Any eq of the form $a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_n y = F(x)$ is called

→ Cauchy- Euler Equation. put $x = e^t$, $t = \ln x$.

Ex 10.4.

Question (1) $(x^2 D^2 + 7xD + 5)y = x^5$ — (1)

Sol:

Let $x = e^t$, $t = \ln x$.

$xD = \Delta$

$x^2 D^2 = \Delta(\Delta - 1)$

put in eq (1)

$(\Delta(\Delta-1) + 7\Delta + 5)y = e^{5t}$

$(\Delta^2 - \Delta + 7\Delta + 5)y = e^{5t}$

$(\Delta^2 + 6\Delta + 5)y = e^{5t}$ ——— (2)

The C.E is

$\Delta^2 + 6\Delta + 5 = 0$

$\Delta^2 + 5\Delta + 1\Delta + 5 = 0$

$\Delta(\Delta + 5) + 1(\Delta + 5) = 0$

$(\Delta + 1)(\Delta + 5) = 0$

$\Delta + 1 = 0 \Rightarrow \Delta = -1$

$\Delta + 5 = 0 \Rightarrow \Delta = -5$

$y_c = c_1 e^{-t} + c_2 e^{-5t}$

$y_p = \frac{1}{\Delta^2 + 6\Delta + 5} \cdot e^{5t}$

$= \frac{1}{(5)^2 + 6(5) + 5} \cdot e^{5t}$

$= \frac{1}{60} e^{5t}$

$$y = c_1 e^{-t} + c_2 e^{-5t} + \frac{e^{5t}}{60} \quad \text{--- (3)}$$

Replacing t by $\ln x$

$$y_1 = c_1 e^{-\ln x} + c_2 e^{-5\ln x} + \frac{e^{5\ln x}}{60}$$

$$= \cancel{c_1 e^{-x}} + \cancel{c_2 e^{-5x}}\; y \Rightarrow c_1 x^{-1} + c_2 x^{-5} + \frac{x^5}{60}$$

Question (2)

$$(x^2D^2 - 3xD + 5)y = x^2 \sin(\ln x) \quad \text{--- (1)}$$

Sol Let $x = e^t$, $t = \ln x$.

$$xD = \Delta.$$

$$x^2D^2 = \Delta(\Delta - 1)$$

$$(\Delta(\Delta-1) - 3\Delta + 5)y = e^{2t} \sin t.$$

$$(\Delta^2 - 4\Delta + 5)y = e^{2t} \sin t \quad \text{--- (2)}$$

The C.F is

$$\Delta^2 - 4\Delta + 5 = 0$$

$\cancel{\Delta^2}\ \cancel{\&}\Delta$ $a = 1$, $b = -4$, $c = 5$

$$\Delta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

$$= \frac{4 \pm \sqrt{16 - 20}}{2}.$$

$$= \frac{4 \pm 2i}{2}.$$

$$= 2 \pm i$$

$$y_c = e^{2t}(c_1 \cos t + c_2 \sin t)$$

$$y_p = \frac{e^{2t} \sin t \cdot 1}{D^2 - 4D + 5} \qquad \cdot e^{2t} \sin t$$

$$= e^{2t} \frac{\sin t}{(D+2)^2 - 4(D+2) + 5} \qquad \text{sin t by exponential shift}$$

$$= e^{2t} \frac{\sin t}{D^2 + \cancel{4D} + 4 - \cancel{4D} - 8 + 5}$$

$$= e^{2t} \frac{\sin t}{D^2 + 1}$$

$$= e^{2t} \, Im \frac{e^{it}}{(D+i)(D-i)}$$

$$= -\frac{e^{2t}}{2} \, Im \, it(\cos t + i \sin t)$$

$$= -\frac{t}{2} e^{2t} \cos t$$

G. S is

$$y = y_c + y_p$$

$$= e^{2t}(c_1 \cos t + c_2 \sin t) - \frac{t}{2} e^{2t} \cos t$$

Question 3

$$\left[x^2D^2-(2m-1)xD+(m^2+n^2)\right]y=n^2x^m\cdot\ln x. \quad \text{— (1)}$$

Sol:

$x=e^t$, $t=\ln x$.

$xD=\Delta$

$x^2D^2=\Delta(\Delta-1)=\Delta^2-\Delta$.

put in (1)

$$\left[\Delta^2-\Delta-(2m-1)\Delta+(m^2+n^2)\right]=n^2te^{mt}$$

$$\left[\Delta^2-2m\Delta+(m^2+n^2)\right]y=n^2te^{mt} \quad \text{— (2)}$$

The C.F is

$$\Delta^2-2m\Delta+(m^2+n^2)=0$$

$$\Delta=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$\Delta=\frac{2m\pm\sqrt{\cancel{4m^2}-\cancel{4m^2}-4n^2}}{2}$$

$$=\frac{2m\pm2ni}{2}$$

$$=m\pm ni$$

$$y_c=(c_1\cos nt+c_2\sin nt)e^{mt}$$

$$y_p=\frac{n^2te^{mt}}{\Delta^2-2m\Delta+m^2+n^2}$$

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$$y_p = \frac{1}{D^2 - 2mD + m^2 + n^2} n^2 t e^{mt}$$

By Exp Shift:-

$$y_p = n^2 e^{mt} \frac{1}{(D+m)^2 - 2m(D+m) + m^2 + n^2} t$$

$$= n^2 e^{mt} \frac{1}{D^2 + \cancel{2Dm} + \cancel{m^2} - \cancel{2mD} - \cancel{2m^2} + \cancel{m^2} + n^2} t$$

$$= n^2 e^{mt} \frac{1}{D^2 + n^2} t$$

$$= \frac{\cancel{n^2} e^{mt}}{\cancel{n^2}} \left[\frac{1}{\left(1 + \frac{D^2}{n^2}\right)} \right] t$$

$$= e^{mt} \left[1 + \frac{D^2}{n^2} \right]^{-1} t$$

$$= e^{mt} \left[1 - \frac{D^2}{n^2} \right] t$$

$$= e^{mt} [t - 0] = t e^{mt}$$

$$y_p = t e^{mt}$$

$$y = y_c + y_p$$

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Question (4) $(4x^2D^2 - 4xD + 3)y = \sin \ln(-x)$

Sol:

Let $-x = e^t$ $\qquad t = \ln(-x)$

$xD = \Delta$

$x^2D^2 = \Delta(\Delta-1) = \Delta^2 - \Delta.$

$(4(\Delta^2 - \Delta) - 4\Delta + 3)y = \sin t$

$(4\Delta^2 - 4\Delta - 4\Delta + 3)y = \sin t.$

$(4\Delta^2 - 8\Delta + 3)y = \sin t$

The C. Eq is

$4\Delta^2 - 8\Delta + 3 = 0$

$4\Delta^2 - 6\Delta - 2\Delta + 3 = 0$

$2\Delta(2\Delta - 3) - 1(2\Delta - 3) = 0$

$(2\Delta - 1)(2\Delta - 3) = 0$

$2\Delta - 1 = 0 \quad | \quad 2\Delta - 3 = 0$

$\Delta = 1/2 \quad | \quad \Delta = 3/2$

$y_c = c_1 e^{3t/2} + c_2 e^{t/2}.$

$$y_p = \frac{\sin t}{4\Delta^2 - 8\Delta + 3}.$$

$$= \frac{\operatorname{Im} e^{it}}{4\Delta^2 - 8\Delta + 3}.$$

$$= \operatorname{Im} \frac{e^{it}}{4(i^2) - 8i + 3}$$

$$= \operatorname{Im} \frac{e^{it}}{-4-8i+3}$$

$$= \operatorname{Im} \frac{e^{it}}{-1-8i}$$

$$= \operatorname{Im} \frac{e^{it}}{-(1+8i)}$$

$$= \operatorname{Im} \frac{e^{it}(1-8i)}{-(1+8i)(1-8i)}$$

$$= \operatorname{Im} \frac{e^{it}(1-8i)}{-(1+64)} \Rightarrow -\frac{1}{65} \operatorname{Im} e^{it}(1-8i)$$

$$= -\frac{1}{65} \operatorname{Im}(\cos t + i \sin t)(1-8i)$$

$$= -\frac{1}{65} \operatorname{Im} (\cos t + i \sin t - 8i \cos t + 8 \sin t)$$

$$y_p = -\frac{1}{65} (\sin t - 8 \cos t)$$

The General sol.

$$y = y_c + y_p$$

$$= c_1 e^{3t/2} + c_2 e^{t/2} - \frac{1}{65} (\sin t - 8 \cos t)$$

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Question 5

$$(x^3D^3+2x^2D^2+2)y = 10x+\frac{10}{x}.$$

Sol:

Let $x=e^t$ $\quad t=\ln x.$

$xD=\Delta.$

$x^2D^2=\Delta(\Delta-1)=\Delta^2-\Delta.$

$x^3D^3=\Delta(\Delta-1)(\Delta-2)=\Delta^3-3\Delta^2+2\Delta.$

$(\Delta^3-3\Delta^2+2\Delta+2\Delta^2-2\Delta+2)y=10e^t+10e^{-t}$

$(\Delta^3-\Delta^2+2)y=10e^t+10e^{-t}$

The C.E is

$\Delta^3-\Delta^2+2=0$

$\cancel{\Delta^3-2\Delta^2+0}$

$$\begin{array}{r|rrrr} & 1 & -1 & 0 & 2 \\ -1 & & -1 & 2 & -2 \\ \hline & 1 & -2 & 2 & 0 \end{array}$$

$\Delta-1=0 \Rightarrow \Delta=1$

$\Delta^2-2\Delta+2=0$

$a=1,\ b=-2,\ c=2.$

$$\Delta=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

$$=\frac{2\pm\sqrt{4-8}}{2}.$$

$$=\frac{2\pm 2i}{2}.$$

$$=1\pm i$$

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$$\Delta = -1, \; 1 \pm i$$

$$y_c = c_1 e^{-t} + (c_2 \cos t + c_3 \sin t) e^{t}$$

$$y_p = \frac{1}{(\Delta+1)(\Delta^2-2\Delta+2)} \, 10 e^{t} + 10 e^{-t}$$

$$= \frac{1}{(\Delta+1)(\Delta-2\Delta+2)} \, 10 e^{t} + \frac{1}{(\Delta+1)(\Delta^2-2\Delta+2)} \, 10 e^{-t}$$

$$= \frac{10 e^{t}}{(1+1)(1-\cancel{2}+\cancel{2})} + \frac{1 \quad t}{\cancel{(\Delta+1)}(\Delta^2+2\Delta+2)} \, 10 e^{-t}$$

$$(1-\cancel{2}+\cancel{2})$$

$$= 5e^{t} + \frac{10t \, e^{-t}}{(-1+2\Delta+2)}$$

$$= 5 e^{t} + \frac{5t e^{-t}}{\cancel{(\Delta+1)}}$$

$$y = y_c + y_p$$

$$= c_1 e^{-t} + (c_2 \cos t + c_3 \sin t) e^{t} + 5e^{t} + 5te^{-t}$$

Question 6: $(x^4D^3+2x^3D^2-x^2D+x)y=1$

Sol:

$$x(x^3D^3+2x^2D^2-xD+1)y=1$$

$$(x^3D^3+2x^2D^2-xD+1)y=\frac{1}{x}.$$

$$x=e^t \qquad t=\ln x.$$

$$xD=\Delta$$

$$x^2D^2=\Delta(\Delta-1)=\Delta^2-\Delta.$$

$$x^3D^3=\Delta(\Delta-1)(\Delta-2)=\Delta^3-3\Delta^2+2\Delta.$$

$$(\Delta^3-3\Delta^2+\cancel{2\Delta}+2\Delta^2-\cancel{2\Delta}-\Delta+1)y=e^{-t}$$

$$(\Delta^3-\Delta^2-\Delta+1)y=e^{-t}$$

The C-F is

$$\Delta^3-\Delta^2-\Delta+1=0$$

$$D+1=0 \Rightarrow D=-1$$

$$\begin{array}{r|rrrr} & 1 & -1 & -1 & 1 \\ -1 & & -1 & 2 & -1 \\ \hline & 1 & -2 & 1 & 0 \end{array}$$

$$\Delta^2-2\Delta+1=0$$

$$(\Delta-1)^2=0$$

$$\Delta=1,1$$

$$y_c=c_1e^{-t}+(c_2+c_3t)e^t$$

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$$y_p = \frac{1}{(\Delta+1)(\Delta^2-2\Delta+1)} e^{-t}$$

$$= \frac{1}{(\Delta+1)(\Delta-1)^2} e^{-t}$$

$$= \frac{t e^{-t}}{1+2+1}$$

$$= \frac{t e^{-t}}{4}.$$

$$y = y_c + y_p.$$

$$y = c_1 e^{-t} + (c_2 + c_3 t) e^{t} + \frac{t e^{-t}}{4}.$$

$$= c_1 x^{-1} + x(c_2 + c_3 \ln x) + \frac{x^{-1}}{4} \ln x.$$

Question (7)

$$(x^3 D^3 + 2x^2 D^2 - 5xD - 15)y = x^4 \quad \text{—} (1)$$

Sol:- $x = e^t$, $t = \ln x.$

$$xD = \Delta.$$

$$x^2 D^2 = \Delta(\Delta-1) = \Delta^2 - \Delta.$$

$$x^3 D^3 = \Delta^3 - 3\Delta^2 + 2\Delta.$$

put in (1)

$$(\Delta^3 - 3\Delta^2 + 2\Delta + 4\Delta^2 - 4\Delta - 5\Delta - 15)y = e^{4t}$$

$$(\Delta^3+\Delta^2-7\Delta-15)y = e^{4t}$$

The C.F is

$$\Delta^3+\Delta^2-7\Delta-15=0$$

3	1	1	-7	-15
		3	12	15
	1	4	5	0

$\Delta-3=0 \Rightarrow \Delta=3$

$\Delta^2+4\Delta+5=0$

$\cancel{\Delta^2+5\Delta}$ $\quad a=1, b=4, c=5$

$$\Delta = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-4\pm\sqrt{16-20}}{2}.$$

$$= \frac{-4\pm\sqrt{-4}}{2}.$$

$$= \frac{-4\pm 2i}{2}. \Rightarrow -2\pm i$$

$$y_c = c_1e^{3t}+e^{-2t}(c_2\cos t+c_3\sin t)$$

$$y_p = \frac{1}{\Delta^3+\Delta^2-7\Delta-15}e^{4t}$$

$$= \frac{1}{64+16-\cancel{7\Delta}-15}e^{4t}$$

$$= \frac{1}{65-28}e^{4t}$$

64
16
80
15
65
28
37

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$$y_p = \frac{e^{4t}}{37}$$

$$y = y_c + y_p.$$

$$y = c_1 e^{3t} + e^{-2t}(c_2 \cos t + c_3 \sin t) \frac{e^{4t}}{37}$$

Question..

$$(x^2D^2 + 2xD - 6)y = 10x^2 \qquad y(1) = 1$$
$$y'(1) = 6.$$

Sol:- $x = e^t$ or $t = \ln x.$

$$xD = \Delta.$$

$$x^2D^2 = \Delta(\Delta-1) = \Delta^2 - \Delta.$$

$$(\Delta^2 - \Delta + 2\Delta - 6)y = 10e^{2t}.$$

$$(\Delta^2 + \Delta - 6)\, y = 10e^{2t}$$

The C.F is

$$\Delta^2 + \Delta - 6 = 0$$

$$\Delta^2 + 3\Delta - 2\Delta - 6 = 0$$

$$\Delta(\Delta+3) - 2(\Delta+3) = 0$$

$$(\Delta-2)(\Delta+3) = 0$$

$$\Delta - 2 = 0 \Rightarrow \Delta = 2.$$

$$\Delta + 3 = 0 \Rightarrow \Delta = -3.$$

$$y_c = c_1 e^{2t} + c_3 e^{-3t}$$

$$y_p = \frac{1}{\Delta^2+\Delta-6} \cdot 10\, e^{2t}$$

$$= \frac{10}{(\Delta+3)(\Delta-2)} \cdot e^{2t}$$

$$= \frac{\cancel{10}^{2} t}{(\cancel{5})} \cdot e^{2t}$$

$$= 2t e^{2t}.$$

$$y = y_c + y_p.$$

$$y = c_1 e^{2t} + c_3 e^{-3t} + 2t e^{2t}$$

$$y = c_1 x^{-3} + c_2 x^2 + 2x^2 \ln x.$$

$$y' = -3c_1 x^{-4} + 2c_2 x + 4x \ln x + 2x.$$

$$y(1) = 1$$

$$c_1 + c_2 = 1 \quad \text{--- } (1)$$

$$y'(1) = -6$$

$$-3c_1 + 2c_2 + 2 = -6 \quad \text{--- } (2).$$

$$-3c_1 + 2c_2 = -8 \quad \text{--- } (2)$$

sol + get

$$c_2 = -1 \quad \Big| \quad c_1 = 2$$

$$y = 2x^{-3} - x^2 + 2x^2 \ln x$$