Discrete Structures

Lecture 11: Sets and Set Operations

based on slides by Jan Stelovsky
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Set Identities

- Identity: $A \cup \emptyset = A = A \cap U$
- Domination: $A \cup U = U, A \cap \emptyset = \emptyset$
- $Idempotent$: $A \cup A = A, A \cap A = A$
- Double complement: $(A^c)^c = A$
- Commutative: $A \cup B = B \cup A, A \cap B = B \cap A$
- Associative:
  - $A \cup (B \cup C) = (A \cup B) \cup C,$
  - $A \cap (B \cap C) = (A \cap B) \cap C$
- Distributive:
  - $A \cap (B \cup C) = (A \cap B) \cup (A \cap C),$
  - $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- Absorption: $A \cup (A \cap B) = A, A \cap (A \cup B) = A$
- Complement: $A \cup A^c = U, A \cap A^c = \emptyset$

De Morgan's Law for Sets

- Exactly analogous to (and provable from) DeMorgan's Law for propositions
  - $(A \cup B)^c = A^c \cap B^c$
  - $(A \cap B)^c = A^c \cup B^c$

Proving Set Identities

- To prove statements about sets, of the form $E1 = E2$ (where the Es are set expressions), here are three useful techniques:
  - 1. Prove $E1 \subseteq E2$ and $E2 \subseteq E1$ separately.
  - 2. Use set builder notation & logical equivalences.
  - 3. Use a membership table.
  - 4. Use a Venn diagrams

Method 1: Mutual Subsets

- Example: Show $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- Part 1: Show $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$
- Assume $x \in A \cap (B \cup C)$, & show $x \in (A \cap B) \cup (A \cap C)$
- We know that $x \in A$, and either $x \in B$ or $x \in C$
    - Case 1: $x \in A \wedge x \in B$. Then $x \in A \cap B$,
    - so $x \in (A \cap B) \cup (A \cap C)$
    - Case 2: $x \in A \wedge x \in C$. Then $x \in A \cap C$,
    - so $x \in (A \cap B) \cup (A \cap C)$
- Therefore, $x \in (A \cap B) \cup (A \cap C)$
- Therefore, $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

Method 1: Mutual Subsets

- Part 2: Show $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$
- Let's assume that $x \in (A \cap B) \cup (A \cap C)$
- $x \in (A \cap B)$ or $x \in (A \cap C)$ By definition of union
- $(x \in A$ and $x \in B)$ or $(x \in A$ and $x \in C)$ By def of intersection
- We can see that, $x \in A$ and $(x \in B$ or $x \in C)$
- And $x \in A$ and $x \in (B \cup C)$ By def of union
- Finally, $x \in A \cap (B \cup C)$ By def of intersection
- Consequently, $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$ is proved

Method 2: Set Builder Notation and Logical Equivalence

- Show that $(A \cap B)^c = A^c \cup B^c$
- $(A \cap B)^c = \{\mathrm{x}|\mathrm{x} \notin A \cap B\}$ definition of complement
- $= \{x|\neg\,(x \in (A \cap B)\}$ definition of does not belong
- $= \{x\,|\neg(x \in A \wedge x \in B)\}$ definition of intersection
- $= \{x|\neg x \in A \vee \neg x \in B\}$ De Morgan's Law (logic)
- $= \{x|x \notin A \vee x \notin B\}$ Definition of does not belong
- $= \{x|x \in A^c \vee x \in B^c\}$ Definition of Complement
- $= \{x|x \in A^c \cup B^c\}$ Definition of Union
- $= A^c \cup B^c$ By set builder Notation

Method 3: Membership Tables

- Analog to truth tables in propositional logic
- Columns for different set expressions
- Rows for all combinations of memberships in constituent sets
- Use “1” to indicate membership in the derived set, “0” for non-membership.
- Prove equivalence with identical columns

Membership Table Example

- Prove that $(A \cup B) - B = A - B$

$\boldsymbol{A}$	$\boldsymbol{B}$	$\boldsymbol{A \cup B}$	$\boldsymbol{(A \cup B) - B}$	$\boldsymbol{A - B}$
1	1	1	0	0
1	0	1	1	1
0	1	1	0	0
0	0	0	0	0

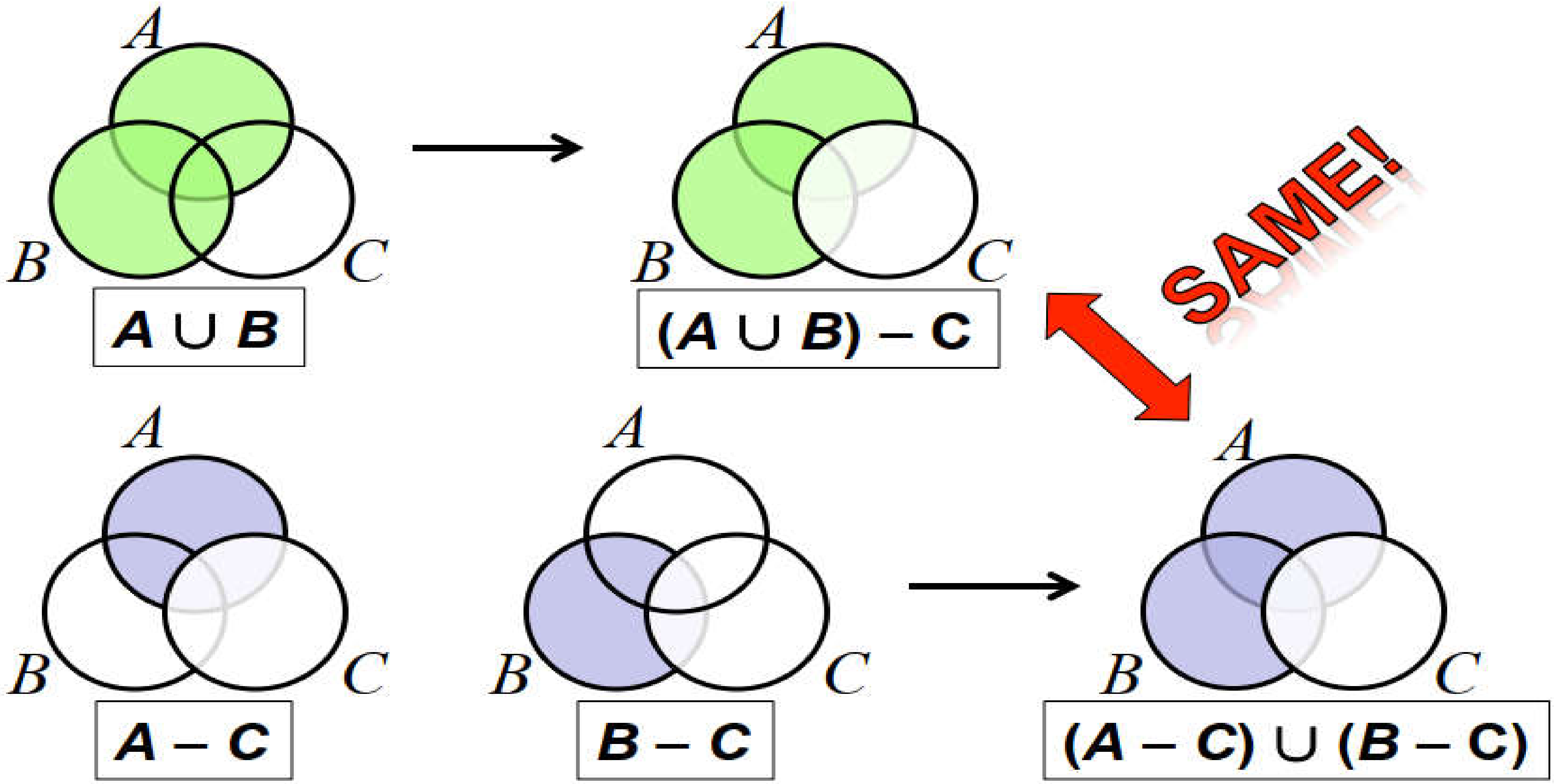
Membership Table Exercise

- Prove $(A \cup B) - C = (A - C) \cup (B - C)$

$\boldsymbol{A}$	$\boldsymbol{B}$	$\boldsymbol{C}$	$\boldsymbol{A \cup B}$	$\boldsymbol{(A \cup B) - C}$	$\boldsymbol{A - C}$	$\boldsymbol{B - C}$	$\boldsymbol{(A - C) \cup (B - C)}$
1	1	1	1	0	0	0	0
1	1	0	1	1	1	1	1
1	0	1	1	0	0	0	0
1	0	0	1	1	1	0	1
0	1	1	1	0	0	0	0
0	1	0	1	1	0	1	1
0	0	1	0	0	0	0	0
0	0	0	0	0	0	0	0

Method 4: Venn Diagrams

- Prove $(A \cup B) - C = (A - C) \cup (B - C)$



Generalized Unions and Intersections

- Since union & intersection are
  - Commutative and
  - Associative,
- we can extend them from operating on pairs of sets A and B to operating on sequences of sets $A_1, \ldots, A_n$, or even on sets of sets, $X = \{A \mid P(A)\}$

Generalized Union

- Binary union operator: $A \cup B$
- n-ary union: $A_1 \cup A_2 \cup \cdots \cup A_n = ((\ldots((A_1 \cup A_2) \cup \cdots) \cup A_n)$
  - (grouping & order is irrelevant)
- "Big U" notation: $\bigcup_{i=1}^{n} A_i$
- More generally, union of the sets A_i $for\ i \in I$: $\bigcup_{i \in I} A_i$
- For infinite number of sets: $\bigcup_{i=1}^{\infty} A_i$

Generalized Union Example

- Let $A_i = \{i, i+1, i+2, \ldots\}$. Then,
- $\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup A_3 \cup \cdots A_n$
- $= \{1,2,3 \ldots\} \cup \{2,3,4 \ldots\} \cup \cdots \cup \{n, n+1, n+2, \ldots\}$
- $= \{1,2,3, \ldots\} = Z^+$
- Let $A_i = \{1,2,3 \ldots i\}$ Then
- $\bigcup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup A_3 \cup \cdots$
- $= \{1\} \cup \{1,2\} \cup \{1,2,3\} \cup \cdots$
- $= \{1,2,3 \ldots\} = Z^+$

Generalized Intersection

- Binary intersection operator: $A \cap B$
- n-ary intersection: $A_1 \cap A_2 \cap \cdots \cap A_n = ((\ldots((A_1 \cap A_2) \cap \cdots) \cap A_n)$
  - (grouping & order is irrelevant)
- "Big arch" notation: $\bigcap_{i=1}^{n} A_i$
- More generally, intersection of the sets A_i $for\ i \in I$: $\bigcap_{i \in I} A_i$
- For infinite number of sets: $\bigcap_{i=1}^{\infty} A_i$

Generalized Intersection Example

- Let $A_i = \{i, i+1, i+2, \dots\}$. Then,
- $\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots A_n$
- $= \{1,2,3 \dots\} \cup \{2,3,4 \dots\} \cup \cdots \cup \{n, n+1, n+2, \dots\}$
- $= \{n, n+1, n+2 \dots\}$
- Let $A_i = \{1,2,3 \dots i\}$ Then
- $\bigcap_{i=1}^{n} A_i = \mathrm{A_i} \cap A_2 \cap \cdots$
- $= \{1\} \cap \{1,2\} \cap \{1,2,3\} \cap \cdots$
- $= \{1\}$

Bit String Representation of Sets

- A frequent theme of this course are methods of representing one discrete structure using another discrete structure of a different type
- For an enumerable universal set U with ordering $x_1, x_2, x_3, \ldots,$ we can represent a finite set $S \subseteq U$ as the finite bit string B = $b_1 b_2 \ldots b_n$ where $b_i = 1 \; if \; x_i \in S$ and $b_i = 0 \; if \; x_i \notin S$
- $U = N, S = \{2,3,5,7,11\}, B = 0011\,0101\,0001$
- In this representation, the set operators, union, intersection and complement are implemented directly by bitwise OR, AND, NOT respectively

Examples of Sets as Bit Strings

- Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, and the ordering of elements of U has the elements in increasing order, then
- $S_1 = \{1, 2, 3, 4, 5\} \Rightarrow B_1 = 11\ 1110\ 0000$
- $S_2 = \{1, 3, 5, 7, 9\} \Rightarrow B_2 = 10\ 1010\ 1010$
- $S_1 \cup S_2 = \{1, 2, 3, 4, 5, 7, 9\} \Rightarrow$ bit string = $11\ 1110\ 1010 = B1 \vee B2$
- $S1 \cap S2 = \{1, 3, 5\} \Rightarrow bit\ string = 10\ 1010\ 0000 = B1 \wedge B2$
- $S_1^c = \{6, 7, 8, 9, 10\} \Rightarrow bit\ string = 00\ 0001\ 1111 = \neg B1$