Discrete Structures

Lecture 10: Sets and Set Operations

based on slides by Jan Stelovsky
based on slides by Dr. Baek and Dr. Still
Originals by Dr. M. P. Frank and Dr. J.L. Gross
Provided by McGraw-Hill

Muhammad Adeel Zahid

Department of Computer Science
Government College University
Faisalabad

- Previously…
  - Literal set {a,b,c} and set-builder notation {x |P(x)}
  - Basic properties: unordered, distinct elements
- Next Topics
  - Infinite Sets
  - $\in$ relational operator and empty set ϕ
  - Venn Diagrams
  - Set Relations $=, \subseteq, \subset, \supset, \not\subset$, etc
  - Cardinality |S| of a set S
  - Power sets P(S)
  - Cartesian product $S \times T$
  - Set operators: $\cup, \cap, -$

Infinite Set

- Conceptually, sets may be infinite (i.e., not finite, without end, unending).
- Symbols for some special infinite sets:
  - N = {0, 1, 2,…} the set of Natural numbers.
  - Z = {…, –2, –1, 0, 1, 2,…} the set of Integers
  - Z^+ = {1, 2, 3,…} the set of positive integers.
  - Q = {p/q | p,q ∈ Z, and q ≠ 0} the set of Rational numbers.
  - R = the set of "Real" numbers.
- "Blackboard Bold" or double-struck font is also often used for these special number sets.

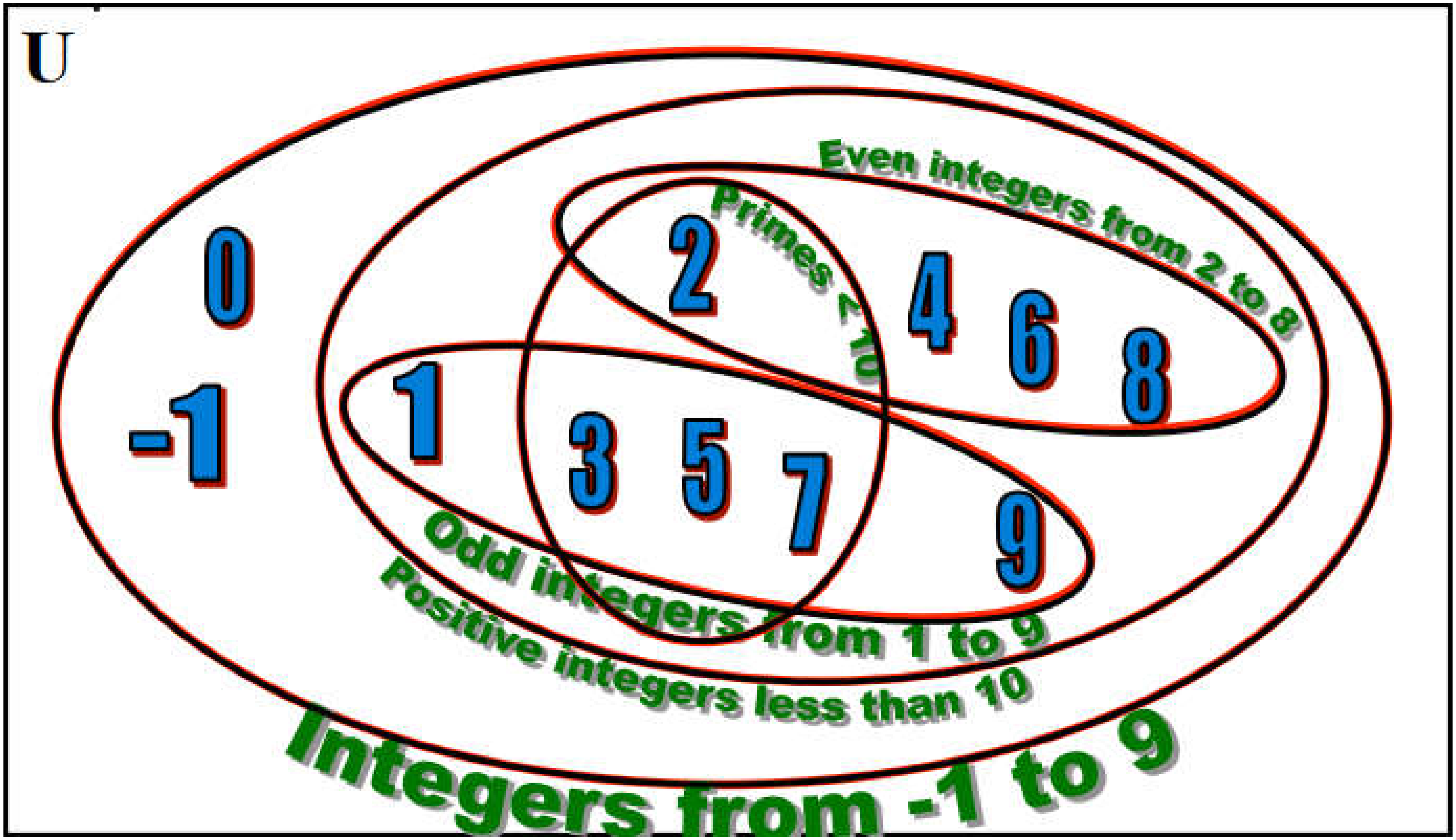
Member of Operator (∈)

- $x \in S$ ("x is in S") is the proposition that object x is an element or member of set S.
  - e.g. 3 ∈ N,
  - a ∈ {x | x is a letter of the alphabet}
- Can define set equality in terms of ∈ relation:
  - $\forall S, T \colon S = T \leftrightarrow [\forall x\, (x \in S \leftrightarrow x \in T)]$
  - Two sets are equal iff they have all the same members
- $x \notin S \equiv \neg(x \in S)$ "x is not in S

The Empty Set (ϕ)

- $\emptyset$ ("null", "the empty set") is the unique set that contains no elements whatsoever.
  - $\emptyset = \{\ \} = \{x \mid \text{False}\}$
- No matter the domain of discourse,
  - we have the axiom $\neg\exists x\colon\ x \in \emptyset$.
- $\{\ \} \neq \{\emptyset\} = \{\ \{\ \}\ \}$
- $\{\emptyset\}$ it isn't empty because it has $\emptyset$ as a member!

Venn Diagrams



Subset and Superset

- $S \subseteq T$ ("S is a subset of T ") means that every element of S is also an element of T
  - $S \subseteq T \equiv \forall x\,(x \in S \rightarrow x \in T)$
  - $\emptyset \subseteq S, S \subseteq S$
- $S \supseteq T$ ("S is a superset of T ") means $T \subseteq S$
- Note $(S = T) \equiv (S \subseteq T \wedge T \subseteq S)$
  - $\equiv \forall x(x \in S \rightarrow x \in T) \wedge \forall x(x \in T \rightarrow x \in S)$
  - $\equiv \forall x(x \in S \leftrightarrow x \in T)$
- $S \nsubseteq T$ means $\neg(S \subseteq T)$, i.e. $\exists x(x \in S \wedge x \notin T)$

Proper (strict) Subsets and Supersets

- $S \subset T$ ("S is a proper subset of T ") means that $S \subseteq T$ but $T \nsubseteq S$
  - Example:
  - $\{1, 2\} \subset \{1, 2, 3\}$
  - $\{1,2,3\} \not\subset \{1,2,3\}$
  - $\{1,2,3\} \subseteq \{1,2,3\}$
- Similarly, $S \supset T$ (S is a proper superset of T) means that $S \supseteq T$ but $T \nsupseteq S$
  - $\{1, 2, 3\} \supset \{1, 2\}$
  - $\{1,2,3\} \not\supset \{1,2,3\}$
  - $\{1,2,3\} \supseteq \{1,2,3\}$

Set as Element of a Set

- The objects that are elements of a set may themselves be sets
  - Example:
  - Let $S = \{x \mid x \subseteq \{1, 2, 3\}\}$
- then $S = \{\ \emptyset,$
$\{1\}, \{2\}, \{3\},$
$\{1, 2\}, \{1, 3\}, \{2, 3\},$
$\{1, 2, 3\}\ \}$
- Note that $1 \neq \{1\} \neq \{\{1\}\}$

Cardinality of a Set

- $|S|$ (read "the cardinality of S") is a measure of how many different elements S has.
  - E.g., $|\emptyset| = 0, |\{1,2,3\}| = 3, |\{a,b\}| = 2,$
  - $|\{\{1,2,3\},\{4,5\}\}| = 2$
- If $|S| \in N$, then we say S is finite.
- Otherwise, we say S is infinite.
- What are some infinite sets we've seen?
  - N, Z, Q, R

The Power Set Operation

- The power set $P(S)$ of a set S is the set of all subsets of S. $P(S) = \{x \mid x \subseteq S\}$.
- Examples
  - $P(\{a, b\}) = \{ \emptyset, \{a\}, \{b\}, \{a, b\} \}$
  - $S = \{0, 1, 2\}$
  - $P(S) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}, \{0, 1, 2\}\}$
  - $P(\emptyset) = \{\emptyset\}$
  - $P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$
- Note that for finite S, $|P(S)| = 2^{|S|}$
- It turns out $\forall S\ (|P(S)| > |S|)$, e.g. $|P(N)| > |N|$

Ordered n-tuples

- These are like sets, except that duplicates matter, and the order makes a difference.
- For $n \in N$, an ordered n-tuple or a sequence or list of length n is written $(a_1, a_2, \dots, a_n)$. Its first element is a_1, its second element is a_2, etc
- Note that $(1, 2) \neq (2, 1) \neq (2, 1, 1)$
- Empty sequence, singlets, pairs, triples, quadruples, quintuples, …, n-tuples
  - $(), (1), (1,2), (a, b, c), (w, x, y, z) \dots$

Cartesian Product of Sets

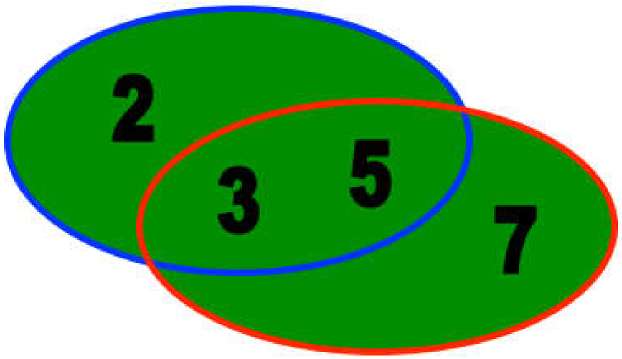
- For sets A and B, their Cartesian product denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$
- Hence, $A \times B = \{ (a, b) \mid a \in A \wedge b \in B \}$
  - E.g. $\{a, b\} \times \{1, 2\} = \{ (a, 1), (a, 2), (b, 1), (b, 2) \}$
- Note that for finite A, B, $|A \times B| = |A||B|$
- Note that the Cartesian product is not commutative: i.e.,
  - $\neg \forall A, B \ (A \times B = B \times A)$
- Extends to $A_1 \times A_2 \times \ldots \times A_n = \{(a_1, a_2, \ldots, a_n) | a_i \in A_i \ for \ i = 1, 2, \ldots, n\}$

The Union Operator (∪)

- For sets A and B, their union $A \cup B$ is the set containing all elements that are either in A, or ("∨") in B (or, of course, in both).
- Formally, $\forall A, B$: $A \cup B = \{x \mid x \in A \lor x \in B\}$
- Note that A∪B is a superset of both A and B
  - in fact, it is the smallest such superset)
- $\forall A, B$: $(A \cup B \supseteq A) \land (A \cup B \supseteq B)$
- Examples:
  - $\{a, b, c\} \cup \{2, 3\} = \{a, b, c, 2, 3\}$
  - $\{2, 3, 5\} \cup \{3, 5, 7\} = \{2, 3, 5, 3, 5, 7\} = \{2, 3, 5, 7\}$

Set Union Example

- $\{2, 3, 5\} \cup \{3, 5, 7\} = \{2, 3, 5, 3, 5, 7\} = \{2, 3, 5, 7\}$

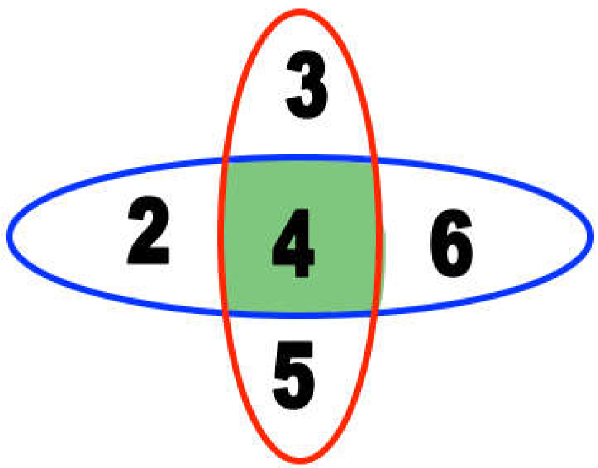


The Intersection Operator

- For sets A and B, their intersection $A \cap B$ is the set containing all elements that are simultaneously in A and ("$\wedge$") in B
- Formally, $\forall A, B$: $A \cap B \; = \; \{x \mid x \in A \; \wedge \; x \in B\}$
- Note that $A \cap B$ is a subset of both A and B
  - In fact it is the largest such subset):
- $\forall A, B$: $(A \cap B \; \subseteq \; A) \; \wedge \; (A \cap B \; \subseteq \; B)$

Intersection Examples

- $\{a, b, c\} \cap \{2, 3\} = \phi$
- $\{2, 4, 6\} \cap \{3, 4, 5\} = \{4\}$



Disjointedness or Exclusive Sets

- Two sets A, B are called disjoint (i.e., unjoined) or mutually exclusive iff their intersection is empty. $(A \cap B = \emptyset)$
- Example: the set of even integers is disjoint with the set of odd integers
- How many elements are in $A \cup B$?
  - $|A \cup B| = |A| + |B| - |A \cap B|$

Inclusion-Exclusion Example

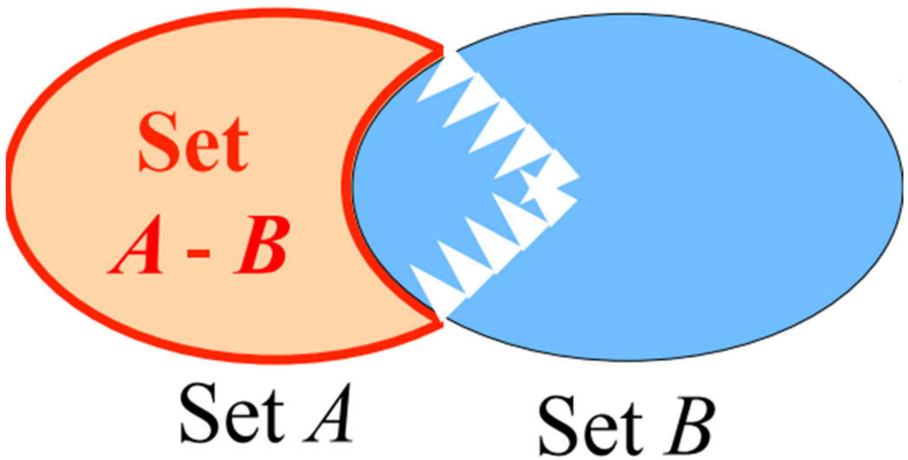
- Example: How many students in the class major in Computer Science or Mathematics?
  - Consider set $E = C \cup M$,
- C = {s | s is a Computer Science major}
- M = {s | s is a Mathematics major}
- Some students are joint majors!
  - $|E| = |C \cup M| = |C| + |M| - |C \cap M|$
  - Remove the intersection cardinality to compensate for double counting

Set Difference

- For sets A and B, the difference of A and B, written $A - B$ or $A \setminus B$, is the set of all elements that are in A but not B
- Formally: $A - B = \{x \mid x \in A \wedge x \notin B\}$
  - $= \{x \mid \neg(x \in A \rightarrow x \in B)\}$
- Also called: The complement of B with respect to A

Set Difference Venn Diagram

- $A - B$ or $A \setminus B$
  - is what's left after B "takes a bite out of A



Set Difference Examples

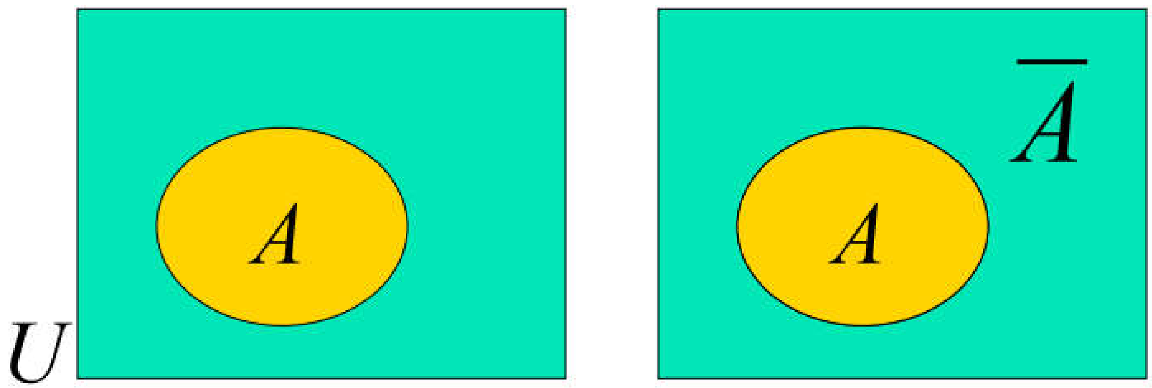
- $\{1, 2, 3, 4, 5, 6\} - \{2, 3, 5, 7, 9, 11\} = \{1,4,6\}$
- $Z - N = \{\dots, -1, 0, 1, 2, \dots\} - \{0, 1, \dots\}$
- $= \{x \mid x \text{ is an integer but not a natural } \#\}$
- $= \{\dots, -3, -2, -1\}$
- $= \{x \mid x \text{ is a negative integer}\}$

Set Complements

- The universe of discourse (or the domain) can itself be considered a set, call it U
- When the context clearly defines U, we say that for any set $A \subseteq U$, the complement of A, written as $\bar{A}$, is the complement of A with respect to U, i.e., it is $U - A$
- E.g., If $U = N$ and $A = \{3,5\}$
  - $\bar{A} = \{0,1,2,4,6,7,\ldots\}$

Set Complement

- An equivalent definition, when U is obvious:
  - $\bar{A} = \{x \mid x \notin A\}$



Interval Notation

- Interval notation is used for real numbers because they cannot be adequately represented otherwise
  - You can represent $N = \{0,1,2 \ldots\}$ but how to represent real numbers between 0 and 1
- $a, b \in R$, and $a < b$ then
  - $(a,b) = \{x \in R \mid a < x < b\}$
  - $[a,b] = \{x \in R \mid a \leq x \leq b\}$
  - $(a,b] = \{x \in R \mid a < x \leq b\}$
  - $(-\infty, b] = \{x \in R \mid x \leq b\}$
  - $[a, \infty) = \{x \in R \mid a \leq x\}$
  - $(a, \infty) = \{x \in R \mid a < x\}$