题号	分析	
10.1	与x无关,所以可以当做常数;因为和的两个极限均存在,所以可以拆分,这样就变成一元函数的极限问题;需要复习极限部分	
10.2	多元函数偏导数在某区域上的正负是不能直接决定该函数在这个区域的单调性的;借助多元函数偏导数的定义,控制变量的桥梁,来建立关系	
10.3	多元函数的连续性;偏导数的定义;公式法;可微	
10.4	二阶偏导	
10.5--10.7	复合函数求偏导,链式求导规则	
10.8	隐函数求偏导	
10.9	隐函数求偏导;根据题问分别求出偏导数,$P(y), P(x)$	
10.10	隐函数求偏导;按照步骤即可	
10.11	对方程求全微分;方程两边分别对x, y求偏导	
10.12	逆推出方程	
10.13	极限,构造1^{∞};微分方程	
10.14	隐函数的无条件极值;	
10.15	显函数的无条件极值	
10.16	闭区域边界上的最值(有条件极值);拉格朗日乘数法	
10.17	闭区域上的最值;闭区间内部无条件极值,闭区间边界有条件极值	
10.18	未完全理解	
10.19		

10.3

$$\begin{cases} (x^2+y^2)sin\frac{1}{\sqrt{x^2+y^2}} & x^2+y^2 \neq 0 \\ 0 & x^2+y^2=0 \end{cases}$$

(1) 判断$f(x,y)$在$(0,0)$处是否连续
根据连续的定义

- \lim_{x \to 0, y \to 0}f(x,y) = f(x_0,y_0)

- \lim_{x \to 0, y \to 0}(x^2 +y^2)sin\frac{1}{\sqrt{x^2 + y^2}} = \lim_{\rho \to 0}\rho^2sin\frac{1}{\rho} = 0 = f(0,0)
    - 此处注意不是等价无穷小,而是有界函数与无穷小的乘积

(2)偏导数f'_x(0, 0)与f'_y(0, 0)是否存在 定义法 f'_x(0, 0)

- f'_x(0, 0) = \lim_{x \to 0}\frac{f(x,0) - f(0,0)}{x-0}
- f'_x(0, 0) = \lim_{x \to 0} x^2 sin\frac{1}{|x|} = 0

(3)判断f'x(x,y), f'y(x,y)是否在$(0,0)$出连续 公式法 f'x(x,y)

- f'x(x,y) = 2x sin\frac{1}{\sqrt{x^2 + y^2}} + (x^2 + y^2)cos\frac{1}{\sqrt{1}{x^2 + y^2}}[{-1}\frac{1}{2}\frac{2x}{\sqrt{{(x^2+y^2)}^3}}]
- f'x(x,y) = 2x sin\frac{1}{\sqrt{x^2 + y^2}} - \frac{x}{\sqrt{x^2 + y^2}}cos\frac{1}{\sqrt{x^2+y^2}}
- 当(x,y) \to (0,0)时
    - 2x sin\frac{1}{\sqrt{x^2 + y^2}} \to 0
    - \frac{x}{\sqrt{x^2 + y^2}}cos\frac{1}{\sqrt{x^2+y^2}}不存在
- 故\lim_{x \to 0, y \to 0}f'_x(x,y)不存在

(4)判断$f(x,y)$在(x,y)处可微 可微的定义 \Delta z = f(0 + \Delta x, 0 + \Delta y) - f(x,y)

- \Delta z = [{(\Delta x)}^2 + {(\Delta y)}^2] \cdot sin\frac{1}{\sqrt{{(\Delta x)}^2 + {(\Delta y)}^2 }} - 0
- A = f'_x(0.0) = 0 , B = f'_y(0.0) = 0
- \lim_{x \to 0, y \to 0} \frac{\Delta z - (A\Delta x + B\Delta x)}{\sqrt{{(\Delta x)}^2 + {(\Delta y)}^2 }} = \lim_{\rho \to 0} \frac{\rho^2 sin\frac{1}{\rho}}{\rho} = 0

10.4

$$\begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

求f''_{xy}(0,0),f''_{yx}(0,0)

解 一阶偏导

- f'_{x}(0, y) = \lim_{x \to 0}\frac{f(x,y) - f(0,y)}{x-0} = \lim_{x \to 0}\frac{\frac{xy(x^2 - y^2)}{x^2 + y^2} -0}{x}= -y
- f'_y(x,0) = \lim_{y \to 0}\frac{f(x,y) - f(x,0)}{y-0} = x

二阶导数

- f''_{xy}(0,0) = \frac{d}{dx}[f'_{x}(0, y)] |_{y=0} = -1
- f''_{yx}(0,0) = 1

10.5

设z = f(e^xsiny,x^2+y^2),其中f具有二阶连续偏导数,求\frac{\partial ^2 z}{\partial x \partial y}

解

\frac{\partial z}{\partial x}

- \frac{\partial z}{\partial x} = f'_1 e^xsiny + f'_22x

\frac{\partial ^2 z}{\partial x \partial y}

- \frac{\partial ^2 z}{\partial x \partial y} = f''_{11}e^xcosy \cdot e^xsiny + f''_{12}2y \cdot e^xsiny+f'_1e^xcosy + f''_{21}e^xcosy \cdot 2x + f''_{22}2y \cdot 2x + f'_2 \cdot 0
    - f''_{21} = f''_{12}

10.7

设函数u=f(x,y)具有二阶连续偏导数,且满足等式4\frac{\partial^2 u}{\partial x^2} + 12\frac{\partial^2 u}{\partial x \partial y} + 5\frac{\partial^2 u}{\partial y^2}, 存在确定a,b的值,使等式在变换\zeta = x + ay, \eta = x + by下化简为\frac{\partial^2 u}{\partial \zeta \partial \eta}= 0 ,求a,b的值

解

- \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \zeta} + \frac{\partial u}{\partial \eta} ,\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \zeta^2} + \frac{\partial^2 u}{\partial \eta^2} + 2\frac{\partial^2 u}{\partial \eta \partial \zeta}
- \frac{\partial u}{\partial x} = a\frac{\partial u}{\partial \zeta} + b\frac{\partial u}{\partial \eta},\frac{\partial^2 u}{\partial y^2} = a^2\frac{\partial^2 u}{\partial \zeta^2} + b^2\frac{\partial^2 u}{\partial \eta^2} + 2ab\frac{\partial^2 u}{\partial \eta \partial \zeta}
- \frac{\partial^2 u}{\partial x \partial y} = a\frac{\partial^2 u}{\partial \zeta^2} + b\frac{\partial^2 u}{\partial \eta^2} + (a + b)\frac{\partial^2 u}{\partial \eta \partial \zeta}

带入

- 4\frac{\partial^2 u}{\partial x^2} + 12\frac{\partial^2 u}{\partial x \partial y} + 5\frac{\partial^2 u}{\partial y^2}

整理然后系数与\frac{\partial^2 u}{\partial \zeta \partial \eta}= 0 相比,对应系数相等

10.8

设y=y(x), z=z(x)是由方程z=xf(x+y)和F(x,y,x) = 0所确定的函数,其中f和F分别具有一阶连续导数和一阶连续偏导数,求\frac{dz}{dx}

解

分别对z=xf(x+y)和F(x,y,x) = 0关于x求导

$$\begin{cases} \frac{dz}{dx} = f + x(1 + \frac{dy}{dx})f' \\ F'_x + F'_y\frac{dy}{dx} + F'_z\frac{dz}{dx} = 0 \end{cases} \text{整理得} \begin{cases} -xf'\frac{dy}{dx} + \frac{dz}{dx} = f + xf' \\ F'_y\frac{dy}{dx} + F'_z\frac{dz}{dx} = -F'_x \end{cases}$$

可得\frac{dz}{dx}

> 消除\frac{dy}{dx},即可得到\frac{dz}{dx},为方便计算可分别用A,B替换

10.9

隐函数求偏导;根据题问分别求出偏导数,$P(y), P(x)$

关于对积分上限函数求导

$\int_x^yP(t)dt = \int_x^0P(t)dt + \int_0^yP(t)dt$

所以对x, y分别求导得P(x),-P(y)

10.11

方法一 求全微分

$F'_1 \cdot d(x+\frac{z}{y}) + F'_2 \cdot d(y + \frac{z}{x})$

10.12

求方程$\frac{\partial^2 z}{\partial x \partial y} = x +y$满足条件$z(x,0) = x, z(0,y) = y^2$的解$z = z(x,y)$

解

- 对$\frac{\partial^2 z}{\partial x \partial y} = x +y$关于y进行积分
- 获得$\frac{\partial z}{\partial x} = xy + \frac{1}{2}y^2 + \varphi(x)_1$
- 对$\frac{\partial z}{\partial x} = xy + \frac{1}{2}y^2 + \varphi(x)_1$关于$x$进行积分
- 获得$z = \frac{1}{2}x^2y + \frac{1}{2}y^2x + \varphi(x) + \psi(y)$
- 由已知$z(x,0) = x, z(0,y) = y^2$
    - 故
        - $\varphi(x) + \psi(0) = x$
        - $\varphi(0) + \psi(y) = y^2$
- 又观察$z(0,0) = 0$
    - 故$\varphi(0) + \psi(0) = 0$
- 所以$\varphi(x) + \psi(0) + \varphi(0) + \psi(y) = x + y^2$

10.13

设函数$f(x, y)$可微,$\frac{\partial f}{\partial x} = -f, f(0,\frac{\pi}{2}) = 1$,且满足$\lim_{n \to \infty}{[\frac{f(0, y + \frac{1}{n})}{f(0,y)}]}^n = e^{coty}$求$f(x, y)$

解

第一步

- 可以看出$n \to \infty, \frac{f(0, y + \frac{1}{n})}{f(0,y)} \to 1$
- 构造1^{∞}极限
    - $\lim_{n \to \infty}{[\frac{f(0, y + \frac{1}{n})}{f(0,y)}]}^n$
    - $\lim_{n \to \infty}{[1 + \frac{f(0, y + \frac{1}{n})}{f(0,y)} -1]}^n$
    - 通分
        - $\lim_{n \to \infty}{[1 + \frac{f(0, y + \frac{1}{n} - f(0,y))}{f(0,y)}]}^n$

        - $\lim_{n \to \infty}\{ {[1 + \frac{f(0, y + \frac{1}{n} - f(0,y))}{f(0,y)}]}^{\frac{f(0,y)}{f(0, y + \frac{1}{n} - f(0,y))}} \}^{\frac{f(0, y + \frac{1}{n} - f(0,y))}{f(0,y)} \cdot n}$
            - $\lim_{n \to \infty}\frac{f(0, y + \frac{1}{n} - f(0,y))}{f(0,y)} = 0$
            - $\lim_{n \to \infty}\frac{f(0, y + \frac{1}{n} - f(0,y))}{f(0,y)} \cdot n$
            - $\lim_{n \to \infty}\frac{f(0, y + \frac{1}{n} - f(0,y))}{\frac{1}{n}} \cdot \frac{1}{f(0,y)} = \frac{f'_y(0,y)}{f(0,y)}$
- 借助$\lim_{t \to 0}{(1 + t)}^{\frac{1}{t}}$
    - $\lim_{n \to \infty}{[\frac{f(0, y + \frac{1}{n})}{f(0,y)}]}^n = e^{\frac{f'_y(0,y)}{f(0,y)}}$
- 故$e^{\frac{f'_y(0,y)}{f(0,y)}} = e^{coty}$
    - 即$\frac{f'_y(0,y)}{f(0,y)} = coty$

第二步

- 对$\frac{\partial f}{\partial x} = -f$两边关于$x$积分
    - $\frac{1}{f} \frac{\partial f}{\partial x} = -1$
    - $ln|f| = -x + C_1(y)$
- 得$f(x,y) = C(y)e^{-x}$,($C(y) = \pm e^{C_1(y)}$)

第三步 综合第一步和第二步

- $f'_y(x,y) = C'(y)e^{-x}$
- 带入(0,y),获得$f'_y(0,y)=C'(y)$
- 由$\frac{f'_y(0,y)}{f(0,y)} = coty$
- 从而$C'(y) = f(0,y)coty$
- 对$C'(y) = f(0,y)coty$两边关于y进行积分
    - $\frac{d[C(y)]}{C(y)} = cotydy$
    - $ln|C(y)| = lna|siny|$,(a>0)
        - 注意此处的a
    - $C(y) = bsiny$,($b = \pm a$)
- 将$f(0,\frac{\pi}{2}) = 1$带入$f(x,y) = C(y)e^{-x} = bsiny \cdot e^{-x}$
- 得b = 1

最后

所以

$f(x,y) = e^{-x}siny$

10.14

设z = z(x,y)是由$x^2-6x+10y^2-2yz-z^2+18=0$所确定的.试求z=z(x,y)的极值点与极值

解

第一步 : 求出可疑点(驻点)

- 对$x^2-6x+10y^2-2yz-z^2+18=0$分别关于$x, y$求偏导
  - $2x - 6y-2yz'_x-2zz'_x = 0$
  - $-6x+20y-2z-2yz'_y-2zz'_y = 0$
- 令$z'_x = 0$, $z'_y=0$
  - $x-3y=0$
  - $-3x+10y-z=0$
- 即
  - $x=3y$
  - $z=y$
- 带入$x^2-6x+10y^2-2yz-z^2+18=0$
  - 得到可疑点
    - $P_1(9,3)$, $z_1=3$
    - $P_2(-9,-3)$, $z_2=-3$

第二步: 用充分条件判断可疑点

10.15

求二元函数$f(x,y) = x^2(2+y^2) + y\ln y$的极值

解

第一步 : 求出可疑点(驻点)

第二步: 用充分条件判断可疑点

10.17

求函数$f(x,y) = x^2 + 2y^2 - x^2y^2$在区域$D = \{(x,y) \mid x^2 + y^2 \leqslant 4,y \geqslant 0\}$上的最大值与最小值

解

第一步:

- 对于区域内部使用无条件极值

第二步:

- 对于区域的边界,使用条件极值
  - 拉格朗日乘法
  - 或直接带入法,把题目的边界方程带入$f(x,y)$
- 设$L_1:x^2+y^2=4(y>0)$,于是有$f(x,y) = f(x), \sqrt{4-x^2} = x^4-5x^2+8$
  - $(x^4-5x^2+8)' = 4x^3-10x$
  - 得到$(0,2)$,$(\sqrt{\frac{5}{2}},\sqrt{\frac{5}{2}}$,$(-\sqrt{\frac{5}{2}},\sqrt{\frac{5}{2}})$

- 设$L_2:y=0$, $(-2 \leqslant x \leqslant 2)$

第三步

- 比较函数值的大小

10.18

设x, y为任意正数,求证$\frac{x^n + y^n}{2} \geqslant {\frac{(x+y)}{2}}^n$

解

- 设x+y=a,则原题转化为在x+y=a条件下, 求$f(x,y) = \frac{x^n + y^n}{2}$, $(0 \leqslant x,y \leqslant a)$的最小值问题
- 拉格朗日乘法,求出最小值

> 未理解

10.19

求函数$u =xy + 2yz$在约束条件$x^2 + y^2 + z^2 = 10$下的最大值和最小值

解

方法一

- 拉格朗日乘法

方法二

- 直接代入法

方法三

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