Discrete Structures

Lecture 13: Functions & Sequences

based on slides by Jan Stelovsky
based on slides by Dr. Baek and Dr. Still
Originals by Dr. M. P. Frank and Dr. J.L. Gross
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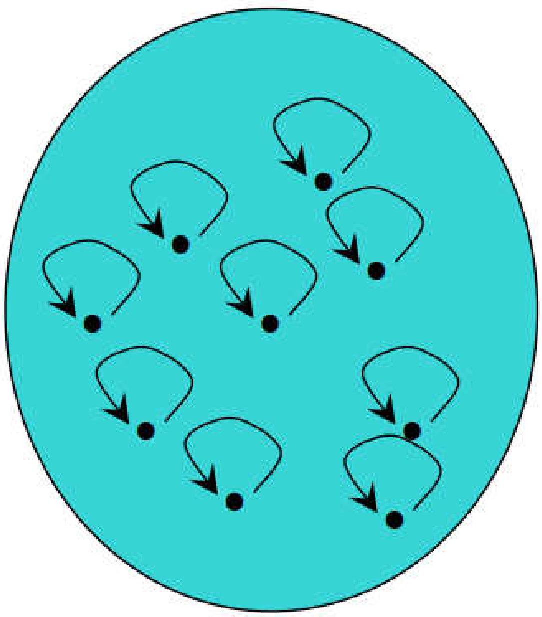
Identity Function

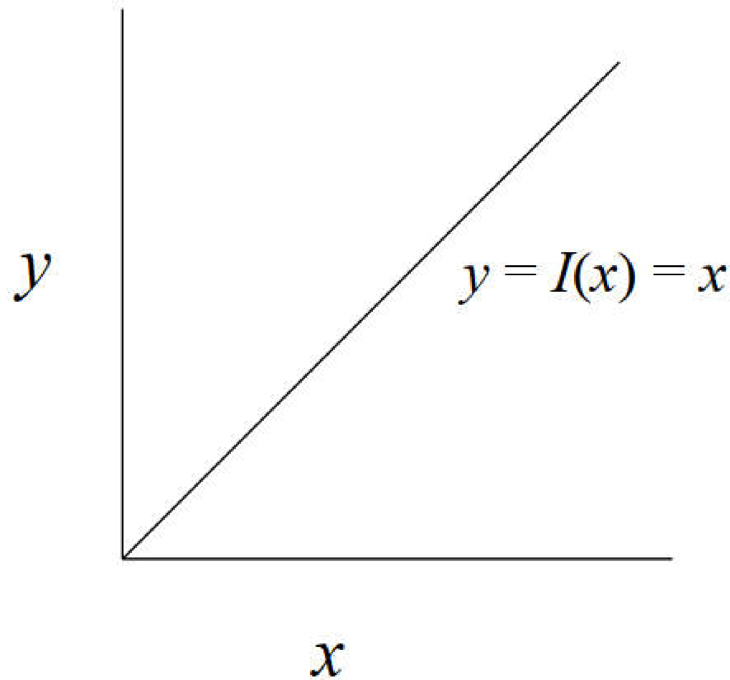
- For any domain A, the identity function $I: A \rightarrow A$ (also written as $I_A, 1, 1_A$) is the unique function such that $\forall a \in A: I(a) = a$.
- Note that the identity function is always both one-to-one and onto (i.e., bijective).
- For a bijection $f : A \rightarrow B$ and its inverse function $f^{-1}: B \rightarrow A$,
  - $f^{-1} \circ f = I_A$
- Some identity functions you've seen:
  - $+\ 0, \times\ 1, \wedge\ T, \vee\ F, \cup\ \emptyset, \cap\ U$

Identity Function Illustration

- Identity Function



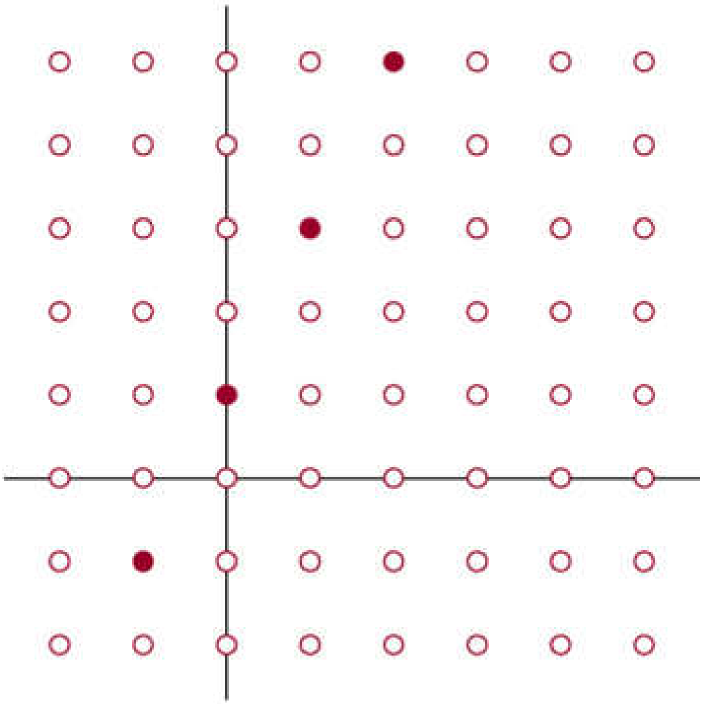
Domain and range



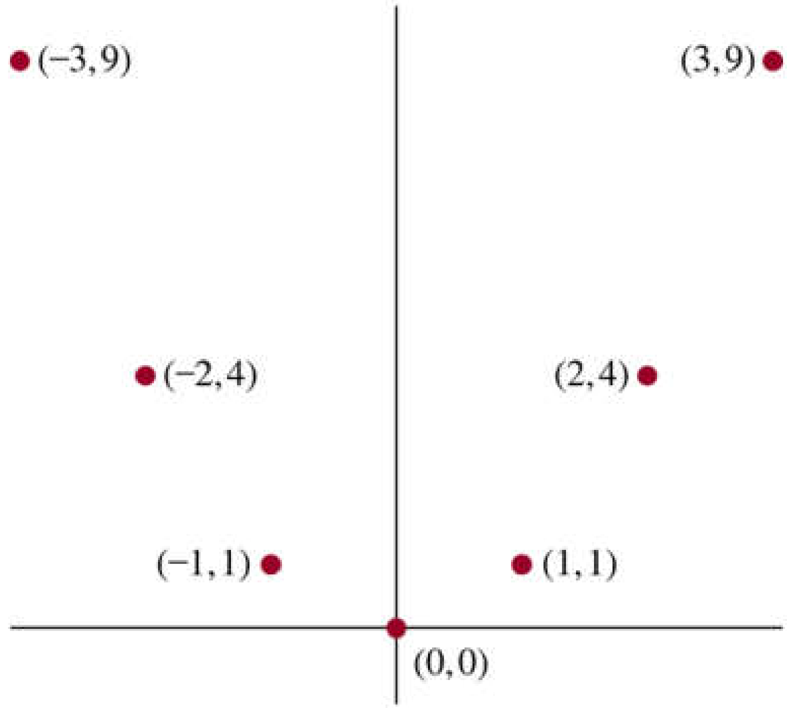
Graphs of Functions

- We can represent a function $f : A \rightarrow B$ as a set of ordered pairs $\{(a, f(a)) \mid a \in A\}$
- Note that $\forall a \in A$, there is only 1 pair (a, b)
- For functions over numbers, we can represent an ordered pair (x, y) as a point on a plane
- A function is then drawn as a curve (set of points), with only one y for each x

Graphs of Functions: Example



The graph of $f(n) = 2n + 1$ from $\mathbf{Z}$ to $\mathbf{Z}$



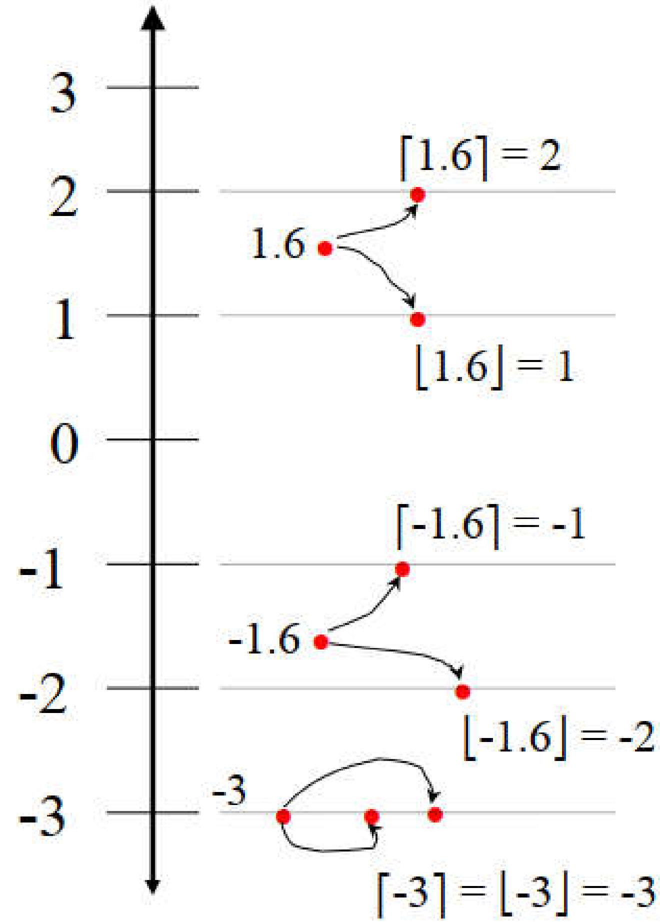
The graph of $f(x) = x^2$ from $\mathbf{Z}$ to $\mathbf{Z}$

Floor and Ceiling Functions

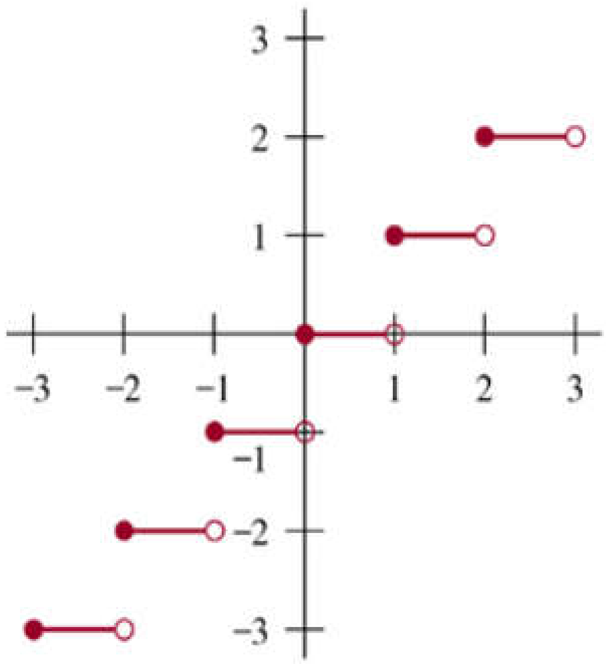
- In discrete math, we frequently use the following two functions over real numbers:
- The floor function $\lfloor\ \rfloor: R \rightarrow Z$, where $\lfloor x \rfloor$ ("floor of x") means the largest integer $\leq x$,
  - i.e., $\lfloor x \rfloor = \max(\{i \in Z \mid i \leq x\})$
- E.g. $\lfloor 2.3 \rfloor = 2$, $\lfloor 5 \rfloor = 5$, $\lfloor -1.2 \rfloor = -2$
- The ceiling function $\lceil \cdot \rceil: R \rightarrow Z$, where $\lceil x \rceil$ ("ceiling of x") means the smallest integer $\geq x$,
  - i.e., $\lceil x \rceil = \min(\{i \in Z \mid i \geq x\})$
- E.g. $\lceil 2.3 \rceil = 3$, $\lceil 5 \rceil = 5$, $\lceil -1.2 \rceil = -1$

Visualizing Floor & Ceiling

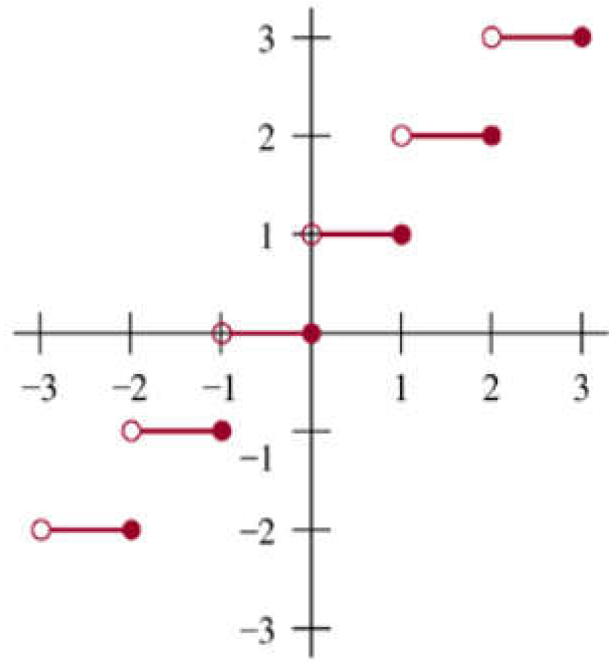
- Real numbers “fall to their floor” or “rise to their ceiling
- Note that if $x \notin Z$,
  - $\lfloor -x \rfloor \neq -\lfloor x \rfloor$
  - $\lceil -x \rceil \neq -\lceil x \rceil$
  - E.g $\lfloor -2.3 \rfloor = -3 \neq -\lfloor 2.3 \rfloor = -2$
- Note that if $x \in Z$,
  - $\lfloor x \rfloor = \lceil x \rceil = x$



Plots with Floor Ceiling Example



$y = \lfloor x \rfloor$



$y = \lceil x \rceil$

Plots with Floor & Ceiling

- Note that for $f(x) = \lfloor x \rfloor$, the graph of f includes the point $(a, 0)$ for all values of a such that $0 \leq a < 1$, but not for the value $a = 1$
- We say that the set of points $(a, 0)$ that is in f does not include its limit or boundary point $(a, 1)$
  - Sets that do not include all of their limit points are called open sets.
- In a plot, we draw a limit point of a curve using an open dot (circle) if the limit point is not on the curve, and with a closed (solid) dot if it is on the curve

Sequences & Summations

- A sequence or series is just like an ordered n-tuple, except:
  - Each element in the sequence has an associated index number
  - A sequence or series may be infinite
- A summation is a compact notation for the sum of the terms in a (possibly infinite) sequence

Sequences

- A sequence or series $\{a_n\}$ is identified with a generating function $f : I \rightarrow S$ for some subset $I \subseteq N$ and for some set S.
- Often we have $I = N \; or \; I = Z+ = N - \{0\}$
- If f is a generating function for a sequence $\{a_n\}$, then for $n \in I$, the symbol a_n denotes $f(n)$, also called term n of the sequence
- The index of a_n is n. (Or, often i is used.)
- A sequence is sometimes denoted by listing its first and/or last few elements, and using ellipsis (…) notation.
- E.g., "$\{a_n\} = 0, 1, 4, 9, 16, 25, \ldots$" is taken to mean $\forall n \in N, a_n = n^2$

Sequence Examples

- Some authors write "the sequence $a_1, a_2, \ldots$" instead of $\{a_n\}$, to ensure that the set of indices is clear.
- An example of an infinite sequence:
- Consider the sequence $\{a_n\} = a_1, a_2, \ldots,$
  - where $(\forall n \geq 1) a_n = f(n) = 1/n$
  - Then, we have $\{a_n\} = 1, 1/2, 1/3, \ldots$
  - Called "harmonic series"

Example With Repetitions

- Like tuples, but unlike sets, a sequence may contain repeated instances of an element.
- Consider the sequence $\{b_n\} = b_0, b_1, \ldots$ (note that 0 is an index) where $b_n = (-1)^n$
- Thus, $\{b_n\} = 1, -1, 1, -1, \ldots$
  - Note repetitions!
- This $\{b_n\}$ denotes an infinite sequence of 1's and $-$ 1's, not the 2-element set $\{1, -1\}$

Geometric Series, Sequence or Progression

- A geometric progression is a sequence of the form

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

- where the initial term a and the common ratio r are real numbers.
- A geometric progression is a discrete analogue of the exponential function $f(x) = ar^x$
- Examples: Assuming $n = 0, 1, 2, \ldots$
- $\{b_n\}$ with $b_n = (-1)^n$ initial term 1, common ratio -1
- $\{c_n\}$ with $c_n = 2 \times 5^n$ initial term 2, common ratio 5
- $\{d_n\}$ with $d_n = 6 \times \left(\frac{1}{3}\right)^n$ initial term 6, common ratio $\frac{1}{3}$

Arithmetic Series, Sequence or Progression

- An arithmetic progression is a sequence of the form
$$a, a + d, a + 2d, \dots, a + nd, \dots$$
- where the initial term a and the common difference d are real numbers.
- An arithmetic progression is a discrete analogue of the linear function $f(x) = a + dx$
- Examples: Assuming $n = 0, 1, 2, \dots$
- $\{s_n\}$ with $s_n = -1 + 4n$ initial term -1, common diff. 4
- $\{t_n\}$ with $t_n = 7 - 3n$ initial term 7, common diff. -3

Recognizing Sequences (I)

- Sometimes, you're given the first few terms of a sequence, and you are asked to
  - find the sequence's generating function,
  - or a procedure to enumerate the sequence
- Examples: What's the next number?
- $1, 2, 3, 4, \ldots$ 5 (the 5^{th} smallest number > 0)
- $1, 3, 5, 7, 9, \ldots$ 11 (the 6^{th} smallest odd number > 0)
- $2, 3, 5, 7, 11, \ldots$ 13 (the 6^{th} smallest prime number)

Recognizing Sequences (II)

- General problems
- Given a sequence, find a formula or a general rule that produced it
- Examples: How can we produce the terms of a sequence if the first 10 terms are
- $1, 2, 2, 3, 3, 3, 4, 4, 4, 4?$
  - Possible match: next five terms would all be 5, the following six terms would all be 6, and so on.
- $5, 11, 17, 23, 29, 35, 41, 47, 53, 59?$
- Possible match: nth term is $5 + 6(n - 1) = 6n - 1$
  - assuming $n = 1, 2, 3, \ldots$

Special Integer Sequences

- A useful technique for finding a rule for generating the terms of a sequence is to compare the terms of a sequence of interest with the terms of a well-known integer sequences (e.g. arithmetic/geometric progressions, perfect squares, perfect cubes, etc.)

TABLE 1 Some Useful Sequences.

nth Term	*First 10 Terms*
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100,… .
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000,… .
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000,… .
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024,… .
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049,… .
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800,… .

Coding: Fibonacci Series

- Series $\{a_n\} = \{1, 1, 2, 3, 5, 8, 13, 21, \dots\}$
- Generating function (recursive definition!):
  - $a_0 = a_1 = 1$ and
  - $a_n = a_{n-1} + a_{n-2}$ for all $n > 1$
- Now let's find the entire series $\{a_n\}$:

```
int [] a = new int [n];
a[0] = 1;
a[1] = 1;
for (int i = 2; i < n; i++) {
        a[i] = a[i-1] + a[i-2];
}
return a;
```

Coding: Factorial Series

- Factorial series $\{a_n\} = \{1, 2, 6, 24, 120, \dots\}$
- Generating function:
  - $a_n = n! = 1 \times 2 \times 3 \times \dots \times n$
- Alternate Generating functions
  - a_0 = 0! = 1
  - $a_n = n! = n \times (n-1)!$ For all $n > 0$
- This time, let's just find the term a_n:

int $a_n = 1;$

for $(int\ i = 1;\ i <= n;\ i++)$ {

$\quad\quad a_n = a_n * i;$

}

$return\ a_n;$

Prime Number Series

- Prime number series $\{a_n\} = 2,3,5,7,11,13 \ldots$
- Generating function
  - Has not yet been discovered
  - Prime numbers are the favorite children of cryptology
  - Unable to produce large prime numbers efficiently
- Check if a number is prime

```
int num=5024556647;
int sqr = math.sqrt(num);
for(i=2; i<=sqr; i++){
  if(num%i == 0)
     return false;
}
return true;
```