PROPOSITIONAL LOGIC

- **Logic** deals with the formalization of natural language and reasoning methods.

- A **proposition** is a sentence that is either true or false, but not both.

- **Propositional logic** is the study of:

  – the *structure* (*syntax*) and

  – the *meaning* (*semantics*) of (simple and complex) propositions.

- Key questions are:

  * How is the truth value of a complex proposition obtained from the truth value of its simpler components?

  * Which propositions represent correct reasoning arguments?

Examples

- *Simple* propositions include:
  - I passed the exam.
  - $5 + 1 = 6$
  - $426 > 1721$
  - It is 52 degrees outside right now.
- Some *complex* propositions are:
  - John is five *and* Mary is six.
  - I passed the exam *or* I did not pass it.
- Sentences which are not propositions include:
  - Did Bill get an 20 on the exam?
  - Go away!

Propositional Formulas

- Propositions are built from **propositional variables**, which represent simple propositions, and symbols representing **logical connectives** such as **and**, **or**, **not**, etc.

- We use the letters p, q, r, s, ... to denote propositional variables and the symbols such as $\wedge$, $\vee$, and $\neg$ to denote the standard logical connectives.

- **Example 1 of proposition:** I passed the exam or I did not pass it.

  Corresponding formula: $p \vee \neg p$

- **Example 2 of proposition:** If I do not pass the exam I will fail the course.

  Corresponding formula: $\neg p \to q$

Truth tables

- A **truth table** for a formula lists all possible "situations" of truth or falsity, depending on the values assigned to the propositional variables of the formula.

- The **semantics** of logical connectives determines how propositional formulas are *evaluated* depending on the truth values assigned to propositional variables.

- $\neg$:

α	$\neg\alpha$
T	F
F	T

- $\wedge$:

α	β	$\alpha \wedge \beta$
T	T	T
T	F	F
F	T	F
F	F	F

- $\vee$:

α	β	$\alpha \vee \beta$
T	T	T
T	F	T
F	T	T
F	F	F

- $\rightarrow$:

α	β	$\alpha \rightarrow \beta$
T	T	T
T	F	F
F	T	T
F	F	T

- $\iff$:

α	β	$\alpha \iff \beta$
T	T	T
T	F	F
F	T	F
F	F	T

- **Example 1:** $p \wedge \neg q$

p	q	$\neg q$	$p \wedge \neg q$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

- **Example 2:** $p \wedge (q \vee r)$ (read "p and, in addition, q or r")

p	q	r	$q \vee r$	$p \wedge (q \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

- How many rows are there in a truth table with n propositional variables?

  For $n = 1$, there are two rows,
  for $n = 2$, there are four rows,
  for $n = 3$, there are eight rows, and so on.

  Do you see a pattern?

Tautologies and Contradictions

- A **tautology** is a formula that is always true, no matter which truth values we assign to its variables.

- $p \vee \neg p$ is a tautology

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

- A **contradiction** is a formula that is always false.

- $p \wedge \neg p$ is a contradiction

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

- Tautologies and contradictions are related.

  Theorem: If α is a tautology (resp. contradiction) then $\neg\alpha$ is a contradiction (resp. tautology).

Logical Equivalence

- If two formulas evaluate to the same truth value in all situations, so that their truth tables are the same, they are said to be **logically equivalent**.

  We write $\alpha \equiv \beta$.

- **Example 1:**

  Is $\neg(p \wedge q)$ logically equivalent to $\neg p \wedge \neg q$?

p	q	$(p \wedge q)$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	F
F	T	F	T	T	F	F
F	F	F	T	T	T	T

  Lines 2 and 3 prove that this is *not* the case.

- **Example 2:**

  Is $\neg(p \wedge q)$ logically equivalent to $\neg p \vee \neg q$?

p	q	$(p \wedge q)$	$\neg(p \wedge q)$	$\neg p$	$\neg q$	$\neg p \vee \neg q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

  Yes, $\neg(p \wedge q) \equiv \neg p \vee \neg q$.

De Morgan's Laws

- There are a number of important equivalences, including the following **De Morgan's Laws**:

  $$\neg(\alpha \wedge \beta) \equiv \neg\alpha \vee \neg\beta$$

  $$\neg(\alpha \vee \beta) \equiv \neg\alpha \wedge \neg\beta$$

  These equivalences can be used to transform a formula into a logically equivalent one of a certain syntactic form (called a "normal form").

Some Logical Equivalences

You can prove each of these:

Commutativity of $\wedge$: $\alpha \wedge \beta \equiv \beta \wedge \alpha$

Commutativity of $\vee$: $\alpha \vee \beta \equiv \beta \vee \alpha$

Associativity of $\wedge$: $\alpha \wedge (\beta \wedge \gamma) \equiv (\alpha \wedge \beta) \wedge \gamma$

Associativity of $\vee$: $\alpha \vee (\beta \vee \gamma) \equiv (\alpha \vee \beta) \vee \gamma$

Idempotence: $\alpha \equiv \alpha \wedge \alpha \equiv \alpha \vee \alpha$

Absorption: $\alpha \equiv \alpha \wedge (\alpha \vee \beta) \equiv \alpha \vee (\alpha \wedge \beta)$

Distributivity of $\wedge$: $\alpha \wedge (\beta \vee \gamma) \equiv (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$

Distributivity of $\vee$: $\alpha \vee (\beta \wedge \gamma) \equiv (\alpha \vee \beta) \wedge (\alpha \vee \gamma)$

De Morgan's Law for $\wedge$: $\neg(\alpha \wedge \beta) \equiv \neg\alpha \vee \neg\beta$

De Morgan's Law for $\vee$: $\neg(\alpha \vee \beta) \equiv \neg\alpha \wedge \neg\beta$

Double Negation: $\alpha \equiv \neg\neg\alpha$

Contradictions: $\alpha \wedge F \equiv F \equiv \alpha \wedge \neg\alpha$

Identities: $\alpha \wedge T \equiv \alpha \equiv \alpha \vee F$

Tautologies: $\alpha \vee \neg\alpha \equiv T \equiv \alpha \vee T$

Modus Ponens

- Instead of using truth tables, we will use syntactic methods to do proofs

- **Modus Ponens** (Latin for "method of affirming") is the following inference rule:

  From $(\alpha \to \beta)$ and α, infer β.

$$\begin{array}{l} \alpha \to \beta \\ \alpha \\ \hline \therefore \beta \end{array}$$

- If we assume that:

  – (1) if it is raining then today must be Monday,

  and

  – (2) it is raining,

  – **then we can conclude** that (3) today must be Monday.

But why?

– Let r denote the proposition "it is raining".

– Let m denote "today must be Monday".

Then $((r \to m) \wedge r) \to m$ is a tautology, so m is a logical consequence of $(r \to m)$ and r.

Modus Tollens

- **Modus Tollens** (Latin for "method of denying"):

  From $(\alpha \to \beta)$ and $\neg\beta$, infer $\neg\alpha$.

$$\begin{array}{l} \alpha \to \beta \\ \neg\beta \\ \hline \therefore \neg\alpha \end{array}$$

Sherlock Holmes in Action

"And now we come to the great question as to the reason why. Robbery has not been the object of this murder, for nothing was taken. Was it politics, or was it a woman? That is the question confronting me. I was inclined from the first to the latter supposition. Political assassins are only too glad to do their work and fly. This murder had, on the contrary, been done most deliberately and the perpetrator had left his tracks all over the room, showing he had been there all the time." – A. Conan Doyle, *A Study in Scarlet*

What did Sherlock Holmes conclude?

Propositions and Premises

- We can break the story into the following propositions:
  - P1: It was robbery.
  - P2: Nothing was taken.
  - P3: It was politics.
  - P4: It was a woman.
  - P5: The assassin left immediately.
  - P6: The assassin left tracks all over the room.
- Holmes identifies the following premises defined on these propositions:
  - $P2 \rightarrow \neg P1$
  - $P2$
  - $P1 \vee (P3 \vee P4)$
  - $P3 \rightarrow P5$
  - $P6 \rightarrow \neg P5$
  - $P6$
- **Conclusion:** $P4$**!**