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Abstract: In this report, the fundamental limits on the rates at which information and energy can be simultaneously transmitted over an additive white Gaussian noise channel are studied under the following assumptions: (a) the channel is memoryless; (b) the number of channel input symbols (constellation size) and block length are finite; and (c) the decoding error probability (DEP) and the energy outage probability (EOP) are bounded away from zero. In particular, it is shown that the limits on the maximum information and energy transmission rates; and the minimum DEP and EOP, are essentially set by the type induced by the code used to perform the transmission. That is, the empirical frequency with which each channel input symbol appears in the codewords. Using this observation, guidelines for optimal constellation design for simultaneous energy and information transmission are presented.

Key-words: Simultaneous information and energy transmission, finite constellations

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Transmission simultanée d'information et d'énergie avec des constellations finies

Résumé : Dans ce rapport, les limites fondamentales des débits auxquelles l'information et l'énergie peuvent être transmises simultanément sur un canal de bruit gaussien blanc additif sont étudiées sous les hypothèses suivantes: (a) le canal est sans mémoire; (b) le nombre de symboles d'entrée du canal (taille de la constellation) et la durée de la transmission sont finis; et (c) la probabilité d'erreur de décodage (DEP) et la probabilité de coupure d'énergie (EOP) sont limitées par des bornes inférieures strictement positives. En particulier, il est montré que les limites sur les débits maximaux de transmission d'information et d'énergie et les DEP et EOP minimaux, sont essentiellement fixés par le *type* induit par le code utilisé pour effectuer la transmission. C'est-à-dire la fréquence empirique à laquelle chaque symbole d'entrée de canal apparaît dans les mots de code. En utilisant cette observation, des recommandations pour la conception optimale des constellations pour la transmission simultanée dinformation et d'énergie sont présentées.

Mots-clés : Transmission simultanée d'information et d'énergie, constellations finies

Contents

1 Introduction

Nikola Tesla suggested that radio frequency signals can be used to simultaneously transmit both information and energy in 1914 [1]. About two centuries later, simultaneous information and energy transmission (SIET), also known as simultaneous wireless information and power transfer (SWIPT) is one of the technologies that might be implemented in 6G communication systems in the near future, c.f., [2]. In the following, for the sake of correctness, the denomination "energy transmission" is preferred against "power transfer", and thus, the acronym SIET is adopted in the remaining of this report.

SIET implies a fundamental trade-off between the amount of energy and information that can be simultaneously transmitted by a signal. This has been the subject of intense research, c.f., [3–6] and [7]. In [3], a capacity-energy function is defined in order to determine the fundamental limit on the information transmission rate subject to the fact that the average energy at the channel output is not smaller than a given threshold. Therein, the underlying assumption is that the communication duration in channel uses is infinitely long. This guarantees that the decoding error probability (DEP) and the energy outage probability (EOP) can be made arbitrarily close to zero, and thus, the focus is only on the information transmission rate and the energy transmission rate. This analysis has been extended to multi-user channels. In this case, the notion of information-energy capacity region generalizes to the set of all information and energy rate tuples that can be simultaneously achieved in the asymptotic block length regime [4]. For instance, the information-energy capacity region of the Gaussian multiple access channel is characterized in [8], whereas the information-energy capacity region of the Gaussian interference channel is approximated in [9].

A first attempt to study the fundamental limits of SIET under the assumption of finite transmission duration with finite constellation sizes is presented in [5]. Therein, the study is restricted to discrete memoryless channels and show that finite transmission duration implies DEPs and EOPs that are bounded away from zero.

This work contributes in this direction and considers the problem of SIET in additive white Gaussian noise channels considering finite transmission duration and finite constellation sizes. The main results in this work highlight the intuition that codes that uniformly use all channel input symbols are associated with high information rates, whereas, codes that exclusively use the channel input symbols that carry the largest amount of energy are associated with high energy rates. More specifically, a characterization of the maximum information and energy rates and minimum DEP and EOP that can be simultaneously achieved is formulated in terms of the type the code induces on the set of channel input symbols. In this work, a type is understood in the sense of the empirical frequency with which each channel input symbol appears in the codewords [10].

2 Notation

The natural, real and complex numbers are denoted by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$, respectively. In particular, $0 \notin \mathbb{N}$. Random variables and random vectors are denoted by uppercase letters and uppercase bold letters. Scalars and vectors are denoted by lowercase letters and lowercase bold letters. The real and imaginary parts of a complex number $c \in \mathbb{C}$ are denoted by $\Re(c)$ and $\Im(c)$. The Kullback-Leibler (KL) divergence between two measures P and Q, with P absolutely continuous with Q, is denoted by $D(P||Q) = \int \log \frac{\mathrm{d}P}{\mathrm{d}Q} \mathrm{d}P$, where $\frac{\mathrm{d}P}{\mathrm{d}Q}$ denotes the Radon-Nykodim derivative of P with respect to Q.

3 System Model

Consider a communication system formed by a transmitter, an information receiver (IR), and an energy harvester (EH). The objective of the transmitter is to simultaneously send information to the IR at a rate of R bits per second; and energy to the EH at a rate of B Joules per second over an additive white Gaussian noise (AWGN) channel. That is, given a channel input $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathsf{T}} \in \mathbb{C}^n$, with $n \in \mathbb{N}$, the outputs of the channel are the random vectors

$$\boldsymbol{Y} = \boldsymbol{x} + \boldsymbol{N}_1, \text{ and} \tag{1a}$$

$$\boldsymbol{Z} = \boldsymbol{x} + \boldsymbol{N}_2, \tag{1b}$$

where n is the duration of the transmission in channel uses; and the vectors $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_n)^{\mathsf{T}} \in \mathbb{C}^n$ and $\boldsymbol{Z} = (Z_1, Z_2, \ldots, Z_n)^{\mathsf{T}} \in \mathbb{C}^n$ are the inputs of the IR and the EH, respectively. The components of the random vectors $\boldsymbol{N}_1 = (N_{1,1}, N_{1,2}, \ldots, N_{1,n})^{\mathsf{T}} \in \mathbb{C}^n$ and $\boldsymbol{N}_2 = (N_{2,1}, N_{2,2}, \ldots, N_{2,n})^{\mathsf{T}} \in \mathbb{C}^n$ are independent and identically distributed. More specifically, for all $(i, j) \in \{1, 2\} \times \{1, 2, \ldots, n\}$, $N_{i,j}$ is a complex circularly symmetric Gaussian random variable whose real and imaginary parts have zero means and variances $\frac{1}{2}\sigma^2$.

That is, for all $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)^{\mathsf{T}} \in \mathbb{C}^n$, for all $\boldsymbol{z} = (z_1, z_2, \ldots, z_n)^{\mathsf{T}} \in \mathbb{C}^n$, and for all $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathsf{T}} \in \mathbb{C}^n$, it holds that the joint probability density function of the channel outputs $(\boldsymbol{Y}, \boldsymbol{Z})$ satisfies $f_{\boldsymbol{Y}\boldsymbol{Z}|\boldsymbol{X}}(\boldsymbol{y}, \boldsymbol{z}|\boldsymbol{x}) = f_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x}) f_{\boldsymbol{Z}|\boldsymbol{X}}(\boldsymbol{z}|\boldsymbol{x})$, where

$$f_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x}) = \prod_{t=1}^{n} f_{Y|X}(y_t|x_t) \text{ and} \tag{2}$$

$$f_{\boldsymbol{Z}|\boldsymbol{X}}(\boldsymbol{z}|\boldsymbol{x}) = \prod_{t=1}^{n} f_{Z|X}(z_t|x_t), \tag{3}$$

and for all $t \in \{1, 2, \ldots, n\}$,

$$f_{Y|X}(y_t|x_t) = \frac{1}{\pi\sigma^2} \exp\left(-\frac{|y_t - x_t|^2}{\sigma^2}\right), \tag{4a}$$

$$= \frac{1}{\pi\sigma^2} \exp\left(-\frac{(\Re(y_t) - \Re(x_t))^2 + (\Im(y_t) - \Im(x_t))^2}{\sigma^2}\right), \tag{4b}$$

$$f_{Z|X}(z_t|x_t) = \frac{1}{\pi\sigma^2} \exp\left(-\frac{|z_t - x_t|^2}{\sigma^2}\right), \tag{4c}$$

$$= \frac{1}{\pi\sigma^2} \exp\left(-\frac{(\Re(z_t) - \Re(x_t))^2 + (\Im(z_t) - \Im(x_t))^2}{\sigma^2}\right). \tag{4d}$$

Within this framework, two tasks must be accomplished: information transmission and energy transmission.

3.1 Information Transmission

Assume that the information transmission takes place using a modulation scheme that uses L symbols. That is, there is a set

$$\mathcal{X} \triangleq \{x^{(1)}, x^{(2)}, \ldots, x^{(L)}\} \subset \mathbb{C} \tag{5}$$

that contains all possible channel input symbols, and

$$L \triangleq |\mathcal{X}|\,. \tag{6}$$

Let M be the number of message indices to be transmitted within n channel uses. That is,

$$M \leqslant 2^{n\lfloor \log_2 L \rfloor}. \tag{7}$$

To reliably transmit a message index, the transmitter uses an (n, M)-code defined as follows.

Definition 3.1. *(n, M)-code: An (n, M)-code for the random transformation in* (1) *is a system:*

$$\{(\boldsymbol{u}(1), \mathcal{D}_1), (\boldsymbol{u}(2), \mathcal{D}_2), \ldots, (\boldsymbol{u}(M), \mathcal{D}_M)\}\,, \tag{8}$$

where, for all $(i, j) \in \{1, 2, \ldots, M\}^2, i \neq j$,

$$\boldsymbol{u}(i) = (u_1(i), u_2(i), \ldots, u_n(i)) \in \mathcal{X}^n, \tag{9a}$$

$$\mathcal{D}_i \cap \mathcal{D}_j = \phi, \tag{9b}$$

$$\bigcup_{i=1}^{M} \mathcal{D}_i \subseteq \mathbb{C}^n, \text{ and} \tag{9c}$$

$$|u_t(i)| \leqslant P, \tag{9d}$$

where P is the peak-power constraint. Assume that the transmitter uses the (n, M)-code

$$\mathscr{C} \triangleq \{(\boldsymbol{u}(1), \mathcal{D}_1), (\boldsymbol{u}(2), \mathcal{D}_2), \ldots, (\boldsymbol{u}(M), \mathcal{D}_M)\}, \tag{10}$$

that satisfies (9). The information rate of any (n, M)-code $\mathscr{C}$ is given by

$$R(\mathscr{C}) = \frac{\log_2 M}{n} \tag{11}$$

in bits per channel use. To transmit the message index i, with $i \in \{1, 2, \ldots, M\}$, the transmitter uses the codeword $\boldsymbol{u}(i) = (u_1(i), u_2(i), \ldots, u_n(i))$. That is, at channel use t, with $t \in \{1, 2, \ldots, n\}$, the transmitter inputs the symbol $u_t(i)$ into the channel. At the end of n channel uses, the IR observes a realization of the random vector $\boldsymbol{Y} = (Y_1, Y_2, \ldots, Y_n)^{\mathsf{T}}$ in (1a). The IR decides that message index j, with $j \in \{1, 2, \ldots, M\}$, was transmitted, if the following event takes place:

$$\boldsymbol{Y} \in \mathcal{D}_j, \tag{12}$$

with $\mathcal{D}_j$ in (10). That is, the set $\mathcal{D}_j \in \mathbb{C}^n$ is the region of correct detection for message index j. Therefore, the DEP associated with the transmission of message index i is

$$\gamma_i(\mathscr{C}) \triangleq 1 - \int_{\mathcal{D}_i} f_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{u}(i))\mathrm{d}\boldsymbol{y}, \tag{13}$$

and the average DEP is given by

$$\gamma(\mathscr{C}) \triangleq \frac{1}{M}\sum_{i=1}^{M} \gamma_i(\mathscr{C}). \tag{14}$$

Using this notation, Definition 3.1 can be refined as follows.

Definition 3.2 ((n, M, ϵ)-codes)**.** *An (n, M)-code for the random transformation in (1), denoted by $\mathscr{C}$, is said to be an (n, M, ϵ)-code if*

$$\gamma(\mathscr{C}) < \epsilon. \tag{15}$$

3.2 Energy Transmission

Given a channel output $z \in \mathbb{C}$, the energy harvested from such channel output is given by a positive monotone increasing circularly symmetric function g given by,

$$g : \mathbb{C} \to [0, +\infty], \tag{16}$$

with $g(0) = 0$. The energy transmission task must ensure that a minimum average energy B is harvested at the EH at the end of n channel uses. Let $\bar{g} : \mathbb{C}^n \to [0, +\infty]$ be a positive function such that given n channel outputs $\boldsymbol{z} = (z_1, z_2, \ldots, z_n)$, the average energy is

$$\bar{g}(\boldsymbol{z}) = \frac{1}{n}\sum_{t=1}^{n} g(z_t), \tag{17}$$

in energy units per channel use. Assume that the transmitter uses the code $\mathscr{C}$ in (10). Then, the EOP associated with the transmission of message index i, with $i \in \{1, 2, \ldots, M\}$, is

$$\theta_i(\mathscr{C}, B) \triangleq \Pr[\bar{g}(\boldsymbol{Z}) < B | \boldsymbol{X} = \boldsymbol{u}(i)], \tag{18}$$

where the probability is with respect to the probability density function $f_{\boldsymbol{Z}|\boldsymbol{X}}$ in (3); and the average EOP is given by

$$\theta(\mathscr{C}, B) \triangleq \frac{1}{M}\sum_{i=1}^{M} \theta_i(\mathscr{C}, B). \tag{19}$$

This leads to the following refinement of Definition 3.2.

Definition 3.3 ($(n, M, \epsilon, B, \delta)$-code)**.** *An (n, M, ϵ)-code for the random transformation in* (1)*, denoted by $\mathscr{C}$, is said to be an $(n, M, \epsilon, B, \delta)$-code if*

$$\theta(\mathscr{C}, B) < \delta. \tag{20}$$

4 Information-Energy Converse Region

The results in this section are presented in terms of the types induced by the codewords of a given code. Given an $(n, M, \epsilon, B, \delta)$-code denoted by $\mathscr{C}$ of the form in (10), the type induced by the codeword $\boldsymbol{u}(i)$, with $i \in \{1, 2, \ldots, M\}$, is a probability mass function (pmf) whose support is $\mathcal{X}$ in (5). Such pmf is denoted by $P_{\boldsymbol{u}(i)}$ and for all $\ell \in \{1, 2, \ldots, L\}$,

$$P_{\boldsymbol{u}(i)}(x^{(\ell)}) \triangleq \frac{1}{n}\sum_{t=1}^{n} \mathbb{1}_{\{u_t(i) = x^{(\ell)}\}}, \tag{21}$$

where $x^{(\ell)}$ is an element of $\mathcal{X}$ in (5). The type induced by all the codewords in $\mathscr{C}$ is also a pmf on the set $\mathcal{X}$ in (5). Such pmf is denoted by $P_{\mathscr{C}}$ and for all $\ell \in \{1, 2, \ldots, L\}$,

$$P_{\mathscr{C}}(x^{(\ell)}) \triangleq \frac{1}{M}\sum_{i=1}^{M} P_{\boldsymbol{u}(i)}(x^{(\ell)}). \tag{22}$$

Using this notation, the main results are essentially upper bounds on the information rate R in (11) and energy rate B in (18); as well as, lower bounds on the average DEP ϵ and average EOP δ for all possible $(n, M, \epsilon, B, \delta)$-codes. These bounds are provided assuming that the L channel input symbols in $\mathcal{X}$ and the block length n are finite.

4.1 Energy Transmission Rate and Average EOP

The following lemma introduces an upper bound on the energy transmission rate that holds for all possible $(n, M, \epsilon, B, \delta)$-codes.

Lemma 4.1. *Given an $(n, M, \epsilon, B, \delta)$-code denoted by $\mathscr{C}$ for the random transformation in* (1) *of the form in* (10)*, the energy transmission rate B satisfies,*

$$B \leq \frac{1}{1-\delta} \sum_{\ell=1}^{L} P_{\mathscr{C}}(x^{(\ell)}) \mathbb{E}\left[g\left(x^{(\ell)} + W\right)\right], \tag{23}$$

where $P_{\mathscr{C}}$ is defined in (22) *and the expectation is with respect to W, which is a complex circularly symmetric Gaussian random variable whose real and imaginary parts have zero means and variances $\frac{1}{2}\sigma^2$.*

Proof. From (18), the following holds:

$$\begin{aligned}
\theta_i(\mathscr{C}, B) &= \Pr\left[\frac{1}{n}\sum_{t=1}^{n} g(u_t(i) + W) < B\right] && (24)\\
&= \Pr\left[\frac{1}{n}\sum_{\ell=1}^{L} n P_{\boldsymbol{u}(i)}(x^{(\ell)}) g(x^{(\ell)} + W) < B\right] && (25)\\
&= \Pr\left[\sum_{\ell=1}^{L} P_{\boldsymbol{u}(i)}(x^{(\ell)}) g(x^{(\ell)} + W) < B\right], && (26)
\end{aligned}$$

where the probability is with respect to the random variable W defined in Lemma 4.1. Plugging (26) in (19) yields,

$$\begin{aligned}
\theta(\mathscr{C}, B) &= \frac{1}{M}\sum_{i=1}^{M} \Pr\left[\sum_{\ell=1}^{L} P_{\boldsymbol{u}(i)}(x^{(\ell)}) g(x^{(\ell)} + W) < B\right] && (27)\\
&= \frac{1}{M}\sum_{i=1}^{M} \left(1 - \Pr\left[\sum_{\ell=1}^{L} P_{\boldsymbol{u}(i)}(x^{(\ell)}) g(x^{(\ell)} + W) \geq B\right]\right) && (28)\\
&\geq \frac{1}{M}\sum_{i=1}^{M} \left(1 - \frac{1}{B}\mathbb{E}\left[\sum_{\ell=1}^{L} P_{\boldsymbol{u}(i)}(x^{(\ell)}) g(x^{(\ell)} + W)\right]\right) && (29)\\
&= 1 - \frac{1}{B\,M}\sum_{i=1}^{M}\sum_{\ell=1}^{L} P_{\boldsymbol{u}(i)}(x^{(\ell)}) \mathbb{E}\left[g(x^{(\ell)} + W)\right] && (30)\\
&= 1 - \frac{1}{B}\sum_{\ell=1}^{L} P_{\mathscr{C}}(x^{(\ell)}) \mathbb{E}\left[g(x^{(\ell)} + W)\right], && (31)
\end{aligned}$$

where the probability in (27) is with respect to the random variable W; the inequality in (29) follows from Markov's inequality and the expectation is with respect to the random variable W; and the equality in (31) follows from (22). Finally, from (20) and (31),

$$\delta \geq 1 - \frac{1}{B}\sum_{\ell=1}^{L} P_{\mathscr{C}}(x^{(\ell)}) \mathbb{E}\left[g(x^{(\ell)} + W)\right], \tag{32}$$

which implies,

$$B \leq \frac{1}{1-\delta}\sum_{\ell=1}^{L} P_{\mathscr{C}}(x^{(\ell)})\mathbb{E}\left[g(x^{(\ell)}+W)\right], \tag{33}$$

and this completes the proof. □

An observation from Lemma 4.1 is that the limit on the energy rate B does not depend on the individual types $P_{\boldsymbol{u}(1)}$, $P_{\boldsymbol{u}(2)}$, ..., $P_{\boldsymbol{u}(M)}$ but on the type induced by all codewords, i.e., $P_{\mathscr{C}}$. The inequality in (32) implies the following corollary.

Corollary 4.2. *Given an $(n, M, \epsilon, B, \delta)$-code denoted by $\mathscr{C}$ for the random transformation in* (1) *of the form in* (10)*, the average EOP δ satisfies,*

$$\delta \geq \left(1 - \frac{1}{B}\sum_{\ell=1}^{L} P_{\mathscr{C}}(x^{(\ell)})\mathbb{E}\left[g(x^{(\ell)}+W\right]\right)^{+}. \tag{34}$$

where $P_{\mathscr{C}}$ is defined in (22) *and the expectation is with respect to W, which is a complex circularly symmetric Gaussian random variable whose real and imaginary parts have zero means and variances $\frac{1}{2}\sigma^2$.*

A particular class of channel input symbols is that of circular constellations.

Definition 4.1 (Circular Constellation)**.** *The set $\mathcal{X}$ in* (5) *is said to form a circular constellation if*

$$\mathcal{X} = \{Ae^{\mathrm{i}\frac{2\pi}{L}k} \subseteq \mathbb{C} : k \in \{0, 1, 2, \ldots, (L-1)\}, A \in (0, P]\}, \tag{35}$$

where i *is the imaginary unit.*

The following corollary introduces an upper bound on the energy transmission rate of codes with circular constellations (Definition 4.1).

Corollary 4.3. *Given an $(n, M, \epsilon, B, \delta)$-code $\mathscr{C}$ for the random transformation in* (1) *of the form in* (10) *with a circular constellation, the energy transmission rate B satisfies*

$$B \leq \frac{1}{1-\delta}\mathbb{E}\left[g(x+W)\right], \tag{36}$$

where $x \in \mathcal{X}$, $P_{\mathscr{C}}$ is defined in (22) *and the expectation is with respect to W, which is a complex circularly symmetric Gaussian random variable whose real and imaginary parts have zero means and variances $\frac{1}{2}\sigma^2$.*

Corollary 4.3 follows immediately from Lemma 4.1 given the fact that for all $(x_1, x_2) \in \mathcal{X}^2$, with $\mathcal{X}$ in (35), it holds that $\mathbb{E}\left[g(x_1+W)\right] = \mathbb{E}\left[g(x_2+W)\right]$, where the random variable W is defined in Lemma 4.1.

4.2 Average Decoding Error Probability

The analysis of the information transmission rate and the average DEP of a given code $\mathscr{C}$ of the form in (10) depends on the choice of the decoding sets $\mathcal{D}_1$, $\mathcal{D}_2$, ..., $\mathcal{D}_M$. Without any loss of generality, assume that for all $i \in \{1, 2, \ldots, M\}$, the decoding set $\mathcal{D}_i$ is written in the form

$$\mathcal{D}_i = \mathcal{D}_{i,1} \times \mathcal{D}_{i,2} \times \ldots \times \mathcal{D}_{i,n}, \tag{37}$$

where for all $t \in \{1, 2, \ldots, n\}$, the set $\mathcal{D}_{i,t}$ is a subset of $\mathbb{C}$.

The following lemma introduces a lower bound on the average DEP that holds for all possible $(n, M, \epsilon, B, \delta)$-codes.

Lemma 4.4. *Given an $(n, M, \epsilon, B, \delta)$-code $\mathscr{C}$ for the random transformation in* (1) *of the form in* (10)*, the average DEP ϵ satisfies,*

$$\epsilon \geq 1 - \frac{1}{M}\sum_{i=1}^{M} \exp\left(- nH(P_{\boldsymbol{u}(i)}) - nD(P_{\boldsymbol{u}(i)}||Q) + n\log\sum_{j=1}^{L}\int_{\mathcal{E}_j} f_{Y|X}(y|x^{(j)})\mathrm{d}y\right), \quad (38)$$

where, $P_{\boldsymbol{u}(i)}$ is the type defined in (21)*; the function Q is a pmf on $\mathcal{X}$ in* (5) *such that for all $i \in \{1, 2, \ldots, L\}$,*

$$Q\left(x^{(i)}\right) = \frac{\int_{\mathcal{E}_i} f_{Y|X}(y|x^{(i)})\mathrm{d}y}{\sum_{j=1}^{L}\int_{\mathcal{E}_j} f_{Y|X}(y|x^{(j)})\mathrm{d}y}; \quad (39)$$

and for all $\ell \in \{1, 2, \ldots, L\}$, the set $\mathcal{E}_\ell$ is

$$\mathcal{E}_\ell = \mathcal{D}_{i^\star, t^\star}, \quad (40)$$

where $(i^\star, t^\star)$ satisfies

$$(i^\star, t^\star) \triangleq \arg \min_{(i,t)\in\{1,2,\ldots,M\}\times\{1,2,\ldots,n\}} \int_{\mathcal{D}_{i,t}} f_{Y|X}(y|x^{(\ell)})\mathrm{d}y. \quad (41)$$

Proof. The DEP associated with message index i, $\gamma_i(\mathscr{C})$ in (13), satisfies

$$\begin{aligned}\gamma_i(\mathscr{C}) &= 1 - \prod_{t=1}^{n}\int_{\mathcal{D}_{i,t}} f_{Y|X}(y|u_t(i))\mathrm{d}y & (42)\\ &\geqslant 1 - \prod_{\ell=1}^{L}\left(\int_{\mathcal{E}_\ell} f_{Y|X}(y|x^{(\ell)})\mathrm{d}y\right)^{nP_{\boldsymbol{u}(i)}(x^{(\ell)})} & (43)\end{aligned}$$

where $P_{\boldsymbol{u}(i)}(x^{(\ell)})$ is defined in (21); the equality in (42) follows from (2) and (13); and the inequality in (43) follows from the definition of $\mathcal{E}_\ell$ in (41). Hence, from (14) and (43), it follows that:

$$\gamma(\mathscr{C}) \geq \frac{1}{M}\sum_{i=1}^{M}\left(1 - \prod_{\ell=1}^{L}\left(\int_{\mathcal{E}_\ell} f_{Y|X}(y|x^{(\ell)})\mathrm{d}y\right)^{nP_{\boldsymbol{u}(i)}(x^{(\ell)})}\right). \quad (44)$$

Evaluating the product in (44), the following holds:

$$
\begin{aligned}
&\prod_{\ell=1}^{L}\left(\int_{\mathcal{E}_\ell} f_{Y|X}(y|x^{(\ell)})\mathrm{d}y\right)^{nP_{\boldsymbol{u}(i)}(x^{(\ell)})}\\
&=\exp\left(\log\prod_{\ell=1}^{L}\left(\int_{\mathcal{E}_\ell} f_{Y|X}(y|x^{(\ell)})\mathrm{d}y\right)^{nP_{\boldsymbol{u}(i)}(x^{(\ell)})}\right), && (45)\\
&=\exp\left(\sum_{\ell=1}^{L} nP_{\boldsymbol{u}(i)}(x^{(\ell)})\log\int_{\mathcal{E}_\ell} f_{Y|X}(y|x^{(\ell)})\mathrm{d}y\right), && (46)\\
&=\exp\left(\sum_{\ell=1}^{L} nP_{\boldsymbol{u}(i)}(x^{(\ell)})\left(\log\frac{\int_{\mathcal{E}_\ell} f_{Y|X}(y|x^{(\ell)})\mathrm{d}y}{\sum_{j=1}^{L}\left(\int_{\mathcal{E}_\ell} f_{Y|X}(y|x^{(j)})\mathrm{d}y\right)}\right.\right.\\
&\quad\left.\left.+\log\sum_{j=1}^{L}\left(\int_{\mathcal{E}_\ell} f_{Y|X}(y|x^{(j)})\mathrm{d}y\right)+\log P_{\boldsymbol{u}(i)}(x^{(\ell)})-\log P_{\boldsymbol{u}(i)}(x^{(\ell)})\right)\right). && (47)
\end{aligned}
$$

Plugging (39) in (47) yields,

$$
\begin{aligned}
&\prod_{\ell=1}^{L}\left(\int_{\mathcal{E}_\ell} f_{Y|X}(y|x^{(\ell)})\mathrm{d}y\right)^{nP_{\boldsymbol{u}(i)}(x^{(\ell)})}\\
&=\exp\left(\sum_{\ell=1}^{L} nP_{\boldsymbol{u}(i)}(x^{(\ell)})\log\frac{Q\left(x^{(\ell)}\right)}{P_{\boldsymbol{u}(i)}(x^{(\ell)})}\right.\\
&\quad\left.+nP_{\boldsymbol{u}(i)}(x^{(\ell)})\log\left(\sum_{j=1}^{L}\int_{\mathcal{E}_\ell} f_{Y|X}(y|x^{(j)})\mathrm{d}y\right)+nP_{\boldsymbol{u}(i)}(x^{(\ell)})\log P_{\boldsymbol{u}(i)}(x^{(\ell)})\right) && (48)\\
&=\exp\left(-nH(P_{\boldsymbol{u}(i)})-nD(P_{\boldsymbol{u}(i)}||Q)+n\log\sum_{j=1}^{L}\int_{\mathcal{E}_\ell} f_{Y|X}(y|x^{(j)})\mathrm{d}y\right), && (49)
\end{aligned}
$$

and plugging (49) into (44) yields,

$$
\gamma(\mathscr{C})\geq 1-\frac{1}{M}\sum_{i=1}^{M}\exp\left(-nH(P_{\boldsymbol{u}(i)})-nD(P_{\boldsymbol{u}(i)}||Q)+n\log\sum_{j=1}^{L}\int_{\mathcal{E}_\ell} f_{Y|X}(y|x^{(j)})\mathrm{d}y\right). \quad (50)
$$

Finally, from (15) and (49), it follows that

$$
\epsilon > 1-\frac{1}{M}\sum_{i=1}^{M}\exp\left(-nH(P_{\boldsymbol{u}(i)})-nD(P_{\boldsymbol{u}(i)}||Q)+n\log\sum_{j=1}^{L}\int_{\mathcal{E}_\ell} f_{Y|X}(y|x^{(j)})\mathrm{d}y\right), \quad (51)
$$

which completes the proof. □

A class of codes that is of particular interest in this study is that of homogeneous codes, which are defined hereunder.

Definition 4.2 (Homogeneous Codes). *An $(n, M, \epsilon, B, \delta)$-code denoted by $\mathscr{C}$ for the random transformation in* (1) *of the form in* (10) *is said to be homogeneous if for all $i \in \{1, 2, \ldots, M\}$ and for all $\ell \in \{1, 2, \ldots, L\}$, it holds that*

$$P_{\boldsymbol{u}(i)}(x^{(\ell)}) = P_{\mathscr{C}}(x^{(\ell)}), \tag{52}$$

and for all $(i, t) \in \{1, 2, \ldots, M\} \times \{1, 2, \ldots, n\}$ for which $u_t(i) = x^{(\ell)}$, for some $\ell \in \{1, 2, \ldots, L\}$, it holds that,

$$\mathcal{D}_{i,t} = \mathcal{E}_{\ell}, \tag{53}$$

where, $P_{\boldsymbol{u}(i)}$ and $P_{\mathscr{C}}$ are the types defined in (21) *and* (22)*, respectively; the sets $\mathcal{D}_{i,t}$ are defined in* (37)*; and the sets $\mathcal{E}_{\ell}$ are defined in* (41).

Homogeneous codes are essentially $(n, M, \epsilon, B, \delta)$-codes that satisfy two conditions. First, a given channel input symbol is used the same number of times in all codewords; and second, every channel input symbol is decoded with the same decoding set independently of the codeword and/or the position in the codeword.

Lemma 4.4 simplifies for the case of homogeneous codes (Defintion 4.2) as follows.

Corollary 4.5. *Given a homogeneous $(n, M, \epsilon, B, \delta)$-code denoted by $\mathscr{C}$ for the random transformation in* (1) *of the form in* (10)*, the DEP ϵ satisfies that*

$$\epsilon > 1 - \exp\Big(- nH(P_{\mathscr{C}}) - nD(P_{\mathscr{C}}||Q) + n \log \sum_{j=1}^{L} \int_{\mathcal{E}_j} f_{Y|X}(y|x^{(j)}) \mathrm{d}y \Big), \tag{54}$$

where, $P_{\mathscr{C}}$ is the type defined in (22)*; the pmf Q is defined in* (39) *and for all $j \in \{1, 2, \ldots, L\}$, the sets $\mathcal{E}_j$ are defined in* (41).

Note that the right-hand side of (54) is minimized when the following condition is met,

$$P_{\mathscr{C}} = Q. \tag{55}$$

This observation leads to the following corollary.

Corollary 4.6. *Given a homogeneous $(n, M, \epsilon, B, \delta)$-code denoted by $\mathscr{C}$ for the random transformation in* (1) *of the form in* (10)*, such that* (55) *is satisfied, the DEP ϵ satisfies that*

$$\epsilon \geqslant 1 - \exp\left(-nH(Q) + n \log \left(\sum_{j=1}^{L} \int_{\mathcal{E}_j} f_{Y|X}(y|x^{(j)}) \mathrm{d}y \right) \right), \tag{56}$$

where the pmf Q is defined in (39) *and for all $j \in \{1, 2, \ldots, L\}$, the sets $\mathcal{E}_j$ are defined in* (41).

Note that the equality in (55) reveals the optimal choice of $P_{\mathscr{C}}$ with respect to the average DEP, which is using the channel input symbols in $\mathcal{X}$ with a probability that is proportional to the probability of correct decoding.

Lemma 4.7. *Given a homogeneous $(n, M, \epsilon, B, \delta)$-code denoted by $\mathscr{C}$ for the random transformation in* (1) *of the form in* (10)*, the DEP ϵ satisfies that*

$$\epsilon \geq 1 - \prod_{\ell=1}^{L} \left(1 - \mathrm{Q}\left(\frac{|x^{(\ell)} - \bar{x}^{(\ell)}|}{\sqrt{2\sigma^2}} - \frac{\sigma}{\sqrt{2}|x^{(\ell)} - \bar{x}^{(\ell)}|} \log\left(\frac{P_{\mathscr{C}}(\bar{x}^{(\ell)})}{P_{\mathscr{C}}(x^{(\ell)})} \right) \right) \right)^{nP_{\mathscr{C}}(x^{(\ell)})}, \tag{57}$$

where, the type $P_{\mathscr{C}}$ is the type defined in (22)*, the real σ^2 is defined in* (4) *and, for all $c \in \{1,2,\ldots,C\}$ and $\ell \in \{1,2,\ldots,L\}$, the complex $\bar{x}^{(\ell)}$ satisfies*

$$\bar{x}^{(\ell)} \in \arg\max_{x \in \mathcal{X}\setminus\{x^{(\ell)}\}} \left(1 - \mathrm{Q}\left(\frac{|x^{(\ell)} - x|}{\sqrt{2\sigma^2}} - \frac{\sigma}{\sqrt{2}|x^{(\ell)} - x|}\log\left(\frac{P_{\mathscr{C}}(x)}{P_{\mathscr{C}}(x^{(\ell)})}\right)\right)\right). \qquad (58)$$

The function Q in (57) *and* (58) *is the* Q *function defined in [11, Chapter* 2*].*

Proof. The DEP associated with message index i, $\gamma_i(\mathscr{C})$ in (13), satisfies

$$\gamma_i(\mathscr{C}) = 1 - \prod_{t=1}^{n} \int_{\mathcal{D}_{i,t}} f_{Y|X}(y|u_t(i))\mathrm{d}y \qquad (59)$$

$$\geqslant 1 - \prod_{\ell=1}^{L} \left(\int_{\mathcal{E}_\ell} f_{Y|X}(y|x^{(\ell)})\mathrm{d}y\right)^{nP_{\boldsymbol{u}(i)}(x^{(\ell)})} \qquad (60)$$

where $P_{\boldsymbol{u}(i)}(x^{(\ell)})$ is defined in (21); the equality in (59) follows from (2) and (13); and the inequality in (60) follows from the definition of $\mathcal{E}_\ell$ in (41). Hence, from (14) and (60), it follows that:

$$\gamma(\mathscr{C}) \geq \frac{1}{M}\sum_{i=1}^{M}\left(1 - \prod_{\ell=1}^{L}\left(\int_{\mathcal{E}_\ell} f_{Y|X}(y|x^{(\ell)})\mathrm{d}y\right)^{nP_{\boldsymbol{u}(i)}(x^{(\ell)})}\right). \qquad (61)$$

For a homogeneous code $\mathscr{C}$, the DEP is given by

$$\gamma(\mathscr{C}) \geq \frac{1}{M}\sum_{i=1}^{M}\left(1 - \prod_{\ell=1}^{L}\left(\int_{\mathcal{E}_\ell} f_{Y|X}(y|x^{(\ell)})\mathrm{d}y\right)^{nP_{\mathscr{C}}(x^{(\ell)})}\right), \qquad (62)$$

$$= 1 - \prod_{\ell=1}^{L}\left(\int_{\mathcal{E}_\ell} f_{Y|X}(y|x^{(\ell)})\mathrm{d}y\right)^{nP_{\mathscr{C}}(x^{(\ell)})}. \qquad (63)$$

For all $\ell, \ell' \in \{1,2,\ldots,L\}^2$ such that $\ell' \neq \ell$, denote by $\mathcal{E}_\ell^{\ell'}$, the following region

$$\mathcal{E}_\ell^{\ell'} = \{y \in \mathbb{C} : P_{\mathscr{C}}(x^{(\ell)})f_{Y|X}(y|x^{(\ell)}) > P_{\mathscr{C}}(x^{(\ell')})f_{Y|X}(y|x^{(\ell')})\}. \qquad (64)$$

From (64) and (63), it follows that

$$\gamma(\mathscr{C}) \geq 1 - \prod_{\ell=1}^{L}\left(\int_{\mathcal{E}_\ell^{\ell'}} f_{Y|X}(y|x^{(\ell)})\mathrm{d}y\right)^{nP_{\mathscr{C}}(x^{(\ell)})}, \qquad (65)$$

$$= 1 - \prod_{\ell=1}^{L}\left(1 - \mathrm{Q}\left(\frac{|x^{(\ell)} - x^{(\ell')}|}{\sqrt{2\sigma^2}} - \frac{\sigma}{\sqrt{2}|x^{(\ell)} - x^{(\ell')}|}\log\left(\frac{P_{\mathscr{C}}(x^{(\ell')})}{P_{\mathscr{C}}(x^{(\ell)})}\right)\right)\right)^{nP_{\mathscr{C}}(x^{(\ell)})}, \qquad (66)$$

$$\geq 1 - \prod_{\ell=1}^{L}\left(1 - \mathrm{Q}\left(\frac{|x^{(\ell)} - \bar{x}^{(\ell)}|}{\sqrt{2\sigma^2}} - \frac{\sigma}{\sqrt{2}|x^{(\ell)} - \bar{x}^{(\ell)}|}\log\left(\frac{P_{\mathscr{C}}(\bar{x}^{(\ell)})}{P_{\mathscr{C}}(x^{(\ell)})}\right)\right)\right)^{nP_{\mathscr{C}}(x^{(\ell)})}, \qquad (67)$$

where, the equality in (66) follows from (65) due to [12, Lemma 20.14.1] and for all $\ell \in \{1,2,\ldots,L\}$, the complex $\bar{x}^{(\ell)}$ satisfies

$$\bar{x}^{(\ell)} \in \arg\max_{x \in \mathcal{X}\setminus\{x^{(\ell)}\}} \left(1 - \mathrm{Q}\left(\frac{|x^{(\ell)} - x|}{\sqrt{2\sigma^2}} - \frac{\sigma}{\sqrt{2}|x^{(\ell)} - x|}\log\left(\frac{P_{\mathscr{C}}(x)}{P_{\mathscr{C}}(x^{(\ell)})}\right)\right)\right). \qquad (68)$$

From (15) and (67), it follows that

$$\epsilon \geq 1 - \prod_{\ell=1}^{L} \left(1 - \mathrm{Q}\left(\frac{|x^{(\ell)} - \bar{x}^{(\ell)}|}{\sqrt{2\sigma^2}} - \frac{\sigma}{\sqrt{2}|x^{(\ell)} - \bar{x}^{(\ell)}|} \log \left(\frac{P_{\mathscr{C}}(\bar{x}^{(\ell)})}{P_{\mathscr{C}}(x^{(\ell)})} \right) \right) \right)^{nP_{\mathscr{C}}(x^{(\ell)})}. \quad (69)$$

This completes the proof. □

4.3 Information Transmission Rate

A first upper bound on the information rate is obtained by upper bounding the number of codewords that a code might possess given the particular types $P_{\boldsymbol{u}(1)}$, $P_{\boldsymbol{u}(2)}$, ..., $P_{\boldsymbol{u}(n)}$; or the average type $P_{\mathscr{C}}$ in (22). The following lemma introduces such a bound for the case of a homogeneous code.

Lemma 4.8. *Given a homogeneous $(n, M, \epsilon, B, \delta)$-code denoted by $\mathscr{C}$ for the random transformation in (1) of the form in (10), the information transmission rate $R(\mathscr{C})$ in (11) is such that*

$$R(\mathscr{C}) \leq \frac{1}{n} \log_2 \left(\frac{n!}{\prod_{\ell=1}^{L} (nP_{\mathscr{C}}(x^{(\ell)}))!} \right) \leq \log_2 L, \quad (70)$$

where, $P_{\mathscr{C}}$ is the type defined in (22).

Proof. The largest number of codewords of length n that can be formed with L different channel input symbols is L^n. In this case, the codewords do not exhibit the type $P_{\mathscr{C}}$. Hence, from (11), in the absence of a constraint on the type $P_{\mathscr{C}}$, the largest rate is $\log_2 L$ bits per channel use. This justifies the inequality in the right-hand side of (70). Alternatively, given a code type $P_{\mathscr{C}}$ that satisfies (52), the number of codewords that can be constructed is upper bounded by

$$\binom{n}{nP_{\mathscr{C}}(x^{(1)})} \binom{n - nP_{\mathscr{C}}(x^{(1)})}{nP_{\mathscr{C}}(x^{(2)})} \cdots \binom{n - \sum_{\ell=1}^{L-1} nP_{\mathscr{C}}(x^{(\ell)})}{nP_{\mathscr{C}}(x^{(L)})} = \frac{n!}{\prod_{\ell=1}^{L} (nP_{\mathscr{C}}(x^{(\ell)}))!}.$$

Therefore, the information rate $R(\mathscr{C})$ in (11) satisfies

$$R(\mathscr{C}) \leq \frac{1}{n} \log_2 \left(\frac{n!}{\prod_{\ell=1}^{L} (nP_{\mathscr{C}}(x^{(\ell)}))!} \right), \quad (71)$$

which completes the proof. □

Lemma 4.8 can be written in terms of the entropy of the type $P_{\mathscr{C}}$ as shown by the following corollary.

Corollary 4.9. *Given a homogeneous $(n, M, \epsilon, B, \delta)$-code denoted by $\mathscr{C}$ for the random transformation in (1) of the form in (10), the information transmission rate $R(\mathscr{C})$ in (11) is such that*

$$\begin{aligned} R(\mathscr{C}) \leq{}& H\left(P_{\mathscr{C}}\right) + \frac{1}{n^2} \left(\frac{1}{12} - \sum_{\ell=1}^{L} \frac{1}{12P_{\mathscr{C}}(x^{(\ell)}) + 1} \right) \\ &+ \frac{1}{n} \left(\log\left(\sqrt{2\pi}\right) - \sum_{\ell=1}^{L} \log \sqrt{2\pi P_{\mathscr{C}}(x^{(\ell)})} \right) - \frac{\log n}{n} \left(\frac{L-1}{2} \right), \end{aligned} \quad (72)$$

where, $P_{\mathscr{C}}$ is the type defined in (22).

Proof. From (70), the following holds

$$R(\mathscr{C}) \leq \frac{1}{n}\log(n!) - \frac{1}{n}\sum_{\ell=1}^{L}\log\left(\left(nP_{\mathscr{C}}(x^{(\ell)})\right)!\right), \tag{73}$$

and using the Stirling's approximation [13] on the factorial terms yields

$$(nP_{\mathscr{C}}(x^{(\ell)}))! \geqslant \sqrt{2\pi}\left(nP_{\mathscr{C}}(x^{(\ell)})\right)^{nP_{\mathscr{C}}(x^{(\ell)})+\frac{1}{2}}\exp\left(-nP_{\mathscr{C}}(x^{(\ell)}) + \frac{1}{12nP_{\mathscr{C}}(x^{(\ell)})+1}\right), \text{ and} \tag{74}$$

$$n! \leqslant \sqrt{2\pi}n^{n+\frac{1}{2}}\exp\left(-n+\frac{1}{12n}\right). \tag{75}$$

From (74) and (75), the following holds,

$$\begin{aligned}\log\left((nP_{\mathscr{C}}(x^{(\ell)}))!\right) &\geq \log\left(\sqrt{2\pi}\right) + \left(nP_{\mathscr{C}}(x^{(\ell)}) + \frac{1}{2}\right)\log(nP_{\mathscr{C}}(x^{(\ell)})) - nP_{\mathscr{C}}(x^{(\ell)}) \\ &\quad + \frac{1}{12nP_{\mathscr{C}}(x^{(\ell)})+1} && (76)\\ &= \log\left(\sqrt{2\pi}\right) + nP_{\mathscr{C}}(x^{(\ell)})\log(P_{\mathscr{C}}(x^{(\ell)})) + \frac{1}{2}\log(P_{\mathscr{C}}(x^{(\ell)})) \\ &\quad + \left(nP_{\mathscr{C}}(x^{(\ell)}) + \frac{1}{2}\right)\log(n) - nP_{\mathscr{C}}(x^{(\ell)}) + \frac{1}{12nP_{\mathscr{C}}(x^{(\ell)})+1} \text{ and }, && (77)\\ \log(n!) &\leq \log\left(\sqrt{2\pi}\right) + \left(n+\frac{1}{2}\right)\log(n) - n + \frac{1}{12n} \\ &= n\log(n) - n + \frac{1}{12n} + \frac{1}{2}\log(2\pi n). && (78)\end{aligned}$$

The sum in (73) satisfies,

$$\begin{aligned}\sum_{\ell=1}^{L}\log\left(\left(nP_{\mathscr{C}}(x^{(\ell)})\right)!\right) &\geq L\log\left(\sqrt{2\pi}\right) - nH\left(P_{\mathscr{C}}\right) + \frac{1}{2}\sum_{\ell=1}^{L}\log(P_{\mathscr{C}}(x^{(\ell)})) + n\log(n) \\ &\quad + \frac{L}{2}\log(n) - n + \sum_{\ell=1}^{L}\frac{1}{12nP_{\mathscr{C}}(x^{(\ell)})+1}. && (79)\end{aligned}$$

Using (78) and (79) in (73) yields,

$$
\begin{aligned}
R(\mathscr{C}) \leq & \log(n) - 1 + \frac{1}{12n^2} + \frac{1}{2n}\log(2\pi n) - \frac{L}{n}\log\left(\sqrt{2\pi}\right) + H\left(P_{\mathscr{C}}\right) - \frac{1}{2n}\sum_{\ell=1}^{L}\log(P_{\mathscr{C}}(x^{(\ell)})) \\
& -\log(n) - \frac{L}{2n}\log(n) + 1 - \frac{1}{n}\sum_{\ell=1}^{L}\frac{1}{12nP_{\mathscr{C}}(x^{(\ell)})+1} \qquad (80) \\
\leq & H\left(P_{\mathscr{C}}\right) + \frac{1}{n^2}\left(\frac{1}{12} - \sum_{\ell=1}^{L}\frac{1}{12P_{\mathscr{C}}(x^{(\ell)})+1}\right) + \frac{1}{2n}\left(\log(2\pi n) - \sum_{\ell=1}^{L}\log(2\pi n P_{\mathscr{C}}(x^{(\ell)}))\right) \qquad (81) \\
= & H\left(P_{\mathscr{C}}\right) + \frac{1}{n^2}\left(\frac{1}{12} - \sum_{\ell=1}^{L}\frac{1}{12P_{\mathscr{C}}(x^{(\ell)})+1}\right) + \frac{1}{n}\left(\log\left(\sqrt{2\pi}\right) - \sum_{\ell=1}^{L}\log\sqrt{2\pi P_{\mathscr{C}}(x^{(\ell)})}\right) \\
& -\frac{\log n}{n}\left(\frac{L-1}{2}\right), \qquad (82)
\end{aligned}
$$

which completes the proof. □

Note that all terms in (72), except the entropy $H\left(P_{\mathscr{C}}\right)$, vanish with the block length n. This implies that the information rate is essentially constrained by the entropy of the channel input symbols. In particular, note that $H\left(P_{\mathscr{C}}\right) \leqslant \log_2 L$.

4.4 Information and Energy Trade-Off

The results presented above lead to the following result for homogeneous codes.

Theorem 4.10. *Given a homogeneous* $(n, M, \epsilon, B, \delta)$*-code denoted by* $\mathscr{C}$ *for the random transformation in* (1) *of the form in* (10), *the following holds*

$$
\begin{aligned}
R(\mathscr{C}) \leq & H\left(P_{\mathscr{C}}\right) + \frac{1}{n^2}\left(\frac{1}{12} - \sum_{\ell=1}^{L}\frac{1}{12P_{\mathscr{C}}(x^{(\ell)})+1}\right) \\
& + \frac{1}{n}\left(\log\left(\sqrt{2\pi}\right) - \sum_{\ell=1}^{L}\log\sqrt{2\pi P_{\mathscr{C}}(x^{(\ell)})}\right) - \frac{\log n}{n}\left(\frac{L-1}{2}\right); \qquad (83)
\end{aligned}
$$

$$
B \leq \frac{1}{1-\delta}\sum_{\ell=1}^{L}P_{\mathscr{C}}(x^{(\ell)})\mathbb{E}\left[g\left(x^{(\ell)}+W\right)\right]; \text{ and} \qquad (84)
$$

$$
\epsilon > 1 - \exp\Big(-nH(P_{\mathscr{C}}) - nD(P_{\mathscr{C}}||Q) + n\log\sum_{j=1}^{L}\int_{\mathcal{E}_j} f_{Y|X}(y|x^{(j)})\mathrm{d}y\Big). \qquad (85)
$$

where, $P_{\mathscr{C}}$ *is the type defined in* (22)*; the pmf* Q *is defined in* (39)*; for all* $j \in \{1, 2, \ldots, L\}$*, the sets* $\mathcal{E}_j$ *are defined in* (41)*; and the expectation in* (84) *is with respect to* W*, which is a complex circularly symmetric Gaussian random variable whose real and imaginary parts have zero means and variances* $\frac{1}{2}\sigma^2$*.*

Proof. The proof of (83) follows from Corollary 4.9; the proof of (84) follows from Lemma 4.1; and the proof of (85) follows from Corollary 4.5. □

Theorem 4.10 quantifies the trade-off between all parameters of a homogeneous $(n, M, \epsilon, B, \delta)$-code for the random transformation in (1). Let such a code be denoted by $\mathscr{C}$ of the form in (10).

Hence, both the upper bound on the information rate in (83) and the upper bound on the energy rate in (84) depend on the type $P_{\mathscr{C}}$. The information rate is essentially upper bounded by the entropy of $P_{\mathscr{C}}$. Hence, codes whose codewords are such that every channel input symbol is used the same number of times are less constrained in terms of information rate. This is the case in which $P_{\mathscr{C}}$ is a uniform distribution. Alternatively, using a uniform type $P_{\mathscr{C}}$ might dramatically constrain the energy transmission rate. For instance, if the constellation is such that for at least one pair $(x_1, x_2) \in \mathcal{X}^2$, with $\mathcal{X}$ in (35), it holds that $\mathbb{E}\left[g(x_1 + W)\right] < \mathbb{E}\left[g(x_2 + W)\right]$ where the random variable W is defined in Lemma 4.1, then using the symbol x_1 equally often as x_2 certainly constraints the energy rate. Codes than might potentially exhibit the largest energy rates are those in which the symbols x that maximize $\mathbb{E}\left[g(x + W)\right]$ are used more often. Clearly this deviates from the uniform distribution and thus, constraints the information rate.

Another interesting trade-off appears between the DEP and EOP. From Corollary 4.2, it follows that codes whose codewords contain mainly channel input symbols x that maximize $\mathbb{E}\left[g(x + W)\right]$ are more reliable from the perspective of energy transmission. Alternatively, from Corollary 4.5, it follows that codes that are more reliable in terms of information transmission are those whose codewords contain more channel input symbols with the smallest DEP. That is, when $P_{\mathscr{C}}$ approaches the pmf Q in (39).

5 Information-Energy Achievable Region

The information-energy capacity region of SIET systems with finite constellations is defined as follows.

Definition 5.1 (Information-Energy Capacity Region). *The information-energy capacity region $\mathcal{C}(n, \epsilon, \delta)$ for the random transformation in* (1) *is the set of all information and energy transmission rate pairs $(R, B) \in \mathbb{R}^2$ for which there exists an $(n, M, \epsilon, B, \delta)$-code $\mathscr{C}$ such that $\frac{\log_2 M}{n} = R(\mathscr{C})$, the average DEP $\lambda(\mathscr{C}) \leq \epsilon$, and, the average EOP $\theta(\mathscr{C}) \leq \delta$.*

5.1 Code Construction

The process of characterizing an achievable information-energy capacity region for SIET with finite constellations begins with the construction of an (n, M)-code. Let the (n, M)-code $\mathscr{C}$ be

$$\mathscr{C} \triangleq \left\{(\boldsymbol{u}(1), \mathcal{D}_1), (\boldsymbol{u}(2), \mathcal{D}_2), \ldots, (\boldsymbol{u}(M), \mathcal{D}_M)\right\}, \tag{86a}$$

The construction of the code begins with the construction of the channel input symbols. The set of channel input symbols is a modulation constellation represented by a finite subset of $\mathbb{C}$. Consider a constellation formed by C *layers*, with $C \in \mathbb{N}$. A layer is a subset of symbols in $\mathbb{C}$ that have the same magnitude. For all $c \in \{1, 2, \ldots, C\}$, denote by $L_c \in \mathbb{N}$ the number of symbols in the c^{th} layer and let $A_c \in \mathbb{R}^+$ and $\alpha_c \in [0, 2\pi]$ be the amplitude and phase shift of the symbols in layer c. Denote such a layer by $\mathcal{U}(A_c, L_c, \alpha_c)$. That is,

$$\mathcal{U}(A_c, L_c, \alpha_c) \triangleq \left\{x_c^{(\ell)} = A_c \exp\left(\mathrm{i}\left(\frac{2\pi}{L_c}\ell + \alpha_c\right)\right) \subseteq \mathbb{C} : \ell \in \{0, 1, 2, \ldots, (L_c - 1)\}\right\}, \tag{86b}$$

where i is the complex unit. Using this notation, the constellation can be described by the following set

$$\mathcal{X} = \bigcup_{c=1}^{C} \mathcal{U}(A_c, L_c, \alpha_c). \tag{86c}$$

Without any loss of generality, assume that

$$A_1 > A_2 > \ldots > A_C. \tag{86d}$$

The symbols in layer c of the form in (86b), are equally spaced along a circle of radius A_c as shown in Figure 1. The constellation induced by the set $\mathcal{X}$ is thus made up of points uniformly distributed along C concentric circles. The total number of symbols L in (6) for $\mathcal{X}$ is

$$L = \sum_{c=1}^{C} L_c. \tag{86e}$$

The construction of the (n, M)-code $\mathscr{C}$ in (86) is as follows. For all $c \in \{1, 2, \ldots, C\}$, let p_c be the frequency with which symbols of the c^{th} layer appear in the code. The resulting probability vector is denoted by

$$\boldsymbol{p} = (p_1, p_2, \ldots, p_C)^{\mathsf{T}}, \tag{86f}$$

where, for all $c \in \{1, 2, \ldots, C\}$,

$$p_c = \frac{1}{Mn} \sum_{\ell=1}^{L_c} \sum_{i=1}^{M} \sum_{t=1}^{n} \mathbb{1}_{\{u_t(i) = x_c^{(\ell)}\}}. \tag{86g}$$

The symbols within a layer are used with the same frequency in $\mathscr{C}$. Hence, for all $c \in \{1, 2, \ldots, C\}$ and for all $\ell \in \{1, 2, \ldots, L_c\}$, the frequency with which the symbol $x_c^{(\ell)}$ appears in $\mathscr{C}$ is

$$P_{\mathscr{C}}(x_c^{(\ell)}) \triangleq \frac{1}{Mn} \sum_{i=1}^{M} \sum_{t=1}^{n} \mathbb{1}_{\{u_t(i) = x_c^{(\ell)}\}}, \tag{86h}$$

$$= \frac{p_c}{L_c}. \tag{86i}$$

The decoding set $\mathcal{G}_c^{(\ell)}$ associated with symbol $x_c^{(\ell)}$ is a circle of radius $r_c \in \mathbb{R}^+$ centered at $x_c^{(\ell)}$. That is,

$$\mathcal{G}_c^{(\ell)} = \left\{ y \in \mathbb{C} : \left| y - x_c^{(\ell)} \right|^2 \leq r_c^2 \right\}, \tag{86j}$$

$$= \left\{ y \in \mathbb{C} : \left(\Re(y) - \Re(x_c^{(\ell)}) \right)^2 + \left(\Im(y) - \Im(x_c^{(\ell)}) \right)^2 \leq r_c^2 \right\}. \tag{86k}$$

The radii r_1, r_2, …, r_C are chosen such that the decoding regions are mutually disjoint. To ensure this, for all $c \in \{1, 2, \ldots, C\}$ the amplitudes A_c in (86b) satisfy the following

$$A_c - A_{c-1} \geq r_c + r_{c-1}. \tag{86l}$$

The vector of these radii is denoted by

$$\boldsymbol{r} = (r_1, r_2, \ldots, r_C)^{\mathsf{T}}. \tag{86m}$$

The decoding region for codeword $\boldsymbol{u}(i)$ is

$$\mathcal{D}_i = \mathcal{D}_{i,1} \times \mathcal{D}_{i,2} \times \ldots \times \mathcal{D}_{i,n}, \tag{86n}$$

where, for all $i \in \{1, 2, \dots, M\}$, $c \in \{1, 2, \dots, C\}$, $t \in \{1, 2, \dots, n\}$, and $\ell \in \{1, 2, \dots, L_c\}$, when $u_t(i) = x_c^{(\ell)}$, then, $\mathcal{D}_{i,t} = \mathcal{G}_c^{(\ell)}$.

This defines a family of (n, M)-codes denoted by

$$\mathsf{C}\left(\mathcal{X}, \boldsymbol{p}, \boldsymbol{r}\right), \tag{87}$$

with constellation $\mathcal{X}$ in (86c), probability distribution $\boldsymbol{p}$ in (86f) and, radii of decoding regions $\boldsymbol{r}$ in (86m).

5.2 Achievable Bounds

This section provides various achievability results for homogeneous codes in the family $\mathsf{C}\left(\mathcal{X}, \boldsymbol{p}, \boldsymbol{r}\right)$.

Lemma 5.1. *Consider an (n, M)-code $\mathscr{C}$ for the random transformation in* (1) *of the form in* (86)*. The code $\mathscr{C}$ is an (n, M, ϵ)-code if the parameters $r_1, r_2, \dots, r_C$ in* (86k) *satisfy:*

$$\frac{1}{M}\sum_{i=1}^{M}\prod_{c=1}^{C}\left(1 - e^{-\frac{r_c^2}{\sigma^2}}\right)^{n\sum_{\ell=1}^{L_c} P_{\boldsymbol{u}(i)}(x_c^{(\ell)})} \geq 1 - \epsilon, \tag{88}$$

where, the type $P_{\boldsymbol{u}(i)}$ is defined in (21)*, the real σ^2 is defined in* (4)*, and $x_c^{(\ell)} \in \mathcal{U}(A_c, L_c, \alpha_c)$, with $\mathcal{U}(A_c, L_c, \alpha_c)$ in* (86b).

Proof. From (44), the average DEP of code $\mathscr{C}$ is given by

$$\gamma(\mathscr{C}) = \frac{1}{M}\sum_{i=1}^{M}\left(1 - \prod_{c=1}^{C}\prod_{\ell=1}^{L_c}\left(\int_{\mathcal{G}_c^{(\ell)}} f_{Y|X}(y|x_c^{(\ell)})\mathrm{d}y\right)^{nP_{\boldsymbol{u}(i)}(x_c^{(\ell)})}\right). \tag{89}$$

From (15), for $\mathscr{C}$ to be an (n, M, ϵ)-code, the following must hold:

$$\gamma\left(\mathscr{C}\right) \leq \epsilon. \tag{90}$$

Using (4) in (89) yields,

$$\gamma\left(\mathscr{C}\right) = 1 - \frac{1}{M}\sum_{i=1}^{M}\prod_{c=1}^{C}\prod_{\ell=1}^{L_c}\left(\int_{\mathcal{G}_c^{(\ell)}} \frac{1}{\pi\sigma^2}\exp\left(-\frac{(\Re(y) - \Re(x_c^{(\ell)}))^2 + (\Im(y) - \Im(x_c^{(\ell)}))^2}{\sigma^2}\right)\mathrm{d}y\right)^{nP_{\boldsymbol{u}(i)}(x_c^{(\ell)})}. \tag{91}$$

Evaluating the integral term in (91) for all $\ell \in \{1, 2, \dots, L_c\}$ yields,

$$\begin{aligned}
&\int_{\mathcal{G}_c^{(\ell)}} \frac{1}{\pi\sigma^2}\exp\left(-\frac{(\Re(y) - \Re(x_c^{(\ell)}))^2 + (\Im(y) - \Im(x_c^{(\ell)}))^2}{\sigma^2}\right)\mathrm{d}y \\
=&\int_{\Im(x_c^{(\ell)}) - r_c}^{\Im(x_c^{(\ell)}) + r_c}\int_{\Re(x_c^{(\ell)}) - \sqrt{r_c^2 - (v - \Im(x_c^{(\ell)}))^2}}^{\Re(x_c^{(\ell)}) + \sqrt{r_c^2 - (v - \Im(x_c^{(\ell)}))^2}} \frac{1}{\pi\sigma^2}e^{-\frac{(u - \Re(x_c^{(\ell)}))^2 + (v - \Im(x_c^{(\ell)}))^2}{\sigma^2}}\mathrm{d}u\mathrm{d}v, && (92)\\
=&\int_{-r_c}^{r_c}\int_{-\sqrt{r_c^2 - v^2}}^{\sqrt{r_c^2 - v^2}} \frac{1}{\pi\sigma^2}e^{-\frac{u^2 + v^2}{\sigma^2}}\mathrm{d}u\mathrm{d}v, && (93)\\
=&\int_{0}^{\pi}\int_{0}^{\frac{r_c}{\sigma}} \frac{1}{2\pi}e^{-\zeta^2}\zeta\mathrm{d}\zeta\mathrm{d}\eta, && (94)\\
=&(1 - e^{-\frac{r_c^2}{\sigma^2}}). && (95)
\end{aligned}$$

The equality in (94) is obtained from the change of variables $u = \sigma\zeta\cos\eta, v = \sigma\zeta\sin\eta$. Plugging (95) in (91) yields,

$$\gamma(\mathscr{C}) = 1 - \frac{1}{M}\sum_{i=1}^{M}\prod_{c=1}^{C}\prod_{\ell=1}^{L_c}\left(1 - e^{-\frac{r_c^2}{\sigma^2}}\right)^{nP_{\boldsymbol{u}(i)}(x_c^{(\ell)})}, \tag{96}$$

$$= 1 - \frac{1}{M}\sum_{i=1}^{M}\prod_{c=1}^{C}\left(1 - e^{-\frac{r_c^2}{\sigma^2}}\right)^{n\sum_{\ell=1}^{L_c} P_{\boldsymbol{u}(i)}(x_c^{(\ell)})}. \tag{97}$$

For $\mathscr{C}$ to be an (n, M, ϵ)-code, from (97) and (90), it follows that,

$$1 - \frac{1}{M}\sum_{i=1}^{M}\prod_{c=1}^{C}\left(1 - e^{-\frac{r_c^2}{\sigma^2}}\right)^{n\sum_{\ell=1}^{L_c} P_{\boldsymbol{u}(i)}(x_c^{(\ell)})} \leq \epsilon, \tag{98}$$

$$\implies \frac{1}{M}\sum_{i=1}^{M}\prod_{c=1}^{C}\left(1 - e^{-\frac{r_c^2}{\sigma^2}}\right)^{n\sum_{\ell=1}^{L_c} P_{\boldsymbol{u}(i)}(x_c^{(\ell)})} \geq 1 - \epsilon, \tag{99}$$

which is the desired result. □

From Definition 4.2, for a homogeneous (n, M)-code $\mathscr{C}$ for the random transformation in (1) of the form in (86), for all $c \in \{1, 2, \ldots, C\}$ and $\ell \in \{1, 2, \ldots, L_c\}$, it holds that:

$$P_{\boldsymbol{u}(i)}(x_c^{(\ell)}) = P_{\mathscr{C}}(x_c^{(\ell)}), \tag{100}$$

$$\implies P_{\boldsymbol{u}(i)}(x_c^{(\ell)}) = \frac{p_c}{L_c}, \tag{101}$$

where, the expression in (101) is obtained by substituting (86i) in (100). The following result for homogeneous codes follows from (101) and Lemma 5.1.

Corollary 5.2. *Consider a homogeneous* (n, M)*-code* $\mathscr{C}$ *for the random transformation in* (1) *of the form in* (86) *with* $\boldsymbol{p} = (p_1, p_2, \ldots, p_C)^{\mathsf{T}}$ *in* (86f)*. The code* $\mathscr{C}$ *is an* (n, M, ϵ)*-code if the parameters* $r_1, r_2, \ldots, r_C$ *in* (86k) *satisfy:*

$$\prod_{c=1}^{C}\left(1 - e^{-\frac{r_c^2}{\sigma^2}}\right)^{np_c} \geq 1 - \epsilon, \tag{102}$$

where, the real σ^2 *is defined in* (4)*, and* $x_c^{(\ell)} \in \mathcal{U}(A_c, L_c, \alpha_c)$*, with* $\mathcal{U}(A_c, L_c, \alpha_c)$ *in* (86b)*.*

Proof. The result follows from Lemma 5.1 and (101). □

For all codewords in an (n, M)-code $\mathscr{C}$ of the form in (86) to have the same DEP, the DEP associated with all the symbols should be the same. This is ensured if the parameters r_c in (86k) are such that, for all $c \in \{1, 2, \ldots, C\}$, $r_c = r$. For such a code $\mathscr{C}$, the average DEP in (97) is given by:

$$
\begin{aligned}
\gamma\left(\mathscr{C}\right) &= 1 - \frac{1}{M}\sum_{i=1}^{M}\prod_{c=1}^{C}\left(1-e^{-\frac{r^2}{\sigma^2}}\right)^{n\sum_{\ell=1}^{L_c}P_{\boldsymbol{u}(i)}(x_c^{(\ell)})}, && (103)\\
&= 1 - \frac{1}{M}\sum_{i=1}^{M}\left(1-e^{-\frac{r^2}{\sigma^2}}\right)^{n\sum_{c=1}^{C}\sum_{\ell=1}^{L_c}P_{\boldsymbol{u}(i)}(x_c^{(\ell)})}, && (104)\\
&= 1 - \frac{1}{M}\sum_{i=1}^{M}\left(1-e^{-\frac{r^2}{\sigma^2}}\right)^{n}, && (105)\\
&= 1 - \left(1-e^{-\frac{r^2}{\sigma^2}}\right)^{n}. && (106)
\end{aligned}
$$

Using (106), the following corollary can be obtained from Lemma 5.1.

Corollary 5.3. *Consider an (n, M)-code $\mathscr{C}$ for the random transformation in* (1) *of the form in* (86) *such that, for all $c \in \{1, 2, \ldots, C\}$, $r_c = r$ in* (86k). *The code $\mathscr{C}$ is an (n, M, ϵ)-code if the parameter r satisfies:*

$$
r \geq \sqrt{\sigma^2 \log\left(\frac{1}{1-(1-\epsilon)^{\frac{1}{n}}}\right)}, \tag{107}
$$

where, the real σ^2 is defined in (4).

Proof. The result follows from Lemma 5.1 and (106). □

Lemma 5.4. *Given an (n, M)-code $\mathscr{C}$ for the random transformation in* (1) *of the form in* (86), *for all $c \in \{1, 2, \ldots, C\}$, and $(\ell_1, \ell_2) \in \{1, 2, \ldots, L_c\}^2$, it holds that*

$$
g(x_c^{(\ell_1)}) = g(x_c^{(\ell_2)}), \tag{108}
$$

where, g is the energy function defined in (16), *and $x_c^{(\ell_1)}, x_c^{(\ell_2)} \in \mathcal{U}(A_c, L_c, \alpha_c)$, with $\mathcal{U}(A_c, L_c, \alpha_c)$ in* (86b).

Lemma 5.4 states that all the symbols that belong to the same layer of $\mathcal{X}$ in (86c) produce the same energy at the EH. This means that, from the perspective of energy harvesting, the symbols in a layer are equivalent and can, therefore, be used with the same frequency within a code. Thus, the symbols that belong to the same layer induce the same type on code $\mathscr{C}$.

Proof. Given $c \in \{1, 2, \ldots, C\}$, let $(\ell_1, \ell_2) \in \{1, 2, \ldots, L_c - 1\}^2$ with $\ell_1 \neq \ell_2$ such that, the symbols $x_c^{(\ell_1)}, x_c^{(\ell_2)} \in \mathcal{U}(A_c, L_c, \alpha_c)$ are as follows:

$$
\begin{aligned}
x_c^{(\ell_1)} &= A_c \exp\left(\mathrm{i}\left(\frac{2\pi}{L_c}\left(\ell_1 - 1\right) + \alpha_c\right)\right), && (109)\\
x_c^{(\ell_2)} &= A_c \exp\left(\mathrm{i}\left(\frac{2\pi}{L_c}\left(\ell_2 - 1\right) + \alpha_c\right)\right). && (110)
\end{aligned}
$$

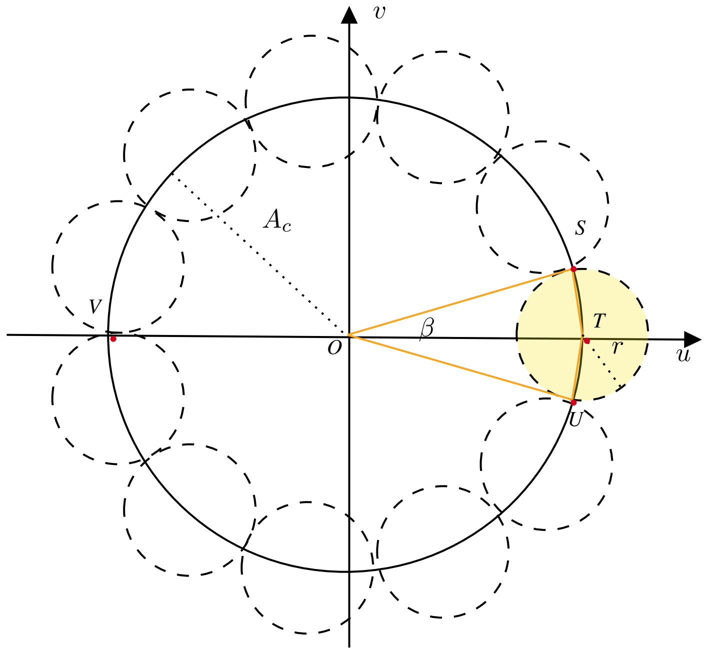

Figure 1: Graphical representation of the symbols in layer c defined in (86b)
.

From (110), the symbol $x_c^{(\ell_2)}$ can be written as follows:

$$x_c^{(\ell_2)} = A_c \exp\left(\mathrm{i}\left(\frac{2\pi}{L_c}(\ell_2 - 1 + \ell_1 - \ell_1) + \alpha_c\right)\right), \tag{111}$$

$$= A_c \exp\left(\mathrm{i}\left(\frac{2\pi}{L_c}(\ell_1 - 1) + \frac{2\pi}{L_c}(\ell_2 - \ell_1) + \alpha_c\right)\right), \tag{112}$$

$$= A_c \exp\left(\mathrm{i}\left(\frac{2\pi}{L_c}(\ell_1 - 1) + \alpha_c\right)\right) \exp\left(\mathrm{i}\left(\frac{2\pi}{L_c}(\ell_2 - \ell_1)\right)\right), \tag{113}$$

$$= \exp\left(\mathrm{i}\left(\frac{2\pi}{L_c}(\ell_2 - \ell_1)\right)\right) x_c^{(\ell_1)}. \tag{114}$$

The energy function g in (16) is circularly symmetric, that is, for all $x \in \mathbb{C}$ and $\alpha \in [0, 2\pi]$ it holds that:

$$g\left(e^{\mathrm{i}\alpha} x\right) = g(x). \tag{115}$$

The energy produced at the EH by the symbol $x_c^{(\ell_2)}$ can be calculated as follows:

$$g\left(x_c^{(\ell_2)}\right) = g\left(\exp\left(\mathrm{i}\left(\frac{2\pi}{L_c}(\ell_2 - \ell_1)\right)\right) x_c^{(\ell_1)}\right), \tag{116}$$

$$= g\left(x_c^{(\ell_1)}\right), \tag{117}$$

where (117) follows from (116) due to (115). This completes the proof. □

Lemma 5.5. *Consider an (n, M)-code $\mathscr{C}$ for the random transformation in* (1) *of the form in* (86)*, with the set of symbols $\mathcal{X}$ in* (86c)*. Then, for all $c \in \{1, 2, \ldots, C\}$, the number of* *symbols in layer c of $\mathcal{X}$ is given by*

$$L_c \leq \left\lfloor \frac{\pi}{2 \arcsin \frac{r_c}{2A_c}} \right\rfloor, \tag{118}$$

where, r_c is the radius of the decoding regions $\mathcal{G}_c^{(1)}, \mathcal{G}_c^{(2)}, \ldots, \mathcal{G}_c^{(L_c)}$ in (86k) *and A_c is the amplitude in* (86b)*.*

Proof. For the (n, M)-code $\mathscr{C}$ for the random transformation in (1) of the form in (86), the radius r_c of the decoding regions in (86k) determines the number of symbols L_c that can be accommodated in layer c of amplitude A_c.

Layer c of the form in (86b) is illustrated in Figure 1. Symbols in layer c are distributed along the circle of radius A_c centered at the origin O. The maximum number of symbols that can be accommodated in layer c is equal to the number of non-overlapping circles of radius r_c corresponding to the decoding regions defined in (86k) that can be placed along the circumference of the circle of radius A_c. From Figure 1, a circle of radius r_c centered at a symbol in layer c subtends angle $\angle SOU = \beta$ at O. Therefore, the maximum number of symbols L_c that can be accommodated along the circle of radius A_c is

$$L_c = \left\lfloor \frac{2\pi}{\beta} \right\rfloor. \tag{119}$$

The following lemma determines the value of the angle β in Figure 1.

Lemma 5.6. *Consider a circle of radius A_c centered at O and a circle of radius r_c centered along the first circle such that $A_c > r_c$. Let β be the angle subtended by the minor arc formed by the points of intersection of the two circles from O. Then, it holds that,*

$$\beta = 4 \arcsin \frac{r_c}{2A_c}. \tag{120}$$

Proof. In Figure 1, consider the circle of radius r_c centered at T and the larger circle of radius A_c centered at the origin O. The smaller circle intersects the larger circle at points S and U. The angle subtended by the major arc $\widehat{SU}$ at O is the reflex angle $2\pi - \beta$. Then, the angle $\angle STU$ subtended by major arc $\widehat{SU}$ at the circumference is given by

$$\angle STU = \frac{2\pi - \beta}{2}, \tag{121}$$

because, the angle subtended by an arc of a circle at its centre is twice of the angle it subtends anywhere on the circle. Line segment TO bisects angles $\angle STU$ and $\angle SOU$. Therefore, angles $\angle STO$ and $\angle SOT$ are given by

$$\angle STO = \frac{2\pi - \beta}{4}, \tag{122}$$

$$\angle SOT = \frac{\beta}{2}. \tag{123}$$

From the triangle $\triangle SOT$, it holds that:

$$\frac{\sin\left(\angle SOT\right)}{ST} = \frac{\sin\left(\angle STO\right)}{SO}, \tag{124}$$

$$\Longrightarrow \frac{\sin\left(\frac{\beta}{2}\right)}{r_c} = \frac{\sin\left(\frac{2\pi-\beta}{4}\right)}{A_c}, \tag{125}$$

$$\Longrightarrow \frac{1}{r_c}\sin\left(\frac{\beta}{2}\right) = \frac{1}{A_c}\sin\left(\frac{\pi}{2} - \frac{\beta}{4}\right), \tag{126}$$

$$\Longrightarrow \frac{2}{r_c}\sin\left(\frac{\beta}{4}\right)\cos\left(\frac{\beta}{4}\right) = \frac{1}{A_c}\cos\left(\frac{\beta}{4}\right), \tag{127}$$

$$\Longrightarrow \sin\left(\frac{\beta}{4}\right) = \frac{r_c}{2A_c}, \tag{128}$$

$$\Longrightarrow \beta = 4\arcsin\frac{r_c}{2A_c}, \tag{129}$$

which is the desired result. □

Substituting the value of β from (120) in (119), the number of symbols in layer c of the set $\mathcal{X}$ is

$$L_c \leq \left\lfloor \frac{\pi}{2\arcsin\frac{r_c}{2A_c}} \right\rfloor. \tag{130}$$

This completes the proof. □

Lemma 5.7. *For an* (n, M)*-code* $\mathscr{C}$ *for the random transformation in* (1) *of the form in* (86)*, the information transmission rate* $R\left(\mathscr{C}\right)$ *satisfies the following:*

$$R\left(\mathscr{C}\right) \leq \log_2 \sum_{c=1}^{C} \left\lfloor \frac{\pi}{2\arcsin\frac{r_c}{2A_c}} \right\rfloor, \tag{131}$$

where, the variable C *is the number of layers in the constellation of* $\mathscr{C}$ *in* (86c)*, and for all* $c \in \{1, 2, \ldots, C\}$*, the radius* r_c *is defined in* (86k)*, and the amplitude* A_c *is defined in* (86b)*.*

Proof. From (11), the information transmission rate for code $\mathscr{C}$ is given by:

$$R\left(\mathscr{C}\right) = \frac{\log_2 M}{n}, \tag{132}$$

$$\leq \frac{\log_2 L^n}{n}, \tag{133}$$

$$= \log_2 L, \tag{134}$$

$$= \log_2 \sum_{c=1}^{C} \left\lfloor \frac{\pi}{2\arcsin\frac{r_c}{2A_c}} \right\rfloor, \tag{135}$$

where (135) is obtained from (134) using Lemma 5.5. This completes the proof. □

Lemma 5.8. *Consider* (n, M)*-codes* $\mathscr{C}$ *and* $\mathscr{C}'$ *for the random transformation in* (1) *of the form in* (86)*. For all* $c \in \{1, 2, \ldots, C\}$ *and* $\ell \in \{1, 2, \ldots, L_c\}$*, denote the symbols in layer* c *of* $\mathscr{C}$ *in* (86b) *by* $x_c^{(\ell)}$ *and those of* $\mathscr{C}'$ *by* $\hat{x}_c^{(\ell)}$ *such that, for some* $\omega \in [0, 2\pi]$*, it holds that,*

$$\hat{x}_c^{(\ell)} = e^{\mathrm{i}\omega} x_c^{(\ell)}, \tag{136}$$

Then, for energy requirement $B \in \mathbb{R}^+$ at the EH defined in (18)*, the average DEP γ in* (14) *and the average EOP θ in* (19) *of $\mathscr{C}$ and $\mathscr{C}'$ satisfy the following:*

$$\gamma(\mathscr{C}) = \gamma(\mathscr{C}'), \text{ and} \tag{137}$$

$$\theta(\mathscr{C}, B) = \theta(\mathscr{C}', B). \tag{138}$$

Proof. From (91), the average DEP for $\mathscr{C}$ is given by

$$\gamma(\mathscr{C}) = 1 - \frac{1}{M}\sum_{i=1}^{M}\prod_{c=1}^{C}\prod_{\ell=1}^{L_c}\left(\int_{\mathcal{G}_c^{(\ell)}} \frac{1}{\pi\sigma^2}\exp\left(-\frac{(\Re(y)-\Re(x_c^{(\ell)}))^2 + (\Im(y)-\Im(x_c^{(\ell)}))^2}{\sigma^2}\right)\mathrm{d}y\right)^{nP_{\boldsymbol{u}(i)}(x_c^{(\ell)})} \tag{139}$$

$$= 1 - \frac{1}{M}\sum_{i=1}^{M}\prod_{c=1}^{C}\prod_{\ell=1}^{L_c}\left(\int_{y:\left|y-x_c^{(\ell)}\right|^2\leq r_c^2} \frac{1}{\pi\sigma^2}\exp\left(-\frac{\left|y-x_c^{(\ell)}\right|^2}{\sigma^2}\right)\mathrm{d}y\right)^{nP_{\boldsymbol{u}(i)}(x_c^{(\ell)})}, \tag{140}$$

where, the expression in (140) follows from (139) due to (4a) and (86j). The change of variable $y = z + x_c^{(\ell)} - \hat{x}_c^{(\ell)}$ in the integral in (140) yields

$$\int_{y:\left|y-x_c^{(\ell)}\right|^2\leq r_c^2} \frac{1}{\pi\sigma^2}\exp\left(-\frac{\left|y-x_c^{(\ell)}\right|^2}{\sigma^2}\right)\mathrm{d}y = \int_{u:\left|z-\hat{x}_c^{(\ell)}\right|^2\leq r_c^2} \frac{1}{\pi\sigma^2}\exp\left(-\frac{\left|z-\hat{x}_c^{(\ell)}\right|^2}{\sigma^2}\right)\mathrm{d}z. \tag{141}$$

The frequency of usage of symbols in equal for symbols $x_c^{(\ell)}$ and $\hat{x}_c^{(\ell)}$. That is,

$$P_{\boldsymbol{u}(i)}(x_c^{(\ell)}) = P_{\boldsymbol{u}(i)}(\hat{x}_c^{(\ell)}). \tag{142}$$

Substituting (141) in (140) yields:

$$\gamma(\mathscr{C}) = 1 - \frac{1}{M}\sum_{i=1}^{M}\prod_{c=1}^{C}\prod_{\ell=1}^{L_c}\left(\int_{z:\left|z-\hat{x}_c^{(\ell)}\right|^2\leq r_c^2} \frac{1}{\pi\sigma^2}\exp\left(-\frac{\left|z-\hat{x}_c^{(\ell)}\right|^2}{\sigma^2}\right)\mathrm{d}z\right)^{nP_{\boldsymbol{u}(i)}(x_c^{(\ell)})}, \tag{143}$$

$$= \gamma(\mathscr{C}'), \tag{144}$$

which completes the proof of (137).

From Lemma 5.4 it holds that, for all $c \in \{1, 2, \ldots, C\}$ and $\ell \in \{1, 2, \ldots, L_c\}$,

$$g\left(\hat{x}_c^{(\ell)}\right) = g\left(x_c^{(\ell)}\right). \tag{145}$$

From (27), the EOP for $\mathscr{C}$ is

$$\theta(\mathscr{C}, B) = \frac{1}{M}\sum_{i=1}^{M}\Pr\left[\sum_{c=1}^{C}\sum_{\ell=1}^{L_c} P_{\boldsymbol{u}(i)}(x_c^{(\ell)}) g(x_c^{(\ell)} + W) < B\right], \tag{146}$$

where, W is the circularly symmetric Gaussian noise, the probability in (146) is with respect to the random variable W. Due to circular symmetry of the noise variable W, for all $\omega \in [0, 2\pi]$ the variables W and $e^{\mathrm{i}\omega}W$ follow the same distribution (Proposition 24.2.6 in [12]). That is,

$$W \stackrel{d}{=} e^{\mathrm{i}\omega}W. \tag{147}$$

Due to (145) and (147), the following holds:

$$\begin{aligned} g\left(x_c^{(\ell)} + W\right) &= g\left(e^{-\mathrm{i}\omega}\hat{x}_c^{(\ell)} + W\right), &(148)\\ &= g\left(e^{-\mathrm{i}\omega}\left(\hat{x}_c^{(\ell)} + e^{\mathrm{i}\omega}W\right)\right), &(149)\\ &= g\left(\hat{x}_c^{(\ell)} + e^{\mathrm{i}\omega}W\right), &(150)\\ &\stackrel{d}{=} g\left(\hat{x}_c^{(\ell)} + W\right). &(151)\end{aligned}$$

From (142) and (151), it follows that

$$\sum_{c=1}^{C}\sum_{\ell=1}^{L_c} P_{\boldsymbol{u}(i)}(x_c^{(\ell)})g(x_c^{(\ell)} + W) \stackrel{d}{=} \sum_{c=1}^{C}\sum_{\ell=1}^{L_c} P_{\boldsymbol{u}(i)}(\hat{x}_c^{(\ell)})g(\hat{x}_c^{(\ell)} + W). \tag{152}$$

From (146) and (152), it follows that

$$\begin{aligned} \theta(\mathscr{C}, B) &= \frac{1}{M}\sum_{i=1}^{M}\Pr\left[\sum_{c=1}^{C}\sum_{\ell=1}^{L_c} P_{\boldsymbol{u}(i)}(x_c^{(\ell)})g(x_c^{(\ell)} + W) < B\right], &(153)\\ &= \frac{1}{M}\sum_{i=1}^{M}\Pr\left[\sum_{c=1}^{C}\sum_{\ell=1}^{L_c} P_{\boldsymbol{u}(i)}(\hat{x}_c^{(\ell)})g(\hat{x}_c^{(\ell)} + W) < B\right], &(154)\\ &= \theta(\mathscr{C}', B). &(155)\end{aligned}$$

This completes the proof. □

Lemma 5.9. *Consider an (n, M, ϵ)-code $\mathscr{C}$ for the random transformation in (1) of the form in (86). The code $\mathscr{C}$ is an $(n, M, \epsilon, B, \delta)$-code if, for all $c \in \{1, 2, \ldots, C\}$, the parameters A_c in (86b) satisfy the following:*

$$\frac{1}{M}\sum_{i=1}^{M}\Pr\left[\sum_{c=1}^{C}\sum_{\ell=1}^{L_c} P_{\boldsymbol{u}(i)}(x_c^{(\ell)})g(A_c + W) < B\right] \leq \delta, \tag{156}$$

where, the probability is with respect to W, which is a complex circularly symmetric Gaussian random variable whose real and imaginary parts have zero means and variances $\frac{1}{2}\sigma^2$, and the type $P_{\boldsymbol{u}(i)}$ is defined in (21).

Proof. From (146), the EOP for code $\mathscr{C}$ is given by:

$$\theta(\mathscr{C}, B) = \frac{1}{M}\sum_{i=1}^{M}\Pr\left[\sum_{c=1}^{C}\sum_{\ell=1}^{L_c} P_{\boldsymbol{u}(i)}(x_c^{(\ell)})g(x_c^{(\ell)} + W) < B\right]. \tag{157}$$

For the code $\mathscr{C}$ to be an $(n, M, \epsilon, B, \delta)$-code, the following holds:

$$\theta(\mathscr{C}, B) \leq \delta. \tag{158}$$

From (158) and (157), it holds that:

$$\delta \geq \frac{1}{M}\sum_{i=1}^{M}\Pr\left[\sum_{c=1}^{C}\sum_{\ell=1}^{L_c} P_{\boldsymbol{u}(i)}(x_c^{(\ell)})g(x_c^{(\ell)}+W) < B\right], \tag{159}$$

$$= \frac{1}{M}\sum_{i=1}^{M}\Pr\left[\sum_{c=1}^{C}\sum_{\ell=1}^{L_c} P_{\boldsymbol{u}(i)}(x_c^{(\ell)})g\left(A_c\exp\left(\mathrm{i}\left(\frac{2\pi}{L_c}(\ell-1)+\alpha_c\right)\right)+W\right) < B\right], \tag{160}$$

$$= \frac{1}{M}\sum_{i=1}^{M}\Pr\Big[\sum_{c=1}^{C}\sum_{\ell=1}^{L_c} P_{\boldsymbol{u}(i)}(x_c^{(\ell)})g\Big(\exp\left(\mathrm{i}\left(\frac{2\pi}{L_c}(\ell-1)+\alpha_c\right)\right)$$
$$\left(A_c+\exp\left(-\mathrm{i}\left(\frac{2\pi}{L_c}(\ell-1)+\alpha_c\right)\right)W\right)\Big) < B\Big], \tag{161}$$

$$= \frac{1}{M}\sum_{i=1}^{M}\Pr\left[\sum_{c=1}^{C}\sum_{\ell=1}^{L_c} P_{\boldsymbol{u}(i)}(x_c^{(\ell)})g\left(A_c+\exp\left(-\mathrm{i}\left(\frac{2\pi}{L_c}(\ell-1)+\alpha_c\right)\right)W\right) < B\right], \tag{162}$$

$$= \frac{1}{M}\sum_{i=1}^{M}\Pr\left[\sum_{c=1}^{C}\sum_{\ell=1}^{L_c} P_{\boldsymbol{u}(i)}(x_c^{(\ell)})g\left(A_c+W\right) < B\right], \tag{163}$$

where, the expression in (162) follows from (161) due to (115), and (163) follows from (162) due to (151). This completes the proof. □

Lemma 5.10. *Consider a homogeneous (n, M, ϵ)-code $\mathscr{C}$ for the random transformation in* (1) *of the form in* (86) *with $\boldsymbol{p} = (p_1, p_2, \ldots, p_C)^{\mathsf{T}}$ in* (86f)*. The code $\mathscr{C}$ is an $(n, M, \epsilon, B, \delta)$-code if, for all $c \in \{1, 2, \ldots, C\}$, the parameters A_c in* (86b) *satisfy the following:*

$$B \leq \frac{1}{1-\delta}\sum_{c=1}^{C} p_c \mathbb{E}_W\left[g\left(A_c+W\right)\right], \tag{164}$$

where, the expectation is with respect to W, which is a complex circularly symmetric Gaussian random variable whose real and imaginary parts have zero means and variances $\frac{1}{2}\sigma^2$.

Proof. For a homogeneous (n, M, ϵ)-code $\mathscr{C}$ for the random transformation in (1) of the form in (86), from Lemma 5.9 and (101), it follows that,

$$\delta \geq \Pr\left[\sum_{c=1}^{C} p_c g\left(A_c+W\right) < B\right]. \tag{165}$$

This implies that,

$$\delta \geq 1-\Pr\left[\sum_{c=1}^{C} p_c g\left(A_c+W\right) \geq B\right], \tag{166}$$

$$\geq 1-\frac{1}{B}\mathbb{E}_W\left[\sum_{c=1}^{C} p_c g\left(A_c+W\right)\right], \tag{167}$$

$$= 1-\frac{1}{B}\sum_{c=1}^{C} p_c \mathbb{E}_W\left[g\left(A_c+W\right)\right], \tag{168}$$

where, the inequality in (167) follows from Markov's inequality. From (168), it follows that

$$B \leq \frac{1}{1-\delta} \sum_{c=1}^{C} p_c \mathbb{E}_W \left[g\left(A_c + W\right)\right]. \tag{169}$$

This completes the proof. □

Lemma 5.11. *Consider an (n, M)-code $\mathscr{C}'$ for the random transformation in (1) of the form in (86) with $\boldsymbol{p} = (p_1, p_2, \ldots, p_C)^{\mathsf{T}}$ in (86f) such that, for all $c \in \{1, 2, \ldots, C\}$,*

$$p_c = \frac{L_c}{L}, \tag{170}$$

where, L_c and L are as defined in (86b) and (86e) respectively. Then, given any other (n, M)-code $\mathscr{C}$ that is identical to $\mathscr{C}'$ except for the probability distribution $\boldsymbol{p}$, it holds that,

$$R\left(\mathscr{C}'\right) \geq R\left(\mathscr{C}\right), \tag{171}$$

where, $R\left(\mathscr{C}'\right)$ and $R\left(\mathscr{C}\right)$ are the information transmission rates for $\mathscr{C}'$ and $\mathscr{C}$, respectively.

Proof. For the (n, M)-code $\mathscr{C}'$ with p_c of the form in (170), it follows from (86i) that, for all $c \in \{1, 2, \ldots, C\}$ and $\ell \in \{1, 2, \ldots, L_c\}$, the frequency with which the symbol $x_c^{(\ell)}$ appears in $\mathscr{C}'$ is

$$P_{\mathscr{C}'}(x_c^{(\ell)}) = \frac{1}{L}, \tag{172}$$

which means that, all the symbols in $\mathcal{X}$ are used with the same frequency in the code $\mathscr{C}'$. Therefore, the number of codewords that can be generated from $\mathcal{X}$ is

$$M = L^n. \tag{173}$$

From (11), the rate of information transmission for $\mathscr{C}'$ is given by

$$R\left(\mathscr{C}'\right) = \frac{\log_2 M}{n}, \tag{174}$$

$$= \frac{\log_2 L^n}{n}, \tag{175}$$

$$= \log_2 L, \tag{176}$$

which is the maximum possible rate of information transmission as shown in Lemma 4.8. Consider an (n, M)-code $\mathscr{C}$ that is identical to $\mathscr{C}'$ with probability distribution $\boldsymbol{p'} \neq \boldsymbol{p}$. From (176) and Lemma 4.8, it holds that,

$$R\left(\mathscr{C}'\right) \geq R\left(\mathscr{C}\right), \tag{177}$$

which completes the proof. □

Lemma 5.12. *Consider an (n, M)-code $\mathscr{C}'$ for the random transformation in (1) of the form in (86) with $\boldsymbol{p} = (p_1, p_2, \ldots, p_C)^{\mathsf{T}}$ in (86f) such that,*

$$p_c = \begin{cases} 1 & \textit{for } c = 1, \\ 0 & \textit{otherwise}. \end{cases} \tag{178}$$

Then, given any other (n, M)*-code* $\mathscr{C}$ *for the random transformation in* (1) *of the form in* (86) *that is identical to* $\mathscr{C}'$ *except for the probability distribution* $\boldsymbol{p}$*, it holds that*

$$\theta\left(\mathscr{C}', B\right) \leq \theta\left(\mathscr{C}, B\right), \tag{179}$$

where $\theta\left(\mathscr{C}', B\right)$ *and* $\theta\left(\mathscr{C}, B\right)$ *are the average EOP for codes* $\mathscr{C}'$ *and* $\mathscr{C}$ *respectively as defined in* (19)*.*

Proof. From (27), the average EOP for the code $\mathscr{C}'$ is

$$\begin{aligned}
\theta\left(\mathscr{C}', B\right) &= \frac{1}{M}\sum_{i=1}^{M}\Pr\left[\sum_{c=1}^{C}\sum_{\ell=1}^{L_c} P_{\boldsymbol{u}(i)}(x_c^{(\ell)})g(x_c^{(\ell)}+W) < B\right], && (180)\\
&= \frac{1}{M}\sum_{i=1}^{M}\Pr\left[\sum_{\ell=1}^{L_1} P_{\boldsymbol{u}(i)}(x_1^{(\ell)})g(x_1^{(\ell)}+W) < B\right], && (181)\\
&= \frac{1}{M}\sum_{i=1}^{M}\Pr\left[\sum_{\ell=1}^{L_1} P_{\boldsymbol{u}(i)}(x_1^{(\ell)})g(x_1^{(1)}+W) < B\right], && (182)\\
&= \frac{1}{M}\sum_{i=1}^{M}\Pr\left[g(x_1^{(1)}+W) < B\right], && (183)\\
&= \Pr\left[g(x_1^{(1)}+W) < B\right], && (184)
\end{aligned}$$

where, the expression in (181) follows from (180) due to (178) and (182) follows from (181) due to Lemma 5.4. Consider another (n, M)-code $\mathscr{C}$ identical to $\mathscr{C}'$ except with probability distribution $\boldsymbol{p} = (p_1, p_2, \ldots, p_C)^{\mathsf{T}}$ such that $p_1 \neq 1$. The EOP for code $\mathscr{C}$ is

$$\begin{aligned}
\theta\left(\mathscr{C}, B\right) &= \frac{1}{M}\sum_{i=1}^{M}\Pr\left[\sum_{c=1}^{C}\sum_{\ell=1}^{L_c} P_{\boldsymbol{u}(i)}(x_c^{(\ell)})g(x_c^{(\ell)}+W) < B\right], && (185)\\
&< \frac{1}{M}\sum_{i=1}^{M}\Pr\left[\sum_{c=1}^{C}\sum_{\ell=1}^{L_c} P_{\boldsymbol{u}(i)}(x_c^{(\ell)})g(x_1^{(1)}+W) < B\right], && (186)\\
&= \frac{1}{M}\sum_{i=1}^{M}\Pr\left[g(x_1^{(1)}+W) < B\right], && (187)\\
&= \Pr\left[g(x_1^{(1)}+W) < B\right], && (188)\\
&= \theta\left(\mathscr{C}', B\right), && (189)
\end{aligned}$$

where, the inequality in (186) follows from (185) due to Lemma 5.4 and (86d). This completes the proof. □

5.3 An Information-Energy Achievable Region

Theorem 5.13. *A homogeneous* (n, M)*-code* $\mathscr{C}$ *for the random transformation in* (1) *of the form in* (86) *is an* $(n, M, \epsilon, B, \delta)$*-code if the number of layers* C*, and for all* $c \in \{1, 2, \ldots, C\}$*,* *the parameters A_c, L_c, r_c, and p_c satisfy the following:*

$$\prod_{c=1}^{C}\left(1-e^{-\frac{r_c^2}{\sigma^2}}\right)^{np_c} \geq 1-\epsilon, \tag{190a}$$

$$L_c \leq \left\lfloor \frac{\pi}{2\arcsin\frac{r_c}{2A_c}} \right\rfloor, \tag{190b}$$

$$M \leq \frac{n!}{\prod_{c=1}^{C}\left((n\frac{p_c}{L_c})!\right)^{L_c}}, \tag{190c}$$

$$B \leq \frac{1}{1-\delta}\sum_{c=1}^{C} p_c \mathbb{E}_W\left[g\left(A_c+W\right)\right], \tag{190d}$$

where, the real σ^2 is defined in (4)*.*

Proof. The result follows from Lemmas 5.10, 5.5, and 5.2, and (7). □

Theorem 5.14. *An (n, M)-code $\mathscr{C}$ for the random transformation in* (1) *of the form in* (86) *is an $(n, M, \epsilon, B, \delta)$-code if the number of layers C, and for all $c \in \{1, 2, \ldots, C\}$, the parameters A_c, L_c, r_c, and p_c satisfy the following:*

$$\prod_{c=1}^{C}\left(1-e^{-\frac{r_c^2}{\sigma^2}}\right)^{np_c} \geq 1-\epsilon, \tag{191a}$$

$$\delta \geq \frac{1}{M}\sum_{i=1}^{M} \Pr\left[\sum_{c=1}^{C}\sum_{\ell=1}^{L_c} P_{\boldsymbol{u}(i)}(x_c^{(\ell)}) g\left(A_c+W\right) < B\right], \tag{191b}$$

$$L_c \leq \left\lfloor \frac{\pi}{2\arcsin\frac{r_c}{2A_c}} \right\rfloor, \tag{191c}$$

$$M \leq \frac{n!}{\prod_{c=1}^{C}\left((n\frac{p_c}{L_c})!\right)^{L_c}}, \tag{191d}$$

where, the type $P_{\boldsymbol{u}(i)}$ is defined in (21)*, the real σ^2 is defined in* (4)*, and $x_c^{(\ell)} \in \mathcal{U}(A_c, L_c, \alpha_c)$, with $\mathcal{U}(A_c, L_c, \alpha_c)$ in* (86b)*.*

Proof. The result follows from Lemmas 5.9, 5.5, and 5.1, and (7). □

6 Examples

6.1 Information-Energy Converse Region

Consider a homogeneous $(n, M, \epsilon, B, \delta)$-code $\mathscr{C}$ (Definition 4.2) for the random transformation in (1) of the form in (10) built upon a 16-QAM constellation. More specifically, the set of channel input symbols $\mathcal{X}$ in (5) is:

$$\begin{aligned}\mathcal{X} = \{&(5+\mathrm{i}5), (-5+\mathrm{i}5), (-5-\mathrm{i}5), (5-\mathrm{i}5), (15+\mathrm{i}5),\\ &(5+\mathrm{i}15), (-5+\mathrm{i}15), (-15+\mathrm{i}5), (-15-\mathrm{i}5), (-5-\mathrm{i}15),\\ &(5-\mathrm{i}15), (15-\mathrm{i}5), (15+\mathrm{i}15), (-15+\mathrm{i}15), (-15-\mathrm{i}15),\\ &(15-\mathrm{i}15)\}.\end{aligned} \tag{192}$$

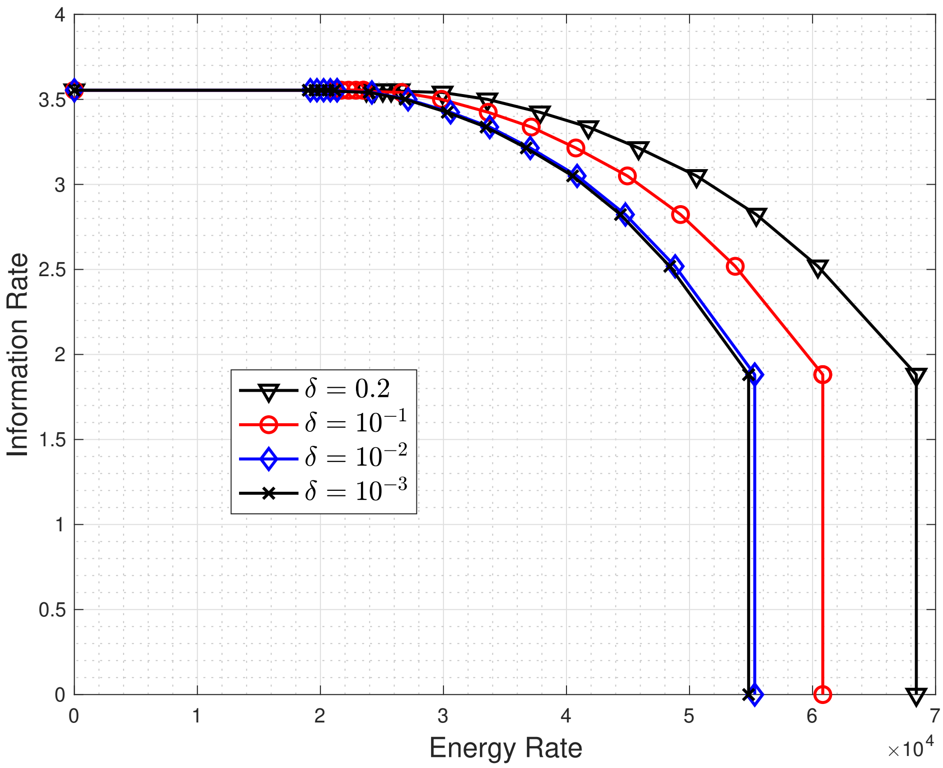

Figure 2: Upper bounds on the information transmission rate in (83) and energy transmission rate in (84) for a homogeneous code built upon a 16-QAM constellation as a function of the EOP δ.

The decoding sets $\mathcal{D}_1$, $\mathcal{D}_2$, ..., $\mathcal{D}_{16}$ in (53) are those obtained from the maximum-à-posteriori (MAP) estimator [11] given the type $P_{\mathscr{C}}$ in (22). The block-length is fixed at $n = 80$ channel uses and the noise variance $\sigma^2 = 1$ in (4). The energy harvested at the EH is calculated taking into account the non-linearities at the receiver described in [14]. In particular, given a type $P_{\mathscr{C}}$, the expression $\sum_{\ell=1}^{L} P_{\mathscr{C}}(x^{(\ell)})\mathbb{E}\left[g\left(x^{(\ell)} + W\right)\right]$ in (84) is obtained by using [14, Proposition 1], which is denoted by P_{del} therein. The choice of parameters while using [14, Proposition 1] is $k_2 = k_4 = 1$ and $|h| = |\tilde{h}| = 1$).

In Figure 2, given a type $P_{\mathscr{C}}$, the upper bound on the information transmission rate and the upper bound on the energy transmission rate in (83) and (84), respectively, are depicted. Each point in the curves corresponds to a different type $P_{\mathscr{C}}$. Therein, it becomes evident that smaller EOPs imply smaller information and energy transmission rates. The maximum information rate achievable with a 16-QAM constellation is four bits per channel use. Nonetheless, when codewords are forced to exhibit the same type (homogeneous code), the information transmission rate does not exceed 3.51 bits per channel use. This maximum holds when the energy transmission rate is smaller than 2×10^4 energy units, which corresponds to the maximum energy that might be delivered by codes whose type $P_{\mathscr{C}}$ approaches the uniform distribution on $\mathcal{X}$. Beyond this value, increasing the energy transmission rate implies deviating from the uniform distribution,

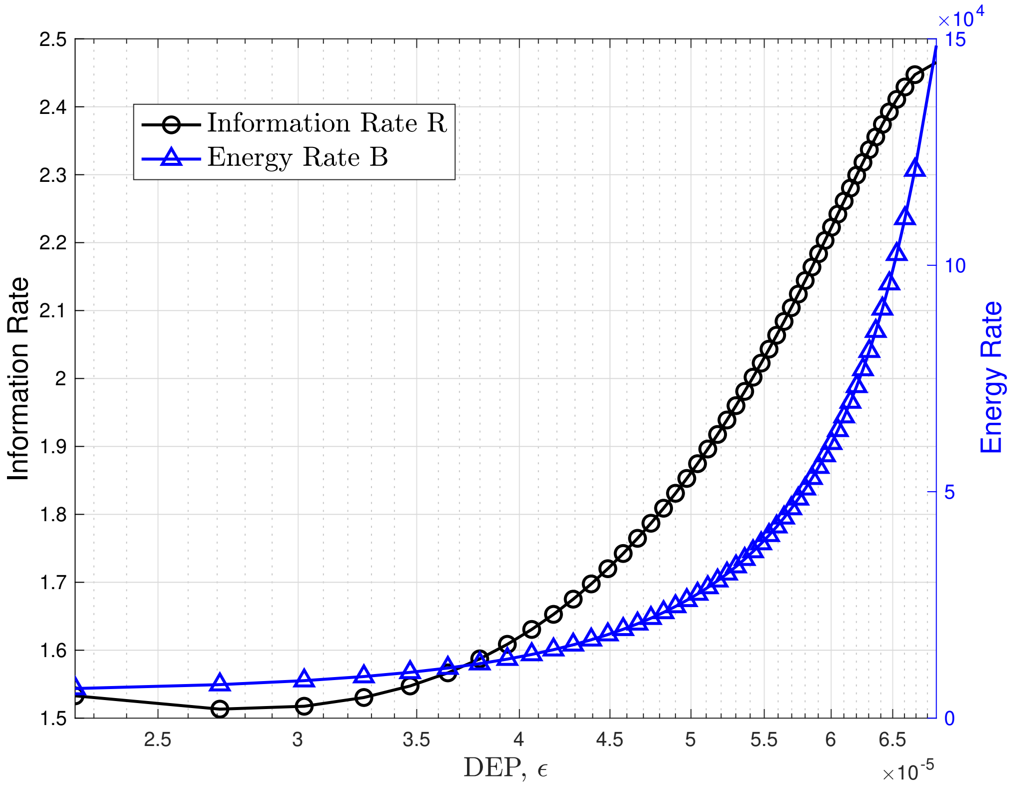

Figure 3: Upper bounds on the information transmission rate in (83) and energy transmission rate in (84) for a homogeneous code built upon a 16-QAM constellation as a function of the lower bound on the DEP in (85).

which penalises the information transmission rate. The maximum energy rate implies types $P_{\mathscr{C}}$ that concentrate on the symbols $(15 + \mathrm{i}15)$, $(-15 + \mathrm{i}15)$, $(-15 - \mathrm{i}15)$, $(15 - \mathrm{i}15)$. A type $P_{\mathscr{C}}$ that approaches the uniform distribution over these four channel inputs induces an upper bound on the information transmission rate of 1.9 bits per channel use. A type $P_{\mathscr{C}}$ that concentrates on one of these channel input symbols induce a zero information rate. Figure 3 illustrates the upper bounds information and energy transmission rates in (83) and (84) as a function of the lower bound on the DEP in (85) when the EOP is kept constant at $\delta = 10^{-4}$ and $n = 80$. More specifically, each point in the curves in Figure 3 for a given value of the DEP is associated with a type $P_{\mathscr{C}}$. This type is used to determine upper bounds on both the information and energy transmission rates, which leads to the curves in Figure 3.

For a higher DEP require types $P_{\mathscr{C}}$ with less constraints, which increase the bounds on both the information and the energy rates.

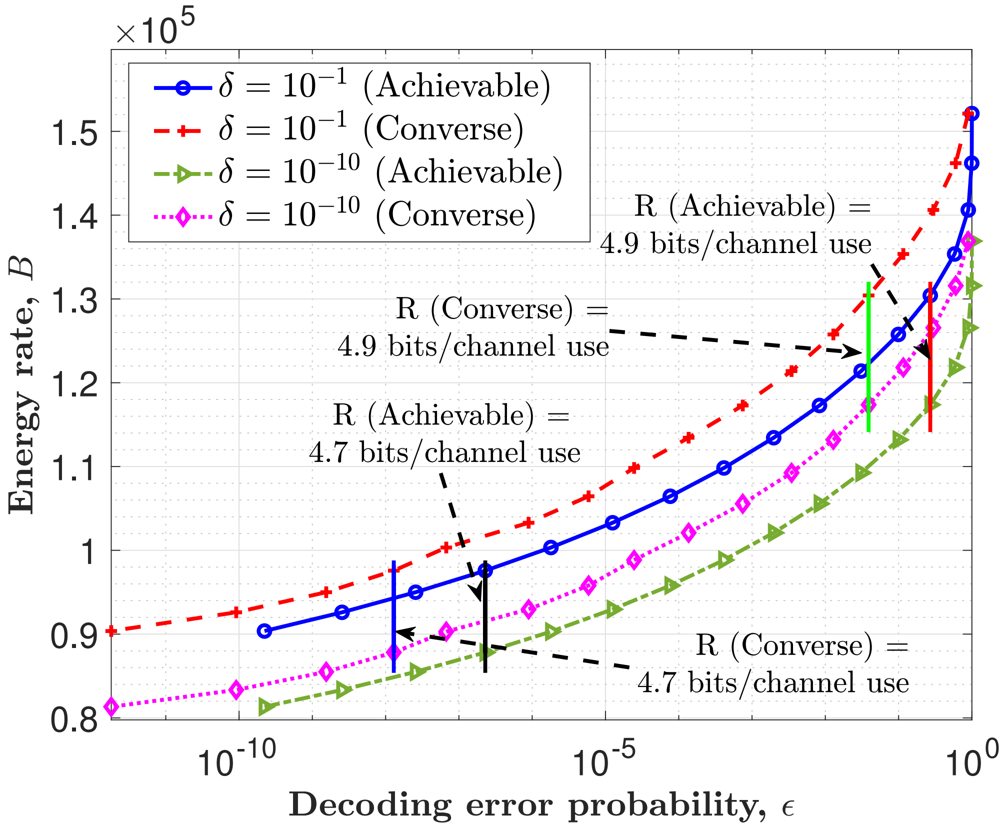

Figure 4: Converse (23) and achievable (190d) bounds on energy transmission rate B for a homogeneous code in the family $\mathsf{C}(\mathcal{X}, \boldsymbol{p}, \boldsymbol{r})$ as a function of the DEP ϵ in (57) and (190a), respectively.

6.2 Information-Energy Achievable Region

6.2.1 Parameter trade-offs

Consider a homogeneous $(n, M, \epsilon, B, \delta)$-code $\mathscr{C}$ in $\mathsf{C}(\mathcal{X}, \boldsymbol{p}, \boldsymbol{r})$ in (87). The constellation $\mathcal{X}$ is of the form in (86c) with number of layers $C = 3$. Duration of the transmission in channel uses is $n = 80$. The energy harvested at the EH takes into consideration the non-linearities of the receiver as suggested in [14, 15]. More specifically, the energy function g in (16) is of the form in [14, Proposition 1].

In Fig. 4, the bound on the achievable energy transmission rate B in (190d) for code $\mathscr{C}$ is plotted as a function of the achievable DEP ϵ in (190a). The figure also shows the converse bound on B in (84) as a function of ϵ in (57). The amplitude of the first layer is $A_1 = 50$. Amplitudes of the second and third layers A_2 and A_3 are determined by the radii of the decoding regions according to (86l). For all $c \in \{1, 2, \ldots, C\}$, the number of symbols in layer c i.e., L_c, is determined by the radii r_c and the amplitudes A_c according to (190b). The probability vector in (86f) is $\boldsymbol{p} = (0.5, 0.3, 0.2)^{\mathsf{T}}$. The points on the curves are generated by varying r_c between 2 and 10.

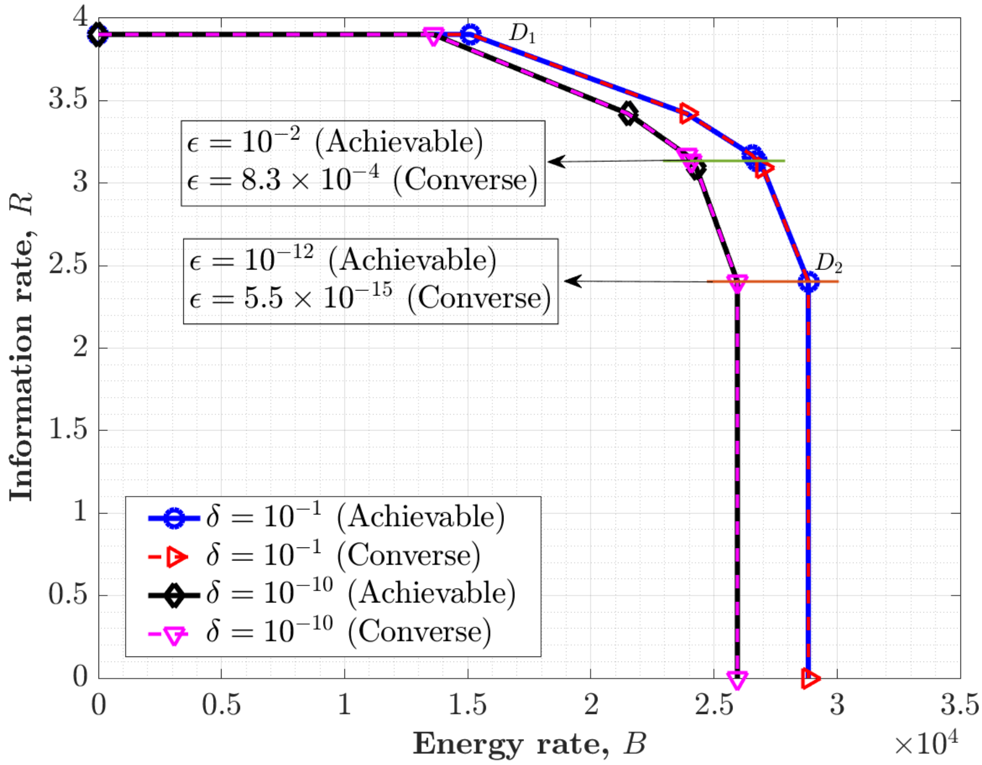

Figure 5: Converse and achievable information-energy regions for homogeneous codes in the family $\mathsf{C}(\mathcal{X}, \boldsymbol{p}, \boldsymbol{r})$.

Fig. 4 shows several trade-offs between the energy transmission rate B, the DEP ϵ, the EOP δ, and the information transmission rate R. Firstly, the energy rate B increases as ϵ increases. This effect is due to the fact that increasing ϵ allows decreasing the radii of the decoding regions $\boldsymbol{r}$ in (86m) according to (190a). At the same time, decrease in r_c allows increasing the amplitudes A_2 and A_3 according to (86l) which increases B according to (190d). Secondly, the energy rate B increases as δ increases. This effect stems from the dependence of B on δ as in (190d). Thirdly, the information rate R increases as ϵ increases. This is because increasing ϵ allows decreasing r_c according to (190a). At the same time, decrease in r_c allows increasing the number of symbols in a layer L_c according to (190b) which increases R according to (190c) and (11).

Fig. 5 shows the converse and achievable information-energy regions of code $\mathscr{C}$ as a function of the EOP δ and the DEP ϵ. The radii of the decoding regions r_c are assumed to be the same for all the layers i.e., for all $c \in \{1, 2, \ldots, C\}$, the radii $r_c = r$ in (86k). The value of r is obtained according to (190a) to satisfy ϵ. The amplitude of the first layer is $A_1 = 30$. Amplitudes of the second and third layers A_2 and A_3 are determined by r according to (86l). The points in Fig. 5 are obtained by varying ϵ and the probability vector $\boldsymbol{p}$ in (86f).

Fig. 5 shows the following trade-offs between the information and energy transmission rates in the converse and achievable curves. Firstly, the maximum achievable information transmission rate is $R = 3.9$ bits/channel use. This R is achieved by a code in which all the symbols in the constellation $\mathcal{X}$ are used with the same frequency. The maximum energy transmission rate that can be achieved at $R = 3.9$ bits/channel use is $B = 1.5 \times 10^4$ energy units. This corresponds to the point D_1 in Fig. 5. Secondly, the maximum achievable B is 2.9×10^4 energy units. This is achieved by a code that exclusively uses the symbols in the first layer i.e., the probability vector $\boldsymbol{p}$ in (86f) is $\boldsymbol{p} = (1, 0, 0)^{\mathsf{T}}$. The maximum R that can be achieved at $B = 2.9 \times 10^4$ energy units is $R = 2.4$ bits/channel use. This corresponds to the point D_2 in Fig. 5. Thirdly, the curves between the points D_1 and D_2 in Fig. 5 show the trade-off between the information and energy transmission rates. As B is increased from 1.5×10^4 energy units at point D_1, R decreases. Similarly, as R is increased from 2.4 bits/channel use at point D_2, B decreases.

6.2.2 Comments on Optimality

The codes constructed in this work are optimal in the sense of the converse results in Section 4 except for the DEP ϵ. Fig.4 shows that the code $\mathscr{C}$ achieves the optimal energy rate B and information rate R as given by the converse results albeit at a higher DEP ϵ. Fig. 5 shows that the converse and achievable information-energy rate curves for $\mathscr{C}$ overlap. However, for the same information and energy rate pair, the DEP for the achievable curves is higher than that of the converse curves. The sub-optimality in DEP arises due to the sub-optimal choice of circular decoding regions in (86k).

The proposed construction provides a method of building codes that meet the given energy and information rate, EOP, and DEP requirements. Building codes that achieve the optimal energy and information rate, EOP, and DEP requires optimizing set of channel input symbols $\mathcal{X}$. However, the problem of optimal input design even for the most well behaved channels [16–18] remains an open problem.

7 Conclusions

In this report, the fundamental trade-offs between information and energy transmission rates have been studied in memoryless AWGN channels under the assumptions that the DEP and EOP are bounded away from zero; and the constellation size and the block length are finite. These trade-offs are essentially translated into constraints over the type that the code must induce, which implies constraints on how often each point in the constellation is used. Using this observation the fundamental limits on both information and energy transmission rates have been characterized for any homogeneous code leading to guidelines on optimal constellation designs for SIET.

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