Shahjalal University of Science & Technology, Sylhet

B.Sc.(Hons.) 1st Year 1st Semester Final Examination, 2011 (Session: 2010-11)

Subject: Mathematics; **Course No.: MAT-102B (for BMB)**; Course Title: Trigonometry, Vectors and Geometry

Full Marks: 70; Credits: 3.0; Total Time: 03 Hours

N.B. Answer **any five** questions from the following. *Marks distributions* are indicated in the right margin.

1. **a)** Represent graphically the set of values of the complex variable z for which $\left|\dfrac{z-3}{z+3}\right| = 2$. **4**

b) State De Moivre's theorem on complex numbers. Use it to solve the complex equation $z^4+z^3+z^2+z+1=0$. Also, represent its solutions in Argand diagram. **5**

c) Using De Moivre's theorem and Euler's formula, prove that **5**

$$\cos\theta+\cos(\theta+\alpha)+\cdots+\cos(\theta+n\alpha)=\frac{\sin\dfrac{(n+1)\alpha}{2}}{\sin\dfrac{\alpha}{2}}\,.\,\cos\left(\theta+\frac{n\alpha}{2}\right).$$

2. **a)** If $x+\dfrac{1}{x}=2\cos\theta$ and $y+\dfrac{1}{y}=2\cos\phi$, then prove that $\cos(\theta+\phi)$ is one of the values of $\dfrac{1}{2}\left(xy+\dfrac{1}{xy}\right)$. **4**

b) Evaluate the sum of the following series: **5 x 2**

(i) $\dfrac{\sin x}{\sin 2x\sin 3x}+\dfrac{\sin x}{\sin 3x\sin 4x}+\dfrac{\sin x}{\sin 4x\sin 5x}+\cdots$ (up to n-th term);

(ii) $x\sin\theta+x^2\sin 2\theta+x^3\sin 3\theta+\cdots$ (up to n-th term).

3. **a)** Define collinearity of vectors. Determine the scalar λ so that the vectors $2\hat{i}-\hat{j}+\hat{k}$, $\hat{i}+2\hat{j}-3\hat{k}$ and $4\hat{i}-\hat{j}+\lambda\hat{k}$ are collinear. **4**

b) Derive the formula $\underline{a}\times(\underline{b}\times\underline{c})=(\underline{a}\,.\,\underline{c})\,\underline{b}-(\underline{a}\,.\,\underline{b})\,\underline{c}$. Also, verify this formula for the vectors $\underline{a}=4\hat{i}-\hat{j}-3\hat{k}$, $\underline{b}=2\hat{i}+5\hat{j}$ and $\underline{c}=\hat{i}+3\hat{j}-2\hat{k}$. **6**

c) For any two vectors $\underline{a}$ and $\underline{b}$, prove that $(\underline{a}\times\underline{b})^2+(\underline{a}\,.\,\underline{b})^2=|\underline{a}|^2\,|\underline{b}|^2$. **4**

4. **a)** For any scalar function $\phi(x,y,z)$, find $\underline{\nabla}\phi$ if (i) $\phi=\ln|\underline{r}|$ and (ii) $\phi=\dfrac{1}{|\underline{r}|}$, where $\underline{r}=x\hat{i}+y\hat{j}+z\hat{k}$. **6**

b) For any vector function $\underline{A}(x,y,z)$, prove that the divergence of the curl of $\underline{A}$ is zero. **4**

c) Find the angle between the surface $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at the point $(2,-1,2)$ **4**

5. **a)** Find the directional derivative of $\varphi=4xz^3-3x^2y^2z$ at the point $(2,-1,2)$ in the direction $2\hat{i}-3\hat{j}+6\hat{k}$. **5**

c) Find the angle between the surface $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at the point $(2, -1, 2)$

5. **a)** Find the directional derivative of $\varphi = 4xz^3 - 3x^2y^2z$ at the point $(2, -1, 2)$ in the direction $2\hat{i} - 3\hat{j} + 6\hat{k}$. **5**

b) Determine the total work done in moving a particle in a force field given by $\underline{F} = 3xy\hat{i} - 5z\hat{j} + 10x\hat{k}$ along the curve $x = t^2 + 1$, $y = 2t^2$, $z = t^3$ from $t = 1$ to $t = 2$. **4**

c) If $\underline{F} = 2y\hat{i} - z\hat{j} + x\hat{k}$, then evaluate $\int_C \underline{F} \times d\underline{r}$ along the curve $x = \cos t$, $y = \sin t$, $z = 2\cos t$ from $t = 0$ to $t = \pi/2$. **5**

6. **a)** Prove that the homogeneous quadratic equation $ax^2 + 2hxy + by^2 = 0$ always represents a pair of straight lines (real or imaginary) through the origin. Also, find the angle between these lines. **5**

b) Show that the straight lines represented by the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ will be equidistant from the origin if $f^4 - g^4 = c\,(bf^2 - ag^2)$. **5**

c) The circles $x^2 + y^2 + 2x + 3y + 1 = 0$ and $x^2 + y^2 + 4x + 3y + 2 = 0$ intersect at the points A and B. Find the equation of the circle whose diameter is AB. **4**

7. **a)** Prove that the polar of any point of the circle $x^2 + y^2 - 2ax - 3a^2 = 0$ with respect to the circle $x^2 + y^2 + 2ax - 3a^2 = 0$ will touch the parabola $y^2 + 4ax = 0$. **5**

b) Show that the locus of poles of normal chords of an ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is $\dfrac{a^6}{x^2} + \dfrac{b^6}{y^2} = (a^2 - b^2)^2$. **5**

c) If the polars of the points (x_1, y_1) and (x_2, y_2) with respect to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ are at right angle, then show that $b^4x_1x_2 + a^4y_1y_2 = 0$. **4**

8. **a)** Find the shortest distance between the lines $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z-3}{4}$ and $\dfrac{x-2}{3} = \dfrac{y-3}{4} = \dfrac{z-4}{5}$. State whether the lines are coplanar or not. **6**

b) Show that the plane $lx + my + nz = 0$ touches the cone $ax^2 + by^2 + cz^2 = 0$ if $\dfrac{l^2}{a} + \dfrac{m^2}{b} + \dfrac{n^2}{c} = 0$. **4**

c) Find the equation to the sphere which passes through the point (α, β, γ) and the circle $z = 0$, $x^2 + y^2 = a^2$. **4**

THE END