Mathematics for Computer Science Summative Assignment 2018-19

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December 4, 2018

1 Discrete Mathematics and Linear Algebra

1.1

Prove, using induction the following holds for all $n \geq 1$.

$$2(\sqrt{n+1}-1) < 1 + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{n}} < 2\sqrt{n}$$

Try $n = 1$,

$$2(\sqrt{1+1}-1) < 1 < 2\sqrt{1}$$
$$0 < 1 < 2$$

So true for $n = 1$.

Assume true for $n = k$,

$$2(\sqrt{k+1}-1) < 1 + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{k}} < 2\sqrt{k}$$

$n = k + 1$,

$$2(\sqrt{k+2}-1) < 1 + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} < 2\sqrt{k+1}$$

Taking the difference of $n = k + 1$ and $n = k$,

$$2(\sqrt{k+2}-1) - 2(\sqrt{k+1}-1) < (1 + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{k}}) + \frac{1}{\sqrt{k+1}} - (1 + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{k}}) < 2\sqrt{k+1} - 2\sqrt{k}$$

we get the following which we want to show is true,

$$2(\sqrt{k+2}-1) - 2(\sqrt{k+1}-1) < \frac{1}{\sqrt{k+1}} < 2\sqrt{k+1} - 2\sqrt{k}$$

Proving the left inequality holds,

$$2(\sqrt{k+2}-1) - 2(\sqrt{k+1}-1) < \frac{1}{\sqrt{k+1}}$$

$$2\sqrt{k+2} - 2\sqrt{k+1} < \frac{1}{\sqrt{k+1}}$$

$$2\sqrt{k+2} \cdot \sqrt{k+1} - 2(k+1) < 1$$

$$2\sqrt{(k+2)(k+1)} < 1 + 2(k+1)$$

$$4(k+2)(k+1) < (2k+3)^2$$
$$4k^2+12k+8 < 4k^2+12k+9$$
$$0 < 1$$

so true.

Proving the right inequality holds,

$$\frac{1}{\sqrt{k+1}} < 2\sqrt{k+1} - 2\sqrt{k}$$
$$1 < 2(k+1) - 2\sqrt{k} \cdot \sqrt{k+1}$$
$$2\sqrt{k(k+1)} < 2(k+1) - 1$$
$$4k(k+1) < (2k+1)^2$$
$$4k^2+4k < 4k^2+4k+1$$
$$0 < 1$$

so true.

Therefore,

$$2(\sqrt{k+2}-1) - 2(\sqrt{k+1}-1) < \frac{1}{\sqrt{k+1}} < 2\sqrt{k+1} - 2\sqrt{k},$$

given true for $n = k$,

$$2(\sqrt{k+1}-1) < 1 + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{k}} < 2\sqrt{k}.$$

now true for $n = k+1$,

$$2(\sqrt{k+2}-1) < 1 + \frac{1}{\sqrt{2}} + ... + \frac{1}{\sqrt{k+1}} < 2\sqrt{k+1}.$$

Given true for $n = 1$ and true for $n = k+1$ then true for all $n \geq 1$.

1.2

Sample space size is $6^4 = 1296$

The probability to roll exactly one 6 can be computed as follows:

$$\binom{4}{1} \cdot \frac{1}{6} \cdot \frac{5}{6}^3 = \frac{500}{1296}$$

The probability to roll 4 different numbers can be computed as follows:

$$\frac{\binom{6}{1} \cdot \binom{5}{1} \cdot \binom{4}{1} \cdot \binom{3}{1}}{1296} = \frac{360}{1296}$$

$$\frac{500}{1296} > \frac{360}{1296}$$

So it is more likely to roll exactly one 6 than 4 different numbers.

1.3

Number of permutations: $P(4,4) = 4! = 24$
Starting from the ordered string ABCD which gives us $X = 0$, we have 3 pairs of adjacent characters: AB, BC, CD. If we reverse the order of one of these pairs we un-order the string and get a permutaion of ABCD that gives us $X = 1$. We have 3 pairs to choose from, so there are 3 permuations of ABCD that give us $X = 1$ (clearly there is one where $X = 0$). Again we have 3 pairs of adjacent characters to choose from, if we select the same pair we did last time then we will go back to the ordered string ABCD. So, to get $X = 2$ we only have 2 adjacent pairs to choose from for each of our 3 permutations that give us $X = 1$. This implies we have 6 permutations that give us $X = 2$. However, if we choose the first pair and the last pair we get the same string, BADC, no matter which order we reversed the first and last adjacent pairs. So actually, we have $X = 2$ for 5 permutations of the string ABCD. Now consider the unordered string DCBA which gives us $X = 6$. Similarly we have 3 pairs of adjacent characters: DC, CB, BA. Again reversing the order of one of these pairs will re-order the list and give us a permutaion of ABCD that gives us $X = 5$. We have 3 pairs to choose from so 3 permutaions of ABCD give us $X = 5$ (again there is clearly one that gives us $X = 6$). We apply the aforementioned idea and the notion that this distribution is inherently symmetrical to determine that there are 5 permutations of ABCD that give us $X = 4$. $24 - 1 - 3 - 5 - 1 - 3 - 5 = 6$ so there are 6 permutations of ABCD that give us $X = 3$. We can now display this information in a table or go straight to a discrete probability distribution for X.

Permutation Matrix:

Permutation	X	Permutation	X	Permutation	X	Permutation	X
ABCD	0	BACD	1	CABD	2	DABC	3
ABDC	1	BADC	2	CADB	3	DACB	4
ACBD	1	BCAD	2	CBAD	3	DBAC	4
ACDB	2	BCDA	3	CBDA	4	DBCA	5
ADBC	2	BDAC	3	CDAB	4	DCAB	5
ADCB	3	BDCA	4	CDBA	5	DCBA	6

Discrete Probability Distribution:

x	0	1	2	3	4	5	6
P(X = x)	$\frac{1}{24}$	$\frac{3}{24}$	$\frac{5}{24}$	$\frac{6}{24}$	$\frac{5}{24}$	$\frac{3}{24}$	$\frac{1}{24}$

$$E(X) = 0 \cdot \frac{1}{24} + 1 \cdot \frac{3}{24} + 2 \cdot \frac{5}{24} + 3 \cdot \frac{6}{24} + 4 \cdot \frac{5}{24} + 5 \cdot \frac{3}{24} + 6 \cdot \frac{1}{24} = 3$$
$$E^2(X) = 0 \cdot \frac{1}{24} + 1^2 \cdot \frac{3}{24} + 2^2 \cdot \frac{5}{24} + 3^2 \cdot \frac{6}{24} + 4^2 \cdot \frac{5}{24} + 5^2 \cdot \frac{3}{24} + 6^2 \cdot \frac{1}{24} = \frac{268}{24}$$
$$VAR(X) = E^2(X) - (E(X))^2 = \frac{268}{24} - 3^2 = \frac{13}{6}$$

1.4

For any complete bipartite graph $K_{i,j}$, we have a set of i nodes V_1 and a set of j nodes V_2, where every node in V_1 is connected to every node in V_2, but none in V_1, and vice versa. For $K_{3,q}$ our set of vertices V_1 has a fixed size i, where $i = 3$. And our set of vertices V_2 has a variable size of q. If $i = j$ then our graph $K_{i,j}$ is clearly KO-reducible with KO-number 1, because each vertex in V_1 selects the verttex in V_2 that is directly opposite and all vertices are knocked out in one step. So when reducing other complete bipartite graphs we want to end up in a situation where $i = j$.

Let S be a (greedy) strategry for KO-reducing a graph $K_{3,q}$:

- if $i = j$ then the KO-number is 1 (a single KO-round is required).

- *Reducing* : if $i < j$ then every vertex in V_1 selects a different vertex in V_2 and every vertex in V_2 selects the same vertex in V_1.
- else if $i > j$ then every vertex in V_1 selects the same vertex in V_2 and every vertex in V_2 selects a different vertex in V_1.
- eliminate selected vertices and their incident edges.
- repeat from *Reducing* until $i = j$ or we have atleast one isolated vertex, at which point $KO = \infty$.
- the KO-number is given by the number of KO-rounds.

1.4.1

Reducing $K_{3,2}$ using strategy S,

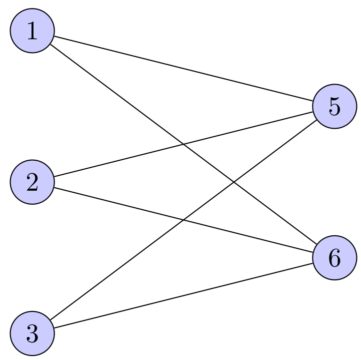

Figure 1: $K_{3,2}$

after 1 iteration of strategy S we get,

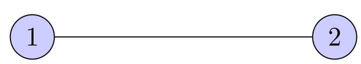

Figure 2: $K_{1,1}$

we now have a situation where $i = j$, so the KO-number for $K_{3,2}$ is 2.

1.4.2

Reducing $K_{3,3}$ using strategy S,

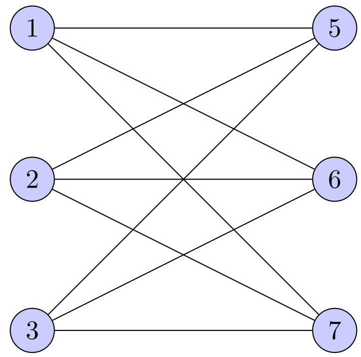

Figure 3: $K_{3,3}$

we have a situation where $i = j$, so the KO-number for $K_{3,3}$ is 1.

1.4.3

Reducing $K_{3,4}$, using strategy S,

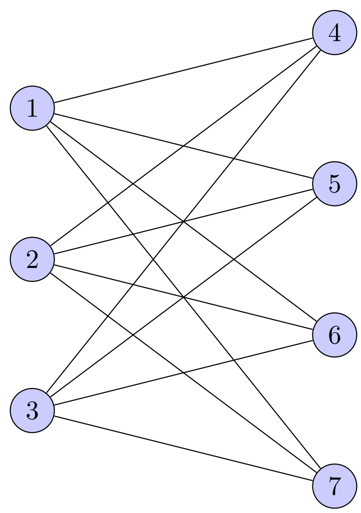

Figure 4: $K_{3,4}$

after one iteration of S we get,

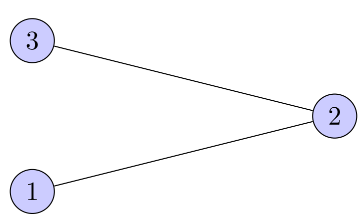

Figure 5: P_3

P_3 is clearly not KO-reducible, so we need to consider a smarter strategy. We want to get a situation where $i = j$ after some KO-round. So, if each vertex in V_1 selects a different vertex in V_2 and 2 of the vertices in V_1 are selected by the vertices in V_2; we get,

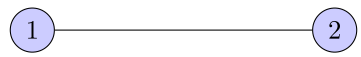

Figure 6: $K_{1,1}$

we now have a situation where $i = j$, so the KO-number for $K_{3,4}$ is 2.

1.4.4

Reducing $K_{3,5}$, using strategy S,

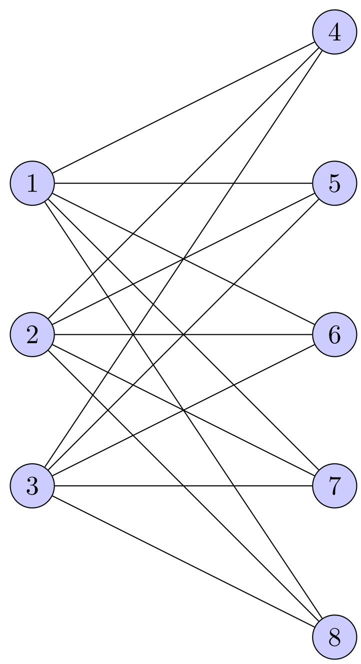

Figure 7: $K_{3,5}$

after one iteration of S we get,

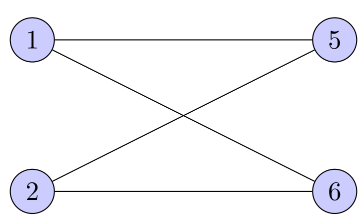

Figure 8: $K_{2,2}$

we now have a situation where $i = j$, so the KO-number for $K_{3,5}$ is 2.

1.4.5

Reducing $K_{3,6}$, using strategy S,

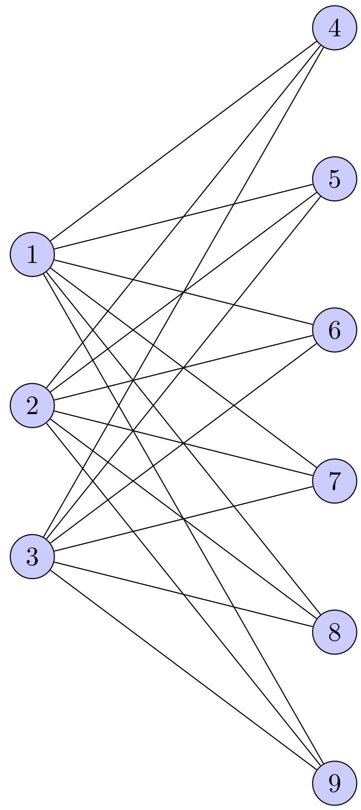

Figure 9: $K_{3,6}$

after one iteration of S we get,

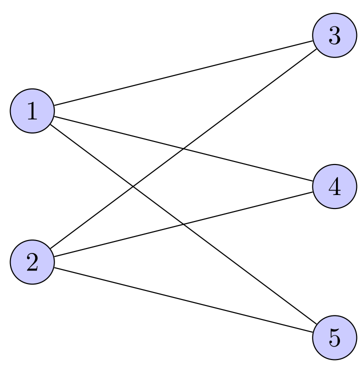

Figure 10: $K_{2,3}$

after another iteration of S we get,

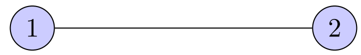

Figure 11: $K_{1,1}$

we now have a situation where $i = j$, so the KO-number for $K_{3,6}$ is 3.

1.4.6

Reducing $K_{3,7}$, using strategy S,

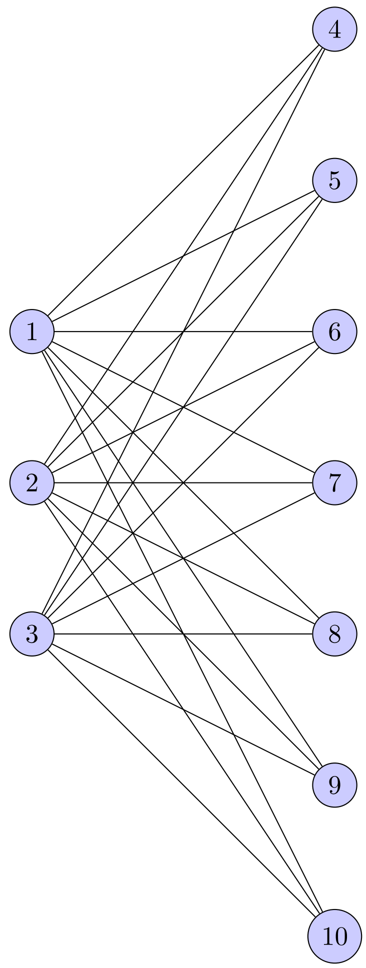

Figure 12: $K_{3,7}$

after one iteration of S we get,

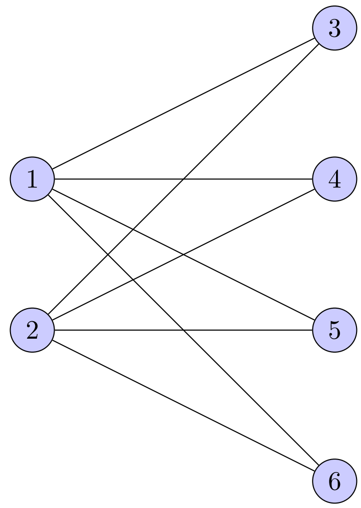

Figure 13: $K_{2,4}$

after another iteration of S we get,

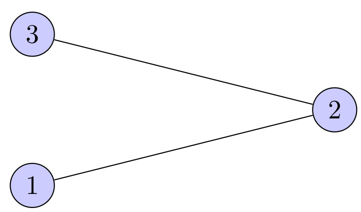

Figure 14: P_3

P_3 is clearly not KO-reducible, so $K_{3,7}$ is not KO-reducible. Actually for any $q \geq 7$, $K_{3,q}$ is not KO-reducible. Because, after we apply our strategy S to the graph $K_{3,q}$ where $q \geq 6$, we are always left with a star graph $S_{(q-5)}$. S_1 is KO-reducible because $S_1 \equiv K_{1,1}$, which is an $i = j$ situtaion, which we know is KO-reducible, so for $q = 6$, $K_{3,q}$ is KO-reducible. However, S_n is clearly not KO-reducible for $n \geq 2$ (we will always get atleast one isolated vertex), so $K_{3,q}$ is not KO-reducible for $q \geq 7$ ($P_3 \equiv S_2$).

So for all $k \geq 7$,

Graph	KO-number
$K_{3,2}$	2
$K_{3,3}$	1
$K_{3,4}$	2
$K_{3,5}$	2
$K_{3,6}$	3
$K_{3,7}$	∞
...	...
$K_{3,k}$	∞

2 Logic and Discrete Structures

2.1

$\varphi = ((a \wedge b) \implies c) \wedge (a \vee b)$

2.1.1

a	b	c	((a ∧	b	)	⟹	c	)	∧	(a ∨	b	)	φ
T	T	T	T	T	T		T	T		T	T	T	T
T	T	F	T	T	T		F	F		T	T	T	F
T	F	T	T	F	F		T	T		T	F	T	T
T	F	F	T	F	F		F	T		T	F	T	T
F	T	T	F	T	F		T	T		F	T	T	T
F	T	F	F	T	F		F	T		F	T	T	T
F	F	T	F	F	F		T	T		F	F	F	F
F	F	F	F	F	F		F	T		F	F	F	F

So in d.n.f, $\varphi = (a \wedge b \wedge c) \vee (a \wedge \neg b \wedge c) \vee (a \wedge \neg b \wedge \neg c) \vee (\neg a \wedge b \wedge c) \vee (\neg a \wedge b \wedge \neg c)$

2.1.2

Negating φ we get:
$\neg\varphi = \neg((a \wedge b \wedge c) \vee (a \wedge \neg b \wedge c) \vee (a \wedge \neg b \wedge \neg c) \vee (\neg a \wedge b \wedge c) \vee (\neg a \wedge b \wedge \neg c))$

by generalised de morgan's laws:
$\neg\varphi = \neg(a \wedge b \wedge c) \wedge \neg(a \wedge \neg b \wedge c) \wedge \neg(a \wedge \neg b \wedge \neg c) \wedge \neg(\neg a \wedge b \wedge c) \wedge \neg(\neg a \wedge b \wedge \neg c)$

by generalised de morgan's laws again we get $\neg\varphi$ in c.n.f:
$\varphi = (\neg a \vee \neg b \vee \neg c) \wedge (\neg a \vee b \vee \neg c) \wedge (\neg a \vee b \vee c) \wedge (a \vee \neg b \vee \neg c) \wedge (a \vee \neg b \vee c)$

2.2

We know the set $\{\neg, \wedge, \vee\}$ is a functionally complete set. To show the set $\{\wedge, \oplus\}$ is functionally complete we will try to construct logically equivalent statements of all the elements in the set $\{\neg, \wedge, \vee\}$ with elements of the set $\{\wedge, \oplus\}$

Obviously, $\wedge$ can be made from the set $\{\wedge, \oplus\}$:
$p \wedge q \equiv p \wedge q$

Making $\vee$:
try $(p \oplus q) \oplus (p \wedge q)$,

p	q	(p ⊕	q	)	⊕	(p ∧	q	)	φ	p ∨	q	Ψ
T	T	T	T	F		T		T	T	T	T	T
T	F	T	F	T		T		F	T	T	F	T
F	T	F	T	T		F		T	T	F	T	T
F	F	F	F	F		F		F	F	F	F	F

where $p \oplus q \equiv \neg(p \iff q)$.

$\varphi \equiv \Psi$, so $(p \oplus q) \oplus (p \wedge q) \equiv p \vee q$
Therefore $\vee$ can be made from the set $\{\wedge, \oplus\}$, we can now use $\vee$ as an operator.

Making $\neg$:
The operator $\neg$ has only one operand, so $\neg$ must be made from one operand p, using the elements from the set $\{\wedge, \oplus\}$, along with $\vee$ which we have just made.

$(p \oplus p)$ is always false, so we can use F as an operand.
$(p \wedge p)$ is always p.
$(p \vee p)$ is always p.
$(p \wedge F)$ is always False.
$(p \vee F)$ is always p.
$(p \oplus F)$is always p.

We can see the operator $\neg$ can't be made from the set $\{\wedge, \oplus\}$, because we are only getting p or False, from the various configurations of operators on the variable p. Therefore $\{\wedge, \oplus\}$ is not a functionally complete set.

2.3

2.3.1

$\neg a \wedge b \wedge (a \wedge (b \implies c)) \vdash c \vee d$

1.	$\neg a \wedge b \wedge (a \vee (b \implies c))$	premise
2.	$\neg a \wedge b$	$\wedge e$
3.	$a \vee (b \implies c)$	$\wedge e$
4.	$\neg a$	$\wedge e$
5.	b	$\wedge e$
6.	a	assume
7.	$\perp$	$\neg e$
8.	$c \vee d$	$\perp e$
9.	$b \implies c$	assume
10.	c	$\implies e$
11.	$c \vee d$	$\vee i$
12.	$c \vee d$	$\vee e$

2.3.2

$a \vee (\neg b \wedge \neg c \wedge \neg d) \vdash (a \vee \neg b) \wedge (a \vee \neg c) \wedge (a \vee \neg d)$

1.	$a \vee (\neg b \wedge \neg c \wedge \neg d)$	premise
2.	a	assume
3.	$a \vee \neg b$	$\vee i$
4.	$\neg b \wedge \neg c \wedge \neg d$	assume
5.	$\neg b$	$\wedge e$
6.	$a \vee \neg b$	$\vee i$
7.	$a \vee \neg b$	$\vee e$
8.	a	assume
9.	$a \vee \neg c$	$\vee i$
10.	$\neg b \wedge \neg c \wedge \neg d$	assume
11.	$\neg c$	$\wedge e$
12.	$a \vee \neg c$	$\vee i$
13.	$a \vee \neg c$	$\vee e$
14.	a	assume
15.	$a \vee \neg d$	$\vee i$
16.	$\neg b \wedge \neg c \wedge \neg d$	assume
17.	$\neg d$	$\wedge e$
18.	$a \vee \neg d$	$\vee i$
19.	$a \vee \neg d$	$\vee e$
20.	$(a \vee \neg b) \wedge (a \vee \neg c)$	$\wedge i$
21.	$(a \vee \neg b) \wedge (a \vee \neg c) \wedge (a \vee \neg d)$	$\wedge i$

2.4

We negate φ and put it in c.n.f, conveniently after φ is negated $\neg\varphi$ is already in c.n.f so:

$\neg\varphi = a \wedge (\neg a \vee \neg b) \wedge (b \vee c) \wedge (\neg c \vee \neg d \vee e) \wedge (e \vee d) \wedge (\neg e \vee \neg c)$

Our set of clauses is:

$a, \neg a \vee \neg b, b \vee c, \neg c \vee \neg d \vee e, e \vee d, \neg e \vee \neg c$

Resolve on a, using: $a, \neg a \vee b$ to get a new set of clauses:

$\neg b, b \vee c, \neg c \vee \neg d \vee e, e \vee d, \neg e \vee \neg c$

Resolve on b, using: $\neg b, b \vee c$ to get a new set of clauses:

$c, \neg c \vee \neg d \vee e, e \vee d, \neg e \vee \neg c$

Resolve on c, using: $c, \neg c \vee \neg d \vee e$ to get a new set of clauses:

$\neg d \vee e, e \vee d, \neg e$

Resolve on e, using: $\neg d \vee e, e \vee d, \neg e$ to get a new set of clauses:

$\neg d, d$

From this we infer the empty clause $\varnothing$ so $\neg\varphi$ is a contradiction by resolution, and therefore φ is a theorem.