Algorithms & Data Structures 2018/19 Coursework

clvp22

January 2, 2019

Question 4.

(a)

Firstly we can say $x^3 + 3x + 2$ is $O(x^3)$ because there exists some k and C such that,

$$x^3 + 3x + 2 \leq C \cdot x^3$$

for every $x \geq k$.
(example; $C = 6$, $k = 0$)

So, $x^3 + 3x + 2$ is $O(x^3)$. And $2x^4$ is $O(x^3 + 3x + 2)$ if and only if $2x^4$ is $O(x^3)$. We now can assume there are some constants k and C such that,

$$2x^4 \leq C \cdot x^3$$

for every $x \geq k$.

So,

$$C \geq 2x$$

however, $2x$ grows monotonically for $x > 0$. This is a contradiction because then no constant value for C can be found for $k > 0$. Therefore $2x^4$ is not $O(x^3)$ and consequently not $O(x^3 + 3x + 2)$.

(b)

For $x \geq 2$, $1 \leq \log x \leq x \leq x^2 \leq x^3$.

So,

$$4x^3 + 2x^2 \cdot \log x + 1 \leq 4x^3 + 2x^2 \cdot x + x^3 = 8x^3$$

for every $x \geq k$.

This inequality holds using $k = 2$ and $C = 8$ as a pair of witnesses,

$$4x^3 + 2x^2 \cdot \log x + 1 \leq C \cdot x^3$$

for every $x \geq k$.

Therefore $4x^3 + 2x^2 \cdot \log x + 1$ is $O(x^3)$.

(c)

$3x^2 + 7x + 1$ is $\omega(x \cdot \log x)$ if and only if $x \cdot \log x$ is $o(3x^2 + 7x + 1)$, which is,

$$\lim_{x \to \infty} \frac{C \cdot x \log x}{3x^2 + 7x + 1}$$

for every $x \geq k$.

For $x \geq 2$, $1 \leq \log x \leq x \leq x^2$.
So $3x^2+7x+1$ grows faster than $x \cdot \log x$ so $\lim_{x \to \infty} \frac{C \cdot x \log x}{3x^2 + 7x + 1}$ holds for $k = 2$ and $C = 1$. Therefore, $x \cdot \log x$ is $o(3x^2 + 7x + 1)$, and consequently $3x^2 + 7x + 1$ is $\omega(x \cdot \log x)$.

(d)

For $x \geq 2$, $\log x \leq x \leq x^2$.

So,

$$x^2 + 4x \geq x^2 = x \cdot x \geq x \cdot \log x$$

for every $x \geq k$.

This inequality holds using $k = 2$ and $C = 1$ as a pair of witnesses,

$$x^2 + 4x \geq C \cdot x \log x$$

Therefore $x^2 + 4x$ is $\Omega(x \cdot \log x)$.

(e)

$f(x) + g(x)$ is $\Theta(f(x) \cdot g(x))$, if $f(x) + g(x)$ is $O(f(x) \cdot g(x))$ and if $f(x) + g(x)$ is $\Omega(f(x) \cdot g(x))$.
Suppose $f(x)$ is $O(g(x))$, or in other words,

$$f(x) \leq C \cdot g(x)$$

We can now ignore $f(x)$ from our previous statement as $g(x)$ dominates $f(x)$. Our question can be reconstructed as follows,

Is $g(x)$ $\Theta(f(x) \cdot g(x))$, for some $f(x)$, where $f(x)$ is $O(g(x))$.

Again, $g(x)$ is $\Theta(f(x) \cdot g(x))$, if $g(x)$ is $O(f(x) \cdot g(x))$ and if $g(x)$ is $\Omega(f(x) \cdot g(x))$.
Or in other words,

$$g(x) \leq C \cdot f(x) \cdot g(x)$$

and

$$g(x) \geq C \cdot f(x) \cdot g(x)$$

so,

$$C_1 \cdot f(x) \cdot g(x) \leq g(x) \leq C_2 \cdot f(x) \cdot g(x)$$

Clearly this inequality can only be satisified if and only if $f(x)$ is a non-zero constant function. If however we said $g(x)$ is $O(f(x))$ then the same principles can be applied and the original statement could only be satisified if and only if $g(x)$ is a non-zero constant funtcion instead. So $f(x) + g(x)$ is not $\Theta(f(x) \cdot g(x))$ for non-constant functions $f(x)$ and $g(x)$.