Any eq of the form $a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_n y = F(x)$ is called

$\rightarrow$ Cauchy-Euler Equation. put $x = e^t$, $t = \ln x$.

Ex 10.4.

Question ① $(x^2D^2 + 7xD + 5)y = x^5$ — ①.

Sol.

Let $x = e^t$, $t = \ln x$.

$xD = \Delta$

$x^2D^2 = \Delta(\Delta - 1)$

put in eq ①

$(\Delta(\Delta-1) + 7\Delta + 5)y = e^{5t}$

$(\Delta^2 - \Delta + 7\Delta + 5)y = e^{5t}$

$(\Delta^2 + 6\Delta + 5)y = e^{5t}$ ——— ②

The C.E is

$\Delta^2 + 6\Delta + 5 = 0$

$\Delta^2 + 5\Delta + \Delta + 5 = 0$

$\Delta(\Delta + 5) + 1(\Delta + 5) = 0$

$(\Delta + 1)(\Delta + 5) = 0$

$\Delta + 1 = 0 \Rightarrow \Delta = -1$

$\Delta + 5 = 0 \Rightarrow \Delta = -5$

$y_c = c_1 e^{-t} + c_2 e^{-5t}$

$y_p = \frac{1}{\Delta^2 + 6\Delta + 5} e^{5t}$

$= \frac{1}{(5)^2 + 6(5) + 5} e^{5t}$

$= \frac{1}{60} e^{5t}$

$$y = c_1 e^{-t} + c_2 e^{-5t} + \frac{e^{5t}}{60} \quad \text{--- (3)}$$

Replacing t by $\ln x$

$$y_1 = c_1 e^{-\ln x} + c_2 e^{-5\ln x} + \frac{e^{5\ln x}}{60}$$

$$= \cancel{c_1 e^{-x}} + \cancel{c_2 e^{-5x}} \; y \Rightarrow c_1 x^{-1} + c_2 x^{-5} + \frac{x^5}{60}$$

Question (2)

$$(x^2D^2 - 3xD + 5)y = x^2 \sin(\ln x) \quad \text{--- (1)}$$

Sol Let $x = e^t$, $t = \ln x$.

$$xD = \Delta$$

$$x^2D^2 = \Delta(\Delta - 1)$$

$$(\Delta(\Delta-1) - 3\Delta + 5)y = e^{2t}\sin t$$

$$(\Delta^2 - 4\Delta + 5)y = e^{2t}\sin t \quad \text{--- (2)}$$

The C.F is

$$\Delta^2 - 4\Delta + 5 = 0$$

$a = 1$, $b = -4$, $c = 5$

$$\Delta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{4 \pm \sqrt{16 - 20}}{2}$$

$$= \frac{4 \pm 2i}{2}$$

$$= 2 \pm i$$

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$$y = c_1 e^{-t} + c_2 e^{-5t} + \frac{e^{5t}}{60} \quad \text{--- (3)}$$

Replacing t by $\ln x$

$$y_1 = c_1 e^{-\ln x} + c_2 e^{-5\ln x} + \frac{e^{5\ln x}}{60}$$

$$= \cancel{c_1 e^{-x}} + \cancel{c_2 e^{-5x}} \; y \Rightarrow c_1 x^{-1} + c_2 x^{-5} + \frac{x^5}{60}$$

Question 2

$$(x^2D^2 - 3xD + 5)y = x^2 \sin(\ln x) \quad \text{--- (1)}$$

Sol Let $x = e^t$, $t = \ln x$.

$$xD = \Delta.$$

$$x^2D^2 = \Delta(\Delta - 1)$$

$$(\Delta(\Delta-1) - 3\Delta + 5)y = e^{2t} \sin t.$$

$$(\Delta^2 - 4\Delta + 5)y = e^{2t} \sin t \quad \text{--- (2)}$$

The C.F is

$$\Delta^2 - 4\Delta + 5 = 0$$

$\cancel{\Delta} \cancel{\Delta}$ $a = 1$, $b = -4$, $c = 5$

$$\Delta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{4 \pm \sqrt{16 - 20}}{2}$$

$$= \frac{4 \pm 2i}{2}$$

$$= 2 \pm i$$

$$y_c = e^{2t}(c_1\cos t + c_2\sin t)$$

$$y_p = \frac{e^{2t}\sin t}{D^2 - 4D + 5} \qquad e^{2t}\sin t$$

$$= e^{2t}\frac{\sin t}{(D+2)^2 - 4(D+2) + 5} \qquad \text{Sit by exponential shift}$$

$$= e^{2t}\frac{\sin t}{D^2 + \cancel{4D} + 4 - \cancel{4D} - 8 + 5}$$

$$= e^{2t}\frac{\sin t}{D^2 + 1}$$

$$= e^{2t}\, Im \frac{e^{it}}{(D+i)(D-i)}$$

$$= -\frac{e^{2t}}{2}\, Im\, it(\cos t + i\sin t)$$

$$= -\frac{t}{2}e^{2t}\cos t$$

G.S is

$$y = y_c + y_p$$

$$= e^{2t}(c_1\cos t + c_2\sin t) - \frac{t}{2}e^{2t}\cos t$$

$$\left[x^2D^2-(2m-1)xD+(m^2+n^2)\right]y = n^2x^m\cdot\ln x. \quad \text{--- (1)}$$

Sol:

$$x = e^t, \quad t = \ln x.$$

$$xD = \Delta$$

$$x^2D^2 = \Delta(\Delta-1) = \Delta^2-\Delta.$$

put in (1)

$$\left[\Delta^2-\Delta-(2m-1)\Delta+(m^2+n^2)\right] = n^2te^{mt}$$

$$\left[\Delta^2-2m\Delta+(m^2+n^2)\right]y = n^2te^{mt} \quad \text{--- (2)}$$

The C.F is

$$\Delta^2-2m\Delta+(m^2+n^2) = 0$$

$$\Delta = \frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

$$\Delta = \frac{2m\pm\sqrt{\cancel{4m^2}-\cancel{4m^2}-4n^2}}{2}.$$

$$= \frac{2m\pm 2ni}{2}.$$

$$= m\pm ni$$

$$y_c = (c_1\cos nt + c_2\sin nt)e^{mt}$$

$$y_p = \frac{n^2te^{mt}}{\Delta^2-2m\Delta+m^2+n^2}$$

$$y_p = \frac{1}{\Delta^2 - 2m\Delta + m^2 + n^2} n^2 t e^{mt}$$

By Exp shift:-

$$y_p = n^2 e^{mt} \frac{1}{(\Delta+m)^2 - 2m(\Delta+m) + m^2 + n^2} t$$

$$= n^2 e^{mt} \frac{1}{\Delta^2 + \cancel{2\Delta m} + \cancel{m^2} - \cancel{2m\Delta} - \cancel{2m^2} + \cancel{m^2} + n^2} t$$

$$= n^2 e^{mt} \frac{1}{\Delta^2 + n^2} t$$

$$= \frac{\cancel{n^2} e^{mt}}{\cancel{n^2}} \left[\frac{1}{\left(1 + \frac{\Delta^2}{n^2}\right)}\right] t$$

$$= e^{mt} \left[1 + \frac{\Delta^2}{n^2}\right]^{-1} t$$

$$= e^{mt} \left[1 - \frac{\Delta^2}{n^2}\right] t$$

$$= e^{mt} [t - 0] = t e^{mt}$$

$$y_p = t e^{mt}$$

$$y = y_c + y_p$$

Question (4) $(4x^2D^2 - 4xD + 3)y = \sin\ln(-x)$

Sol:

Let $-x = e^t$

$t = \ln(-x)$

$xD = \Delta$

$x^2D^2 = \Delta(\Delta-1) = \Delta^2 - \Delta$.

$(4(\Delta^2 - \Delta) - 4\Delta + 3)y = \sin t$.

$(4\Delta^2 - 4\Delta - 4\Delta + 3)y = \sin t$.

$(4\Delta^2 - 8\Delta + 3)y = \sin t$

The C. EQ is

$4\Delta^2 - 8\Delta + 3 = 0$

$4\Delta^2 - 6\Delta - 2\Delta + 3 = 0$

$2\Delta(2\Delta - 3) - 1(2\Delta - 3) = 0$

$(2\Delta - 1)(2\Delta - 3) = 0$

$2\Delta - 1 = 0$ | $2\Delta - 3 = 0$

$\Delta = 1/2$ | $\Delta = 3/2$

$y_c = c_1 e^{3t/2} + c_2 e^{t/2}$.

$$y_p = \frac{\sin t}{4\Delta^2 - 8\Delta + 3}.$$

$$= \frac{\operatorname{Im} e^{it}}{4\Delta^2 - 8\Delta + 3}.$$

$$= \operatorname{Im} \frac{e^{it}}{4(i^2) - 8i + 3}$$

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$$= Im \frac{e^{it}}{-4-8i+3}$$

$$= Im \frac{e^{it}}{-1-8i}$$

$$= Im \frac{e^{it}}{-(1+8i)}$$

$$= Im \frac{e^{it}(1-8i)}{-(1+8i)(1-8i)}$$

$$= Im \frac{e^{it}(1-8i)}{-(1+64)} \Rightarrow -\frac{1}{65} Im\, e^{it}(1-8i)$$

$$= -\frac{1}{65} Im (\cos t + i \sin t)(1-8i)$$

$$= -\frac{1}{65} Im (\cos t + i\sin t - 8i\cos t + 8\sin t)$$

$$y_p = -\frac{1}{65}(\sin t - 8\cos t)$$

The General Sol.

$$y = y_c + y_p$$

$$= c_1 e^{3t/2} + c_2 e^{t/2} - \frac{1}{65}(\sin t - 8\cos t)$$

Question 5

$$(x^3D^3 + 2x^2D^2 + 2)y = 10x + \frac{10}{x}.$$

Sol:-

Let $x = e^t$ $\quad t = \ln x.$

$xD = \Delta.$

$x^2D^2 = \Delta(\Delta-1) = \Delta^2 - \Delta.$

$x^3D^3 = \Delta(\Delta-1)(\Delta-2) = \Delta^3 - 3\Delta^2 + 2\Delta.$

$(\Delta^3 - 3\Delta^2 + 2\Delta + 2\Delta^2 - 2\Delta + 2)y = 10e^t + 10e^{-t}$

$(\Delta^3 - \Delta^2 + 2)y = 10e^t + 10e^{-t}$

The C.E is

$\Delta^3 - \Delta^2 + 2 = 0$

$\cancel{\Delta^3 - 2\Delta^2}$

$\Delta - 1 = 0 \Rightarrow \Delta = 1$

$$\begin{array}{r|rrrr} -1 & 1 & -1 & 0 & 2 \\ & & -1 & 2 & -2 \\ \hline & 1 & -2 & 2 & 0 \end{array}$$

$\Delta^2 - 2\Delta + 2 = 0$

$a = 1,\ b = -2,\ c = 2.$

$$\Delta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

$$= \frac{2 \pm \sqrt{4-8}}{2}.$$

$$= \frac{2 \pm 2i}{2}.$$

$$= 1 \pm i$$

$$D = -1, \quad 1 \pm i$$

$$y_c = c_1 e^{-t} + (c_2 \cos t + c_3 \sin t)\, e^{t}$$

$$y_p = \frac{1}{(D+1)(D^2-2D+2)} \, 10e^{t} + 10e^{-t}$$

$$= \frac{1}{(D+1)(D-2D+2)} \, 10e^{t} + \frac{1}{(D+1)(D^2-2D+2)} \, 10e^{-t}$$

$$= \frac{10e^{t}}{(1+1)(1-\cancel{2}+\cancel{2})} + \frac{1 \quad t}{\cancel{(D+1)}(+D^2+2D+2)} \, 10e^{-t}$$

$$(1-\cancel{2}+\cancel{2})$$

$$= 5e^{t} + \frac{10t\, e^{-t}}{(-1+2D+2)}$$

$$= 5e^{t} + \frac{5te^{-t}}{\cancel{(2D+1)}}$$

$$y = y_c + y_p$$

$$= c_1 e^{-t} + (c_2 \cos t + c_3 \sin t)\, e^{t} + 5e^{t} + 5te^{-t}$$

Question 6: $(x^4D^3 + 2x^3D^2 - x^2D + x)y = 1$

Sol:

$x(x^3D^3 + 2x^2D^2 - xD + 1)y = 1$

$(x^3D^3 + 2x^2D^2 - xD + 1)y = \frac{1}{x}$

$x = e^t \qquad t = \ln x$

$xD = \Delta$

$x^2D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta$

$x^3D^3 = \Delta(\Delta-1)(\Delta-2) = \Delta^3 - 3\Delta^2 + 2\Delta$

$(\Delta^3 - 3\Delta^2 + \cancel{2\Delta} + 2\Delta^2 - \cancel{2\Delta} - \Delta + 1)y = e^{-t}$

$(\Delta^3 - \Delta^2 - \Delta + 1)y = e^{-t}$

The C.F is

$\Delta^3 - \Delta^2 - \Delta + 1 = 0$

$D + 1 = 0 \Rightarrow D = -1$

-1	1	-1	-1	1
		-1	2	-1
	1	-2	1	0

$\Delta^2 - 2\Delta + 1 = 0$

$(\Delta - 1)^2 = 0$

$\Delta = 1, 1$

$y_c = c_1 e^{-t} + (c_2 + c_3 t)e^t$

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$$y_p = \frac{1}{(\Delta+1)(\Delta^2-2\Delta+1)} e^{-t}$$

$$= \frac{1}{(\Delta+1)(\Delta-1)^2} e^{-t}$$

$$= \frac{t\,e^{-t}}{1+2+1}$$

$$= \frac{t\,e^{-t}}{4}.$$

$$y = y_c + y_p.$$

$$y = c_1 e^{-t} + (c_2 + c_3 t) e^{t} + \frac{t\,e^{-t}}{4}.$$

$$= c_1 x^{-1} + x\,(c_2 + c_3 \ln x) + \frac{x^{-1}}{4} \ln x.$$

Question 7

$$(x^3 D^3 + 2x^2 D^2 - 5xD - 15)y = x^4 \quad \text{--- (1)}$$

Sol:-

$$x = e^t \quad , \quad t = \ln x.$$

$$xD = \Delta.$$

$$x^2 D^2 = \Delta(\Delta-1) = \Delta^2 - \Delta.$$

$$x^3 D^3 = \Delta^3 - 3\Delta^2 + 2\Delta.$$

put in (1)

$$(\Delta^3 - 3\Delta^2 + 2\Delta + 4\Delta^2 - 4\Delta - 5\Delta - 15)y = e^{4t}$$

$$(\Delta^3 + \Delta^2 - 7\Delta - 15)y = e^{4t}$$

The C.F is

$$\Delta^3 + \Delta^2 - 7\Delta - 15 = 0$$

$$\begin{array}{c|cccc} 3 & 1 & 1 & -7 & -15 \\ & & 3 & 12 & 15 \\ \hline & 1 & 4 & 5 & 0 \end{array}$$

$$\Delta - 3 = 0 \Rightarrow \Delta = 3$$

$$\Delta^2 + 4\Delta + 5 = 0$$

$\cancel{\Delta^2 + 5\Delta}$ $a = 1,\ b = 4,\ c = 5$

$$\Delta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-4 \pm \sqrt{16 - 20}}{2}.$$

$$= \frac{-4 \pm \sqrt{-4}}{2}.$$

$$= \frac{-4 \pm 2i}{2}. \Rightarrow -2 \pm i$$

$$y_c = c_1 e^{3t} + e^{-2t}(c_2 \cos t + c_3 \sin t$$

$$y_p = \frac{1}{\Delta^3 + \Delta^2 - 7\Delta - 15} e^{4t}$$

$$= \frac{1}{64 + 16 - 7\Delta - 15} e^{4t}$$

$$\begin{array}{r} 64 \\ 16 \\ \hline 80 \\ 15 \\ \hline 65 \\ 28 \\ \hline 37 \end{array}$$

$$= \frac{1}{\quad} e^{4t}$$

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$$y_p = \frac{e^{4t}}{37}$$

$$y = y_c + y_p.$$

$$y = c_1 e^{3t} + e^{-2t}(c_2 \cos t + c_3 \sin t) \frac{e^{4t}}{37}$$

Question..

$$(x^2D^2 + 2xD - 6)y = 10x^2 \qquad y(1) = 1$$
$$y'(1) = 6.$$

Sol.. $x = e^t$ or $t = \ln x.$

$$xD = \Delta.$$

$$x^2D^2 = \Delta(\Delta - 1) = \Delta^2 - \Delta.$$

$$(\Delta^2 - \Delta + 2\Delta - 6)y = 10e^{2t}.$$

$$(\Delta^2 + \Delta - 6)\, y = 10e^{2t}$$

The C.F is

$$\Delta^2 + \Delta - 6 = 0$$

$$\Delta^2 + 3\Delta - 2\Delta - 6 = 0$$

$$\Delta(\Delta + 3) - 2(\Delta + 3) = 0$$

$$(\Delta - 2)(\Delta + 3) = 0$$

$$\Delta - 2 = 0 \Rightarrow \Delta = 2.$$

$$\Delta + 3 = 0 \Rightarrow \Delta = -3.$$

$$y_c = c_1 e^{2t} + c_3 e^{-3t}$$

$$y_p = \frac{1}{\Delta^2+\Delta-6} 10 e^{2t}$$

$$= \frac{10}{(\Delta+3)(\Delta-2)} e^{2t}$$

$$= \frac{\cancel{10}\,2t}{(\cancel{5})} \cdot e^{2t}$$

$$= 2te^{2t}$$

$$y = y_c + y_p.$$

$$y = c_1 e^{2t} + c_3 e^{-3t} + 2te^{2t}$$

$$y = c_1 x^{-3} + c_2 x^2 + 2x^2 \ln x.$$

$$y' = -3c_1 x^{-4} + 2c_2 x + 4x \ln x + 2x.$$

$$y(1) = 1$$

$$c_1 + c_2 = 1 \quad \text{---} \quad (1)$$

$$y'(1) = -6$$

$$-3c_1 + 2c_2 + 2 = -6 \quad \text{---} \quad (2)$$

$$-3c_1 + 2c_2 = -8 \quad \text{---} \quad (2)$$

sol + get

$$c_2 = -1 \quad | \quad c_1 = 2$$

$$y = 2x^{-3} - x^2 + 2x^2 \ln x$$