(Differential Equation)

Types of Variables:

There are two types of variable.

(i) Dependent variable.

(ii) Independent variable.

Differential equation:

Containing of

An equation involving derivative of dependent variable w.r.t one or more independent variable is called differential equation.

e.g

$$\frac{dy}{dx} + y\sin x = 0$$

where y is a dependent variable and x is independent variable.

Types of differential equation.

(i) Ordinary diff eq.

(ii) Partial diff eq.

(iii) Order of diff eq.

(iv) Degree of diff eq.

(v) Ordinary linear diff eq.

(i) Ordinary differential equation:

The differential equation involving derivative of dependent variable w.r.t single independent variable is called ordinary differential equation.

e.g

$$\frac{dy}{dx} + y \sin x = 0$$

(ii) Partial differential equation.

The differential equation involving derivative of dependent variable w.r.t more than one independent variable is called partial differential equation.

e.g

$$\frac{\partial y}{\partial x} + \frac{\partial y}{\partial z} = 0$$

[redacted]

$$\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial z^2} + \frac{\partial^2 y}{\partial u^2} = 0$$

[redacted]

(iii) Order of differential equation.

The order of a differential equation is the order of the heighest derivative term in that equation.

e.g

$$\frac{d^2y}{dx^2} + 2x\left(\frac{dy}{dx}\right)^3 = 0$$

Above eq. has order '2'.

(iv) Degree of diff - eq.

The degree of a diff-eq is degree of highest order derivative term e.g

$$\frac{d^2y}{dx^2} + 2x\left(\frac{dy}{dx}\right)^2 = 0$$

The above equation has degree 1.

(v) Ordinary linear differential eq/.

An ordinary eq/ of the form
$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots\ldots \frac{d^ny}{dx^n}\right)$ is said to be linear if F is a linear function of $\left(x, \frac{dy}{dx}, \ldots\right)$

Condition for linear diff-eq/.

(i) A diff-eq/ is said to be linear if the degree of dependent variable or degree of all its derivation is one otherwise it's non-linear.

$$y^2\frac{dy}{dx} + 5x = 0 \longrightarrow$$ Non-linear as y has degree 2.

$$x^2\frac{dy}{dx} + 5x = 0 \longrightarrow$$ Linear

(ii) There should not be Product of dependent variable with its own derivative or with itself.

$$y^2\frac{dy}{dx} + 5x = 0 \longrightarrow \text{Non-linear}$$
$$y\frac{d^2y}{dx^2} + \sin x = 0 \longrightarrow \text{Non-linear}$$

$$x\frac{dy}{dx} + 5x = 0 \longrightarrow \text{linear}$$

(iii) There should not any trignometric function of defendent variable (y).

$$\frac{d^2y}{dx^2} + \cot y \quad = \text{Non-linear}$$

$$\frac{d^2y}{dx^2} + \tan x = 0 \qquad \text{Linear.}$$

Homogenious Differential Equation.

An equation of the form.

$$Mdx + Ndy = 0$$

is called homogenious D.E, if degree of all terms is Same.

$$\overbrace{x^{1} y^{2}}^{3} + 2\overbrace{y^{1} x^{2}}^{3} + 3\overbrace{x^{1} y^{1} z^{1}}^{3}$$

(9.7) **Definition.** A **solution** (or **integral**) of a differential equation is a relation between the variables, not containing derivatives, such that this relation and the derivatives obtained from it satisfy the given differential equation identically. For example,

The equation

$$\frac{dy}{dx} = -\lambda y \quad \text{has a solution}$$

$$y = c\,e^{-\lambda x}, \quad \text{where } c \text{ is an arbitrary constant.}$$

The equation

$$\frac{d^2y}{dx^2} + y = 0 \qquad \text{has solutions}$$

$$y = A\cos x, \quad y = B\sin x \ \text{ and } \ y = A\cos x + B\sin x,$$

where A and B are arbitrary constants.

A solution of a differential equation which contains as many arbitrary constants as the order of the equation is called **general solution** (or **integral**) of the differential equation. A solution obtained from the general solution by giving particular values to the constants is called a **particular solution** (or **integral**). The graph of a particular integral is called an **integral curve** of the differential equation.

INITIAL AND BOUNDARY CONDITIONS

We have observed that general solution of a differential equation contains the same number of arbitrary constants as is the order of the equation. Sometimes we need to find the solutions of differential equations subject to supplementary conditions. Two types of conditions will be often encountered.

(9.9) **Definition. (Initial Conditions).** It is often required to find the solution of a differential equation subject to certain conditions. If the conditions relate to one value of the independent variable such as $y = y_0$ at $x = x_0$ (written as $y(x_0) = y_0$) and $\dfrac{dy}{dx} = y'(x_0)$ at $x = x_0$, where x_0 belongs to some interval $]\,\alpha, \beta\,[$ then they are called **initial conditions** (or **one-point boundary conditions**) and x_0 is called the **initial point**. An **initial value problem** consists of a differential equation (of any order) together with a collection of initial conditions that must be satisfied by the solution of the differential equation and derivatives at the initial point.

(9.10) **Definition. (Boundary Conditions).** The problem of finding the solution of a differential equation such that all the associated constraints relate to two different values of the independent variable is called a **two-point boundary value problem** (or simply a **boundary value problem**). The associated supplementary **boundary conditions** are called **two-point boundary conditions**.

SEPARABLE EQUATIONS

(9.11) Definition. A differential equation of the type

$$F(x)\,G(y)\,dx + f(x)\,g(y)\,dy = 0 \qquad (1)$$

is called an equation with **separable variables** or simply a **separable equation**. Equation (1) may be written as

$$\frac{F(x)}{f(x)}\,dx + \frac{g(y)}{G(y)}\,dy = 0$$

which can be easily integrated.

HOMOGENEOUS EQUATIONS

(9.12) Definition. A function $f(x, y)$ is called **homogeneous** of degree n if

$$f(tx, ty) \quad = \quad t^n f(x, y),$$

where t is a nonzero real number. Thus $\sqrt{xy}$, $\dfrac{x^{10}+y^{10}}{x^2+y^2}$ and $\sin\left(\dfrac{x}{y}\right)$ are homogeneous function of degree 1, 8 and 0 respectively. (Check!)

A first order differential equation

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

is said to be **homogeneous** if f is a homogeneous function of any degree. If (1) is written in the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

then it is called homogeneous if $M(x, y)$ and $N(x, y)$ are homogeneous functions of the same degree.

The equation

$$\frac{dy}{dx} = \ln x - \ln y + \frac{x+y}{x-y} = \ln\frac{1}{y/x} + \frac{1 + y/x}{1 - y/x}$$

is homogeneous, but

$$\frac{dy}{dx} = \frac{y^3 + 2xy}{x^2} = y\left(\frac{y}{x}\right)^2 + 2\left(\frac{y}{x}\right)$$

is not homogeneous.

EXACT EQUATIONS

(9.15) Definition. The expression

$$M(x, y)\,dx + N(x, y)\,dy \tag{1}$$

is called an **exact differential** if there exists a continuously differentiable function $f(x, y)$ of two real variables x and y such that the expression equals the total differential df. We know from calculus that

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$

Thus, if (1) is exact then

$$M(x, y) = \frac{\partial f}{\partial x} = f_x \quad \text{and} \quad N(x, y) = \frac{\partial f}{\partial y} = f_y.$$

If (1) is an exact differential then the differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is called an **exact equation**.

(.16) Theorem. The differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0 \qquad (1)$$

is an exact equation if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

where the functions $M(x, y)$ and $N(x, y)$ have continuous first order partial derivatives.

Proof. Suppose that the equation (1) is exact so that $M\,dx + N\,dy$ is an exact differential. By definition, there exists a function $f(x, y)$ such that

$$M(x, y) = \frac{\partial f}{\partial x} = f_x \quad \text{and} \quad N(x, y) = \frac{\partial f}{\partial y} = f_y.$$

Then $$M_y = \frac{\partial M}{\partial y} = f_{xy} = \frac{\partial^2 f}{\partial y\,\partial x}$$

and $$N_x = \frac{\partial N}{\partial x} = f_{yx} = \frac{\partial^2 f}{\partial x\,\partial y}.$$

Since M and N possess continuous first order partial derivatives, we have $f_{xy} = f_{yx}$ and, therefore

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{as desired.}$$

The proof of the converse is omitted since it is beyond our scope.

(Another Method To Solve Exact eq.)

If D.E is Exact then we can apply this formula

$$\int M\,dx + \int (\text{term of } N \text{ free of } x)\,dy = C$$

INTEGRATING FACTORS

(9.18) Definition. If the differential equation

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

is not exact but when it is multiplied by a function $\mu(x, y)$ and the resulting equation

$$\mu(x, y)\,M(x, y)\,dx + \mu(x, y)\,N(x, y)\,dy = 0$$

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exact, then $\mu(x, y)$ is called an **integrating factor (I.F.)** of the differential equation (1). The number of integrating factors of an equation may be infinite.

We list below (without proofs) some rules to find the integrating factors of equations of special types.

Rule I. If $M(x, y)\,dx + N(x, y)\,dy = 0$ (1)

is not exact and $\dfrac{M_y - N_x}{N} = P$, where P is a function of x only, then (1) has an integrating factor $\mu(x)$ which also depends on x. $\mu(x)$ is solution of the differential equation

$$\frac{d\mu}{dx} = P\mu$$

i.e.,

$$\mu(x) = \exp \int P\,dx.$$

Note that $M_y = \dfrac{\partial M}{\partial y}$, $N_x = \dfrac{\partial N}{\partial x}$.

Rule II. If $\dfrac{N_x - M_y}{M} = Q$, where Q is a function of y only, then the differential equation $M\,dx + N\,dy = 0$ has an integrating factor

$$\mu(y) = \exp \int Q\,dy.$$

Rule III. If $M\,dx + N\,dy = 0$ (1)

is homogeneous and $xM + yN \neq 0$, then

$\dfrac{1}{xM + yN}$ is an I.F. of (1).

Rule IV. If $M\,dx + N\,dy = 0$ is of the form

$$yf(xy)\,dx + xg(xy)\,dy = 0$$

and $xM - yN \neq 0$, then

$\dfrac{1}{xM - yN}$ is an I.F. of (1).

(Integrating Factors)

If differential equation.

$$M(x,y)dx + N(x,y)dy = 0$$

is not exact but when it multiplied by a function $\mu(x,y)$ and the resulting equation

$$\mu(x,y)M(x,y)dx + \mu(x,y)N(x,y)dy = 0$$

is exact, then $\mu(x,y)$ is called an integrating factors (I.F) of the differential equation.

=> The number of integrating factors of an equation may be infinite.

Rule (i)

$$\frac{M_y - N_x}{N} = P(x)$$

Here "P" should be only function of 'x'.

Then, Rule (i) is True.

$$I.F = e^{\int P dx}$$

Then multiply I.F with Given D.E & given D.E will become Exact.

Note

$M_y = \frac{\partial M}{\partial y}$

$N_x = \frac{\partial N}{\partial x}$

Rule (ii)

$$\frac{N_x - M_y}{M} = Q(y)$$

Here "Q" should be only function of 'y'.

Then Rule (ii) is true.

$$I.F = e^{\int Q dy}$$

Rule-(iii)

Rule III is applicable to only homogenious D.E

$$I.F = \frac{1}{xM + yN}$$

Rule-(iv)

if D.E is of the form $y(f(x))dx + x(g(x))dy = 0$

Then

$$I.F = \frac{1}{xM - yN}$$

Solve it by I.F

LINEAR EQUATIONS

(9.19) Definition. A first order ordinary differential equation is **linear** in the dependent variable y and the independent variable x if it is or can be written in the form

$$\frac{dy}{dx} + P(x)\,y = Q(x), \qquad (1)$$

where P and Q are functions of x.

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Exercise # 9.6

Linear Differential Equation.

A Differential Equation of the form

$$\frac{dy}{dx} + P(x).y = Q(x) \Rightarrow \text{called Linear D.E}$$

Then it's I.F will be

$$I.F = e^{\int P(x)dx}$$

Then It's solution can be obtained by solving this eq/u.

$$d(y.I.F) = I.F.Q(x)$$

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Another form of Linear Differential Equation.

$$\frac{dx}{dy} + P(y) \cdot x = Q(y)$$

Then it's I.F will be

$$I.F = e^{\int P(y)\,dy}$$

THE BERNOULLI EQUATION[1]

(9.21) Definition. An equation of the form

$$\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n \tag{1}$$

is called the **Bernoulli differential equation**. This equation is linear if $n = 0$ or 1. If n is not zero or 1, then (1) is reducible to a linear equation. Dividing by y^n, (1) becomes

$$y^{-n}\frac{dy}{dx} + P(x)\,y^{1-n} = Q(x). \tag{2}$$

In (2), put $v = y^{1-n}$ then it reduces to

$$\frac{dv}{dx} + (1-n)\,P(x)\,v = (1-n)\,Q(x)$$

which is linear in v.

Note. Consider the equation

$$f'(y)\frac{dy}{dx} + P(x)\,f(y) = Q(x)$$

Letting $v = f(y)$, this equation becomes

$$\frac{dv}{dx} + P(x)\,v = Q(x)$$

which is linear in v.

ORTHOGONAL TRAJECTORIES

(9.22) (Rectangular Coordinates) Definition. It has been observed that the general solution of a first order differential equation contains one arbitrary constant. When this constant is assigned different values, one obtains a one-parameter family of curves. Each of these curves represents a particular solution of the given differential equation.

On the other hand, given a one-parameter family of curves

$$f(x, y, c) = 0, \qquad (1)$$

c being parameter, then each member of the family is a particular solution of some differential equation. In fact, this differential equation is obtained by elimination of the parameter c between (1) and the relation obtained by differentiating (1) w.r.t x.

Let $f(x, y, c) = 0$ and $F(x, y, k) = 0$ be two families of curves with parameters c and k. If each curve in either family is intersected orthogonally by every curve in the other family, then each family is said to be **orthogonal trajectory** of the other. Recall that two curves are said to be orthogonal (intersect orthogonally) if their tangents at the point of intersection are perpendicular to each other.

For example, the families of curves given by

$$x^2 + y^2 = c^2 \quad \Rightarrow \quad f(x, y, c) \equiv x^2 + y^2 - c^2 = 0$$

and

$$y = kx \quad \Rightarrow \quad F(x, y, k) \equiv y - kx = 0$$

are orthogonal as illustrated graphically below: