Discrete Structures

Lecture 12: Functions

based on slides by Jan Stelovsky
based on slides by Dr. Baek and Dr. Still
Originals by Dr. M. P. Frank and Dr. J.L. Gross
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Muhammad Adeel Zahid
Department of Computer Science
Government College University
Faisalabad

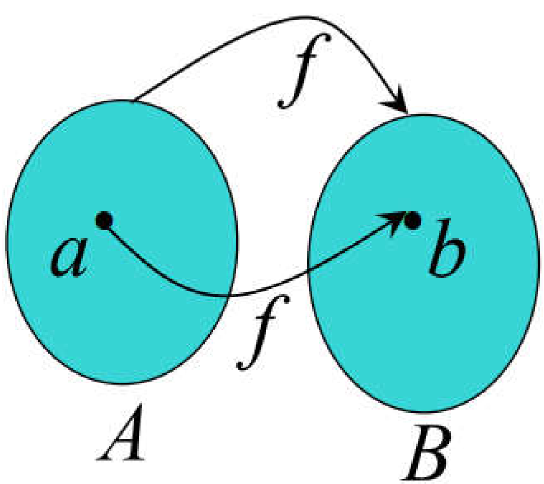
Functions

- From calculus, you are familiar with the concept of a real-valued function f, which assigns to each number $x \in R$ a value $y = f(x)$, where $y \in R$.

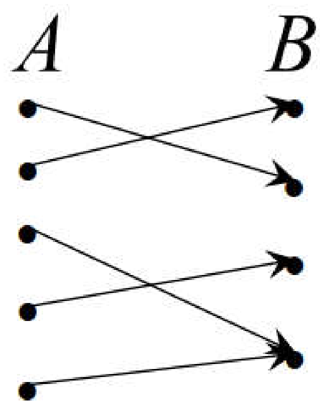
- But, the notion of a function can also be naturally generalized to the concept of assigning elements of any set to elements of any set. (Also known as a map.)

Function: Formal Definition

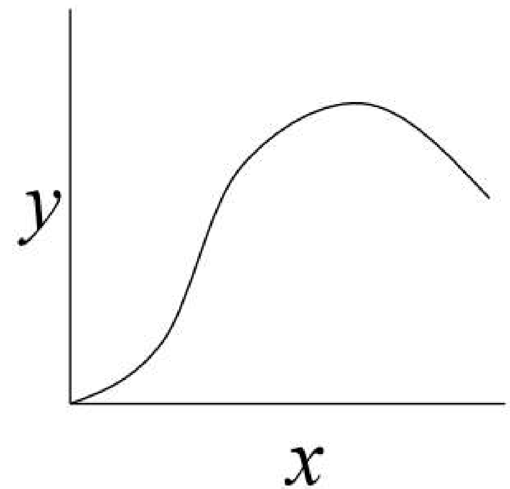
- For any sets A and B, we say that a function (or “mapping”) f from A to B ($f : A \rightarrow B$) is a particular assignment of exactly one element $f(x) \in B$ to each element $x \in A$.
- Functions can be represented graphically in several ways:



Like Venn diagrams



Bipartite Graph



Plot

Terminology

- If it is written that $f : A \rightarrow B$, and $f(a) = b$ (where $a \in A$ and $b \in B$), then we say:
  - A is the domain of f
  - B is the codomain of f
  - b is the image of a under f
    - a can not have more than 1 image
  - a is a pre-image of b under f
    - b may have more than 1 pre-image
- The range $R \subseteq B$ of f is $R = \{b \mid \exists a\, f(a) = b\}$

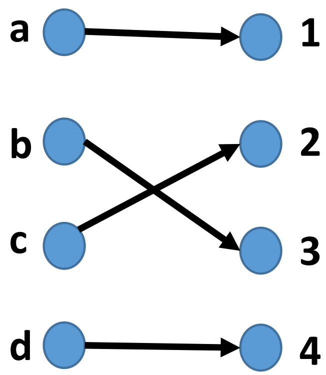
Range vs Codomain

- The range of a function might not be its whole codomain
- The codomain is the set that the function is declared to map all domain values into
- The range is the particular set of values in the codomain that the function actually maps elements of the domain to

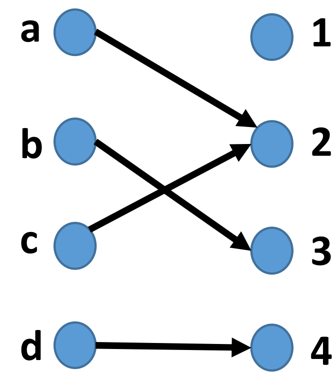
Example

- Suppose I declare that: "f is a function mapping students in this class to the set of grades $\{A, B, C, D, F\}$"
- At this point, you know f's codomain is: $\{A, B, C, D, F\}$ and its range is unknown
- Suppose the grades turn out all As and Bs
- Then the range of f is $\{A, B\}$, but its codomain is still $\{A, B, C, D, F\}$

Function and Non-Function

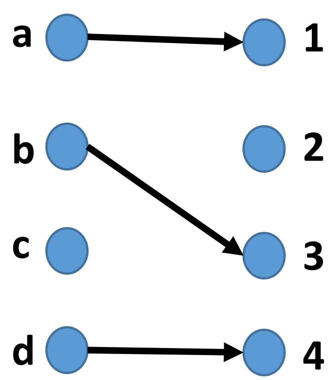


Valid
Function

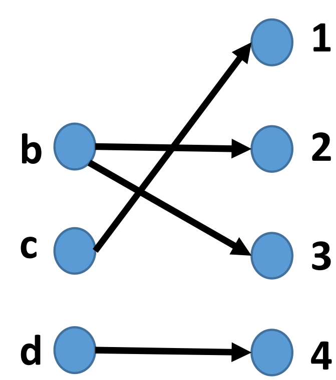


Valid
Function

Not a valid
function because
c does not map to
any value



Not a valid
function because
c maps to more
than 1 value



Function Operators

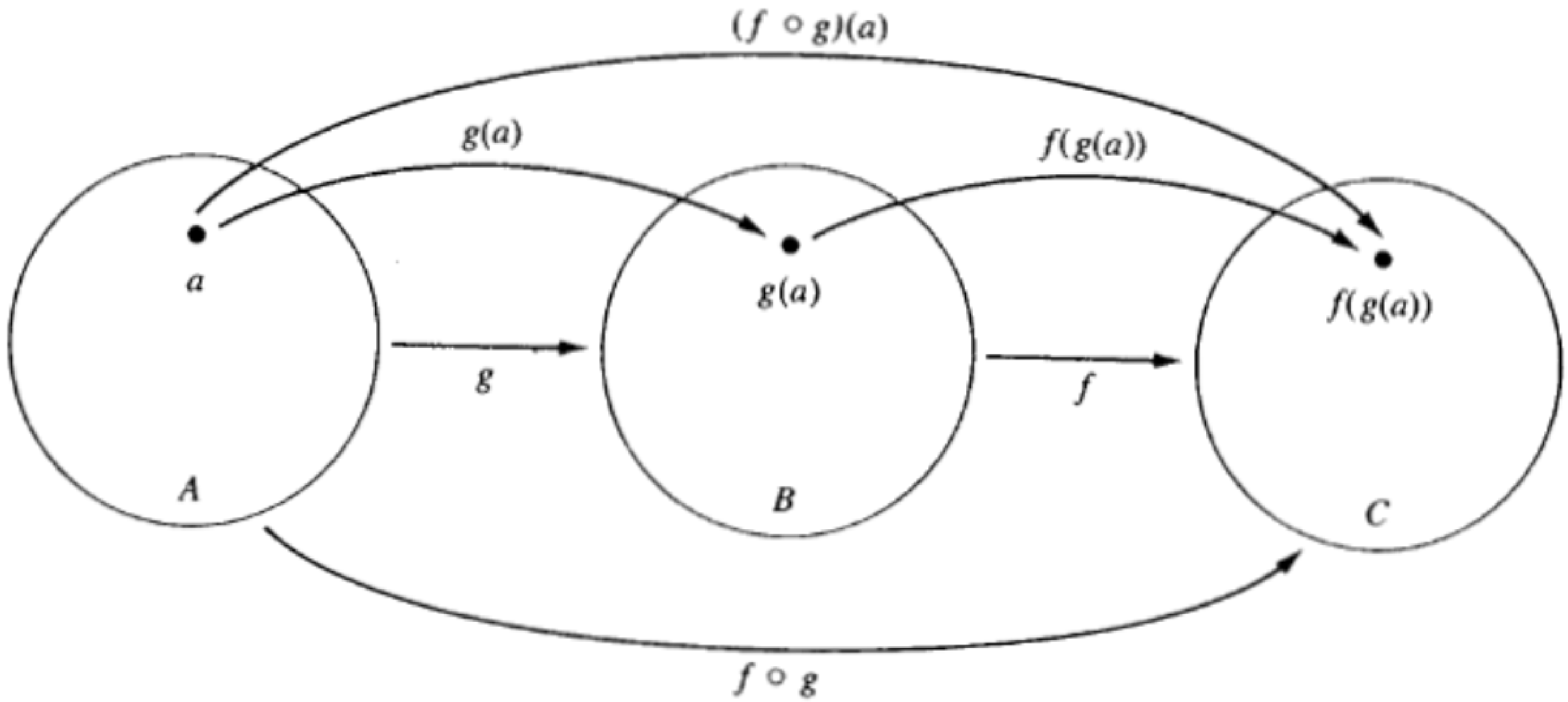
- $+$,$\times$ (“plus”,“times”) are binary operators over R. (Normal addition & multiplication.)
- Therefore, we can also add and multiply two real-valued functions $f, g\colon R \rightarrow R$
  - $(f + g)\colon R \rightarrow R$, where $(f + g)(x) = f(x) + g(x)$
  - $(fg)\colon R \rightarrow R$, where $(fg)(x) = f(x)g(x)$
- Example:
- Let f and g be functions from R to R such that $f(x) = x^2$ and $g(x) = x - x^2$
- What are the functions $f + g$ and fg?
  - $(f+g)(x) = f(x) + g(x) = x^2 + x - x^2 = x$
  - $(fg)(x) = f(x)g(x) = (x^2)(x - x^2) = x^3 - x^4$

Function Composition Operator

- For functions $g\colon A \rightarrow B$ and $f : B \rightarrow C$, there is a special operator called compose ("$\circ$")
- It composes (creates) a new function from f and g by applying f to the result of applying g
- We say $(f \circ g)\colon A \rightarrow C$, where $(f \circ g)(a) = f(g(a))$
- **Note:** $f \circ g$ cannot be defined unless range of g is a subset of the domain of f.
- Note $g(a) \in B$, so $f(g(a))$ is defined and $\in C$
- Note that $\circ$ is non-commuting. (Like Cartesian $\times$, but unlike $+, \wedge, \cup$) (Generally, $f \circ g \neq g \circ f$)

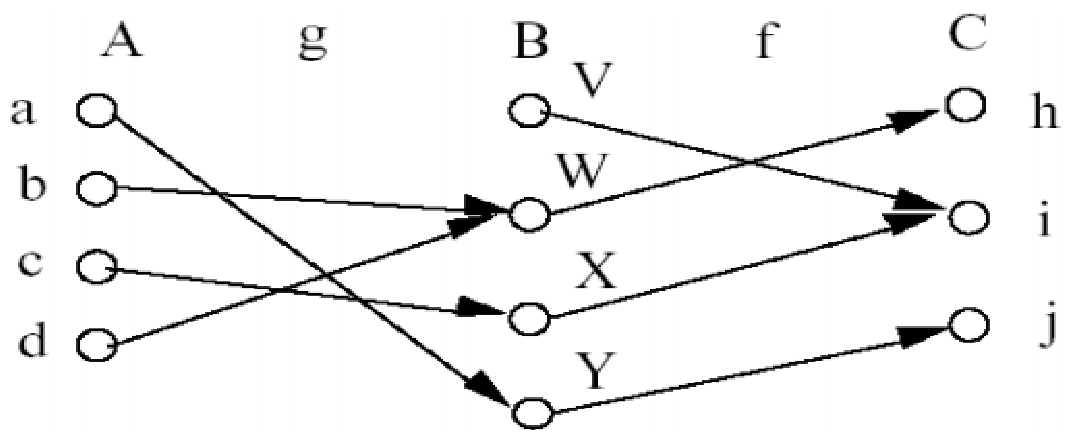
Function Composition

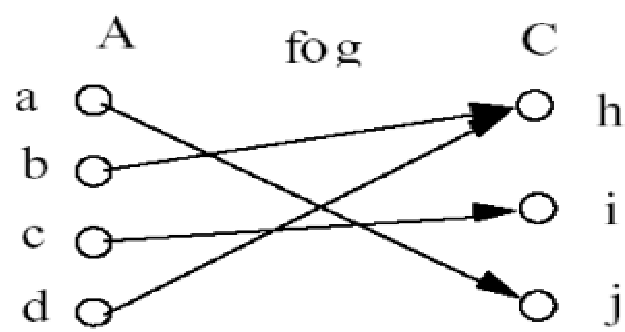
- $g\colon A \to B, f\colon B \to C$



Function Composition

- $g: A \rightarrow B, f : B \rightarrow C$





Function Composition Example

- Example: Let $g: \{a, b, c\} \rightarrow \{a, b, c\}$ such that
  - $g(a) = b, g(b) = c, g(c) = a$
- Let $f : \{a, b, c\} \rightarrow \{1, 2, 3\}$ such that
  - $f(a) = 3, f(b) = 2, f(c) = 1$
- What is the composition of f and g, and what is the composition of g and f ?
- $f \circ g$: $\{a, b, c\} \rightarrow \{1, 2, 3\}$ such that
- $(f \circ g)(a) = 2, (f \circ g)(b) = 1, (f \circ g)(c) = 3$
  - $(f \circ g)(a) = f(g(a)) = f(b) = 2$
  - $(f \circ g)(b) = f(g(b)) = f(c) = 1$
  - $(f \circ g)(c) = f(g(c)) = f(a) = 3$
- $g \circ f$ is not defined (why?): $(g \circ f)(x) = g(f(x))$
  - $f(a) = 3$, but $g(3)$ is not defined
  - $f(b) = 2$, but $g(2)$ is not defined
  - $f(c) = 1$, but $g(1)$ is not defined
- Range of f is not subset of domain of g
  - Range of $f = \{1,2,3\}$
  - Domain of $g = \{a, b, c\}$

Function Composition Example

- If $f(x) = x^2$ and $g(x) = 2x + 1$, then what is the composition of f and g, and what is the composition of g and f ?

$$\begin{aligned}(f \circ g)(x) &= f(g(x)) \\ &= f(2x+1) \\ &= (2x+1)^2\end{aligned}$$

$$\begin{aligned}(g \circ f)(x) &= g(f(x)) \\ &= g(x^2) \\ &= 2x^2 + 1\end{aligned}$$

- Note that $f \circ g \neq g \circ f$. $(4x^2 + 4x + 1 \neq 2x^2 + 1)$

Images of Sets Under Functions

- Given $f\,:\,A \to B$, and $S \subseteq A$,
- The image of S under f is simply the set of all images (under f) of the elements of S

$$f(S) = \{f(t) \mid t \in S\}$$
$$= \{b \mid \exists t \in S\colon f(t) = b\}$$

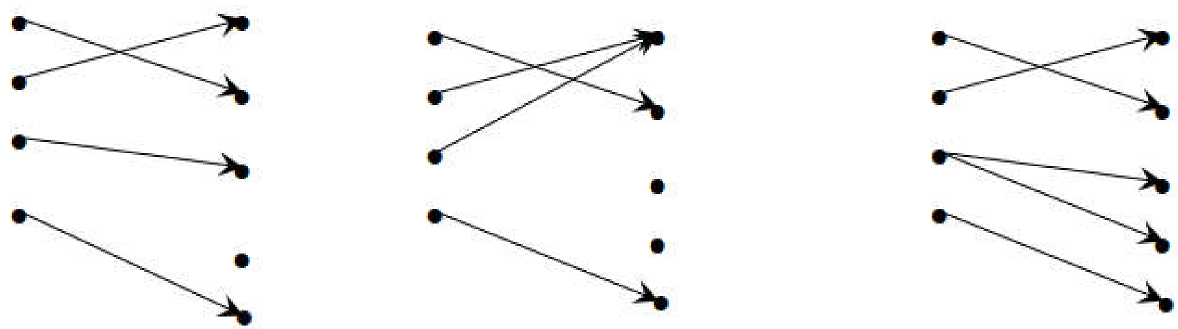
- Note the range of f can be defined as simply the image (under f) of f's domain

One-to-One Function

- A function f is one-to-one (1–1), or **injective**, or an **injection**, iff $f(a) = f(b)$ implies that $a = b$ for all a and b in the domain of f (i.e. every element of its range has only 1 pre-image)
- Formally, given $f : A \rightarrow B$,
  - "f is injective": $\forall a, b\ (f(a) = f(b) \rightarrow a = b)$ or equivalently
  - $\forall a, b\ (a \neq b \rightarrow f(a) \neq f(b))$
- Only one element of the domain is mapped to any given one element of the range
- Domain & range have the same cardinality
- What about codomain?

One-to-One Illustration

- Bipartite (2-part) graph representations of functions that are (or not) one-to-one:



One-to-one Not one-to-one Not even a function!

One-to-one Functions

- Example:
  - Is the function $f : \{a, b, c, d\} \rightarrow \{1, 2, 3, 4, 5\}$ with
- $f(a) = 4, f(b) = 5, f(c) = 1$, and $f(d) = 3$ one-to-one?
- Yes, it is one-to-one
  - No element of the range is assigned more than once to any element of the domain
- Example:
- Let $f : Z \rightarrow Z$ such that $f(x) = x^2$. Is f one-to-one?
  - No, it is not because $f(-2) = f(2) = 4$ even though $2 \neq -2$
  - Generally for this function $f(x) = f(-x)$ even though $x \neq -x$

Sufficient Conditions for one-to-oneness

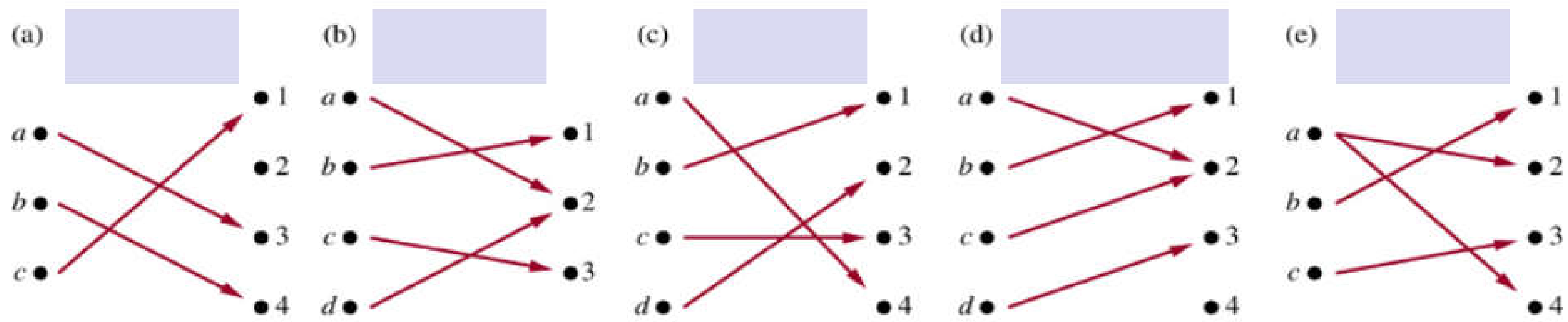
- For functions f over numbers, we say:
- f is strictly (or monotonically) increasing iff $x > y \rightarrow f(x) > f(y)$ for all x, y in domain;
- f is strictly (or monotonically) decreasing iff $x > y \rightarrow f(x) < f(y)$ for all x, y in domain;
- If f is either strictly increasing or strictly decreasing, then f is one-to-one.
- E.g. x^3
  - $(-2)^3 = -8$
  - $(-1)^3 = -1$
  - $(0)^3 = 0$
  - $(1)^3 = 1$
  - $(2)^3 = 8$

Onto (Surjective) Functions

- A function $f : A \rightarrow B$ is onto or **surjective** or a **surjection** iff for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$
  - $(\forall b \in B, \exists a \in A: f(a) = b)$ (i.e. its range is equal to its codomain)
- Think: An onto function maps the set A onto (over, covering) the entirety of the set B, not just over a piece of it
- E.g., for domain & codomain R, x^3 is onto, whereas x^2 isn't. (Why not?)
  - Because x^2 does not map any real number to any negative number hence the codomain and range are not equal

Illustration of Onto

- Some functions that are, or are not, onto their codomains:



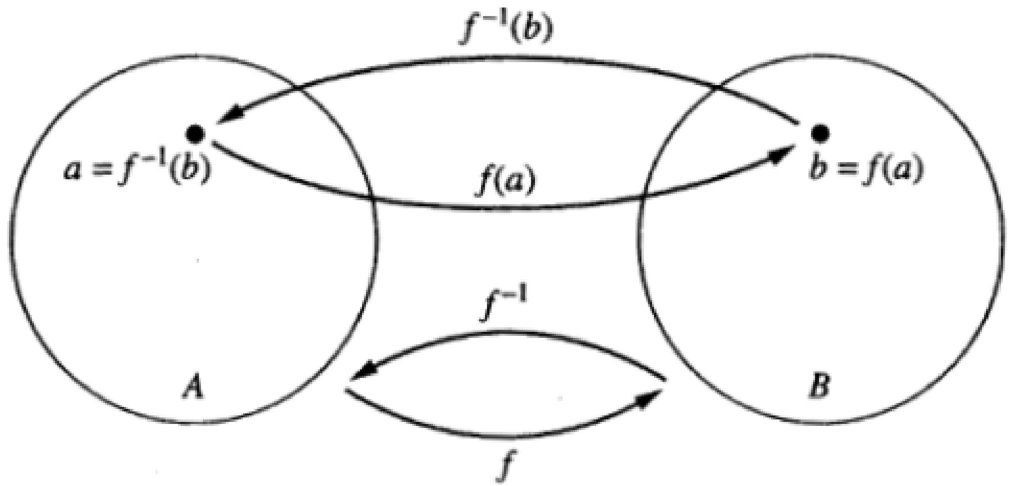
- Example13: Is the function f(x) = x + 1 from the set of integers to the set of integers onto?
  - Yes, because for every integer $n \in Z$(codomain), we have $p \in Z$(domain) such that $f(p) = n$

Bijections and Inversible Functions

- A function f is said to be a one-to-one correspondence, or a **bijection**, or **reversible**, or **invertible**, iff it is both one-to-one and onto
- Let $f : A \to B$ be a bijection
- The inverse function of f is the function that assigns to an element $b \in B$ the unique element $a \in A$ such that $f(a) = b$
- The inverse function of f is denoted by $f^{-1}: B \to A$
- Hence, $f^{-1}(b) = a$ when $f(a) = b$

Inverse Function Illustration

- Let $f\colon A \rightarrow B$ be a bijection



Inversible Function Examples

- Example: Let $f : \{a, b, c\} \rightarrow \{1, 2, 3\}$ such that $f(a) = 2, f(b) = 3, f(c) = 1$ Is f invertible, and if it is, what is its inverse?
- Yes. $f^{-1}(1) = c, f^{-1}(2) = a, f^{-1}(3) = b$

- Example: Let f be the function from R to R with $f(x) = x^2$ Is f invertible?
- No. f is neither a one-to-one function and nor an onto function
  - $f(-2) = f(2)$
- So it's not invertible

Mappings in Java

- A discrete function can be represented by a Map interface or HashMap class in Java programming language
  - $Map\ map < Integer, String >= \ new\ HashMap < Integer, String >$ $() ;$
- Here, the domain is Integer, the codomain is String
- We can construct such a mapping by putting all pairs $\{a, f(a)\}$ into our map. (a is the key, $f(a)$ is the value.)
  - $map.put(2, \text{"Jan"});$
- $for\ (Kid\ kid: kids)\ \{map.put(kid.id, kid.name);\}$
- If we put another pair with the same key, it will overwrite the previous pair

Images, Range, Bijection in Java

- $map.keys()$ returns the image
  - it's a Java Set!
- $map.values()$ returns the range
  - it's a Java Set!
- Is a map a bijection?
  - Iff the cardinalities of the image and range are the same:
- $if\ (map.keys().size() == map.values().size())\ \{$
  - $System.out.println($"map is a bijection"$);$
- $\}$