5.3 HW

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1 Problem 3

Q : Prove by induction on the number of elements in the finite set B that: If A is denumerable and B is finite, then $A \cup B$ is denumerable.

Answer: By theorem since A is denumerable, then $A \cup \{x\}$ is denumerable, where x is a finite element. This proves our base case.

Suppose that $k \in \{0, 1, 2, ...\}$ and every set C with up to k elements has the property that $A \cup C$ is denumerable.

Further suppose that $\overline{\overline{B}} = k + 1$.

Let $g : B \to \mathbb{N}_{k+1}$ be a bijection.

Let $x \in B$ be the element for which $g(x) = k + 1$.

Let $D = B - \{x\}$. Then $g|_D = D \to \mathbb{N}_k$ is a bijection. The inductive hypothesis applies because $\overline{\overline{D}} = k$, so $A \cup D$ is denumerable.

And the union of a denumerable set and a finite element is denumerable, see supra, so $A \cup D \cup \{x\} = A \cup B$ is denumerable.

2 Problem 4

Q : Complete the proof of Theorem 5.3.6 by showing that the function h as defined below is one-to-one and onto $A \cup B$. Where $f : \mathbb{N} \xrightarrow[\text{onto}]{\text{1-1}} A$, $g : \mathbb{N} \xrightarrow[\text{onto}]{\text{1-1}} B$, and

$$h(n) = \begin{cases} f(\frac{n+1}{2}) & \text{if } n \text{ is odd} \\ g(\frac{n}{2}) & \text{if } n \text{ is even} \end{cases} \tag{1}$$

Answer: We will first show that the function is a surjection. Then, we will show that h is one-to-one. Finally, we will conclude that $h : \mathbb{N} \xrightarrow[\text{onto}]{\text{1-1}} A \cup B$.

At the outset, we note that while the domains of the functions f and g are given as the natural numbers, we may use any denumerable set as their domain while preserving the bijection; for example $f : \mathbb{O}^+ \xrightarrow[\text{onto}]{\text{1-1}} A$, $g : \mathbb{E}^+ \xrightarrow[\text{onto}]{\text{1-1}} B$, where $\text{Dom}(g) = \mathbb{E}^+$, $\text{Dom}(f) = \mathbb{O}^+$, and $\text{Dom}(g) \cap \text{Dom}(f) = \emptyset$.

Moreover, we may understand the function h as $h = f \cup g$. We may do this because $h : (\mathbb{E}^+ \cup \mathbb{O}^+) = (\mathbb{N} \cup \mathbb{N}) = \mathbb{N} \to A \cup B$.

Lastly, we note that since $\text{Rng}(f) = A$ and $\text{Rng}(g) = B$ where $A \cap B = \emptyset$, then $\text{Rng}(f) \cap \text{Rng}(g) = \emptyset$.

Surjection: To show that h is a surjection onto the union of A and B we simply map every element of A to the positive odd numbers, and every element of B to the positive even numbers. We can do this because there is a bijection from the positive even and odd numbers to any denumerable set, such as A and B. And since the union of the positive even and odd numbers is the natural numbers, we have covered the whole domain of h.

Therefore, h is a surjection from the natural numbers to $A \cup B$.

Injection: We show that an injection exists between the natural numbers and $A \cup B$. Suppose $n, m \in \mathbb{N}$. There are three cases, either m and n are odd, both are even, or one is odd and the other is even which we treat as a single case without loss of generality.

1. *n and m are both odd*: then $h(n) = f(n) = f(m) = h(m)$. Since f is a bijection to the odd/natural numbers, $n = m$.
2. *n and m are both even*: then $h(n) = g(n) = g(m) = h(m)$. Since g is a bijection to the even/natural numbers, $n = m$.
3. *n is odd and m is even*: then $h(n) = f(n) = h(m) = g(m)$, $h(n) \in A$ and $h(m) \in B$. Since $A \cap B = \emptyset$, $h(n) \neq h(m)$. This case is impossible.

Consequently, h is also an injection from the natural numbers to $A \cup B$, so h is a bijection from the natural numbers to the set $A \cup B$. En summa, $h : \mathbb{N} \xrightarrow[\text{onto}]{\text{1-1}} A \cup B$.

3 Problem 10a

Q : If $A \subseteq B$ and B is denumerable, then A is denumerable

Answer: *False.* Suppose: $A = \{17, 19\}$, $B = \mathbb{W}$, where $A \subseteq B$. Clearly A is a finite set that contains only two elements, and it is a subset of B. Thus, the proposition is false.

4 Problem 10b

Q : If $A \subseteq B$ and A is denumerable, then B is denumerable

Answer: *False.* Suppose: $A = \mathbb{W}$ and $B = \mathbb{R}$, where clearly $A \subseteq B$ because $\mathbb{W} \subset \mathbb{R}$. Since B contains the real numbers, it is uncountably infinite and thus not denumerable.

5 Problem 10c

Q : If A and B are denumerable, then the set $A - B$ is denumerable.

Answer: *False*. Suppose: $A = \mathbb{W}$ and that $B = \mathbb{Q}$, so $A - B = \emptyset$. By definition the empty set ($\emptyset$) is finite, so it is not denumerable.