THE BERNOULLI EQUATION[1]

(9.21) Definition. An equation of the form

$$\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n$$

is called the **Bernoulli differential equation**. This equation is linear if $n = 0$ or 1. If n not zero or 1, then (1) is reducible to a linear equation. Dividing by y^n, (1) becomes

$$y^{-n}\frac{dy}{dx} + P(x)\,y^{1-n} = Q(x).$$

In (2), put $v = y^{1-n}$ then it reduces to

$$\frac{dv}{dx} + (1-n)\,P(x)\,v = (1-n)\,Q(x)$$

which is linear in v.

Note. Consider the equation

$$f'(y)\frac{dy}{dx} + P(x)f(y) = Q(x)$$

Letting $v = f(y)$, this equation becomes

$$\frac{dv}{dx} + P(x)\,v = Q(x)$$

which is linear in v.

Example 26. Solve: $\dfrac{dy}{dx} + \dfrac{xy}{1-x^2} = xy^{\frac{1}{2}}$.

Solution. Dividing by $y^{\frac{1}{2}}$, (1) becomes

$$y^{-\frac{1}{2}}\frac{dy}{dx} + \frac{x}{1-x^2}\,y^{\frac{1}{2}} = x.$$

Put $\quad y^{\frac{1}{2}} = v$

or $\quad \dfrac{1}{2}y^{-\frac{1}{2}}\dfrac{dy}{dx} = \dfrac{dv}{dx}$

Then (2) reduces to

$$\frac{dv}{dx} + \frac{x}{2(1-x^2)}\,v = \frac{x}{2}.$$

1. After the name of Swiss mathematician Ja...

This is linear in v.

$$\text{I.F.} = \exp\left[\int \frac{x}{2(1-x^2)}dx\right] = \exp\left[\frac{-1}{4}\ln(1-x^2)\right] = (1-x^2)^{-\frac{1}{4}}.$$

Multiplying (3) by $(1-x^2)^{-\frac{1}{4}}$, we get

$$(1-x^2)^{-\frac{1}{4}}\frac{dv}{dx} + \frac{x}{2(1-x^2)^{5/4}}v = \frac{x}{2(1-x^2)^{1/4}}$$

or
$$\frac{d}{dx}\left[(1-x^2)^{-\frac{1}{4}}v\right] = \frac{-1}{4}\left[-2x(1-x^2)^{-\frac{1}{4}}\right]$$

Integrating, we have

$$v(1-x^2)^{-\frac{1}{4}} = \frac{-1}{4}\frac{(1-x^2)^{3/4}}{3/4} + c$$

or
$$v = c(1-x)^{-\frac{1}{4}} - \frac{1-x^2}{3}$$

or
$$y^{\frac{1}{2}} = c(1-x^2)^{-\frac{1}{4}} - \frac{1-x^2}{3}$$

or

required solution of (1).

EXERCISE 9.6

Q:- $x\frac{dy}{dx} + y = y^2 \ln x$ —— (1)

$$\frac{dy}{dx} + \frac{y}{x} = y^2 \frac{\ln x}{x}$$

$$\frac{dy}{dx} + \frac{y}{x} = y^2 \frac{\ln x}{x}$$

$\div$ by y^2

$$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = \frac{\ln x}{x}$$

Let $\frac{1}{y} = t$

$$-\frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$$

$$-\frac{dt}{dx} + \frac{t}{x} = +\frac{\ln x}{x}$$

$$\frac{dt}{dx} - \frac{t}{x} = -\frac{\ln x}{x} \quad \text{—— (2)}$$

$$I.F = e^{\int -\frac{1}{x}dx} \Rightarrow e^{-\ln x} \Rightarrow e^{\ln x^{-1}}$$

$$I.F = \frac{1}{x}$$

$\times$ eq (2) by $1/x$.

$$\frac{1}{x}\frac{dt}{dx} + \frac{t}{x^2} = -\frac{\ln x}{x^2}$$

$$\frac{d}{dx}\left(\frac{t}{x}\right) = -\frac{\ln x}{x^2}$$

Integrate

$$\frac{t}{x} = \frac{1}{x}\ln x - \int \frac{dx}{x^2}$$

$$\frac{t}{x} = \frac{1}{x}\ln x + 1/x + C$$

$$\frac{1}{xy} = \frac{\ln x}{x} + \frac{1}{x}$$

$$\boxed{\frac{1}{y} = \ln x + 1 + C}$$

Question $\frac{dy}{dx} + y = xy^3$ — (1)

$\div$ by y^3 $\quad y^{-3}\frac{dy}{dx} + y^{-2} = x$ $\quad \boxed{y^{-n}\frac{dy}{dx} + P(x)\,y^{1-n} = Q(x)}$

Let $t = y^{1-3} = y^{-2}$

$$t = y^{-2}$$

$$\frac{dt}{dx} = -2y^{-3}\frac{dy}{dx}$$

$$-\frac{1}{2}\frac{dt}{dx} = y^{-3}\frac{dy}{dx}$$

$$-\frac{1}{2}\frac{dt}{dx} + t = x$$

$$\frac{dt}{dx} - 2t = -2x \quad \text{— (2)}$$

$$I.F = e^{\int -2\,dx} = e^{-2x}$$

$\times$ I.F eq (2)

$$e^{-2x}\frac{dt}{dx} - 2t \cdot e^{-2x} = -2x\,e^{-2x}$$

$$\frac{d}{dx}\left(t e^{-2x}\right) = -2x\,e^{-2x}$$

Integrate.

$$t e^{-2x} = -2\left[x\frac{e^{-2x}}{-2} - \int \frac{e^{-2x}}{-2}\,dx\right] + c$$

$$= x e^{-2x} + e^{-2x}/2 + c$$

$$\frac{e^{-2x}}{y^2} = x e^{-2x} + \frac{e^{-2x}}{2} + c$$

$$1/y^2 = x + \frac{1}{2} + c\,e^{2x}$$

Question $\frac{dy}{dx} + y = x y^3$ — (1)

$\div$ by y^3 $\quad y^{-3}\frac{dy}{dx} + y^{-2} = x$

Let $t = y^{1-3} = y^{-2}$

$$\left| y^{-n}\frac{dy}{dx} + P(x)\, y^{1-n} = Q(x) \right.$$

$$t = y^{-2}$$

$$\frac{dt}{dx} = -2y^{-3}\frac{dy}{dx}.$$

$$-\frac{1}{2}\frac{dt}{dx} = y^{-3}\frac{dy}{dx}.$$

$$-\frac{1}{2}\frac{dt}{dx} + t = x$$

$$\frac{dt}{dx} - 2t = -2x. \quad \text{— (2)}$$

$$I.F = e^{\int -2\,dx} = e^{-2x}.$$

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$$e^{-2x}\frac{dt}{dx} - 2t \cdot e^{-2x} = -2x e^{-2x}.$$

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$$t e^{-2x} = -2\left[x\frac{e^{-2x}}{-2} - \int \frac{e^{-2x}}{-2}\,dx\right] + C.$$

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$$\frac{e^{-2x}}{y^2} = x e^{-2x} + \frac{e^{-2x}}{2} + c.$$

$$1/y^2 = x + \frac{1}{2} + c e^{2x}$$

Q:- $x\frac{dy}{dx} - 2x^2y = y\ln y$

Sol $\frac{dy}{dx} - 2xy = \frac{y\ln y}{x}$ — ①

This is an eq of type.

$\frac{dy}{dx} + P(x)\,y = \phi(x)\,y\ln y$

$\frac{1}{y}\frac{dy}{dx} - 2x = \frac{\ln y}{x}$

In such eq $\Rightarrow \frac{1}{y}\frac{dy}{dx} - \frac{\ln y}{x} = 2x$ — ①'

Let $t = \ln y$

$\frac{1}{y}\frac{dy}{dx} = \frac{dt}{dx}$.

①' become

$\frac{dt}{dx} - \frac{t}{x} = 2x$ — ②

$I.F = e^{-\int \frac{1}{x}dx} \Rightarrow e^{-\ln x} \Rightarrow e^{\ln x^{-1}}$

$I.F = \frac{1}{x}$.

× eq ② by I.F

$\frac{d}{dx}\left(\frac{t}{x}\right) = 2$.

Integrate.

$\frac{t}{x} = 2x + c$.

$t = 2x^2 + cx$.

$\ln y = 2x^2 + cx$.