Final Exam Part 1 of 2

Gerald Sufleta

December 2022

1 Problem

Prove that a relation R on a set A is asymmetric if and only if it is antisymmetric and irreflexive.

2 Solution

We will prove this by breaking out the biconditional into its component conditionals, namely:

$$\text{Asymmetric} \implies (\text{Antisymmetric} \wedge \text{Irreflexive}) \tag{1}$$

$$(\text{Antisymmetric} \wedge \text{Irreflexive}) \implies \text{Asymmetric} \tag{2}$$

2.1 Conditional 1

Here, we will prove (1) by assuming R is asymmetric, then showing it is antisymmetric and irreflexive.

If R is asymmetric, then by definition:

$$(\forall x, y \in A)(x \; R \; y \implies y \not R \; x) \tag{2.1.1}$$

This is logically equivalent to:

$$(\forall x, y \in A)(x \not R \; y \vee y \not R \; x) \tag{2.1.2}$$

To show that this implies R is irreflexive, let us assume for some $x, y \in A$ that $x = y$. Then (2.1.2) reduces to:

$$(x \not R \; x \vee x \not R \; x) \equiv x \not R \; x$$

Thus, R is irreflexive.

Next, we show R is antisymmetric. For a relation to be antisymmetric, the following conditional must be true:

$$(\forall x, y \in A)(x \; R \; y \wedge y \; R \; x) \implies (x = y) \tag{2.1.3}$$

However, according to (2.1.2) which we assumed to be true, the antecedent of (2.1.3) can never be true since either ($x \not R y$) or ($y \not R x$) is true. As a result, the implication of (2.1.3) as a whole will always be true. Therefore, R is antisymmetric.

2.2 Conditional 2

Next, we show that (2) is true. We will assume R is both antisymmetric and irreflexive, and show that as a consequence (2.1.1) holds.

Since R is irreflexive, by definition we know:

$$(\forall x \in A)(x \not R x) \tag{2.2.1}$$

Further, since we assumed R is antisymmetric we know (2.1.3) holds in this case, too. It is helpful to re-write (2.1.3) into the equivalent logical statement:

$$(\forall x, y \in A)[(x = y) \vee x \not R y \vee y \not R x] \tag{2.2.2}$$

Now, suppose for some $x, y \in A$ that $x = y$. This reduces (2.1.1) to:

$$(\forall x \in A)(x \; R \; x \implies x \not R x) \tag{2.2.3}$$

However, since we assumed R is irreflexive, the antecedent of (2.2.3) is always false meaning the implication is always true. Therefore, when $x = y$, the relation R is asymmetric.

Next, suppose for some $x, y \in A$ that $x \neq y$. From our assumption in (2.2.2) combined with the fact that $x \neq y$, we know that at least one of ($x \not R y$) or ($y \not R x$) is true. Consequently, in (2.1.1) either the antecedent is false, the consequent is true, or both; thus, the implication is always true resulting in R being asymmetric.

Therefore, in both cases R is asymmetric.

2.3 Conclusion

En summa, R is asymmetric if and only if it is also irreflexive and antisymmetric.