Discrete Structures

Lecture 15: Sequences & Summations

based on slides by Jan Stelovsky
based on slides by Dr. Baek and Dr. Still
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Summation Notation

- Given a sequence $\{a_n\}$, an integer lower bound (or limit) $j \geq 0$, and an integer upper bound $k \geq j$, then the summation of $\{a_n\}$ from a_j to a_k is written and defined as follows:

$$\sum_{i=j}^{k} a_i = a_j + a_{j+1} + \cdots a_k$$

- Here i is called the index of the summation (i could be replaced by any other letter e.g

$$\sum_{i=j}^{k} a_i = \sum_{m=j}^{k} a_m = \sum_{l=j}^{k} a_l$$

Generalized Summations

- For an infinite sequence, we write:

$$\sum_{i=j}^{\infty} a_i$$

- To sum a function over all members of a set $X = \{x_1, x_2, \ldots\}$:

$$\sum_{x \in X} f(x) = f(x_1) + f(x_2) + \cdots$$

- Or, if $X = \{x \mid P(x)\}$, we may just write:

$$\sum_{\mathrm{P(x)}} f(x) = f(x_1) + f(x_2) + \cdots$$

Example

$$\sum_{i=2}^{4}(i^2+1) = (2^2+1)+(3^2+1)+(4^2+1) = 5+10+17 = 32$$

- How do we represent $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\frac{1}{100}$ in concise form

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\frac{1}{100} = \sum_{i=1}^{100}\frac{1}{i}$$

More Examples

- An infinite sequence with a finite sum:

$$\sum_{i=0}^{\infty} 2^{-i} = 2^0 + 2^{-1} + \cdots = 1 + \frac{1}{2} + \frac{1}{4} + \cdots = 2$$

- Using a predicate to define a set of elements to sum over:

$$\sum_{Prime(x) \land x<10} x^2 = 2^2 + 3^2 + 5^2 + 7^2$$
$$= 4 + 9 + 25 + 49$$
$$= 87$$

Summation Manipulations

- Summing constant values

$$\sum_{n=i}^{j} c = c.(j - i + 1)$$

$$\sum_{n=1}^{3} 3 = (3 - 1 + 1).3 = 9$$

$$\sum_{n=-1}^{2} 2i = 2i.(2 - (-1) + 1)$$

Summation Manipulations

- Distributive Law

$$\sum_{n=i}^{j} c.f(n) = c\sum_{n=i}^{j} f(n)$$

$$\sum_{n=1}^{3} 4.n^2 = 4.1^2 + 4.2^2 + 4.3^2$$

$$= 4.(1^2 + 2^2 + 3^2)$$

$$= 4.\sum_{n=1}^{3} n^2$$

Summation Manipulations

- An application of commutativity

$$\sum_{n=i}^{j}\big(f(n)+g(n)\big)=\sum_{n=i}^{j}f(n)+\sum_{n=i}^{j}g(n)$$

$$\sum_{n=2}^{4}(n+2n)=(2+2\times 2)+(3+2\times 3)+(4+2\times 4)$$

$$=(2+3+4)+(2\times 2+2\times 3+2\times 4)$$

$$=\sum_{n=2}^{4}n+\sum_{n=2}^{4}2n$$

Index Shifting

$$\sum_{i=j}^{m} f(i) = \sum_{k=j+n}^{m+n} f(k-n)$$

$$\sum_{i=1}^{4} i^2 = 1^2 + 2^2 + 3^2 + 4^2$$

- Let $k = i + 2$, then $i = k - 2$

$$\sum_{k=1+2}^{4+2} (k-2)^2 = \sum_{k=3}^{6} (k-2)^2$$

$$= (3-2)^2 + (4-2)^2 + (5-2)^2 + (6-2)^2$$

Summation Manipulations

- Sequence Splitting

$$\sum_{i=j}^{k} f(i) = \sum_{i=j}^{m} f(i) + \sum_{i=m+1}^{k} f(i) \qquad if\ j \leq m < k$$

$$\sum_{i=0}^{4} i^3 = 0^3 + 1^3 + 2^3 + 3^3 + 4^3$$

$$= (0^3 + 1^3 + 2^3) + (3^3 + 4^3)$$

$$= \sum_{i=0}^{2} i^3 + \sum_{i=3}^{4} i^3$$

Summation Manipulations

- Order Reversal

$$\sum_{i=0}^{k} f(i) = \sum_{i=0}^{k} f(k-i)$$

$$\sum_{i=0}^{3} i^3 = 0^3 + 1^3 + 2^3 + 3^3$$

$$= (3-0)^3 + (3-1)^3 + (3-2)^3 + (3-3)^3$$

$$= \sum_{i=0}^{3} (3-i)^3$$

Geometric Progression

- A geometric progression is a sequence of the form $a, ar, ar^2, ar^3, \ldots, ar^n, \ldots$ where $a, r \in R$
- The sum of such a sequence is given by

$$S = \sum_{i=0}^{n} ar^i$$

- How do we solve this, if we have n=1000 for instance?
  - Can we derive a closed form solution?

Geometric Sum Derivation

$$S = \sum_{i=0}^{n} ar^{i}$$

$$rS = r\sum_{i=0}^{n} ar^{i} = \sum_{i=0}^{n} ar^{i+1}$$

$$= \sum_{j=1}^{n+1} ar^{1+(j-1)} = \sum_{j=1}^{n+1} ar^{j}$$

$$= \sum_{j=1}^{n} ar^{j} + \sum_{j=n+1}^{n+1} ar^{j} = (\sum_{j=1}^{n} ar^{j}) + (ar^{n+1})$$

Geometric Sum Derivation

$$rS = \left(\sum_{j=1}^{n} ar^{j}\right) + ar^{n+1} = (ar^{0} - ar^{0}) + \left(\sum_{j=1}^{n} ar^{j}\right) + ar^{n+1}$$

$$= ar^{0} + \left(\sum_{j=1}^{n} ar^{j}\right) + ar^{n+1} - ar^{0}$$

$$= \left(\sum_{j=0}^{0} ar^{j}\right) + \left(\sum_{j=1}^{n} ar^{j}\right) + ar^{n+1} - a$$

$$rS = \left(\sum_{j=0}^{n} ar^{j}\right) + a(r^{n+1} - 1) = S + a(r^{n+1} - 1)$$

Geometric Sum Derivation

$$rS = S + a(r^{n+1} - 1)$$
$$rS - S = a(r^{n+1} - 1)$$
$$S(r - 1) = a(r^{n+1} - 1)$$
$$S = \frac{a(r^{n+1} - 1)}{r - 1} \; where \; r \neq 1$$

- When r=1

$$S = \sum_{i=0}^{n} ar^{i} = \sum_{i=0}^{n} a1^{i} = \sum_{i=0}^{n} a = (n + 1)a$$

Sum Numbers from 1 to n

- What if you are asked to add numbers from 1 to 100 or from 1 to any integer number n?
- We will need to evaluate the following summation

$$\sum_{i=1}^{n} i$$

- But, do we have simple closed form formula?
  - Yes, discovered by Gauss at age 10!

Gauss Trick for Summing Numbers (1 to n)

- Consider the sum

$$1 + 2 + 3 + \cdots \left(\frac{n}{2}\right) + \left(\left(\frac{n}{2}\right) + 1\right) + \cdots + (n-2) + (n-1) + n$$

- Sum ith number and $(n-i+1)th$ number
  - When $i = 1$, the sum is $n + 1$
  - When $i = 2$, the sum is $(n-1) + 2 \;=\; n + 1$
  - When $i = 3$, the sum is $(n-2) + 3 \;=\; n + 1$
  - When $i = \frac{n}{2}$, the sum is $\left(\frac{n}{2}\right) + \left(\left(\frac{n}{2}\right) + 1\right) \;=\; n + 1$
- There are $\frac{n}{2}$ such pairs that add to $n + 1$ and the sum can be reduced to $\left(\frac{n}{2}\right)(n+1) = \frac{n(n+1)}{2}$
- Can we prove this?

Derivation of Gauss Trick

- Let's consider the case $k = \frac{n}{2}$ or $n = 2k$ for integer k, i.e n is even

$$\sum_{i=1}^{n} i = \sum_{i=1}^{2\mathrm{k}} \mathrm{i} = (\sum_{i=1}^{k} i) + (\sum_{i=k+1}^{n} i) = (\sum_{i=1}^{k} i) + \sum_{j=0}^{n-(k+1)} j + (k+1)$$

$$= (\sum_{i=1}^{k} i) + \sum_{j=0}^{n-(k+1)} ((n-(k+1)-j) + (k+1)$$

$$= (\sum_{i=1}^{k} i) + \sum_{j=0}^{n-(k+1)} (n-j) = (\sum_{i=1}^{k} i) + \sum_{l=1}^{n-k} n-(l-1)$$

$$= (\sum_{i=1}^{k} i) + \sum_{l=1}^{n-k} n+1-l = \sum_{i=1}^{k} i + \sum_{l=1}^{k} n+1-l$$

Derivation of Gauss Trick

$$\sum_{i=1}^{n} i = \sum_{i=1}^{k} i + \sum_{i=1}^{k} n + 1 - i = \sum_{i=1}^{k} (i + n + 1 - i)$$

$$= \sum_{i=1}^{k} (n + 1) = k(n + 1) = \left(\frac{n}{2}\right)(n + 1)$$

$$= \frac{n(n + 1)}{2}$$

- So, you need only one multiplication and then cut it into half to add numbers in range 1 to n
  - Also works for odd numbers and can be proven for odd numbers too!

Useful Closed Form Expressions

TABLE 2 **Some Useful Summation Formulae.**

Sum	*Closed Form*	
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$	Geometric sequence
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$	Gauss' trick
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$	Quadratic series
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$	Cubic series
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$	
$\sum_{k=1}^{\infty}, kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$	

Using Shortcuts for Evaluating Expressions

- Example: Evaluate $\sum_{k=50}^{100} k^2$

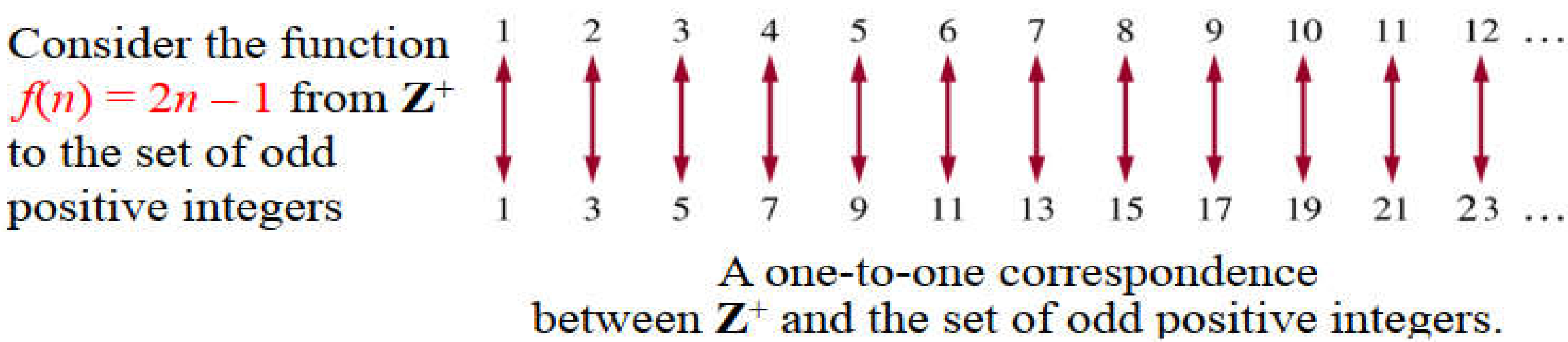
$$\sum_{k=1}^{100} k^2 = \sum_{k=1}^{49} k^2 + \sum_{k=50}^{100} k^2$$

$$\sum_{k=50}^{100} k^2 = \sum_{k=1}^{100} k^2 - \sum_{k=1}^{49} k^2$$

$$= \frac{100 \times 101 \times 201}{6} - \frac{49 \times 50 \times 99}{6} = 297{,}925$$

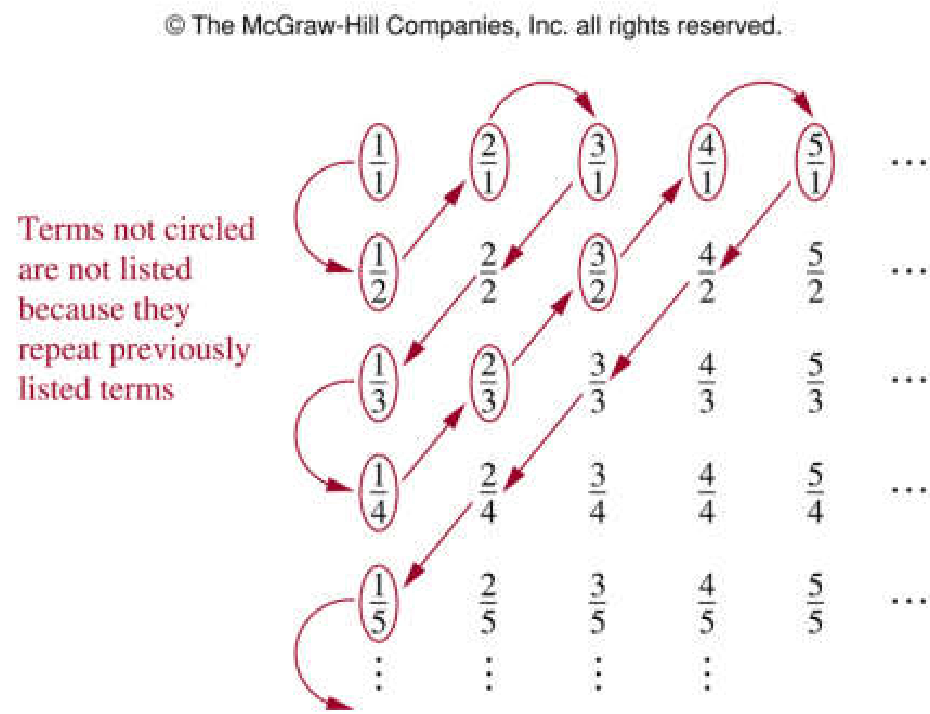
Cardinality

- The sets A and B have the same cardinality if and only if there is a one-to-one correspondence from A to B.
- A set that is either finite or has the same cardinality as the set of positive integers is called countable
- A set that is not countable is called uncountable
- Example: Show that the set of odd positive integers is a countable set.



Cardinality

- An infinite set S is countable iff it is possible to list the elements of the set in a sequence (indexed by the positive integers)
    - $a_1, a_2, \dots, a_n, \dots$ is one-to-one mapping $f\colon Z + \rightarrow\ S$ where $a_1 = f(1), a_2 = f\ (2), \dots, a_n = f(n), \dots$
- Example: Show that the set of positive rational numbers is countable



Summation Manipulations

- Useful Identities

$$\sum_{i=j}^{k} f(i) = \sum_{i=j}^{m} f(i) + \sum_{i=m+1}^{k} f(i) \quad if\ j \le m < k\ (sequence\ splitting)$$

$$\sum_{i=0}^{k} f(i) = \sum_{i=0}^{k} f(k-i) \quad (order\ reversal)$$

$$\sum_{i=1}^{2k} f(i) = \sum_{i=1}^{k} \big(f(2i-1) + f(2i)\big) \quad Grouping$$

Nested Summations

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}(\sum_{j=1}^{3} ij)$$
$$= \sum_{i=1}^{4} i + 2i + 3i = \sum_{i=1}^{4} 6i$$
$$= 6(1) + 6(2) + 6(3) + 6(4)$$
$$= 6 + 12 + 18 + 24 = 60$$

- Verify that

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{j=1}^{3}(\sum_{i=1}^{4} ij) \ = 60$$

Nested Summations

$$\sum_{S(x)} \sum_{T(y)} f(x, y) = \sum_{T(y)} \sum_{S(x)} f(x, y)$$

$$\sum_{i=k}^{l} \sum_{j=m}^{n} f(i, j) = \sum_{j=m}^{n} \sum_{i=k}^{l} f(i, j)$$

Summation’s Conclusion

- You should know
  - how to read, write and evaluate summation expressions like

$$\sum_{i=j}^{k} a_i \qquad \sum_{i=j}^{\infty} a_i \qquad \sum_{x \in X} f(x) \qquad \sum_{P(x)} f(x)$$

  - Summation manipulations laws, we covered
  - Closed-form formulas and how to use them