

Deep Generative Models

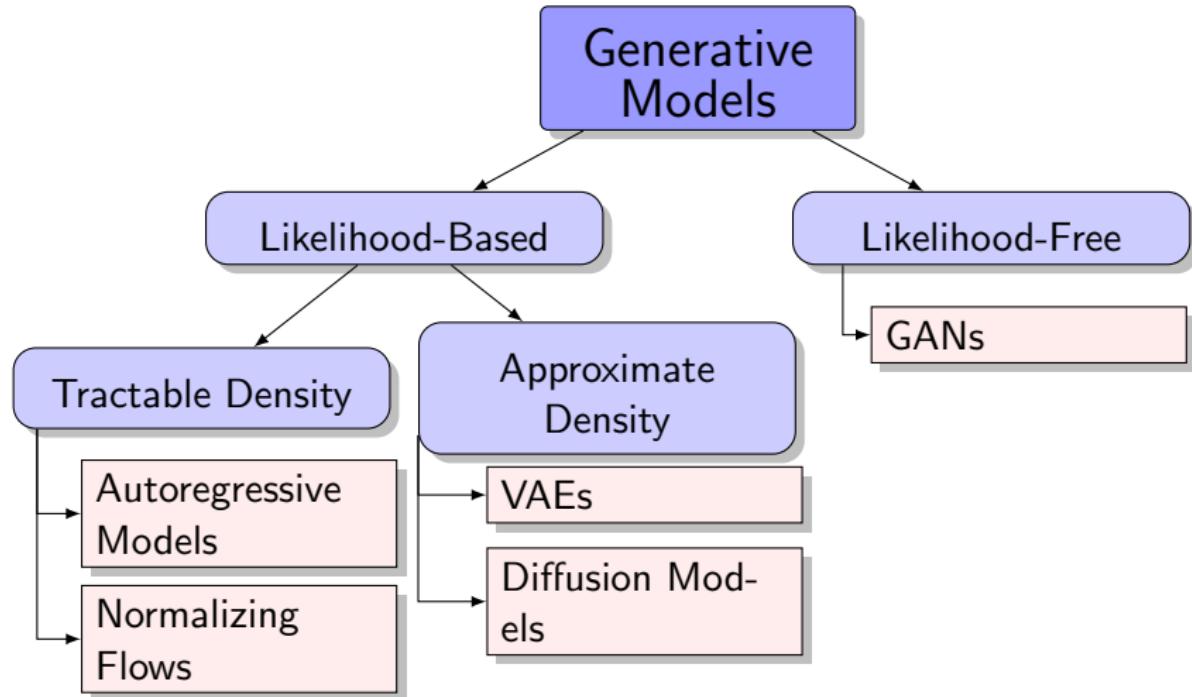
Lecture 1

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2025, Autumn

Generative Models Zoo



Outline

1. Generative Models Overview
2. Course Tricks
3. Problem Statement
4. Divergence Minimization Framework
5. Autoregressive Modeling

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VAE – The First Scalable Approach for Image Generation



DCGAN – The First Convolutional GAN for Image Generation



StyleGAN – High-Quality Face Generation



Karras T., Laine S., Aila T. A Style-Based Generator Architecture for Generative Adversarial Networks, 2018

Language Modeling at Scale

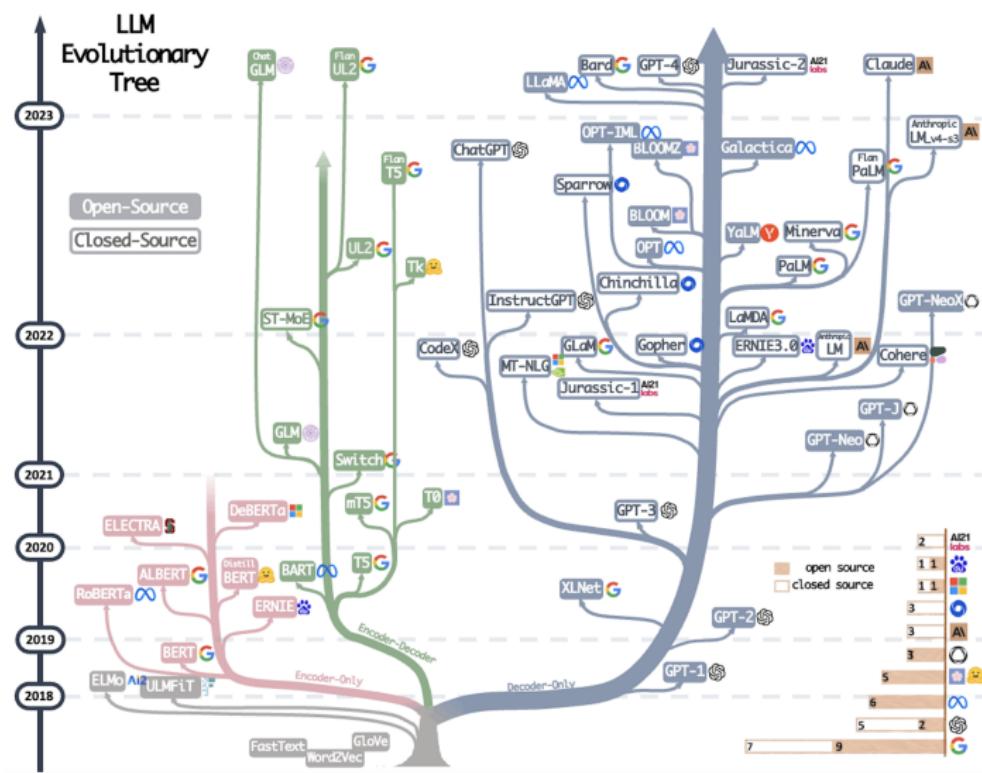
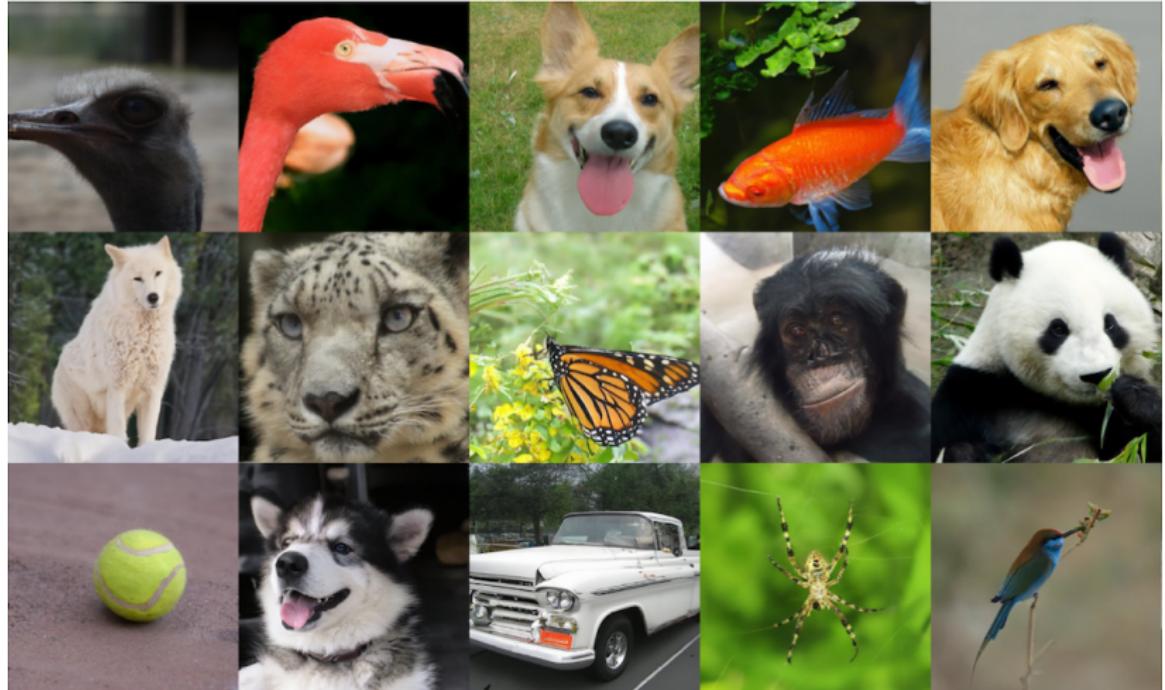


Image credit:

<https://blog.biocomm.ai/2023/05/14/open-source-proliferation-lm-evolutionary-tree/>

Denoising Diffusion Probabilistic Model



Midjourney – Impressive Text-to-Image Results



Image credit: <https://www.midjourney.com/explore>

Sora – Video Generation



Image credit: <https://openai.com/index/sora>

GPT4o Image Editing

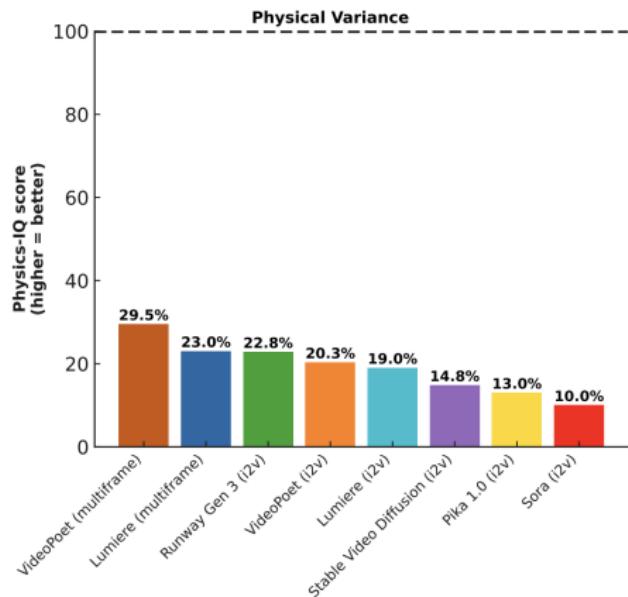
Prompt: Give this cat a detective hat and a monocle



Image credit: <https://openai.com/index/introducing-4o-image-generation/>

Open Problems in Generative Models

- ▶ Video generation
- ▶ 3D scene generation
- ▶ Understanding of physical processes
- ▶ Multimodal end-to-end models



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Course Tricks I

Log-Derivative Trick

Given a differentiable function $p : \mathbb{R}^m \rightarrow \mathbb{R}$ (usually density function),

$$\nabla \log p(\mathbf{x}) = \frac{1}{p(\mathbf{x})} \cdot \nabla p(\mathbf{x}).$$

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Jensen's Inequality

If $\mathbf{x} \in \mathbb{R}^m$ is a continuous random variable with density $p(\mathbf{x})$ and $f : \mathbb{R}^m \rightarrow \mathbb{R}$ is convex, then

$$\mathbb{E}[f(\mathbf{x})] \geq f(\mathbb{E}[\mathbf{x}]).$$

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Monte Carlo Estimation

Let $\mathbf{x} \in \mathbb{R}^m$ be a continuous random variable with density $p(\mathbf{x})$, and $\mathbf{f} : \mathbb{R}^m \rightarrow \mathbb{R}^d$ be any vector-valued function. Then,

$$\mathbb{E}_{p(\mathbf{x})}\mathbf{f}(\mathbf{x}) = \int p(\mathbf{x})\mathbf{f}(\mathbf{x})d\mathbf{x} \approx \frac{1}{n} \sum_{i=1}^n \mathbf{f}(\mathbf{x}_i), \quad \text{where } \mathbf{x}_i \sim p(\mathbf{x}).$$

Course Tricks II

Change of Variables Theorem (CoV)

Let $\mathbf{x} \in \mathbb{R}^m$ be a random vector with density $p(\mathbf{x})$, and let $\mathbf{f} : \mathbb{R}^m \rightarrow \mathbb{R}^m$ be a C^1 -diffeomorphism (\mathbf{f} and \mathbf{f}^{-1} are continuously differentiable mappings). If $\mathbf{z} = \mathbf{f}(\mathbf{x})$, then

$$p(\mathbf{x}) = p(\mathbf{z}) |\det(\mathbf{J}_\mathbf{f})| = p(\mathbf{z}) \left| \det \left(\frac{\partial \mathbf{z}}{\partial \mathbf{x}} \right) \right|$$

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Proof (1D)

Assume f is monotonically increasing.

$$F_Y(y) = P(Y \leq y) = P(x \leq f^{-1}(y)) = F_X(f^{-1}(y))$$

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$$p(y) = \frac{dF_Y(y)}{dy} = \frac{dF_X(f^{-1}(y))}{dy} = \frac{dF_X(x)}{dx} \frac{df^{-1}(y)}{dy} = p(x) \frac{df^{-1}(y)}{dy}$$

Course Tricks III

Law of the Unconscious Statistician (LOTUS)

Let $\mathbf{x} \in \mathbb{R}^m$ be a continuous random variable with density $p(\mathbf{x})$, and let $\mathbf{f} : \mathbb{R}^m \rightarrow \mathbb{R}^m$ be measurable. If $\mathbf{y} = \mathbf{f}(\mathbf{x})$, then

$$\mathbb{E}_{p(\mathbf{y})}\mathbf{g}(\mathbf{y}) = \int p(\mathbf{y})\mathbf{g}(\mathbf{y})d\mathbf{y}$$

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Dirac Delta Function

Any deterministic variable \mathbf{x}_0 can be interpreted as a random variable with density $p(\mathbf{x}) = \delta(\mathbf{x} - \mathbf{x}_0)$.

$$\delta(\mathbf{x}) = \begin{cases} +\infty, & \mathbf{x} = \mathbf{x}_0 \\ 0, & \mathbf{x} \neq \mathbf{x}_0 \end{cases} \quad \int \delta(\mathbf{x})d\mathbf{x} = 1$$

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The data is high-dimensional and complex. For example, image datasets live in $\mathbb{R}^{\text{width} \times \text{height} \times \text{channels}}$. The curse of dimensionality makes accurately estimating $p_{\text{data}}(\mathbf{x})$ infeasible.

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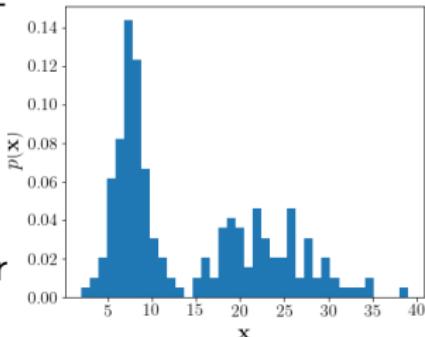
Note: here we use a strong assumption that our data is continuous (thus avoiding the domain of texts).

Histogram as a Generative Model

Assume $x \sim \text{Categorical}(\pi)$. The histogram model is fully characterized by

$$\hat{\pi}_k = \hat{\pi}(x = k) = \frac{\sum_{i=1}^n [x_i = k]}{n}.$$

Curse of dimensionality: The number of bins rises exponentially.

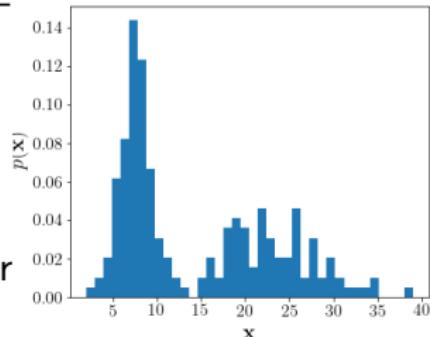


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MNIST example: 28×28 grayscale images, with each image $\mathbf{x} = (x_1, \dots, x_{784})$, $x_i \in \{0, 1\}$:

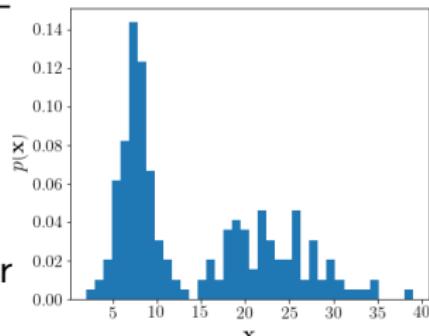
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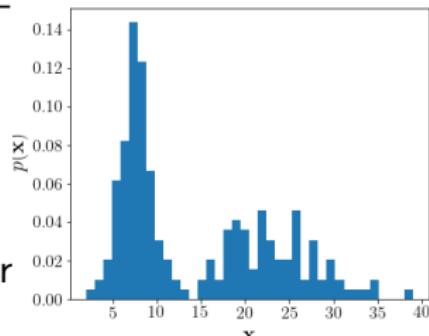
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Question: How many parameters are required in these cases?

$$p_{\text{data}}(\mathbf{x}) = p(x_1) \cdot p(x_2) \cdot \dots \cdot p(x_m);$$

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Conditional Models

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- ▶ $\mathbf{y} = \emptyset, \mathbf{x} = \text{image} \Rightarrow \text{unconditional image model}$
- ▶ $\mathbf{y} = \text{class label}, \mathbf{x} = \text{image} \Rightarrow \text{class-conditional image model}$
- ▶ $\mathbf{y} = \text{text prompt}, \mathbf{x} = \text{image} \Rightarrow \text{text-to-image model}$
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- ▶ $\mathbf{y} = \text{image}, \mathbf{x} = \text{text} \Rightarrow \text{image-to-text (image captioning) model}$
- ▶ $\mathbf{y} = \text{English text}, \mathbf{x} = \text{Russian text} \Rightarrow \text{sequence-to-sequence model (machine translation) model}$
- ▶ $\mathbf{y} = \text{sound}, \mathbf{x} = \text{text} \Rightarrow \text{speech-to-text (automatic speech recognition) model}$
- ▶ $\mathbf{y} = \text{text}, \mathbf{x} = \text{sound} \Rightarrow \text{text-to-speech model}$

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What is a Divergence?

Let \mathcal{P} be the set of all probability distributions. A mapping $D : \mathcal{P} \times \mathcal{P} \rightarrow \mathbb{R}$ is called a **divergence** if

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Divergence Minimization Problem

$$\min_{\theta} D(p_{\text{data}} \| p_{\theta})$$

where $p_{\text{data}}(\mathbf{x})$ is the true data distribution and $p_\theta(\mathbf{x})$ is the model distribution.

Forward KL vs Reverse KL (Kullback-Leibler Divergence)

Forward KL

$$\text{KL}(p_{\text{data}} \| p_{\theta}) = \int p_{\text{data}}(\mathbf{x}) \log \frac{p_{\text{data}}(\mathbf{x})}{p_{\theta}(\mathbf{x})} d\mathbf{x} \rightarrow \min_{\theta}$$

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Maximum Likelihood Estimation (MLE)

Let $\{\mathbf{x}_i\}_{i=1}^n$ be i.i.d. observed samples.

$$\theta^* = \arg \max_{\theta} \prod_{i=1}^n p_{\theta}(\mathbf{x}_i) = \arg \max_{\theta} \sum_{i=1}^n \log p_{\theta}(\mathbf{x}_i).$$

Forward KL vs Reverse KL: MLE as Forward KL

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$$\begin{aligned}\text{KL}(p_{\text{data}} \| p_{\theta}) &= \int p_{\text{data}}(\mathbf{x}) \log \frac{p_{\text{data}}(\mathbf{x})}{p_{\theta}(\mathbf{x})} d\mathbf{x} \\ &= \int p_{\text{data}}(\mathbf{x}) \log p_{\text{data}}(\mathbf{x}) d\mathbf{x} - \int p_{\text{data}}(\mathbf{x}) \log p_{\theta}(\mathbf{x}) d\mathbf{x}\end{aligned}$$

Forward KL vs Reverse KL: MLE as Forward KL

Forward KL

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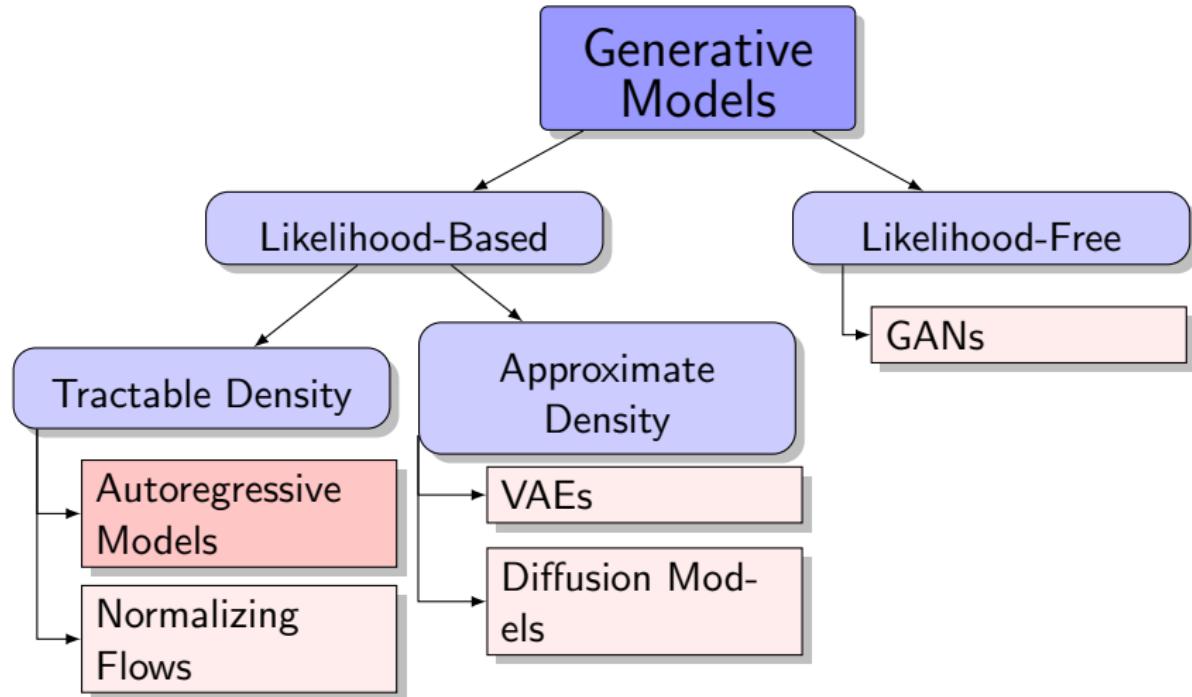
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Outline

1. Generative Models Overview
2. Course Tricks
3. Problem Statement
4. Divergence Minimization Framework
5. Autoregressive Modeling

Generative Models Zoo



Autoregressive Modeling

MLE Problem

$$\theta^* = \arg \max_{\theta} \prod_{i=1}^n p_{\theta}(\mathbf{x}_i) = \arg \max_{\theta} \sum_{i=1}^n \log p_{\theta}(\mathbf{x}_i)$$

Autoregressive Modeling

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- ▶ Thus, efficient computation of both $\log p_{\theta}(\mathbf{x})$ and its gradient $\frac{\partial \log p_{\theta}(\mathbf{x})}{\partial \theta}$ is crucial.

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Likelihood as a Product of Conditionals

For $\mathbf{x} = (x_1, \dots, x_m)$, $\mathbf{x}_{1:j} = (x_1, \dots, x_j)$,

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- ▶ Each conditional $p_{\theta}(x_j | \mathbf{x}_{1:j-1})$ can be modeled using a neural network.
- ▶ Modeling all conditionals separately isn't feasible. To address this, we share parameters across all conditionals.

Autoregressive Models: MLP

For large j , the conditional $p_{\theta}(x_j | \mathbf{x}_{1:j-1})$ becomes intractable as the history $\mathbf{x}_{1:j-1}$ grows variable-length.

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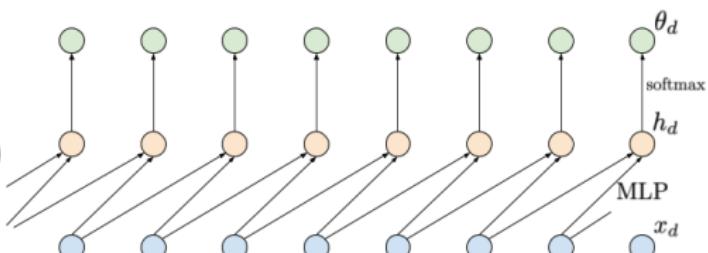
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Example

- ▶ $d = 2$
- ▶ $x_j \in \{0, 255\}$
- ▶ $\mathbf{h}_j = \text{MLP}_{\theta}(x_{j-1}, x_{j-2})$
- ▶ $\pi_j = \text{softmax}(\mathbf{h}_j)$
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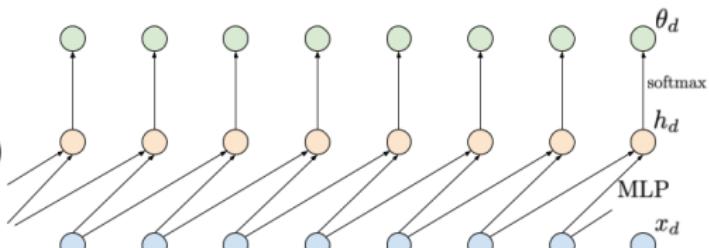
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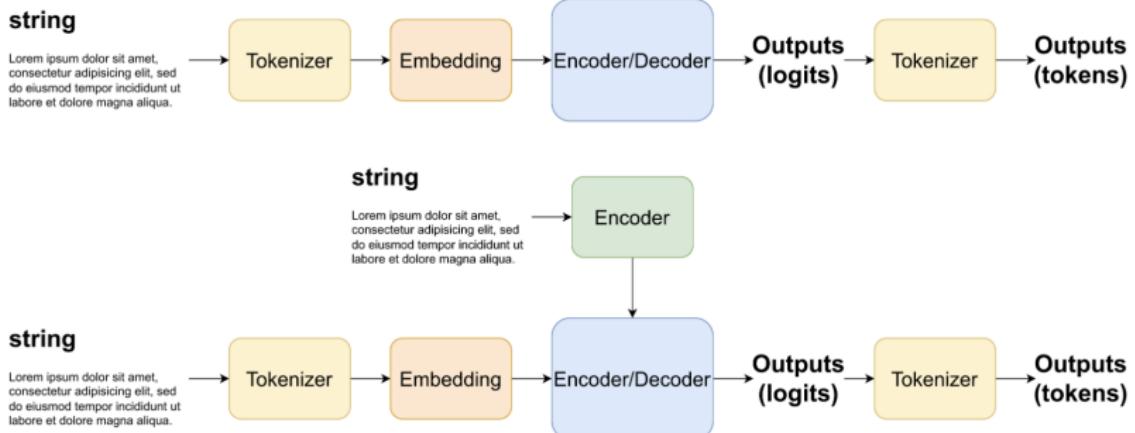
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Can we also model continuous-valued data, not just the discrete case?

Autoregressive Models: LLM

$$p_{\theta}(x_j | \mathbf{x}_{1:j-1}) = p_{\theta}(x_j | \mathbf{x}_{j-d:j-1}), \quad d \text{ is the context window.}$$



Autoregressive Models for Images

How do we model the distribution $p_{\text{data}}(\mathbf{x})$ of natural images?

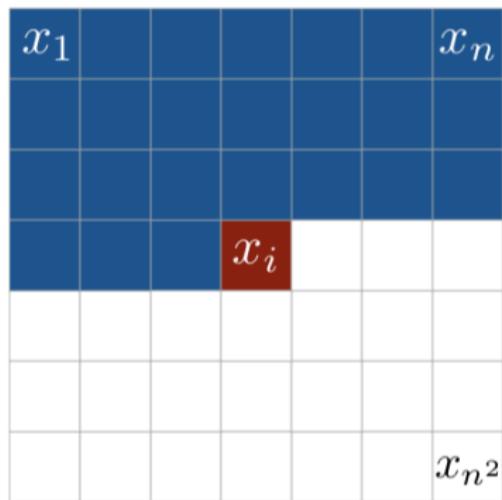
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Autoregressive Models for Images

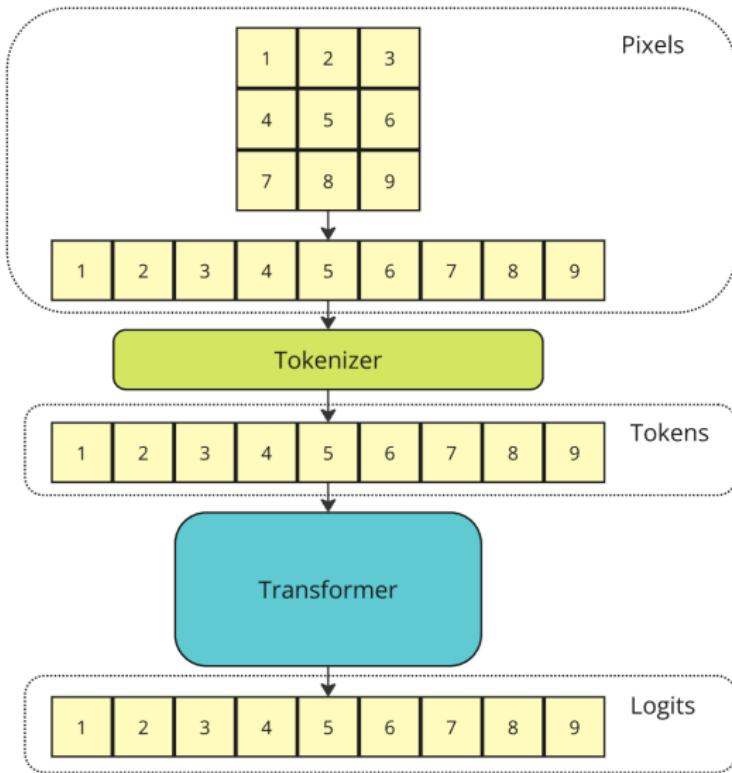
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- ▶ A pixel ordering must be selected; the raster scan is a standard choice.
- ▶ RGB channel dependencies can be modeled explicitly as well.



Autoregressive Models: ImageGPT



Summary

- ▶ Our target is to approximate the data distribution both for density estimation and for generation.
- ▶ The divergence minimization framework offers a principled way to learn distributions that match the data.
- ▶ Minimizing the forward KL divergence is equivalent to maximum likelihood estimation.
- ▶ Autoregressive models decompose the joint distribution as a product of conditionals.
- ▶ Autoregressive sampling is simple, but inherently sequential.
- ▶ Joint density evaluation multiplies all conditional probabilities $p_{\theta}(x_j | \mathbf{x}_{1:j-1})$.
- ▶ ImageGPT applies a transformer architecture to sequences of raster-ordered image pixels.