-) + \lambda s - \lambda s)^2 = \\ \sum_{s \leq t} (N(s) - N(s-))^2. \end{aligned}$$

Now if there is a jump at s the term in the sum equals 1, and if not, it equals 0. So this sum counts the number of events that occurred up to t, in other words it equals $N(t)$ itself. This is true for any counting process.